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We consider the affine variety $\mathscr{Z}_{2,2}^{m,n}$ of first-order jets over $\mathscr{Z}_2^{m,n}$, where $\mathscr{Z}_2^{m,n}$ is the classical determinantal variety given by the vanishing of all 2×2 minors of a generic $m \times n$ matrix. When $2 < m \leq n$, this jet scheme $\mathscr{Z}_{2,2}^{m,n}$ has two irreducible components: a trivial component, isomorphic to an affine space, and a nontrivial component that is the closure of the jets supported over the smooth locus of $\mathscr{Z}_2^{m,n}$. This second component is referred to as the *principal component* of $\mathscr{Z}_{2,2}^{m,n}$; it is, in fact, a cone and can also be regarded as a projective subvariety of $\mathbb{P}^{2mn-1}$. We prove that the degree of the principal component of $\mathscr{Z}_{2,2}^{m,n}$ is the square of the degree of $\mathscr{Z}_2^{m,n}$ and, more generally, the Hilbert series of the principal component of $\mathscr{Z}_{2,2}^{m,n}$ is the square of the Hilbert series of $\mathscr{Z}_2^{m,n}$. As an application, we compute the a-invariant of the principal component of $\mathscr{Z}_{2,2}^{m,n}$ and show that the principal component of $\mathscr{Z}_{2,2}^{m,n}$ is Gorenstein if and only if $m = n$.

1. Introduction

Let $\mathbb{F}$ be an algebraically closed field and m, n, r be integers with $1 \leq r \leq m \leq n$. Let $\mathscr{Z}_r^{m,n}$ denote the affine variety in $\mathbb{A}_{\mathbb{F}}^{mn}$ defined by the vanishing of all $r \times r$ minors of an $m \times n$ matrix whose entries are independent indeterminates over $\mathbb{F}$. Equivalently $\mathscr{Z}_r^{m,n}$ is the locus of $m \times n$ matrices over $\mathbb{F}$ of rank $< r$. This is a classical and well-studied object and a number of its properties are known. For example, we know that $\mathscr{Z}_r^{m,n}$ is irreducible, rational, arithmetically Cohen–Macaulay and projectively normal. Moreover the multiplicity of $\mathscr{Z}_r^{m,n}$ (at its vertex, since $\mathscr{Z}_r^{m,n}$ is evidently a cone) or, equivalently, the degree of the corresponding projective subvariety of $\mathbb{P}_{\mathbb{F}}^{mn-1}$ is given by the following elegant formula (see [Abhyankar 1988, Remarks 20.18 and 20.19] or [Ghorpade 1994, Corollary 6.2]; see also [Herzog and Trung 1992] for an alternative proof and [Arbarello et al. 1985, Chapter 2, §4] or [Ghorpade and Krattenthaler 2004, p. 352] for an alternative approach and a different formula):

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$$e(\mathscr{L}_r^{m,n}) = \det_{1\le i,j\le r-1}\left(\binom{m+n-i-j}{m-i}\right). \tag{1}$$

More generally, the Hilbert series of $\mathscr{L}_r^{m,n}$ (or, more precisely, of the corresponding projective subvariety of $\mathbb{P}_{\mathbb{F}}^{mn-1}$) is also known and is explicitly given by

$$\frac{\sum_{k\ge 0} h_k t^k}{(1-t)^d}, \tag{2}$$

where $d = (r-1)(m+n-r+1)$ is the dimension of $\mathscr{L}_r^{m,n}$ (as an affine variety), and the coefficients h_k are given by sums of binomial determinants as follows:

$$h_k = \sum_{k_1+\cdots+k_{r-1}=k} \det_{1\le i,j\le r-1}\left(\binom{m-i}{k_i}\binom{n-j}{k_i+i-j}\right).$$

For a proof of this formula, we refer to [Ghorpade 1996] (see also [Galligo 1985] and [Conca and Herzog 1994]). Using this, or otherwise (see [Svanes 1974]), it can be shown that $\mathscr{L}_r^{m,n}$ is Gorenstein if and only if $m = n$. Moreover one can also show that the a-invariant of the (homogeneous) coordinate ring of $\mathscr{L}_r^{m,n}$ (which, by definition, is the least degree of a generator of its graded canonical module) is $n(1-r)$; see, e.g., [Gräbe 1988] or [Ghorpade 1996, Theorem 4].

We now turn to jet schemes, which have been of much recent interest due in large part to Nash's suggestion [1995] that jet schemes should give information about singularities of the base; see, e.g., [Mustaţă 2001; 2002; Ein and Mustaţă 2009]. If $\mathscr{L}$ is a scheme of finite type over $\mathbb{F}$ and k a positive integer, then a $(k-1)$-jet on $\mathscr{L}$ is a morphism $\operatorname{Spec}\mathbb{F}[t]/(t^k) \to \mathscr{L}$. The set of $(k-1)$-jets on $\mathscr{L}$ forms a scheme of finite type over $\mathbb{F}$, denoted $\mathscr{J}_{k-1}(\mathscr{L})$ and called the $(k-1)$-th jet scheme of $\mathscr{L}$. A little more concretely, suppose $\mathscr{L}$ is the affine scheme $\operatorname{Spec} S/I$ defined by the ideal $I = \langle f_1, \dots, f_s\rangle$ in the polynomial ring $S = \mathbb{F}[X_1, \dots, X_N]$. Consider independent indeterminates t and $X_i^{(\ell)}$ ($i = 1, \dots, N$ and $\ell = 0, \dots, k-1$) over $\mathbb{F}$ and the corresponding polynomial ring $S^{(k)}$ in the Nk variables $X_i^{(\ell)}$. For each $j = 1, \dots, s$, the polynomial

$$f_j\big(X_1^{(0)} + tX_1^{(1)} + \cdots + t^{k-1}X_1^{(k-1)}, \dots, X_N^{(0)} + tX_N^{(1)} + \cdots + t^{k-1}X_N^{(k-1)}\big)$$

is of the form

$$f_j^{(0)} + tf_j^{(1)} + \cdots + t^{k-1}f_j^{(k-1)} \quad \text{modulo } \langle t^k\rangle$$

for unique $f_j^{(\ell)} \in S^{(k)}$ ($0 \le \ell < k$). Then $\mathscr{J}_{k-1}(\mathscr{L})$ is the affine scheme $\operatorname{Spec} S^{(k)}/I'$, where I' is the ideal generated by all $f_j^{(\ell)}$, $1 \le j \le s$, $0 \le \ell < k$, (Often in the literature, authors conflate the algebraic set in $\mathbb{A}^{Nk}$ consisting of the zeros of the polynomials $f_j^{(\ell)}$ with $\mathscr{J}_{k-1}(\mathscr{L})$ itself. This is generally harmless, especially when considering topological properties such as components, since the points of this algebraic set correspond bijectively with the set of closed points of $\mathcal{J}_{k-1}(\mathcal{X})$ as $\mathbb{F}$ is algebraically closed, and the set of closed points of an affine scheme is dense in the scheme. See [Liu 2002, Chapter 2, Remark 3.49], for instance.)

When $\mathcal{X}$ is smooth of dimension d, the jet scheme $\mathcal{J}_{k-1}(\mathcal{X})$ is known to be smooth of dimension kd. In general, $\mathcal{J}_{k-1}(\mathcal{X})$ can have multiple irreducible components, and these include a principal component that corresponds to the closure of the set of jets supported over the smooth points of the base scheme $\mathcal{X}$. These components are usually quite complicated and interesting. In fact, very little seems to be known about the structure of these components and their numerical invariants such as multiplicities. For example, even when $\mathcal{X}$ is a monomial scheme such as the one given by $X_1 X_2 \cdots X_e = 0$, where $e \le N$, determining the irreducible components and the multiplicity of $\mathcal{J}_{k-1}(\mathcal{X})$ appears to require some effort; see, e.g., [Goward and Smith 2006] and [Yuen 2007b]. Irreducible components of jet schemes of toric surfaces are discussed in [Mourtada 2011], while the irreducibility of jet schemes of the commuting matrix pairs scheme is discussed in [Sethuraman and Šivic 2009]. In a more recent work [Bruschek et al. 2011], the Hilbert series of arc spaces (that are, in a sense, limits of k-th jet schemes as $k \to \infty$) of seemingly simple objects such as the double line $y^2 = 0$ are shown to have connections with the Rogers–Ramanujan identities.

Now determinantal varieties such as $\mathcal{X}_r^{m,n}$ above are natural examples of singular algebraic varieties, and it is not surprising that the study of their jet schemes has been of considerable interest. This was done first by Košir and Sethuraman [2005a; 2005b] (see also [Yuen 2007a]). To describe the related results, henceforth we fix positive integers r, k, m, n with $r \le m \le n$, and let $\mathcal{X}_{r,k}^{m,n}$ denote the $(k-1)$-th jet scheme on $\mathcal{X}_r^{m,n}$. It was shown in [Košir and Sethuraman 2005a] that $\mathcal{X}_{r,k}^{m,n}$ is irreducible of codimension $k(n-m+1)$ when $r = m$, and if $r < m$, then it can have $\ge 1 + \lfloor k/2 \rfloor$ irreducible components with equality when $r = 2$ or $k = 2$. A more unified result was obtained in [Docampo 2013], showing that $\mathcal{X}_{r,k}^{m,n}$ has exactly $k + 1 - \lceil k/r \rceil$ irreducible components. At any rate, the best understood case with multiple components is $\mathcal{X}_{2,2}^{m,n}$, where $2 < m \le n$. In this case $\mathcal{X}_{2,2}^{m,n} = Z_0 \cup Z_1$, where Z_1 is isomorphic to $\mathbb{A}^{mn}$ while Z_0 is the principal component which is the closure of the jets supported over the smooth points of the base variety $\mathcal{X}_2^{m,n}$. Here it will be convenient to consider $2mn$ indeterminates, denoted $x_{i,j}, y_{i,j}$ for $1 \le i \le m$, $1 \le j \le n$, and the corresponding polynomial ring $R = \mathbb{F}[x_{i,j}, y_{i,j} : 1 \le i \le m, 1 \le j \le n]$. Also let $\mathcal{I} = \mathcal{I}_{2,2}^{m,n}$ and $\mathcal{I}_0$ denote, respectively, the ideals of R corresponding to the jet scheme $\mathcal{X}_{2,2}^{m,n}$ and its principal component Z_0. In [Košir and Sethuraman 2005b], it was shown that both $\mathcal{I}$ and $\mathcal{I}_0$ are homogeneous radical ideals of R (so that $\mathcal{I}_0$ is prime), and moreover their Gröbner bases were explicitly determined. The leading term ideal $\mathrm{LT}(\mathcal{I}_0)$ of $\mathcal{I}_0$ with respect to this Gröbner basis is generated by squarefree monomials and hence $R/\mathrm{LT}(\mathcal{I}_0)$ is the Stanley–Reisner ring of a simplicial complex Δ_0. Jonov [2011] subsequently studied this simplicial complex. He showed that Δ_0 is shellable and thus deduced that $R/\mathcal{I}_0$ is Cohen–Macaulay. (This last result was independently obtained in [Smith and Weyman 2007] as well, using a geometric technique for computing syzygies.) Jonov also found a formula for the multiplicity of $R/\mathcal{I}_0$, namely,

$$e(R/\mathcal{I}_0) = \sum_{\substack{i=1 \\ (i,j)\neq(m,n)}}^{m} \sum_{j=1}^{n} \binom{m+n-i-j}{m-i} \det \begin{pmatrix} \binom{i+n-2}{i-1} & \binom{m+j-2}{m-1} \\ \binom{i+n-3}{i-2} & \binom{m+j-3}{m-2} \end{pmatrix}. \tag{3}$$

Equation (3) above is the starting point of the present paper. We first show that the right side of this equation simplifies remarkably to yield the pretty result

$$e(R/\mathcal{I}_0) = \binom{m+n-2}{m-1}^2 = e(\mathcal{Z}_2^{m,n})^2.$$

(this was already alluded to in [Jonov 2011, Remark 2.8]). Next we proceed to determine the Hilbert series of $R/\mathcal{I}_0$ or of the principal component Z_0. We use the well-known connections between the Hilbert series of $R/\mathcal{I}_0$, that of $R/\mathrm{LT}(\mathcal{I}_0)$ and the shelling of the facets of the simplicial complex Δ_0 obtained in [Jonov 2011]. With some effort we are led to an initial formula for the Hilbert series of $R/\mathcal{I}_0$, which is enormously complicated and involves multiple sums of products of binomials in the same vein as the right side of (3). But we persist with the combinatorics and are eventually rewarded with the main result of this paper. Namely, just like the multiplicity, the Hilbert series of $R/\mathcal{I}_0$ is precisely the square of the Hilbert series of the base determinantal variety $\mathcal{Z}_2^{m,n}$. As a corollary of this, we are able to determine the a-invariant of $R/\mathcal{I}_0$ and the Hilbert series of its graded canonical module. Moreover we show that, as in the case of classical determinantal varieties, Z_0 is Gorenstein if and only if $m = n$.

