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ON STABLE COMMUTATOR LENGTH IN HYPERELLIPTIC MAPPING CLASS GROUPS

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We give a new upper bound on the stable commutator length of Dehn twists in hyperelliptic mapping class groups and determine the stable commutator length of some elements. We also calculate values and the defects of homogeneous quasimorphisms derived from ω-signatures and show that they are linearly independent in the mapping class groups of pointed 2-spheres when the number of points is small.

1. Introduction

The aim of this paper is to investigate stable commutator length in hyperelliptic mapping class groups and in mapping class groups of pointed 2-spheres. Given a group G and an element $x \in [G, G]$, the commutator length of x, denoted by $\mathrm{cl}_G(x)$, is the smallest number of commutators in G whose product is x, and the stable commutator length of x is defined by the limit $\mathrm{scl}_G(x) := \lim_{n\to\infty} \mathrm{cl}_G(x^n)/n$ (see Definition 2.1 for details).

We investigate stable commutator length in two groups, $\mathcal{M}_0^m$ and $\mathcal{H}_g$. Let m be a positive integer greater than 3. Choose m distinct points $\{q_i\}_{i=1}^m$ in a 2-sphere S^2. Let $\mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$ denote the set of all orientation-preserving diffeomorphisms in S^2 which preserve $\{q_i\}_{i=1}^m$ setwise with the C^∞-topology. We define the mapping class group of the m-pointed 2-sphere by $\mathcal{M}_0^m = \pi_0 \mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$. Let Σ_g be a closed connected oriented surface of genus $g \geq 1$. An involution $\iota : \Sigma_g \to \Sigma_g$ defined as in Figure 1 is called the hyperelliptic involution.

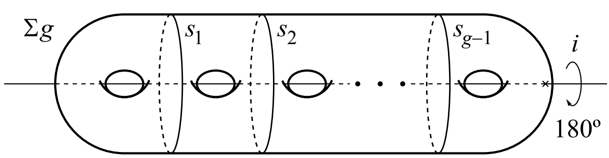

Figure 1. Hyperelliptic involution ι and the curves $s_1, \ldots, s_{g-1}$.

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Let $\mathcal{M}_g$ denote the mapping class group of Σ_g, that is, the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g, and let $\mathcal{H}_g$ be the centralizer of the isotopy class of a hyperelliptic involution in $\mathcal{M}_g$, which is called the hyperelliptic mapping class group of genus g. Note that $\mathcal{M}_g = \mathcal{H}_g$ when $g = 1, 2$. Since there exists a surjective homomorphism $\mathcal{P} : \mathcal{H}_g \to \mathcal{M}_0^{2g+2}$ with finite kernel (see Lemma 3.3 and the paragraph before Remark 3.7), these two groups have the same stable commutator length.

Let s_0 be a nonseparating curve on Σ_g satisfying $\iota(s_0) = s_0$, and let s_h be a separating curve in Figure 1 for $h = 1, \ldots, g-1$. We denote by t_{s_j} the Dehn twist about s_j for $j = 0, 1, \ldots, g-1$. In general, it is difficult to compute stable commutator length, but those of some mapping classes are known. In the mapping class group of a compact oriented surface with connected boundary, Baykur, Korkmaz and the second author [Baykur et al. 2013] determined the commutator length of the Dehn twist about a boundary curve. In the mapping class group of a closed oriented surface, interesting lower bounds on scl of Dehn twists are obtained using gauge theory. Endo and Kotschick [2001], and Korkmaz [2004] proved that $1/(18g-6) \le \mathrm{scl}_{\mathcal{M}_g}(t_{s_j})$ for $j = 0, 1, \ldots, g-1$. For technical reasons, this result is stated in [Endo and Kotschick 2001] only for separating curves. This technical assumption is removed in [Korkmaz 2004]. The second author [Monden 2012] also showed that $1/(8g+4) \le \mathrm{scl}_{\mathcal{H}_g}(t_{s_0})$ and

$$\frac{h(g-h)}{g(2g+1)} \le \mathrm{scl}_{\mathcal{H}_g}(t_{s_h}) \quad \text{for } h = 1, \ldots, g-1.$$

Stable commutator length on a group is closely related to functions on the group called homogeneous quasimorphisms through Bavard's duality theorem. Homogeneous quasimorphisms are homomorphisms up to bounded error called the defect (see Definition 2.2 for details). By Bavard's theorem, if we obtain a homogeneous quasimorphism on the group and calculate its defect, we also obtain a lower bound on stable commutator length. Actually, Bestvina and Fujiwara [2002, Theorem 12] proved that the spaces of homogeneous quasimorphisms on $\mathcal{M}_g$ and $\mathcal{M}_0^m$ are infinite-dimensional when $g \ge 2$ and $m \ge 5$, respectively. However it is hard to compute explicit values of these quasimorphisms and their defects. To compute stable commutator length, we consider computable quasimorphisms derived from ω-signature in [Gambaudo and Ghys 2005] on symmetric mapping class groups.

In Section 3, we review symmetric mapping class groups, which are defined by Birman and Hilden as generalizations of hyperelliptic mapping class groups. We reconsider cobounding functions of ω-signatures as a series of quasimorphisms $\phi_{m,j}$ on a symmetric mapping class group $\pi_0 C_g(t)$. Since there exists a surjective homomorphism $\mathcal{P} : \pi_0 C_g(t) \to \mathcal{M}_0^m$ with finite kernel, the homogenizations $\bar{\phi}_{m,j}$

induce homogeneous quasimorphisms on $\mathcal{M}_0^m$. Let $\sigma_i \in \mathcal{M}_0^m$ be a half twist which permutes the i-th point and the $(i+1)$-th point. We denote by $\tilde{\sigma}_i \in \pi_0 C_g(t)$ a lift of σ_i, which will be defined on page 333.

In Section 6, we calculate $\phi_{m,j}$ and their homogenizations $\bar{\phi}_{m,j}$.

Theorem 1.1. *Let r be an integer such that $2 \le r \le m$. Then:*

(i) $$\phi_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{r-1}) = \frac{2(r-1)j(m-j)}{m(m-1)}.$$

(ii) $$\bar{\phi}_{m,j}(\sigma_1 \cdots \sigma_{r-1}) = -\frac{2}{r}\left\{\frac{jr(m-j)(m-r)}{m^2(m-1)} + \left(\frac{rj}{m} - \left[\frac{rj}{m}\right] - \frac{1}{2}\right)^2 - \frac{1}{4}\right\},$$

where $[x]$ denotes the greatest integer $\le x$.

Since this requires straightforward and lengthy calculations to prove, we leave it until the last section. A computer calculation shows that the $([m/2]-1)\times([m/2]-1)$ matrix whose (i, j)-entry is $\bar{\phi}_{m,j+1}(\sigma_1 \cdots \sigma_i)$ is nonsingular when $4 \le m \le 30$. Thus we have:

Corollary 1.2. *The set $\{\bar{\phi}_{m,j}\}_{j=2}^{[m/2]}$ is linearly independent when $4 \le m \le 30$.*

In Section 4, we calculate the defects of the homogenizations of these quasimorphisms. In particular, we determine the defect of $\bar{\phi}_{m,m/2}$ when m is even. Actually $\bar{\phi}_{m,m/2}$ is the same as the homogenization of the Meyer function on the hyperelliptic mapping class group $\mathcal{H}_g$.

Theorem 1.3. *Let $D(\phi_{m,j})$ and $D(\bar{\phi}_{m,j})$ be the defects of the quasimorphisms $\phi_{m,j}$ and $\bar{\phi}_{m,j}$, respectively.*

(i) *For $j = 1, 2, \ldots, [m/2]$,*

$$D(\bar{\phi}_{m,j}) \le D(\phi_{m,j}) \le m-2.$$

(ii) *When m is even and $j = m/2$,*

$$D(\bar{\phi}_{m,m/2}) = m-2.$$

Remark 1.4. If $\phi : G \to \mathbb{R}$ is a quasimorphism and $\bar{\phi} : G \to \mathbb{R}$ is its homogenization, they satisfy

$$D(\bar{\phi}) \le 2D(\phi)$$

(see [Calegari 2009] Corollary 2.59). We will claim in Lemma 4.1 that, when ϕ is antisymmetric and a class function, they satisfy the sharper inequality

$$D(\bar{\phi}) \le D(\phi).$$

Note that when $g = 2$, the hyperelliptic mapping class group $\mathcal{H}_2$ coincides with $\mathcal{M}_2$. We may think of the lift of $\sigma_i \in \mathcal{M}_0^6$ for $i = 1, 2, 3, 4, 5$ to $\mathcal{M}_2$ as the Dehn twist t_{c_i} along the simple closed curve c_i in Figure 2 (see page 333). Similarly the Dehn twist $t_{s_1} \in \mathcal{M}_2$ can be considered as a lift of $(\sigma_1\sigma_2)^6 \in \mathcal{M}_0^6$ by the chain relation (see Lemma 2.8). Since Theorem 1.1(ii) implies $\bar{\phi}_{6,2}((\sigma_1\sigma_2)^6) = -8/5$ and Theorem 1.3(i) implies $D(\bar{\phi}_{6,2}) \le 4$, by applying Bavard's duality theorem, we have:

Corollary 1.5. $$\frac{1}{5} \le \mathrm{scl}_{\mathcal{M}_2}(t_{s_1}).$$

To the best of our knowledge, for $g \ge 2$, there is not known an element x in $\mathcal{H}_g$ (or $\mathcal{M}_g$) such that $\mathrm{scl}(x)$ is nonzero and can be computed explicitly. By Theorem 1.3(ii), we can determine the stable commutator length of the following element in $\mathcal{H}_g$.

Theorem 1.6. *Let $d_2^+, d_2^-, \dots, d_{g-1}^+, d_{g-1}^-$ be simple closed curves in Figure 7. Let c be a nonseparating simple closed curve satisfying $\iota(c) = c$ which is disjoint from d_i^+, d_i^- and s_h $(i = 1, \dots, g, h = 1, \dots, g-1)$. For $g \ge 2$,*

$$\mathrm{scl}_{\mathcal{H}_g}\bigl(t_c^{2g+8}(t_{d_2^+}t_{d_2^-}\cdots t_{d_{g-1}^+}t_{d_{g-1}^-})^2(t_{s_1}\cdots t_{s_{g-1}})^{-1}\bigr) = \frac{1}{2}.$$

In particular, if $g = 2$, then we have $\mathrm{scl}_{\mathcal{H}_2}(t_c^{12}t_{s_1}^{-1}) = 1/2$.

Next we consider upper bounds on stable commutator length. Korkmaz [2004] also gave the upper bound $\mathrm{scl}_{\mathcal{M}_g}(t_{s_0}) \le 3/20$ for $g \ge 2$. In the case of $g = 2$, the second author [Monden 2012] showed $\mathrm{scl}_{\mathcal{M}_2}(t_{s_0}) < \mathrm{scl}_{\mathcal{M}_2}(t_{s_1})$. However these upper bounds do not depend on g. On the other hand, Kotschick [2008] proved that there is an estimate $\mathrm{scl}_{\mathcal{M}_g}(t_{s_0}) = O(1/g)$ by using the so-called "Munchhausen trick".

In Section 5, we give the following upper bounds.

Theorem 1.7. *Let s_0 be a nonseparating curve on Σ_g, and let G_g be either $\mathcal{M}_g$ or $\mathcal{H}_g$. For all $g \ge 1$, we have*

$$\mathrm{scl}_{G_g}(t_{s_0}) \le \frac{1}{2\{2g+3+(1/g)\}}.$$

2. Preliminaries

Stable commutator lengths and quasimorphisms. Let G denote a group, and let $[G, G]$ denote the commutator subgroup, which is the subgroup of G generated by all commutators $[x, y] = xyx^{-1}y^{-1}$ for $x, y \in G$.

