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MODULAR TRANSFORMATIONS INVOLVING THE MORDELL INTEGRAL IN RAMANUJAN'S LOST NOTEBOOK

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For my teacher Bruce C. Berndt on his 75th birthday.

In his "lost notebook" (p. 202 of the 1988 edition), S. Ramanujan recorded modular transformations involving the Mordell integral, q-hypergeometric series, and generalized Lambert series. He gave no proofs; here we prove these formulas and use them to derive modular transformations of third-order mock theta functions. Mordell's formula, the properties of q-hypergeometric series and Appell–Lerch sums play central roles in the proofs.

1. Introduction

For a complex number q with $|q| < 1$, we define the notation

$$(a;q)_\infty := \prod_{m=0}^{\infty}(1-aq^m) \quad \text{and} \quad (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty} \text{ for any integer } n.$$

L. J. Mordell [1920; 1933] studied the integral

$$\int_{-\infty}^{\infty} \frac{e^{at^2+bt}}{e^{ct}+d}\,dt,$$

where $\Re(a) < 0$. This integral appeared in the work of L. Kronecker [1889a; 1889b] and B. Riemann (as described by C. L. Siegel [1932]). However, Mordell was the first to analyze its behavior relative to modular transformations, so we refer to it as the Mordell integral. In [Mordell 1920] he derived the formula

$$(1) \quad \int_{-\infty}^{\infty} \frac{e^{\pi i\tau t^2-2\pi xt}}{e^{2\pi t}-e^{2\pi i\theta}}\,dt = e^{-\pi i(\theta^2\tau+2\theta x+2\theta)} \frac{F[(x+\theta\tau)/\tau, -1/\tau] + i\tau F(x+\theta\tau,\tau)}{\tau\theta_{11}(x+\theta\tau,\tau)}.$$

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where, for $\Im(\tau) > 0$ and setting $q = e^{\pi i\tau}$,

$$iF(x,\tau) := \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+m+1/4} e^{(2m+1)\pi ix}}{1+q^{2m+1}},$$

$$i\theta_{11}(x,\tau) := \sum_{m=-\infty}^{\infty} (-1)^m q^{m^2+m+1/4} e^{(2m+1)\pi ix}.$$

To get (1), he mainly used functional equations satisfied by the functions $F(x,\tau)$ and $\theta_{11}(x,\tau)$.

S. Ramanujan studied definite integrals and recorded modular transformations involving the Mordell integral. In his lost notebook [1988, p. 9], he stated two modular transformations involving Mordell integrals and his tenth-order mock theta functions $\phi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}/(q;q^2)_{n+1}$ and $\psi(q) := \sum_{n=0}^{\infty} q^{(n+1)(n+2)/2}/(q;q^2)_{n+1}$:

$$\int_0^\infty \frac{e^{-\pi nx^2}}{\cosh\frac{2\pi x}{\sqrt{5}} + \frac{1+\sqrt{5}}{4}}\,dx + \frac{1}{\sqrt{n}} e^{\frac{\pi}{5n}} \psi(-e^{-\frac{\pi}{n}}) = \sqrt{\frac{5+\sqrt{5}}{2}} e^{-\frac{\pi n}{5}} \phi(-e^{-\pi n}) - \frac{\sqrt{5}+1}{2\sqrt{n}} e^{-\frac{\pi}{5n}} \phi(-e^{\frac{\pi}{n}}),$$

$$\int_0^\infty \frac{e^{-\pi nx^2}}{\cosh\frac{2\pi x}{\sqrt{5}} + \frac{1-\sqrt{5}}{4}}\,dx + \frac{1}{\sqrt{n}} e^{\frac{\pi}{5n}} \psi(-e^{-\frac{\pi}{n}}) = -\sqrt{\frac{5-\sqrt{5}}{2}} e^{\frac{\pi n}{5}} \phi(-e^{-\pi n}) + \frac{\sqrt{5}-1}{2\sqrt{n}} e^{-\frac{\pi}{5n}} \phi(-e^{\frac{\pi}{n}}).$$

In [Choi 2002], we proved these equations. In the lost notebook Ramanujan [1988, p. 202] also wrote (without proofs) two equations involving a Mordell integral, hypergeometric series and generalized Lambert series. Namely, for $q_1 = e^{-\frac{\pi}{3n}}$ and $q = e^{-3\pi n}$,

$$\frac{2}{\sqrt{3}} \int_0^\infty \frac{e^{-\frac{\pi nx^2}{3}} \cos \pi tx}{e^{\frac{2\pi x}{3}} + 1 + e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{1}{18}} \sum_{m=1}^{\infty} \frac{q^{\frac{(2m-1)^2}{6}}}{(-e^{\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m (-e^{-\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m} + \frac{e^{-\frac{3\pi t^2}{4n}} q_1^{\frac{1}{2}}}{\sqrt{n}} \sum_{m=1}^{\infty} \frac{q_1^{\frac{3}{2}(2m-1)^2}}{(-e^{\frac{\pi it}{n}} q_1^3; q_1^6)_m (-e^{-\frac{\pi it}{n}} q_1^3; q_1^6)_m}$$

$$= \frac{q^{-\frac{1}{36}}}{(q^{\frac{2}{3}}; q^{\frac{2}{3}})_\infty} \left\{ \sum_{m=1}^{\infty} (-1)^{m+1} q^{\frac{(2m-1)^2}{4}} \left(\frac{1}{1+e^{\pi t} q^{\frac{2m-1}{3}}} + \frac{1}{1+e^{-\pi t} q^{\frac{2m-1}{3}}} - 1 \right) + \frac{e^{-\frac{3\pi t^2}{4n}}}{n} \sum_{m=1}^{\infty} (-1)^{m+1} q_1^{\frac{9}{4}(2m-1)^2} \left(\frac{1}{1+e^{\frac{\pi it}{n}} q_1^{3(2m-1)}} + \frac{1}{1+e^{-\frac{\pi it}{n}} q_1^{3(2m-1)}} - 1 \right) \right\}.$$

We prove these equations in this paper. Proving these identities is equivalent to proving the following two theorems.

Theorem 1. *For a positive number n, set $q = e^{-3\pi n}$ and $q_1 = e^{-\frac{\pi}{3n}}$. For a number t such that $\Re(\frac{t}{n}) \pm \frac{2}{3} \notin \mathbb{Z}$ and $\Re(\frac{t}{n}) \pm \frac{4}{3} \notin \mathbb{Z}$, we have*

$$\frac{2}{\sqrt{3}} \int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}} \cos \pi t x}{e^{\frac{2\pi x}{3}} + 1 + e^{-\frac{2\pi x}{3}}}\, dx = q^{\frac{1}{18}} \sum_{m=1}^{\infty} \frac{q^{\frac{(2m-1)^2}{6}}}{(-e^{\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m (-e^{-\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m} + \frac{e^{-\frac{3\pi t^2}{4n}} q_1^{\frac{1}{2}}}{\sqrt{n}} \sum_{m=1}^{\infty} \frac{q_1^{\frac{3}{2}(2m-1)^2}}{(-e^{\frac{\pi i t}{n}} q_1^3; q_1^6)_m (-e^{-\frac{\pi i t}{n}} q_1^3; q_1^6)_m}.$$

Theorem 2. *For a positive number n, set $q = e^{-3\pi n}$ and $q_1 = e^{-\frac{\pi}{3n}}$. We have*

$$q^{\frac{1}{18}} \sum_{m=1}^{\infty} \frac{q^{\frac{(2m-1)^2}{6}}}{(-e^{\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m (-e^{-\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m} + \frac{e^{-\frac{3\pi t^2}{4n}} q_1^{\frac{1}{2}}}{\sqrt{n}} \sum_{m=1}^{\infty} \frac{q_1^{\frac{3}{2}(2m-1)^2}}{(-e^{\frac{\pi i t}{n}} q_1^3; q_1^6)_m (-e^{-\frac{\pi i t}{n}} q_1^3; q_1^6)_m}$$
$$= \frac{q^{-\frac{1}{36}}}{(q^{\frac{2}{3}}; q^{\frac{2}{3}})_\infty} \left\{ \sum_{m=1}^{\infty} (-1)^{m+1} q^{\frac{(2m-1)^2}{4}} \left(\frac{1}{1+e^{\pi t} q^{\frac{2m-1}{3}}} + \frac{1}{1+e^{-\pi t} q^{\frac{2m-1}{3}}} - 1 \right) + \frac{e^{-\frac{3\pi t^2}{4n}}}{n} \sum_{m=1}^{\infty} (-1)^{m+1} q_1^{\frac{9}{4}(2m-1)^2} \left(\frac{1}{1+e^{\frac{\pi i t}{n}} q_1^{3(2m-1)}} + \frac{1}{1+e^{-\frac{\pi i t}{n}} q_1^{3(2m-1)}} - 1 \right) \right\}.$$

G. E. Andrews [1981] also studied modular transformations consisting of the Mordell integral and the three functions

$$M_1(q) := \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+n}}{1+q^{2n}}, \quad M_2(q) := \sum_{n=-\infty}^{\infty} \frac{q^{2n^2-n}}{1+q^{2n-1}},$$
$$M_3(q) := \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+2n}}{1+q^{2n+1}}.$$

These functions are related to the classical theta functions $\vartheta_2(0,q)$ and $\vartheta_4(0,q)$, and the first two of them appear in Ramanujan's lost notebook.

In [Choi 2011], we made the definition

$$f(\alpha, z; q) := \sum_{m=0}^{\infty} \frac{q^{m^2-3m} \alpha^m z^{2m}}{(-z;q)_m (-\frac{\alpha z}{q}; q)_m}.$$

If we let $\alpha = z = q$, we see that $f(q,q;q)$ is one of Ramanujan's famous third-order mock theta functions, $f(q)$, from his letter [Berndt and Rankin 1995]. We can rewrite the right-hand side of the equation in Theorem 1 in terms of $f(\alpha, z; q)$, namely,

$$q^{\frac{2}{9}} f(e^{-2\pi t} q^{\frac{2}{3}}, e^{\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}}) + \frac{e^{-\frac{3\pi t^2}{4n}} q_1^2}{\sqrt{n}} f(e^{-\frac{2\pi i t}{n}} q_1^6, e^{\frac{\pi i t}{n}} q_1^3; q_1^6).$$

Ramanujan's equations involve the hypergeometric series

$$\sum_{m=1}^{\infty} \frac{q^{\frac{(2m-1)^2}{6}}}{(-e^{\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m (-e^{-\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m},$$

$$\sum_{m=1}^{\infty} \frac{q_1^{\frac{3}{2}(2m-1)^2}}{(-e^{\frac{\pi i t}{n}} q_1^3; q_1^6)_m (-e^{-\frac{\pi i t}{n}} q_1^3; q_1^6)_m}.$$

These are special cases of the function

(2) $$g_3(z, q) := \sum_{m=1}^{\infty} \frac{q^{m(m-1)}}{(z; q)_m (z^{-1} q; q)_m}.$$

Andrews and F. G. Garvan [1989] called attention to what they called the "mock theta conjectures", which roughly say that Ramanujan's fifth-order mock theta functions are not, in fact, theta functions. These were proved by D. Hickerson [1988]; though he did not use the function (2) in the proof, he remarked that he could express the conjectures in terms of it. Since then g_3 and a couple of other so-called *universal mock theta functions* have acquired a central role in the study of mock theta functions; see the survey by B. Gordon and R. McIntosh [2012] for discussion.

