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MAXIMAL ESTIMATES FOR SCHRÖDINGER EQUATIONS WITH INVERSE-SQUARE POTENTIAL

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We consider maximal estimates for the solution to an initial value problem of the linear Schrödinger equation with a singular potential. We show a result about the pointwise convergence of solutions to this special variable coefficient Schrödinger equation with initial data $u_0 \in H^s(\mathbb{R}^n)$ for $s > \frac{1}{2}$ or radial initial data $u_0 \in H^s(\mathbb{R}^n)$ for $s \geqslant \frac{1}{4}$ and that the solution does not converge when $s < \frac{1}{4}$.

1. Introduction and statement of main result

We study the maximal estimates for the solution to an initial value problem of the linear Schrödinger equation with an inverse square potential. More precisely, we consider the Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u + \dfrac{a}{|x|^2}u = 0 & (t,x) \in \mathbb{R}\times\mathbb{R}^n \setminus \{0\},\ a > -(n-2)^2/4, \\ u(x,0) = u_0(x). \end{cases} \tag{1-1}$$

The scale-covariance elliptic operator $P_a := -\Delta + a/|x|^2$ appearing in (1-1) plays a key role in many problems of physics and geometry. The heat and wave flows for the elliptic operator P_a have been studied in the theory of combustion (see [Vazquez and Zuazua 2000]) and in wave propagation on conic manifolds (see [Cheeger and Taylor 1982]). The Schrödinger equation (1-1) arises in the study of quantum mechanics [Kalf et al. 1975]. There has been a lot of interest in developing Strichartz estimates both for the Schrödinger and wave equations with the inverse square potential; we refer the reader to [Burq et al. 2003; 2004; Planchon et al. 2003a; 2003b; Miao et al. 2013]. However, as far as we know, there are few results about the maximal estimates associated with the operator P_a, which arises in the study of pointwise convergence problem for the Schrödinger and wave equations with the inverse square potential. In this paper, we aim to address some maximal estimates in the special settings associated with the operator P_a. As a

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direct consequence, we obtain the pointwise convergence result for $u_0 \in H^s(\mathbb{R}^n)$ with $s > \frac{1}{2}$.

In the case of the free Schrödinger equation without potential, i.e., $a = 0$, there is a large amount of literature on developing maximal estimates for its solution, which can be formally written as

$$u(t,x) = e^{it\Delta}u_0(x) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi - t|\xi|^2)}\hat{u}_0(\xi)\,\mathrm{d}\xi.$$

When $n = 1$, Carleson [1980] proved that the convergence result holds in the sense that $\lim_{t\to 0} u(t) = u_0$ for a.e. x when $u_0 \in H^s(\mathbb{R})$ with $s \geqslant \frac{1}{4}$. Dahlberg and Kenig [1982] showed that the result is sharp in the sense that the solution does not converge when $s < \frac{1}{4}$. When $n \geqslant 2$, Sjölin [1987] and Vega [1988] independently proved convergence results when $u_0 \in H^s(\mathbb{R}^n)$ with $s > \frac{1}{2}$. It follows from the construction of [Dahlberg and Kenig 1982; Vega 1988] that the solution does not converge when $s < \frac{1}{4}$. When $n = 2$, Bourgain [1995] showed that there is a certain $s < \frac{1}{2}$ such that the convergence result holds, and this result was improved by Moyua, Vargas and Vega [Moyua et al. 1996]. Having shown the bilinear restriction estimates for paraboloids, Tao and Vargas [2000] and Tao [2003] showed convergence for $s > \frac{15}{32}$ and $s > \frac{2}{5}$ respectively. This was improved further to $s > \frac{3}{8}$ in [Lee 2006; Shao 2010]. Very recently, Bourgain [2013] made some progress in high dimension $n \geqslant 2$ to show that the convergence result holds for $s > \frac{1}{2} - \frac{1}{4n}$ when $n \geqslant 1$ and that the convergence result needs $s \geqslant (n-2)/(2n)$ when $n \geqslant 5$.

