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# BOUNDARY LIMITS FOR FRACTIONAL POISSON $a$-EXTENSIONS OF $L^{p}$ BOUNDARY FUNCTIONS IN A CONE 

Lei Qiao and Tao Zhao


#### Abstract

If one replaces the Poisson kernel of a cone by the Poisson $a$-kernel, then normalized Poisson integrals with respect to the stationary Schrödinger operator converge along approach regions wider than the ordinary nontangential cones. In this paper we present new and simplified proofs of these results. We also generalize the result by Mizuta and Shimomura to the smooth cones.


## 1. Introduction and main results

Let $\mathbb{R}$ and $\mathbb{R}_{+}$be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbb{R}^{n}(n \geq 2)$ the $n$-dimensional Euclidean space. A point in $\mathbb{R}^{n}$ is denoted by $P=\left(X, x_{n}\right), X=\left(x_{1}, x_{2}, \ldots, x_{n-1}\right)$. The Euclidean distance of two points $P$ and $Q$ in $\mathbb{R}^{n}$ is denoted by $|P-Q|$. Also $|P-O|$, with $O$ the origin of $\mathbb{R}^{n}$, is simply denoted by $|P|$. The boundary, the closure and the complement of a set $S$ in $\mathbb{R}^{n}$ are denoted by $\partial S, \bar{S}$ and $S^{c}$, respectively.

We introduce a system of spherical coordinates $(r, \Theta), \Theta=\left(\theta_{1}, \theta_{2}, \ldots, \theta_{n-1}\right)$, in $\mathbb{R}^{n}$ which are related to cartesian coordinates $\left(x_{1}, x_{2}, \ldots, x_{n-1}, x_{n}\right)$ by

$$
x_{1}=r\left(\prod_{j=1}^{n-1} \sin \theta_{j}\right), \quad x_{n}=r \cos \theta_{1}
$$

for $n \geq 2$, and for $n \geq 3$,

$$
x_{n-m+1}=r\left(\prod_{j=1}^{m-1} \sin \theta_{j}\right) \cos \theta_{m} \quad(2 \leq m \leq n-1)
$$

where $0 \leq r<+\infty,-\pi / 2 \leq \theta_{n-1}<3 \pi / 2$, and if $n \geq 3$, then $0 \leq \theta_{j} \leq \pi$ ( $1 \leq j \leq n-2$ ).

[^0]The unit sphere and the upper unit half-sphere are denoted by $S^{n-1}$ and $S_{+}^{n-1}$, respectively. For simplicity, a point $(1, \Theta)$ on $S^{n-1}$ and the set $\{\Theta ;(1, \Theta) \in \Omega\}$, for a set $\Omega \subset S^{n-1}$, are often identified with $\Theta$ and $\Omega$, respectively. For two sets $\Xi \subset \mathbb{R}_{+}$ and $\Omega \subset S^{n-1}$, the set $\left\{(r, \Theta) \in \mathbb{R}^{n} ; r \in \Xi,(1, \Theta) \in \Omega\right\}$ in $\mathbb{R}^{n}$ is simply denoted by $\Xi \times \Omega$. In particular, the half-space $\mathbb{R}_{+} \times S_{+}^{n-1}=\left\{\left(X, x_{n}\right) \in \mathbb{R}^{n} ; x_{n}>0\right\}$ will be denoted by $T_{n}$.

By $C_{n}(\Omega)$, we denote the set $\mathbb{R}_{+} \times \Omega$ in $\mathbb{R}^{n}$ with the domain $\Omega$ on $S^{n-1}$. We call it a cone. Then $T_{n}$ is a special cone obtained by putting $\Omega=S_{+}^{n-1}$. We denote the sets $I \times \Omega$ and $I \times \partial \Omega$, with $I$ an interval on $\mathbb{R}$, by $C_{n}(\Omega ; I)$ and $S_{n}(\Omega ; I)$. By $S_{n}(\Omega)$, we denote $S_{n}\left(\Omega ;(0,+\infty)\right.$ ), which is $\partial C_{n}(\Omega)-\{O\}$.