The proofs given here are completely elementary but highly combinatorial and rather intricate. Heuristically it appears to us that up to some flat deformation (such as the Gröbner deformation of $\mathcal{I}_0$ to $\mathrm{LT}(\mathcal{I}_0)$, which preserves the Hilbert series), the coordinate ring of the principal component (suitably deformed) should look like the tensor product of the coordinate ring of the base (similarly deformed) with itself. (This would reflect the fact that, at the smooth points, the base variety locally looks like its tangent space.) It would follow then that the Hilbert series of the principal component is the square of that of $\mathcal{Z}_2^{m,n}$. We emphasize that this is only heuristics (with all of its ever-present dangers); nevertheless we suspect that analogous results relating the Hilbert series of the principal component to that of the base scheme should hold more generally for all $\mathcal{Z}_{r,k}^{m,n}$, and possibly also for jet schemes over a wider class of affine base schemes. We do not know how to prove this, and leave it open for investigation.

2. Binomials and lattice paths

In this section we collect some preliminaries concerning binomial coefficients, alterations of summations, and lattice paths. These will be useful in the sequel.

2.1. ***Binomials.*** To begin with, let us recall that the binomial coefficient $\binom{s}{a}$ is defined for any integer parameters s, a (and with the standard convention that the empty product is taken as 1) as follows:

$$\binom{s}{a} = \begin{cases} \dfrac{s(s-1)\cdots(s-a+1)}{a!} & \text{if } a \geq 0, \\ 0 & \text{if } a < 0. \end{cases}$$

In fact, this definition makes sense not only for any $s \in \mathbb{Z}$ but also for s in any overring of $\mathbb{Z}$ and in particular, s can be an indeterminate over $\mathbb{Q}$ in which case $\binom{s}{a}$ is a polynomial in s of degree a, provided $a \geq 0$. Now let $s, a \in \mathbb{Z}$. Note that

$$\binom{s}{a} = 0 \iff \text{either } a < 0 \text{ or } a > s \geq 0. \tag{4}$$

One has to be careful with the validity of some of the familiar identities; for example,

$$\binom{s}{a} = \binom{s}{s-a} \iff \text{either } s \geq 0 \text{ or } s < a < 0, \tag{5}$$

whereas some standard identities such as the Pascal triangle identity or its alternative equivalent version below are valid for arbitrary integer parameters:

$$\binom{s}{a-1} + \binom{s}{a} = \binom{s+1}{a} \quad \text{and} \quad \binom{s+a}{a} + \binom{s+a}{a+1} = \binom{s+a+1}{a+1}. \tag{6}$$

The equivalence of the two identities above follows from the simple fact below, which is also valid for arbitrary integer parameters:

$$\binom{s+a}{a} = (-1)^a \binom{-s-1}{a}, \quad \text{that is,} \quad \binom{s}{a} = (-1)^a \binom{a-s-1}{a}. \tag{7}$$

We now record some basic facts, which are often used in later sections. Proofs are easy and are briefly outlined for the sake of completeness.

Lemma 1. *For any $e, s, t \in \mathbb{Z}$ with $s \leq t$, we have*

$$\sum_{s<d\leq t} \binom{d}{e} = \binom{t+1}{e+1} - \binom{s+1}{e+1}.$$

Proof. Induct on $t - s$, using the first identity in (6) to rewrite $\binom{t+1}{e+1}$. □

The following result is a version of the so-called Chu–Vandermonde identity.

Lemma 2. *For any $s, t, \alpha, \beta \in \mathbb{Z}$, we have*

$$\sum_{j\in\mathbb{Z}} \binom{s}{\alpha+j}\binom{t}{\beta-j} = \binom{s+t}{\alpha+\beta} \tag{8}$$

and

$$\sum_{j\in\mathbb{Z}} \binom{s+\alpha+j}{\alpha+j}\binom{t+\beta-j}{\beta-j} = \binom{s+t+\alpha+\beta+1}{\alpha+\beta}, \tag{9}$$

where, in view of (4), the summation on the left in (8) as well as in (9) is essentially finite in the sense that all except finitely many summands are zero.

Proof. Let X be an indeterminate over $\mathbb{Q}$. Use the binomial theorem, namely,

$$(1+X)^d = \sum_{i=0}^{\infty} \binom{d}{i} X^i,$$

which is valid in the formal power series ring $\mathbb{Q}[\![X]\!]$ for any $d \in \mathbb{Z}$, and compare the coefficients of $X^{\alpha+\beta}$ on the two sides of the identity $(1+X)^s(1+X)^t = (1+X)^{s+t}$ to obtain (8). Now (8) and (7) imply (9). □

2.2. ***Alterations of summations.*** As in (8) and (9) above, we will often deal with summations that are *essentially finite*, by which we mean that the parameters in the sum range over an infinite set, but the summand is zero for all except finitely many values of parameters, and so the summation is, in fact, finite. It is, however, very useful that the parameters range freely over a seemingly infinite set so that useful alterations such as the ones listed below can be readily made. These are too obvious to be stated as lemmas and proved formally. But for ease of reference, we record below some elementary transformations of essentially finite summations. In what follows, $f : \mathbb{Z}^2 \to \mathbb{Q}$ will denote a rational-valued function of two integer parameters with the property that the *support* of f, namely, the set $\{(s_1, s_2) \in \mathbb{Z}^2 : f(s_1, s_2) \neq 0\}$ is finite or more generally, it is *diagonally finite*, that is, for each $k \in \mathbb{Z}$, the set $\{(s_1, s_2) \in \mathbb{Z}^2 : s_1 + s_2 = k \text{ and } f(s_1, s_2) \neq 0\}$ is finite. In this case, for any $\nu \in \mathbb{Z}$ and any $\alpha, \beta \in \mathbb{Z}$ such that $\alpha + \beta = \nu$, we have

$$\sum_{s_1+s_2=k-\nu} f(s_1, s_2) = \sum_{t_1+t_2=k} f(t_1 - \alpha, t_2 - \beta), \tag{10}$$

where writing $s_1 + s_2 = k - \nu$ below the first summation indicates that the sum is over all $(s_1, s_2) \in \mathbb{Z}^2$ satisfying $s_1 + s_2 = k - \nu$. A similar meaning applies for the second summation and in fact, for all such summations appearing in the sequel. Since the "diagonal condition" $t_1 + t_2 = k$ is symmetric, we also have

$$\sum_{t_1+t_2=k} f(t_1, t_2) = \sum_{t_1+t_2=k} f(t_2, t_1). \tag{11}$$

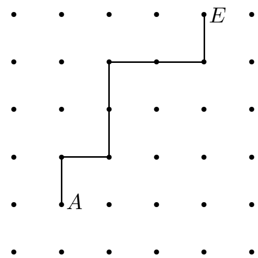

Figure 1. A lattice path from $A = (1, 1)$ to $E = (4, 5)$.

Thus, for example, using (10) and (11), we find

$$\sum_{t_1+t_2=k} f(t_1, t_2) = \sum_{t_1+t_2=k} f(t_2+1, t_1-1) = \sum_{t_1+t_2=k} f(t_1+1, t_2-1).$$

2.3. ***Lattice paths.*** Let $A = (a, a')$ and $E = (e, e')$ be points in the integer lattice $\mathbb{Z}^2$. By a *lattice path* from A to E we mean a finite sequence $L = (P_0, P_1, \ldots, P_t)$ of points in $\mathbb{Z}^2$ with $P_0 = A$, $P_t = E$ and

$$P_i - P_{i-1} = (1, 0) \text{ or } (0, 1) \quad \text{for } i = 1, \ldots, t.$$

The lattice path L can and will be identified with its point set $\{P_j : 0 \le j \le t\}$; indeed L is obtained by simply arranging the elements of this set in a lexicographic order. The point $A = P_0$ is called the *initial point* of L while $E = P_t$ is called the *end point* of L. We say that a point P_j is a *NE-turn* of the lattice path L if $0 < j < t$ and $P_j - P_{j-1} = (0, 1)$ while $P_{j+1} - P_j = (1, 0)$. Note that a lattice path is also determined by its NE turns.

In more intuitive terms, a lattice path consists of vertical or horizontal steps of length 1, and a NE-turn is simply a northeast turn. For example, a lattice path from $A = (1, 1)$ to $E = (4, 5)$ may be depicted as in Figure 1, and it may be noted that the points $(1, 2)$ and $(2, 4)$ are its NE turns.

If we let $\mathcal{P}(A \to E)$ denote the set of lattice paths from $A = (a, a')$ to $E = (e, e')$ and, for any $k \in \mathbb{Z}$, let $\mathcal{P}_k(A \to E)$ denote the subset of $\mathcal{P}(A \to E)$ consisting of lattice paths with exactly k NE turns, then it is easily seen that

$$|\mathcal{P}(A \to E)| = \binom{e-a+e'-a'}{e-a},$$
$$|\mathcal{P}_k(A \to E)| = \binom{e-a}{k}\binom{e'-a'}{k}, \tag{12}$$

where as usual, for a finite set $\mathcal{P}$, we denote by $|\mathcal{P}|$ the cardinality of $\mathcal{P}$. Given any two d-tuples $\mathcal{A} = (A_1, \ldots, A_d)$ and $\mathcal{E} = (E_1, \ldots, E_d)$ of points in $\mathbb{Z}^2$, by a *lattice path* from $\mathcal{A}$ to $\mathcal{E}$ we mean a d-tuple $\mathcal{L} = (L_1, \ldots, L_d)$, where L_r is a lattice path from A_r to E_r, for $1 \le r \le d$. We call $\mathcal{L}$ to be *nonintersecting* if no two of the paths $L_1, \dots, L_d$ have a point in common. We say that $\mathscr{L}$ has k NE turns if the total number of NE turns in the d paths $L_1, \dots, L_d$ is k. The set of nonintersecting lattice paths from $\mathscr{A} = (A_1, \dots, A_d)$ to $\mathscr{E} = (E_1, \dots, E_d)$ will be denoted by $\mathscr{P}(A_1 \to E_1, \dots, A_d \to E_d)$ or simply by $\mathscr{P}(\mathscr{A} \to \mathscr{E})$, and its subset consisting of nonintersecting lattice paths with exactly k NE turns will be denoted by $\mathscr{P}_k(A_1 \to E_1, \dots, A_d \to E_d)$ or simply by $\mathscr{P}_k(\mathscr{A} \to \mathscr{E})$.