In the situation when $a \neq 0$, (1-1) can be viewed as a special Schrödinger equation with variable singular coefficients. The potential prevents us from using the Fourier transform to give the expression of the solution. With the motivation of regarding the potential term as a perturbation on angular direction in [Burq et al. 2003; Planchon et al. 2003b; Miao et al. 2013], we express the solution by using the Hankel transform of radial functions and spherical harmonics. Instead of the Fourier transform, we utilize the Hankel transform and modify the argument of [Vega 1988] to show that the pointwise convergence result holds when the initial data $u_0 \in H^s(\mathbb{R}^n)$ for $s > \frac{1}{2}$, or when radial initial data $u_0 \in H^s(\mathbb{R}^n)$ for $s \geqslant \frac{1}{4}$, and that the solution does not converge when $s < \frac{1}{4}$.

Let u be the solution to (1-1); we define the maximal function by

$$u^*(x) = \sup_{|t|>0} |u(x,t)|. \tag{1-2}$$

Our main theorems are the following.

Theorem 1.1. *Let $\beta > 1$, $n \geqslant 2$ and $s > \frac{1}{2}$. Then*

$$\int_{\mathbb{R}^n} |u^*(x)|^2 \frac{\mathrm{d}x}{(1+|x|)^\beta} \leqslant C\|u_0\|^2_{H^s(\mathbb{R}^n)}. \tag{1-3}$$

As a direct consequence of Theorem 1.1, we have:

Corollary 1.1. *Let $u_0 \in H^s(\mathbb{R}^n)$ with $s > \frac{1}{2}$ and $n \geqslant 2$. Then*

$$\lim_{t\to 0} u(t,x) = u_0(x), \quad a.e.\ x \in \mathbb{R}^n. \tag{1-4}$$

Theorem 1.2. *Let B^n be the open unit ball in $\mathbb{R}^n$. Assume that there exists a constant C independent of u_0 such that*

$$\int_{B^n} |u^*(x)|^2\,\mathrm{d}x \leqslant C\|u_0\|^2_{H^s(\mathbb{R}^n)} \quad \text{for all } u_0(x) \in H^s(\mathbb{R}^n). \tag{1-5}$$

Then $s \geqslant \frac{1}{4}$.

With this in mind, Theorem 1.1 is far from being sharp. Assuming that the initial data possesses additional angular regularity, we have:

Theorem 1.3. *Let B^n be the open unit ball in $\mathbb{R}^n$ and $\epsilon > 0$. There exists a constant C independent of u_0 such that*

$$\int_{B^n} |u^*(x)|^2\,\mathrm{d}x \leqslant C\|u_0\|^2_{H_r^{\frac{1}{4}} H_\theta^{\frac{n-1}{2}+\epsilon}}, \tag{1-6}$$

where for $s, s' \geqslant 0$,

$$H_r^s H_\theta^{s'} = \Big\{ g : \|g\|_{H_r^s H_\theta^{s'}} := \big\|(1-\Delta_\theta)^{\frac{s'}{2}}((1-\Delta)^{\frac{s}{2}} g)\big\|_{L^2_{r^{n-1}\mathrm{d}r}(\mathbb{R}^+;L^2_\theta(\mathbb{S}^{n-1}))} < \infty \Big\}.$$

Here Δ_θ denotes the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$.

Remark 1.1. i) This result implies that the pointwise convergence of solutions to (1-1) holds for radial initial data $u_0 \in H^s(\mathbb{R}^n)$ with $s \geqslant \frac{1}{4}$.

ii) This result is an analogue of [Cho et al. 2006, Theorem 1.1]. We remark that the parameter ϵ there should be corrected to $\epsilon > \frac{1}{2}$ rather than $\epsilon > 0$. Thus, we generalize and improve the result of Cho et al. by making use of a finer result proved in [Gigante and Soria 2008].