For positive functions $h_{1}$ and $h_{2}$, we say that $h_{1} \lesssim h_{2}$ if $h_{1} \leq M h_{2}$ for some constant $M>0$. If $h_{1} \lesssim h_{2}$ and $h_{2} \lesssim h_{1}$, we say that $h_{1} \approx h_{2}$.

This article is devoted to the stationary Schrödinger operator

$$
\mathrm{SSE}_{a}=-\Delta+a(P) I
$$

where $\Delta$ is the Laplace operator and $I$ is the identity operator. We assume hereafter that the potential $a(P)$ is a nonnegative, locally integrable function in $C_{n}(\Omega)$, namely, $0 \leq a \in L_{\mathrm{loc}}^{b}\left(C_{n}(\Omega)\right)$, with $b>n / 2$ if $n \geq 4$, and with $b=2$ if $n=2$ or 3 . We denote this class of potentials by $\mathscr{A}$.

If $a \in \mathscr{A}$, then the operator $\operatorname{SSE}_{a}$ can be extended in the usual way from the space $C_{0}^{\infty}\left(C_{n}(\Omega)\right)$ to an essentially self-adjoint operator on $L^{2}\left(C_{n}(\Omega)\right)$ (see [Reed and Simon 1979, Chapter 13]). We shall denote the extended operator by $\operatorname{SSE}_{a}$ as well. The latter has Green function $G_{\Omega}^{a}(P, Q)$ vanishing almost everywhere at the boundary and possessing all the analytic properties. For $|P-Q| \rightarrow 0$, we normalize it such that $c_{n} G_{\Omega}^{a}(P, Q) \approx-\log |P-Q|$ when $n=2$, or $c_{n} G_{\Omega}^{a}(P, Q) \approx|P-Q|^{2-n}$ when $n \geq 3$. Here $c_{2}=2 \pi, c_{n}=(n-2) s_{n}$ when $n \geq 3$, and $s_{n}$ is the surface area $2 \pi^{n / 2}(\Gamma(n / 2))^{-1}$ of $S^{n-1}$. The Green function $G_{\Omega}^{a}(P, Q)$ is positive on $C_{n}(\Omega)$ and its inner normal derivative $\partial G_{\Omega}^{a}(P, Q) / \partial n_{Q} \geq 0$. We denote this derivative by $\mathrm{PI}_{\Omega}^{a}(P, Q)$, which is called the Poisson $a$-kernel with respect to $C_{n}(\Omega)$. Then the Poisson $a$-integral $\mathrm{PI}_{\Omega}^{a} f(P)\left(P \in C_{n}(\Omega)\right)$ is defined by

$$
\mathrm{PI}_{\Omega}^{a} f(P)=\int_{S_{n}(\Omega)} \mathrm{PI}_{\Omega}^{a}(P, Q) f(Q) d \sigma_{Q}
$$

where

$$
\mathrm{PI}_{\Omega}^{a}(P, Q)=\frac{\partial}{\partial n_{Q}} G_{\Omega}^{a}(P, Q)
$$

$f \in L^{p}\left(\partial C_{n}(\Omega)\right)(1 \leq p<\infty)$ and $d \sigma_{Q}$ is the surface area element on $S_{n}(\Omega)$.
Remark 1 [Yoshida 1991]. Let $\Omega=S_{+}^{n-1}$ and $a=0$. Then

$$
G_{S_{+}^{n-1}}^{0}(P, Q)= \begin{cases}\log \left|P-Q^{*}\right|-\log |P-Q| & n=2 \\ |P-Q|^{2-n}-\left|P-Q^{*}\right|^{2-n} & n \geq 3\end{cases}
$$

where $Q^{*}=\left(Y,-y_{n}\right)$; that is, $Q^{*}$ is the mirror image of $Q=\left(Y, y_{n}\right)$ with respect to $\partial T_{n}$. Hence, for the two points $P=\left(X, x_{n}\right) \in T_{n}$ and $Q=\left(Y, y_{n}\right) \in \partial T_{n}$, we have

$$
\mathrm{PI}_{S_{+}^{n-1}}^{0}(P, Q)= \begin{cases}2|P-Q|^{-2} x_{n} & n=2 \\ 2(n-2)|P-Q|^{-n} x_{n} & n \geq 3\end{cases}
$$

