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ON THE ATKIN POLYNOMIALS

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We identify the Atkin polynomials in terms of associated Jacobi polynomials. Our identification then takes advantage of the theory of orthogonal polynomials and their asymptotics to establish many new properties of the Atkin polynomials. This shows that corecursive polynomials may lead to interesting sets of orthogonal polynomials.

1. Introduction

In unpublished work, Oliver Atkin introduced a family of orthogonal polynomials with fascinating number-theoretic properties: They are the unique family of monic orthogonal polynomials corresponding to a unique scalar product on the space of polynomials in the modular j-invariant for which all Hecke operators are self-adjoint. Furthermore, their reductions modulo a prime $p \geq 5$ are also very significant in the theory of elliptic curves, as they match the supersingular polynomial at p whenever the degrees agree. For all the number-theoretic definitions, as well as beautiful proofs of these and other facts about the Atkin polynomials, we refer the reader to the excellent [Kaneko and Zagier 1998], where Atkin's results were popularized, simplified, and expanded upon.

The Atkin polynomials are generated by the recurrence relation

$$A_{n+1}(x) = \left[x - 24\frac{144n^2-29}{(2n+1)(2n-1)}\right]A_n(x) - 36\frac{(12n-13)(12n-7)(12n-5)(12n+1)}{n(n-1)(2n-1)^2}A_{n-1}(x), \quad n > 1, \tag{1-1}$$

through the initial conditions

$$A_0(x) = 1, \quad A_1(x) = x - 720, \quad A_2(x) = x^2 - 1640x + 269280. \tag{1-2}$$

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The polynomials $\{A_n(x)\}$ are orthogonal with respect to an absolutely continuous measure supported on [0, 1728] (see Section 7).

In this paper we show that the Atkin polynomials are related to the associated Jacobi polynomials of Wimp [1987] and of Ismail and Masson [1991]. This identification leads to many new properties of the polynomials $\{A_n(x)\}$.

It is worth pointing out that the way the Atkin polynomials are defined, that is, defining $P_0(x)$, $P_1(x)$ and $P_2(x)$, then using a recurrence relation to generate the rest, is not unusual in the literature on orthogonal polynomials. The idea is to start with two monic polynomials, $P_k(x)$ and $P_{k+1}(x)$, of degrees k and $k+1$, respectively, with real, simple and interlacing zeros. Then use the division algorithm to generate the monic polynomials $P_n(x)$, $0 \le n < k$; we are guaranteed to have a sequence of monic orthogonal polynomials $\{P_j(x) : 0 \le j \le k+1\}$. Now use any three-term recurrence relation of the form

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) + \beta_n P_{n-1}(x), \tag{1-3}$$

where $\alpha_n \in \mathbb{R}$ and $\beta_n > 0$ for $n > k$, to generate the polynomials $\{P_n(x)\}$ for $n > k+1$. The construction above is referred to as "Wendroff's Theorem" in the orthogonal polynomial literature. The interested reader may consult [Ismail 2009] or [Chihara 1978] for the precise statement and the detailed proof of Wendroff's theorem. This is also related to the concept of corecursive polynomials [Chihara 1978].

In Section 2, we recall some preliminary facts about associated Jacobi polynomials and orthogonal polynomials in general. In Section 3, we obtain a representation of (a scaled version of) the Atkin polynomials as a linear combination of the associated Jacobi polynomials of Wimp [1987] and of Ismail and Masson [1991]. Building on that, we provide an explicit representation of the coefficients of the Atkin polynomials in Section 4, a representation in terms of certain hypergeometric functions and an asymptotic expansion in Section 5, and a generating function identity in Section 6. Lastly, in Section 7 we give an explicit description of the weight function in terms of certain ${}_2F_1$ functions.

We shall follow the standard notation for hypergeometric functions and orthogonal polynomials as in [Andrews et al. 1999; Ismail 2009; Luke 1969; Rainville 1960; Szegő 1975]. In particular we use $F\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix} \,\middle|\, z\right)$ to mean to mean ${}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix} \,\middle|\, z\right)$.

2. Preliminaries

Let $\{\lambda_n\}$ and $\{\mu_n\}$ be the birth and death rates of a birth and death process; that is, $\lambda_n > 0$ and $\mu_{n+1} > 0$ for all $n \ge 0$, with $\mu_0 \ge 0$. Such a process generates a sequence of orthogonal polynomials through a three-term recurrence relation

$$-xQ_n(x) = \lambda_n Q_{n+1}(x) - (\lambda_n+\mu_n)Q_n(x) + \mu_n Q_{n-1}(x), \quad n > 0, \tag{2-1}$$

with the initial conditions

$$Q_0(x)=1, \qquad Q_1(x)=(\lambda_0+\mu_0-x)/\lambda_0. \tag{2-2}$$

The corresponding monic polynomials satisfy

$$x\widetilde{Q}_n(x)=\widetilde{Q}_{n+1}+(\lambda_n+\mu_n)\widetilde{Q}_n(x)-\lambda_{n-1}\mu_n\widetilde{Q}_{n-1}(x), \tag{2-3}$$

with $\widetilde{Q}_0(x)=1$, $\widetilde{Q}_1(x)=x-\lambda_0-\mu_0$. When $\mu_0\neq 0$ there is a second natural birth and death process with birth rates $\{\lambda_n\}$ and death rates $\{\tilde{\mu}_n\}$ with $\tilde{\mu}_n=\mu_n$ for $n>0$ but $\tilde{\mu}_0=0$ [Ismail et al. 1988]. The latter birth and death generate a second family of orthogonal polynomials satisfying (2-1) but with initial conditions $Q_0(x)=1$, $Q_1(x)=(\lambda_0-x)/\lambda_0$. This observation is due to Ismail, Letessier and Valent [Ismail et al. 1988].

The associated polynomials of $\{Q_n(x)\}$ correspond to the birth and death rates $\{\lambda_{n+c}\}$ and death rates $\{\mu_{n+c}\}$, when such rates are well defined. Since we consider $c\geq 0$, usually $\mu_c>0$. Thus we usually have two families of associated polynomials. One is defined when μ_c is defined from the pattern of μ_n. When $\mu_c\neq 0$, a second family arises if μ_{n+c}, when $n=0$ is interpreted as zero.

Recall that the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$ can be defined by the three-term recurrence relation

$$\begin{aligned}2(n+1)(n+\alpha+\beta+1)(\alpha+\beta+2n)P_{n+1}^{(\alpha,\beta)}(x)\\ =(\alpha+\beta+2n+1)\big[(\alpha^2-\beta^2)+x(\alpha+\beta+2n+2)(\alpha+\beta+2n)\big]P_n^{(\alpha,\beta)}(x)\\ -2(\alpha+n)(\beta+n)(\alpha+\beta+2n+2)P_{n-1}^{(\alpha,\beta)}(x),\end{aligned} \tag{2-4}$$

for $n\geq 0$, with $P_{-1}^{(\alpha,\beta)}(x)=0$, $P_0^{(\alpha,\beta)}(x)=1$. We now set

$$\begin{aligned}V_n^{(\alpha,\beta)}(x)&=\frac{n!\,(\alpha+\beta+1)_n}{(\alpha+\beta+1)_{2n}}P_n^{(\alpha,\beta)}(2x-1)\\ &=\frac{n!}{(n+\alpha+\beta+1)_n}P_n^{(\alpha,\beta)}(2x-1).\end{aligned} \tag{2-5}$$

One can easily verify that the polynomials $\{V_n^{(\alpha,\beta)}(x)\}$ are monic birth and death process polynomials $\widetilde{Q}_n$, with rates

$$\begin{aligned}\lambda_n&=\frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},\\ \mu_n&=\frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.\end{aligned} \tag{2-6}$$