Now we introduce some notation. We use $A \lesssim B$ to denote the statement that $A \leqslant CB$ for some large constant C which may vary from line to line and depend on various parameters; and similarly use $A \ll B$ to denote the statement $A \leqslant C^{-1}B$. We employ $A \sim B$ to denote the statement that $A \lesssim B \lesssim A$. If the constant C depends on a special parameter other than the above, we shall denote it explicitly by subscripts. We briefly write $A+\epsilon$ as $A+$ or $A-\epsilon$ as $A-$ for $0 < \epsilon \ll 1$. Throughout this paper, pairs of conjugate indices are written as p, p', where $1/p + 1/p' = 1$ with $1 \leqslant p \leqslant \infty$.

This paper is organized as follows. In Section 2, we mainly revisit the properties of the Bessel functions and the Hankel transform associated with $-\Delta + a/|x|^2$. Section 3 is devoted to the proofs of the theorems.

2. Preliminaries

We list some results about the Hankel transform and the Bessel functions and then show a characterization of the Sobolev norm in the Hankel transform version.

We begin by recalling the expansion formula with respect to spherical harmonics. For details, we refer to [Stein and Weiss 1971]. For the sake of convenience, let

$$\xi=\rho\omega \quad \text{and} \quad x=r\theta \quad \text{with } \omega,\theta\in\mathbb{S}^{n-1}. \tag{2-1}$$

For any $g\in L^2(\mathbb{R}^n)$, the expansion formula with respect to the spherical harmonics yields

$$g(x)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} a_{k,\ell}(r)Y_{k,\ell}(\theta),$$

where

$$\{Y_{k,1},\dots,Y_{k,d(k)}\}$$

is the orthogonal basis of the space of spherical harmonics of degree k on $\mathbb{S}^{n-1}$, called $\mathcal{H}^k$, having dimension

$$d(k)=\frac{2k+n-2}{k}C_{n+k-3}^{k-1}\simeq\langle k\rangle^{n-2}.$$

We remark that for $n=2$, the dimension of $\mathcal{H}^k$ is independent of k. Obviously, we have the orthogonal decomposition

$$L^2(\mathbb{S}^{n-1})=\bigoplus_{k=0}^{\infty}\mathcal{H}^k.$$

By orthogonality, it gives

$$\|g(x)\|_{L^2_\theta(\mathbb{S}^{n-1})}=\|a_{k,\ell}(r)\|_{\ell^2_{k,\ell}}. \tag{2-2}$$

From $-\Delta_\theta Y_{k,\ell}(\theta)=k(k+n-2)Y_{k,\ell}(\theta)$, the fractional power of $1-\Delta_\theta$ can be written explicitly [Machihara et al. 2005] as

$$(1-\Delta_\theta)^{\frac{s}{2}}g(x)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}(1+k(k+n-2))^{\frac{s}{2}}a_{k,\ell}(r)Y_{k,\ell}(\theta). \tag{2-3}$$

We will need the Fourier transform of $a_{k,\ell}(r)Y_{k,\ell}(\theta)$. Theorem 3.10 of [Stein and Weiss 1971] asserts the Hankel transform formula

$$\hat{g}(\rho\omega)\sim\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} i^kY_{k,\ell}(\omega)\rho^{-\frac{n-2}{2}}\int_0^\infty J_{k+\frac{n-2}{2}}(2\pi r\rho)a_{k,\ell}(r)r^{\frac{n}{2}}\,dr. \tag{2-4}$$

Here the Bessel function $J_k(r)$ of order k is defined by the integral

$$J_k(r)=\frac{(\frac{r}{2})^k}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1}e^{isr}(1-s^2)^{\frac{2k-1}{2}}\,ds\quad\text{with } k>-\tfrac{1}{2}\text{ and } r>0.$$

A simple computation gives the rough estimates

$$|J_k(r)|\leqslant\frac{Cr^k}{2^k\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})}\left(1+\frac{1}{k+\frac{1}{2}}\right),\tag{2-5}$$

where C is an absolute constant. This estimate will be mainly used when $r\lesssim 1$. Another well-known asymptotic expansion for the Bessel function is

$$J_k(r)=r^{-\frac{1}{2}}\sqrt{\frac{2}{\pi}}\cos\left(r-\frac{k\pi}{2}-\frac{\pi}{4}\right)+O_k(r^{-\frac{3}{2}})\quad\text{as } r\to\infty,\tag{2-6}$$

but with a constant depending on k (see [Stein and Weiss 1971]). As pointed out in [Stein 1993], if one seeks a uniform bound for large r and k, the best one can do is $|J_k(r)|\leqslant Cr^{-\frac{1}{3}}$. One will find that this decay doesn't lead to the desired result.