The function g_3 also satisfies certain modular transformations [Gordon and McIntosh 2012]. For $q = e^{-\alpha}$, $q_1 = e^{-\pi^2/\alpha}$, and

$$h_3(e^{2\pi i r}, q) := \frac{4 \sin^2 \pi r}{(q; q)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{\frac{m(3m+1)}{2}}}{(1 - e^{2\pi i r} q^m)(1 - e^{-2\pi i r} q^m)},$$

one of the modular transformations satisfied by g_3 is

$$q^{\frac{3}{2} r(1-r) - \frac{1}{24}} g_3(q^r, q) = \sqrt{\frac{\pi}{2\alpha}} \csc \pi r \; q_1^{-\frac{1}{6}} h_3(e^{2\pi i r}, q_1^4), \\ - \sqrt{\frac{3\alpha}{2\pi}} \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\cosh(3r-1)\alpha x + \cosh(3r-2)\alpha x}{\cosh \frac{3}{2}\alpha x} \, dx.$$

With the function $g_3(z, q)$, we can rewrite the right-hand side of Ramanujan's first equation (page 60) as

$$q^{\frac{2}{9}} g_3(-e^{\pi t} q^{\frac{1}{3}}, q^{\frac{2}{3}}) + \frac{e^{-\frac{3\pi t^2}{4n}} q_1^2}{\sqrt{n}} g_3(-e^{\frac{\pi i t}{n}} q_1^3; q_1^6).$$

Ramanujan listed four third-order mock theta functions $f(q)$, $\phi(q)$, $\psi(q)$, and $\chi(q)$ in his last letter to G. H. Hardy [Berndt and Rankin 1995]. G. N. Watson [1936] later added three further third-order mock theta functions $\omega(q)$, $\upsilon(q)$ and $\rho(q)$, and derived modular transformations for the seven third-order mock theta functions using Cauchy's theorem. One of the modular transformations is

$$q^{-\frac{1}{24}} f(q) = 2\sqrt{\frac{2\pi}{\alpha}} q_1^{\frac{4}{3}} \omega(q_1^2) + 4\sqrt{\frac{3\alpha}{2\pi}} \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\sinh \alpha x}{\sinh \frac{3}{2}\alpha x} \, dx$$

where $q = e^{-\alpha}$ and $q_1 = -\pi^2/\alpha$. Gordon and McIntosh [2003; 2012] introduced two more third-order mock theta functions $\xi(q)$ and $\rho(q)$ and their modular transformations.

In his thesis, S. Zwegers [2002] studied the normalized Appell–Lerch sum which is defined by

$$\mu(u, v; \tau) = \frac{1}{f(-e^{2\pi i v}, -e^{2\pi i \tau - 2\pi i v})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m e^{\pi i m(m+1)\tau + 2\pi i m v}}{1 - e^{2\pi i m \tau + 2\pi i u}}$$

where $u,\ v \notin \mathbb{Z}\tau + \mathbb{Z}$ and $\tau \in \mathcal{H}$. He showed the symmetry property, the elliptic transformation properties, and the modular transformation properties satisfied by the normalized Appell–Lerch sum. One of the modular transformation properties contains the Mordell integral, namely,

$$\left(\frac{\tau}{i}\right)^{-\frac{1}{2}} e^{\frac{\pi i (u-v)^2}{\tau}} \mu\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) = -\mu(u, v; \tau) + \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{\pi i x^2 \tau - 2\pi x(u-v)}}{\cosh \pi x} \, dx.$$

With these properties, Zwegers explained that $\mu(u, v; \tau)$ behaves nearly like a Jacobi form of weight $1/2$ in two variables.

Recently, B. Chern and R. C. Rhoades [2012] proved the modular transformation

$$\begin{aligned} \widetilde{R}(z;\tau) - \frac{e^{\frac{3\pi i z^2}{\tau}}}{\sqrt{i\tau}} \widetilde{R}\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) &= e^{-\frac{\pi i \tau}{3}} i \sin 2\pi z \int_{-\infty}^{\infty} e^{3\pi i \tau x^2 - 6\pi z x} \frac{\cosh 2\pi \tau x}{\cosh \pi x} \, dx \\ &\quad + e^{-\frac{\pi i \tau}{3}} \cos 2\pi z \int_{-\infty}^{\infty} e^{3\pi i \tau x^2 - 6\pi z x} \frac{\sinh 2\pi \tau x}{\cosh \pi x} \, dx \end{aligned}$$

where

$$\widetilde{R}(z;\tau) := \frac{i e^{\frac{\pi i \tau}{12}}}{2 \sin \pi z} \sum_{m=0}^{\infty} \frac{e^{2\pi i \tau m^2}}{(e^{2\pi i (z+\tau)}; e^{2\pi i \tau})_m (e^{-2\pi i (z-\tau)}; e^{2\pi i \tau})_m}.$$

They employed the results in Zwegers' thesis [2002] to prove this equation. By results in [Garvan 1988], we can rewrite $\widetilde{R}$ in terms of g_3:

$$\begin{aligned}\widetilde{R}(z;\tau)&\\ &= \frac{ie^{\frac{\pi i\tau}{12}}}{2\sin\pi z}(1-e^{2\pi iz})\left(1+e^{2\pi iz}\sum_{m=1}^{\infty}\frac{e^{2\pi i\tau m(m-1)}}{(e^{2\pi iz};e^{2\pi i\tau})_m(e^{-2\pi i(z-\tau)};e^{2\pi i\tau})_m}\right)\\ &= \frac{ie^{\frac{\pi i\tau}{12}}}{2\sin\pi z}(1-e^{2\pi iz})(1+e^{2\pi iz}g_3(e^{2\pi iz},e^{2\pi i\tau})).\end{aligned}$$

In their paper, Chern and Rhoades [2012] also discussed and proved two more identities involving the Mordell integral and partial theta functions. In this paper, Ramanujan's theta function $f(a,b)$ is used instead of the Jacobi theta functions. The definition of Ramanujan's theta functions is, for $|ab|<1$,

$$f(a,b) := \sum_{m=-\infty}^{\infty} a^{m(m+1)/2}b^{m(m-1)/2}.$$

By the Jacobi triple product identity, this equals $(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$.

In Section 2, we introduce Lemmas 1 and 2. The identities in these lemmas include generalized Lambert series which are the Appell–Lerch sums. The transformation for the Appell–Lerch sum in [Zwegers 2002] plays a central role in the proofs of Lemmas 1 and 2. In Section 3, we prove Theorem 1 twice with distinct methods. We first prove Theorem 1 by using Lemmas 1 and 2, Mordell's formula, the modular transformation for a theta function θ_{11}, and the evaluations of the contour integrals. Secondly, we prove Theorem 1 by proving

$$(3)\qquad \frac{1}{\sqrt{3}}\int_{-\infty}^{\infty}\frac{e^{\frac{\pi i\tau x^2}{3}-2\pi zx}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = g(z;\tau)+\frac{e^{\frac{3\pi iz^2}{\tau}}}{\sqrt{-i\tau}}g\left(-\frac{z}{\tau};-\frac{1}{\tau}\right)$$

where

$$g(z;\tau) := \frac{e^{\frac{2\pi i\tau}{3}}}{(e^{2\pi i\tau};e^{2\pi i\tau})_\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{3\pi i\tau m(m+1)}}{1+e^{2\pi iz+2\pi i\tau(m+\frac{1}{2})}}.$$

To prove the equation above, we discuss the elliptic transformation properties of $g(z;\tau)$, evaluate the contour integrals, and employ Liouville's theorem. In Section 4, we prove Theorem 2 by using Equation (6) and some results in the first proof of Theorem 1. In Section 5, with Theorem 1, we derive modular transformations for third-order mock theta functions which are similar to the modular transformations for tenth-order mock theta functions in the lost notebook [1988, p. 9].

2. Lemmas

To prove Theorems 1 and 2, we require the following lemmas.

Lemma 1. *For a complex number q with $|q| < 1$, we have*

$$\sum_{m=1}^{\infty} \frac{q^{\frac{2m(m-1)}{3}}}{(-tq^{\frac{1}{3}};q^{\frac{2}{3}})_m(-t^{-1}q^{\frac{1}{3}};q^{\frac{2}{3}})_m}$$
$$= \frac{tq^{\frac{2}{3}}}{f(-t^3q^{\frac{4}{3}},-t^{-3}q^{\frac{2}{3}})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+\frac{7}{3}m} t^{3m}}{1+q^{2m+1}}$$
$$+ \frac{t^{-1}q^{\frac{2}{3}}}{f(-t^{-3}q^{\frac{4}{3}},-t^{3}q^{\frac{2}{3}})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+\frac{7}{3}m} t^{-3m}}{1+q^{2m+1}} + \frac{(q^2;q^2)_\infty^3}{(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty f(t^3q,t^{-3}q)}$$
$$+ \frac{q^{\frac{1}{3}}(q^2;q^2)_\infty^3 f(q^{\frac{1}{3}},q^{\frac{5}{3}})}{(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty f(q,q) f(t^{-3}q,t^3q)} \left(\frac{t^{-2} f(-t^{-3}q^2,-t^3)}{f(-t^3q^{\frac{4}{3}},-t^{-3}q^{\frac{2}{3}})} + \frac{t^2 f(-t^3q^2,-t^{-3})}{f(-t^{-3}q^{\frac{4}{3}},-t^3q^{\frac{2}{3}})} \right).$$

Proof. Garvan [1988] showed that, for $|q| < |z| < |q|^{-1}$ and $z \neq 1$,

$$z^{-1}\left(-1 + \sum_{m=0}^{\infty} \frac{q^{m^2}}{(z;q)_{m+1}(q/z;q)_m}\right) = \frac{1}{(q;q)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{\frac{3m(m+1)}{2}}}{1-q^m z}. \tag{4}$$

Hickerson [1988, p. 649] remarked that

$$z^{-1}\left(-1 + \sum_{m=0}^{\infty} \frac{q^{m^2}}{(z;q)_{m+1}(q/z;q)_m}\right) = \sum_{m=1}^{\infty} \frac{q^{m(m-1)}}{(z;q)_m(z^{-1}q;q)_m}. \tag{5}$$

Combining the two results above, we have

$$\sum_{m=1}^{\infty} \frac{q^{m(m-1)}}{(z;q)_m(z^{-1}q;q)_m} = \frac{1}{(q;q)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{\frac{3m(m+1)}{2}}}{1-zq^m}. \tag{6}$$