Let $\Omega$ be a domain on $S^{n-1}$ with smooth boundary. Consider the Dirichlet problem

$$
\begin{aligned}
\left(\Lambda_{n}+\lambda\right) \varphi=0 & \text { on } \Omega \\
\varphi=0 & \text { on } \partial \Omega
\end{aligned}
$$

where $\Lambda_{n}$ is the spherical part of the Laplace operator

$$
\Delta_{n}=\frac{n-1}{r} \frac{\partial}{\partial r}+\frac{\partial^{2}}{\partial r^{2}}+\frac{\Lambda_{n}}{r^{2}}
$$

We denote the least positive eigenvalue of this boundary value problem by $\lambda$ and the normalized positive eigenfunction corresponding to $\lambda$ by $\varphi(\Theta) ; \int_{\Omega} \varphi^{2}(\Theta) d \sigma_{\Theta}=1$, where $d \sigma_{\Theta}$ is the surface area on $S^{n-1}$.

To simplify our consideration in the following, we shall assume that if $n \geq 3$, then $\Omega$ is a $C^{2, \alpha}$-domain $(0<\alpha<1)$ on $S^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g., see [Gilbarg and Trudinger 1977, pp. 88-89] for the definition of $C^{2, \alpha}$-domain). Then by modifying Miranda's method [1970, pp. 7-8], we can prove the inequality (see [Yoshida 1991, p. 373])

$$
\begin{equation*}
\varphi(\Theta) \approx \operatorname{dist}(\Theta, \partial \Omega) \quad(\Theta \in \Omega) \tag{1-1}
\end{equation*}
$$

For any $(1, \Theta) \in \Omega$, we have (see [Courant and Hilbert 1953])

$$
\varphi(\Theta) \approx \operatorname{dist}\left((1, \Theta), \partial C_{n}(\Omega)\right)
$$

which yields that

$$
\begin{equation*}
\delta(P) \approx r \varphi(\Theta) \tag{1-2}
\end{equation*}
$$

where $\delta(P)=\operatorname{dist}\left(P, \partial C_{n}(\Omega)\right)$ and $P=(r, \Theta) \in C_{n}(\Omega)$.
Solutions of the ordinary differential equation

$$
\begin{equation*}
-Q^{\prime \prime}(r)-\frac{n-1}{r} Q^{\prime}(r)+\left(\frac{\lambda}{r^{2}}+a(r)\right) Q(r)=0, \quad 0<r<\infty \tag{1-3}
\end{equation*}
$$

with a parameter $\lambda$ play an essential role in these questions. It is known (see, for example, [Verzhbinskii and Maz'ya 1971]) that if the potential $a$ belongs to $\mathscr{A}$, then (1-3) has a fundamental system of positive solutions $\{V, W\}$ such that $V$ is nondecreasing with

$$
0 \leq V(0+) \leq V(r) \nearrow \infty \quad \text { as } r \rightarrow+\infty
$$

and $W$ is monotonically decreasing with

$$
+\infty=W(0+)>W(r) \searrow 0 \quad \text { as } r \rightarrow+\infty .
$$

We will also consider the class $\mathscr{B}$, consisting of the potentials $a \in \mathscr{A}$ such that there exists a finite limit $\lim _{r \rightarrow \infty} r^{2} a(r)=k \in[0, \infty)$ and moreover $r^{-1}\left|r^{2} a(r)-k\right| \in$ $L(1, \infty)$. If $a \in \mathscr{B}$, then the (sub)superfunctions are continuous (see [Simon 1982]).

In the rest of paper, we assume that $a \in \mathscr{B}$ and we shall suppress this assumption for simplicity.