Wimp [1987] considered the recurrence relation obtained by formally replacing n by $n+c$ in (2-4), and he showed that the new relation has two linearly independent solutions $P_n^{(\alpha,\beta)}(x;c)$ and $P_{n-1}^{(\alpha,\beta)}(x;c+1)$. Ismail and Masson [1991] identified the birth and death rates corresponding to that three-term recurrence relation and provided two linearly independent solutions $P_n^{(\alpha,\beta)}(x;c)$ and $\mathcal{P}_n^{(\alpha,\beta)}(x;c)$. They then used the notation

$$R_n^{(\alpha,\beta)}(x;c) = P_n^{(\alpha,\beta)}(2x-1;c), \quad \mathcal{R}_n^{(\alpha,\beta)}(x;c) = \mathcal{P}_n^{(\alpha,\beta)}(2x-1;c). \tag{2-7}$$

We shall use the notation

$$\begin{aligned} V_n^{(\alpha,\beta)}(x;c) &= \frac{(c+1)_n(\alpha+\beta+c+1)_n}{(\alpha+\beta+2c+1)_{2n}} R_n^{(\alpha,\beta)}(x;c), \\ \mathcal{V}_n^{(\alpha,\beta)}(x;c) &= \frac{(c+1)_n(\alpha+\beta+c+1)_n}{(\alpha+\beta+2c+1)_{2n}} \mathcal{R}_n^{(\alpha,\beta)}(x;c). \end{aligned} \tag{2-8}$$

To lighten our notation, we shall occasionally omit the parameters when the context is clear. We consider the birth and death rates

$$\begin{aligned} \lambda_n &= \frac{(n+c+\beta+1)(n+c+\alpha+\beta+1)}{(2n+2c+\alpha+\beta+1)(2n+2c+\alpha+\beta+2)}, \quad n \geq 0, \\ \mu_n &= \frac{(n+c)(n+c+\alpha)}{(2n+2c+\alpha+\beta)(2n+2c+\alpha+\beta+1)}, \quad n > 0, \end{aligned} \tag{2-9}$$

with

$$\mu_0 := \begin{cases} \dfrac{c(c+\alpha)}{(2c+\alpha+\beta)(2c+\alpha+\beta+1)} & \text{for } V, \\ 0 & \text{for } \mathcal{V}. \end{cases} \tag{2-10}$$

3. The Atkin polynomials

In order to compare the Atkin polynomials with other results in the literature we need to renormalize them. Let

$$A_n(1728y) = (1728)^n \mathcal{A}_n(y). \tag{3-1}$$

The polynomials $\mathcal{A}_n$ are now generated by the recurrence

$$\begin{aligned} \mathcal{A}_{n+1}(x) &= \left[x - \frac{2\left(n^2-\frac{29}{144}\right)}{4n^2-1}\right]\mathcal{A}_n(x) - \frac{\left(n-\frac{13}{12}\right)\left(n-\frac{7}{12}\right)\left(n-\frac{5}{12}\right)\left(n+\frac{1}{12}\right)}{2n(2n-1)^2(2n-2)}\mathcal{A}_{n-1}(x) \end{aligned} \tag{3-2}$$

for $n > 1$. The initial conditions are

$$\mathcal{A}_0(x) = 1, \quad \mathcal{A}_1(x) = x - \tfrac{5}{12}, \quad \mathcal{A}_2(x) = x^2 - \tfrac{205}{216}x + \tfrac{935}{10368}. \tag{3-3}$$

Kaneko and Zagier [1998] wrote the recurrence relation (1-1) in the monic form (2-3). Indeed their (19), when written in terms of the $\mathcal{A}_n$, corresponds to (2-3) with

$$\lambda_n = \frac{\left(n-\frac{1}{12}\right)\left(n+\frac{5}{12}\right)}{2n(2n+1)}, \quad \mu_n = \frac{\left(n-\frac{5}{12}\right)\left(n+\frac{1}{12}\right)}{2n(2n-1)}. \tag{3-4}$$

From (2-8), we see that $V_n^{(\alpha,\beta)}(x; c)$ and $\mathcal{V}_n^{(\alpha,\beta)}(x; c)$ satisfy the second-order difference equation

$$T_{n+1}(x) = \left(x + \frac{\alpha^2 - \beta^2 - (2n+2c+\alpha+\beta)(2n+2c+\alpha+\beta+2)}{2(2n+2c+\alpha+\beta)(2n+2c+\alpha+\beta+2)} \right) T_n(x) - \frac{(n+c)(n+c+\alpha)(n+c+\beta)(n+c+\alpha+\beta)}{(2n+2c+\alpha+\beta-1)(2n+2c+\alpha+\beta)^2(2n+2c+\alpha+\beta+1)} T_{n-1}(x) \tag{3-5}$$

for $n \geq 1$, with the initial conditions $V_0 = \mathcal{V}_0 = 1$ and

$$\begin{aligned} V_1^{(\alpha,\beta)}(x; c) &= x - (\lambda_0 + \mu_0), \\ \mathcal{V}_1^{(\alpha,\beta)}(x; c) &= x - \lambda_0, \end{aligned} \tag{3-6}$$

where λ_n and μ_n are defined as in (2-9)–(2-10). On the other hand, we see that the sequence $\{\mathcal{A}_{n+1}(x)\}_{n=-1}^{\infty}$ is a solution of the second-order difference equation

$$T_{n+1}(x) = \left(x - \frac{7 + 36(2n+1)(2n+3)}{72(2n+1)(2n+3)} \right) T_n(x) - \frac{\left(n - \frac{1}{12}\right)\left(n + \frac{5}{12}\right)\left(n + \frac{7}{12}\right)\left(n + \frac{13}{12}\right)}{(2n)(2n+1)^2(2n+2)} T_{n-1}(x) \tag{3-7}$$

for $n \geq 1$. It is not hard to check that (3-7) is identical to (3-5) in exactly four cases, namely,

$$(\alpha, \beta, c) \in S := \left\{ \left(-\tfrac{1}{2}, -\tfrac{2}{3}, \tfrac{13}{12}\right), \left(\tfrac{1}{2}, -\tfrac{2}{3}, \tfrac{7}{12}\right), \left(-\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{5}{12}\right), \left(\tfrac{1}{2}, \tfrac{2}{3}, -\tfrac{1}{12}\right) \right\}.$$

Theorem 3.1. *For $n \geq 0$ and $(\alpha, \beta, c) \in S$, we have the following representations for $\mathcal{A}_{n+1}(x)$:*

$$\mathcal{A}_{n+1}(x) = \left(x - \tfrac{5}{12}\right) V_n^{(\alpha,\beta)}(x; c) - \tfrac{91}{384} V_{n-1}^{(\alpha,\beta)}(x; c+1), \tag{3-8}$$

$$\mathcal{A}_{n+1}(x) = (x - 8) V_n^{(-1/2,2/3)}\left(x; \tfrac{5}{12}\right) + \tfrac{91}{12} \mathcal{V}_n^{(-1/2,2/3)}\left(x; \tfrac{5}{12}\right), \tag{3-9}$$

$$\mathcal{A}_{n+1}(x) = x V_n^{(1/2,-2/3)}\left(x; \tfrac{7}{12}\right) - \tfrac{5}{12} \mathcal{V}_n^{(1/2,-2/3)}\left(x; \tfrac{7}{12}\right). \tag{3-10}$$

Proof. It is straightforward to check that for any $(\alpha, \beta, c) \in S$,

$$\{V_n^{(\alpha,\beta)}(x; c), \mathcal{V}_n^{(\alpha,\beta)}(x; c)\}$$

is a basis of solutions of (3-7), and the same is true for

$$\{V_n^{(\alpha,\beta)}(x; c), V_{n-1}^{(\alpha,\beta)}(x; c+1)\}.$$

The results follow by simple linear algebra on the equations corresponding to $n = 0$ and $n = 1$. □

