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NORMAL FAMILIES OF HOLOMORPHIC MAPPINGS INTO COMPLEX PROJECTIVE SPACE CONCERNING SHARED HYPERPLANES

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We prove new criteria for normality for holomorphic mappings into the complex projective space using the generalized Zalcman lemma. This improves previous results in one complex variable. An example is included to complement our theory.

1. Introduction

Recall that a family $\mathcal{F}$ of meromorphic functions on a plane domain $D \subset \mathbb{C}$ is *normal* on D if every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on D (with respect to the spherical metric) to a meromorphic function or to ∞.

The following Picard-type theorem is a consequence of the second main theorem of value distribution theory.

Theorem A [Bergweiler 2006, pp. 78–80] . *Let f be a meromorphic function on the complex plane $\mathbb{C}$. If there exist three mutually distinct points a_1, a_2 and a_3 on the Riemann sphere such that $f(z) - a_j$ (for $j = 1, 2, 3$) has no zero on the complex plane then $f(z)$ is a constant.*

A heuristic principle, bearing Bloch's name and playing an important role in the theory of normal families, says that if the only meromorphic function with a certain property are constant, then a family of meromorphic functions in a plane domain possessing this property is likely to be normal [Bergweiler 2006, pp. 78–80]. For example, the Montel-type theorem associated with Theorem A is true:

Theorem B [Bergweiler 2006, pp. 78–80]. *Let $\mathcal{F}$ be a family of meromorphic functions on a plane domain D. Suppose that there exist three mutually distinct points a_1, a_2 and a_3 on the Riemann sphere such that $f(z) - a_j$ (for $j = 1, 2, 3$) has no zero on D for each $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family on D.*

We say that two meromorphic functions f and g on a domain D *share the value* a ($a = \infty$ is allowed) if $f^{-1}(a) = g^{-1}(a)$ as sets (ignoring multiplicities). There

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are many results concerning this notion in value distribution theory, such as R. Nevanlinna's famous theorem [1926] that two meromorphic functions on the complex plane sharing five distinct values coincide identically. (The number 5 cannot be reduced, as the pair e^z, e^{-z}, with shared values $0, 1, -1, \infty$, demonstrates; but Nevanlinna [1926] also showed that if four values are shared and the multiplicities with which these each of these values is taken are the same for the two functions, the two functions differ only by a Möbius transformation. The condition that the multiplicities are the same cannot be relaxed; see [Gundersen 1979].)

More generally, the maximum modulus principle and Montel's theorem yield this extension of Theorem B:

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions on a plane domain D. Suppose that there exist three mutually distinct points a_1, a_2 and a_3 on the Riemann sphere such that for each $f, g \in \mathcal{F}$, f and g share a_j (for $j = 1, 2, 3$) on D. Then $\mathcal{F}$ is normal on D.*

The following question arises naturally from Theorem C. *Suppose two families of meromorphic functions share some values a_j. If one is normal, is the other normal?* Recently the problem was solved by Pang and Liu, who showed that if two families of meromorphic functions share four values, the normality of one family implies the normality of the other. They also gave a counterexample to show that the number 4 is sharp.

Theorem D [Liu et al. 2013]. *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions on a plane domain D. Suppose that there exist four mutually distinct points a_1, a_2, a_3 and a_4 on the Riemann sphere such that for each $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that f and g share a_j for $j = 1, \dots, 4$ on D. If $\mathcal{G}$ is normal on D, then $\mathcal{F}$ is also normal on D.*

The classical Zalcman lemma plays a central role in normal family theory of one complex variable. On the other hand, the study of normal families for holomorphic mappings was initiated by H. Wu in his well-known paper in Acta Math [1967]. Much attention has been given to find the correct generalization of Zalcman's result to several complex variables. In this paper we prove some new normality criteria for holomorphic mappings from plane domains into $\mathbb{P}^s(\mathbb{C})$ using the generalized Zalcman lemma. An example will be included to complement our theory.

2. Basic notions and main results

Basic notions. We start with relevant definitions. For details see [Mai et al. 2005; Shabat 1985, pp. 99–106; Ru 2001, pp. 99–102].