We also see this equation in [Gordon and McIntosh 2012, p. 104]. Now, replacing q and z by $q^{2/3}$ and $-tq^{1/3}$, respectively, (6) becomes

$$\sum_{m=1}^{\infty} \frac{q^{\frac{2m(m-1)}{3}}}{(-tq^{\frac{1}{3}};q^{\frac{2}{3}})_m(-t^{-1}q^{\frac{1}{3}};q^{\frac{2}{3}})_m} = \frac{1}{(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)}}{1+q^{\frac{2}{3}m+\frac{1}{3}}t}. \tag{7}$$

In [Choi 2004, p. 378], the author showed that

$$\sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)}}{1-q^{2m}z} = \frac{(q^2;q^2)_\infty^2}{(z;q^2)_\infty(q^2/z;q^2)_\infty}, \tag{8}$$

which was also recorded by Ramanujan [1988, p. 59] in the lost notebook without proofs. Using (8) with z replaced by $-t^3q$, the Jacobi triple product identity and a straightforward calculation show that

$$\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m(m+1)}}{1+q^{\frac{2}{3}m+\frac{1}{3}}t}=\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m(m+1)}(1-q^{\frac{2}{3}m+\frac{1}{3}}t+q^{\frac{4}{3}m+\frac{2}{3}}t^2)}{1+q^{2m+1}t^3}$$
$$=\frac{(q^2;q^2)_\infty^3}{f(t^3q,t^{-3}q)}+\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m^2+\frac{7}{3}m+\frac{2}{3}}t^{-2}}{1+q^{2m+1}t^{-3}}$$
$$+\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m^2+\frac{7}{3}m+\frac{2}{3}}t^{2}}{1+q^{2m+1}t^{3}}. \tag{9}$$

The two sums on the right side of the equation above are Appell–Lerch sums. In his thesis, Zwegers [2002] showed that the normalized Appell–Lerch sum satisfies

$$\frac{z}{f(-hz,-\frac{q}{hz})}\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{\frac{m(m+1)}{2}}(hz)^m}{1-q^m tz}$$
$$-\frac{1}{f(-h,-\frac{q}{h})}\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{\frac{m(m+1)}{2}}}{h^m 1-q^m t}$$
$$=-\frac{(q;q)_\infty^3 f(-htz,-\frac{q}{htz})f(-z,-\frac{q}{z})}{f(-t,-\frac{q}{t})f(-h,-\frac{q}{h})f(-tz,-\frac{q}{tz})f(-hz,-\frac{q}{hz})} \tag{10}$$

where $q=e^{2\pi i\tau}$, $h=e^{2\pi i\nu}$, $t=e^{2\pi iu}$ and $z=e^{2\pi iz'}$, such that $\nu,\ u,\ z'\notin\mathbb{Z}$ and $u,\ \nu,\ u+z',\ \nu+z'\notin\mathbb{Z}\tau+\mathbb{Z}$. Hence, employing the Jacobi triple product identity, using (7) and (9), applying (10) with q, t, h, and z replaced by q^2, $-q$, $t^{-3}q^{4/3}$, and t^3, respectively, then again with q, t, h, and z replaced by q^2, $-q$, $t^3q^{4/3}$, and t^{-3}, respectively, and employing the fact that $f(q^{7/3},q^{-1/3})=q^{1/3}f(q^{1/3},q^{5/3})$, we obtain Lemma 1 after a slight rearrangement. □

Lemma 2. *Set $\omega=e^{\frac{2\pi i}{3}}$. For a complex number q with $|q|<1$, we have*

$$q^2\sum_{m=1}^{\infty}\frac{q^{6m(m-1)}}{(-tq^3;q^6)_m(-t^{-1}q^3;q^6)_m}$$
$$+\frac{i}{\sqrt{3}}\left\{\frac{1}{f(-\omega^2t^{-1}q^2,-\omega^{-2}t)}\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m^2+m}\omega^{2m}t^{-m}}{1+q^{2m+1}}\right.$$
$$\left.+\frac{1}{f(-\omega^2tq^2,-\omega^{-2}t^{-1})}\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m^2+m}\omega^{2m}t^{m}}{1+q^{2m+1}}-1\right\}$$
$$=-\frac{(q^2;q^2)_\infty^3}{3(q^6;q^6)_\infty f(tq,t^{-1}q)}$$
$$+\frac{i}{\sqrt{3}}\frac{(q^2;q^2)_\infty^4 f(-t,-t^{-1}q^2)}{(1-\omega^2)f(q,q)f(q,q^5)f(tq,t^{-1}q)}$$
$$\times\left(\frac{1}{f(-\omega^2t^{-1}q^2,-\omega t)}-\frac{t^{-1}}{f(-\omega^2tq^2,-\omega t^{-1})}\right). \tag{11}$$

Proof. We first consider the left side of (11). Employing the Jacobi triple product identity, we easily verify that

$$f(-\omega^2,-\omega q^2)=(1-\omega^2)(q^6;q^6)_\infty,\quad f(\omega^2 q,\omega q)=\frac{(q^2;q^2)(q^6;q^6)_\infty}{f(q,q^5)},$$
$$f(-\omega^2 t^{-1},-\omega t q^2)=-\omega^2 t^{-1} f(-\omega^2 t^{-1}q^2,-\omega t).$$

Then, using (10) with q, h, t, and z replaced by q^2, $\omega^2 t^{-1}$, $-q$, and t, respectively, applying the equations above, and rearranging terms, we obtain

$$\begin{aligned}(12)\quad &\frac{1}{f(-\omega^2 t^{-1}q^2,-\omega t)}\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m(m+1)}\omega^{2m}t^{-m}}{1+q^{2m+1}}\\ &=\frac{1}{(1-\omega)(q^6;q^6)_\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m(m+1)}\omega^{2m}}{1+q^{2m+1}t}\\ &\quad+\frac{(q^2;q^2)_\infty^4 f(-t,-t^{-1}q^2)}{(1-\omega^2)f(q,q)f(q,q^5)f(-\omega^2t^{-1}q^2,-\omega t)f(tq,t^{-1}q)}.\end{aligned}$$

A straightforward calculation and the Jacobi triple product identity lead us to

$$\begin{aligned}(13)\quad &\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m(m+1)}\omega^{2m}}{1+q^{2m+1}t^{-1}}\\ &=\sum_{m=-\infty}^{\infty}\frac{(-1)^{m+1} q^{m(m+1)}\omega^{m+1}(q^{2m+1}t+1-1)}{1+q^{2m+1}t}\\ &=\sum_{m=-\infty}^{\infty}(-1)^{m+1}q^{m(m+1)}\omega^{m+1}+\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m(m+1)}\omega^{m+1}}{1+q^{2m+1}t}a\\ &=(1-\omega)(q^6;q^6)_\infty+\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m(m+1)}\omega^{m+1}}{1+q^{2m+1}t}.\end{aligned}$$

Using (12), again using (12) with t replaced by t^{-1}, and applying (13), we obtain

$$\begin{aligned}(14)\quad &\frac{1}{f(-\omega^2 t^{-1}q^2,-\omega t)}\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m(m+1)}\omega^{2m}t^{-m}}{1+q^{2m+1}}\\ &\quad+\frac{1}{f(-\omega^2 tq^2,-\omega t^{-1})}\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m(m+1)}\omega^{2m}t^{m}}{1+q^{2m+1}}-1\\ &=\frac{1}{(1-\omega)(q^6;q^6)_\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{m(m+1)}(\omega^{2m}+\omega^{m+1})}{1+q^{2m+1}t}\\ &\quad+\frac{(q^2;q^2)_\infty^4 f(-t,-t^{-1}q^2)}{(1-\omega^2)f(q,q)f(q,q^5)f(tq,t^{-1}q)}\\ &\qquad\times\left(\frac{1}{f(-\omega^2t^{-1}q^2,-\omega t)}-\frac{t^{-1}}{f(-\omega^2 tq^2,-\omega t^{-1})}\right).\end{aligned}$$

We now consider

$$\sum_{m=1}^{\infty} \frac{q^{6m(m-1)+2}}{(-tq^3;q^6)_m(-t^{-1}q^3;q^6)_m} + \frac{i}{\sqrt{3}(1-\omega)(q^6;q^6)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)}(\omega^{2m}+\omega^{m+1})}{1+q^{2m+1}t}. \tag{15}$$

An elementary calculation shows that $(1+\omega)/(1-\omega) = i/\sqrt{3}$ and $\omega^2/(1-\omega) = -i/\sqrt{3}$. Using these, we find that

$$\begin{aligned} \frac{1}{1-\omega} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)}(\omega^{2m}+\omega^{m+1})}{1+q^{2m+1}t} &= \frac{i}{\sqrt{3}} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{9m^2+3m}}{1+q^{6m+1}t} + \frac{2i}{\sqrt{3}} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{9m^2+9m+2}}{1+q^{6m+3}t} \\ &\quad + \frac{i}{\sqrt{3}} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{9m^2+15m+6}}{1+q^{6m+5}t}. \end{aligned} \tag{16}$$

Therefore, applying (6) with q and z replaced by q^6 and $-tq^3$, respectively, (16), and (8) with z replaced by $-tq$, we deduce that (15) equals

$$\begin{aligned} &\frac{1}{(q^6;q^6)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{9m^2+9m+2}}{1+q^{6m+3}t} - \frac{1}{3(q^6;q^6)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{9m^2+3m}}{1+q^{6m+1}t} \\ &\qquad - \frac{2}{3(q^6;q^6)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{9m^2+9m+2}}{1+q^{6m+3}t} \\ &\qquad - \frac{1}{3(q^6;q^6)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{9m^2+15m+6}}{1+q^{6m+5}t} \\ &= -\frac{1}{3(q^6;q^6)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+m}}{1+q^{2m+1}t} \\ &= -\frac{1}{3(q^6;q^6)_\infty} \frac{(q^2;q^2)_\infty^3}{f(tq,t^{-1}q)}. \end{aligned} \tag{17}$$

In conclusion, by combining (14) and (17) we have derived Lemma 2. □

3. The proofs of the first identity

First proof of Theorem 1. By a simple calculation and integration by substitution, we obtain

$$
\begin{aligned}
(18)\quad & \int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cos \pi t x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx \\
&= \frac{\sqrt{3}}{4i} e^{-\frac{3\pi t^2}{4n}} \left\{ e^{\frac{\pi i}{3}} \int_{-\infty - i\frac{t}{2n}}^{\infty - i\frac{t}{2n}} \frac{e^{-3\pi n y^2}}{e^{2\pi(y+i\frac{t}{2n})}+e^{\frac{\pi i}{3}}}\,dy \right. \\
&\quad + e^{\frac{\pi i}{3}} \int_{-\infty + i\frac{t}{2n}}^{\infty + i\frac{t}{2n}} \frac{e^{-3\pi n y^2}}{e^{2\pi(y-i\frac{t}{2n})}+e^{\frac{\pi i}{3}}}\,dy \\
&\quad - e^{-\frac{\pi i}{3}} \int_{-\infty - i\frac{t}{2n}}^{\infty - i\frac{t}{2n}} \frac{e^{-3\pi n y^2}}{e^{2\pi(y+i\frac{t}{2n})}+e^{-\frac{\pi i}{3}}}\,dy \\
&\quad \left. - e^{-\frac{\pi i}{3}} \int_{-\infty + i\frac{t}{2n}}^{\infty + i\frac{t}{2n}} \frac{e^{-3\pi n y^2}}{e^{2\pi(y-i\frac{t}{2n})}+e^{-\frac{\pi i}{3}}}\,dy \right\}.
\end{aligned}
$$