We note that $V_n^{(\alpha,\beta)}(x;c)$ is the same for the four triples in S, whereas we have two possibilities for $\mathcal{V}_n^{(\alpha,\beta)}(x;c)$, depending on whether $\beta=\frac{2}{3}$ or $\beta=-\frac{2}{3}$. For convenience we explicitly write down the first few of these polynomials:

$$
\begin{aligned}
&V_0^{(\alpha,\beta)}(x;c)=1,\\
&V_1^{(\alpha,\beta)}(x;c)=x-\tfrac{115}{216},\\
&V_2^{(\alpha,\beta)}(x;c)=x^2-\tfrac{187}{180}x+\tfrac{11621}{55296};
\end{aligned}
\tag{3-11}
$$

$$
\begin{aligned}
&V_{-1}^{(\alpha,\beta)}(x;c+1)=0,\\
&V_0^{(\alpha,\beta)}(x;c+1)=1,\\
&V_1^{(\alpha,\beta)}(x;c+1)=x-\tfrac{547}{1080};
\end{aligned}
\tag{3-12}
$$

$$
\begin{aligned}
&\mathcal{V}_0^{(1/2,-2/3)}\big(x;\tfrac{7}{12}\big)=\mathcal{V}_0^{(-1/2,-2/3)}\big(x;\tfrac{13}{12}\big)=1,\\
&\mathcal{V}_1^{(1/2,-2/3)}\big(x;\tfrac{7}{12}\big)=\mathcal{V}_1^{(-1/2,-2/3)}\big(x;\tfrac{13}{12}\big)=x-\tfrac{187}{864},\\
&\mathcal{V}_2^{(1/2,-2/3)}\big(x;\tfrac{7}{12}\big)=\mathcal{V}_2^{(-1/2,-2/3)}\big(x;\tfrac{13}{12}\big)=x^2-\tfrac{347}{480}x+\tfrac{124729}{2488320};
\end{aligned}
\tag{3-13}
$$

$$
\begin{aligned}
&\mathcal{V}_0^{(1/2,2/3)}\big(x;\tfrac{-1}{12}\big)=\mathcal{V}_0^{(-1/2,2/3)}\big(x;\tfrac{5}{12}\big)=1,\\
&\mathcal{V}_1^{(1/2,2/3)}\big(x;\tfrac{-1}{12}\big)=\mathcal{V}_1^{(-1/2,2/3)}\big(x;\tfrac{5}{12}\big)=x-\tfrac{475}{864},\\
&\mathcal{V}_2^{(1/2,2/3)}\big(x;\tfrac{-1}{12}\big)=\mathcal{V}_2^{(-1/2,2/3)}\big(x;\tfrac{5}{12}\big)=x^2-\tfrac{169}{160}x+\tfrac{108965}{497664}.
\end{aligned}
\tag{3-14}
$$

One can check the first few cases of Theorem 3.1 using the equalities

$$
\begin{aligned}
&\mathcal{A}_1(x)=x-\tfrac{5}{12},\\
&\mathcal{A}_2(x)=x^2-\tfrac{205}{216}x+\tfrac{935}{10368},\\
&\mathcal{A}_3(x)=x^3-\tfrac{131}{90}x^2+\tfrac{28277}{55296}x-\tfrac{124729}{5971968}.
\end{aligned}
\tag{3-15}
$$

4. Explicit representations

Wimp [1987, p. 987] gave an explicit formula for $R_n^{(\alpha,\beta)}(x;c)$. When translated in terms of the V_n polynomials it becomes

$$
\begin{aligned}
V_n^{(\alpha,\beta)}(x;c)=(-1)^n&\frac{(c+1)_n(\beta+c+1)_n}{(\alpha+\beta+2c+n+1)_n\,n!}\\
&\times\sum_{k=0}^{n}\frac{(-n)_k(n+2c+\alpha+\beta+1)_k}{(c+1)_k(c+\beta+1)_k}x^k\\
&\times{}_4F_3\left(\begin{matrix}k-n,\,n+k+\alpha+\beta+2c+1,\,c+\beta,\,c\\ k+\beta+c+1,\,k+c+1,\,\alpha+\beta+2c\end{matrix}\,\middle|\,1\right).
\end{aligned}
\tag{4-1}
$$

On the other hand, Ismail and Masson [1991, Theorem 3.3] gave a similar formula for $\mathcal{R}_n^{(\alpha,\beta)}(x;c)$, which leads to

$$(4\text{-}2)\quad \mathcal{V}_n^{(\alpha,\beta)}(x;c) = (-1)^n \frac{(c+1)_n(\beta+c+1)_n}{(\alpha+\beta+2c+n+1)_n\, n!} \times \sum_{k=0}^{n} \frac{(-n)_k(n+2c+\alpha+\beta+1)_k}{(c+1)_k(c+\beta+1)_k} x^k \times {}_4F_3\left(\begin{matrix} k-n,\, n+k+\alpha+\beta+2c+1,\, c+\beta+1,\, c \\ k+\beta+c+1,\, k+c+1,\, \alpha+\beta+2c+1 \end{matrix} \,\middle|\, 1\right).$$

The following theorem establishes an analogous representation of $\mathcal{A}_{n+1}(x)$:

Theorem 4.1. *For $n \geq 0$, we have*

$$(4\text{-}3)\quad \mathcal{A}_{n+1}(x) = \frac{\left(\frac{19}{12}\right)_n\left(\frac{11}{12}\right)_n}{(n+2)_n(-n)_n} \times \Bigg[{}_3F_2\left(\begin{matrix} -n,\, n+2,\, \frac{7}{12} \\ \frac{19}{12},\, 2 \end{matrix} \,\middle|\, 1\right) + \sum_{k=0}^{n} \frac{(-n)_k(n+2)_k}{\left(\frac{19}{12}\right)_k\left(\frac{11}{12}\right)_k} x^{k+1} \left\{ \frac{6}{5}\, {}_4F_3\left(\begin{matrix} k-n,\, n+k+2,\, \frac{11}{12},\, \frac{-5}{12} \\ k+\frac{11}{12},\, k+\frac{19}{12},\, 1 \end{matrix} \,\middle|\, 1\right) - \frac{1}{5}\, {}_4F_3\left(\begin{matrix} k-n,\, n+k+2,\, \frac{-1}{12},\, \frac{-5}{12} \\ k+\frac{11}{12},\, k+\frac{19}{12},\, 1 \end{matrix} \,\middle|\, 1\right) \right\} \Bigg].$$

Proof. From (3-10) we have

$$\mathcal{A}_{n+1}(x) = x\, V_n^{(1/2,-2/3)}\left(x;\, \tfrac{7}{12}\right) - \tfrac{5}{12} \mathcal{V}_n^{(1/2,-2/3)}\left(x;\, \tfrac{7}{12}\right), \quad n \geq 0;$$

we see that the coefficient of x^{k+1} in $\mathcal{A}_{n+1}(x)$ is given by

$$(4\text{-}4)\quad (-1)^n \frac{\left(\frac{19}{12}\right)_n\left(\frac{11}{12}\right)_n}{(n+2)_n\, n!} \frac{(-n)_k(n+2)_k}{\left(\frac{19}{12}\right)_k\left(\frac{11}{12}\right)_k} \times \Bigg[{}_4F_3\left(\begin{matrix} k-n,\, n+k+2,\, \frac{-1}{12},\, \frac{7}{12} \\ k+\frac{11}{12},\, k+\frac{19}{12},\, 1 \end{matrix} \,\middle|\, 1\right) - \frac{5}{12} \frac{(k-n)(n+k+2)}{\left(k+\frac{19}{12}\right)\left(k+\frac{11}{12}\right)} {}_4F_3\left(\begin{matrix} k+1-n,\, n+k+3,\, \frac{11}{12},\, \frac{7}{12} \\ k+1+\frac{11}{12},\, k+1+\frac{19}{12},\, 2 \end{matrix} \,\middle|\, 1\right) \Bigg].$$