Let $\mathbb{P}^s(\mathbb{C})$ be a complex s-dimensional projective space and $\rho : \mathbb{C}^{s+1} \backslash \{0\} \to \mathbb{P}^s(\mathbb{C})$ be the standard projective mapping. A subset H of $\mathbb{P}^s(\mathbb{C})$ is called a hyperplane if there is a s-dimensional linear subspace $\widetilde{H}$ of $\mathbb{C}^{s+1}$ such that

$$\rho(\widetilde{H} - \{0\}) = H.$$

For a fixed system of homogeneous coordinates $Z = [Z_0 : Z_1 : \cdots : Z_s]$, a hyperplane H of $\mathbb{P}^s(\mathbb{C})$ can be written as

$$H = \big\{[Z_0 : Z_1 : \cdots : Z_s] \in \mathbb{P}^s(\mathbb{C}) \mid \langle Z, \alpha\rangle = 0\big\},$$

where

$$\langle Z, \alpha\rangle := a_0 Z_0 + \cdots + a_s Z_s$$

and $\alpha = (a_0, \ldots, a_s) \in \mathbb{C}^{s+1}$ is a nonzero vector. We write it as

$$H = \{\langle Z, \alpha\rangle = 0\}$$

for convenience. In particular, we can take $\alpha \in B$, where B is the set of Euclidean unit vectors in $\mathbb{C}^{s+1}$.

Let $H_1, \ldots, H_{s+1}$ be hyperplanes in $\mathbb{P}^s(\mathbb{C})$. Let $\alpha_j = (a_{j0}, \ldots, a_{js}) \in B$ be such that

$$H_j = \{\langle Z, \alpha_j\rangle = 0\}$$

for $j = 1, \ldots, s+1$. Define

$$D(H_1, \ldots, H_{s+1}) := |\det(\alpha_1^T, \ldots, \alpha_{s+1}^T)|$$

which only depends on H_j but does not depend on the choice of $\alpha_j \in B$.

Definition 2.1. Let $H_1, \ldots, H_q$, with $q \geq s+1$, be hyperplanes in $\mathbb{P}^s(\mathbb{C})$. Define

$$D(H_1, \ldots, H_q) := \prod_{1 \leq j_1 < \cdots < j_{s+1} \leq q} |\det(\alpha_{j_1}^T, \ldots, \alpha_{j_{s+1}}^T)|.$$

We say the hyperplane family $H_1, \ldots, H_q$, $q \geq s+1$, in $\mathbb{P}^s(\mathbb{C})$ is in general position if $D(H_1, \ldots, H_q) > 0$.

Let M and N be connected Hermitian manifolds of dimension m and s with Hermitian metrics h_M and h_N, respectively. The space $\mathscr{C}(M; N)$ of continuous mappings between M and N endowed with the compact-open topology is second countable so that a metric can be furnished in $\mathscr{C}(M; N)$ which induces the compact-open topology.

Remark 2.2. A sequence $\{f_n\}$ in $\mathscr{C}(M; N)$ converges to f in $\mathscr{C}(M; N)$ in this topology if and only if $\{f_n\}$ converges to f uniformly on compact subset of M.

The space $\mathscr{H}(M; N)$ of holomorphic mappings from M into N is a closed subspace of $\mathscr{C}(M; N)$.

Definition 2.3. A family $\mathcal{F} \subset \mathcal{H}(M; N)$ is called normal on M if any sequence in $\mathcal{F}$ contains a subsequence which is relatively compact in $\mathcal{H}(M; N)$, that is, if any sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges to $f \in \mathcal{H}(M; N)$ uniformly on every compact subset of M.

Throughout this paper, we consider the special case where M is a plane domain and N is a complex projective space.

Let $f : D \to \mathbb{P}^s(\mathbb{C})$ be a holomorphic map and U be an open set in D. Any holomorphic mapping $\tilde{f} : U \to \mathbb{C}^{s+1}$ such that $\rho \circ \tilde{f}(z) \equiv f(z)$ in U is called a representation of f on U. For a fixed system of homogeneous coordinates $[Z_0 : Z_1 : \cdots : Z_s]$ we set

$$V_i = \big\{[Z_0 : Z_1 : \cdots : Z_s] \mid Z_i \neq 0\big\}, \quad \text{for } i = 0, \ldots, s.$$

Then every $a \in D$ has a neighborhood U of a such that $f(U) \subset V_i$ for some i, and f has a representation

$$\tilde{f} = (f_0, \ldots, f_{i-1}, 1, f_{i+1}, \ldots, f_s)$$

on U with holomorphic functions $f_0, \ldots, f_{i-1}, f_{i+1}, \ldots, f_s$.