Proposition 3. *Let d be a positive integer and let $A_r = (a_r, a'_r)$ and $E_r = (e_r, e'_r)$, $r = 1, \dots, d$, be points in $\mathbb{Z}^2$. Also let $\mathscr{A} = (A_1, \dots, A_d)$ and $\mathscr{E} = (E_1, \dots, E_d)$.*

(i) *Suppose*

$$a_1 \le \cdots \le a_d, \quad e_1 \le \cdots \le e_d \quad \textit{and} \quad a'_1 \ge \cdots \ge a'_d, \quad e'_1 \ge \cdots \ge e'_d.$$

Then the number of nonintersecting lattice paths from $\mathscr{A}$ to $\mathscr{E}$ is equal to

$$\det\left(\binom{e_j - a_i + e'_j - a'_i}{e_j - a_i}_{1\le i,j\le d}\right) \tag{13}$$

(ii) *Let $k \in \mathbb{Z}$ and suppose*

$$a_1 \le \cdots \le a_d, \quad e_1 < \cdots < e_d \quad \textit{and} \quad a'_1 > \cdots > a'_d, \quad e'_1 \ge \cdots \ge e'_d.$$

Then the number of nonintersecting lattice paths from $\mathscr{A}$ to $\mathscr{E}$ with exactly k NE turns is equal to

$$\sum_{k_1+\cdots+k_d=k} \det\left(\binom{e_j - a_i + i - j}{k_i + i - j}\binom{e'_j - a'_i - i + j}{k_i}_{1\le i,j\le d}\right) \tag{14}$$

Part (i) of the above proposition is due to Gessel and Viennot [1985, Theorem 1], although some of the ideas can be traced back to Chaundy [1932], Karlin and McGregor [1959], and Lindström [1973]. The statement here is a little more general than that of [Gessel and Viennot 1985], and a proof can be found, for example, in [Ghorpade 2001, §3] or [Krattenthaler 1995b, §2.2]. Part (ii) was proved independently by Modak [1992], Krattenthaler [1995a] and Kulkarni [1996] (see also [Ghorpade 1996]), although the hypothesis in [Modak 1992] and [Kulkarni 1996] on the coordinates of the initial and the end points is slightly more restrictive than in (ii) above where we follow [Krattenthaler 1995a, Theorem 1]. The following consequence is frequently used in Section 4.

Corollary 4. *For any $a, b, c, d, s \in \mathbb{Z}$ with $a < c$ and $b \ge d$, the cardinality of $\mathscr{P}_s((1, 2) \to (a, b), (1, 1) \to (c, d))$ is given by*

$$\sum_{s_1+s_2=s} \binom{a-1}{s_1}\binom{b-2}{s_1}\binom{c-1}{s_2}\binom{d-1}{s_2} - \binom{a}{s_2+1}\binom{b-2}{s_2}\binom{c-2}{s_1-1}\binom{d-1}{s_1}.$$

Proof. This is just a special case of part (ii) of Proposition 3. $\square$

3. Multiplicity

As in the Introduction, we fix in the remainder of this paper an algebraically closed field $\mathbb{F}$ and integers m, n with $2 < m \leq n$. Also let $x_{i,j}$, $y_{i,j}$, $1 \leq i \leq m$, $1 \leq j \leq n$, be independent indeterminates over $\mathbb{F}$. Denote by V_x the set

$$\{x_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$$

of the "x-variables", and by V_y a similar set of the "y-variables". Let $V = V_x \cup V_y$ and let $R = \mathbb{F}[V]$ be the corresponding polynomial ring in $2mn$ variables; also let $R_x = \mathbb{F}[V_x]$ and $R_y = \mathbb{F}[V_y]$ be the corresponding polynomial rings in mn variables. By the *support* of a monomial F in R, denoted $\operatorname{supp}(F)$, we mean the subset of V consisting of the variables appearing in F. Clearly a monomial F in R can be uniquely written as

$$F = F_x F_y, \quad \text{where } F_x, F_y \text{ are monomials with } F_x \in R_x \text{ and } F_y \in R_y, \tag{15}$$

and moreover F is squarefree if and only if both F_x and F_y are squarefree. Note that squarefree monomials can be identified with their supports, and in particular, faces of a simplicial complex Δ with vertex set V can be viewed as squarefree monomials in R. With this in view, we may not distinguish between a squarefree monomial and its support, and we may sometimes write $x_{i,j} \in G$ rather than $x_{i,j} \mid G$ when G is a squarefree monomial in R and $x_{i,j}$ is a variable appearing in it. A monomial G in R_x will be called a *lattice path monomial* in R_x if there is a positive integer t and variables $x_{i_1,j_1}, \ldots, x_{i_t,j_t}$ in V_x such that

$$G = \prod_{s=1}^{t} x_{i_s,j_s} \quad \text{with } (i_s - i_{s-1},\, j_s - j_{s-1}) = (1, 0) \text{ or } (0, 1) \text{ for } 1 < s \leq t. \tag{16}$$

In this case G is called a lattice path monomial from x_{i_1,j_1} to x_{i_t,j_t}, and we will refer to x_{i_1,j_1} as the *leader* of G and denote it by $\mu(G)$. Note that $\mu(G) = x_{i_1,j_1}$ depends only on G (and not on the given ordering of the variables appearing in it) since (i_1, j_1) is lexicographically the least among the pairs (i, j) for which $x_{i,j} \in \operatorname{supp}(G)$. A variable x_{i_s,j_s} in $\operatorname{supp}(G)$ will be called an *ES-turn* of G if $1 < s < t$, $i_s = i_{s-1}$, and $j_s = j_{s+1}$. Analogously a variable x_{i_s,j_s} in $\operatorname{supp}(G)$ will be called a *SE-turn* of G if $1 < s < t$, $j_s = j_{s-1}$, and $i_s = i_{s+1}$. Moreover we will call a variable x_{i_s,j_s} in $\operatorname{supp}(G)$ the *midpoint of a segment* in G if $1 < s < t$ and either $i_{s-1} = i_s = i_{s+1}$ (horizontal segment) or $j_{s-1} = j_s = j_{s+1}$ (vertical segment). It may be noted that a variable x_{i_s,j_s} with $1 < s < t$ is either an ES-turn or a SE-turn or the midpoint of a segment in G.

Evidently lattice path monomials in R_x correspond to lattice paths in the sense of Section 2.3 if we turn the $m \times n$ rectangular matrix $(x_{i,j})$ left by 90° and identify the variable $x_{i,j}$ with the lattice point (i, j). In this way leaders correspond to initial

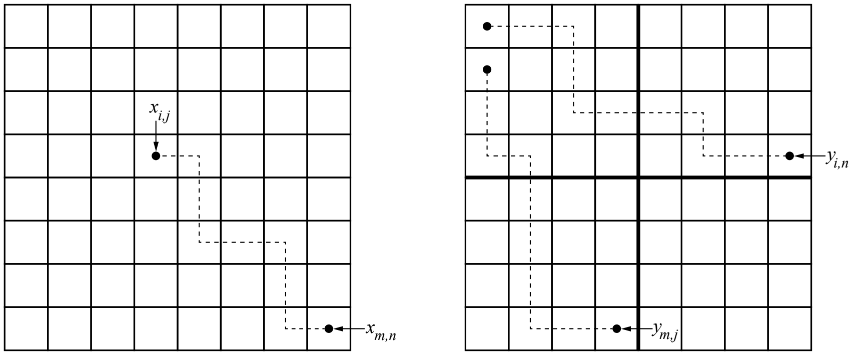

Figure 2. Lattice path monomials F_x and $F_y = F_y^{\mathsf{U}} F_y^{\mathsf{L}}$ in Proposition 5.

points while ES turns correspond to NE turns. Lattice path monomials in R_y together with their leaders, ES turns, SE turns, and midpoints of segments are similarly defined (and similarly identified with lattice paths in the sense of Section 2.3).

We have noted in the introduction that a Gröbner basis (with respect to reverse lexicographic order on monomials with the $2mn$ variables arranged suitably) of the ideal $\mathcal{I}$ of the variety $\mathfrak{X}_{2,2}^{m,n}$ of first-order jets over $\mathfrak{X}_2^{m,n}$, as well as of the ideal $\mathcal{I}_0$ of the principal component Z_0 of $\mathfrak{X}_{2,2}^{m,n}$, was determined in [Košir and Sethuraman 2005b]. As a consequence, one can write down the generators of the leading term ideal of $\mathcal{I}_0$ (see [Jonov 2011, Proposition 1.1]), say $\mathrm{LT}(\mathcal{I}_0)$, and deduce that $R/\mathrm{LT}(\mathcal{I}_0)$ is the Stanley–Reisner ring of a simplicial complex Δ_0 with V as its set of vertices. A precise description of the facets of Δ_0 was given by Jonov [2011, §2], and we recall it below.

Proposition 5. *A squarefree monomial F, decomposed as in (15) above, is a facet of Δ_0 if and only if there is a unique $(i, j) \in \mathbb{Z}^2$, with $1 \le i \le m$, $1 \le j \le n$, such that $(i, j) \ne (m, n)$ and F_x is a lattice path monomial from $x_{i,j}$ to $x_{m,n}$, whereas $F_y = F_y^{\mathsf{U}} F_y^{\mathsf{L}}$, where F_y^{U} is a lattice path monomial from $y_{1,1}$ to $y_{i,n}$, F_y^{L} is a lattice path monomial from $y_{2,1}$ to $y_{m,j}$, and the supports of F_y^{U} and F_y^{L} are disjoint.*

The lattice path monomials F_x and $F_y = F_y^{\mathsf{U}} F_y^{\mathsf{L}}$ are illustrated in Figure 2 by the corresponding "paths" in rectangular matrices.

Using Proposition 5 together with the first identity in (12) and part (i) of Proposition 3, Jonov showed that the simplicial complex Δ_0 is pure (i.e., all its facets have the same dimension) and deduced the dimension and the formula stated in the introduction for the multiplicity of the coordinate ring $R/\mathcal{I}_0$ of Z_0.

Corollary 6. *The (Krull) dimension of $R/\mathcal{I}_0$ is $2(m + n - 1)$ and the multiplicity of $R/\mathcal{I}_0$ is given by (3).*

Now here is the pretty result about the multiplicity that was alluded to in the introduction, namely, that the formula (3) admits a remarkable simplification.