The coefficient of y^m in

$$ {}_4F_3\left(\begin{matrix} k-n, n+k+2, \frac{-1}{12}, \frac{7}{12} \\ k+\frac{11}{12}, k+\frac{19}{12}, 1 \end{matrix} \,\middle|\, y\right) - \frac{5y}{12}\frac{(k-n)(n+k+2)}{\left(k+\frac{19}{12}\right)\left(k+\frac{11}{12}\right)}{}_4F_3\left(\begin{matrix} k+1-n, n+k+3, \frac{11}{12}, \frac{7}{12} \\ k+1+\frac{11}{12}, k+1+\frac{19}{12}, 2 \end{matrix} \,\middle|\, y\right) $$

is

$$ \frac{(k-n)_m(n+k+2)_m\left(\frac{-1}{12}\right)_m\left(\frac{7}{12}\right)_m}{\left(k+\frac{11}{12}\right)_m\left(k+\frac{19}{12}\right)_m(m!)^2} - \frac{5m}{12}\frac{(k-n)(n+k+2)}{\left(k+\frac{19}{12}\right)\left(k+\frac{11}{12}\right)}\frac{(k-n+1)_{m-1}(n+k+3)_{m-1}\left(\frac{11}{12}\right)_{m-1}\left(\frac{7}{12}\right)_{m-1}}{\left(k+1+\frac{11}{12}\right)_{m-1}\left(k+1+\frac{19}{12}\right)_{m-1}(2)_{m-1}(m)!}. $$

Using the identity $(z)_m = z(z+1)_{m-1}$, we get that this coefficient is

$$ \frac{(k-n)_m(n+k+2)_m\left(\frac{-1}{12}\right)_m\left(\frac{7}{12}\right)_m}{\left(k+\frac{11}{12}\right)_m\left(k+\frac{19}{12}\right)_m(m!)^2}\left(1+\frac{\left(\frac{-5}{12}\right)(-12m)}{\left(\frac{7}{12}+m-1\right)}\right) \tag{4-5} $$

$$ = \frac{(k-n)_m(n+k+2)_m\left(\frac{-1}{12}\right)_m\left(\frac{-5}{12}\right)_m}{\left(k+\frac{11}{12}\right)_m\left(k+\frac{19}{12}\right)_m(m!)^2}\left(1-\frac{72}{5}m\right) $$

$$ = \frac{1}{5}\frac{(k-n)_m(n+k+2)_m\left(\frac{-5}{12}\right)_m}{\left(k+\frac{11}{12}\right)_m\left(k+\frac{19}{12}\right)_m(m!)^2}\left(6\left(\frac{11}{12}\right)_m-\left(\frac{-1}{12}\right)_m\right). $$

In the last equality, we used

$$ \left(\tfrac{-1}{12}\right)_m\left(m-\tfrac{5}{72}\right) = \left(\tfrac{-1}{12}\right)_m\left[\left(m-\tfrac{1}{12}\right)+\tfrac{1}{72}\right] = -\tfrac{1}{12}\left(\tfrac{11}{12}\right)_m+\tfrac{1}{72}\left(\tfrac{-1}{12}\right)_m. $$

It now follows that

$$ \left[{}_4F_3\left(\begin{matrix} k-n, n+k+2, \frac{-1}{12}, \frac{7}{12} \\ k+\frac{11}{12}, k+\frac{19}{12}, 1 \end{matrix} \,\middle|\, y\right) - \frac{5y}{12}\frac{(k-n)(n+k+2)}{\left(k+\frac{19}{12}\right)\left(k+\frac{11}{12}\right)}{}_4F_3\left(\begin{matrix} k+1-n, n+k+3, \frac{11}{12}, \frac{7}{12} \\ k+1+\frac{11}{12}, k+1+\frac{19}{12}, 2 \end{matrix} \,\middle|\, y\right)\right] $$
$$ = \tfrac{6}{5}\,{}_4F_3\left(\begin{matrix} k-n, n+k+2, \frac{11}{12}, \frac{-5}{12} \\ k+\frac{11}{12}, k+\frac{19}{12}, 1 \end{matrix} \,\middle|\, y\right) - \tfrac{1}{5}\,{}_4F_3\left(\begin{matrix} k-n, n+k+2, \frac{-1}{12}, \frac{-5}{12} \\ k+\frac{11}{12}, k+\frac{19}{12}, 1 \end{matrix} \,\middle|\, y\right). \tag{4-6} $$

The result now follows by substituting (4-6) with $y = 1$ into (4-4) . □

Remark 4.2. There is another explicit representation of a somewhat different form than (4-3) for the Atkin polynomials. Indeed, it follows from Theorem 4(ii) in [Kaneko and Zagier 1998] that

$$\mathcal{A}_n(x)=\sum_{i=0}^{n}\sum_{m=0}^{i}(-1)^m\binom{\frac{-1}{12}}{i-m}\binom{\frac{-5}{12}}{i-m}\binom{n+\frac{1}{12}}{m}\binom{n-\frac{7}{12}}{m}\binom{2n-1}{m}^{-1}x^{n-i}. \tag{4-7}$$

5. Asymptotics

Wimp [1987, Proof of Theorem 1] showed that the functions u_n and y_n (u_n and v_n in his notation) defined by

$$\begin{aligned} u_n^{(\alpha,\beta)}(x;c)&=(-1)^n\frac{\Gamma(n+\beta+c+1)}{\Gamma(n+c+1)}F\left(\begin{matrix}-n-c,\,n+\alpha+\beta+c+1\\1+\beta\end{matrix}\,\middle|\,x\right),\\ y_n^{(\alpha,\beta)}(x;c)&=(-1)^n\frac{\Gamma(n+\alpha+c+1)}{\Gamma(n+\alpha+\beta+c+1)}F\left(\begin{matrix}-n-\beta-c,\,n+\alpha+c+1\\1-\beta\end{matrix}\,\middle|\,x\right),\end{aligned} \tag{5-1}$$

satisfy the same recurrence relation satisfied by R_n and $\mathcal{R}_n$, and thus the latter can be represented as linear combinations of the former. We shall slightly modify these functions so as to replace the gamma factors by rising factorials (thus getting rational rather than transcendental coefficients when the parameters are rational) as follows. Set

$$\begin{aligned} U_n^{(\alpha,\beta)}(x;c)&=\frac{\Gamma(c+1)}{\Gamma(\beta+c+1)}u_n^{(\alpha,\beta)}(x;c),\\ Y_n^{(\alpha,\beta)}(x;c)&=\frac{\Gamma(\alpha+\beta+c+1)}{\Gamma(\alpha+c+1)}y_n^{(\alpha,\beta)}(x;c).\end{aligned} \tag{5-2}$$

Thus we have

$$\begin{aligned} U_n^{(\alpha,\beta)}(x;c)&=(-1)^n\frac{(\beta+c+1)_n}{(c+1)_n}F\left(\begin{matrix}-n-c,\,n+\alpha+\beta+c+1\\1+\beta\end{matrix}\,\middle|\,x\right),\\ Y_n^{(\alpha,\beta)}(x;c)&=(-1)^n\frac{(\alpha+c+1)_n}{(\alpha+\beta+c+1)_n}F\left(\begin{matrix}-n-\beta-c,\,n+\alpha+c+1\\1-\beta\end{matrix}\,\middle|\,x\right).\end{aligned} \tag{5-3}$$