Definition 2.4. For an open subset U of D we call a representation $\tilde{f} = (f_0, \ldots, f_s)$ a reduced representation of f on U if $f_0, \ldots, f_s$ are holomorphic functions on U and have no common zero.

Remark 2.5. Every holomorphic map of D into $\mathbb{P}^s(\mathbb{C})$ has a reduced representation on some neighborhood of each point in D. Moreover, let $\tilde{f} = (f_0, \ldots, f_s)$ be a reduced representation of f. For an arbitrary nowhere zero holomorphic function h, $(f_0 h, \ldots, f_s h)$ is also a reduced representation of f. Conversely, for every reduced representation $(g_0, \ldots, g_s)$ of f, each g_i can be written as $g_i = h f_i$ for some nowhere zero holomorphic function h.

Remark 2.6. Every $f \in \mathcal{H}(D; \mathbb{P}^s(\mathbb{C}))$ has a reduced representation on the totality of D [Fujimoto 1974].

We now give the definition of sharing hyperplanes, which extends the definition of sharing values. Take $f \in \mathcal{H}(D; \mathbb{P}^s(\mathbb{C}))$. Let $H = \{\langle Z, \alpha\rangle = 0\}$ be a hyperplane in $\mathbb{P}^s(\mathbb{C})$, where $\alpha = (a_0, \ldots, a_s) \in \mathbb{C}^{s+1} - \{0\}$. Let $\tilde{f} = (f_0, \ldots, f_s)$ be a reduced representation of f. We consider the holomorphic function on D

$$\langle \tilde{f}(z), H\rangle := a_0 f_0 + \cdots + a_s f_s.$$

Definition 2.7. Suppose f, g are in $\mathcal{H}(D; \mathbb{P}^s(\mathbb{C}))$ and H is a hyperplane in $\mathbb{P}^s(\mathbb{C})$. If there exist some (thus all) reduced representations of f and g respectively such that $\langle \tilde{f}(z), H\rangle$ and $\langle \tilde{g}(z), H\rangle$ share 0 on D, we say that f and g share H on D.

By Remark 2.5, $\langle \tilde{f}(z), H\rangle = 0$ is indeed independent of the choice of the reduced representation of f. Therefore sharing hyperplanes is well defined.

We will use the notation $\langle f(z), H\rangle$ when some properties are independent of the choice of the reduced representation of f. For example, we can say that $\langle f(z), H\rangle$ has finite zeros on D.

H. Fujimoto [1974] gave the relation between m-convergence and quasiregularity. In the case of holomorphic maps, we have the following properties. Suppose $\{f_n\} \subset \mathcal{H}(D; \mathbb{P}^s(\mathbb{C}))$; then $\{f_n\}$ converges uniformly on compact subsets of D to a holomorphic mapping f of D into $\mathbb{P}^s(\mathbb{C})$ if and only if, for any $a \in D$, each f_n has a reduced representation

$$\tilde{f}_n = (f_{n0}, f_{n1}, \dots, f_{ns})$$

on some fixed neighborhood U of a in D such that $\{f_{ni}\}$ converges uniformly on compact subsets of U to a holomorphic function f_i on U, $i = 0, 1, \dots, s$, with the property that

$$\tilde{f} = (f_0, f_1, \dots, f_s)$$

is a reduced representation of f on U.

Main results. Here we shall improve both Theorem C and Theorem D and obtain the following results.

