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## BOUNDARY LIMITS FOR FRACTIONAL POISSON $a$-EXTENSIONS OF $L^p$ BOUNDARY FUNCTIONS IN A CONE


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# BOUNDARY LIMITS FOR FRACTIONAL POISSON $a$-EXTENSIONS OF $L^p$ BOUNDARY FUNCTIONS IN A CONE

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**If one replaces the Poisson kernel of a cone by the Poisson $a$-kernel, then normalized Poisson integrals with respect to the stationary Schrödinger operator converge along approach regions wider than the ordinary non-tangential cones. In this paper we present new and simplified proofs of these results. We also generalize the result by Mizuta and Shimomura to the smooth cones.**

## 1. Introduction and main results

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbb{R}^n$ $(n \geq 2)$ the $n$-dimensional Euclidean space. A point in $\mathbb{R}^n$ is denoted by $P = (X, x_n)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance of two points $P$ and $Q$ in $\mathbb{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$, with $O$ the origin of $\mathbb{R}^n$, is simply denoted by $|P|$. The boundary, the closure and the complement of a set $S$ in $\mathbb{R}^n$ are denoted by $\partial S$, $\overline{S}$ and $S^c$, respectively.

We introduce a system of spherical coordinates $(r, \Theta)$, $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbb{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, x_n)$ by

$$x_1 = r\left(\prod_{j=1}^{n-1} \sin\theta_j\right), \qquad x_n = r\cos\theta_1,$$

for $n \geq 2$, and for $n \geq 3$,

$$x_{n-m+1} = r\left(\prod_{j=1}^{m-1} \sin\theta_j\right)\cos\theta_m \quad (2 \leq m \leq n-1),$$

where $0 \leq r < +\infty$, $-\pi/2 \leq \theta_{n-1} < 3\pi/2$, and if $n \geq 3$, then $0 \leq \theta_j \leq \pi$ $(1 \leq j \leq n-2)$.

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The unit sphere and the upper unit half-sphere are denoted by $S^{n-1}$ and $S_+^{n-1}$, respectively. For simplicity, a point $(1, \Theta)$ on $S^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$, for a set $\Omega \subset S^{n-1}$, are often identified with $\Theta$ and $\Omega$, respectively. For two sets $\Xi \subset \mathbb{R}_+$ and $\Omega \subset S^{n-1}$, the set $\{(r, \Theta) \in \mathbb{R}^n; r \in \Xi, (1, \Theta) \in \Omega\}$ in $\mathbb{R}^n$ is simply denoted by $\Xi \times \Omega$. In particular, the half-space $\mathbb{R}_+ \times S_+^{n-1} = \{(X, x_n) \in \mathbb{R}^n; x_n > 0\}$ will be denoted by $T_n$.

By $C_n(\Omega)$, we denote the set $\mathbb{R}_+ \times \Omega$ in $\mathbb{R}^n$ with the domain $\Omega$ on $S^{n-1}$. We call it a cone. Then $T_n$ is a special cone obtained by putting $\Omega = S_+^{n-1}$. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$, with $I$ an interval on $\mathbb{R}$, by $C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega)$, we denote $S_n(\Omega; (0, +\infty))$, which is $\partial C_n(\Omega) - \{O\}$.

For positive functions $h_1$ and $h_2$, we say that $h_1 \lesssim h_2$ if $h_1 \le M h_2$ for some constant $M > 0$. If $h_1 \lesssim h_2$ and $h_2 \lesssim h_1$, we say that $h_1 \approx h_2$.

This article is devoted to the stationary Schrödinger operator

$$\mathrm{SSE}_a = -\Delta + a(P)I,$$

where $\Delta$ is the Laplace operator and $I$ is the identity operator. We assume hereafter that the potential $a(P)$ is a nonnegative, locally integrable function in $C_n(\Omega)$, namely, $0 \le a \in L^b_{\mathrm{loc}}(C_n(\Omega))$, with $b > n/2$ if $n \ge 4$, and with $b = 2$ if $n = 2$ or 3. We denote this class of potentials by $\mathscr{A}$.