Theorem 7. *The multiplicity of $R/\mathcal{I}_0$ is given by*

$$e(R/\mathcal{I}_0) = \binom{m+n-2}{m-1}^2. \tag{17}$$

Proof. For $1 \le i \le m$ and $1 \le j \le n$, let $\Delta_{i,j}$ denote the 2×2 determinant in (3). Observe that if $(i, j) = (m, n)$, then $\Delta_{i,j} = 0$. Thus, by expanding this determinant and rearranging the summands, we can write

$$e(R/\mathcal{I}_0) = \sum_{i=1}^{m} \binom{i+n-2}{i-1} \sum_{j=1}^{n} S_{i,j} - \sum_{i=1}^{m} \binom{i+n-3}{i-2} \sum_{j=1}^{n} T_{i,j},$$

where, for $1 \le i \le m$ and $1 \le j \le n$, we have put

$$S_{i,j} = \binom{m+n-i-j}{m-i}\binom{m+j-3}{m-2} \quad \text{and} \quad T_{i,j} = \binom{m+n-i-j}{m-i}\binom{m+j-2}{m-1}.$$

Rewriting $S_{i,j}$ using (5) and then noting that the resulting product is zero if $j < 1$ or $j > n$, thanks to (4), we see from Equation (9) in Lemma 2 that

$$\sum_{j=1}^{n} S_{i,j} = \sum_{j} \binom{m+n-i-j}{n-j}\binom{m+j-3}{j-1} = \binom{2m+n-i-2}{n-1},$$

for each $i = 1, \dots, m$. In a similar manner,

$$\sum_{j=1}^{n} T_{i,j} = \sum_{j} \binom{m+n-i-j}{n-j}\binom{m+j-2}{j-1} = \binom{2m+n-i-1}{n-1},$$

for each $i = 1, \dots, m$. It follows that $e(R/\mathcal{I}_0)$ is given by the telescoping sum

$$e(R/\mathcal{I}_0) = \sum_{i=1}^{m} (a_i - a_{i-1}), \quad \text{where } a_i := \binom{i+n-2}{i-1}\binom{2m+n-i-2}{n-1},$$

for $0 \le i \le m$. Since $a_0 = 0$ and $a_m = \binom{m+n-2}{m-1}^2$, we obtain the desired result. $\square$

It may be noted that in view of (1) and (17), the multiplicity of the principal component Z_0 is precisely the square of the multiplicity of the base variety $\mathcal{Z}_2^{m,n}$.

4. Hilbert series

Let us begin by recalling that a *shelling* of a pure simplicial complex Δ is a linear ordering $F_1, \dots, F_e$ of its facets such that for all positive integers i, j, with $j < i \le e$, there exist some $v \in F_i \setminus F_j$ and some positive integer $k < i$ such that $F_i \setminus F_k = \{v\}$. Given such a shelling and any $t \in \{1, \dots, e\}$, we let

$$c(F_t) = \{v \in F_t : \text{there exists } s < t \text{ such that } F_t \setminus F_s = \{v\}\}.$$

Elements of $c(F_t)$ will be referred to as the *corners* of F_t. It may be noted that $c(F_t)$ is nonempty if and only if $t > 1$. Recall also that a simplicial complex Δ is said to be *shellable* if it is pure and it has a shelling. The following result is well known (see [Bruns and Conca 2003, Theorem 6.3]).

Proposition 8. *Let Δ be a shellable simplicial complex and let R_Δ denote its Stanley–Reisner ring. Then*:

(i) *R_Δ is Cohen–Macaulay and its* (*Krull*) *dimension* $\dim R_\Delta$ *is* $1+\dim\Delta$.

(ii) *Suppose $d = \dim R_\Delta$ and $F_1, \dots, F_e$ is a shelling of Δ. Then the Hilbert series of R_Δ is given by*

$$\frac{\sum_{j\geq 0} h_j z^j}{(1-z)^d}, \quad \textit{where } h_j = \left|\{t \in \{1,\dots,e\} : |c(F_t)| = j\}\right| \textit{ for } j \geq 0.$$

Jonov [2011] showed that the simplicial complex Δ_0 mentioned in the previous section is shellable and concluded using part (i) of Proposition 8 that the coordinate ring of $R/\mathcal{I}_0$ of the principal component Z_0 of $\mathfrak{X}_{2,2}^{m,n}$ is Cohen–Macaulay. We shall now proceed to use part (ii) of Proposition 8 to determine the Hilbert series of $R/\mathcal{I}_0$. We will use the notation and terminology introduced at the beginning of Section 3. Further we introduce the following "antilexicographic" linear order on V_x, that is, on the x-variables. For any $x_{a,b}, x_{c,d} \in V_x$, define

$$x_{a,b} \prec x_{c,d} \iff \text{either } a > c \quad \text{or} \quad a = c \text{ and } b > d.$$

Given a lattice path monomial G as in (16), the *spread* of G, denoted $\mathrm{sp}(G)$, is the set of variables that are on or below the corresponding lattice path; more precisely,

$$\mathrm{sp}(G) = \{x_{a,b} : i_s \leq a \leq m \text{ and } 1 \leq b \leq j_s \text{ for some } s = 1, \dots, t\}.$$

The notion of spread is defined for lattice path monomials in R_y in exactly the same manner. It may be observed that if G, H are lattice path monomials (both in R_x or both in R_y), then the condition $\mathrm{sp}(G) \subseteq \mathrm{sp}(H)$ means, roughly speaking, that H is to the right of G; moreover, if $\mu(G) = \mu(H)$ and $\mathrm{sp}(G) = \mathrm{sp}(H)$, then we must have $G = H$.

Notice that the lattice path monomials F_y^{U} and F_y^{L} of Proposition 5 have the property that $\mathrm{sp}(F_y^{\mathsf{L}}) \subseteq \mathrm{sp}(F_y^{\mathsf{U}})$.

Following [Jonov 2011], we now define a partial order on the facets of Δ_0.

Definition 9. For any facets P, Q of Δ_0 with decompositions $P = P_x P_y^{\mathsf{U}} P_y^{\mathsf{L}}$ and $Q = Q_x Q_y^{\mathsf{U}} Q_y^{\mathsf{L}}$ as in Proposition 5, define $P < Q$ if one of the following four conditions hold: (i) $\mu(P_x) \prec \mu(Q_x)$, (ii) $\mu(P_x) = \mu(Q_x)$ and $\mathrm{sp}(P_x) \subsetneq \mathrm{sp}(Q_x)$, (iii) $P_x = Q_x$ and $\mathrm{sp}(P_y^{\mathsf{U}}) \subsetneq \mathrm{sp}(Q_y^{\mathsf{U}})$, (iv) $P_x = Q_x$, $P_y^{\mathsf{U}} = Q_y^{\mathsf{U}}$ and $\mathrm{sp}(P_y^{\mathsf{L}}) \subsetneq \mathrm{sp}(Q_y^{\mathsf{L}})$.

The next result is a consequence of [Jonov 2011, Theorem 3.2] and its proof.

Proposition 10. *The relation $<$ in Definition 9 defines a partial order and any extension of it to a total order on the facets of Δ_0 gives a shelling of Δ_0.*

The terminology of ES turns can be extended from lattice path monomials to facets of Δ_0 as follows. For any facet F of Δ_0 having a decomposition $F = F_x F_y^{\mathsf{U}} F_y^{\mathsf{L}}$ as in Proposition 5, by an *ES-turn* of F we shall mean an ES-turn of either F_x or F_y^{L} or F_y^{U}. It turns out that the corners of a facet of Δ_0 are essentially its ES turns or the leader of its x-component. There are, however, some subtleties involved and a precise relation is given below.

Lemma 11. *Let F be a facet of Δ_0 and $F = F_x F_y^{\mathsf{U}} F_y^{\mathsf{L}}$ be its decomposition as in Proposition 5. Also let $v \in V$ be a vertex of Δ_0. Then:*

(i) *If $v \in c(F)$, then either $v = \mu(F_x)$ or v is an ES-turn of F. In particular, $x_{m,n} \notin c(F)$ and $y_{m,n} \notin c(F)$.*

(ii) *If $\mu(F_x) = x_{i,j}$, with $(i, j) \neq (m, n-1)$, then $\mu(F_x) \in c(F)$. Moreover $x_{m,n-1} \notin c(F)$.*

(iii) *If v is an ES-turn of F_x, then $v \in c(F)$.*

(iv) *If v is an ES-turn of F_y^{U} or of F_y^{L}, then $v \in c(F)$, except when v is an ES-turn of F_y^{U} such that $v = y_{1,2}$ or when v is an ES-turn of F_y^{U} such that $v = y_{m-1,j+1}$ and $\mu(F_x) = x_{m,j}$ for some $j < n$.*

Proof. (i) Let $P = P_x P_y^{\mathsf{U}} P_y^{\mathsf{L}}$ be a facet of Δ_0 such that $F \setminus P = \{v\}$ and $F > P$. The latter implies that one of the four possibilities in Definition 9 must arise. First suppose $\mu(P_x) \prec \mu(F_x)$. Then $\mu(F_x)$ is a vertex of F that is smaller than $\mu(P_x)$ in the standard lexicographic order, and hence $\mu(F_x) \notin P_x$; consequently $v = \mu(F_x)$, and we are done. Now suppose $\mu(P_x) = \mu(F_x)$ and $\mathrm{sp}(P_x) \subsetneq \mathrm{sp}(F_x)$. Then $P_x \neq F_x$ and hence $F_x \setminus P_x = \{v\}$. Note that since $\mu(F_x)$ and $x_{m,n}$ are in P_x, the vertex v is an ES-turn, SE-turn, or the midpoint of a segment of F_x. In case it is the midpoint of a segment of F_x, the other two vertices in that segment must be in P_x, and since P_x is a lattice path monomial, we see that $v \in P_x$, which is a contradiction. Also if $v = x_{k,l}$ (say) is a SE-turn of F_x, then $x_{k-1,l}$ and $x_{k,l+1}$ must be in F_x and hence in P_x. But then P_x must contain $x_{k-1,l+1}$, which is a contradiction since $x_{k-1,l+1} \notin \mathrm{sp}(F_x)$. It follows that v is an ES-turn of F_x. Next suppose $P_x = F_x$ and $\mathrm{sp}(P_y^{\mathsf{U}}) \subsetneq \mathrm{sp}(F_y^{\mathsf{U}})$. Then $F_y^{\mathsf{U}} \setminus P_y^{\mathsf{U}} = \{v\}$. Since $\mu(P_x) = \mu(F_x)$, in view of Proposition 5, we see that the initial and the terminal variables of P_y^{U} and F_y^{U} coincide, and so v is neither of these. Arguing as in the preceding case, we can rule out the possibilities that v is a SE-turn or the midpoint of a segment of F_y^{U}. Hence v is an ES-turn of F_y^{U}. In a similar manner, we see that if $P_x = F_x$, $P_y^{\mathsf{U}} = F_y^{\mathsf{U}}$ and $\mathrm{sp}(P_y^{\mathsf{L}}) \subsetneq \mathrm{sp}(F_y^{\mathsf{L}})$, then v is a ES-turn of F_y^{L}. Thus (i) is proved.

(ii) Let $\mu(F_x) = x_{i,j}$ with $(i, j) \neq (m, n-1)$. Then either $x_{i,j+1} \in F_x$ or $x_{i+1,j} \in F_x$. First suppose $x_{i,j+1} \in F_x$. We define a new facet P as follows. Let $P_x = F_x \setminus \{x_{i,j}\}$

and $P_y^{\mathsf{L}} = F_y^{\mathsf{L}} \cup \{y_{m,j+1}\}$. To define P_y^{U}, we take $P_y^{\mathsf{U}} = F_y^{\mathsf{U}}$ in the case $y_{m,j+1} \notin F_y^{\mathsf{U}}$. If $y_{m,j+1} \in F_y^{\mathsf{U}}$, then this must mean that $i = m$, and hence $j < n-1$. We therefore define $P_y^{\mathsf{U}} = (F_y^{\mathsf{U}} \setminus \{y_{m,j+1}\}) \cup \{y_{m-1,j+2}\}$. Observe that $P = P_x P_y^{\mathsf{U}} P_y^{\mathsf{L}}$ is a facet of Δ_0 and since $\mu(P_x) \prec \mu(F_x)$, we have $P < F$. It follows that $\mu(F_x) \in c(F)$. Next suppose $x_{i+1,j} \in F_x$. We first assume that $(i, j) \neq (m-1, n)$. Now define a new facet P as follows. First we let $P_x = F_x \setminus \{x_{i,j}\}$. If $y_{i+1,n} \notin F_y^{\mathsf{L}}$, then we let $P_y^{\mathsf{U}} = F_y^{\mathsf{U}} \cup \{y_{i+1,n}\}$ and $P_y^{\mathsf{L}} = F_y^{\mathsf{L}}$. If $y_{i+1,n} \in F_y^{\mathsf{L}}$, then j must equal n. If now $i \le m-2$, then we let $P_y^{\mathsf{L}} = (F_y^{\mathsf{L}} \setminus \{y_{i+1,n}\}) \cup \{y_{i+2,n-1}\}$. We are left with the special case $i = m-1$, $j = n$. Here we let $P_x = \{x_{m,n-1},\ x_{m,n}\}$, $P_y^{\mathsf{U}} = F_y^{\mathsf{U}} \cup \{y_{m,n}\}$, and $P_y^{\mathsf{L}} = F_y^{\mathsf{L}} \setminus \{y_{m,n}\}$. In all three cases, it is easy to verify that $P = P_x P_y^{\mathsf{U}} P_y^{\mathsf{L}}$ is a facet of Δ_0 such that $F \setminus P = \{x_{i,j}\}$ and $P < F$. Consequently $\mu(F_x) \in c(F)$. Finally we show that $x_{m,n-1} \notin c(F)$. Assume, on the contrary, that there is a facet P of Δ_0 such that $F \setminus P = \{x_{m,n-1}\}$. By (i) above, $\mu(F) = x_{m,n-1}$ because there can be no ES-turn at $x_{m,n-1}$. In view of Proposition 5, P must contain at least one variable other than $x_{m,n}$, and since $x_{m,n-1} \notin P$, it follows that $x_{m-1,n} \in P$. This forces $\mu(F_x) \prec \mu(P_x)$, which violates the fact that $P < F$. Thus (ii) is proved.