Note that since the factors multiplied by u_n and y_n in (5-2) are independent of n, U_n and Y_n satisfy the same recurrence as R_n and $\mathcal{R}_n$. Indeed, after a simple Kummer transformation, Formula (28) on p. 988 of [Wimp 1987] can be written as

$$\begin{aligned} R_n=\frac{(\beta+c)(\alpha+\beta+c)}{\beta(\alpha+\beta+2c)}&F\left(\begin{matrix}c,\,1-(\alpha+\beta+c)\\1-\beta\end{matrix}\,\middle|\,x\right)U_n\\ &-\frac{c(\alpha+c)}{\beta(\alpha+\beta+2c)}F\left(\begin{matrix}\beta+c,\,1-(\alpha+c)\\1+\beta\end{matrix}\,\middle|\,x\right)Y_n.\end{aligned} \tag{5-4}$$

Similarly, Theorem 3.10 of [Ismail and Masson 1991] leads to

$$\mathcal{R}_n = F\left(\begin{matrix} c,\, -(\alpha+\beta+c) \\ -\beta \end{matrix}\,\middle|\, x\right) U_n - \frac{c(\alpha+c)}{\beta(\beta+1)} x F\left(\begin{matrix} 1+\beta+c,\, 1-(\alpha+c) \\ 2+\beta \end{matrix}\,\middle|\, x\right) Y_n. \tag{5-5}$$

The following theorem provides the analogous representation for the Atkin polynomials:

Theorem 5.1. *Let U_n and Y_n be as in* (5-3), *and set*

$$\begin{aligned} \widetilde{U}_n^{(\alpha,\beta)}(x;\, c) &= \frac{(c+1)_n(\alpha+\beta+c+1)_n}{(\alpha+\beta+2c+1)_{2n}} U_n^{(\alpha,\beta)}(x;\, c), \\ \widetilde{Y}_n^{(\alpha,\beta)}(x;\, c) &= \frac{(c+1)_n(\alpha+\beta+c+1)_n}{(\alpha+\beta+2c+1)_{2n}} Y_n^{(\alpha,\beta)}(x;\, c). \end{aligned} \tag{5-6}$$

Then we have

$$\mathcal{A}_{n+1}(x) = C(x)\widetilde{U}_n^{(1/2,-2/3)}\big(x;\, \tfrac{7}{12}\big) + D(x)\widetilde{Y}_n^{(1/2,-2/3)}\big(x;\, \tfrac{7}{12}\big), \quad n \geq 0, \tag{5-7}$$

with $C(x)$ and $D(x)$ given by

$$\begin{aligned} C(x) &:= \frac{-1}{60}\left(24F\left(\begin{matrix} \frac{-5}{12},\, \frac{-5}{12} \\ \frac{-1}{3} \end{matrix}\,\middle|\, x\right) + F\left(\begin{matrix} \frac{-5}{12},\, \frac{-5}{12} \\ \frac{2}{3} \end{matrix}\,\middle|\, x\right)\right), \\ D(x) &:= \frac{91}{384}x\left(4F\left(\begin{matrix} \frac{-1}{12},\, \frac{-1}{12} \\ \frac{1}{3} \end{matrix}\,\middle|\, x\right) - 5F\left(\begin{matrix} \frac{11}{12},\, \frac{-1}{12} \\ \frac{4}{3} \end{matrix}\,\middle|\, x\right)\right). \end{aligned} \tag{5-8}$$

Proof. From (5-4) and (5-5), we see that

$$\begin{aligned} & xR_n^{(1/2,-2/3)}\big(x;\, \tfrac{7}{12}\big) - \tfrac{5}{12}\mathcal{R}_n^{(1/2,-2/3)}\big(x;\, \tfrac{7}{12}\big) \\ &= \frac{5}{12}U_n^{(1/2,-2/3)}\big(x;\, \tfrac{7}{12}\big)\left(\frac{\big(\frac{-1}{12}\big)}{\big(-\frac{2}{3}\big)} x F\left(\begin{matrix} \frac{7}{12},\, \frac{7}{12} \\ \frac{5}{3} \end{matrix}\,\middle|\, x\right) - F\left(\begin{matrix} \frac{7}{12},\, \frac{-5}{12} \\ \frac{2}{3} \end{matrix}\,\middle|\, x\right)\right) \\ &\quad - xY_n^{(1/2,-2/3)}\big(x;\, \tfrac{7}{12}\big)\left(\frac{\big(\frac{7}{12}\big)\big(\frac{13}{12}\big)}{\big(-\frac{2}{3}\big)} F\left(\begin{matrix} \frac{-1}{12},\, \frac{-1}{12} \\ \frac{1}{3} \end{matrix}\,\middle|\, x\right) - \frac{5}{12}\frac{\big(\frac{7}{12}\big)\big(\frac{13}{12}\big)}{\big(-\frac{2}{3}\big)\big(\frac{1}{3}\big)} F\left(\begin{matrix} \frac{11}{12},\, \frac{-1}{12} \\ \frac{4}{3} \end{matrix}\,\middle|\, x\right)\right). \end{aligned} \tag{5-9}$$

Expanding the hypergeometric series in powers of x, we get, after some computation,

$$(5\text{-}10)\quad x R_n^{(1/2,-2/3)}\big(x;\tfrac{7}{12}\big)-\tfrac{5}{12}\mathcal{R}_n^{(1/2,-2/3)}\big(x;\tfrac{7}{12}\big)$$
$$=\frac{-1}{60}\left(24F\left(\begin{matrix}\frac{-5}{12},\frac{-5}{12}\\ \frac{-1}{3}\end{matrix}\,\middle|\,x\right)+F\left(\begin{matrix}\frac{-5}{12},\frac{-5}{12}\\ \frac{2}{3}\end{matrix}\,\middle|\,x\right)\right)U_n^{(1/2,-2/3)}\left(x;\frac{7}{12}\right)$$
$$+\frac{91}{384}x\left(4F\left(\begin{matrix}\frac{-1}{12},\frac{-1}{12}\\ \frac{1}{3}\end{matrix}\,\middle|\,x\right)-5F\left(\begin{matrix}\frac{11}{12},\frac{-1}{12}\\ \frac{4}{3}\end{matrix}\,\middle|\,x\right)\right)Y_n^{(1/2,-2/3)}\left(x;\frac{7}{12}\right),$$

and the result follows from (3-10). □

Theorem 5.1 enables us to obtain an asymptotic formula for the Atkin polynomials:

Theorem 5.2. *Let $C(x)$ and $D(x)$ be as in* (5-8). *For fixed $\theta\in(0,\pi/2)$, the following asymptotic formula holds as $n\to\infty$:*

$$(5\text{-}11)\quad \mathcal{A}_{n+1}(\sin^2\theta)$$
$$\sim\frac{(-1)^n}{2^{2n+1}(\cos\theta)(\sin\theta)^{\frac{7}{6}}}\left[C(\sin^2\theta)\frac{\Gamma\big(\frac{1}{3}\big)(\sin\theta)^{\frac{2}{3}}}{\Gamma\big(\frac{11}{12}\big)\Gamma\big(\frac{17}{12}\big)}\cos\left[2(n-1)\theta+\frac{\pi}{12}\right]\right.$$
$$\left.+D(\sin^2\theta)\frac{\Gamma\big(\frac{5}{3}\big)}{\Gamma\big(\frac{13}{12}\big)\Gamma\big(\frac{19}{12}\big)}\cos\left[2(n-1)\theta-\frac{7\pi}{12}\right]\right]$$