If $a \in \mathscr{A}$, then the operator $\mathrm{SSE}_a$ can be extended in the usual way from the space $C_0^\infty(C_n(\Omega))$ to an essentially self-adjoint operator on $L^2(C_n(\Omega))$ (see [Reed and Simon 1979, Chapter 13]). We shall denote the extended operator by $\mathrm{SSE}_a$ as well. The latter has Green function $G^a_\Omega(P, Q)$ vanishing almost everywhere at the boundary and possessing all the analytic properties. For $|P - Q| \to 0$, we normalize it such that $c_n G^a_\Omega(P, Q) \approx -\log|P - Q|$ when $n = 2$, or $c_n G^a_\Omega(P, Q) \approx |P - Q|^{2-n}$ when $n \ge 3$. Here $c_2 = 2\pi$, $c_n = (n-2)s_n$ when $n \ge 3$, and $s_n$ is the surface area $2\pi^{n/2}(\Gamma(n/2))^{-1}$ of $S^{n-1}$. The Green function $G^a_\Omega(P, Q)$ is positive on $C_n(\Omega)$ and its inner normal derivative $\partial G^a_\Omega(P, Q)/\partial n_Q \ge 0$. We denote this derivative by $\mathrm{PI}^a_\Omega(P, Q)$, which is called the Poisson $a$-kernel with respect to $C_n(\Omega)$. Then the Poisson $a$-integral $\mathrm{PI}^a_\Omega f(P)$ $(P \in C_n(\Omega))$ is defined by

$$\mathrm{PI}^a_\Omega f(P) = \int_{S_n(\Omega)} \mathrm{PI}^a_\Omega(P, Q) f(Q)\, d\sigma_Q,$$

where

$$\mathrm{PI}^a_\Omega(P, Q) = \frac{\partial}{\partial n_Q} G^a_\Omega(P, Q),$$

$f \in L^p(\partial C_n(\Omega))$ $(1 \le p < \infty)$ and $d\sigma_Q$ is the surface area element on $S_n(\Omega)$.

**Remark 1** [Yoshida 1991]. Let $\Omega = S_+^{n-1}$ and $a = 0$. Then

$$G^0_{S_+^{n-1}}(P, Q) = \begin{cases} \log|P - Q^*| - \log|P - Q| & n = 2, \\ |P - Q|^{2-n} - |P - Q^*|^{2-n} & n \ge 3, \end{cases}$$

where $Q^* = (Y, -y_n)$; that is, $Q^*$ is the mirror image of $Q = (Y, y_n)$ with respect to $\partial T_n$. Hence, for the two points $P = (X, x_n) \in T_n$ and $Q = (Y, y_n) \in \partial T_n$, we have

$$\mathrm{PI}^0_{S_+^{n-1}}(P, Q) = \begin{cases} 2|P-Q|^{-2}x_n & n = 2, \\ 2(n-2)|P-Q|^{-n}x_n & n \geq 3. \end{cases}$$

Let $\Omega$ be a domain on $S^{n-1}$ with smooth boundary. Consider the Dirichlet problem

$$\begin{aligned} (\Lambda_n + \lambda)\varphi = 0 &\quad \text{on } \Omega, \\ \varphi = 0 &\quad \text{on } \partial\Omega, \end{aligned}$$

where $\Lambda_n$ is the spherical part of the Laplace operator

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\Lambda_n}{r^2}.$$

We denote the least positive eigenvalue of this boundary value problem by $\lambda$ and the normalized positive eigenfunction corresponding to $\lambda$ by $\varphi(\Theta)$; $\int_\Omega \varphi^2(\Theta)\, d\sigma_\Theta = 1$, where $d\sigma_\Theta$ is the surface area on $S^{n-1}$.

To simplify our consideration in the following, we shall assume that if $n \geq 3$, then $\Omega$ is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $S^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g., see [Gilbarg and Trudinger 1977, pp. 88–89] for the definition of $C^{2,\alpha}$-domain). Then by modifying Miranda's method [1970, pp. 7–8], we can prove the inequality (see [Yoshida 1991, p. 373])

$$\varphi(\Theta) \approx \operatorname{dist}(\Theta, \partial\Omega) \quad (\Theta \in \Omega). \tag{1-1}$$

For any $(1, \Theta) \in \Omega$, we have (see [Courant and Hilbert 1953])

$$\varphi(\Theta) \approx \operatorname{dist}\big((1, \Theta), \partial C_n(\Omega)\big),$$

which yields that

$$\delta(P) \approx r\varphi(\Theta), \tag{1-2}$$

where $\delta(P) = \operatorname{dist}(P, \partial C_n(\Omega))$ and $P = (r, \Theta) \in C_n(\Omega)$.