Theorem 2.8. *Suppose $\mathcal{F} \subset \mathcal{H}(D; \mathbb{P}^s(\mathbb{C}))$. Let $H_1, \dots, H_q$, with $q \geq 2s+1$, be hyperplanes in $\mathbb{P}^s(\mathbb{C})$ located in general position. Suppose that for each $f, g \in \mathcal{F}$, f and g share H_j on D, for $j = 1, \dots, q$. Then $\mathcal{F}$ is normal on D.*

Corollary 2.9. *Suppose $\mathcal{F} \subset \mathcal{H}(D; \mathbb{P}^s(\mathbb{C}))$. Let $H_1, \dots, H_q$, with $q \geq 2s+1$, be hyperplanes in $\mathbb{P}^s(\mathbb{C})$ located in general position. Suppose that for each $f \in \mathcal{F}$, f omits H_j on D, for $j = 1, \dots, q$. Then $\mathcal{F}$ is normal on D.*

Proof. Each H_j $(j = 1, \dots, q)$ is a shared value of all $f \in \mathcal{F}$, since $f^{-1}(H_j) = \varnothing$. Thus, the family $\mathcal{F}$ satisfies the assumptions of Theorem 2.8. $\square$

Theorem 2.10. *Suppose $\mathcal{F}, \mathcal{G} \subset \mathcal{H}(D; \mathbb{P}^s(\mathbb{C}))$. Let $q \geq 3s+1$ be a integer, and suppose the following three conditions are satisfied:*

(i) *For each $f \in \mathcal{F}$, there exist $g \in \mathcal{G}$ and q hyperplanes $H_{1,f}, \dots, H_{q,f}$ (which may depend on f) such that f and g share $H_{j,f}$ on D, for $j = 1, \dots, q$.*

(ii) $\inf\{D(H_{1,f}, \dots, H_{q,f}) : f \in \mathcal{F}\} > 0$.

(iii) *$\mathcal{G}$ is normal on D.*

Then $\mathcal{F}$ is a normal family on D.

By Theorem 2.10 we immediately have the following corollary.

Corollary 2.11. *Suppose $\mathcal{F}, \mathcal{G} \subset \mathcal{H}(D; \mathbb{P}^s(\mathbb{C}))$. Let $H_1, \dots, H_q$, with $q \geq 3s+1$, be hyperplanes in $\mathbb{P}^s(\mathbb{C})$ located in general position. Suppose that for each $f \in \mathcal{F}$ there exists $g \in \mathcal{G}$ such that f and g share H_j on D, $j = 1, \dots, q$. If $\mathcal{G}$ is normal on D, then $\mathcal{F}$ is also normal on D.*

The following example shows that the number $3s+1$ in Theorem 2.10 is sharp when $s = 2$.

Example 1. Let Δ be the unit disk. Let $\mathcal{F} = \{f_n(z)\}$, where

$$f_n(z) = \left[\sqrt{-1}\cos nz : \sin nz : \sin nz\right].$$

We denote by $z_{n,1}, z_{n,2}, \dots, z_{n,k_n}$ the zeros of $\sin nz$ in Δ. Let $\mathcal{G} = \{g_n(z)\}$, where

$$g_n(z) = \left[1 : \prod_{1 \leq i \leq k_n} \frac{z - z_{n,i}}{1 - \bar{z}_{n,i} z} : \prod_{1 \leq i \leq k_n} \frac{z - z_{n,i}}{1 - \bar{z}_{n,i} z}\right].$$

Let

$$\begin{aligned}
H_1 &= \{[Z_0 : Z_1 : Z_2] \mid 3Z_0 + Z_1 + 2Z_2 = 0\},\\
H_2 &= \{[Z_0 : Z_1 : Z_2] \mid -5Z_0 + Z_1 + 4Z_2 = 0\},\\
H_3 &= \{[Z_0 : Z_1 : Z_2] \mid 7Z_0 + Z_1 + 6Z_2 = 0\},\\
H_4 &= \{[Z_0 : Z_1 : Z_2] \mid -9Z_0 + Z_1 + 8Z_2 = 0\},\\
H_5 &= \{[Z_0 : Z_1 : Z_2] \mid Z_2 = 0\},\\
H_6 &= \{[Z_0 : Z_1 : Z_2] \mid Z_1 = 0\}.
\end{aligned}$$

Then these hyperplanes are in general position.

One can verify that f_n and g_n share H_j on Δ for $j = 1, \dots, 6$. Clearly, $\mathcal{G}$ is normal on Δ. However, $\mathcal{F}$ fails to be normal on any neighborhood of 0 by Lemma 3.2 in next section.