Proof. We start by recalling the following asymptotic formula, due to Watson, [Luke 1969, (8), p. 237] (all of our asymptotic formulas will be as $n\to\infty$).

$$(5\text{-}12)\quad F\left(\begin{matrix}b-n,n+a\\ d\end{matrix}\,\middle|\,\sin^2\theta\right)$$
$$\sim\frac{\Gamma(d)n^{-d+\frac{1}{2}}}{\sqrt{\pi}}\,\frac{(\cos\theta)^{d-a-b-\frac{1}{2}}}{(\sin\theta)^{d-\frac{1}{2}}}\cos\left[2n\theta+(a-b)\theta-\frac{\pi}{2}\left(d-\frac{1}{2}\right)\right]$$

for fixed $\theta\in(0,\pi)$. Note that Stirling's formula can be written as

$$(5\text{-}13)\qquad \Gamma(n+a)\sim\sqrt{2\pi}\,n^{n+a-\frac{1}{2}}e^{-n}\quad\text{as } n\to\infty,$$

from which we deduce

$$\frac{\Gamma(n+a)}{\Gamma(n+b)}\sim n^{a-b}.$$

Hence

$$u_n(\sin^2\theta)\sim\frac{(-1)^n\Gamma(1+\beta)(\cos\theta)^{-\alpha-\frac{1}{2}}}{\sqrt{\pi n}(\sin\theta)^{\beta+\frac{1}{2}}}\cos\left[2n\theta-(\alpha+\beta+2c+1)\theta-\frac{\pi}{2}\left(\beta+\frac{1}{2}\right)\right],$$
$$y_n(\sin^2\theta)\sim\frac{(-1)^n\Gamma(1-\beta)(\cos\theta)^{-\alpha-\frac{1}{2}}}{\sqrt{\pi n}(\sin\theta)^{-\beta+\frac{1}{2}}}\cos\left[2n\theta-(\alpha+\beta+2c+1)\theta+\frac{\pi}{2}\left(\beta-\frac{1}{2}\right)\right].$$

Also, from (5-13) we get

$$\text{(5-14)}\quad \frac{(c+1)_n(\alpha+\beta+c+1)_n}{(\alpha+\beta+2c+1)_{2n}} = \frac{\Gamma(\alpha+\beta+2c+1)}{\Gamma(c+1)\Gamma(\alpha+\beta+c+1)}\frac{\Gamma(n+c+1)\Gamma(n+\alpha+\beta+c+1)}{\Gamma(2n+\alpha+\beta+2c+1)} \sim \frac{\Gamma(\alpha+\beta+2c+1)}{\Gamma(c+1)\Gamma(\alpha+\beta+c+1)}\sqrt{\pi n}\left(\tfrac{1}{2}\right)^{2n+\alpha+\beta+2c}.$$

Substituting $(\alpha, \beta, c) = \left(\frac{1}{2}, \frac{-2}{3}, \frac{7}{12}\right)$, we see that

$$\text{(5-15)}\quad \begin{aligned} \widetilde{U}_n(\sin^2\theta) &\sim \frac{(-1)^n\Gamma\left(\frac{1}{3}\right)(\sin\theta)^{1/6}}{2^{2n+1}\cos\theta\,\Gamma\left(\frac{11}{12}\right)\Gamma\left(\frac{17}{12}\right)}\cos\left[2(n-1)\theta+\frac{\pi}{12}\right],\\ \widetilde{Y}_n(\sin^2\theta) &\sim \frac{(-1)^n\Gamma\left(\frac{5}{3}\right)(\sin\theta)^{-7/6}}{2^{2n+1}\cos\theta\,\Gamma\left(\frac{13}{12}\right)\Gamma\left(\frac{19}{12}\right)}\cos\left[2(n-1)\theta-\frac{7\pi}{12}\right], \end{aligned}$$

and the result follows from (5-7) and (5-8). □

6. Generating functions

We start by recalling a remarkable identity of Flensted-Jensen and Koornwinder [1973]. The interested reader could also consult [Wimp 1987] for more details on various other authors who presented variants of this identity as well as other proofs.

Lemma 6.1. *Let t, x, a, b, d be complex numbers with $x \notin [1, \infty)$ and*

$$\text{(6-1)}\quad |t| < \frac{1}{|\sqrt{x}+\sqrt{x-1}|^2}.$$

Then

$$\text{(6-2)}\quad \sum_{n=0}^{\infty}\frac{(d+a)_n(b)_n}{(a+b+1)_n}F\left(\begin{matrix}-n-a, n+b\\ d\end{matrix}\,\middle|\, x\right)\frac{(-t)^n}{n!} = \left(\frac{z_2-t}{z_2+t}\right)^{a+d}\left(\frac{2}{z_2-t}\right)^{b}F\left(\begin{matrix}-a, b\\ d\end{matrix}\,\middle|\,\frac{t+z_1}{2t}\right)F\left(\begin{matrix}a+d, a+1\\ a+b+1\end{matrix}\,\middle|\,\frac{2t}{t+z_2}\right),$$

where $z_1 = 1-\sqrt{(1+t)^2-4xt}$ and $z_2 = 1+\sqrt{(1+t)^2-4xt}$.

To simplify notation we shall write, for $t \neq 0$,

$$\text{(6-3)}\quad \begin{aligned} \delta &= \frac{t+z_1}{2t} = \frac{(1+t)-\sqrt{(1+t)^2-4xt}}{2t},\\ \epsilon &= \frac{t+z_2}{2t} = \frac{(1+t)+\sqrt{(1+t)^2-4xt}}{2t}. \end{aligned}$$

We clearly have

$$t(y-\delta)(y-\epsilon)=ty^2-(1+t)y+x. \tag{6-4}$$

Obviously $z_2+t=2t\epsilon$, and we also have $z_2-t=2t(\epsilon-1)$. Furthermore we have $\delta\epsilon=x/t$. Thus we can rewrite (6-2) for $x\neq 0$ as

$$\sum_{n=0}^{\infty}\frac{(d+a)_n(b)_n}{(a+b+1)_n}F\left(\begin{matrix}-n-a,\,n+b\\ d\end{matrix}\,\middle|\,x\right)\frac{(-t)^n}{n!} \\ =\frac{(x-t\delta)^{a+d-b}\delta^b}{x^{a+d}}F\left(\begin{matrix}-a,\,b\\ d\end{matrix}\,\middle|\,\delta\right)F\left(\begin{matrix}a+d,\,a+1\\ a+b+1\end{matrix}\,\middle|\,\frac{t}{x}\delta\right). \tag{6-5}$$

The following proposition provides a generating function for U_n and Y_n:

Proposition 6.2. *Let U_n and Y_n be as in* (5-2). *Let t and x be such that $x\notin[1,\infty)$ is nonzero and $|t(\sqrt{x}+\sqrt{x-1})^2|<1$, and set δ as in* (6-3). *Then*:

$$\sum_{n=0}^{\infty}\frac{(\alpha+\beta+c+1)_n(c+1)_n}{(\alpha+\beta+2c+2)_n}U_n(x)\frac{t^n}{n!} \\ =\frac{\delta^{\alpha+\beta+c+1}}{x^{\beta+c+1}(x-t\delta)^{\alpha}}F\left(\begin{matrix}-c,\,\alpha+\beta+c+1\\ 1+\beta\end{matrix}\,\middle|\,\delta\right)F\left(\begin{matrix}\beta+c+1,\,c+1\\ \alpha+\beta+2c+2\end{matrix}\,\middle|\,\frac{t\delta}{x}\right), \tag{6-6}$$

$$\sum_{n=0}^{\infty}\frac{(\alpha+\beta+c+1)_n(c+1)_n}{(\alpha+\beta+2c+2)_n}Y_n(x)\frac{t^n}{n!} \\ =\frac{\delta^{\alpha+c+1}}{x^{c+1}(x-t\delta)^{\alpha}}F\left(\begin{matrix}-\beta-c,\,\alpha+c+1\\ 1-\beta\end{matrix}\,\middle|\,\delta\right)F\left(\begin{matrix}c+1,\,\beta+c+1\\ \alpha+\beta+2c+2\end{matrix}\,\middle|\,\frac{t\delta}{x}\right). \tag{6-7}$$