(iii) Let $v = x_{k,l}$ be an ES-turn of F_x. Define $P_x = F_x \setminus \{x_{k,l}\} \cup \{x_{k+1,l-1}\}$ and $P_y = F_y$. It is clear that $P = P_x P_y$ is a facet of Δ_0 such that $P < F$ and $F \setminus P = \{v\}$. This proves (iii).

(iv) First suppose $v = y_{k,l}$ is an ES-turn of F_y^{L}. Then $k < m$ and $l > 1$. Define $P_x = F_x$, $P_y^{\mathsf{U}} = F_y^{\mathsf{U}}$, and $P_y^{\mathsf{L}} = F_y^{\mathsf{L}} \setminus \{y_{k,l}\} \cup \{y_{k+1,l-1}\}$. It is easy to see that $P = P_x P_y^{\mathsf{U}} P_y^{\mathsf{L}}$ is facet of Δ_0 such that $P < F$ and $F \setminus P = \{v\}$. Next suppose $v = y_{k,l}$ is an ES-turn of F_y^{U}. Then once again $k < m$ and $l > 1$. In case $y_{k+1,l-1}$ is not in F_y^{L}, we define $P_x = F_x$, $P_y^{\mathsf{L}} = F_y^{\mathsf{L}}$, and $P_y^{\mathsf{U}} = F_y^{\mathsf{U}} \setminus \{y_{k,l}\} \cup \{y_{k+1,l-1}\}$, whereas in case $y_{k+1,l-1}$ is in F_y^{L} and also $k < m-1$ and $l > 2$, we define $P_x = F_x$, $P_y^{\mathsf{U}} = F_y^{\mathsf{U}} \setminus \{y_{k,l}\} \cup \{y_{k+1,l-1}\}$, and $P_y^{\mathsf{L}} = F_y^{\mathsf{L}} \setminus \{y_{k+1,l-1}\} \cup \{y_{k+2,l-2}\}$. We verify that in both the cases, $P = P_x P_y^{\mathsf{U}} P_y^{\mathsf{L}}$ is a facet of Δ_0 such that $P < F$ and $F \setminus P = \{v\}$.

When $l = 2$, it is easy to see that $v = y_{k,2}$ can be an ES-turn of F_y^{U} only when $k = 1$ lest F_y^{U} and F_y^{L} intersect at $y_{k,1}$. We now show that $y_{1,2}$ is not a corner of F. Suppose that $P = P_x P_y^{\mathsf{U}} P_y^{\mathsf{L}}$ is a facet of Δ_0 such that $F \setminus P = \{v\}$, $v = y_{1,2}$ and $F > P$. By Proposition 5, P_y^{U} must start at $y_{1,1}$ and P_y^{L} must start at $y_{2,1}$. For P_y^{U} to avoid $v = y_{1,2}$, it must be the case that P_y^{U} contains $y_{2,1}$. But this contradicts the fact that P_y^{U} and P_y^{L} do not intersect.

We are left with the situation where $k = m-1$ and $v = y_{k,l}$ is an ES-turn of F_y^{U} and moreover $y_{m,l-1} \in F_y^{\mathsf{L}}$. Now since F_y^{U} has an ES-turn at $y_{m-1,l}$, we see that $l > 1$ and both $y_{m-1,l-1}$ and $y_{m,l}$ are in F_y^{U}. In particular, $y_{m,l} \notin F_y^{\mathsf{L}}$ and since $y_{m,l-1} \in F_y^{\mathsf{L}}$, in view of Proposition 5, it follows that F_y^{L} ends at $y_{m,l-1}$, while F_y^{U} ends at $y_{m,n}$ and also that $\mu(F_x) = x_{m,l-1}$. Now if there were a facet $P = P_x P_y^{\mathsf{U}} P_y^{\mathsf{L}}$ of Δ_0 such that $F \setminus P = \{v\}$ and $F > P$, then $P_x = F_x$ and $P_y^{\mathsf{L}} = F_y^{\mathsf{L}}$, whereas

$F_y^{\mathsf{U}} \setminus P_y^{\mathsf{U}} = \{y_{m-1,l}\}$. But then P_y^{U} is a lattice path monomial that contains both $y_{m-1,l-1}$ and $y_{m,l}$ and does not contain $y_{m-1,l}$; so it must contain $y_{m,l-1}$. This is a contradiction since $y_{m,l-1} \in F_y^{\mathsf{L}} = P_y^{\mathsf{L}}$ and the monomials P_y^{U} and P_y^{L} have no variable in common. This completes the proof. □

For any integers i, j, k with $k \geq 0$, $1 \leq i \leq m$ and $1 \leq j \leq n$, we define $C_{i,j}^k$ to be the number of facets $F = F_x F_y$ of Δ_0 such that $\mu(F_x) = x_{i,j}$ and F has exactly k ES turns that are in $c(F)$. We state a useful consequence of Lemma 11:

Corollary 12. *The Hilbert series of the coordinate ring $R/\mathcal{I}_0$ of the principal component Z_0 of $\mathcal{Z}_{2,2}^{m,n}$ is given by*

$$\frac{\sum_{k\geq 0} h_k z^k}{(1-z)^{2(m+n-1)}}, \tag{18}$$

where $h_0 = 1$, and for $k \geq 1$,

$$h_k = C_{m,n-1}^k + \sum_{\substack{(i,j)\neq(m,n-1)\\(i,j)\neq(m,n)}} C_{i,j}^{k-1}, \tag{19}$$

where the last sum is over all pairs (i, j) of integers satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$, with $(i, j) \neq (m, n-1)$ and $(i, j) \neq (m, n)$.

Proof. It is well-known that the (Krull) dimension as well as the Hilbert series of $R/\mathcal{I}_0$ coincides with that of $R/\mathrm{LT}(\mathcal{I}_0)$ (see, e.g., [Bruns and Conca 2003, §3]), where $\mathrm{LT}(\mathcal{I}_0)$ denotes the leading term ideal of $\mathcal{I}_0$ as in [Košir and Sethuraman 2005b] and [Jonov 2011, Proposition 1.1]. Now Δ_0 is precisely the simplicial complex such that $R/\mathrm{LT}(\mathcal{I}_0)$ is the Stanley–Reisner ring of Δ_0. Thus it follows from Corollary 6 and part (ii) of Proposition 8 that the Hilbert series of $R/\mathcal{I}_0$ is given by (18), where $h_0 = 1$, and for $k \geq 1$,

$$h_k = \big|\{F : F \text{ a facet of } \Delta_0 \text{ with } |c(F)| = k\}\big|.$$

Partitioning the facets $F = F_x F_y$ in the above set in accordance with the values of $\mu(F_x)$ and noting from Proposition 5 that $\mu(F_x) \neq (m, n)$, and then applying Lemma 11, we obtain the desired result. □

We have seen in Section 3 that lattice path monomials can be related to lattice paths in the sense of Section 2.3 if we rotate to the left by 90° and identify the variable $x_{i,j}$ with the point (i, j) of $\mathbb{Z}^2$. Also recall that for any $(a, a'), (e, e') \in \mathbb{Z}^2$ and $s \in \mathbb{Z}$, we denote by $\mathcal{P}_s((a, a') \to (e, e'))$ the set of lattice paths from (a, a') to (e, e') with s NE turns. Likewise if $(a_i, a_i'), (e_i, e_i') \in \mathbb{Z}^2$ for $i = 1, 2$ and $s \in \mathbb{Z}$, then by $\mathcal{P}_s((a_1, a_1') \to (e_1, e_1'), (a_2, a_2') \to (e_2, e_2'))$ we denote the set of pairs (L_1, L_2) of nonintersecting lattice paths such that L_i is from (a_i, a_i') to (e_i, e_i') for $i = 1, 2$,

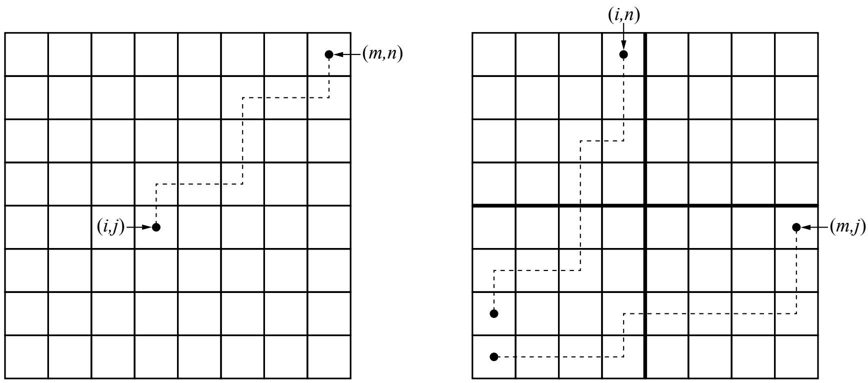

Figure 3. Lattice paths L and (L_1, L_2) corresponding to F_x and $(F_y^{\mathsf{U}}, F_y^{\mathsf{L}})$.

and the paths L_1 and L_2 together have exactly s NE turns. Evidently these sets are empty (and hence of cardinality 0) when $s < 0$.

Lemma 13. *Let $s, i, j \in \mathbb{Z}$ with $s \geq 0$, $1 \leq i \leq m$ and $1 \leq j \leq n$.*

(i) *If $i \neq m$, then*

$$C^s_{i,j} = \sum_{s_1+s_2=s} \left|\mathcal{P}_{s_1}((i, j) \to (m, n))\right| \left|\mathcal{P}_{s_2}((1, 2) \to (i, n), (1, 1) \to (m, j))\right|,$$

where the sum is over pairs (s_1, s_2) of nonnegative integers with $s_1 + s_2 = s$.