3. Some lemmas

The following is the general version of the Zalcman lemma.

Lemma 3.1 [Thai et al. 2003]. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain Ω in $\mathbb{C}^m$ into $\mathbb{P}^s(\mathbb{C})$. The family $\mathcal{F}$ is not normal on Ω if and only if there exist sequences $\{f_n\} \subset \mathcal{F}$, $\{z_n\} \subset \Omega$ with $z_n \to z_0 \in \Omega$, and $\{\rho_n\}$ with $\rho_n > 0$ and $\rho_n \to 0$ such that*

$$h_n(\xi) := f_n(z_n + \rho_n \xi)$$

converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping h of $\mathbb{C}$ into $\mathbb{P}^s(\mathbb{C})$.

Lemma 3.2 [Osserman and Ru 1997]. *Let M be a Riemann surface, and $f_n : M \to \mathbb{P}^s(\mathbb{C})$ be a sequence of holomorphic maps converging uniformly on every compact subset of M to a holomorphic map $f : M \to \mathbb{P}^s(\mathbb{C})$. Given $a, b \in \mathbb{P}^s(\mathbb{C}^*)$, let $f_{a,b}$ be the meromorphic function defined by*

$$f_{a,b} = \frac{\langle \tilde{f}, \alpha \rangle}{\langle \tilde{f}, \beta \rangle},$$

where $\tilde{f}$ is a reduced representation of f on U, and $\alpha, \beta \in (\mathbb{C}^{s+1})^*$ are such that $a = \rho(\alpha), b = \rho(\beta)$. Assume that $\beta(\tilde{f}) \not\equiv 0$ on some U. Let $p \in M$ be such that $\beta(\tilde{f})(p) \neq 0$, and U_p be a neighborhood of p such that $\beta(\tilde{f})(z) \neq 0$ for $z \in U_p$. Then $\{f_{n_{a,b}}\}$ converges uniformly on U_p to the meromorphic function $f_{a,b}$.

Let $\mu > 0$ be an integer. The holomorphic map $f \in \mathcal{H}(\mathbb{C}; \mathbb{P}^s(\mathbb{C}))$ is said to be ramified over a hyperplane $H = \{\langle Z, \alpha \rangle = 0\}$ with multiplicity at least μ if all zeros of $\langle f(z), \alpha \rangle = 0$ have orders at least μ, where $\tilde{f}$ is a local reduced representation of f (it is easy to check that this definition is independent of the choice of reduced representation). If either the image of f completely omits H or $f(\mathbb{C}) \subseteq H$, we shall say that f is ramified over H with multiplicity ∞.

Nochka [1983] improved the result of Green [1977] and proved H. Cartan's conjecture.

Lemma 3.3 [Nochka 1983]. *Suppose that $q (\geq 2s + 1)$ hyperplanes $H_1, \ldots, H_q$ are given in general position in $\mathbb{P}^s(\mathbb{C})$, along with q positive integers $m_1, \ldots, m_q$ (some of them may be ∞). If*

$$\sum_{j=1}^{q} \Big(1 - \frac{s}{m_j}\Big) > s + 1,$$

then there does not exist a nonconstant holomorphic mapping $f : \mathbb{C} \to \mathbb{P}^s(\mathbb{C})$ such that f intersects H_j with multiplicity at least m_j, $j = 1, \ldots, q$.

Lemma 3.4 (first main theorem [Fujimoto 1993, Corollary 3.1.16]). *Let $f : \mathbb{C} \to \mathbb{P}^s(\mathbb{C})$ be a holomorphic map. Let H be a hyperplane in $\mathbb{P}^s(\mathbb{C})$. If $f(\mathbb{C}) \not\subseteq H$, then*

$$T_f(r) = m_f(r, H) + N_f(r, H) + O(1).$$

The second main theorem about linearly degenerated case is also required.