Definition 2.1. For $x \in [G, G]$, the commutator length $\mathrm{cl}_G(x)$ of x is the least number of commutators in G whose product is equal to x. The stable commutator length of x, denoted $\mathrm{scl}(x)$, is the limit

$$\mathrm{scl}_G(x) = \lim_{n\to\infty} \frac{\mathrm{cl}_G(x^n)}{n}.$$

For each fixed x, the function $n \mapsto \mathrm{cl}_G(x^n)$ is nonnegative and

$$\mathrm{cl}_G(x^{m+n}) \le \mathrm{cl}_G(x^m) + \mathrm{cl}_G(x^n).$$

Hence this limit exists. If x is not in $[G, G]$ but has a power x^m which is, define $\mathrm{scl}_G(x) = \mathrm{scl}_G(x^m)/m$. We also define $\mathrm{scl}_G(x) = \infty$ if no power of x is contained in $[G, G]$ (we refer the reader to [Calegari 2009] for the details of the theory of the stable commutator length).

Definition 2.2. A quasimorphism is a function $\phi : G \to \mathbb{R}$ for which there is a least constant $D(\phi) \ge 0$ such that

$$|\phi(xy) - \phi(x) - \phi(y)| \le D(\phi),$$

for all $x, y \in G$. We call $D(\phi)$ the defect of ϕ. A quasimorphism is homogeneous if it satisfies the additional property $\phi(x^n) = n\phi(x)$ for all $x \in G$ and $n \in \mathbb{Z}$.

We recall the following basic facts. Let ϕ be a quasimorphism on G. For each $x \in G$, define

$$\bar{\phi}(a) := \lim_{n\to\infty} \frac{\phi(x^n)}{n}.$$

The limit exists and defines a homogeneous quasimorphism. Homogeneous quasimorphisms have the following properties, shown for example in [Calegari 2009, Section 5.5.2] and [Kotschick 2008, Lemma 2.1(1)].

Lemma 2.3. *Let ϕ be a homogeneous quasimorphism on G. For all $x, y \in G$,*

(i) $\phi(x) = \phi(yxy^{-1})$,

(ii) $xy = yx \implies \phi(xy) = \phi(x) + \phi(y)$.

Theorem 2.4 (Bavard's duality theorem [1991]). *Let Q be the set of homogeneous quasimorphisms on G with positive defects. For any $x \in [G, G]$, we have*

$$\mathrm{scl}_G(x) = \sup_{\phi\in Q} \frac{|\phi(x)|}{2D(\phi)}.$$

Mapping class groups. For $g \ge 1$, the abelianizations of the mapping class group $\mathcal{M}_g$ of the surface Σ_g and its subgroup $\mathcal{H}_g$ are finite (see [Powell 1978]). Therefore all elements of $\mathcal{M}_g$ and $\mathcal{H}_g$ have powers that are products of commutators. Dehn showed that the mapping class group $\mathcal{M}_g$ is generated by Dehn twists along nonseparating simple closed curves. We review some relations between them. Hereafter we do not distinguish a simple closed curve in Σ_g and its isotopy class. The following relations are well known. See, for example, [Farb and Margalit 2012, Sections 3.3, 3.5.1, 5.1.4, and 4.4.1].

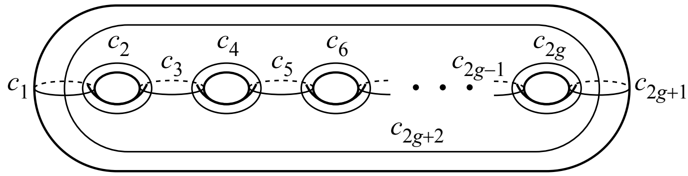

Figure 2. The curves $c_1, c_2, \dots, c_{2g+2}$.

Lemma 2.5. *Let c be a simple closed curve in Σ_g and $f \in \mathcal{M}_g$. Then we have*

$$t_{f(c)} = f t_c f^{-1}.$$

From this lemma, the values of scl and homogeneous quasimorphisms on the Dehn twists about nonseparating simple closed curves are constant.

Lemma 2.6. *Let c and d be simple closed curves in Σ_g.*

(i) *If c is disjoint from d, then $t_c t_d = t_d t_c$.*

(ii) *If c intersects d in one point transversely, then $t_c t_d t_c = t_d t_c t_d$.*

Lemma 2.7 (hyperelliptic involution). *Let $c_1, \dots, c_{2g+1}$ be nonseparating curves in Σ_g as in Figure 2. We call the product*

$$\iota := t_{c_{2g+1}} t_{c_{2g}} \cdots t_{c_2} t_{c_1} t_{c_1} t_{c_2} \cdots t_{c_{2g}} t_{c_{2g+1}}$$

the hyperelliptic involution. For $g = 1$, the hyperelliptic involution ι equals $t_{c_1} t_{c_2} t_{c_1} t_{c_1} t_{c_2} t_{c_1}$, where c_1 and c_2 are respectively the meridian and longitude of Σ_1.

Lemma 2.8 (chain relation). *For a positive integer n, let $a_1, a_2, \dots, a_n$ be a sequence of simple closed curves in Σ_g such that a_i and a_j are disjoint if $|i - j| \geq 2$ and a_i and a_{i+1} intersect at one point.*

When n is odd, a regular neighborhood of $a_1 \cup a_2 \cup \cdots \cup a_n$ is a subsurface of genus $(n-1)/2$ with two boundary components, denoted by d_1 and d_2. Then

$$(t_{a_n} \cdots t_{a_2} t_{a_1})^{n+1} = t_{d_1} t_{d_2}.$$

When n is even, a regular neighborhood of $a_1 \cup a_2 \cup \cdots \cup a_n$ is a subsurface of genus $n/2$ with connected boundary, denoted by d. Then

$$(t_{a_n} \cdots t_{a_2} t_{a_1})^{2(n+1)} = t_d.$$

Meyer's signature cocycle. Let X be a compact oriented $(4n+2)$-manifold for nonnegative integer n, and let Γ be a local system on X such that $\Gamma(x)$ is a finite-dimensional real or complex vector space for $x \in X$. If we are given a regular antisymmetric (respectively, skew-hermitian) form $\Gamma \otimes \Gamma \to \mathbb{R}$ (respectively, $\Gamma \otimes \Gamma \to \mathbb{C}$), we have a symmetric (respectively, hermitian) form on $H_{2n+1}(X; \Gamma)$

as in [Meyer 1972, p. 12]. For simplicity, we only explain the complex case. It is defined by

$$\begin{aligned} H_{2n+1}(X;\Gamma)\otimes H_{2n+1}(X;\Gamma) &\cong H^{2n+1}(X,\partial X;\Gamma)\otimes H^{2n+1}(X,\partial X;\Gamma) \\ &\xrightarrow{\cup} H^{4n+2}(X,\partial X;\Gamma\otimes\Gamma) \\ &\to H^{4n+2}(X,\partial X;\mathbb{C}) \\ &\xrightarrow{[X,\partial X]} \mathbb{C}, \end{aligned}$$

where the first row is defined by the Poincaré duality, the second row is defined by the cup product of the base space, the third row comes from the skew-hermitian form of Γ as above, and the fourth row is the evaluation by the fundamental class of X. Meyer showed additivity of signatures with respect to this hermitian form (more strongly, he showed Wall's nonadditivity formula for G-signatures of homology groups with local coefficients).

Theorem 2.9 [Meyer 1972, Satz I.3.2]. *Let X and Γ be as above. Assume that X is obtained by gluing two compact oriented $(4n+2)$-manifold X_- and X_+ along some boundary components.*

Then we have

$$\operatorname{Sign}(H_{2n+1}(X;\Gamma)) = \operatorname{Sign}(H_{2n+1}(X_-;\Gamma|_{X_-})) + \operatorname{Sign}(H_{2n+1}(X_+;\Gamma|_{X_+})).$$

Consider the case when X is a pair of pants, which we denote by P. Let α and β be loops in P as in Figure 3, left.

For $\varphi, \psi \in \mathcal{M}_g$, there exists a Σ_g-bundle $E_{\varphi,\psi}$ on P whose monodromies along α and β are φ and ψ, respectively. This is unique up to bundle isomorphism. In this setting, the intersection form on the local system $H_1(\Sigma_g;\mathbb{R})$ induces the symmetric form on $H_1(P; H_1(\Sigma_g;\mathbb{R}))$. Meyer showed that the signature of this symmetric form on $H_1(P; H_1(\Sigma_g;\mathbb{R}))$ coincides with that of $E_{\varphi,\psi}$. Moreover he explicitly described it in terms of the action of the mapping class group on $H_1(\Sigma_g;\mathbb{R})$ as

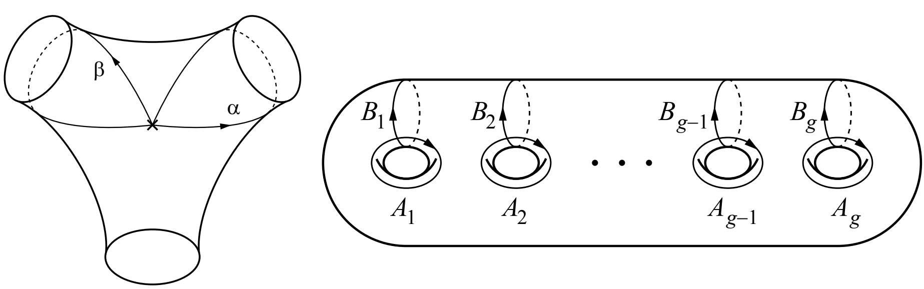

Figure 3. Left: loops in a pair of pants. Right: a symplectic basis of $H_1(\Sigma_g;\mathbb{Z})$.

follows. Fix the symplectic basis $\{A_i, B_i\}_{i=1}^{g}$ of $H_1(\Sigma_g; \mathbb{Z})$ as in Figure 3, right; then the action induces a homomorphism $\rho : \mathcal{M}_g \to \mathrm{Sp}(2g; \mathbb{Z})$. Let I denote the identity matrix of rank g and define

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

For symplectic matrices A and B of rank $2g$, define the vector space

$$V_{A,B} = \bigl\{(v, w) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} \mid (A^{-1} - I)v + (B - I)w = 0\bigr\}.$$

Consider the symmetric bilinear form

$$\langle\ ,\ \rangle_{A,B} : V_{A,B} \times V_{A,B} \to \mathbb{R}$$

on $V_{A,B}$ defined by

$$\langle(v_1, w_1), (v_2, w_2)\rangle_{A,B} := (v_1 + w_1)^T J(I - B)w_2.$$

Then, the space $V_{A,B}$ coincides with $H_1(P; H_1(\Sigma_g; \mathbb{R}))$, and the above form $\langle\ ,\ \rangle_{\rho(\varphi),\rho(\psi)}$ corresponds to the symmetric form on $H_1(P; H_1(\Sigma_g; \mathbb{R}))$.

Meyer's signature cocycle $\tau_g : \mathcal{M}_g \times \mathcal{M}_g \to \mathbb{Z}$ is the map defined by

$$(\varphi, \psi) \mapsto \mathrm{Sign}(\langle\ ,\ \rangle_{\rho(\varphi),\rho(\psi)}),$$

which is known to be a bounded 2-cocycle by Theorem 2.9. When we restrict it to the hyperelliptic mapping class group $\mathcal{H}_g$, it represents the trivial cohomology class in $H^2(\mathcal{H}_g; \mathbb{Q})$. Since the first homology $H_1(\mathcal{H}_g; \mathbb{Q})$ is trivial, the cobounding function $\phi_g : \mathcal{H}_g \to \mathbb{Q}$ of τ_g is unique. It is a quasimorphism, called the Meyer function. Endo [2000] computed it to investigate signatures of fibered 4-manifolds called hyperelliptic Lefschetz fibrations. Morifuji [2003] relates it to the eta invariants of mapping tori and the Casson invariants of integral homology 3-spheres.

3. Cobounding functions of the Meyer's signature cocycles on symmetric mapping class groups

As in the introduction, let m be a positive integer greater than 3 and $\{q_i\}_{i=1}^m$ be m distinct points in a 2-sphere S^2. Choose a base point $* \in S^2 - \{q_i\}_{i=1}^m$, and denote by $\alpha_i \in \pi_1(S^2 - \{q_i\}_{i=1}^m, *)$ a loop which rounds the point q_i clockwise as in Figure 4.