(ii) *If $1 < j < n - 1$, then*

$$C^s_{m,j} = \sum_{p=1}^{m-1} \sum_{q=j+1}^{n-1} \left|\mathcal{P}_{s-1}((1, 2) \to (p, q), (1, 1) \to (m, j))\right| + \sum_{p=1}^{m-2} \left|\mathcal{P}_{s-1}((1, 2) \to (p, j), (1, 1) \to (m, j))\right| + \left|\mathcal{P}_{s}((1, 2) \to (m-1, j), (1, 1) \to (m, j))\right|.$$

(iii) $C^s_{m,1} = \binom{n-2}{s}\binom{m-1}{s}$ *and*

$$C^s_{m,n-1} = \sum_{p=1}^{m-2} \left|\mathcal{P}_{s-1}((1, 2) \to (p, n-1), (1, 1) \to (m, n-1))\right| + \left|\mathcal{P}_{s}((1, 2) \to (m-1, n-1), (1, 1) \to (m, n-1))\right|.$$

Proof. Let $i, j \in \mathbb{Z}$ with $1 \leq i \leq m$, $1 \leq j \leq n$, and $(i, j) \neq (m, n)$. By a 90° rotation to the left, we see from Proposition 5 that the facets $F = F_x F_y$ of Δ_0 with $\mu(F_x) = x_{i,j}$ are in one-to-one correspondence with the triples (L, L_1^*, L_2^*)

of lattice paths, where L is from (i, j) to (m, n), while L_1^* is from $(1, 1)$ to (i, n) and L_2^* is from $(2, 1)$ to (m, j), and moreover L_1^*, L_2^* are nonintersecting. We will now modify L_1^*, L_2^* slightly keeping in mind the hypothesis in Corollary 4. To this end, first note that $(1, 2) \in L_1^*$ since $2 < m \leq n$. Thus if we let $L_1 := L_1^* \setminus \{(1, 1)\}$ and $L_2 := L_2^* \cup \{(1, 1)\}$, then (L_1^*, L_2^*) and (L_1, L_2) are pairs of nonintersecting lattice paths that determine each other and have exactly the same NE turns, except that if L_1^* had a NE turn at $(1, 2)$, then L_1 will not have a NE turn at $(1, 2)$. Note though that, by Lemma 11 (iv), $y_{1,2}$ is not a corner of any facet, and this switch will therefore not affect the count of corners. Consequently the facets $F = F_x F_y$ of Δ_0 with $\mu(F_x) = x_{i,j}$ are in one-to-one correspondence with

$$\mathcal{P}((i, j) \to (m, n)) \times \mathcal{P}((1, 2) \to (i, n), (1, 1) \to (m, j)).$$

The lattice paths L and (L_1, L_2) corresponding to the components F_x and $(F_y^{\sqcup}, F_y^{\llcorner})$ of the facet $F = F_x F_y$ are illustrated in Figure 3; these may be compared with Figure 2 that depicts the lattice path monomials F_x and $F_y = F_y^{\sqcup} F_y^{\llcorner}$.

(i) Suppose $i \neq m$. Then, from Lemma 11, we see that, for every facet $F = F_x F_y$ of Δ_0 with $\mu(F_x) = x_{i,j}$, all the ES turns of F_x, $F_y^{\sqcup}$ or $F_y^{\llcorner}$ that are in $c(F)$ correspond to the NE turns of the corresponding lattice paths L, L_1 or L_2. From this, we readily obtain the formula in (i).

(ii) Suppose $i = m$ and $1 < j < n - 1$. Then for a facet $F = F_x F_y$ of Δ_0 with $\mu(F_x) = x_{m,j}$, the lattice path L corresponding to F_x is from (m, j) to (m, n) and evidently this has no NE turns. Consider in $\mathcal{P}((1, 2) \to (i, n), (1, 1) \to (m, j))$ the pair (L_1, L_2) corresponding to $(F_y^{\sqcup}, F_y^{\llcorner})$. Suppose the last NE-turn of L_1 is at $(p, q + 1)$. Note that if $q < j$, then we must have $(m, j) \in L_1$, which contradicts the fact that L_1, L_2 are nonintersecting. Thus $1 \leq p \leq m - 1$ and $j \leq q < n$. Moreover if $q = j$, then by part (iv) of Lemma 11, we see that either $p \leq m - 2$ or the NE-turn $(p, q + 1)$ is not in $c(F)$. It follows that L_1 can be replaced by its truncation $\tilde{L}_1$, which is a lattice path from $(1, 2)$ to (p, q) such that $\tilde{L}_1$ and L_2 are nonintersecting. Moreover the number of NE turns of $\tilde{L}_1$ in $c(F)$ are exactly one less than the number of NE turns of L_1 in $c(F)$, except when $(p, q) = (m - 1, j)$ in which case they are the same. Thus by varying (p, q) over an appropriate range, we obtain the formula in (ii).

(iii) If $(i, j) = (m, 1)$ and $F = F_x F_y$ is a facet of Δ_0 with $\mu(F_x) = x_{m,1}$, then the path L corresponding to F_x as well as the path L_2 corresponding to $F_y^{\llcorner}$ have no NE turns. Moreover every NE-turn of the path $L_1 \in \mathcal{P}((1, 2) \to (m, n))$ corresponding to $F_y^{\sqcup}$ is necessarily in $c(F)$, thanks to Lemma 11. Thus, in view of (12), we see that $C_{m,1}^s = \binom{n-2}{s}\binom{m-1}{s}$. Finally if $(i, j) = (m, n - 1)$, then arguing as in (ii) above, we see that for a facet $F = F_x F_y$ of Δ_0 with $\mu(F_x) = x_{m,n-1}$, the lattice path L corresponding to F_x has no NE turns and the last NE-turn of the lattice path L_1 corresponding to $F_y^{\sqcup}$ must be (p, n) for some $p = 1, \ldots, m - 1$. Moreover by Lemma 11, this turn is counted as a corner (i.e., $x_{p,n} \in c(F)$) if and only if $p < m-1$. Thus upon replacing L_1 by its truncation up to $(p, n-1)$, we obtain the desired formula for $C^s_{m,n-1}$ in (iii). $\square$

We can already use the results obtained thus far to write down an explicit formula for the Hilbert series of the graded ring $R/\mathcal{I}_0$ corresponding to Z_0. Indeed it suffices to combine Corollary 12, Lemma 13, and Corollary 4. However the resulting formula is much too complicated and we will instead use results in Section 2 for simplifying various terms in (19) so as to eventually arrive at an elegant formula for (18).

Lemma 14. *Let k be a positive integer. Then $C^k_{m,n-1}$ is equal to*

$$\sum_{t_1+t_2=k} \binom{m-2}{t_1}\binom{n-2}{t_1}\binom{m-1}{t_2}\binom{n-2}{t_2} - \binom{m-1}{t_2+1}\binom{n-2}{t_2}\binom{m-2}{t_1-1}\binom{n-2}{t_1}.$$

Proof. For $s \in \mathbb{Z}$, let $f(s) := \binom{m-1}{s}\binom{n-2}{s}$ and $g(s) := \binom{m-2}{s-1}\binom{n-2}{s}$. By Corollary 4,

$$\sum_{p=1}^{m-2} \left|\mathcal{P}_{k-1}((1,2) \to (p, n-1), (1,1) \to (m, n-1))\right| \tag{20}$$

$$= \sum_{p=1}^{m-2} \sum_{s_1+s_2=k-1} \binom{p-1}{s_1}\binom{n-3}{s_1} f(s_2) - \binom{p}{s_2+1}\binom{n-3}{s_2} g(s_1)$$

$$= \sum_{s_1+s_2=k-1} \left(\sum_{p'=0}^{m-3}\binom{p'}{s_1}\right)\binom{n-3}{s_1} f(s_2) - \left(\sum_{p=1}^{m-2}\binom{p}{s_2+1}\right)\binom{n-3}{s_2} g(s_1)$$

$$= \sum_{s_1+s_2=k-1} \binom{m-2}{s_1+1}\binom{n-3}{s_1} f(s_2) - \binom{m-1}{s_2+2}\binom{n-3}{s_2} g(s_1)$$

$$= \sum_{t_1+t_2=k} \binom{m-2}{t_1}\binom{n-3}{t_1-1} f(t_2) - \binom{m-1}{t_2+1}\binom{n-3}{t_2-1} g(t_1),$$

where the penultimate equality follows from Lemma 1 since $\binom{0}{s_1+1} = 0 = \binom{1}{s_2+2}$ for $s_1, s_2 \geq 0$, and also since $\binom{n-3}{s_1} f(s_2) = 0 = \binom{n-3}{s_2} g(s_1)$ if $s_1 < 0$ or $s_2 < 0$, while the last equality follows by altering the summations (twice!) as in (10). On the other hand, by Corollary 4, $\left|\mathcal{P}_k((1,2) \to (m-1, n-1), (1,1) \to (m, n-1))\right|$ is equal to

$$\sum_{t_1+t_2=k} \binom{m-2}{t_1}\binom{n-3}{t_1} f(t_2) - \binom{m-1}{t_2+1}\binom{n-3}{t_2} g(t_1). \tag{21}$$

Now combining (20) and (21), using (6), and then using part (iii) of Lemma 13, we obtain the desired result. $\square$

Lemma 15. *Let k be a positive integer. Then* $\sum_{i=1}^{m-1}\sum_{j=1}^{n} C_{i,j}^{k-1}$ *is equal to*

$$\sum_{t_1+t_2=k}\binom{m}{t_2}\binom{n}{t_1+1}\binom{m-1}{t_1}\binom{n-2}{t_2-1}-\binom{m-1}{t_1}\binom{n}{t_2}\binom{m-1}{t_2-1}\binom{n-2}{t_1}.$$

Proof. Using (12) and part (i) of Lemma 13, we see that $\sum_{i=1}^{m-1}\sum_{j=1}^{n} C_{i,j}^{k-1}$ equals

$$\sum_{i=1}^{m-1}\sum_{j=1}^{n}\sum_{k_1+k_2=k-1}\binom{m-i}{k_1}\binom{n-j}{k_1}\big|\mathcal{P}_{k_2}((1,2)\to(i,n),(1,1)\to(m,j))\big|.$$

Applying Corollary 4 and then suitably interchanging summations and noting that the summands below are zero if $k_1<0$ or $s_1<0$ or $s_2<0$, this can be written as

$$\sum_{\substack{k_1+s_1+s_2=k-1\\ k_1,s_1,s_2\geq 0}} M_1N_1\binom{m-1}{s_2}\binom{n-2}{s_1}-M_2N_2\binom{m-2}{s_1-1}\binom{n-2}{s_2}, \tag{22}$$

where, for any given $k_1, s_1, s_2 \geq 0$, we have temporarily put

$$M_1=\sum_{i=1}^{m-1}\binom{m-i}{k_1}\binom{i-1}{s_1},\qquad N_1=\sum_{j=1}^{n}\binom{n-j}{k_1}\binom{j-1}{s_2}=\binom{n}{k_1+s_2+1},$$

$$M_2=\sum_{i=1}^{m-1}\binom{m-i}{k_1}\binom{i}{s_2+1},\qquad N_2=\sum_{j=1}^{n}\binom{n-j}{k_1}\binom{j-1}{s_1}=\binom{n}{k_1+s_1+1},$$

and where the simplified expressions for N_1, N_2 follow by rewriting each of the summands in N_1 and N_2 using (5), invoking (4) (noting that $k_1, s_1, s_2 \geq 0$), and then applying (9) for suitable values of s, t, α and β. A similar simplification is possible in M_1 and M_2 if we add and subtract the term corresponding to $i=m$, and in view of (4), this is only necessary if $k_1=0$. Thus

$$M_1=\binom{m}{k_1+s_1+1}-\delta_{0,k_1}\binom{m-1}{s_1}\quad\text{and}\quad M_2=\binom{m+1}{k_1+s_2+2}-\delta_{0,k_1}\binom{m}{s_2+1},$$

where δ is the Kronecker delta. Substituting the simplified values of M_1, N_1, M_2, N_2 in (22), and letting

$$A(s_1,s_2):=\binom{m-1}{s_2}\binom{n-2}{s_1},\qquad B(s_1,s_2):=\binom{m-2}{s_1-1}\binom{n-2}{s_2}$$

for $s_1, s_2\in\mathbb{Z}$, we see that (22) is of the form E_3+S_3, where

$$E_3=\sum_{\substack{k_1+s_1+s_2=k-1\\ k_1,s_1,s_2\geq 0}}\binom{m}{k-s_2}\binom{n}{k-s_1}A(s_1,s_2)-\binom{m+1}{k-s_1+1}\binom{n}{k-s_2}B(s_1,s_2),$$

and S_3 is the part where the Kronecker delta is nonzero:

$$S_3 = \sum_{s_1+s_2=k-1} \binom{m}{s_2+1}\binom{n}{s_1+1}B(s_1,s_2) - \binom{m-1}{s_1}\binom{n}{s_2+1}A(s_1,s_2).$$