We now recall the properties of Bessel function $J_k(r)$ in [Stein 1993; Stempak 2000].

Lemma 2.1 (asymptotics of the Bessel function). *Assume that $k\in\mathbb{N}$ and $k\gg 1$. Let $J_k(r)$ be the Bessel function of order k defined as above. There exist a large constant C and small constant c independent of k and r satisfying these conditions*:

- *When $r\leqslant\frac{k}{2}$,*

$$|J_k(r)|\leqslant Ce^{-c(k+r)}.\tag{2-7}$$

- *When $\frac{k}{2}\leqslant r\leqslant 2k$,*

$$|J_k(r)|\leqslant Ck^{-\frac{1}{3}}(k^{-\frac{1}{3}}|r-k|+1)^{-\frac{1}{4}}.\tag{2-8}$$

- *When $r\geqslant 2k$,*

$$J_k(r)=r^{-\frac{1}{2}}\sum_{\pm}a_{\pm}(r,k)e^{\pm ir}+E(r,k),\tag{2-9}$$

where $|a_{\pm}(r,k)|\leqslant C$ and $|E(r,k)|\leqslant Cr^{-1}$.

As a consequence of Lemma 2.1, we have:

Lemma 2.2. *Let $R\gg 1$. There exists a constant C independent of k, R such that*

$$\int_R^{2R}|J_k(r)|^2\,\mathrm{d}r\leqslant C.\tag{2-10}$$

Proof. To prove (2-10), we write

$$\int_R^{2R} |J_k(r)|^2\,\mathrm{d}r = \int_{I_1} |J_k(r)|^2\,\mathrm{d}r + \int_{I_2} |J_k(r)|^2\,\mathrm{d}r + \int_{I_3} |J_k(r)|^2\,\mathrm{d}r$$

where $I_1 = [R, 2R] \cap [0, \frac{k}{2}]$, $I_2 = [R, 2R] \cap \left[\frac{k}{2}, 2k\right]$ and $I_3 = [R, 2R] \cap [2k, \infty]$. By (2-7) and (2-9), we have

$$\int_{I_1} |J_k(r)|^2\,\mathrm{d}r \leqslant C \int_{I_1} e^{-cr}\,\mathrm{d}r \leqslant Ce^{-cR}, \tag{2-11}$$

and

$$\int_{I_3} |J_k(r)|^2\,\mathrm{d}r \leqslant C. \tag{2-12}$$

On the other hand, one has by (2-8)

$$\int_{[\frac{k}{2},2k]} |J_k(r)|^2\,\mathrm{d}r \leqslant C \int_{[\frac{k}{2},2k]} k^{-\frac{2}{3}}(1 + k^{-\frac{1}{3}}|r-k|)^{-\frac{1}{2}}\,\mathrm{d}r \leqslant C.$$

Observing that $[R, 2R] \cap [\frac{k}{2}, 2k] = \varnothing$ unless $R \sim k$, we obtain

$$\int_{I_2} |J_k(r)|^2\,\mathrm{d}r \leqslant C. \tag{2-13}$$

This together with (2-11) and (2-12) yields (2-10). □

For simplicity, we define

$$\mu(k) = \frac{n-2}{2} + k \quad \text{and} \quad \nu(k) = \sqrt{\mu^2(k) + a} \quad \text{with } a > -(n-2)^2/4. \tag{2-14}$$

We sometime write ν instead of $\nu(k)$. Let f be a Schwartz function defined on $\mathbb{R}^n$. We define the Hankel transform of order ν by

$$(\mathcal{H}_\nu f)(\xi) = \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) f(r\omega) r^{n-1}\,\mathrm{d}r, \tag{2-15}$$

where $\rho = |\xi|$, $\omega = \xi/|\xi|$ and J_ν is the Bessel function of order ν. In particular, if the function f is radial, then we have

$$(\mathcal{H}_\nu f)(\rho) = \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) f(r) r^{n-1}\,\mathrm{d}r. \tag{2-16}$$