For an integer d such that $d \geq 2$ and $d \mid m$, define a homomorphism

$$\pi_1(S^2 - \{q_i\}_{i=1}^m) \to \mathbb{Z}/d\mathbb{Z}$$

by mapping each generator α_i to $1 \in \mathbb{Z}/d\mathbb{Z}$. This homomorphism induces a d-cyclic branched covering $p_d : \Sigma_h \to S^2$ with m branched points, where Σ_h is a closed oriented surface of genus h. Applying the Riemann–Hurwitz formula, we have

$h=(d-1)(m-2)/2$. We denote by $t:\Sigma_h\to\Sigma_h$ the deck transformation which corresponds to the generator $1\in\mathbb{Z}/d\mathbb{Z}$.

Let η denote the d-th root of unity $\exp(2\pi\boldsymbol{i}/d)$, where $\boldsymbol{i}$ is a square root of -1. The first homology $H_1(\Sigma_h;\mathbb{C})$ decomposes into a direct sum $\bigoplus_{j=1}^{d-1}V^{\eta^j}$, where V^z is an eigenspace whose eigenvalue is $z\in\mathbb{C}$. Note that V^1 is trivial since the quotient space $\Sigma_g/\langle t\rangle$ is a 2-sphere, where $\langle t\rangle$ denotes the cyclic group generated by the deck transformation t. We also denote by $C_h(t)$ the centralizer of t in the diffeomorphism group $\mathrm{Diff}_+\,\Sigma_h$. We call the path-connected component $\pi_0C_h(t)$ the symmetric mapping class group of the covering p, which is defined by Birman and Hilden [1973].

In this section, we introduce 2-cocycles on the symmetric mapping class group $\pi_0C_h(t)$, derived from the nonadditivity formula for signatures. These are almost the same as the ω-signatures defined in [Gambaudo and Ghys 2005].

Let us consider an oriented Σ_h-bundle $E_{\varphi,\psi}$ over P whose structure group is contained in $C_h(t)$, and monodromies along α and β are φ and ψ in $\pi_0C_h(t)$, respectively. Since coordinate transformations commute with the deck transformation t, we can define a fiberwise $\mathbb{Z}/d\mathbb{Z}$-action on $E_{\varphi,\psi}$. Since the structure group is in $C_h(t)$, not only $H_1(\Sigma_h;\mathbb{C})$ but also each eigenspace V^{η^j} is a local system on P. We can extend the intersection form as a skew-hermitian form $H_1(\Sigma_h;\mathbb{C})\otimes H_1(\Sigma_h;\mathbb{C})\to\mathbb{C}$ defined by

$$(x_1+x_2\boldsymbol{i})\cdot(y_1+y_2\boldsymbol{i})=x_1\cdot y_1+x_2\cdot y_2+(x_1\cdot y_2-x_2\cdot y_1)\boldsymbol{i}.$$

For $v\in V^{\eta^j}$ and $w\in V^{\eta^k}$ ($1\le j\le d-1$, $1\le k\le d-1$),

$$\begin{aligned}(tv)\cdot w&=(\omega^j v)\cdot w=\omega^{-j}(v\cdot w),\\ (tv)\cdot w&=v\cdot(t^{-1}w)=v\cdot(\omega^{-k}w)=\omega^{-k}(v\cdot w).\end{aligned}$$

Since ω^{-j} is not equal to ω^{-k}, we have $v\cdot w=0$. Hence, $H_1(\Sigma_h;\mathbb{C})$ decomposes into an orthogonal sum of subspaces $\{V^{\omega^j}\}_{j=1}^{d-1}$. By restricting the intersection form on $H_1(\Sigma_h;\mathbb{C})$ to V^{η^j}, we can define a hermitian form on $H_1(P;V^{\eta^j})$. By Theorem 2.9, we have a 2-cocycle on $\pi_0C_h(t)$ as follows.

Lemma 3.1. *Let j be an integer such that $1\le j\le m-1$. The map*

$$\tau_{m,d,j}:\pi_0C_h(t)\times\pi_0C_h(t)\to\mathbb{Z}$$

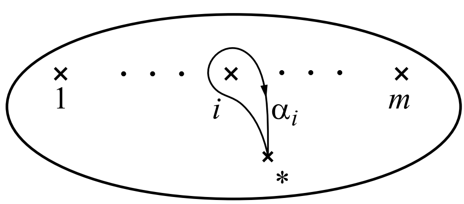

Figure 4. A loop α_i.

defined by

$$\tau_{m,d,j}(\varphi,\psi)=\mathrm{Sign}(H_1(P;V^{\eta^j}))$$

is a 2-cocycle, where V^{η^j} is the local system on P induced from the oriented Σ_h-bundle $E_{\varphi,\psi}\to P$.

Proof. The proof is the same as for [Meyer 1972, p. 43, Equation (0)]. Applying additivity of signatures to two oriented Σ_h-bundles on P, we can see that $\tau_{m,d,j}$ satisfies

$$\tau_{m,d,j}(\varphi_1,\varphi_2)+\tau_{m,d,j}(\varphi_1\varphi_2,\varphi_3)=\tau_{m,d,j}(\varphi_1,\varphi_2\varphi_3)+\tau_{m,d,j}(\varphi_2,\varphi_3),$$

for $\varphi_1,\varphi_2,\varphi_3\in\pi_0C_h(t)$. □

Since the deck transformation t acts on $H^1(P,\partial P;V^{\eta^j})$ by multiplication of η^j, we can calculate $\mathbb{Z}/d\mathbb{Z}$-signature as

$$\mathrm{Sign}(H_1(P;V^{\eta^j}),t^k)=\eta^{kj}\,\mathrm{Sign}(H_1(P;V^{\eta^j}))=\eta^{kj}\tau_{m,d,j}(\varphi,\psi),$$

for $0\le k\le m-1$. Moreover Meyer [1972, Satz I.2.2] proved $\mathrm{Sign}(E_{\varphi,\psi},t^k)=\mathrm{Sign}(H_1(P;H^1(\Sigma_h;\mathbb{C})),t^k)$. Hence we have:

Lemma 3.2. *For* $0\le k\le m-1$,

$$\mathrm{Sign}(E_{\varphi,\psi},t^k)=\sum_{j=1}^{d-1}\eta^{kj}\tau_{m,d,j}(\varphi,\psi).$$

The symmetric mapping class groups. A diffeomorphism $f:\Sigma_h\to\Sigma_h$ in $C_h(t)$ induces a diffeomorphism $\bar f:S^2\to S^2$ which satisfies the commutative diagram

$$\begin{array}{ccc}\Sigma_h & \xrightarrow{f} & \Sigma_h\\ {\scriptstyle p_d}\big\downarrow & & {\scriptstyle p_d}\big\downarrow\\ S^2 & \xrightarrow{\bar f} & S^2.\end{array}$$

Moreover since $\bar f$ satisfies $p_d^{-1}(q)=p_d^{-1}(\bar f(q))$ for any $q\in S^2$, we have

$$\bar f\in\mathrm{Diff}_+(S^2,\{q_i\}_{i=1}^m).$$

Therefore we have a natural homomorphism $\mathcal{P}:\pi_0C_h(t)\to\mathcal{M}_0^m$ which maps $[f]$ to $[\bar f]$. By a method similar to [Birman and Hilden 1971, Theorem 1] (see also [Birman and Hilden 1973, Section 5]), we have:

Lemma 3.3. *Let $m\ge 4$. The sequence*

$$1\longrightarrow\mathbb{Z}/d\mathbb{Z}\longrightarrow\pi_0C_h(t)\xrightarrow{\mathcal{P}}\mathcal{M}_0^m\longrightarrow 1$$

is exact.

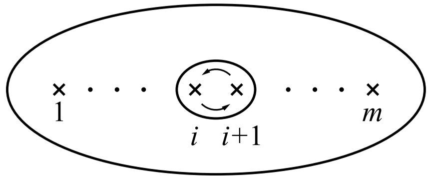

Figure 5. The diffeomorphism s_i.

Let $s_i : S^2 \to S^2$ be a half twist of the disk which exchanges the points q_i and q_{i+1} as in Figure 5.

We denote by $\sigma_i \in \mathcal{M}_0^m$ the mapping class represented by s_i. By lifting s_i, we have a unique diffeomorphism $\tilde{s}_i : \Sigma_h \to \Sigma_h$ which satisfies $\operatorname{supp} \tilde{s}_i = p_d^{-1}(\operatorname{supp} s_i)$. Let us denote the mapping class of $\tilde{s}_i$ by $\tilde{\sigma}_i \in \pi_0 C_h(t)$. Note that when $d = 2$, $\tilde{\sigma}_i$ is the Dehn twist along a nonseparating simple closed curve.

Lemma 3.4. *The set $\{\tilde{\sigma}_i\}_{i=1}^{m-1} \subset \pi_0 C_h(t)$ generates the group $\pi_0 C_h(t)$.*

Proof. Since $\{\sigma_i\}_{i=1}^{m-1}$ generates the group $\mathcal{M}_0^m$, it suffices to represent $[t] \in \pi_0 C_h(t)$ as a product of $\{\sigma_i\}_{i=1}^{m-1}$. Let $C_h^{(*)}(t)$ denote the subgroup of $C_h(t)$ defined by $C_h^{(*)}(t) = \{f \in C_h(t) \mid f(p_d^{-1}(*)) = p_d^{-1}(*)\}$. In this proof, by abuse of terminology, we use the term "Dehn twist" both for a diffeomorphism and for a mapping class. The diffeomorphism $s_1 \cdots s_{m-2} s_{m-1}^2 s_{m-2} \cdots s_1$ in $\mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$ is isotopic to the product of Dehn twists $t_c^{-1} t_{c'}$ in Figure 6, and it is also isotopic to the Dehn twist t_d^{-1}.

Therefore the lift $\tilde{s}_1 \cdots \tilde{s}_{m-2} \tilde{s}_{m-1}^2 \tilde{s}_{m-2} \cdots \tilde{s}_1$ is isotopic to some lift $\tilde{f}_1 : \Sigma_h \to \Sigma_h$ of t_d^{-1}. Since we can choose the isotopy in $\mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$ so that it does not move $*$, the lift $\tilde{f}_1$ fixes $p^{-1}(*)$ pointwise. Let D be the closed disk which is bounded by d and contains $*$, and let $\tilde{f}_2$ denote the lift of t_d which satisfies $\operatorname{supp} \tilde{f}_2 \subset p^{-1}(D)$. Since $f_1 f_2$ is a lift of the identity map of S^2, and the action of $\tilde{f}_2$ on $p^{-1}(*)$ coincides with that of t, we have $\tilde{f}_1 \tilde{f}_2 = t \in \mathrm{Diff}_+ \Sigma_h$. Since t_d is isotopic to the identity map in $\mathrm{Diff}_+ \Sigma_h$, we have $[\tilde{f}_2] = 1 \in \pi_0 C_h(t)$. Thus we obtain

$$\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-2} \tilde{\sigma}_{m-1}^2 \tilde{\sigma}_{m-2} \cdots \tilde{\sigma}_1 = [\tilde{f}_1] = [\tilde{f}_1 \tilde{f}_2] = [t] \in \pi_0 C_h(t). \qquad \square$$

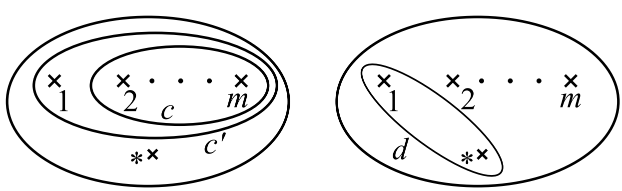

Figure 6. The curves c, c', d.

The cobounding function of the cocycles $\tau_{m,d,j}$. Recall that, for an integer d with $d \mid m$, we have a covering space $p_d : \Sigma_h \to S^2$. Let $g = (m-1)(m-2)/2$. If we consider the case when $d = m$, p_d is the m-cyclic covering on S^2 whose genus of the covering surface is g. Thus we identify it with the surface Σ_g, and denote the covering by $p : \Sigma_g \to S^2$.