Proof. From (5-1), we see that

$$\frac{\Gamma(c+1)}{\Gamma(\beta+c+1)}\frac{(\alpha+\beta+c+1)_n(c+1)_n}{(\alpha+\beta+2c+2)_n}u_n \\ =\frac{(c+\beta+1)_n(\alpha+\beta+c+1)_n}{(\alpha+\beta+2c+2)_n}F\left(\begin{matrix}-n-c,\,n+\alpha+\beta+c+1\\ 1+\beta\end{matrix}\,\middle|\,x\right) \tag{6-8}$$

and

$$\frac{\Gamma(\alpha+\beta+c+1)}{\Gamma(\alpha+c+1)}\frac{(\alpha+\beta+c+1)_n(c+1)_n}{(\alpha+\beta+2c+2)_n}y_n \\ =\frac{(c+1)_n(\alpha+c+1)_n}{(\alpha+\beta+2c+2)_n}F\left(\begin{matrix}-n-\beta-c,\,n+\alpha+c+1\\ 1-\beta\end{matrix}\,\middle|\,x\right). \tag{6-9}$$

The identities (6-6) and (6-7) follow from applying (6-5) with the choices $(a,b,d)=(c,\alpha+\beta+c+1,\beta+1)$ and $(a,b,d)=(\beta+c,\alpha+c+1,1-\beta)$, respectively. □

Remark 6.3. The result in Proposition 6.2 is essentially due to Wimp. However, we take this opportunity to correct a misprint in the statement of Theorem 5 in [Wimp 1987]: In the first line of page 999, the parameter "$\gamma+c+\beta$" should be replaced by "$\gamma+c-\beta$" (in our notation, the later is $\alpha+c+1$ while the former would be $\alpha+c+1+2\beta$, which indeed doesn't ever seem to figure in the theory).

We next obtain a generating function identity for the Atkin polynomials scaled by a rather unexpected appearance of the Catalan numbers. The right-hand side of the generating series has four summands; each is up to a relatively simple multiple a product of three hypergeometric functions in the variables x, δ and $1/\epsilon = t\delta/x$.

Theorem 6.4. *Let $C(x)$ and $D(x)$ be as in* (5-8), *and δ as in* (6-3). *Furthermore, let $\left\{C_n = 1/(n+1)\binom{2n}{n}\right\}_n$ denote the sequence of Catalan numbers.*

(1) *For $0 < x < 1$ and $|t| < 1$, we have*

$$\sum_{n=0}^{\infty} C_{n+1}\mathcal{A}_{n+1}(x)t^n = \frac{\delta^{17/12}}{x^{11/12}\sqrt{x-t\delta}} F\left(\begin{matrix}\frac{11}{12}, \frac{19}{12}\\ 3\end{matrix}\,\middle|\, \frac{t\delta}{x}\right) \times\left[C(x)F\left(\begin{matrix}\frac{-7}{12}, \frac{17}{12}\\ \frac{1}{3}\end{matrix}\,\middle|\,\delta\right) + D(x)\left(\frac{x}{\delta}\right)^{2/3} F\left(\begin{matrix}\frac{1}{12}, \frac{25}{12}\\ \frac{5}{3}\end{matrix}\,\middle|\,\delta\right)\right]. \tag{6-10}$$

(2) *For $|t| < 1$, we have*

$$\sum_{n=0}^{\infty} C_{n+1}\mathcal{A}_{n+1}(0)(-t)^n = \tfrac{-5}{12} F\left(\begin{matrix}\frac{11}{12}, \frac{17}{12}\\ 3\end{matrix}\,\middle|\, t\right), \tag{6-11}$$

and consequently for $n \geq 0$, we have

$$\mathcal{A}_{n+1}(0) = (-1)^n\left(\tfrac{-5}{12}\right)\frac{\left(\frac{11}{12}\right)_n\left(\frac{17}{12}\right)_n}{(2n+1)!}. \tag{6-12}$$

Proof. Note that for $0 \leq x < 1$, we have

$$|\sqrt{x}+\sqrt{x-1}|^2 = |\sqrt{x}+i\sqrt{1-x}|^2 = 1,$$

so (6-1) indeed translates into $|t| < 1$. Now, using (5-6), we see that

$$\begin{aligned} \frac{(\alpha+\beta+c+1)_n(c+1)_n}{(\alpha+\beta+2c+2)_n} U_n &= \frac{(\alpha+\beta+2c+1)_{2n}}{(\alpha+\beta+2c+2)_n}\widetilde{U}_n \\ &= (\alpha+\beta+2c+1)\frac{(\alpha+\beta+2c+1+n)_n}{(\alpha+\beta+2c+1+n)}\widetilde{U}_n, \end{aligned} \tag{6-13}$$

with a similar identity for Y_n, and (6-10) now follows from (5-7) and Proposition 6.2 by substituting $(\alpha, \beta, c) = \left(\frac{1}{2}, \frac{-2}{3}, \frac{7}{12}\right)$.

When $x = 0$ and $|t| < 1$, then, in the notation of (6-2), we have $t + z_1 = 0$ and $z_2 + t = 2(1+t)$, and hence $z_2 - t = 2$. Furthermore, we have $C(0) = \frac{-5}{12}$ and

$D(0) = 0$. It thus follows from (6-2) and (5-7) that

$$2\sum_{n=0}^{\infty}\binom{2n+1}{n}\mathcal{A}_{n+1}(0)\frac{t^n}{2+n} = \frac{-5}{12(1+t)^{11/12}}F\left(\begin{matrix}\frac{11}{12}, \frac{19}{12}\\ 3\end{matrix}\,\middle|\, \frac{t}{1+t}\right). \tag{6-14}$$

Replacing t with $(-t)$ and applying the Pfaff–Kummer transformation [Erdélyi et al. 1953, Formula (2) on p. 105], we obtain (6-11), from which (6-12) follows by comparing coefficients and simplifying. □

Remark 6.5. Formula (6-12) can also be obtained directly from the defining recursion of the Atkin polynomials, as in Proposition 6 of [Kaneko and Zagier 1998]. In that same proposition, and again using only the defining recurrence (1-1), Kaneko and Zagier also obtain a formula equivalent to

$$\mathcal{A}_{n+1}(1) = \frac{7}{12}\frac{\left(\frac{11}{12}\right)_n\left(\frac{19}{12}\right)_n}{(2n+1)!}. \tag{6-15}$$

Taking a hint from (6-11), it is straightforward to prove directly from (6-15) that for $|t| < 1$, we have

$$\sum_{n=0}^{\infty} C_{n+1}\mathcal{A}_{n+1}(1)t^n = \frac{7}{12}F\left(\begin{matrix}\frac{11}{12}, \frac{19}{12}\\ 3\end{matrix}\,\middle|\, t\right). \tag{6-16}$$

Alternatively, one can prove (6-16) in a manner similar to (6-11), bearing in mind that we have $C(1) = D(1) = 0$, whereas $\widetilde{U}_n^{(1/2,-2/3)}\big(x;\frac{7}{12}\big)$ and $\widetilde{V}_n^{(1/2,-2/3)}\big(x;\frac{7}{12}\big)$ have simple poles at $x = 1$, and thus their product is to be interpreted in the limit $x \to 1^-$ as the derivative of the former multiplied by the residue of the latter.