Lemma 3.5 (degenerate second main theorem [Ru 2001, Theorem A3.4.4]). *Let $f = [f_0 : \cdots : f_s] : \mathbb{C} \to \mathbb{P}^s(\mathbb{C})$ be a holomorphic map whose image is contained in some k-dimensional subspace but not in any subspace of dimension lower than k. Let $H_1, \ldots, H_q$ be hyperplanes in general position. Assume that $f(\mathbb{C}) \not\subseteq H_j$, for*

$j = 1, \dots, q$. Then the inequality

$$\sum_{j=1}^{q} m_f(r, H_j) + \frac{n+1}{k+1} N(R_f, r) \leq (2n - k + 1)T_f(r) + O(\log T_f(r))$$

holds for all r outside a set E with finite Lebesgue measure. Here $N(R_f, r)$ is the ramification term.

Lemma 3.6 [Fujimoto 1974]. *Let $f \in \mathcal{H}(\mathbb{C}; \mathbb{P}^s(\mathbb{C}))$. The map f is rational, namely, f is representable as $f = [f_0 : \cdots : f_s]$ with polynomial f_i, $i = 0, \dots, s$, if and only if*

$$\lim_{r\to\infty} \frac{T_f(r)}{\log r} < \infty.$$

Lemma 3.7. *Let $f \in \mathcal{H}(\mathbb{C}; \mathbb{P}^s(\mathbb{C}))$, and $H_1, \dots, H_{2s+1}$ be hyperplanes in $\mathbb{P}^s(\mathbb{C})$ located in general position. If for each hyperplane H_j, $j = 1, \dots, 2s+1$, either $f(\mathbb{C}) \subset H_j$ or $\langle f(z), H_j\rangle$ has finite zeros in $\mathbb{C}$ (no zero point is allowed), then the map f is rational.*

Proof. Let $\tilde{f} = (f_0, \dots, f_s)$ be a reduced representation of f on $\mathbb{C}$. We set the rank of the vector group $\{f_0, \dots, f_s\}$ to be $k+1$, with $0 \leq k \leq s$. Thus, $f(\mathbb{C})$ is contained in some k-dimensional subspace of $\mathbb{P}^s(\mathbb{C})$ but not in any subspace of dimension lower than k.

Let I be a subset of $\{1, \dots, 2s+1\}$ such that i is in I if and only if $f(\mathbb{C}) \subset H_i$, and let

$$X_I = \bigcap_{i\in I} H_i.$$

We can identify X_I with a projective space of dimension $s - k_1$, where $k_1 = \#I$. So $0 \leq k_1 \leq s - k$. According to the definition, the restrictions of

$$H_j^* := H_j \bigcap X_I, \quad j \notin I$$

are hyperplanes which are still in general position in $X_I = \mathbb{P}^{s-k_1}(\mathbb{C})$.

Applying Lemma 3.5 to $f = [f_0 : \cdots : f_s] : \mathbb{C} \to \mathbb{P}^{s-k_1}(\mathbb{C})$ and the hyperplanes H_j^*, $j \notin I$, and using the first main theorem about holomorphic curves, it follows that the inequality

$$(2s - k_1 + 1)T_f(r) \leq \sum_{j\notin I} N_f(r, H_j^*) + (2(s - k_1) - k + 1)T_f(r) + O(\log T_f(r))$$

holds for all r outside a set with finite Lebesgue measure. Since $\langle f(z), H_j^*\rangle$ has finite zeros in $\mathbb{C}$, this yields the inequality

$$(k_1 + k)T_f(r) \leq O(\log T_f(r)) + O(\log r).$$

If $k = k_1 = 0$, the rank of the vector group $\{f_0, \dots, f_s\}$ is 1, which means that f is a constant map.

If $k_1 + k > 0$. Together with Lemma 3.6, the above inequality implies that f is rational. Hence, the lemma is proved. □