Since the quotient space $\Sigma_g/\langle t^d\rangle$ is also a d-cyclic covering of S^2 with m branched points, we can identify $\Sigma_h \cong \Sigma_g/\langle t^d\rangle$. Since a diffeomorphism $f \in C_g(t)$ induces a diffeomorphism $\bar{f}$ on $\Sigma_g/\langle t^d\rangle$ which commutes with t, we have a natural homomorphism $\mathcal{P} : \pi_0 C_g(t) \to \pi_0 C_h(t)$ which maps $[f]$ to $[\bar{f}]$. Since $H^*(\pi_0 C_h(t); \mathbb{Q}) \cong H^*(\mathcal{M}_0^m; \mathbb{Q})$, and $H^*(\mathcal{M}_0^m; \mathbb{Q})$ is trivial (see [Cohen 1987] Corollary 2.2), there exists a unique cobounding function of $\tau_{m,d,j}$. Denote it by $\phi_{m,d,j} : \pi_0 C_h(t) \to \mathbb{Q}$. Since $\tau_{m,d,j}$ is bounded, the cobounding function $\phi_{m,d,j}$ is a quasimorphism.

Remark 3.5. Gambaudo and Ghys [2005] already constructed almost the same quasimorphisms on the mapping class groups of pointed disks, called ω-signatures. They calculated the value of their quasimorphisms for an element similar to

$$\tilde{\sigma}_1\tilde{\sigma}_2\cdots\tilde{\sigma}_{r-1} \in \pi_0 C_h(t)$$

in [Gambaudo and Ghys 2005, Proposition 5.2].

Remark 3.6. This construction is also similar to higher-order signature cocycles in Cochran, Harvey and Horn's paper [Cochran et al. 2012]. They considered von Neumann signatures of surface bundles whose fibers are nonfinite regular coverings on a surface with boundary.

Let us recall a natural homomorphism $\pi_0 C_h(t) \to \mathcal{M}_h$ defined by forgetting symmetries of mapping classes. It maps a mapping class $[f] \in \pi_0 C_h(t)$ to $[f] \in \mathcal{M}_h$, and is injective as shown in Birman and Hilden [1973, Theorem 1]. In particular, if we consider the case when m is even and the double covering $p_2 : \Sigma_h \to S^2$, this homomorphism induces isomorphism between $\pi_0 C_h(t)$ and $\mathcal{H}_h$. In this case, the eigenspace V^{-1} coincides with $H_1(\Sigma_h; \mathbb{C})$. Thus we have:

Remark 3.7. When m is even, $\phi_{m,2,1} : \pi_0 C_h(t) \to \mathbb{Q}$ is equal to the Meyer function $\phi_h : \mathcal{H}_h \to \mathbb{Q}$ on the hyperelliptic mapping class group, under the natural isomorphism $\pi_0 C_h(t) \cong \mathcal{H}_h$.

Lemma 3.8. *For $1 \le j \le d-1$ and $\varphi \in \pi_0 C_g(t)$,*

$$\phi_{m,m,mj/d}(\varphi) = \phi_{m,d,j}(\mathcal{P}(\varphi)).$$

Proof. Since $H_1(\pi_0 C_g(t); \mathbb{Q})$ is trivial, it suffices to show that

$$\tau_{m,m,mj/d}(\varphi, \psi) = \tau_{m,d,j}(\mathcal{P}(\varphi), \mathcal{P}(\psi)) \quad \text{for } \varphi, \psi \in \pi_0 C_g(t).$$

If $f : E \to P$ is an oriented Σ_g-bundle with structure group $C_g(t)$, the induced map $\bar{f} : E/\langle t^d \rangle \to P$ is an oriented Σ_h-bundle with structure group $C_h(t)$. If we denote the monodromies of f along α and β by φ and ψ, the ones of $\bar{f}$ are $\mathcal{P}(\varphi)$ and $\mathcal{P}(\psi)$.

Let ω be the m-th root of unity $\exp(2\pi \boldsymbol{i}/m)$, and let $q_d : \Sigma_g \to \Sigma_g/\langle t^d \rangle$ denote the projection. To distinguish eigenspaces of $H_1(\Sigma_g; \mathbb{C})$ and $H_1(\Sigma_h; \mathbb{C})$ of the action by t, we denote them by $(V_g)^z$ and $(V_h)^z$ instead of V^z, respectively. The projection q_d induces the isomorphism $H_1(\Sigma_g; \mathbb{C})^{\langle t^d \rangle} \cong H_1(\Sigma_h; \mathbb{C})$. Moreover we have $(V_g)^{\omega^{mj/d}} \cong (V_h)^{\eta^j}$. Hence it also induces a natural isomorphism between $H_1(P; (V_g)^{\omega^{mj/d}})$ and $H_1(P; (V_h)^{\eta^j})$, where $(V_g)^{\omega^{mj/d}}$ and $(V_h)^{\eta^j}$ are local systems coming from f and $\bar{f}$.

Let $\tilde{a}$, $\tilde{b}$ be loops in $\Sigma_g - \{q_i\}_{i=1}^m$. We may assume that $q_d(\tilde{a}) \cup q_d(\tilde{b})$ has no triple point. Then the intersection number $[q_d(\tilde{a})] \cdot [q_d(\tilde{b})]$ in Σ_h coincides with $[q_d^{-1}(q_d(\tilde{a}))] \cdot [\tilde{b}]$ in Σ_g. Hence we have

$$\begin{aligned}\sum_{i=0}^{m/d-1} [(t^{di})_* \tilde{a}] \cdot \sum_{j=0}^{m/d-1} [(t^{dj})_* \tilde{b}] &= \sum_{i=0}^{m/d-1} \sum_{j=0}^{m/d-1} [(t^{di-dj})_* \tilde{a}] \cdot [\tilde{b}] \\ &= \frac{m}{d}[q_d^{-1}(q_d(\tilde{a}))] \cdot [\tilde{b}] = \frac{m}{d}[q_d(\tilde{a})] \cdot [q_d(\tilde{b})].\end{aligned}$$

Therefore the isomorphism $H_1(\Sigma_g; \mathbb{C})^{\langle t^d \rangle} \cong H_1(\Sigma_h; \mathbb{C})$ induced by the quotient map $q_d : \Sigma_g \to \Sigma_h$ preserves the intersection form up to constant multiple. Thus it also preserves the intersection forms on $H_1(P; (V_g)^{\omega^{mj/d}})$ and $H_1(P; (V_h)^{\eta^j})$, and we obtain

$$\begin{aligned}\tau_{m,m,mj/d}(\varphi, \psi) &= \mathrm{Sign}(H_1(P; (V_g)^{\omega^{mj/d}})) \\ &= \mathrm{Sign}(H_1(P; (V_h)^{\eta^j})) \\ &= \tau_{m,d,j}(\mathcal{P}(\varphi), \mathcal{P}(\psi)).\end{aligned}$$

□

By Lemma 3.8, it suffices to consider the case when $d = m$. We shorten $\tau_{m,m,j}$ and $\phi_{m,m,j}$ to $\tau_{m,j}$ and $\phi_{m,j}$.

Lemma 3.9.

$$\phi_{m,j}(\varphi) = \phi_{m,m-j}(\varphi).$$

Proof. By taking complex conjugates, we have an isomorphism $i : V^{\omega^j} \cong V^{\omega^{m-j}}$. Moreover it induces the isomorphism $i_* : H_1(P; V^{\omega^j}) \cong H_1(P; V^{\omega^{m-j}})$.

Let us denote the hermitian form on $H_1(P; V^{\omega^j})$ by $\langle\ ,\ \rangle_j$. By the definition of the hermitian form, we have $\langle x, y \rangle_j = \overline{\langle i_* x, i_* y \rangle_{m-j}}$ for $x, y \in H_1(P; V^{\omega^j})$, where $\bar{z}$ is a complex conjugate of $z \in \mathbb{C}$. Thus the signatures of the hermitian forms $\langle\ ,\ \rangle_j$ and $\langle\ ,\ \rangle_{m-j}$ coincide, and the cobounding functions of $\tau_{m,j}$ and $\tau_{m,m-j}$ also coincide. □

4. Defects of homogeneous quasimorphisms

In this section, we will prove Theorems 1.3 and 1.6. On page 336, we give an inequality between the defects of a quasimorphism and its homogenization when the quasimorphism is antisymmetric and a class function (Lemma 4.1) and prove Theorem 1.3(i). On page 337, we prove Theorem 1.3(ii) by giving a lower bound on the defect of $\phi_{m,m/2} : \pi_0 C_g(t) \to \mathbb{R}$, which is the cobounding function of the 2-cocycle $\tau_{m,m/2}$. On page 337, we prove Theorem 1.6.

Proof of Theorem 1.3(i). Endo [2000, Proposition 3.1] showed that the Meyer function $\phi_g : \mathcal{H}_g \to \mathbb{Q}$ satisfies the conditions in Lemma 4.1. The quasimorphisms $\bar{\phi}_{m,j}$ also satisfy these conditions.

Turaev [1985] defined another 2-cocycle on the symplectic group. Endo and Nagami [2005, Proposition A.3] showed that Turaev's cocycle coincides with the Meyer cocycle up to sign. Since Turaev's cocycle is defined by the signature on a vector space of rank less than or equal to $m-2$. A similar argument shows $D(\phi_{m,j}) \le m-2$. Thus Theorem 1.3(i) follows from Lemma 4.1 below.

Lemma 4.1. *Let G be a group, and $\phi : G \to \mathbb{R}$ a quasimorphism satisfying*

$$\phi(xyx^{-1}) = \phi(y), \quad \phi(x^{-1}) = -\phi(x).$$

Then we have

$$D(\bar{\phi}) \le D(\phi),$$

where $\bar{\phi}$ is the homogenization of ϕ.

Proof of Lemma 4.1. Without loss of generality, we may assume that the quasimorphism $\phi : G \to \mathbb{R}$ is antisymmetric:

$$\phi(x^{-1}) = -\phi(x).$$

Otherwise pass to the antisymmetrization $\phi' : G \to \mathbb{R}$ defined by

$$\phi'(x) = \frac{\phi(x) - \phi(x^{-1})}{2},$$

which satisfies

$$D(\phi') \le D(\phi), \text{ and } \bar{\phi}' = \bar{\phi}.$$

For any $x, y \in G$, we have

$$\begin{aligned}\phi([x, y]) &= |\phi([x, y]) - \phi(y) + \phi(y)| \\ &= |\phi(xyx^{-1}y^{-1}) - \phi(xyx^{-1}) - \phi(y^{-1})| \le D(\phi).\end{aligned}$$

Thus for any $g \in [G, G]$,

$$|\phi(g)| \le (2\,\mathrm{cl}(g) - 1) D(\phi).$$

As observed by Bavard [1991, Lemma 3.6],

$$\mathrm{cl}(x^n y^n (xy)^{-n}) \leq \frac{n}{2},$$

for every $n \geq 0$. Therefore we have $|\phi(x^n y^n (xy)^{-n})| \leq (n-1)D(\phi)$. Hence

$$\begin{aligned} |\delta\bar{\phi}(x, y)| &= \lim_{n\to\infty}\left|\frac{\phi(x^n)+\phi(y^n)-\phi(x^n y^n)}{n}\right| \\ &= \lim_{n\to\infty}\left|\frac{\phi(x^n y^n (xy)^{-n})}{n}\right| \leq D(\phi). \end{aligned}$$

□

Proof of Theorem 1.3(ii). Let m be an even number greater than or equal to 4. By Remark 3.7, we consider the Meyer function ϕ_g on the hyperelliptic mapping class group $\mathcal{H}_g$ instead of $\phi_{m,m/2}$.

Lemma 4.2 [Barge and Ghys 1992, Proposition 3.5]. *For any* $A \in \mathrm{Sp}(2g; \mathbb{Z})$,

$$\mathrm{Sign}(\langle\ ,\ \rangle_{A^k,A}) = \mathrm{Sign}\left(-J\sum_{i=1}^{k}(A^i - A^{-i})\right).$$

Let c_i, d_i^+, and d_i^- denote the simple closed curves in Figure 7. For simplicity, we also denote by c_i, d_i^+, and d_i^- the Dehn twists along these curves.

To prove Theorem 1.3(ii), it suffices to show the following.