For a sufficiently large positive number d, we consider the integral

$$
\int_\gamma \frac{e^{-3\pi n y^2}}{e^{2\pi(y+i\frac{t}{2n})}+e^{\frac{\pi i}{3}}}\,dy
$$

taken around the rectangle γ whose vertices are at the points $\pm d$ and $\pm d - i\frac{t}{2n}$. We easily verify that

$$
\frac{e^{-3\pi n y^2}}{e^{2\pi(y+i\frac{t}{2n})}+e^{\frac{\pi i}{3}}}
$$

has simple poles at $i(-\frac{t}{2n}+\frac{2}{3}+k)$ for integers k. Assume that $\Re(\frac{t}{n})>0$. Since $\Re(\frac{t}{2n}) \pm \frac{1}{3} \notin \mathbb{Z}$ and $\Re(\frac{t}{2n}) \pm \frac{2}{3} \notin \mathbb{Z}$, after some elementary manipulation and employing Cauchy's residue theorem, we find that

$$
\begin{aligned}
&\int_{-d-i\frac{t}{2n}}^{d-i\frac{t}{2n}} \frac{e^{-3\pi n y^2}}{e^{2\pi(y+i\frac{t}{2n})}+e^{\frac{\pi i}{3}}}\,dy \\
&\qquad = \sum_{0\le k<\Re\frac{t}{2n}-\frac{2}{3}} i e^{\frac{2\pi i}{3}+\frac{3\pi t^2}{4n}-2\pi t-3\pi t k} q^{-\frac{4}{9}-k^2-\frac{4}{3}k} \\
&\qquad\qquad + \left(\int_{-d}^{d} + \int_{d}^{d-i\frac{t}{2n}} + \int_{-d-i\frac{t}{2n}}^{-d} \right) \frac{e^{-3\pi n y^2}}{e^{2\pi(y+i\frac{t}{2n})}+e^{\frac{\pi i}{3}}}\,dy.
\end{aligned}
$$

Since

$$
\left| \left(\int_{d}^{d-i\frac{t}{2n}} + \int_{-d-i\frac{t}{2n}}^{-d} \right) \frac{e^{-3\pi n y^2}}{e^{2\pi(y+i\frac{t}{2n})}+e^{\frac{\pi i}{3}}}\,dy \right| \le 2\frac{e^{-3\pi n(d^2-(\frac{t}{2n})^2)}}{e^{2\pi d}-1}\frac{|t|}{n},
$$

we find that the sum of these integrals tends to 0 as d tends to ∞. Thus, letting $d \to \infty$ we verify that

$$\int_{-\infty-i\frac{t}{2n}}^{\infty-i\frac{t}{2n}} \frac{e^{-3\pi ny^2}}{e^{2\pi(y+i\frac{t}{2n})}+e^{\frac{\pi i}{3}}}\,dy$$
$$=\sum_{0\le k<\Re\frac{t}{2n}-\frac{2}{3}} ie^{\frac{2\pi i}{3}+\frac{3\pi t^2}{4n}-2\pi t-3\pi tk}q^{-\frac{4}{9}-k^2-\frac{4}{3}k}+\int_{-\infty}^{\infty}\frac{e^{-3\pi ny^2}}{e^{2\pi(y+i\frac{t}{2n})}+e^{\frac{\pi i}{3}}}\,dy.$$

We can also establish similar results for the other three integrals in (18) for $\Re(\frac{t}{n})>0$. We then apply these results to (18) and collect the sums to obtain

$$-\sum_{0\le k<\Re\frac{t}{2n}-\frac{2}{3}} e^{-2\pi t-3\pi tk}q^{-\frac{4}{9}-k^2-\frac{4}{3}k}-\sum_{-\Re\frac{t}{2n}-\frac{2}{3}<k\le -1} e^{2\pi t+3\pi tk}q^{-\frac{4}{9}-k^2-\frac{4}{3}k}$$
$$+\sum_{0\le k<\Re\frac{t}{2n}-\frac{1}{3}} e^{-\pi t-3\pi tk}q^{-\frac{1}{9}-k^2-\frac{2}{3}k}+\sum_{-\Re\frac{t}{2n}-\frac{1}{3}<k\le -1} e^{\pi t+3\pi tk}q^{-\frac{1}{9}-k^2-\frac{2}{3}k}.$$

Replacing k by $-k-1$ in the second and fourth sums above, we find that the four sums above cancel. Thus, for $\Re(\frac{t}{n})>0$,

$$\int_0^{\infty}\frac{e^{-\frac{\pi nx^2}{3}}\cos\pi tx}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx \tag{19}$$
$$=\frac{\sqrt{3}}{4i}e^{-\frac{3\pi t^2}{4n}}\left\{e^{\frac{\pi i}{3}}\int_{-\infty}^{\infty}\left(\frac{e^{-3\pi ny^2}}{e^{2\pi(y+i\frac{t}{2n})}+e^{\frac{\pi i}{3}}}+\frac{e^{-3\pi ny^2}}{e^{2\pi(y-i\frac{t}{2n})}+e^{\frac{\pi i}{3}}}\right)dy\right.$$
$$\left.-e^{-\frac{\pi i}{3}}\int_{-\infty}^{\infty}\left(\frac{e^{-3\pi ny^2}}{e^{2\pi(y+i\frac{t}{2n})}+e^{-\frac{\pi i}{3}}}+\frac{e^{-3\pi ny^2}}{e^{2\pi(y-i\frac{t}{2n})}+e^{-\frac{\pi i}{3}}}\right)dy\right\}.$$

For $\Re(\frac{t}{n})<0$, a similar process also brings us to (19). Also, for $\Re(\frac{t}{n})=0$, we directly derive (19) from (18). Therefore, for any positive number n and any number t such that $\Re(\frac{t}{n})\pm\frac{1}{3}\notin\mathbb{Z}$ and $\Re(\frac{t}{n})\pm\frac{2}{3}\notin\mathbb{Z}$, we obtain (19).

Next we must evaluate the integrals on the right side of (19). We need the modular transformation formula for θ_{11}

$$\theta_{11}\left(\frac{x}{\tau},-\frac{1}{\tau}\right)=-i\sqrt{-i\tau}\,e^{\pi ix^2/\tau}\theta_{11}(x,\tau). \tag{20}$$

Additionally, $F(x,\tau)$ and θ_{11} satisfy the transformation formulas

$$\theta_{11}(x,\tau)=-\theta_{11}(x+1,\tau)=-\theta_{11}(-x,\tau)=-e^{\pi i(2x+\tau)}\theta_{11}(x+\tau,\tau),$$
$$F(x,\tau)=-F(x+1,\tau)=-F(x+\tau,\tau)+\theta_{11}(x,\tau)=-F(-x+\tau,\tau)$$
$$=F(-x,\tau)+\theta_{11}(x,\tau).$$

We employ these formulas to evaluate the four integrals on the right-hand side of (19). Recall that $q_1=e^{-\frac{\pi}{3n}}$ and $q=e^{-3\pi n}$. We first consider the first integral on the right side of (19). Replacing τ, x, and θ by $3in$, 0, and $\frac{2}{3}-\frac{t}{2n}$, respectively, in Mordell's formula (1), we find that

$$\int_{-\infty}^{\infty} \frac{e^{-3\pi n y^2}}{e^{2\pi(y+i\frac{t}{2n})}-e^{\frac{4\pi i}{3}}}\,dy = e^{-\frac{\pi i t}{n}} \int_{-\infty}^{\infty} \frac{e^{-3\pi n y^2}}{e^{2\pi y}-e^{2\pi i(\frac{2}{3}-\frac{t}{2n})}}\,dy$$
$$= e^{\frac{3\pi t^2}{4n}-2\pi t-\frac{4\pi i}{3}} q^{-\frac{4}{9}} \frac{F(\frac{2}{3}-\frac{t}{2n},-\frac{1}{3in})-3nF(2in-i\frac{3t}{2},3in)}{3in\theta_{11}(2in-i\frac{3t}{2},3in)}.$$

We are able to establish a similar result for each of the remaining three integrals. From (20), we deduce that

$$\theta_{11}\Big(2in-\tfrac{3}{2}it,3in\Big) = \frac{i}{\sqrt{3n}} e^{\frac{4}{3}\pi n-2\pi t+\frac{3\pi t^2}{4n}} \theta_{11}\Big(\frac{2}{3}-\frac{t}{2n},-\frac{1}{3in}\Big),$$
$$\theta_{11}\Big(in-\tfrac{3}{2}it,3in\Big) = \frac{i}{\sqrt{3n}} e^{\frac{1}{3}\pi n-\pi t+\frac{3\pi t^2}{4n}} \theta_{11}\Big(\frac{1}{3}-\frac{t}{2n},-\frac{1}{3in}\Big).$$

Using the evaluations of the four integrals, employing the above modular transformations for θ_{11} and the formulas satisfied by θ_{11} and F, simplifying terms, and employing the definitions of θ_{11} and F, we obtain

$$(21)\quad \frac{2}{\sqrt{3}} \int_0^{\infty} \frac{e^{-\frac{\pi n x^2}{3}}\cos \pi t x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx$$
$$= \frac{e^{\pi t} q^{\frac{8}{9}}}{f(-e^{3\pi t}q^{\frac{4}{3}},-e^{-3\pi t}q^{\frac{2}{3}})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+\frac{7}{3}m} e^{3m\pi t}}{1+q^{2m+1}}$$
$$+ \frac{e^{-\pi t} q^{\frac{8}{9}}}{f(-e^{-3\pi t}q^{\frac{4}{3}},-e^{3\pi t}q^{\frac{2}{3}})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+\frac{7}{3}m} e^{-3m\pi t}}{1+q^{2m+1}}$$
$$\times \Bigg\{ \frac{1}{f(-\omega^2 e^{-\frac{\pi i t}{n}} q_1^2, -\omega^{-2} e^{\frac{\pi i t}{n}})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2+m} \omega^{2m} e^{-\frac{m\pi i t}{n}}}{1+q_1^{2m+1}}$$
$$+ \frac{1}{f(-\omega^2 e^{\frac{\pi i t}{n}} q_1^2, -\omega^{-2} e^{-\frac{\pi i t}{n}})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2+m} \omega^{2m} e^{\frac{m\pi i t}{n}}}{1+q_1^{2m+1}} - 1 \Bigg\}.$$

We are now ready to complete the proof. Use Lemma 1 with t replaced by $e^{\pi t}$ and employ Lemma 2 with q and t replaced by q_1 and $e^{\frac{\pi i t}{n}}$, respectively. After some elementary manipulations, we find that the sum of the new left-hand sides of Lemma 1 and (11) equals