4. Proofs of the theorems

Proof of Theorem 2.8. Fix $g \in \mathcal{F}$. Suppose that $\mathcal{F}$ is not normal on some point $z_0 \in D$. Suppose there are k hyperplanes H_{j_l}, $l = 1, \dots, k$, such that

$$g(z_0) \in \bigcap_{l=1}^{k} H_{j_l}.$$

Then $k \leq s$. For otherwise $k \geq s + 1$, and because $H_1, \dots, H_q$, $q \geq 2s + 1$, are hyperplanes in $\mathbb{P}^s(\mathbb{C})$ located in general position, it follows that $g = [0 : 0 : \dots : 0]$. This is a contradiction. Therefore, $k \leq s$. Without loss of generality, we assume that there exists a neighborhood $U(z_0) \subset D$ such that for $l = 1, \dots, k_1$,

$$g(U(z_0)) \subset H_l,$$

for $\mu = k_1 + 1, \dots, k$,

$$g(U(z_0)) \cap H_\mu = \{g(z_0)\},$$

and for $\nu = k + 1, \dots, 2s + 1$,

$$g(D(z_0)) \cap H_\nu = \phi.$$

In other words, these hyperplanes are divided into three groups.

Observing that normality is a local property, we may suppose that $U(z_0)$ is the unit disk Δ, and $z_0 = 0$. Then by Lemma 3.1 there exist points z_n with $z_n \to z_0 \in D$, positive numbers ρ_n with $\rho_n \to 0$, and functions $f_n \in \mathcal{F}$ such that

$$h_n(\xi) := f_n(z_n + \rho_n \xi)$$

converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping h of $\mathbb{C}$ into $\mathbb{P}^s(\mathbb{C})$. Here $\xi \in \mathbb{C}$ satisfies $z_n + \rho_n \xi \in \Delta$.

We consider two cases.

If $z_n/\rho_n \to \infty$, then for each $\xi \in \mathbb{C}$, $z_n + \rho_n \xi \neq z_0$ when n is large enough. It follows that for $i = k_1 + 1, \dots, 2s + 1$,

$$\langle f_n(z_n + \rho_n \xi), H_i \rangle \neq 0.$$

The Hurwitz theorem implies that for $i = k_1 + 1, \dots, 2s + 1$, $\langle h(\xi), H_i \rangle \neq 0$ or $\langle h(\xi), H_i \rangle \equiv 0$. Thus, $\langle h(\xi), H_j \rangle \neq 0$ or $\langle h(\xi), H_j \rangle \equiv 0$ for $j = 1, \dots, 2s + 1$. By Lemma 3.3, h is a constant holomorphic mapping. This contradicts the claim that

h is a nonconstant holomorphic mapping.

If $z_n/\rho_n \not\to \infty$, taking a subsequence and renumbering, we may assume that $z_n/\rho_n \to c$, $c \in \mathbb{C}$. Then

$$f_n(\rho_n\xi) = h_n\Big(\xi - \frac{z_n}{\rho_n}\Big)$$

converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping $h(\xi - c)$. Since for each hyperplane H_j, $j = 1, \dots, 2s+1$, either $h(\mathbb{C}) \subset H_j$ or $\langle h(\xi - c)(z), H_j\rangle$ has finite zeros in $\mathbb{C}$, $h(\xi - c)$ is rational by Lemma 3.7. Since $h(\xi - c)$ is a holomorphic mapping, there exist some constants c_ν, with $\nu = k+1, \dots, 2s+1$, such that

$$\langle h(\xi - c)(z), H_\nu\rangle \equiv c_\nu.$$

Note that $2s - k + 1 \geq s + 1$, and $\{H_j\}$ are in general position. Hence we see that $h(\xi - c)$ is a constant map. Again, this a contradiction. And hence the family $\mathcal{F}$ is normal on D. □

Proof of Theorem 2.10. If $\mathcal{F}$ is not normal on D, then by Lemma 3.1, there exist points $z_n \to z_0 \in D$, positive numbers $\rho_n \to 0$ and functions $f_n \in \mathcal{F}$, such that

$$h_n(\xi) := f_n(z_n + \rho_n\xi),$$

where $\xi \in \mathbb{C}$ satisfies $z_n + \rho_n\xi \in D$, converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping h of $\mathbb{C}$ into $\mathbb{P}^s(\mathbb{C})$.