Denote

$$
\iota_{k}^{ \pm}=\frac{2-n \pm \sqrt{(n-2)^{2}+4(k+\lambda)}}{2}
$$

then the solutions to (1-3) have the asymptotics (see [Hartman 1964])

$$
\begin{equation*}
V(r) \approx r^{l_{k}^{+}}, \quad W(r) \approx r^{\iota_{k}}, \quad \text { as } r \rightarrow \infty \tag{1-4}
\end{equation*}
$$

Let $u(r, \Theta)$ be a function on $C_{n}(\Omega)$. For any given $r \in \mathbb{R}_{+}$, the integral

$$
\int_{\Omega} u(r, \Theta) \varphi(\Theta) d S_{1}
$$

is denoted by $N_{u}(r)$, when it exists. The finite or infinite limit

$$
\lim _{r \rightarrow \infty} V^{-1}(r) N_{u}(r)
$$

is denoted by $U_{u}$, when it exists.
We fix an open, nonempty and bounded set $G \subset \partial C_{n}(\Omega)$. In $C_{n}(\Omega)$, we normalize the extension, with respect to $G$, by

$$
\mathscr{P}_{\Omega}^{a} f(P)=\frac{\mathrm{PI}_{\Omega}^{a} f(P)}{\mathrm{PI}_{\Omega}^{a} \chi_{G}(P)}
$$

Let

$$
\Gamma(\zeta)=\left\{P=(r, \Theta) \in C_{n}(\Omega):|(r, \Theta)-\zeta| \lesssim \delta(P)\right\}
$$

be a nontangential cone in $C_{n}(\Omega)$ with vertex $\zeta \in \partial C_{n}(\Omega)$.
We define

$$
\aleph_{p}(f, l, P)=\left(\frac{1}{l^{n-1}} \int_{B(P, l)}|f(Q)|^{p} d \sigma_{Q}\right)^{1 / p}
$$

and

$$
\mathbb{E}_{f}^{p}(G)=\left\{P \in G: \aleph_{p}(f-f(P), l, P) \rightarrow 0 \text { as } l \rightarrow 0\right\}
$$

Note that if $f \in L^{p}\left(\partial C_{n}(\Omega)\right)$, then $\left|G \backslash \mathbb{E}_{f}^{p}(G)\right|=0$ (almost every point is a Lebesgue point).

In the proof we need inequalities between the Green function $G_{\Omega}^{a}(P, Q)$ and that of the Laplacian, hereafter denoted by $G_{\Omega}(P, Q)$. It is well known that, for any potential $a(P) \geq 0$,

$$
\begin{equation*}
G_{\Omega}^{a}(P, Q) \leq G_{\Omega}(P, Q) \tag{1-5}
\end{equation*}
$$

The inverse inequality is much more elaborate. Cranston, Fabes and Zhao (see [Cranston et al. 1988]; the case $n=2$ is implicitly contained in [Cranston 1989]) have proved

$$
\begin{equation*}
G_{\Omega}^{a}(P, Q) \geq M(\Omega) G_{\Omega}(P, Q) \tag{1-6}
\end{equation*}
$$

where $M(\Omega)=M(\Omega, a(P))$ is a positive constant and does not depend on points $P$ and $Q$ in $C_{n}(\Omega)$. If $a=0$, then obviously $M(\Omega) \equiv 1$.

So we have

$$
G_{\Omega}^{a}(P, Q) \approx G_{\Omega}(P, Q)
$$

from (1-5) and (1-6), which yields that

$$
\begin{equation*}
\mathrm{PI}_{\Omega}^{a}(P, Q) \approx \mathrm{PI}_{\Omega}(P, Q) \tag{1-7}
\end{equation*}
$$

Now we state our results, which are due to Qiao [2012] in the case $a=0$ by the remark. For related results in the half-space and the unit disc, we refer readers to [Mizuta and Shimomura 2003, Theorem 3; Sjögren 1984; 1997; Rönning 1997; Brundin 1999].

Theorem 2. Let $1 \leq p<\infty$ and $f \in L^{p}\left(\partial C_{n}(\Omega)\right)$. Then, for any $\zeta \in \mathbb{E}_{f}^{p}(G)$ (in particular, for a.e. $\zeta \in G$ ), one has that $\mathscr{P}_{\Omega}^{a} f(P) \rightarrow f(\zeta)$ as $P \rightarrow \zeta$ along $\Gamma(\zeta)$.