(22)

$$
\begin{aligned}
&q^{\frac{2}{9}} \sum_{m=1}^{\infty} \frac{q^{\frac{2m(m-1)}{3}}}{(-e^{\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m (-e^{-\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m} \\
&\quad + \frac{e^{-\frac{3\pi t^2}{4n}} q_1^2}{\sqrt{n}} \sum_{m=1}^{\infty} \frac{q_1^{6m(m-1)}}{(-e^{i\frac{\pi t}{n}} q_1^3; q_1^6)_m (-e^{-i\frac{\pi t}{n}} q_1^3; q_1^6)_m} \\
&\quad - \frac{e^{\pi t} q^{\frac{8}{9}}}{f(-e^{3\pi t} q^{\frac{4}{3}}, -e^{-3\pi t} q^{\frac{2}{3}})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+\frac{7}{3}m} e^{3m\pi t}}{1+q^{2m+1}} \\
&\quad - \frac{e^{-\pi t} q^{\frac{8}{9}}}{f(-e^{-3\pi t} q^{\frac{4}{3}}, -e^{3\pi t} q^{\frac{2}{3}})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+\frac{7}{3}m} e^{-3m\pi t}}{1+q^{2m+1}} \\
&\quad + i \frac{e^{-\frac{3\pi t^2}{4n}}}{\sqrt{3n}} \left\{ \frac{1}{f(-\omega^2 e^{-\frac{\pi i t}{n}} q_1^2, -\omega^{-2} e^{\frac{\pi i t}{n}})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2+m} \omega^{2m} e^{-\frac{m\pi i t}{n}}}{1+q_1^{2m+1}} \right. \\
&\qquad\qquad \left. + \frac{1}{f(-\omega^2 e^{\frac{\pi i t}{n}} q_1^2, -\omega^{-2} e^{-\frac{\pi i t}{n}})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2+m} \omega^{2m} e^{\frac{m\pi i t}{n}}}{1+q_1^{2m+1}} - 1 \right\}
\end{aligned}
$$

and the sum of the new right-hand sides of Lemma 1 and (11) equals

(23)

$$
\begin{aligned}
&q^{\frac{2}{9}} \left\{ \frac{(q^2;q^2)_\infty^3}{(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty f(e^{3\pi t} q, e^{-3\pi t} q)} + \frac{q^{\frac{1}{3}} (q^2;q^2)_\infty^3 f(q^{\frac{1}{3}}, q^{\frac{5}{3}}) f(-e^{-3\pi t} q^2, -e^{3\pi t})}{(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty f(q,q) f(e^{-3\pi t} q, e^{3\pi t} q)} \right. \\
&\qquad\qquad \left. \times \left(\frac{e^{-2\pi t}}{f(-e^{3\pi t} q^{\frac{4}{3}}, -e^{-3\pi t} q^{\frac{2}{3}})} - \frac{e^{-\pi t}}{f(-e^{-3\pi t} q^{\frac{4}{3}}, -e^{3\pi t} q^{\frac{2}{3}})} \right) \right\} \\
&\quad + \frac{e^{-\frac{3\pi t^2}{4n}}}{\sqrt{n}} \left\{ - \frac{(q_1^2;q_1^2)_\infty^3}{3(q_1^6;q_1^6)_\infty f(e^{\frac{\pi i t}{n}} q_1, e^{-\frac{\pi i t}{n}} q_1)} \right. \\
&\qquad + \frac{i}{\sqrt{3}} \frac{(q_1^2;q_1^2)_\infty^4 f(-e^{\frac{\pi i t}{n}}, -e^{-\frac{\pi i t}{n}} q_1^2)}{(1-\omega^2) f(q_1,q_1) f(q_1,q_1^5) f(e^{\frac{\pi i t}{n}} q_1, e^{-\frac{\pi i t}{n}} q_1)} \\
&\qquad\qquad \left. \times \left(\frac{1}{f(-\omega^2 e^{-\frac{\pi i t}{n}} q_1^2, -\omega e^{\frac{\pi i t}{n}})} - \frac{e^{-\frac{\pi i t}{n}}}{f(-\omega^2 e^{\frac{\pi i t}{n}} q_1^2, -\omega e^{-\frac{\pi i t}{n}})} \right) \right\}.
\end{aligned}
$$

Next we prove that (23) is identically equal to zero. Using the definition of θ_{11}, the Jacobi triple product identity, and the transformation formula (20) for θ_{11}, we deduce the following formula for Ramanujan's theta function f:

$$
f(-e^{2\pi i(x+\tau)}, -e^{-2\pi i x}) = \frac{i}{\sqrt{-i\tau}} e^{-\pi i (x + \frac{\tau}{4} + \frac{x^2-x}{\tau} + \frac{1}{4\tau})} f(-e^{2\pi i \frac{x-1}{\tau}}, -e^{-2\pi i \frac{x}{\tau}}).
$$

Set $\tau = 3in$, and recall that $q_1 = e^{-\frac{\pi}{3n}}$ and $q = e^{-3\pi n}$ to obtain

(24)
$$f(-e^{2\pi ix}q^2, -e^{-2\pi ix}) = \frac{i}{\sqrt{3n}} q^{-\frac{1}{4}} e^{-\pi ix - \frac{\pi}{3n}(x-\frac{1}{2})^2} f(-e^{\frac{2\pi x}{3n}} q_1^2, -e^{-\frac{2\pi x}{3n}}).$$

Since $\lim_{x\to 0}(1-e^{-\frac{2\pi}{3n}x})/(1-e^{-2\pi ix}) = -i/(3n)$, dividing both sides of (24) by $1-e^{-2\pi ix}$ and tending x to 0 leads us to find that

$$(q^2;q^2)_\infty^3 = \frac{1}{3n\sqrt{3n}} q^{-\frac{1}{4}} q_1^{\frac{1}{4}} (q_1^2;q_1^2)_\infty^3. \tag{25}$$

Applying (24) twice with $x = \frac{1}{2} - \frac{3}{2}in - \frac{3}{2}it$ and $x = -in$, respectively, (25), and the fact that $i\sqrt{3}(1-\omega)e^{-\frac{\pi i}{3}} = 3$, and employing the Jacobi triple product identity, we obtain

$$\frac{q^{\frac{2}{9}}(q^2;q^2)_\infty^3}{(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty f(e^{3\pi t}q, e^{-3\pi t}q)} = \frac{e^{-\frac{3\pi t^2}{4n}}}{\sqrt{n}} \frac{(q_1^2;q_1^2)_\infty^3}{3(q_1^6;q_1^6)_\infty f(e^{\frac{\pi it}{n}}q_1, e^{-\frac{\pi it}{n}}q_1)}. \tag{26}$$

Applying (24) with x replaced by $\frac{1}{2}-\frac{1}{2}in$, $\frac{3it}{2}$, $\frac{1}{2}-\frac{3}{2}in$, $-\frac{3}{2}it-in$, and $\frac{3}{2}it-in$, respectively, and employing the Jacobi triple product,we obtain

(27)
$$f(q^{\frac{5}{3}}, q^{\frac{1}{3}}) = \frac{1}{\sqrt{3n}} q^{-\frac{1}{9}} f(-e^{-\frac{\pi i}{3}}q_1, -e^{\frac{\pi i}{3}}q_1) = \frac{1}{\sqrt{3n}} q^{-\frac{1}{9}} \frac{(q_1^2;q_1^2)_\infty (q_1^6;q_1^6)_\infty}{f(q_1,q_1^5)},$$

$$f(-e^{-3\pi t}q^2, -e^{3\pi t}) = -\frac{iq^{-\frac{1}{4}}q_1^{\frac{1}{4}}e^{\frac{3\pi t}{2}+\frac{3\pi t^2}{4n}-\frac{\pi it}{2n}}}{\sqrt{3n}} f(-e^{\frac{\pi it}{n}}, -e^{-\frac{\pi it}{n}}q_1^2), \tag{28}$$

$$f(q,q) = \frac{1}{\sqrt{3n}} f(q_1,q_1), \tag{29}$$

$$\begin{aligned} f(-e^{3\pi t}q^{\frac{4}{3}}, -e^{-3\pi t}q^{\frac{2}{3}}) &\\ = \frac{iq^{-\frac{1}{36}}q_1^{\frac{1}{4}}e^{-\frac{\pi i}{3}-\frac{\pi t}{2}+\frac{3\pi t^2}{4n}-\frac{\pi it}{2n}}}{\sqrt{3n}} &f(-e^{-\frac{\pi it}{n}}\omega^2 q_1^2, -e^{\frac{\pi it}{n}}\omega), \end{aligned} \tag{30}$$

$$\begin{aligned} f(-e^{-3\pi t}q^{\frac{4}{3}}, -e^{3\pi t}q^{\frac{2}{3}}) &\\ = \frac{iq^{-\frac{1}{36}}q_1^{\frac{1}{4}}e^{-\frac{\pi i}{3}+\frac{\pi t}{2}+\frac{3\pi t^2}{4n}+\frac{\pi it}{2n}}}{\sqrt{3n}} &f(-e^{\frac{\pi it}{n}}\omega^2 q_1^2, -e^{-\frac{\pi it}{n}}\omega). \end{aligned} \tag{31}$$

Employing (26)–(31) and using the fact that $e^{\frac{\pi i}{3}} = \sqrt{3}i/(1-\omega^2)$, we conclude that

$$
\begin{aligned}
(32)\quad & \frac{q^{\frac{5}{9}}(q^2;q^2)_\infty^3 f(q^{\frac{1}{3}},q^{\frac{5}{3}}) f(-e^{-3\pi t}q^2,-e^{3\pi t})}{(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty f(q,q) f(e^{-3\pi t}q,e^{3\pi t}q)} \\
&\qquad \times \left(\frac{e^{-2\pi t}}{f(-e^{3\pi t}q^{\frac{4}{3}},-e^{-3\pi t}q^{\frac{2}{3}})} - \frac{e^{-\pi t}}{f(-e^{-3\pi t}q^{\frac{4}{3}},-e^{3\pi t}q^{\frac{2}{3}})}\right) \\
&= -\frac{e^{-\frac{3\pi t^2}{4n}}}{\sqrt{n}} \frac{i}{\sqrt{3}} \frac{(q_1^2;q_1^2)_\infty^4 f(-e^{\frac{\pi i t}{n}},-e^{-\frac{\pi i t}{n}}q_1^2)}{(1-\omega^2) f(q_1,q_1) f(q_1,q_1^5) f(e^{\frac{\pi i t}{n}}q_1,e^{-\frac{\pi i t}{n}}q_1)} \\
&\qquad \times \left(\frac{1}{f(-\omega^2 e^{-\frac{\pi i t}{n}}q_1^2,-\omega e^{\frac{\pi i t}{n}})} - \frac{e^{-\frac{\pi i t}{n}}}{f(-\omega^2 e^{\frac{\pi i t}{n}}q_1^2,-\omega e^{-\frac{\pi i t}{n}})}\right).
\end{aligned}
$$