By condition (i), there exist q hyperplane sequences $\{H_{j,f_n}\}_{n=1}^\infty$ and $\{g_n\} \subset \mathcal{G}$ such that for $z \in D$, $j = 1, \dots, q$,

$$\langle g_n(z), H_{j,f_n}\rangle = 0$$

whenever

$$\langle f_n(z), H_{j,f_n}\rangle = 0, \quad z \in D.$$

For $j = 1, \dots, q$, take $\{\alpha_{jn}\}_{n=1}^\infty \subset B$ satisfying

$$H_{j,f_n} = \{\langle Z, \alpha_{jn}\rangle = 0\}.$$

Since B is a compact subset of $\mathbb{C}^{s+1}$, there exist $\alpha_j = (a_{j0}, \dots, a_{js}) \in B$ for $j = 1, \dots, q$, and subsequences which (to avoid complication in notation) we again call $\{\alpha_{jn}\}$ satisfying that $\alpha_{jn} \to \alpha_j$ as $n \to \infty$. Let

$$H_j = \{\langle Z, \alpha_j\rangle = 0\}$$

be hyperplanes of $\mathbb{P}^s(\mathbb{C})$, $j = 1, \dots, q$. From condition (i), it follows that

$$D(H_1, \dots, H_q) \geq \liminf_{n\to\infty} D(H_{1,f_n}, \dots, H_{q,f_n}) > 0.$$

Thus, the hyperplanes H_j, $j = 1, \dots, q$, are located in general position.

Claim. There exist at most $2s$ *hyperplanes such that for each hyperplane* H*, either the image of* h *completely omits* H *or* $h(\mathbb{C}) \subset H$*.* If not, Lemma 3.3 shows that h is a constant holomorphic mapping, which is a contradiction. So there exist at least $s+1$ hyperplanes of H_j, $j = 1, \dots, q$, such that for $i = 1, \dots, s+1$, $h(\mathbb{C}) \cap H_i \neq \phi$ and $h(\mathbb{C}) \nsubseteq H_i$.

For a fixed $i \in \{1, \dots, s+1\}$, suppose that $\xi_i \in h(\mathbb{C}) \cap H_i$. Choose a small neighborhood $U(\xi_i)$ of ξ_i such that $h(\mathbb{C}) \cap H_i = \{\xi_i\}$. Hence $\langle \tilde{h}(\xi_i), H\rangle = 0$ and $\langle \tilde{h}(\xi_i), H\rangle \not\equiv 0$, where $\tilde{h}$ is a local reduced representation. Since h_n converges uniformly to h on $U(\xi_i)$, h_n has a local reduced representation $\tilde{h}_n = (h_{n0}, \dots, h_{ns})$ such that $\tilde{h}_n$ uniformly converges to a reduced representation $\tilde{h} = (h_0, \dots, h_s)$ of h on $U(\xi_i)$. Obviously, h_{nk} converges uniformly to h_k on $U(\xi_i)$ for each $k = 0, \dots, s$. Therefore $\langle \tilde{h}_n(\xi), \alpha_{in}\rangle$ converges uniformly to $\langle \tilde{h}(\xi), \alpha_i\rangle$ on $U(\xi_i)$. By the Hurwitz theorem, there exist $\xi_{in} \to \xi_i$ such that $\langle \tilde{h}_n(\xi_{in}), \alpha_{in}\rangle = 0$, that is, $\langle \tilde{f}_n(z_n + \rho_n \xi_{in}), \alpha_{in}\rangle = 0$.

On the other hand, applying condition (iii), we can find subsequences of $\{g_n\}$ (again denoted by themselves) such that g_n converges uniformly to g on D, where g is a holomorphic mapping of D into $\mathbb{P}^s(\mathbb{C})$. As we noted earlier, g_n has a local reduced representation $\tilde{g}_n = (g_{n0}, \dots, g_{ns})$ such that $\tilde{g}_n$ uniformly converges to a reduced representation $\tilde{g} = (g_0, \dots, g_s)$ of g on $U(z_0)$. It follows that

$$\langle \tilde{g}_n(z_n + \rho_n \xi_{in}), \alpha_{in}\rangle = 0.$$

As $n \to \infty$, we have

$$\langle \tilde{g}(z_0), \alpha_i\rangle = 0.$$

So there exist $s+1$ hyperplanes H_i, $i = 1, \dots, s+1$, which intersect at one point $\rho(\tilde{g}(z_0))$. This contradicts the claim that the hyperplanes H_j, $j = 1, \dots, q$, are located in general position. This finishes the proof. □

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