Lemma 4.3. $\delta\bar{\phi}_g(c_2^2 c_4^2 \cdots c_{2g}^2,\ d_1^+ d_1^- d_2^+ d_2^- \cdots d_g^+ d_g^-) = -2g.$

Proof of Lemma 4.3. Since the pairs (c_i, c_j), $(d_i^+ d_i^-, d_j^+ d_j^-)$, and $(c_i, d_j^+ d_j^-)$ mutually commute when $i \neq j$, the expression in the lemma equals

$$\begin{aligned} &\bar{\phi}_g(c_2^2 c_4^2 \cdots c_{2g}^2) + \bar{\phi}_g(d_1^+ d_1^- d_2^+ d_2^- \cdots d_g^+ d_g^-) - \bar{\phi}_g(c_2^2 d_1^+ d_1^- c_4^2 d_2^+ d_2^- \cdots c_{2g}^2 d_g^+ d_g^-) \\ &\qquad = \sum_{i=1}^{g}(\bar{\phi}_g(c_{2i}^2) + \bar{\phi}_g(d_i^+ d_i^-) - \bar{\phi}_g(c_{2i}^2 d_i^+ d_i^-)). \end{aligned}$$

Hence it suffices to prove $\bar{\phi}_g(c_{2i}^2)+\bar{\phi}_g(d_i^+ d_i^-)-\bar{\phi}_g(c_{2i}^2 d_i^+ d_i^-) = -2$ for $1 \leq i \leq g$.

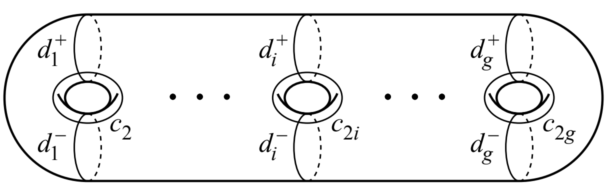

Figure 7. Curves in Σ_g.

Since $\rho(d_i^+) = \rho(d_i^-)$, we have

$$\begin{aligned}
&\bar{\phi}_g(c_{2i}^2) + \bar{\phi}_g(d_i^+ d_i^-) - \bar{\phi}_g(c_{2i}^2 d_i^+ d_i^-) \\
&\quad = -\lim_{n\to\infty} \frac{1}{n}\big\{\phi_g((c_{2i}^2 d_i^+ d_i^-)^n) - \phi_g((c_{2i}^2)^n) - \phi_g((d_i^+ d_i^-)^n)\big\} \\
&\quad = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1}\big\{\tau_g((c_{2i}^2 d_i^+ d_i^-)^k, c_{2i}^2 d_i^+ d_i^-) - \tau_g(c_{2i}^{2i}, c_{2i}^2) - \tau_g((d_i^+ d_i^-)^i, d_i^+ d_i^-)\big\} \\
&\qquad\qquad + \tau_g(c_{2i}^2, d_i^+ d_i^-) \\
&\quad = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1}\big\{\tau_g((c_{2i}^2 (d_i^+)^2)^k, c_{2i}^2 (d_i^+)^2)\tau_g(c_{2i}^{2i}, c_{2i}^2) - \tau_g((d_i^+)^{2i}, (d_i^+)^2)\big\} \\
&\qquad\qquad + \tau_g(c_{2i}^2, (d_i^+)^2).
\end{aligned}$$

There exists a mapping class ψ_i such that $\psi_i c_{2i} \psi_i^{-1} = c_2$ and $\psi_i d_i^+ \psi_i^{-1} = d_i^+$ for $i = 2, \ldots, g$. Since the Meyer cocycle satisfies the property

$$\tau_g(xyx^{-1}, xzx^{-1}) = \tau_g(y, z)$$

for $x, y, z \in \mathcal{M}_g$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{n}\{\tau_g((c_{2i}^2 (d_i^+)^2)^k, c_{2i}^2 (d_i^+)^2) - \tau_g(c_{2i}^{2i}, c_{2i}^2) - \tau_g((d_i^+)^{2i}, (d_i^+)^2)\} \\
&\qquad\qquad + \tau_g(c_{2i}^2, (d_i^+)^2) \\
&\quad = \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{n}\{\tau_g((c_2^2 (d_1^+)^2)^k, c_2^2 (d_1^+)^2) - \tau_g(c_2^{2i}, c_2^2) - \tau_g((d_1^+)^{2i}, (d_1^+)^2)\} \\
&\qquad\qquad + \tau_g(c_2^2, (d_1^+)^2).
\end{aligned}$$

Let us consider the case when $g = 1$. Since $\rho(c_2^2) = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$, $\rho((d_1^+)^2) = \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right)$ and $\rho(c_2^2 (d_1^+)^2) = \left(\begin{smallmatrix} -3 & 2 \\ -2 & 1 \end{smallmatrix}\right)$, we have

$$-J\sum_{k=1}^{n}(\rho(c_2^{2k}) - \rho(c_2^{-2k})) = \begin{pmatrix} 0 & 0 \\ 0 & 2n(n+1) \end{pmatrix},$$

$$-J\sum_{k=1}^{n}(\rho(d_1^+)^{2k} - \rho(d_1^+)^{-2k}) = \begin{pmatrix} 2n(n+1) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
-J\sum_{k=1}^{n}(\rho((c_2^2 (d_1^+)^2)^k) - \rho((c_2^2 (d_1^+)^2)^{-k})) &= \sum_{k=1}^{n} 4k(-1)^k \left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right) \\
&= \{(-1)^n(2n+1) - 1\}\left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right).
\end{aligned}$$

By Lemma 4.2 we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{\tau_g(c_2^{2k},c_2^2)}{n}=\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{\tau_g((d_1^+)^{2k},(d_1^+)^2)}{n}=1, \tag{1}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{\tau_g((c_2^2(d_1^+)^2)^k,c_2^2(d_1^+)^2)}{n}=0. \tag{2}$$

When $g\geq 2$, the same calculation also shows (1). It is an easy calculation to show that

$$\tau_g(c_2^2,d_1^+d_1^-)=0.$$

Therefore we obtain

$$\bar{\phi}_g(c_{2i}^2)+\bar{\phi}_g(d_i^+d_i^-)-\bar{\phi}_g(c_{2i}^2d_i^+d_i^-)=-2. \qquad \square$$

In the same way as for (1), we have $\tau_g(s_0^i,s_0)=1$. Hence we obtain

$$\bar{\phi}_g(s_0)=-\lim_{n\to\infty}\frac{\sum_{i=1}^{n-1}\tau_g(s_0^i,s_0)}{n}+\phi_g(s_0)=-1+\phi_g(s_0)$$

and

$$\bar{\phi}_g(s_h)=\phi_g(s_0).$$

By [Endo 2000, Lemmas 3.3 and 3.5], we have

$$\bar{\phi}_g(t_{s_0})=-\frac{g}{2g+1},\quad\text{and}\quad\bar{\phi}_g(t_{s_h})=-\frac{4h(g-h)}{2g+1}.$$

Remark 4.4. By Theorems 1.3 and 2.4, $\bar{\phi}_g$ gives the lower bounds for $\mathrm{scl}_{\mathcal{H}_g}(t_{s_h})$ $(j=0,\dots,g-1)$ corresponding to ones given in [Monden 2012].

Remark 4.5. By Theorems 1.7 and 2.4 and Remark 4.4, we have $\mathrm{scl}_{\mathcal{M}_1}(t_c)=\frac{1}{12}$. Let $\rho:\mathcal{M}_1\cong \mathrm{SL}(2,\mathbb{Z})\to\mathrm{PSL}(2,\mathbb{Z})$ be the natural quotient map. It is easily seen that for all $x\in\mathcal{M}_1$, $\mathrm{scl}_{\mathcal{M}_1}(x)=\mathrm{scl}_{\mathrm{PSL}(2,\mathbb{Z})}(\rho(x))$. Louwsma [2011] determined $\mathrm{scl}_{\mathrm{PSL}(2,\mathbb{Z})}(y)=\frac{1}{12}$ for $y=\rho(t_c)$.

Proof of Theorem 1.6. If $x\in\mathcal{H}_g$ satisfies $|\bar{\phi}_g(x)|=D(\bar{\phi}_g)$ and $|\phi(x)|\leq D(\phi)$ for any homogeneous quasimorphism $\phi:\mathcal{H}_g\to\mathbb{R}$, we obtain $\mathrm{scl}(x)=\frac{1}{2}$ by Bavard's duality theorem (Theorem 2.4). We will show that

$$x=c^{2g+8}(d_2^+d_2^-\cdots d_{g-1}^+d_{g-1}^-)^2(s_1\cdots s_{g-1})^{-1}$$

satisfies this property.

Firstly we will prove

$$\sum_{j=1}^{g-1}\phi(s_j)=\sum_{i=1}^{g}(\phi(c_{2i}^2d_i^-d_i^+)-\phi(d_i^-d_i^+))$$

for any homogeneous quasimorphism $\phi : \mathcal{H}_g \to \mathbb{R}$. By Lemma 2.8, we have

$$(d_1^+ c_2 d_1^-)^4 = s_1, \quad (d_g^+ c_{2g} d_g^-)^4 = s_{g-1},$$
$$(d_i^+ c_{2i} d_i^-)^4 = s_{i-1} s_i \quad (i = 2, \ldots, g-1).$$

Since c_{2i} commutes with s_j, $(c_2 d_1^- d_1^+ c_2 d_1^- d_1^+)^2 = s_1$, $(c_{2i} d_i^- d_i^+ c_{2i} d_i^- d_i^+)^2 = s_{i-1} s_i$, and $(c_{2g} d_g^- d_g^+ c_{2g} d_g^- d_g^+)^2 = s_{g-1}$. By Lemma 2.6, $c_{2i} d_i^- d_i^+ c_{2i}$ commutes with $d_i^- d_i^+$ for $i = 1, \ldots, g$, as is easy to check. Therefore $(c_2 d_1^- d_1^+ c_2)^2 = s_1 (d_1^- d_1^+)^{-2}$, $(c_{2i} d_i^- d_i^+ c_{2i})^2 = s_{i-1} s_i (d_i^- d_i^+)^{-2}$, and $(c_{2g} d_g^- d_g^+ c_{2g})^2 = s_{g-1} (d_g^- d_g^+)^{-2}$. These equations give

$$2\phi(c_2^2 d_1^- d_1^+) = \phi(s_1) - 2\phi(d_1^- d_1^+),$$
$$2\phi(c_{2i}^2 d_i^- d_i^+) = \phi(s_{i-1}) + \phi(s_i) - 2\phi(d_i^- d_i^+),$$
$$2\phi(c_{2g}^2 d_g^- d_g^+) = \phi(s_{g-1}) - 2\phi(d_g^- d_g^+).$$

Thus we obtain $\sum_{j=1}^{g-1} \phi(s_j) = \sum_{i=1}^{g} (\phi(c_{2i}^2 d_i^- d_i^+) - \phi(d_i^- d_i^+))$.

Secondly we will prove $\bar{\phi}_g(x) = D(\bar{\phi}_g)$. The curves $c, s_1, \ldots, s_{g-1}, d_2^+, d_2^-, \ldots, d_{g-1}^+, d_{g-1}^-$ are mutually disjoint, and c_i is conjugate to c. Hence, by Lemma 2.3(i) and (ii), we have

$$\begin{aligned}\phi(x) &= (g+4)\phi(c^2) + 2\sum_{i=2}^{g-1} \phi(d_i^+ d_i^-) - \sum_{j=1}^{g-1} \phi(s_i) \\ &= \sum_{i=1}^{g} (\phi(c_{2i}^2) + \phi(d_i^+ d_i^-) - \phi(c_{2i}^2 d_i^- d_i^+)).\end{aligned}$$

In the proof of Lemma 4.3, we showed

$$\sum_{i=1}^{g} (\bar{\phi}_g(c_{2i}^2) + \bar{\phi}_g(d_i^+ d_i^-) - \bar{\phi}_g(c_{2i}^2 d_i^- d_i^+)) = -2g = -D(\bar{\phi}_g).$$

Thus we obtain $|\bar{\phi}_g(x)| = D(\bar{\phi}_g)$.