Solutions of the ordinary differential equation

$$-Q''(r) - \frac{n-1}{r}Q'(r) + \left(\frac{\lambda}{r^2} + a(r)\right)Q(r) = 0, \quad 0 < r < \infty, \tag{1-3}$$

with a parameter $\lambda$ play an essential role in these questions. It is known (see, for example, [Verzhbinskii and Maz'ya 1971]) that if the potential $a$ belongs to $\mathscr{A}$, then (1-3) has a fundamental system of positive solutions $\{V, W\}$ such that $V$ is nondecreasing with

$$0 \leq V(0+) \leq V(r) \nearrow \infty \quad \text{as } r \to +\infty,$$

and $W$ is monotonically decreasing with

$$+\infty = W(0+) > W(r) \searrow 0 \quad \text{as } r \to +\infty.$$

We will also consider the class $\mathcal{B}$, consisting of the potentials $a \in \mathcal{A}$ such that there exists a finite limit $\lim_{r\to\infty} r^2 a(r) = k \in [0, \infty)$ and moreover $r^{-1}|r^2 a(r) - k| \in L(1, \infty)$. If $a \in \mathcal{B}$, then the (sub)superfunctions are continuous (see [Simon 1982]).

In the rest of paper, we assume that $a \in \mathcal{B}$ and we shall suppress this assumption for simplicity.

Denote

$$\iota_k^{\pm} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(k+\lambda)}}{2};$$

then the solutions to (1-3) have the asymptotics (see [Hartman 1964])

$$V(r) \approx r^{\iota_k^+}, \quad W(r) \approx r^{\iota_k^-}, \quad \text{as } r \to \infty. \tag{1-4}$$

Let $u(r, \Theta)$ be a function on $C_n(\Omega)$. For any given $r \in \mathbb{R}_+$, the integral

$$\int_\Omega u(r, \Theta)\varphi(\Theta)\, dS_1,$$

is denoted by $N_u(r)$, when it exists. The finite or infinite limit

$$\lim_{r\to\infty} V^{-1}(r) N_u(r)$$

is denoted by $\mathcal{U}_u$, when it exists.

We fix an open, nonempty and bounded set $G \subset \partial C_n(\Omega)$. In $C_n(\Omega)$, we normalize the extension, with respect to $G$, by

$$\mathcal{P}^a_\Omega f(P) = \frac{\mathrm{PI}^a_\Omega f(P)}{\mathrm{PI}^a_\Omega \chi_G(P)}.$$

Let

$$\Gamma(\zeta) = \{P = (r, \Theta) \in C_n(\Omega) : |(r, \Theta) - \zeta| \lesssim \delta(P)\}$$

be a nontangential cone in $C_n(\Omega)$ with vertex $\zeta \in \partial C_n(\Omega)$.

We define

$$\aleph_p(f, l, P) = \left(\frac{1}{l^{n-1}} \int_{B(P,l)} |f(Q)|^p \, d\sigma_Q\right)^{1/p}$$

and

$$\mathbb{E}^p_f(G) = \{P \in G : \aleph_p(f - f(P), l, P) \to 0 \text{ as } l \to 0\}.$$

Note that if $f \in L^p(\partial C_n(\Omega))$, then $|G \backslash \mathbb{E}^p_f(G)| = 0$ (almost every point is a Lebesgue point).

In the proof we need inequalities between the Green function $G^a_\Omega(P,Q)$ and that of the Laplacian, hereafter denoted by $G_\Omega(P,Q)$. It is well known that, for any potential $a(P)\ge 0$,

$$G^a_\Omega(P,Q)\le G_\Omega(P,Q). \tag{1-5}$$

The inverse inequality is much more elaborate. Cranston, Fabes and Zhao (see [Cranston et al. 1988]; the case $n=2$ is implicitly contained in [Cranston 1989]) have proved

$$G^a_\Omega(P,Q)\ge M(\Omega)G_\Omega(P,Q), \tag{1-6}$$

where $M(\Omega)=M(\Omega,a(P))$ is a positive constant and does not depend on points $P$ and $Q$ in $C_n(\Omega)$. If $a=0$, then obviously $M(\Omega)\equiv 1$.