## 2. Some lemmas

Lemma 1. For any $P=(r, \Theta) \in C_{n}(\Omega)$ and any $Q=(t, \Phi) \in S_{n}(\Omega)$ satisfying $0<t / r \leq \frac{4}{5}\left(r e s p .0<r / t \leq \frac{4}{5}\right)$,

$$
\begin{align*}
\mathrm{PI}_{\Omega}^{a}(P, Q) & \approx t^{-1} V(t) W(r) \varphi(\Theta)  \tag{2-1}\\
\left(r e s p . \mathrm{PI}_{\Omega}^{a}(P, Q)\right. & \left.\approx V(r) t^{-1} W(t) \varphi(\Theta)\right) \tag{2-2}
\end{align*}
$$

For any $P=(r, \Theta) \in C_{n}(\Omega)$ and any $Q=(t, \Phi) \in S_{n}(\Omega ;(4 r / 5,5 r / 4))$,

$$
\begin{equation*}
\mathrm{PI}_{\Omega}^{a}(P, Q) \approx \frac{r \varphi(\Theta)}{|P-Q|^{n}} \tag{2-3}
\end{equation*}
$$

Proof. These immediately follow from [A. Escassut and Yang 2008, Chapter 11], [Essén and Lewis 1973, Lemma 2], [Azarin 1969, Lemma 4 and Remark] and (1-7).

Lemma 2. $\mathrm{PI}_{\Omega}^{a} 1(P)=O(1)$ as $P \rightarrow \zeta \in G$.
Proof. Write

$$
\operatorname{PI}_{\Omega}^{a} 1(P)=\int_{E_{1}}+\int_{E_{2}}+\int_{E_{3}}=U_{1}(P)+U_{2}(P)+U_{3}(P)
$$

where

$$
E_{1}=S_{n}\left(\Omega ;\left(0, \frac{4}{5} r\right]\right), \quad E_{2}=S_{n}\left(\Omega ;\left[\frac{5}{4} r, \infty\right)\right), \quad E_{3}=S_{n}\left(\Omega ;\left(\frac{4}{5} r, \frac{5}{4} r\right)\right)
$$

By (1-4), (2-1) and (2-2), we have the estimates

$$
\begin{align*}
& U_{1}(P) \approx W(r) \varphi(\Theta) \int_{E_{1}} t^{t_{k}^{+}-1} d \sigma_{Q} \approx-\frac{s_{n}}{\iota_{k}^{-}} W\left(\frac{5}{4}\right) \varphi(\Theta)  \tag{2-4}\\
& U_{2}(P) \approx \frac{s_{n}}{\iota_{k}^{+}} V\left(\frac{4}{5}\right) \varphi(\Theta) \tag{2-5}
\end{align*}
$$

Next we shall estimate $U_{3}(P)$. Take a sufficiently small positive number $k$ such that

$$
S_{n}\left(\Omega ;\left(\frac{4}{5} r, \frac{5}{4} r\right)\right) \subset \bigcup_{P=(r, \Theta) \in \Lambda(k)} B\left(P, \frac{1}{2} r\right)
$$

where

$$
\Lambda(k)=\left\{P=(r, \Theta) \in C_{n}(\Omega): \inf _{z \in \partial \Omega}|(1, \Theta)-(1, z)|<k, 0<r<\infty\right\}
$$

Since $P \rightarrow \zeta \in G$, we only consider the case $P \in \Lambda(k)$. Now, put

$$
H_{i}(P)=\left\{Q \in E_{3}: 2^{i-1} \delta(P) \leq|P-Q|<2^{i} \delta(P)\right\}
$$

Since $S_{n}(\Omega) \cap\left\{Q \in \mathbb{R}^{n}:|P-Q|<\delta(P)\right\}=\varnothing$, we have by (1-5) and (2-3) that

$$
U_{3}(P) \approx \sum_{i=1}^{i(P)} \int_{H_{i}(P)} \frac{r \varphi(\Theta)}{|P-Q|^{n}} d \sigma_{Q}
$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1} \delta(P) \leq r / 2<2^{i(P)} \delta(P)$.
By (1-2) we have

$$
\int_{H_{i}(P)} \frac{r \varphi(\Theta)}{|P-Q|^{n}} d \sigma_{Q} \approx r \varphi(\Theta) \int_{H_{i}(P)} \frac{1}{\delta(P)} d \sigma_{Q}=\frac{r \varphi(\Theta)}{\delta(P)} \frac{s_{n}}{2^{i(P)}} \approx \frac{s_{n}}{2^{i(P)}}
$$

for $i=1,2, \ldots, i(P)$.
So

$$
\begin{equation*}
U_{3}(P) \approx O(1) \tag{2-6}
\end{equation*}
$$

Combining (2-4)-(2-6), Lemma 2 is proved.