As a result, combining (26) and (32), we know that (23) equals 0. Thus, (22) equals 0. Therefore, comparing (21) and (22), we have proved Theorem 1. □

Second proof of Theorem 1. We now consider the equation

$$
(33)\qquad \frac{1}{\sqrt{3}}\int_{-\infty}^{\infty} \frac{e^{\frac{\pi i \tau x^2}{3}-2\pi z x}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = g(z;\tau) + \frac{e^{\frac{3\pi i z^2}{\tau}}}{\sqrt{-i\tau}} g\left(-\frac{z}{\tau};-\frac{1}{\tau}\right)
$$

where

$$
g(z;\tau) := \frac{e^{\frac{2\pi i \tau}{3}}}{(e^{2\pi i \tau};e^{2\pi i \tau})_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m e^{3\pi i \tau m(m+1)}}{1+e^{2\pi i z + 2\pi i \tau (m+\frac{1}{2})}}.
$$

Comparing the definitions of g and g_3, we find that

$$
g(z;\tau) = e^{\frac{2\pi i \tau}{3}} g_3(e^{2\pi i z + \pi i(\tau+1)}, e^{2\pi i \tau}).
$$

We now set $\tau = in$, $q = e^{-2\pi n}$, and $q_1 = e^{-\frac{2\pi}{n}}$. Using (6) with z replaced by $e^{2\pi i z}$, we get

$$
\begin{aligned}
g(z;in) &= \frac{q^{\frac{1}{3}}}{(q;q)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{\frac{3m(m+1)}{2}}}{1+e^{2\pi i z} q^{(m+\frac{1}{2})}} \\
&= q^{\frac{1}{3}} \sum_{m=1}^{\infty} \frac{q^{m(m-1)}}{(-e^{2\pi i z}q^{\frac{1}{2}};q)_m (-e^{-2\pi i z}q^{\frac{1}{2}};q)_m}.
\end{aligned}
$$

Similarly, we obtain

$$
g\left(\frac{iz}{n};\frac{i}{n}\right) = q_1^{\frac{1}{3}} \sum_{m=1}^{\infty} \frac{q_1^{m(m-1)}}{(-e^{-\frac{2\pi z}{n}}q_1^{\frac{1}{2}};q_1)_m (-e^{\frac{2\pi z}{n}}q_1^{\frac{1}{2}};q_1)_m}.
$$

Applying these results to (33), we easily verify that proving (33) is equivalent to proving the equation in Theorem 1. So, we prove (33) instead of Theorem 1.

We first discuss the right-hand side of (33). From the definition of $g(z;\tau)$, we see that $g(z;\tau)$ is a meromorphic function of z with simple poles in $(\frac{1}{2}+\mathbb{Z})\tau+\frac{1}{2}+\mathbb{Z}$. By a direct calculation, we can determine that its residue at $-\frac{1}{2}\tau-\frac{1}{2}$ is $-q^{1/3}/(2\pi i(q;q)_\infty)$. We will find two functional equations for the function $g(z;\tau)$. By the definition of $g(z;\tau)$, we easily get

$$g(z+1;\tau)=g(z;\tau). \tag{34}$$

Using the definition of $g(z;\tau)$ and the Jacobi triple product identity, we obtain

$$\begin{aligned}
g(z+\tau;\tau)&=\frac{e^{\frac{5\pi i\tau}{3}+2\pi iz}}{(e^{2\pi i\tau};e^{2\pi i\tau})_\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{\pi i\tau(3m^2-m)}}{1+e^{2\pi iz+2\pi i\tau(m+\frac{1}{2})}}\\
&=e^{\frac{5\pi i\tau}{3}+2\pi iz}-\frac{e^{\frac{8\pi i\tau}{3}+4\pi iz}}{(e^{2\pi i\tau};e^{2\pi i\tau})_\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{\pi i\tau(3m^2+m)}}{1+e^{2\pi iz+2\pi i\tau(m+\frac{1}{2})}}\\
&=e^{\frac{5\pi i\tau}{3}+2\pi iz}-e^{\frac{8\pi i\tau}{3}+4\pi iz}+e^{3\pi i\tau+6\pi iz}g(z;\tau).
\end{aligned}$$

In particular,

$$g(z+\tau;\tau)-e^{3\pi i\tau+6\pi iz}g(z;\tau)=e^{\frac{5\pi i\tau}{3}+2\pi iz}-e^{\frac{8\pi i\tau}{3}+4\pi iz}. \tag{35}$$

Let $G(z;\tau)$ denote the right-hand side of (33). Then, using the functional equations (34) and (35), we get

$$\begin{aligned}
G(z+1;\tau)&=g(z+1;\tau)+\frac{e^{\frac{3\pi i(z+1)^2}{\tau}}}{\sqrt{-i\tau}}g\Big(-\frac{z+1}{\tau};-\frac{1}{\tau}\Big)\\
&=g(z;\tau)+\frac{e^{\frac{3\pi i(z+1)^2}{\tau}}}{\sqrt{-i\tau}}\Big\{e^{-\frac{5\pi i}{3\tau}-\frac{2\pi iz}{\tau}}-e^{-\frac{8\pi i}{3\tau}-\frac{4\pi iz}{\tau}}\\
&\qquad+e^{-\frac{3\pi i}{\tau}-\frac{6\pi iz}{\tau}}g\Big(-\frac{z}{\tau};-\frac{1}{\tau}\Big)\Big\}.
\end{aligned}$$

Thus,

$$G(z+1;\tau)-G(z;\tau)=\frac{e^{\frac{3\pi i(z+1)^2}{\tau}}}{\sqrt{-i\tau}}\Big(e^{-\frac{5\pi i}{3\tau}-\frac{2\pi iz}{\tau}}-e^{-\frac{8\pi i}{3\tau}-\frac{4\pi iz}{\tau}}\Big). \tag{36}$$

Again, using the functional equations (34) and (35), we obtain

$$\begin{aligned}
G(z+\tau;\tau)&=g(z+\tau;\tau)+\frac{e^{\frac{3\pi i(z+\tau)^2}{\tau}}}{\sqrt{-i\tau}}g\Big(-\frac{z+\tau}{\tau};-\frac{1}{\tau}\Big)\\
&=e^{\frac{5\pi i\tau}{3}+2\pi iz}-e^{\frac{8\pi i\tau}{3}+4\pi iz}+e^{3\pi i\tau+6\pi iz}g(z;\tau)\\
&\qquad+\frac{e^{\frac{3\pi iz^2}{\tau}+6\pi iz+3\pi i\tau}}{\sqrt{-i\tau}}g\Big(-\frac{z}{\tau};-\frac{1}{\tau}\Big).
\end{aligned}$$

So,

$$G(z+\tau;\tau)-e^{3\pi i\tau+6\pi iz}G(z;\tau)=e^{\frac{5\pi i\tau}{3}+2\pi iz}-e^{\frac{8\pi i\tau}{3}+4\pi iz}. \tag{37}$$

Therefore, $G(z;\tau)$ satisfies the functional equations (36) and (37). Recall that the residue of the function $g(z;\tau)$ at $-\frac{1}{2}\tau-\frac{1}{2}$ is $-q^{1/3}/(2\pi i(q;q)_\infty)$. A simple calculation shows that the residue of the function $\left(e^{3\pi iz^2/\tau}/\sqrt{-i\tau}\right)g\left(-\frac{z}{\tau};-\frac{1}{\tau}\right)$ at $-\frac{1}{2}\tau-\frac{1}{2}$ is $q^{1/3}/(2\pi i(q;q)_\infty)$. Using these results and the two functional equations satisfied by $G(z;\tau)$, we easily verify that $G(z;\tau)$ is a holomorphic function of z.

Now we discuss the left-hand side of (33). Let $H(z;\tau)$ denote the left-hand side of (33). Then, by the definition of $H(z;\tau)$, we get

$$\begin{aligned}H(z+1;\tau)-H(z;\tau)&=\frac{1}{\sqrt{3}}\int_{-\infty}^{\infty}\frac{e^{\frac{\pi i\tau x^2}{3}-2\pi zx}(e^{-2\pi x}-1)}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx\\&=\frac{e^{\frac{\pi i(3z+2)^2}{3\tau}}}{\sqrt{3}}\int_{-\infty}^{\infty}e^{\frac{\pi i\tau}{3}\{x+\frac{i}{\tau}(3z+2)\}^2}\,dx\\&\qquad-\frac{e^{\frac{\pi i(3z+1)^2}{3\tau}}}{\sqrt{3}}\int_{-\infty}^{\infty}e^{\frac{\pi i\tau}{3}\{x+\frac{i}{\tau}(3z+1)\}^2}\,dx.\end{aligned}$$

Recall that $\tau=in$. If z is a real number then we easily show that each of two integrals equals $\sqrt{3i/\tau}$. Assume that z is a complex number such that $\Im(z)\neq 0$. We consider the first integral on the right-hand side of the equation above.

$$\int_{-\infty}^{\infty}e^{\frac{\pi i\tau}{3}\{x+\frac{i}{\tau}(3z+2)\}^2}\,dx=\int_{-\infty+\frac{i}{\tau}(3z+2)}^{\infty+\frac{i}{\tau}(3z+2)}e^{\frac{\pi i\tau}{3}x^2}\,dx.$$

For a positive number t, we consider the integral

$$\int_\gamma e^{\frac{\pi i\tau}{3}x^2}\,dx$$

taken around the rectangle γ whose vertices are at the points $\pm t$ and $\pm t+\frac{i}{\tau}(3z+2)$. By Cauchy's residue theorem, we easily get that the integral above equals 0. We first evaluate

$$\int_{-t}^{-t+\frac{i}{\tau}(3z+2)}e^{\frac{\pi i\tau}{3}x^2}\,dx.$$

Let $z=a+bi$ where a and b are real and $b\neq 0$. We only need to consider three cases: $(3a+2)/n=0$, $(3a+2)/n>0$, and $(3a+2)/n<0$. If $(3a+2)/n=0$, then

$$\left|\int_{-t}^{-t+\frac{i}{\tau}(3z+2)}e^{\frac{\pi i\tau}{3}x^2}\,dx\right|\leq\frac{3b}{n}e^{-\frac{\pi n}{3}t^2+\frac{3\pi b^2}{n}}.$$

Thus, $\int_{-t}^{-t+\frac{i}{\tau}(3z+2)} e^{\frac{\pi i \tau}{3}x^2}\,dx$ tends to 0 as t tends to ∞. If $(3a+2)/n$ is positive (or negative) then there is a real number c such that $-t < c < -t + (3a+2)/n$ (or $-t + (3a+2)/n < c < -t$) and

$$\left|\int_{-t}^{-t+\frac{i}{\tau}(3z+2)} e^{\frac{\pi i \tau}{3}x^2}\,dx\right| \le \frac{\sqrt{(3a+2)^2+9b^2}}{n} e^{-\frac{\pi n}{3}c^2+\frac{3\pi b^2}{n}}.$$