If $f(x) = \sum\limits_{k=0}^{\infty} \sum\limits_{\ell=1}^{d(k)} a_{k,\ell}(r) Y_{k,\ell}(\theta)$, it follows from (2-4) that

$$\hat{f}(\xi) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} 2\pi i^k Y_{k,\ell}(\omega) \big(\mathcal{H}_{\mu(k)} a_{k,\ell}\big)(\rho). \tag{2-17}$$

The following properties of the Hankel transform are obtained in [Burq et al. 2003; Planchon et al. 2003b].

Lemma 2.3. *Let $\mathcal{H}_\nu$ be as above and set*

$$A_{\nu(k)} := -\partial_r^2 - \frac{n-1}{r}\partial_r + \Big[\nu^2(k) - \Big(\frac{n-2}{2}\Big)^2\Big] r^{-2}.$$

(i) $\mathcal{H}_\nu = \mathcal{H}_\nu^{-1}$.

(ii) $\mathcal{H}_\nu$ *is self-adjoint, i.e.,* $\mathcal{H}_\nu = \mathcal{H}_\nu^*$.

(iii) $\mathcal{H}_\nu$ *is an* L^2 *isometry, i.e.,* $\|\mathcal{H}_\nu \phi\|_{L^2_\xi} = \|\phi\|_{L^2_x}$.

(iv) $\mathcal{H}_\nu(A_\nu \phi)(\xi) = |\xi|^2 (\mathcal{H}_\nu \phi)(\xi)$, *for* $\phi \in L^2$.

We next recall an almost orthogonality inequality. Denote by P_j and $\tilde{P}_j$ the usual dyadic frequency localization at $|\xi| \sim 2^j$ and the localization with respect to $(-\Delta + a/|x|^2)^{\frac{1}{2}}$. We define the projectors $M_{jj'} = P_j \tilde{P}_{j'}$ and $N_{jj'} = \tilde{P}_j P_{j'}$. More precisely, let f be in the k-th harmonic subspace; then

$$P_j f = \mathcal{H}_{\mu(k)} \beta_j \mathcal{H}_{\mu(k)} f \quad \text{and} \quad \tilde{P}_j f = \mathcal{H}_{\nu(k)} \beta_j \mathcal{H}_{\nu(k)} f,$$

where $\beta_j(\xi) = \beta(2^{-j}|\xi|)$ with $\beta \in C_0^\infty(\mathbb{R}^+)$ supported in $\big[\frac{1}{2}, 2\big]$.

Lemma 2.4 (almost orthogonality inequality [Burq et al. 2003]). *Let $f \in L^2(\mathbb{R}^n)$. There exists a constant C independent of j, j' such that*

$$\|M_{jj'} f\|_{L^2(\mathbb{R}^n)}, \|N_{jj'} f\|_{L^2(\mathbb{R}^n)} \leqslant C 2^{-\epsilon|j-j'|} \|f\|_{L^2(\mathbb{R}^n)}, \tag{2-18}$$

where $\epsilon < 1 + \min\big\{\frac{n-2}{2}, \big(\frac{(n-2)^2}{4} + a\big)^{\frac{1}{2}}\big\}$.

As a consequence, we have:

Lemma 2.5. *Let $f \in L^2(\mathbb{R}^n)$ be given by*

$$f(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}(r) Y_{k,\ell}(\theta).$$

Then for $0 \leqslant s < 1 + \min\big\{\frac{n-2}{2}, \big(\frac{(n-2)^2}{4} + a\big)^{\frac{1}{2}}\big\}$ *and* $s' \geqslant 0$,

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{M \in 2^{\mathbb{Z}}} M^{2s} (1+k)^{2s'} \Big\| b_{k,\ell}(\rho) \chi\Big(\frac{\rho}{M}\Big) \rho^{\frac{n-1}{2}} \Big\|_{L^2_\rho}^2 \sim \|f\|^2_{\dot{H}^s_r H^{s'}_\theta}, \tag{2-19}$$

where $b_{k,\ell}(\rho) = (\mathcal{H}_{\nu(k)} a_{k,\ell})(\rho)$ *and* $\chi \in C_0^\infty(\mathbb{R}^n)$ *such that* $\operatorname{supp} \chi \subset \big[\frac{1}{2}, 1\big]$.