Lastly we prove $\phi(x) \leq D(\phi)$ for any homogeneous quasimorphism $\phi : \mathcal{H}_g \to \mathbb{R}$:

$$\begin{aligned}D(\phi) &\geq |\delta(c_2^2 \cdots c_{2g}^2, d_1^+ d_1^- \cdots d_g^+ d_g^-)| \\ &= \left|\phi(c_2^2 \cdots c_{2g}^2) + \phi(d_1^+ d_1^- \cdots d_g^+ d_g^-) - \phi(c_2^2 \cdots c_{2g}^2 d_1^+ d_1^- \cdots d_g^+ d_g^-)\right| \\ &= \left|\phi(c_2^2 \cdots c_{2g}^2) + \phi(d_1^+ d_1^- \cdots d_g^+ d_g^-) - \phi((c_2^2 d_1^+ d_1^-) \cdots (c_{2g}^2 d_g^+ d_g^-))\right| \\ &= \left|\sum_{i=1}^{g} (\phi(c_{2i}^2) + \phi(d_i^+ d_i^-) - \phi(c_{2i}^2 d_i^+ d_i^-))\right| = |\phi(x)|. \qquad \square\end{aligned}$$

5. Proof of Theorem 1.7

Let $c_1, \dots, c_{2g+2}$ be nonseparating simple closed curves on Σ_g as in Figure 2 and let ϕ be a homogeneous quasimorphism on $\mathcal{H}_g$. For simplicity of notation, we write t_i instead of t_{c_i} By $\iota = \iota^{-1}$, we have $t_{2g+1}^2 t_{2g} \cdots t_2 t_1^2 = (t_{2g} \cdots t_2)^{-1} \iota^{-1}$. Since each of the two boundary components of a regular neighborhood of $c_2 \cup c_3 \cup \cdots \cup c_{2g}$ is c_{2g+2} by Lemma 2.8, we have $(t_{2g} \cdots t_2)^{2g} = t_{2g+2}^2$. Note that this relation holds in $\mathcal{H}_g$. Therefore, by Definition 2.2 and Lemma 2.3, we have

$$\phi(t_{2g+1}^2 t_{2g} \cdots t_2 t_1^2) = -\phi(t_{2g} \cdots t_2) + \phi(\iota^{-1}) = -\frac{1}{g}\phi(t_{2g+2}). \tag{3}$$

Applying Lemma 2.3(i) and 2.6(i), we can move the factors with single and double underlines alternatively as follows.

$$\begin{aligned}
\phi(t_{2g+1}^2 t_{2g} \cdots t_3 t_2 \underline{t_1^2}) &= \phi(\underline{t_1^2} t_{2g+1}^2 t_{2g} \cdots t_3 t_2) && \text{(by Lemma 2.3)}\\
&= \phi(t_{2g+1}^2 t_{2g} \cdots t_3 \underline{t_1^2}\underline{\underline{t_2}}) && \text{(by Lemma 2.6)}\\
&= \phi(\underline{\underline{t_2}} t_{2g+1}^2 t_{2g} \cdots t_4 t_3 t_1^2) && \text{(by Lemma 2.3)}\\
&= \phi(t_{2g+1}^2 t_{2g} \cdots t_4 \underline{\underline{t_2}}\underline{t_3 t_1^2}) && \text{(by Lemma 2.6)}\\
&= \phi(t_{2g+1}^2 t_{2g} \cdots t_6 t_5 \underline{t_3 t_1^2}\underline{\underline{t_4 t_2}}) && \text{(by Lemma 2.3 and 2.6)}\\
&= \phi(t_{2g+1}^2 t_{2g} \cdots t_6 \underline{\underline{t_4 t_2}}\underline{t_5 t_3 t_1^2}) && \text{(by Lemmas 2.3 and 2.6)}\\
&= \phi(t_{2g+1}^2 t_{2g} \cdots t_7 \underline{t_5 t_3 t_1^2}\underline{\underline{t_6 t_4 t_2}}) && \text{(by Lemmas 2.3 and 2.6)}\\
&= \phi(t_{2g+1}^2 t_{2g} \cdots t_8 \underline{\underline{t_6 t_4 t_2}}\underline{t_7 t_5 t_3 t_1^2}) && \text{(by Lemmas 2.3 and 2.6)}\\
&= \phi(t_{2g+1}^2 t_{2g} \cdots t_9 \underline{t_7 t_5 t_3 t_1^2}\underline{\underline{t_8 t_6 t_4 t_2}}) && \text{(by Lemmas 2.3 and 2.6)}\\
&\;\;\vdots\\
&= \phi\big((t_{2g+1}^2 t_{2g-1} \cdots t_5 t_3 t_1^2)(t_{2g} t_{2g-4} \cdots t_4 t_2)\big).
\end{aligned}$$

From Definition 2.2 and Equation (3),

$$\begin{aligned}
D(\phi) &\geq |\phi((t_{2g+1}^2 \cdots t_3 t_1^2)(t_{2g} \cdots t_4 t_2)) - \phi(t_{2g+1}^2 \cdots t_3 t_1^2) - \phi(t_{2g} \cdots t_4 t_2)|\\
&= \left| -\frac{1}{g}\phi(t_{2g+2}) - \phi(t_{2g+1}^2 \cdots t_3 t_1^2) - \phi(t_{2g} \cdots t_4 t_2) \right|,
\end{aligned}$$

where $D(\phi)$ is the defect of ϕ. From Lemmas 2.3, 2.5 and 2.6 we have

$$D(\phi) \geq \left| \frac{1}{g}\phi(t_1) + (g+3)\phi(t_1) + g\phi(t_1) \right| = (2g+3+\frac{1}{g})|\phi(t_1)|.$$

By Theorem 2.4 we have $\mathrm{scl}_{\mathcal{H}_g}(t_1) \leq \dfrac{1}{(2(2g+3+1/g))}$. This completes the proof of Theorem 1.7. □

Remark 5.1. By a similar argument to the proof of Theorem 1.7, we can show that

$$\mathrm{scl}_{\mathcal{M}_0^m}(\sigma_1) = \frac{1}{2\{m+1+2/(m-2)\}} \quad \text{for all } m \geq 4.$$

6. Calculation of quasimorphisms

In this section, we prove Theorem 1.1. To prove it, we perform a straightforward and elementary calculation of the hermitian form $\langle\ ,\ \rangle_{\tilde{\sigma}^k,\tilde{\sigma}}$ on the eigenspace V^{ω^j}.

Let $p : \Sigma_g \to S^2$ be the regular branched m-cyclic covering on S^2 with m branched points as on page 333. Choose a point in $p^{-1}(*)$, and denote it by $\tilde{*} \in \Sigma_g$. We denote by $\tilde{\alpha}_i$ the lift of α_i which starts at $\tilde{*}$. Note that $\tilde{\alpha}_i t(\tilde{\alpha}_{i+1})^{-1}$ is a loop in Σ_g while $\tilde{\alpha}_i$ is an arc. We denote by $e_i(k) \in H_1(\Sigma_g;\mathbb{Z})$ the homology class represented by $t^k(\tilde{\alpha}_i t(\tilde{\alpha}_{i+1})^{-1})$.

Lemma 6.1. *The homology classes $\{e_i(k)\}_{\substack{1\leq i\leq m-2\\0\leq k\leq m-2}}$ form a basis of $H_1(\Sigma_g;\mathbb{Z})$.*

Proof. We use the Schreier method. Let T denote a Schreier transversal $T=\{\alpha_1^k\}_{i=0}^{m-1}$ and S a generating set $S=\{\alpha_i\}_{i=1}^{m-1}$ of $\pi_1(S^2-\{q_i\}_{i=1}^m)$. Then the subgroup $\pi_1(\Sigma_g-\{p^{-1}(q_i)\}_{i=1}^m)$ is generated by

$$\{(rs(\overline{rs})^{-1} \mid r\in T, s\in S\} = \{\alpha_1^k\alpha_i\alpha_1^{-k-1}\}_{\substack{2\leq i\leq m-1\\0\leq k\leq m-2}} \cup \{\alpha_1^{m-1}\alpha_i\}_{1\leq i\leq m-1}.$$

By van Kampen's theorem, the group $\pi_1(\Sigma_g)$ is obtained by adding the relation $\alpha_i^m=1$ to $\pi_1(\Sigma_g-\{p^{-1}(q_i)\}_{i=1}^m)$. Thus, the set $\{\alpha_1^k\alpha_i\alpha_{i+1}^{-1}\alpha_1^{-k}\}_{i,k}$, where from now through the end of the proof we have $1\leq i\leq m-2$ and $0\leq k\leq m-2$, generates the group $\pi_1(\Sigma_g)$. This implies that $\{e_i(k)\}_{i,k}$ is a generating set of $H_1(\Sigma_g;\mathbb{Z})$.

By the Riemann–Hurwitz formula, $H_1(\Sigma_g;\mathbb{Z})$ is a free module of rank equal to $2g=(m-1)(m-2)$, and this is equal to the order of the set $\{e_i(k)\}_{i,k}$. Therefore the set $\{e_i(k)\}_{i,k}$ is a basis of the free module $H_1(\Sigma_g;\mathbb{Z})$. □

The intersection form and the action of $\tilde{\sigma}_i$. Let j be an integer with $1\leq j\leq m-1$. Firstly we find a basis of $V^{\omega^j}\subset H_1(\Sigma_g;\mathbb{C})$ and calculate intersection numbers.

Lemma 6.2. *The intersection numbers of $\{e_i(k)\}_{\substack{1\leq i\leq m-2\\0\leq k\leq m-2}}$ are*

$$e_i(k)\cdot e_{i'}(k) = \begin{cases} -1 & \text{if } i=i'-1,\\ 1 & \text{if } i=i'+1,\\ 0 & \text{otherwise,}\end{cases} \qquad e_i(k)\cdot e_{i'}(k+1) = \begin{cases} -1 & \text{if } i=i',\\ 1 & \text{if } i=i'-1,\\ 0 & \text{otherwise,}\end{cases}$$

$$e_i(k)\cdot e_{i'}(k-1) = \begin{cases} -1 & \text{if } i=i',\\ 1 & \text{if } i=i'+1,\\ 0 & \text{otherwise,}\end{cases} \qquad e_i(k)\cdot e_{i'}(k') = 0 \quad \text{if } |k-k'|\geq 2.$$

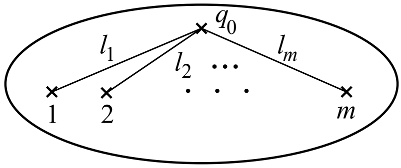

Figure 8. The paths $l_1, l_2, \ldots, l_m$.

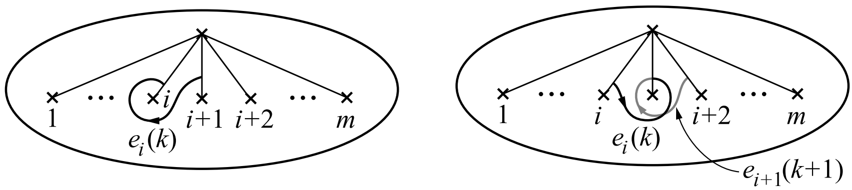

Figure 9. Left: the k-th copy. Right: the $(k+1)$-th copy.

Proof. We only prove the equality $e_i(k) \cdot e_{i+1}(k+1) = 1$ since the other cases are proved in the same way.

Let l_i be the paths as in Figure 8. Consider m copies of the 2-sphere cut along the l_i, and number them from 1 to m. (For convenience, copy 1 will also be called copy $m+1$.) Gluing the left-hand side of l_i in the k-th copy to the right-hand side of l_i in the $(k+1)$-th copy for $k = 1, 2, \ldots, m$, we obtain a closed connected surface homeomorphic to Σ_g, and it is naturally a covering space on S^2. As in Figure 9, the loops representing $e_i(k)$ and $e_{i+1}(k+1)$ intersect once positively in the $(k+1)$-th copy. Hence we have $e_i(k) \cdot e_{i+1}(k+1) = 1$. □

For $1 \le i \le m-2$, we define $w_i = \sum_{k=0}^{m-1} \omega^{-jk} e_i(k)$. Since $te_i(k) = e_i(k+1)$ for $1 \le k \le m-2$ and $e_i(m-1) = -\sum_{k=0}^{m-2} e_i(k)$, we have $w_i \in V^{\omega^j}$, and the set $\{w_i\}_{i=1}^{m-2}$ is a basis of V^{ω^j}.