Altering the summation as in (10), we see that S_3 can be written as

$$\sum_{t_1+t_2=k} \binom{m}{t_2}\binom{n}{t_1+1}\binom{m-2}{t_1-1}\binom{n-2}{t_2-1} - \binom{m-1}{t_1}\binom{n}{t_2}\binom{m-1}{t_2-1}\binom{n-2}{t_1}. \tag{23}$$

On the other hand, in view of (4) and (11), we can write

$$E_3 = \sum_{\ell=0}^{k-1}\sum_{s_1+s_2=\ell} \binom{m}{k-s_1}\binom{n}{k-s_2}A(s_2,s_1) - \binom{m+1}{k-s_1+1}\binom{n}{k-s_2}B(s_1,s_2).$$

By (6), we have

$$\binom{m+1}{k-s_1+1} = \binom{m}{k-s_1} + \binom{m}{k-(s_1-1)}.$$

Using this to split the second summand in E_3 into two parts and combining one of the parts with the first summand in E_3 and then applying (6) once again, we see that

$$E_3 = \sum_{\ell=0}^{k-1}\sum_{s_1+s_2=\ell} f(s_1,s_2) - f(s_1-1,s_2),$$

where

$$f(s_1,s_2) := \binom{m}{k-s_1}\binom{n}{k-s_2}\binom{m-2}{s_1}\binom{n-2}{s_2}$$

for $s_1, s_2 \in \mathbb{Z}$. Now in view of (10), we find that E_3 is given by the telescoping sum

$$E_3 = \sum_{\ell=0}^{k-1} F_\ell - F_{\ell-1}, \quad \text{where } F_\ell := \sum_{s_1+s_2=\ell} f(s_1,s_2) \text{ for } \ell \in \mathbb{Z}.$$

From the definition of f, we see that $F_{-1} = 0$, and thus $E_3 = F_{k-1}$, that is,

$$E_3 = \sum_{s_1+s_2=k-1} \binom{m}{k-s_1}\binom{n}{k-s_2}\binom{m-2}{s_1}\binom{n-2}{s_2}.$$

Now we can replace $k-s_1, k-s_2$ by s_2+1, s_1+1, respectively, in the above summand, and then alter the summation using (10) to obtain

$$E_3 = \sum_{t_1+t_2=k} \binom{m}{t_2}\binom{n}{t_1+1}\binom{m-2}{t_1}\binom{n-2}{t_2-1}. \tag{24}$$

Finally, by adding (24) and (23) termwise and using (6), we obtain the desired formula for $E_3 + S_3$, i.e., for $\sum_{i=1}^{m-1}\sum_{j=1}^{n} C_{i,j}^{k-1}$. □

Lemma 16. *Let k be a positive integer. Then* $\sum_{j=1}^{n-2} C_{m,j}^{k-1}$ *is equal to*

$$\sum_{t_1+t_2=k} \binom{m-1}{t_1}\binom{n-2}{t_1}\binom{m-1}{t_2-1}\binom{n-2}{t_2} - \binom{m}{t_2+1}\binom{n-2}{t_2}\binom{m-2}{t_1-2}\binom{n-2}{t_1}.$$

Proof. The desired result is easily verified when $n \le 3$ and so we assume that $n > 3$. For $j, s \in \mathbb{Z}$, let

$$f_j(s) := \binom{m-1}{s}\binom{j-1}{s}, \quad g_j(s) := \binom{m-2}{s-1}\binom{j-1}{s}.$$

In view of parts (iii) and (ii) of Lemma 13 together with (4) and Corollary 4, we see that

$$C_{m,1}^{k-1} = \binom{n-2}{k-1}\binom{m-1}{k-1} \quad \text{and} \quad \sum_{j=2}^{n-2} C_{m,j}^{k-1} = S_4 + S_5 + S_6, \tag{25}$$

where

$$S_4 = \sum_{j=2}^{n-2}\sum_{p=1}^{m-1}\sum_{q=j+1}^{n-1}\sum_{\substack{s_1+s_2=k-2\\ s_1,s_2\ge 0}} \binom{p-1}{s_1}\binom{q-2}{s_1} f_j(s_2) - \binom{p}{s_2+1}\binom{q-2}{s_2} g_j(s_1),$$

$$S_5 = \sum_{j=2}^{n-2}\sum_{p=1}^{m-2}\sum_{\substack{s_1+s_2=k-2\\ s_1,s_2\ge 0}} \binom{p-1}{s_1}\binom{j-2}{s_1} f_j(s_2) - \binom{p}{s_2+1}\binom{j-2}{s_2} g_j(s_1),$$

$$S_6 = \sum_{j=2}^{n-2}\sum_{s_1+s_2=k-1} \binom{m-2}{s_1}\binom{j-2}{s_1} f_j(s_2) - \binom{m-1}{s_2+1}\binom{j-2}{s_2} g_j(s_1).$$

Interchanging s_1 and s_2 in the second summand for S_6 as in (11), we can write

$$S_6 = \sum_{s_1+s_2=k-1} \lambda(s_1, s_2)\left(\binom{m-2}{s_1}\binom{m-1}{s_2} - \binom{m-1}{s_1+1}\binom{m-2}{s_2-1}\right), \tag{26}$$

where, for $s_1, s_2 \in \mathbb{Z}$, we let

$$\lambda(s_1, s_2) := \sum_{j=2}^{n-2} \binom{j-2}{s_1}\binom{j-1}{s_2}.$$

Next, by Lemma 1,

$$\sum_{p=1}^{m-2} \binom{p-1}{s_1} = \binom{m-2}{s_1+1} \quad \text{and} \quad \sum_{p=1}^{m-2} \binom{p}{s_2+1} = \binom{m-1}{s_2+2} \quad \text{for } s_1, s_2 \ge 0.$$

Consequently, by interchanging summations and rearranging terms, we find

$$S_5 = \sum_{j=2}^{n-2} \sum_{\substack{s_1+s_2=k-2\\ s_1,s_2\ge 0}} \binom{m-2}{s_1+1}\binom{j-2}{s_1} f_j(s_2) - \binom{m-1}{s_2+2}\binom{j-2}{s_2} g_j(s_1) \tag{27}$$
$$= \sum_{s_1+s_2=k-2} \lambda(s_1,s_2)\left(\binom{m-2}{s_1+1}\binom{m-1}{s_2} - \binom{m-1}{s_1+2}\binom{m-2}{s_2-1}\right)$$
$$= \sum_{s_1+s_2=k-1} \lambda(s_1-1,s_2)\left(\binom{m-2}{s_1}\binom{m-1}{s_2} - \binom{m-1}{s_1+1}\binom{m-2}{s_2-1}\right),$$

where the penultimate equality follows from (4) and (11) by interchanging s_1 and s_2 in the second summand of the preceding formula, while the last equality follows from (10). Now, using (6), we easily see that

$$\lambda(s_1-1,s_2)+\lambda(s_1,s_2)=\nu(s_1,s_2) \quad \text{for any } s_1,s_2\in\mathbb{Z},$$

where

$$\nu(s_1,s_2) := \sum_{j=2}^{n-2}\binom{j-1}{s_1}\binom{j-1}{s_2}.$$

Hence we can combine (27) and (26) to obtain

$$S_5+S_6 = \sum_{s_1+s_2=k-1} \nu(s_1,s_2)\left(\binom{m-2}{s_1}\binom{m-1}{s_2} - \binom{m-1}{s_1+1}\binom{m-2}{s_2-1}\right). \tag{28}$$

It remains to consider S_4 or rather $C_{m,1}^{k-1}+S_4$. This is a little more complicated, but it can be handled using arguments similar to those in the proof of Lemma 15 as follows. First, by interchanging summations and using Lemma 1, we find

$$S_4 = \sum_{j=2}^{n-2} \sum_{\substack{s_1+s_2=k-2\\ s_1,s_2\ge 0}} \binom{m-1}{s_1+1}\theta(s_1) f_j(s_2) - \binom{m}{s_2+2}\theta(s_2) g_j(s_1),$$

where, for $s\in\mathbb{Z}$, we have let

$$\theta(s) := \binom{n-2}{s+1} - \binom{j-1}{s+1}.$$

Now observe that if $s_1<0$ or $s_2<0$, then $\theta(s_1)f_j(s_2)=0=\theta(s_2)g_j(s_1)$. Thus we may drop the condition $s_1,s_2\ge 0$ in the above expression for S_4, and then alter each of the two summations over (s_1,s_2) using (10) to write

$$S_4 = \sum_{j=2}^{n-2} \sum_{s_1+s_2=k-1} \binom{m-1}{s_1}\theta(s_1-1) f_j(s_2) - \binom{m}{s_2+1}\theta(s_2-1) g_j(s_1).$$

Next we collate the terms involving j and bring the summation over j inside, and note that, by Lemma 1, $\sum_{j=2}^{n-2}\binom{j-1}{s}=\binom{n-2}{s+1}-\delta_{0,s}$ for any $s\geq 0$. This yields

$$\begin{aligned}S_4=\sum_{s_1+s_2=k-1}&\binom{m-1}{s_1}\binom{n-2}{s_1}\binom{m-1}{s_2}\left(\binom{n-2}{s_2+1}-\delta_{0,s_2}\right)\\&-\binom{m}{s_2+1}\binom{n-2}{s_2}\binom{m-2}{s_1-1}\left(\binom{n-2}{s_1+1}-\delta_{0,s_1}\right)\\&-\binom{m-1}{s_1}\binom{m-1}{s_2}\nu(s_1,s_2)+\binom{m}{s_2+1}\binom{m-2}{s_1-1}\nu(s_1,s_2).\end{aligned}$$

Since $\binom{m-2}{s_1-1}=0$ when $s_1=0$, the only contribution of the terms involving Kronecker delta is when $s_2=0$, and it is $-\binom{m-1}{k-1}\binom{n-2}{k-1}$, that is, precisely $-C_{m,1}^{k-1}$. It follows that $C_{m,1}^{k-1}+S_4=S_4^*+E_4$, where

$$S_4^*=\sum_{s_1+s_2=k-1}\binom{m-1}{s_1}\binom{n-2}{s_1}\binom{m-1}{s_2}\binom{n-2}{s_2+1}-\binom{m}{s_2+1}\binom{n-2}{s_2}\binom{m-2}{s_1-1}\binom{n-2}{s_1+1}$$

and

$$\begin{aligned}E_4&=\sum_{s_1+s_2=k-1}\nu(s_1,s_2)\left(\binom{m}{s_2+1}\binom{m-2}{s_1-1}-\binom{m-1}{s_1}\binom{m-1}{s_2}\right)\\&=\sum_{s_1+s_2=k-1}\nu(s_1,s_2)\left(\binom{m}{s_1+1}\binom{m-2}{s_2-1}-\binom{m-1}{s_1}\binom{m-1}{s_2}\right),\end{aligned}\tag{29}$$

where the last equality follows by interchanging s_1 and s_2, while noting that ν is symmetric in s_1,s_2.