Lemma 3. $\mathrm{PI}_{\Omega}^{a} \chi_{G}(P)=\mathrm{PI}_{\Omega}^{a} 1(P)+O(1)$ as $P \rightarrow \zeta \in G$.
Proof. In fact, we only need to prove

$$
\begin{equation*}
U_{4}(P)=\int_{S_{n}(\Omega)-G} \operatorname{PI}_{\Omega}^{a}(P, Q) d \sigma_{Q} \lesssim O(1) \tag{2-7}
\end{equation*}
$$

Write

$$
\begin{aligned}
U_{4}(P) & =\int_{\left(S_{n}(\Omega)-G\right) \cap E_{1}}+\int_{\left(S_{n}(\Omega)-G\right) \cap E_{2}}+\int_{\left(S_{n}(\Omega)-G\right) \cap E_{3}} \\
& =U_{5}(P)+U_{6}(P)+U_{7}(P) .
\end{aligned}
$$

Obviously

$$
\begin{align*}
& U_{5}(P) \lesssim U_{1}(P) \approx O(1)  \tag{2-8}\\
& U_{6}(P) \lesssim U_{2}(P) \approx O(1) \tag{2-9}
\end{align*}
$$

Further, we have by (2-3) that

$$
\begin{align*}
U_{7}(P) & \approx r \varphi(\Theta) \int_{\left(S_{n}(\Omega)-G\right) \cap E_{3}} \frac{1}{|P-Q|^{n}} d \sigma_{Q}  \tag{2-10}\\
& \lesssim \frac{s_{n}}{d}|\zeta| \varphi(\Theta) \quad(P \rightarrow \zeta \in G),
\end{align*}
$$

where $d=\inf _{Q \in \partial C_{n}(\Omega)-G}|Q-\zeta|$.
Combining (2-8)-(2-10), (2-7) holds, which gives the conclusion.

## 3. Proof of the theorem

As $P \rightarrow \zeta \in G$,

$$
\mathrm{PI}_{\Omega}^{a} \chi_{G}(P)=O(1) \neq 0
$$

from Lemmas 2 and 3.
Now let $f \in L^{p}\left(\partial C_{n}(\Omega)\right)$ and $\zeta \in \mathbb{E}_{f}^{p}(G)$ be given. We may, without loss of generality, assume that $f(\zeta)=0$. Furthermore we assume that $P=(r, \Theta) \in \Gamma(\zeta)$. Let $s=|(r, \Theta)-\zeta|$. We write

$$
\begin{aligned}
\mathrm{PI}_{\Omega}^{a} f(P) & =\int_{E_{1}}+\int_{E_{2}}+\int_{E_{3} \cap B(\zeta, 2 s)}+\int_{E_{3} \cap B^{c}(\zeta, 2 s)} \\
& =V_{1} f(P)+V_{2} f(P)+V_{3} f(P)+V_{4} f(P) .
\end{aligned}
$$

By using Hölder's inequality, (1-4), (2-1) and (2-2), we have the estimates

$$
\begin{aligned}
& \left|V_{1} f(P)\right| \lesssim W(r) \varphi(\Theta) \int_{E_{1}} t^{\alpha-1} f(Q) d \sigma_{Q} \lesssim r^{(1-n) / p}\|f\|_{p}, \\
& \left|V_{2} f(P)\right| \lesssim r^{(1-n) / p}\|f\|_{p} .
\end{aligned}
$$