So we have

$$G^a_\Omega(P,Q)\approx G_\Omega(P,Q),$$

from (1-5) and (1-6), which yields that

$$\mathrm{PI}^a_\Omega(P,Q)\approx \mathrm{PI}_\Omega(P,Q). \tag{1-7}$$

Now we state our results, which are due to Qiao [2012] in the case $a=0$ by the remark. For related results in the half-space and the unit disc, we refer readers to [Mizuta and Shimomura 2003, Theorem 3; Sjögren 1984; 1997; Rönning 1997; Brundin 1999].

**Theorem 2.** *Let $1\le p<\infty$ and $f\in L^p(\partial C_n(\Omega))$. Then, for any $\zeta\in \mathbb{E}^p_f(G)$ (in particular, for a.e. $\zeta\in G$), one has that $\mathcal{P}^a_\Omega f(P)\to f(\zeta)$ as $P\to\zeta$ along $\Gamma(\zeta)$.*

## 2. Some lemmas

**Lemma 1.** *For any $P=(r,\Theta)\in C_n(\Omega)$ and any $Q=(t,\Phi)\in S_n(\Omega)$ satisfying $0<t/r\le\frac{4}{5}$ (resp. $0<r/t\le\frac{4}{5}$),*

$$\mathrm{PI}^a_\Omega(P,Q)\approx t^{-1}V(t)W(r)\varphi(\Theta) \tag{2-1}$$

$$(\textit{resp. } \mathrm{PI}^a_\Omega(P,Q)\approx V(r)t^{-1}W(t)\varphi(\Theta)). \tag{2-2}$$

*For any $P=(r,\Theta)\in C_n(\Omega)$ and any $Q=(t,\Phi)\in S_n(\Omega;(4r/5,5r/4))$,*

$$\mathrm{PI}^a_\Omega(P,Q)\approx \frac{r\varphi(\Theta)}{|P-Q|^n}, \tag{2-3}$$

*Proof.* These immediately follow from [A. Escassut and Yang 2008, Chapter 11], [Essén and Lewis 1973, Lemma 2], [Azarin 1969, Lemma 4 and Remark] and (1-7). □

**Lemma 2.** $\mathrm{PI}_\Omega^a 1(P) = O(1)$ *as* $P \to \zeta \in G$.

*Proof.* Write

$$\mathrm{PI}_\Omega^a 1(P) = \int_{E_1} + \int_{E_2} + \int_{E_3} = U_1(P) + U_2(P) + U_3(P),$$

where

$$E_1 = S_n\big(\Omega; \big(0, \tfrac{4}{5}r\big]\big), \quad E_2 = S_n\big(\Omega; \big[\tfrac{5}{4}r, \infty\big)\big), \quad E_3 = S_n\big(\Omega; \big(\tfrac{4}{5}r, \tfrac{5}{4}r\big)\big).$$

By (1-4), (2-1) and (2-2), we have the estimates

$$U_1(P) \approx W(r)\varphi(\Theta) \int_{E_1} t^{\iota_k^+ - 1}\, d\sigma_Q \approx -\frac{s_n}{\iota_k^-} W\Big(\frac{5}{4}\Big)\varphi(\Theta), \tag{2-4}$$

$$U_2(P) \approx \frac{s_n}{\iota_k^+} V\Big(\frac{4}{5}\Big)\varphi(\Theta). \tag{2-5}$$

Next we shall estimate $U_3(P)$. Take a sufficiently small positive number $k$ such that

$$S_n\big(\Omega; \big(\tfrac{4}{5}r, \tfrac{5}{4}r\big)\big) \subset \bigcup_{P=(r,\Theta)\in\Lambda(k)} B\big(P, \tfrac{1}{2}r\big),$$

where

$$\Lambda(k) = \Big\{ P = (r, \Theta) \in C_n(\Omega) : \inf_{z\in\partial\Omega} |(1, \Theta) - (1, z)| < k, 0 < r < \infty \Big\}.$$