Now combining (28) and (29), and then, making an easy calculation using (6), we see that

$$E_4+S_5+S_6=\sum_{s_1+s_2=k-1}\nu(s_1,s_2)\left(\binom{m-1}{s_1}\binom{m-2}{s_2-1}-\binom{m-2}{s_1-1}\binom{m-1}{s_2}\right)=0,$$

where the last equality follows by interchanging s_1 and s_2 in one of the summations above. Thus $\sum_{j=1}^{n-2}C_{m,j}^{k-1}=S_4^*$. Finally, using (10), we readily see that S_4^* is precisely the desired formula in the statement of the lemma. □

Corollary 17. *Let k be a positive integer. Then $C_{m,n-1}^{k}+\sum_{j=1}^{n-2}C_{m,j}^{k-1}$ is equal to*

$$\sum_{t_1+t_2=k}\binom{m-1}{t_1}\binom{n-2}{t_1}\binom{m-1}{t_2}\binom{n-2}{t_2}-\binom{m-1}{t_2+1}\binom{n-2}{t_2}\binom{m-1}{t_1-1}\binom{n-2}{t_1}.$$

Proof. Consider the formula for $\sum_{j=1}^{n-2}C_{m,j}^{k-1}$ given by Lemma 16. This is a difference of two summations over $(t_1, t_2) \in \mathbb{Z}^2$ with $t_1 + t_2 = k$. Alter the first of these summations by interchanging t_1 and t_2, while putting $\binom{m}{t_2+1} = \binom{m-1}{t_2} + \binom{m-1}{t_2+1}$ in the second summation to split it into two summations. Then, using (6), we readily see that the formula for $\sum_{j=1}^{n-2} C_{m,j}^{k-1}$ becomes

$$\sum_{t_1+t_2=k} \binom{m-2}{t_1-1}\binom{n-2}{t_1}\binom{m-1}{t_2}\binom{n-2}{t_2} - \binom{m-1}{t_2+1}\binom{n-2}{t_2}\binom{m-2}{t_1-2}\binom{n-2}{t_1}.$$

This can be added termwise, using (6) once again, with the formula for $C_{m,n-1}^{k}$ given by Lemma 14, to obtain the desired result. $\square$

We are now ready for our main theorem.

Theorem 18. *The Hilbert series of $R/\mathcal{I}_0$ is given by*

$$\left(\frac{\sum_{e=0}^{m-1}\binom{m-1}{e}\binom{n-1}{e}z^e}{(1-z)^{m+n-1}}\right)^2. \tag{30}$$

Proof. First note that (30) is of the form $(1-z)^{-2(m+n-1)}\sum_{k=0}^{2m-2} h_k^* z^k$, where

$$h_k^* = \sum_{t_1+t_2=k} \binom{m-1}{t_1}\binom{n-1}{t_1}\binom{m-1}{t_2}\binom{n-1}{t_2} \quad \text{for } k \in \mathbb{Z}. \tag{31}$$

On the other hand, by Corollary 12, we see that the Hilbert series of $R/\mathcal{I}_0$ is given by $(1-z)^{-2(m+n-1)}\sum_{k\geq 0} h_k z^k$, where $h_0 = 1$, and

$$h_k = \left(C_{m,n-1}^{k} + \sum_{j=1}^{n-2} C_{m,j}^{k-1}\right) + \sum_{i=1}^{m-1}\sum_{j=1}^{n} C_{i,j}^{k-1} \quad \text{for } k \geq 1. \tag{32}$$

It is clear that $h_0^* = 1 = h_0$ and so it suffices to show that $h_k^* = h_k$ for all $k \geq 1$. In view of Corollary 17 and Lemma 15, this is equivalent to showing that

$$\sum_{t_1+t_2=k} P_1(t_1,t_2) - P_2(t_1,t_2) + P_3(t_1,t_2) - P_4(t_1,t_2) - P(t_1,t_2) = 0 \quad \text{for } k \geq 1,$$

where $P_i(t_1, t_2)$ for $i = 1, \ldots, 4$, and $P(t_1, t_2)$ are the relevant summands, namely,

$$P_1(t_1,t_2) := \binom{m-1}{t_1}\binom{n-2}{t_1}\binom{m-1}{t_2}\binom{n-2}{t_2},$$

$$P_2(t_1,t_2) := \binom{m-1}{t_2+1}\binom{n-2}{t_2}\binom{m-1}{t_1-1}\binom{n-2}{t_1},$$

$$P_3(t_1,t_2) := \binom{m}{t_2}\binom{n}{t_1+1}\binom{m-1}{t_1}\binom{n-2}{t_2-1},$$

$$P_4(t_1,t_2) := \binom{m-1}{t_1}\binom{n}{t_2}\binom{m-1}{t_2-1}\binom{n-2}{t_1},$$

and

$$P(t_1,t_2) := \binom{m-1}{t_1}\binom{n-1}{t_1}\binom{m-1}{t_2}\binom{n-1}{t_2}$$

for $t_1, t_2 \in \mathbb{Z}$. To this end, we will make an extensive use of alterations as in (10) and (11); more specifically, the fact that

$$\sum_{t_1+t_2=k} f(t_1,t_2) = \sum_{t_1+t_2=k} f(t_2,t_1) = \sum_{t_1+t_2=k} f(t_1+1,t_2-1) = \sum_{t_1+t_2=k} f(t_2+1,t_1-1)$$

for any $f : \mathbb{Z}^2 \to \mathbb{Q}$ with finite support and any $k \in \mathbb{Z}$. Now fix any positive integer k and any $(t_1, t_2) \in \mathbb{Z}^2$ with $t_1 + t_2 = k$. Observe that

$$P_3(t_1-1,t_2+1) - P_4(t_2,t_1) = \binom{m-1}{t_2+1}\binom{n}{t_1}\binom{m-1}{t_1-1}\binom{n-2}{t_2}.$$

Using (6) twice, we may substitute $\binom{n-2}{t_1}+\binom{n-2}{t_1-1}+\binom{n-1}{t_1-1}$ for $\binom{n}{t_1}$ in the right-hand side of the above identity to obtain

$$-P_2(t_1,t_2) + P_3(t_1-1,t_2+1) - P_4(t_2,t_1) = Q_1(t_1,t_2) + Q_2(t_1,t_2),$$

where

$$Q_1(t_1,t_2) := \binom{m-1}{t_2+1}\binom{n-2}{t_1-1}\binom{m-1}{t_1-1}\binom{n-2}{t_2},$$

$$Q_2(t_1,t_2) := \binom{m-1}{t_2+1}\binom{n-1}{t_1-1}\binom{m-1}{t_1-1}\binom{n-2}{t_2}.$$

Finally observe that $P_1(t_1,t_2) + Q_1(t_1+1, t_2-1) + Q_2(t_2+1, t_1-1) = P(t_1,t_2)$. This yields the desired result. $\square$

It may be noted that in view of (2) and (30), the Hilbert series of the principal component Z_0 is precisely the square of the Hilbert series of the base variety $\mathcal{Z}_2^{m,n}$, and, as such, Theorem 7 could be deduced as a consequence of Theorem 18.

As an application of Theorem 18, we will now compute the a-invariant of the coordinate ring $R/\mathcal{I}_0$ of the principal component Z_0 of $\mathcal{Z}_{2,2}^{m,n}$ and determine when Z_0 is Gorenstein. Recall that if A is a finitely generated, positively graded Cohen–Macaulay algebra over a field, then A admits a graded canonical module ω_A and the a-invariant of A is defined as the negative of the least degree of a generator of ω_A. If the Hilbert series of A is given by $H_A(z) = h(z)/(1-z)^d$, where $d = \dim A$ and $h(z) \in \mathbb{Q}[z]$ with $h(1) \neq 0$, then the a-invariant of A is the order of the pole of $H_A(z)$ at infinity, which is $-(d - \deg h(z))$. Moreover the Hilbert series of ω_A is given by $H_{\omega_A}(z) = (-1)^d H_A(z^{-1})$. As a general reference for these notions and results, one may consult [Bruns and Herzog 1993], especially Sections 3.6 and 4.4. The following result is an analogue of a theorem of Gräbe [1988] (see also

[Ghorpade 1996, Theorem 4]) for classical determinantal varieties which says that if $1 \le r \le m \le n$, then the a-invariant of (the coordinate ring of) $\mathscr{Z}_r^{m,n}$ is $-(r-1)n$.

Corollary 19. *The a-invariant of $R/\mathscr{I}_0$ is equal to $-2n$ and the Hilbert series of the graded canonical module of $R/\mathscr{I}_0$ is given by*

$$\left(\frac{\sum_{e=0}^{m-1}\binom{m-1}{e}\binom{n-1}{e}z^{m+n-e-1}}{(1-z)^{m+n-1}}\right)^2. \tag{33}$$

Proof. We know from [Jonov 2011, Theorem 1.2] that $A = R/\mathscr{I}_0$ is Cohen–Macaulay and it is obviously a finitely generated, positively graded $\mathbb{F}$-algebra. Moreover, by Theorem 18, the Hilbert series of A is given by $h_0(z)/(1-z)^{2(m+n-1)}$, where

$$h_0(z) = \left(\sum_{e=0}^{m-1}\binom{m-1}{e}\binom{n-1}{e}z^e\right)^2.$$

Since $2 \le m \le n$, we see that $h_0(z)$ is a polynomial in z of degree $2(m-1)$, with leading coefficient $\binom{n-1}{m-1}^2$, and all other coefficients nonnegative integers; in particular, $h_0(1) \ne 0$. Hence the a-invariant of $A = R/\mathscr{I}_0$ is

$$2(m-1) - 2(m+n-1) = -2n,$$

and also that the Hilbert series of ω_A is given by (33). □

The following result is an analogue of a theorem of Svanes [1974] (see also [Conca and Herzog 1994]) for classical determinantal varieties which says that for any $r \ge 1$, (the coordinate ring of) $\mathscr{Z}_r^{m,n}$ is Gorenstein if and only if $m = n$.

Corollary 20. *The coordinate ring $R/\mathscr{I}_0$ of Z_0 is Gorenstein if and only if $m = n$.*

Proof. By [Jonov 2011, Theorem 1.2] and [Košir and Sethuraman 2005b, Proposition 3.3], $A = R/\mathscr{I}_0$ is a Cohen–Macaulay domain. Hence from a well-known result of Stanley [1978, Theorem 4.4] (see also [Bruns and Herzog 1993, Corollary 4.4.6]), we see that A is Gorenstein if and only if $H_A(z) = (-1)^d z^a H_A(z^{-1})$ for some $a \in \mathbb{Z}$. Moreover, in this case, the integer a is necessarily the a-invariant of A. Thus, from Corollary 19, we see that $R/\mathscr{I}_0$ is Gorenstein if and only if

$$\left(\sum_{e=0}^{m-1}\binom{m-1}{e}\binom{n-1}{e}z^e\right)^2 = \left(\sum_{e=0}^{m-1}\binom{m-1}{e}\binom{n-1}{e}z^{m-1-e}\right)^2.$$

Since both the polynomials inside the square brackets on the two sides of the above equality have positive leading coefficients, it follows that $R/\mathscr{I}_0$ is Gorenstein if and only if $\binom{n-1}{e} = \binom{n-1}{m-1-e}$ for all $e = 0, 1, \ldots, m-1$. Since $1 < m-1 \le n-1$, the latter clearly holds if and only if $m = n$. □

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