Thus, the integral $\int_{-t}^{-t+\frac{i}{\tau}(3z+2)} e^{\frac{\pi i \tau}{3}x^2}\,dx$ tends to 0 as t tends to ∞. Similarly, $\int_{t}^{t+\frac{i}{\tau}(3z+2)} e^{\frac{\pi i \tau}{3}x^2}\,dx$ tends to 0 as t tends to ∞. Therefore, we see that

$$\int_{-\infty+\frac{i}{\tau}(3z+2)}^{\infty+\frac{i}{\tau}(3z+2)} e^{\frac{\pi i \tau}{3}x^2}\,dx = \int_{-\infty}^{\infty} e^{\frac{\pi i \tau}{3}x^2}\,dx = \sqrt{\frac{3i}{\tau}}.$$

After a simple calculation, we obtain

$$H(z+1;\tau) - H(z;\tau) = \frac{e^{\frac{3\pi i (z+1)^2}{\tau}}}{\sqrt{-i\tau}}\left(e^{-\frac{5\pi i}{3\tau}-\frac{2\pi i z}{\tau}} - e^{-\frac{8\pi i}{3\tau}-\frac{4\pi i z}{\tau}}\right).$$

Next, we discuss $e^{-3\pi i \tau - 6\pi i z}H(z+\tau;\tau) - H(z;\tau)$. After simple calculations and integration by substitution, we get

$$\begin{aligned} &e^{-3\pi i \tau - 6\pi i z}H(z+\tau;\tau) - H(z;\tau) \\ &\quad= \frac{1}{\sqrt{3}}\int_{-\infty+3i}^{\infty+3i} \frac{e^{\frac{\pi i \tau x^2}{3}-2\pi z x}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx - \frac{1}{\sqrt{3}}\int_{-\infty}^{\infty} \frac{e^{\frac{\pi i \tau x^2}{3}-2\pi z x}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx. \end{aligned}$$

For a positive number s, we consider the integral

$$\int_{\delta} \frac{e^{\frac{\pi i \tau x^2}{3}-2\pi z x}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx$$

taken around the rectangle δ whose vertices are at the points $\pm s$ and $\pm s+3i$. By Cauchy's residue theorem, after some elementary algebra, we find that the integral above equals $\sqrt{3}e^{-\frac{\pi i \tau}{3}-2\pi i z}(1-e^{-\pi i \tau - 2\pi i z})$. We first evaluate

$$\int_{s}^{s+3i} \frac{e^{\frac{\pi i \tau x^2}{3}-2\pi z x}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx.$$

We again recall that $\tau = in$. Since, for any y such that $0 < y < 3$,

$$\left|\frac{e^{\frac{\pi i \tau (s+yi)^2}{3}-2\pi z(s+yi)}}{e^{\frac{2\pi(s+yi)}{3}}+1+e^{-\frac{2\pi(s+yi)}{3}}}\right|$$

tends to 0 as s tends to ∞, we easily find that the integral above tends to 0 as s tends to ∞. Similarly, we deduce that

$$\int_{-s}^{-s+3i} \frac{e^{\frac{\pi i \tau x^2}{3} - 2\pi z x}}{e^{\frac{2\pi x}{3}} + 1 + e^{-\frac{2\pi x}{3}}}\, dx$$

tends to 0 as s tends to ∞. Therefore, we obtain

$$e^{-3\pi i\tau - 6\pi i z} H(z+\tau;\tau) - H(z;\tau) = e^{-\frac{\pi i \tau}{3} - 2\pi i z}(1 - e^{-\pi i \tau - 2\pi i z}).$$

After some elementary algebra and elementary manipulation, we find that $H(z;\tau)$ also satisfies the same functional equations as $G(z;\tau)$ and is a holomorphic function of z.

Let $F(z;\tau) := H(z;\tau) - G(z;\tau)$. Then, by the functional equations satisfied by H and G, we obtain

$$F(z+1;\tau) = F(z;\tau) \quad \text{and} \quad F(z+\tau;\tau) = e^{3\pi i \tau + 6\pi i z} F(z;\tau).$$

Let T be a set of complex numbers such that for any $t \in T$, $0 \le \Re(t) \le 1$ and $0 \le \Im(t) \le n$. Since T is a compact set, $F(z;\tau)$ is bounded on T. For any $t' \in \mathbb{C} \setminus T$, there are two integers k and l and a complex number t such that $t \in T$ and $t' = t + k\tau + l$. Thus, using repeatedly the functional equations satisfied by $F(z;\tau)$,

$$F(t') = F(t + k\tau + l) = F(t + k\tau) = e^{-3\pi n k^2 + 6\pi i t k} F(t).$$

Hence,

$$|F(t')| = e^{-3\pi n k^2 - 6\pi k \Im t} |F(t)| \le e^{-3\pi n\{(|k|-1)^2 - 1\}} |F(t)|. \tag{38}$$

So, we are able to say that F is bounded on $\mathbb{C}$. Therefore, by Liouville's theorem, $F(z;\tau)$ is a constant. In (38), F tends to 0 as k tends to ∞. This implies that $F(z;\tau) = 0$. Finally, we have proved (33). □

4. The proof of the second identity

Proof of Theorem 2. Employing (6) with a moderate modification, we derive

(39)
$$\sum_{m=1}^{\infty} \frac{q^{\frac{2m(m-1)}{3}}}{(-e^{\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m (-e^{-\pi t} q^{\frac{1}{3}}; q^{\frac{2}{3}})_m} = \frac{1}{(q^{\frac{2}{3}}; q^{\frac{2}{3}})_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)}}{1 + q^{\frac{2}{3}m + \frac{1}{3}} e^{\pi t}},$$

(40)
$$\sum_{m=1}^{\infty} \frac{q_1^{6m(m-1)}}{(-e^{\frac{\pi i t}{n}} q_1^3; q_1^6)_m (-e^{-\frac{\pi i t}{n}} q_1^3; q_1^6)_m} = \frac{1}{(q_1^6; q_1^6)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{9m^2+9m}}{1 + q_1^{6m+3} e^{\frac{\pi i t}{n}}}.$$

By a straightforward calculation, we easily find that
(41)

$$\begin{aligned}\sum_{m=-\infty}^{\infty} &\frac{(-1)^m q^{m(m+1)}}{1+q^{\frac{2}{3}m+\frac{1}{3}}e^{\pi t}} \\ &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} q^{m(m-1)}}{1+q^{\frac{2}{3}m-\frac{1}{3}}e^{\pi t}} + \sum_{m=1}^{\infty} \frac{(-1)^m q^{m(m-1)}}{1+q^{-\frac{2}{3}m+\frac{1}{3}}e^{\pi t}} \\ &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} q^{m(m-1)}}{1+q^{\frac{2}{3}m-\frac{1}{3}}e^{\pi t}} + \sum_{m=1}^{\infty} \frac{(-1)^m q^{m(m-1)}(q^{\frac{2}{3}m-\frac{1}{3}}e^{-\pi t}+1-1)}{1+q^{\frac{2}{3}m-\frac{1}{3}}e^{-\pi t}} \\ &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} q^{m(m-1)}}{1+q^{\frac{2}{3}m-\frac{1}{3}}e^{\pi t}} + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} q^{m(m-1)}}{1+q^{\frac{2}{3}m-\frac{1}{3}}e^{-\pi t}} - \sum_{m=1}^{\infty} (-1)^{m+1} q^{m(m-1)}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}(42)\quad \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{9m^2+9m}}{1+q_1^{6m+3}e^{\frac{\pi i t}{n}}} &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} q_1^{9m^2-9m}}{1+q_1^{6m-3}e^{\frac{\pi i t}{n}}} \\ &\quad + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} q_1^{9m^2-9m}}{1+q_1^{6m-3}e^{-\frac{\pi i t}{n}}} - \sum_{m=1}^{\infty} (-1)^{m+1} q_1^{9m^2-9m}.\end{aligned}$$

We previously derived that

$$(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty = \frac{i(1-\omega)}{\sqrt{3n}} q^{-\frac{1}{36}} q_1^{\frac{1}{4}} e^{-\frac{\pi i}{3}} (q_1^6;q_1^6)_\infty \quad \text{and} \quad i(1-\omega)e^{-\frac{\pi i}{3}} = \sqrt{3}.$$

Thus, using these, we obtain

$$(43)\qquad \frac{q^{-\frac{1}{36}}}{(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty} \frac{q_1^{\frac{9}{4}}}{n} = \frac{q_1^2}{\sqrt{n}\,(q_1^6;q_1^6)_\infty}.$$

Combining equations (39)–(43) completes the proof. □

5. Modular transformations derived from Ramanujan's identity

In this section, we derive modular transformations for third-order mock theta functions from Theorem 1.

Ramanujan's third-order mock theta functions are defined by

$$f(q) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(-q;q)_m^2}, \quad \phi(q) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(-q^2;q^2)_m}, \quad \psi(q) = \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q;q^2)_m},$$

$$\chi(q) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(-\omega q;q)_m(-\omega^2;q)_m}.$$

Watson's third-order mock theta functions are defined by

$$\omega(q)=\sum_{m=1}^{\infty}\frac{q^{2m(m-1)}}{(q;q^2)_m^2},\quad \upsilon(q)=\sum_{m=1}^{\infty}\frac{q^{m(m-1)}}{(-q;q^2)_m},$$
$$\rho(q)=\sum_{m=1}^{\infty}\frac{q^{2m(m-1)}}{(\omega q;q^2)_m(\omega^2 q;q^2)_m}.$$

Gordon and McIntosh's third-order mock theta functions are defined by

$$\xi(q)=1+2\sum_{m=1}^{\infty}\frac{q^{6m(m-1)}}{(q;q^6)_m(q^5;q^6)_m},\quad \sigma(q)=\sum_{m=1}^{\infty}\frac{q^{3m(m-1)}}{(-q;q^3)_m(-q^2;q^3)_m}.$$

To apply Theorem 1 directly to these functions, we first need new representations for Ramanujan's mock theta functions. With his formula for basic hypergeometric series, Watson [1936] gave new representations for $\phi(q)$ and $\psi(q)$, namely,

$$\phi(q)=\frac{1}{(q;q)_\infty}\left(1+2\sum_{m=1}^{\infty}\frac{(-1)^m(1+q^m)q^{m(3m+1)/2}}{1+q^{2m}}\right), \tag{44}$$

$$\psi(q)=\frac{1}{(q^4;q^4)_\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{6m(m+1)+1}}{1-q^{4m+1}}. \tag{45}$$

Then, using the definition of $f(q)$ and applying (5) with z replaced by -1, we deduce that

$$f(q)=2-2\sum_{m=1}^{\infty}\frac{q^{m(m-1)}}{(-1;q)_m(-q;q)_m}.$$

Using the definition of $\phi(q)$ and applying (5) with z replaced by i and $-i$, respectively, we obtain

$$\phi(q)=(1-i)\left(1+i\sum_{m=1}^{\infty}\frac{q^{m(m-1)}}{(i;q)_m(-iq;q)_m}\right)$$
$$=(1+i)\left(1-i\sum_{m=1}^{\infty}\frac{q^{m(m-1)}}{(-i;q)_m(iq;q)_m}\right).$$

Using (45) and applying (6) with q and z replaced by q^4 and q, respectively, we deduce that

$$\psi(q)=q\sum_{m=1}^{\infty}\frac{q^{4m(m-1)}}{(q;q^4)_m(q^3;q^4)_m}.$$

Using the definition of $\chi(q)$ and applying (5) with z replaced by $-\omega$ and $-\omega^2$, respectively, we have

$$\chi(q) = (1+\omega)\left(1-\omega\sum_{m=1}^{\infty}\frac{q^{m(m-1)}}{(-\omega;q)_m(-\omega^2q;q)_m}\right)$$
$$= (1+\omega^2)\left(1-\omega^2\sum_{m=1}^{\infty}\frac{q^{m(m-1)}}{(-\omega^2;q)_m(-\omega q;q)_m}\right).$$

We are now ready to derive modular transformations from Theorem 1. We record here the ones which are derived directly from Theorem 1 and expressed in terms of Mordell integrals and third-order mock theta functions. Similar modular transformations can be found in [Gordon and McIntosh 2012].