Lemma 6.3. *The intersection numbers of* $\{w_i\}_{1 \le i \le m-2}$ *are*

$$w_i \cdot w_{i'} = \begin{cases} d(1-\omega^j) & \text{if } i = i'+1, \\ d(-\omega^{-j}+\omega^j) & \text{if } i = i', \\ d(\omega^{-j}-1) & \text{if } i = i'-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 6.2, we have

$$w_i \cdot w_i = \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} \omega^{j(k-l)} e_i(k) \cdot e_i(l) = d(-\omega^{-j}+\omega^j),$$

$$w_i \cdot w_{i+1} = \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} \omega^{j(k-l)} e_i(k) \cdot e_{i+1}(l) = d(\omega^{-j}-1),$$

and $w_i \cdot w_k = 0$ when $|i-k| \geq 2$. □

Let $\tilde{\sigma} = \tilde{\sigma}_1 \cdots \tilde{\sigma}_{r-1}$. We next find eigenvectors in V^{ω^j} relative to the action by $\tilde{\sigma}$.

Lemma 6.4. *Let i be an integer such that $1 \leq i \leq m-1$. Then we have*

$$(\tilde{\sigma}_i)_* e_l(k) = \begin{cases} e_l(k) + e_{l+1}(k) & \text{if } 2 \leq i \leq m-1, \text{ and } l = i-1, \\ -e_l(k-1) & \text{if } l = i, \\ e_{l-1}(k-1) + e_l(k) & \text{if } l = i+1, \\ e_l(k) & \text{if } l \neq i-1, i, i+1. \end{cases}$$

Proof. Recall that $e_i(k)$ is the homology class represented by the loop $\tilde{\alpha}_1^k \tilde{\alpha}_i \tilde{\alpha}_{i+1}^{-1} \tilde{\alpha}_1^{-k}$.

In the fundamental group $\pi_1(S^2 - \{q_i\}_{i=1}^m)$, we have

$$\begin{aligned} (\sigma_i)_*(\alpha_{i-1}\alpha_i^{-1}) &= \alpha_{i-1}\alpha_{i+1}^{-1} = (\alpha_{i-1}\alpha_i^{-1})(\alpha_i\alpha_{i+1}^{-1}), \\ (\sigma_i)_*(\alpha_i\alpha_{i+1}^{-1}) &= \alpha_{i+1}(\alpha_{i+1}^{-1}\alpha_i\alpha_{i+1})^{-1} = \alpha_{i+1}^{-1}(\alpha_i\alpha_{i+1}^{-1})^{-1}\alpha_{i+1}, \\ (\sigma_i)_*(\alpha_{i+1}\alpha_{i+2}^{-1}) &= (\alpha_{i+1}^{-1}\alpha_i\alpha_{i+1})\alpha_{i+2}^{-1} = \alpha_{i+1}^{-1}(\alpha_i\alpha_{i+1}^{-1})\alpha_{i+1}(\alpha_{i+1}\alpha_{i+2}^{-1}). \end{aligned}$$

By lifting these loops to the covering space Σ_g, we obtain what we want. □

By Lemma 6.4, the matrix representations of the actions of $\{\tilde{\sigma}_i\}_{i=1}^{m-1}$ on V^{ω^j} with respect to the basis $\{w_i\}_{1 \leq i \leq m-2}$ are calculated as

$$(\tilde{\sigma}_1)_* = \begin{pmatrix} -\omega^{-j} & \omega^{-j} & O \\ 0 & 1 & O \\ O & O & I_{m-4} \end{pmatrix}, \qquad (\tilde{\sigma}_i)_* = \begin{pmatrix} I_{i-1} & O & O \\ O & L & O \\ O & O & I_{m-i-4} \end{pmatrix},$$

$$(\tilde{\sigma}_{m-2})_* = \begin{pmatrix} I_{m-4} & O & O \\ O & 1 & O \\ O & 1 & -\omega^{-j} \end{pmatrix}, \quad (\tilde{\sigma}_{m-1})_* = \begin{pmatrix} I_{m-3} & v \\ O & -1 + \sum_{k=1}^{m-2} \omega^{-jk} \end{pmatrix},$$

where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -\omega^{-j} & \omega^{-j} \\ 0 & 0 & 1 \end{pmatrix}, \quad v = \left(1, 1+\omega^{-j}, \dots, \sum_{k=0}^{m-3} \omega^{-jk}\right)^T.$$

Let r be an integer with $2 \leq r \leq m$, and put

$$e_r'(k) = [\tilde{a}_1^k \tilde{a}_r (\tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_r)^{-1} \tilde{a}_1^{-1} (\tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_r) \tilde{a}_1^{-k}].$$

By Lemma 6.4, we have

$$\begin{aligned} \tilde{\sigma}_* e_i(k) &= e_{i+1}(k), \qquad \text{when } 1 \leq i \leq r-2, \\ \tilde{\sigma}_* e_r(k) &= -e_r'(k) + e_r(k), \\ \tilde{\sigma}_* e_{r-1}(k) &= e_r'(k), \\ \tilde{\sigma}_* e_r'(k) &= e_1(k-r+1). \end{aligned}$$

The sum $w'_r := \sum_{k=0}^{m-1} \omega^{-jk} e'_r(k)$ is contained in V^{ω^j}. For $i = 1, 2, \dots, r-2$, we have

$$\tilde{\sigma}_* w_i = \tilde{\sigma}_* \sum_{k=0}^{m-1} \omega^{-jk} e_i(k) = \sum_{k=0}^{m-1} \omega^{-jk} e_{i+1}(k) = w_{i+1},$$

$$\tilde{\sigma}_* w_{r-1} = \sum_{k=0}^{m-1} \omega^{-jk} (e_{r-1}(k) + e'_r(k)) = w_{r-1} + w'_r,$$

$$\tilde{\sigma}_* w'_r = \sum_{k=0}^{m-1} \omega^{-jk} e_1(k-r+1) = \sum_{k=0}^{m-1} \omega^{-j(k+r-1)} e_1(k) = \omega^{-(r-1)j} w_1.$$

Let $\zeta = \exp(2\pi \boldsymbol{i}/r)$ and $v_i = \sum_{k=1}^{r-1} \omega^{(k-1)j} \zeta^{-(k-1)i} w_k + \omega^{(r-1)j} \zeta^{-(r-1)i} w'_r$. Then

$$\begin{aligned}
\tilde{\sigma}_* v_i &= \sum_{k=1}^{r-1} \omega^{(k-1)j} \zeta^{-(k-1)i} (\tilde{\sigma})_* w_k + \omega^{(r-1)j} \zeta^{-(r-1)i} (\tilde{\sigma})_* w'_r \\
&= \sum_{k=1}^{r-2} \omega^{(k-1)j} \zeta^{-(k-1)i} w_{k+1} + \omega^{(r-2)j} \zeta^{-(r-2)i} w'_r + \omega^{(r-1)j} \zeta^{-(r-1)i} \omega^{-pj} w_1 \\
&= \omega^{-j} \zeta^{i} \left(\sum_{k=1}^{r-1} \omega^{(k-1)j} \zeta^{-(k-1)i} w_k + \omega^{(r-1)j} \zeta^{-(r-1)i} w'_r \right) \\
&= (\omega^{-j} \zeta^{i}) v_i.
\end{aligned}$$

Hence v_i is an eigenvector with eigenvalue $\omega^{-j}\zeta^{i}$ with respect to the action by $\tilde{\sigma}$. Note that the subspace generated by $\{w_i\}_{i=1}^{r-1}$ coincides with one generated by $\{v_i\}_{i=1}^{r-1}$. Since $\tilde{\sigma}$ acts trivially on $\{w_i\}_{i=r+1}^{m-1}$, they are also eigenvectors with eigenvalue 0. Moreover the set $\{v_i\}_{i=1}^{r-1} \cup \{w_i\}_{i=r+1}^{m-2}$ is linearly independent.

Lemma 6.5. *Let i, i' be integers such that $1 \le i \le r-1$ and $1 \le i' \le r-1$. Then we have*

$$v_i \cdot v_{i'} = \begin{cases} 8rd\boldsymbol{i} \sin \dfrac{\pi i}{r} \sin \dfrac{\pi j}{m} \sin \pi \left(\dfrac{i}{r} - \dfrac{j}{m} \right) & \text{if } i = i', \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since the action of the mapping class group $\pi_0 C_g(t)$ preserves the intersection form,

$$\begin{aligned}
v_i \cdot v_i &= \sum_{k=0}^{r-1} \sum_{l=0}^{r-1} \omega^{(l-k)j} \zeta^{-(l-k)i} (\tilde{\sigma}_*^k w_1 \cdot \tilde{\sigma}_*^l w_1) \\
&= \sum_{k=0}^{r-1} \sum_{l=0}^{r-1} \omega^{(l-k)j} \zeta^{-(l-k)i} (w_2 \cdot \tilde{\sigma}_*^{l-k+1} w_1).
\end{aligned}$$

Thus Lemma 6.3 implies

$$\begin{aligned} v_i \cdot v_i &= \omega^{(r-1)j}\zeta^{-(r-1)i}(w_2 \cdot \tilde{\sigma}_*^r w_1) + \omega^j \zeta^{-i}(r-1)(w_2 \cdot \tilde{\sigma}_*^2 w_1) + r(w_2 \cdot \tilde{\sigma}_* w_1) \\ &\qquad + \omega^{-j}\zeta^i (r-1)(w_2 \cdot w_1) + \omega^{-(r-1)j}\zeta^{(r-1)i}(w_2 \cdot \tilde{\sigma}_*^{-r+2} w_1) \\ &= r\{(\omega^{-j}\zeta^i) w_2 \cdot w_1 + (\omega^j \zeta^{-i}) w_2 \cdot w_3 + w_2 \cdot w_2\} \\ &= 8rd\boldsymbol{i} \sin\frac{\pi i}{r} \sin\frac{\pi j}{m} \sin\pi\Big(\frac{i}{r} - \frac{j}{m}\Big). \end{aligned}$$

□

Calculation of ω-signatures and the cobounding functions $\phi_{m,j}$. Lastly we will calculate the hermitian form $\langle\ ,\ \rangle_{\tilde{\sigma}^k,\tilde{\sigma}}$ and the ω-signature. We have already found the set of eigenvectors $\{v_i\}_{i=1}^{r-1} \cup \{w_i\}_{i=r+1}^{m-2}$ with respect to the action by $\tilde{\sigma}$ which is linearly independent. Since $\dim V^{\omega^j} = m-2$, we need to find another eigenvector.

Lemma 6.6. $$\sum_{k=1}^{rm} \tau(\tilde{\sigma}^k, \tilde{\sigma}) = rm - 2|mi - rj|.$$

Proof. We first consider the case when rj/m is not an integer. Put

$$\beta = \sum_{i=1}^{r} w_i - \frac{1}{r}\sum_{k=1}^{r} \frac{1}{1-\omega^j\zeta^{-k}} v_k.$$

The subspace generated by $\{v_i\}_{i=1}^{r-1}$ and that generated by $\{w_i\}_{i=1}^{r-1}$ coincide. Thus the set $\{v_i\}_{i=1}^{r-1}$, β, $\{w_i\}_{i=r+1}^{m-2}$ forms a basis of V^{ω^j} when $1 \le r \le m-2$, and the set $\{v_i\}_{i=1}^{m-2}$ forms a basis of V^{ω^j} when $r = m-1$. We have

$$\begin{aligned} \tilde{\sigma}_*\beta &= \sum_{i=2}^{r} w_i - \frac{1}{r}\sum_{k=1}^{r} \frac{\omega^j\zeta^{-k}}{1-\omega^j\zeta^{-k}} v_k \\ &= \sum_{i=2}^{r} w_i + \frac{1}{r}\sum_{k=1}^{r} v_k - \frac{1}{r}\sum_{k=1}^{r} \frac{1}{1-\omega^j\zeta^{-k}} v_k \\ &= \sum_{i=1}^{r} w_i - \frac{1}{r}\sum_{k=1}^{r} \frac{1}{1-\omega^j\zeta^{-k}} v_k = \beta. \end{aligned}$$

Note that β and $\{w_i\}_{i=r+1}^{m-2}$ are in the annihilator of the hermitian form $\langle\ ,\ \rangle_{\tilde{\sigma}^k,\tilde{\sigma}}$ since they have eigenvalue 1 with respect to the action by $\tilde{\sigma}$.