Since $P \to \zeta \in G$, we only consider the case $P \in \Lambda(k)$. Now, put

$$H_i(P) = \big\{ Q \in E_3 : 2^{i-1}\delta(P) \le |P - Q| < 2^i \delta(P) \big\}.$$

Since $S_n(\Omega) \cap \{Q \in \mathbb{R}^n : |P - Q| < \delta(P)\} = \varnothing$, we have by (1-5) and (2-3) that

$$U_3(P) \approx \sum_{i=1}^{i(P)} \int_{H_i(P)} \frac{r\varphi(\Theta)}{|P - Q|^n}\, d\sigma_Q,$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P) \le r/2 < 2^{i(P)}\delta(P)$.

By (1-2) we have

$$\int_{H_i(P)} \frac{r\varphi(\Theta)}{|P - Q|^n}\, d\sigma_Q \approx r\varphi(\Theta) \int_{H_i(P)} \frac{1}{\delta(P)}\, d\sigma_Q = \frac{r\varphi(\Theta)}{\delta(P)} \frac{s_n}{2^{i(P)}} \approx \frac{s_n}{2^{i(P)}},$$

for $i = 1, 2, \ldots, i(P)$.

So

$$U_3(P) \approx O(1). \tag{2-6}$$

Combining (2-4)–(2-6), Lemma 2 is proved. $\square$

**Lemma 3.** $\mathrm{PI}^a_\Omega\, \chi_G(P) = \mathrm{PI}^a_\Omega\, 1(P) + O(1)$ *as* $P \to \zeta \in G$.

*Proof.* In fact, we only need to prove

$$U_4(P) = \int_{S_n(\Omega)-G} \mathrm{PI}^a_\Omega(P, Q)\, d\sigma_Q \lesssim O(1). \tag{2-7}$$

Write

$$\begin{aligned} U_4(P) &= \int_{(S_n(\Omega)-G)\cap E_1} + \int_{(S_n(\Omega)-G)\cap E_2} + \int_{(S_n(\Omega)-G)\cap E_3} \\ &= U_5(P) + U_6(P) + U_7(P). \end{aligned}$$

Obviously

$$U_5(P) \lesssim U_1(P) \approx O(1), \tag{2-8}$$

$$U_6(P) \lesssim U_2(P) \approx O(1). \tag{2-9}$$

Further, we have by (2-3) that

$$\begin{aligned} U_7(P) &\approx r\varphi(\Theta) \int_{(S_n(\Omega)-G)\cap E_3} \frac{1}{|P-Q|^n}\, d\sigma_Q \\ &\lesssim \frac{s_n}{d} |\zeta| \varphi(\Theta) \quad (P \to \zeta \in G), \end{aligned} \tag{2-10}$$

where $d = \inf_{Q \in \partial C_n(\Omega)-G} |Q - \zeta|$.

Combining (2-8)–(2-10), (2-7) holds, which gives the conclusion. □

## 3. Proof of the theorem

As $P \to \zeta \in G$,

$$\mathrm{PI}^a_\Omega\, \chi_G(P) = O(1) \neq 0,$$

from Lemmas 2 and 3.

Now let $f \in L^p(\partial C_n(\Omega))$ and $\zeta \in \mathbb{E}^p_f(G)$ be given. We may, without loss of generality, assume that $f(\zeta) = 0$. Furthermore we assume that $P = (r, \Theta) \in \Gamma(\zeta)$. Let $s = |(r, \Theta) - \zeta|$. We write

$$\begin{aligned} \mathrm{PI}^a_\Omega\, f(P) &= \int_{E_1} + \int_{E_2} + \int_{E_3 \cap B(\zeta, 2s)} + \int_{E_3 \cap B^c(\zeta, 2s)} \\ &= V_1 f(P) + V_2 f(P) + V_3 f(P) + V_4 f(P). \end{aligned}$$

By using Hölder's inequality, (1-4), (2-1) and (2-2), we have the estimates

$$\begin{aligned} |V_1 f(P)| &\lesssim W(r)\varphi(\Theta) \int_{E_1} t^{\alpha-1} f(Q)\, d\sigma_Q \lesssim r^{(1-n)/p} \|f\|_p, \\ |V_2 f(P)| &\lesssim r^{(1-n)/p} \|f\|_p. \end{aligned}$$