Using Theorem 1 with t replaced by $n-\frac{i}{2}$ and $n+\frac{i}{2}$, respectively, we obtain

$$\frac{2}{\sqrt{3}}\int_0^{\infty}\frac{e^{-\frac{\pi n x^2}{3}}\cos\pi(n-\frac{i}{2})x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}\left(\frac{\phi(q^{\frac{2}{3}})}{1+i}+i\right)-\frac{\sqrt{2}q^{\frac{1}{4}}q_1^{-\frac{1}{16}}}{(1+i)\sqrt{n}}\psi(q_1^{\frac{3}{2}}),$$
$$\frac{2}{\sqrt{3}}\int_0^{\infty}\frac{e^{-\frac{\pi n x^2}{3}}\cos\pi(n+\frac{i}{2})x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}\left(\frac{\phi(q^{\frac{2}{3}})}{1-i}-i\right)-\frac{\sqrt{2}q^{\frac{1}{4}}q_1^{-\frac{1}{16}}}{(1-i)\sqrt{n}}\psi(q_1^{\frac{3}{2}}).$$

Adding the two results above and calculating straightforwardly, we have

$$\text{(46)}\quad \frac{4}{\sqrt{3}}\int_0^{\infty}\frac{e^{-\frac{\pi n x^2}{3}}\cos\pi n x\cosh\frac{\pi x}{2}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}\phi(q^{\frac{2}{3}})-\sqrt{\frac{2}{n}}q^{\frac{1}{4}}q_1^{-\frac{1}{16}}\psi(q_1^{\frac{3}{2}}).$$

Using Theorem 1 with t replaced by $\frac{n}{2}-i$ and $-\frac{n}{2}-i$, respectively, we obtain

$$\frac{2}{\sqrt{3}}\int_0^{\infty}\frac{e^{-\frac{\pi n x^2}{3}}\cos\pi(\frac{n}{2}-i)x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx$$
$$= q^{\frac{1}{18}}\psi(q^{\frac{1}{6}})-\frac{q^{\frac{1}{16}}q_1^{-\frac{1}{4}}}{\sqrt{2n}}\phi(q_1^6)+\frac{q^{\frac{1}{16}}q_1^{-\frac{1}{4}}e^{\pi i/4}}{\sqrt{n}},$$
$$\frac{2}{\sqrt{3}}\int_0^{\infty}\frac{e^{-\frac{\pi n x^2}{3}}\cos\pi(\frac{n}{2}+i)x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx$$
$$= q^{\frac{1}{18}}\psi(q^{\frac{1}{6}})-\frac{q^{\frac{1}{16}}q_1^{-\frac{1}{4}}}{\sqrt{2n}}\phi(q_1^6)+\frac{q^{\frac{1}{16}}q_1^{-\frac{1}{4}}e^{-\pi i/4}}{\sqrt{n}}.$$

Adding the two results above and calculating straightforwardly, we find that

$$\text{(47)}\quad \frac{2}{\sqrt{3}}\int_0^{\infty}\frac{e^{-\frac{\pi n x^2}{3}}\cos\frac{\pi n x}{2}\cosh\pi x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{1}{18}}\psi(q^{\frac{1}{6}})-\frac{q^{\frac{1}{16}}q_1^{-\frac{1}{4}}}{\sqrt{2n}}(\phi(q_1^6)-1).$$

Using Theorem 1 with t replaced by $n+\frac{2}{3}i$ and $-n+\frac{2}{3}i$, respectively, we obtain

$$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cos\pi(n+\frac{2}{3}i)x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}(\omega^2+\chi(q^{\frac{1}{3}})) - \frac{q^{\frac{1}{4}}q_1}{2\sqrt{n}}(\xi(q_1)-1),$$
$$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cos\pi(n-\frac{2}{3}i)x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}(\omega+\chi(q^{\frac{1}{3}})) - \frac{q^{\frac{1}{4}}q_1}{2\sqrt{n}}(\xi(q_1)-1).$$

Adding the two results above and calculating straightforwardly, we find that

$$\frac{4}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cos\pi n x\cosh\frac{2\pi x}{3}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}(2\chi(q^{\frac{1}{3}})-1) - \frac{q^{\frac{1}{4}}q_1}{\sqrt{n}}(\xi(q_1)-1).$$

Using Theorem 1 with t replaced by n, $\frac{n}{2}$, i, 0, $\frac{i}{2}$, $-\frac{i}{3}$, $\frac{2i}{3}$, and $\frac{n}{3}$, respectively, we obtain,

$$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cos\pi n x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}\big(1-\tfrac{1}{2}f(q)\big) + \frac{q^{\frac{1}{4}}q_1^2}{\sqrt{n}}\omega(q_1^3),$$

(48) $$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cos\frac{\pi n x}{2}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = -q^{\frac{1}{18}}\psi(-q^{\frac{1}{6}}) + \frac{q^{\frac{1}{16}}q_1^2}{\sqrt{n}}\upsilon(q_1^6)$$

$$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cosh\pi x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}\omega(q^{\frac{1}{3}}) + \frac{q_1^{-\frac{1}{4}}}{\sqrt{n}}\big(1-\tfrac{1}{2}f(q_1^6)\big),$$

(49) $$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}\omega(-q^{\frac{1}{3}}) + \frac{q_1^2}{\sqrt{n}}\omega(-q_1^3),$$

(50) $$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cosh\frac{\pi x}{2}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}\upsilon(q^{\frac{2}{3}}) - \frac{q_1^{-\frac{1}{16}}}{\sqrt{n}}\psi(-q_1^{\frac{3}{2}}),$$

$$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cosh\frac{\pi x}{3}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}\rho(q^{\frac{1}{3}}) + \frac{q_1^{\frac{7}{4}}}{\sqrt{n}}\sigma(q_1^2),$$

$$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cosh\frac{2\pi x}{3}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}\rho(-q^{\frac{1}{3}}) + \frac{q_1}{2\sqrt{n}}(\xi(-q_1)-1),$$

$$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cos\frac{\pi n x}{3}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = q^{\frac{2}{9}}\sigma(q^{\frac{2}{9}}) + \frac{q^{\frac{1}{36}}q_1^2}{\sqrt{n}}\rho(q_1^3).$$

Here, using (46)–(50), we give evaluations for specific Mordell integrals and new representations for Ramanujan's third-order mock theta functions ϕ, ψ, and ω.

Replacing n by $\frac{1}{2}$ in (46), we obtain

$$\frac{4}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi x^2}{6}}\cos\frac{\pi x}{2}\cosh\frac{\pi x}{2}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = e^{-\frac{\pi}{3}}\{\phi(e^{-\pi})-2q\psi(e^{-\pi})\}. \tag{51}$$

Replacing n by 2 in (47) and multiplying 2 to both sides of (47), we find

$$\frac{4}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{2\pi x^2}{3}}\cos\pi x\cosh\pi x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = e^{-\frac{\pi}{3}}\{2q\psi(e^{-\pi})-\phi(e^{-\pi})+1\}. \tag{52}$$

Adding (51) and (52), we obtain

$$\frac{4}{\sqrt{3}}\left(\int_0^\infty \frac{e^{-\frac{\pi x^2}{6}}\cos\frac{\pi x}{2}\cosh\frac{\pi x}{2}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx+\int_0^\infty \frac{e^{-\frac{2\pi x^2}{3}}\cos\pi x\cosh\pi x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx\right)=e^{-\frac{\pi}{3}}.$$

Replacing n by 2 in (48) and replacing n by $\frac{1}{2}$ in (50), we obtain

$$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{2\pi x^2}{3}}\cos\pi x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = -e^{-\frac{\pi}{3}}\psi(-e^{-\pi})+\frac{e^{-\frac{17}{24}\pi}}{\sqrt{2}}\upsilon(e^{-\pi}) \tag{53}$$

$$\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi x^2}{6}}\cosh\frac{\pi x}{2}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = e^{-\frac{\pi}{3}}\upsilon(e^{-\pi})-\sqrt{2}e^{\frac{\pi}{24}}\psi(-e^{-\pi}). \tag{54}$$

Comparing (53) and (54), we have

$$\int_0^\infty \frac{e^{-\frac{2\pi x^2}{3}}\cos\pi x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx = \frac{e^{-\frac{3}{8}\pi}}{\sqrt{2}}\int_0^\infty \frac{e^{-\frac{\pi x^2}{6}}\cosh\frac{\pi x}{2}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx.$$

Ramanujan [1988] recorded

$$\phi(q)+2\psi(q)=\frac{(-q;-q)_\infty}{(q;-q)_\infty^2}, \tag{55}$$

which was proved by Watson [1936]. The right-hand side of (55) can be expressed in terms of theta functions. Using (51), (52), and (55), we obtain new representations for ϕ and ψ which are

$$\phi(e^{-\pi})=\frac{2}{\sqrt{3}}e^{\frac{\pi}{3}}\int_0^\infty \frac{e^{-\frac{\pi x^2}{6}}\cos\frac{\pi x}{2}\cosh\frac{\pi x}{2}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx+\frac{1}{2}\frac{(-e^{-\pi};-e^{-\pi})_\infty}{(e^{-\pi};-e^{-\pi})_\infty^2}$$

$$\psi(e^{-\pi})=-\frac{1}{\sqrt{3}}e^{\frac{\pi}{3}}\int_0^\infty \frac{e^{-\frac{2\pi x^2}{3}}\cos\pi x\cosh\pi x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx+\frac{1}{4}\frac{(-e^{-\pi};-e^{-\pi})_\infty}{(e^{-\pi};-e^{-\pi})_\infty^2}.$$

Replacing n by 1 in (49), we obtain a new representation for ω, namely,

$$\omega(-e^{-\pi})=\frac{1}{\sqrt{3}}e^{\frac{2\pi}{3}}\int_0^\infty \frac{e^{-\frac{\pi x^2}{3}}}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}\,dx.$$

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