By Lemma 4.2, we have

$$\begin{aligned} \tau(\tilde{\sigma}^k, \tilde{\sigma}) &= \sum_{i=1}^{r} \operatorname{sign}\langle v_i, v_i\rangle_{\tilde{\sigma}^k,\tilde{\sigma}} = -\sum_{i=1}^{r} \operatorname{sign}\Big((v_i \cdot v_i)\sum_{l=1}^{k}((\omega^{-j}\zeta^i)^l - (\omega^j\zeta^{-i})^l)\Big) \\ &= -\sum_{i=1}^{r} \operatorname{sign}\Big((v_i \cdot v_i)(1-\omega^j\zeta^{-i})(1-\omega^{-j}\zeta^i)\sum_{l=1}^{k}((\omega^{-j}\zeta^i)^l - (\omega^j\zeta^{-i})^l)\Big). \end{aligned}$$

By the equation

$$(1-\omega^j\zeta^{-i})(1-\omega^{-j}\zeta^{i})\sum_{l=1}^{k}((\omega^{-j}\zeta^{i})^l-(\omega^j\zeta^{-i})^l)$$
$$=8\boldsymbol{i}\sin\Big(-\frac{\pi(k+1)j}{m}+\frac{\pi(k+1)i}{r}\Big)\sin\Big(-\frac{\pi kj}{m}+\frac{\pi ki}{r}\Big)\sin\Big(-\frac{\pi j}{m}+\frac{\pi i}{r}\Big)$$

and Lemma 6.5, we have

$$\tau((\tilde{\sigma})^k,\tilde{\sigma})=\sum_{i=1}^{r-1}\operatorname{sign}\Big(\sin k\pi\Big(\frac{i}{r}-\frac{j}{m}\Big)\sin(k+1)\pi\Big(\frac{i}{r}-\frac{j}{m}\Big)\Big).$$

Since rj/m is not an integer, $i/r-j/m$ is not zero. Thus we obtain

$$\sum_{k=1}^{rm}\tau((\tilde{\sigma})^k,\tilde{\sigma})=\sum_{i=1}^{r-1}\sum_{k=1}^{rm}\operatorname{sign}\Big(\sin k\pi\Big(\frac{i}{r}-\frac{j}{m}\Big)\sin(k+1)\pi\Big(\frac{i}{r}-\frac{j}{m}\Big)\Big)$$
$$=\sum_{i=1}^{r-1}(rm-2|mi-rj|).$$

Next consider the case when rj/m is an integer and $1\le r\le m-1$. Denote this integer rj/m by i_0. Then, the eigenvalue of v_{i_0} is 1, and v_{i_0} and $\{w_i\}_{i=r+1}^{m-2}$ are in the annihilator of $\langle\ ,\ \rangle_{\tilde{\sigma}^k,\tilde{\sigma}}$. If we put

$$\beta'=\sum_{i=1}^{r}w_i-\frac{1}{r}\sum_{\substack{1\le k\le r\\ k\ne i_0}}\frac{1}{1-\omega^j\zeta^{-k}}v_k,$$

the set of the homology classes $\{v_i\}_{i=1}^{r-1}$, β', $\{w_i\}_{i=r+1}^{m-2}$ forms a basis of V^{ω^j}. We have

$$\tilde{\sigma}\beta'=\sum_{i=2}^{r}w_i-\frac{1}{r}\sum_{\substack{1\le k\le r\\ k\ne i_0}}\frac{\omega^j\zeta^{-k}}{1-\omega^j\zeta^{-k}}v_k$$
$$=\sum_{i=2}^{r}w_i+\frac{1}{r}\sum_{\substack{1\le k\le r\\ k\ne i_0}}v_k-\frac{1}{r}\sum_{\substack{1\le k\le r\\ k\ne i_0}}\frac{1}{1-\omega^j\zeta^{-k}}v_k$$
$$=\sum_{i=1}^{r}w_i-\frac{1}{r}v_{i_0}-\frac{1}{r}\sum_{\substack{1\le k\le r\\ k\ne i_0}}\frac{1}{1-\omega^j\zeta^{-k}}v_k=\beta'-\frac{1}{r}v_{i_0}.$$

By Lemma 4.2,

$$\langle\beta',\beta'\rangle_{\tilde{\sigma}^k,\tilde{\sigma}}=\beta'\cdot\frac{1}{r}\sum_{i=1}^{k}2i\,v_{i_0}=\frac{k(k+1)}{r}\sum_{i=1}^{r}w_i\cdot v_{i_0}.$$

Since the eigenvalues of $\{v_i\}_{i=1}^{r-1}$ are different from 1, the intersection $v_i \cdot v_{i_0} = 0$ for $1 \leq i \leq r-1$. Since the subspace generated by $\{w_i\}_{i=1}^{r-1}$ and that generated by $\{v_i\}_{i=1}^{r-1}$ coincide, we also have $w_i \cdot v_{i_0} = 0$. Thus we have

$$\begin{aligned}\frac{r}{k(k+1)}\langle \beta', \beta'\rangle_{\tilde{\sigma}^k,\tilde{\sigma}} &= w_r \cdot v_{i_0} = w_r \cdot (w_{r-1} + w_r') \\ &= w_r \cdot \left(w_{r-1} - \sum_{k=0}^{r-1} \omega^{(k-r)j} w_k \right) \\ &= (1-\omega^{-j}) w_r \cdot w_{r-1} = (1-\omega^{-j})(1-\omega^j) > 0.\end{aligned}$$

Moreover since $v_i \cdot v_{i_0} = 0$, Lemma 4.2 implies $\langle v_i, \beta'\rangle_{\tilde{\sigma}^k,\tilde{\sigma}} = 0$ for $1 \leq i \leq r-1$. Therefore we have

$$\begin{aligned}\sum_{k=1}^{rm} \tau(\tilde{\sigma}^k, \tilde{\sigma}) &= \sum_{k=1}^{rm}\left(\sum_{i=1}^{k} \operatorname{sign}(\langle v_i, v_i\rangle_{\tilde{\sigma}^k,\tilde{\sigma}}) + \operatorname{sign}(\langle \beta', \beta'\rangle_{\tilde{\sigma}^k,\tilde{\sigma}})\right) \\ &= \sum_{k=1}^{rm}\left(\sum_{\substack{1\leq i\leq r-1 \\ i\neq i_0}} \operatorname{sign}\left(\sin k\pi\left(\frac{i}{r} - \frac{j}{m}\right)\sin(k+1)\pi\left(\frac{i}{r}-\frac{j}{m}\right)\right) + 1\right) \\ &= \sum_{\substack{1\leq i\leq r-1 \\ i\neq i_0}} (rm - 2|mi - rj|) + rm = \sum_{i=1}^{r-1}(rm - 2|mi-rj|).\end{aligned}$$

In the case when $r = m$, the set $\{v_i\}_{i=1}^{r-2}$ forms a basis of V^{ω^j}. By a similar calculation, we can also prove what we want. $\square$

Lemma 6.7. *For $r = 2, 3, \ldots, m$,*

$$\phi_{m,j}(\tilde{\sigma}) - \bar{\phi}_{m,j}(\tilde{\sigma}) = \frac{2}{r}\left\{\left(\frac{rj}{m} - \left[\frac{rj}{m}\right] - \frac{1}{2}\right)^2 - \frac{r^2 j(m-j)}{m^2} - \frac{1}{4}\right\}.$$

Proof.
$$\tau(\tilde{\sigma}^k, \tilde{\sigma}) = \sum_{i=1}^{r-1} \operatorname{sign}\left(\sin k\pi\left(\frac{i}{r}-\frac{j}{m}\right)\sin(k+1)\pi\left(\frac{i}{r}-\frac{j}{m}\right)\right).$$

Since we have $\tau(\tilde{\sigma}^{k+rm}, \tilde{\sigma}) = \tau(\tilde{\sigma}^k, \tilde{\sigma})$,

$$\begin{aligned}\phi_{m,j}(\tilde{\sigma}) - \bar{\phi}_{m,j}(\tilde{\sigma}) &= \frac{1}{rm}\sum_{k=1}^{rm}\tau(\tilde{\sigma}^k,\tilde{\sigma}) = \frac{1}{rm}\sum_{i=1}^{r-1}(rm - 2|mi - rj|) \\ &= r - 1 - \frac{2}{rm}\left(\sum_{i=1}^{[\frac{rj}{m}]}(rj - mi) + \sum_{[\frac{rj}{m}]+1}^{r-1}(mi - rj)\right) \\ &= \frac{2}{r}\left\{\left(\frac{rj}{m} - \left[\frac{rj}{m}\right] - \frac{1}{2}\right)^2 + \frac{r^2 j(m-j)}{m^2} - \frac{1}{4}\right\}.\end{aligned}$$
$\square$

Proof of Theorem 1.1. Applying Lemma 6.7 to the case when $r = m$, we have

$$\phi_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}) - \bar{\phi}_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}) = \frac{2j(m-j)}{m}.$$

Since

$$\bar{\phi}_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}) = \frac{1}{m}\bar{\phi}_{m,j}((\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1})^m) = 0,$$

we have

$$\phi_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}) = \frac{2j(m-j)}{m}.$$

Put $\varphi = \tilde{\sigma}_1\tilde{\sigma}_3 \cdots \tilde{\sigma}_{m-1}$, $\psi = \tilde{\sigma}_2\tilde{\sigma}_4 \cdots \tilde{\sigma}_{m-2}$ when m is even, and $\varphi = \tilde{\sigma}_1\tilde{\sigma}_3 \cdots \tilde{\sigma}_{m-2}$, $\psi = \tilde{\sigma}_2\tilde{\sigma}_4 \cdots \tilde{\sigma}_{m-1}$, when m is odd. As we saw in Section 5, $\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}$ is conjugate to $\varphi\psi$. By direct computation, if $(\varphi_*^{-1} - I_{2g})x + (\psi_* - I_{2g})y = 0$ for $x, y \in V^{\omega^j}$, we have $(\varphi_*^{-1} - I_{2g})x = (\psi_* - I_{2g})y = 0$. Hence we have $\tau_g(\varphi, \psi) = 0$.

In the same way, for $i = 1, 2, \ldots, [(m-1)/2]$, we have

$$\tau_g(\tilde{\sigma}_1\tilde{\sigma}_3 \cdots \tilde{\sigma}_{2i+1}, \tilde{\sigma}_2\tilde{\sigma}_4 \cdots \tilde{\sigma}_{2i}) = \tau_g(\tilde{\sigma}_1\tilde{\sigma}_3 \cdots \tilde{\sigma}_{2i+1}, \tilde{\sigma}_2\tilde{\sigma}_4 \cdots \tilde{\sigma}_{2i+2}) = 0,$$
$$\tau_g(\tilde{\sigma}_1\tilde{\sigma}_3 \cdots \tilde{\sigma}_{2i-1}, \tilde{\sigma}_{2i+1}) = \tau_g(\tilde{\sigma}_2\tilde{\sigma}_4 \cdots \tilde{\sigma}_{2i}, \tilde{\sigma}_{2i+2}) = 0.$$

Thus

$$\phi_{m,j}(\tilde{\sigma}) = (r-1)\phi_{m,j}(\tilde{\sigma}_1) = \frac{r-1}{m-1}\phi_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}) = \frac{2(r-1)j(m-j)}{m(m-1)}.$$

Hence we obtain

$$\begin{aligned}
\bar{\phi}_{m,j}(\sigma_1 \cdots \sigma_{r-1}) &= \bar{\phi}_{m,j}(\tilde{\sigma}) \\
&= \phi_{m,j}(\tilde{\sigma}) - (\phi_{m,j}(\tilde{\sigma}) - \bar{\phi}_{m,j}(\tilde{\sigma})) \\
&= -\frac{2}{r}\left\{\frac{jr(m-j)(m-r)}{m^2(m-1)} + \left(\frac{rj}{m} - \left[\frac{rj}{m}\right] - \frac{1}{2}\right)^2 - \frac{1}{4}\right\}. \qquad \square
\end{aligned}$$

By the values of $\bar{\phi}_{m,1}$, we see:

Remark 6.8. Let r be an integer such that $2 \le r \le m$. Then

$$\bar{\phi}_{m,1}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{r-1}) = 0.$$

However we do not know whether the quasimorphism $\bar{\phi}_{m,1}$ is trivial or not.

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