7. The weight function for the Atkin polynomials

Kaneko and Zagier [1998] gave the weight function for the Atkin polynomials $A_n(j)$ on [0, 1728] as

$$w(j) = \frac{6}{\pi}\theta'(j), \tag{7-1}$$

where $\theta : [0, 1728] \to [\pi/3, \pi/2]$ is the inverse of the monotone increasing function $\theta \mapsto j(e^{i\theta})$, where $j(\tau)$ is the usual modular j-invariant from the theory of modular forms. In this section we derive an explicit description of the weight function in terms of hypergeometric series. Formula (25) on p. 20 of [Erdélyi et al. 1953] states that an inverse for the scaled j-invariant given by

$$J(z) = \frac{j(z)}{1728}$$

is obtainable by the formula

$$z = e^{2\pi i/3} \frac{F - \lambda e^{i\pi/3} J^{1/3} F^*}{F - \lambda e^{-i\pi/3} J^{1/3} F^*}, \tag{7-2}$$

where

$$\begin{aligned} F(J) &= {}_2F_1\left(\begin{matrix} \frac{1}{12}, \frac{1}{12} \\ \frac{2}{3} \end{matrix} \,\middle|\, J \right), \\ F^*(J) &= {}_2F_1\left(\begin{matrix} \frac{5}{12}, \frac{5}{12} \\ \frac{4}{3} \end{matrix} \,\middle|\, J \right), \\ \lambda &= \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{7}{12}\right)} = (2-\sqrt{3}) \frac{\Gamma\left(\frac{2}{3}\right)\Gamma^2\left(\frac{11}{12}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma^2\left(\frac{7}{12}\right)}. \end{aligned} \tag{7-3}$$

We must note that this is one inverse of many as J is invariant under modular transformations. This particular formula gives, easily, that $z(0) = e^{2\pi i/3}$. In order to use the same intervals as in [Kaneko and Zagier 1998], we consider another inverse, corresponding to applying $z \mapsto -1/z$, thus obtaining

$$z(J) = e^{\pi i/3} \frac{F - \lambda e^{-i\pi/3} J^{1/3} F^*}{F - \lambda e^{i\pi/3} J^{1/3} F^*}. \tag{7-4}$$

It is straightforward to verify that using (7-4), we get $z(0) = e^{\pi i/3}$ and $z(1) = e^{\pi i/2}$. For $0 \le J \le 1$, F and F^* are computed in terms of the converging hypergeometric series and hence are real. Thus in the ratio

$$\frac{F(J) - \lambda e^{-i\pi/3} J^{1/3} F^*(J)}{F(J) - \lambda e^{i\pi/3} J^{1/3} F^*(J)}$$

the denominator is the complex conjugate of the numerator. Hence the ratio has absolute value equal to 1, and is of the form $e^{i\rho}$. We will show below that $0 \le \rho \le \pi/6$. Thus an explicit description of the function $\theta(j) : [0, 1728] \to [\pi/3, \pi/2]$ is given by $\theta(j) = \phi(j/1728)$, where $\phi(J) : [0, 1] \to [\pi/3, \pi/2]$ is defined by

$$\phi(J) = \frac{\pi}{3} - i \log\left(\frac{F(J) - \lambda e^{-i\pi/3} J^{1/3} F^*(J)}{F(J) - \lambda e^{i\pi/3} J^{1/3} F^*(J)} \right) = \frac{\pi}{3} + \rho(J),$$

and we have

$$\begin{aligned} \phi'(J) &= -i \frac{F(J) - \lambda e^{i\pi/3} J^{1/3} F^*(J)}{F(J) - \lambda e^{-i\pi/3} J^{1/3} F^*(J)} \left(\frac{F(J) - \lambda e^{-i\pi/3} J^{1/3} F^*(J)}{F(J) - \lambda e^{i\pi/3} J^{1/3} F^*(J)} \right)' \\ &= -i \frac{W(J)}{|F(J) - \lambda e^{-i\pi/3} J^{1/3} F^*(J)|^2}, \end{aligned} \tag{7-5}$$

where $W(J)$ is given explicitly by

$$
\begin{aligned}
(7\text{-}6)\quad & W(J) \\
&= \big(F(J)-\lambda e^{i\pi/3}J^{1/3}F^*(J)\big)\big(F'(J)-\lambda e^{-i\pi/3}J^{1/3}(F^*)'(J)-\tfrac{\lambda}{3}e^{-i\pi/3}J^{-2/3}F^*(J)\big) \\
&\quad -\big(F(J)-\lambda e^{-i\pi/3}J^{1/3}F^*(J)\big)\big(F'(J)-\lambda e^{i\pi/3}J^{1/3}(F^*)'(J)-\tfrac{\lambda}{3}e^{i\pi/3}J^{-2/3}F^*(J)\big) \\
&= \tfrac{\lambda}{3}J^{-2/3}i\sqrt{3}\big(F(J)F^*(J)+3JF(J)(F^*)'(J)-3JF'(J)F^*(J)\big).
\end{aligned}
$$

We also note that W is the Wronskian of two linearly independent solutions for the equation

$$z(1-z)\frac{d^2u}{dz^2}+(c-(1+a+b)z)\frac{du}{dz}-abu=0,$$

where here $a=b=\frac{1}{12}$ and $c=\frac{2}{3}$. It follows that W itself satisfies the equation

$$z(1-z)\frac{dW}{dz}=((a+b+1)z-c)W. \tag{7-7}$$

On the open interval (0, 1), (7-7) has solution

$$W(J)=BJ^{-2/3}(1-J)^{-1/2}. \tag{7-8}$$

To determine the constant B we compare the coefficient of $J^{-2/3}$ in (7-8) and (7-6) to get

$$B=\frac{i\lambda}{\sqrt{3}}.$$

Hence

$$\phi'(J)=\frac{\lambda}{\sqrt{3}}\frac{J^{-2/3}(1-J)^{-1/2}}{|F(J)-\lambda e^{-i\pi/3}J^{1/3}F^*(J)|^2}. \tag{7-9}$$

The fact that the derivative is positive for $0\le J\le 1$ implies that $\phi(J)$ is monotone increasing, and hence that it is bounded between $\phi(0)$ and $\phi(1)$, as we claimed above.

Note that

$$
\begin{aligned}
w(j)&=\frac{6}{\pi}\theta'(j)=\frac{6}{1728\pi}\phi'\left(\frac{j}{1728}\right) \\
&=\frac{6\lambda}{1728\pi\sqrt{3}}\frac{12(12^2j^{-2/3})((1728-j)^{-1/2}12^{3/2})}{\big|12F\big(\frac{j}{1728}\big)-\lambda e^{-i\pi/3}j^{1/3}F^*\big(\frac{j}{1728}\big)\big|^2}.
\end{aligned} \tag{7-10}
$$

We have thus proved the following theorem:

Theorem 7.1. *Let* λ *be as in* (7-3). *Then the normalized weight function for the Atkin polynomials* $A_n(j)$ *on the interval* [0, 1728] *is given by*

$$w(j)=\frac{144\lambda}{\pi}\frac{j^{-2/3}(1728-j)^{-1/2}}{\left|12F\left(\begin{matrix}\frac{1}{12},\frac{1}{12}\\ \frac{2}{3}\end{matrix}\,\middle|\,\frac{j}{1728}\right)-\lambda e^{-i\pi/3}j^{1/3}F\left(\begin{matrix}\frac{5}{12},\frac{5}{12}\\ \frac{4}{3}\end{matrix}\,\middle|\,\frac{j}{1728}\right)\right|^2}. \tag{7-11}$$

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