Similar to the estimate of $U_3(P)$ in Lemma 2, we only consider the following inequality by (1-2):

$$\int_{H_i(P)} \frac{r\varphi(\Theta)}{|P-Q|^n}\,d\sigma_Q \approx r\varphi(\Theta)\int_{H_i(P)} \frac{1}{(2^{i-1}\delta(P))^n}\,d\sigma_Q$$
$$\lesssim r^{\iota_0^+}\varphi(\Theta)\int_{E_2} t^{\iota_0^- -1}|f(Q)|\,d\sigma_Q \lesssim r^{(1-n)/p}\|f\|_p,$$

for $i=0,1,2,\dots,i(P)$, which is similar to the estimate of $V_2 f(P)$.

So

$$|V_3 f(P)| \lesssim r^{(1-n)/p}\|f\|_p.$$

Notice that $|P-Q|>\frac{1}{2}|\zeta-Q|$ in the case $Q\in E_3\cap B^c(\zeta,2s)$. By (1-2) and (2-3), we have

$$\begin{aligned}
|V_4 f(P)| &\lesssim \delta(P)\int_{E_3\cap B^c(\zeta,2s)} \frac{|f(Q)|}{|P-Q|^n}\,d\sigma_Q\\
&\lesssim \delta(P)\sum_{i=1}^{\infty}\int_{E_3\cap(B(\zeta,2^{i+1}s)\setminus B(\zeta,2^i s))} \frac{|f(Q)|}{|\zeta-Q|^n}\,d\sigma_Q\\
&\lesssim \delta(P)\sum_{i=1}^{\infty}\Big(\frac{1}{2^i s}\Big)^n\int_{E_3\cap B(\zeta,2^{i+1}s)} |f(Q)|\,d\sigma_Q\\
&\lesssim \delta(P)\sum_{i=1}^{\infty}\aleph_1(f,2^{i+1}s,\zeta) \lesssim \delta(P)\sum_{i=1}^{\infty}\int_{2^{i+1}s}^{2^{i+2}s}\frac{\aleph_1(f,l,\zeta)}{l}\,dl\\
&\lesssim \delta(P)\int_s^{\infty}\frac{\aleph_1(f,l,\zeta)}{l}\,dl \lesssim \delta(P)\int_{\delta(P)}^{\infty}\frac{\aleph_1(f,l,\zeta)}{l}\,dl.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
|\mathscr{P}_\Omega f(P)| &\lesssim \frac{1}{O(1)}\big[|V_1 f(P)|+|V_2 f(P)|+|V_3 f(P)|+|V_4 f(P)|\big]\\
&\lesssim r^{(1-n)/p}\|f\|_p+\delta(P)\int_{\delta(P)}^{\infty}\frac{\aleph_1(f,l,\zeta)}{l}\,dl.
\end{aligned}$$

Using the fact that $s\lesssim\delta(P)\lesssim r\varphi(\Theta)$, we get

$$|\mathscr{P}_\Omega f(P)| \lesssim \aleph_1(f,2s,\zeta)+\delta(P)\int_{\delta(P)}^{\infty}\frac{\aleph_1(f,l,\zeta)}{l}\,dl.$$

It is clear that

$$\int_{\delta(P)}^{\infty}\frac{\aleph_1(f,l,\zeta)}{l}\,dl$$

is a convergent integral, since

$$\frac{\aleph_1(f,l,\zeta)}{l} \lesssim s^{-1-n} s^{n/q} \|f\|_p \lesssim s^{-1-n/p} \|f\|_p,$$

from Hölder's inequality.

Now, as $\delta(P) \to 0$, we also have $s \to 0$. Since $f(\zeta) = 0$ and since we have assumed that $\zeta \in \mathbb{E}^p_f(G)$ (and thus that $\zeta \in \mathbb{E}^1_f(G)$), it follows that $\mathscr{P}^a_\Omega f(P) \to 0 = f(\zeta)$ as $P = (r, \Theta) \to \zeta$ along $\Gamma(\zeta)$. This concludes the proof.

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