Similar to the estimate of $U_{3}(P)$ in Lemma 2, we only consider the following inequality by (1-2):

$$
\begin{aligned}
\int_{H_{i}(P)} \frac{r \varphi(\Theta)}{|P-Q|^{n}} d \sigma_{Q} & \approx r \varphi(\Theta) \int_{H_{i}(P)} \frac{1}{\left(2^{i-1} \delta(P)\right)^{n}} d \sigma_{Q} \\
& \lesssim r^{\iota_{0}^{+}} \varphi(\Theta) \int_{E_{2}} t^{\iota_{0}^{-}-1}|f(Q)| d \sigma_{Q} \lesssim r^{(1-n) / p}\|f\|_{p}
\end{aligned}
$$

for $i=0,1,2, \ldots, i(P)$, which is similar to the estimate of $V_{2} f(P)$.
So

$$
\left|V_{3} f(P)\right| \lesssim r^{(1-n) / p}\|f\|_{p}
$$

Notice that $|P-Q|>\frac{1}{2}|\zeta-Q|$ in the case $Q \in E_{3} \cap B^{c}(\zeta, 2 s)$. By (1-2) and (2-3), we have

$$
\begin{aligned}
\left|V_{4} f(P)\right| & \lesssim \delta(P) \int_{E_{3} \cap B^{c}(\zeta, 2 s)} \frac{|f(Q)|}{|P-Q|^{n}} d \sigma_{Q} \\
& \lesssim \delta(P) \sum_{i=1}^{\infty} \int_{E_{3} \cap\left(B\left(\zeta, 2^{i+1} s\right) \backslash B\left(\zeta, 2^{i} s\right)\right)} \frac{|f(Q)|}{|\zeta-Q|^{n}} d \sigma_{Q} \\
& \lesssim \delta(P) \sum_{i=1}^{\infty}\left(\frac{1}{2^{i} s}\right)^{n} \int_{E_{3} \cap B\left(\zeta, 2^{i+1} s\right)}|f(Q)| d \sigma_{Q} \\
& \lesssim \delta(P) \sum_{i=1}^{\infty} \aleph_{1}\left(f, 2^{i+1} s, \zeta\right) \lesssim \delta(P) \sum_{i=1}^{\infty} \int_{2^{i+1} s}^{2^{i+2} s} \frac{\aleph_{1}(f, l, \zeta)}{l} d l \\
& \lesssim \delta(P) \int_{s}^{\infty} \frac{\aleph_{1}(f, l, \zeta)}{l} d l \lesssim \delta(P) \int_{\delta(P)}^{\infty} \frac{\aleph_{1}(f, l, \zeta)}{l} d l
\end{aligned}
$$

Thus it follows that

$$
\begin{aligned}
\left|\mathscr{P}_{\Omega} f(P)\right| & \lesssim \frac{1}{O(1)}\left[\left|V_{1} f(P)\right|+\left|V_{2} f(P)\right|+\left|V_{3} f(P)\right|+\left|V_{4} f(P)\right|\right] \\
& \lesssim r^{(1-n) / p}\|f\|_{p}+\delta(P) \int_{\delta(P)}^{\infty} \frac{\aleph_{1}(f, l, \zeta)}{l} d l
\end{aligned}
$$

Using the fact that $s \lesssim \delta(P) \lesssim r \varphi(\Theta)$, we get

$$
\left|\mathscr{P}_{\Omega} f(P)\right| \lesssim \aleph_{1}(f, 2 s, \zeta)+\delta(P) \int_{\delta(P)}^{\infty} \frac{\aleph_{1}(f, l, \zeta)}{l} d l
$$

It is clear that

$$
\int_{\delta(P)}^{\infty} \frac{\aleph_{1}(f, l, \zeta)}{l} d l
$$

is a convergent integral, since

$$
\frac{\aleph_{1}(f, l, \zeta)}{l} \lesssim s^{-1-n} s^{n / q}\|f\|_{p} \lesssim s^{-1-n / p}\|f\|_{p}
$$

from Hölder's inequality.
Now, as $\delta(P) \rightarrow 0$, we also have $s \rightarrow 0$. Since $f(\zeta)=0$ and since we have assumed that $\zeta \in \mathbb{E}_{f}^{p}(G)$ (and thus that $\zeta \in \mathbb{E}_{f}^{1}(G)$ ), it follows that $\mathscr{P}_{\Omega}^{a} f(P) \rightarrow$ $0=f(\zeta)$ as $P=(r, \Theta) \rightarrow \zeta$ along $\Gamma(\zeta)$. This concludes the proof.

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