Proof. Note that $-\Delta_\theta Y_{k,\ell} = k(k+n-2) Y_{k,\ell}$. By Lemma 2.3, we have

$$\begin{aligned}\|f\|^2_{\dot H^0_r H^{s'}_\theta} &\sim \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}(1+k)^{2s'}\|a_{k,\ell}(r)\|^2_{L^2_{r^{n-1}\,\mathrm{d}r}(\mathbb{R}^+)}\|Y_{k,\ell}(\theta)\|^2_{L^2_\theta}\\ &\sim \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}(1+k)^{2s'}\|b_{k,\ell}(\rho)\|^2_{L^2_{\rho^{n-1}\,\mathrm{d}\rho}(\mathbb{R}^+)}.\end{aligned}$$

By (2-3), it suffices to show (2-19) with $s'=0$. By Lemma 2.3, we have

$$\begin{aligned}\Big\|b_{k,\ell}(\rho)\chi\big(\tfrac{\rho}{M}\big)\rho^{\frac{n-1}{2}}\Big\|_{L^2_\rho} &= \Big\|\chi\big(\tfrac{\rho}{M}\big)\mathcal{H}_\nu[Y_{k,l}(\theta)a_{k,\ell}(r)](\xi)\Big\|_{L^2_\xi}\\ &= \Big\|\mathcal{H}_\nu\big[\chi\big(\tfrac{\rho}{M}\big)\mathcal{H}_\nu(Y_{k,l}(\theta)a_{k,\ell}(r))(\xi)\big]\Big\|_{L^2_x}.\end{aligned}$$

This yields, by letting $j=\log_2 M$,

$$\begin{aligned}\Big\|b_{k,\ell}(\rho)\chi\big(\tfrac{\rho}{M}\big)\rho^{\frac{n-1}{2}}\Big\|_{L^2_\rho} &= \Big\|\big[\mathcal{H}_\nu\chi\big(\tfrac{\rho}{M}\big)\mathcal{H}_\nu\big](Y_{k,l}(\theta)a_{k,\ell}(r))\Big\|_{L^2_x}\\ &= \big\|\tilde P_j(Y_{k,l}(\theta)a_{k,\ell}(r))\big\|_{L^2_x}.\end{aligned}$$

Let $g_{k,\ell}(x)=Y_{k,l}(\theta)a_{k,\ell}(r)$ and $\overline{P_{j'}}=P_{j'-1}+P_{j'}+P_{j'+1}$. We have by the triangle inequality and Lemma 2.4

$$\begin{aligned}\text{LHS of (2-19)} &= \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\sum_{j\in\mathbb{Z}}2^{2sj}\big\|\tilde P_j g_{k,\ell}\big\|^2_{L^2_x}\\ &\lesssim \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\sum_{j\in\mathbb{Z}}2^{2sj}\Big(\sum_{j'}\big\|\tilde P_j\overline{P_{j'}}P_{j'}g_{k,\ell}\big\|_{L^2_x}\Big)^2\\ &\lesssim \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\sum_{j\in\mathbb{Z}}2^{2sj}\Big(\sum_{j'}2^{-\epsilon|j-j'|}\big\|P_{j'}g_{k,\ell}\big\|_{L^2_x}\Big)^2,\end{aligned}$$

where $s<\epsilon<1+\min\big\{\frac{n-2}{2},\big(\frac{(n-2)^2}{4}+a\big)^{\frac12}\big\}$. Let $0<\epsilon_1\ll 1$ be such that $\epsilon_2:=\epsilon-\epsilon_1>s$; then the LHS of (2-19) is bounded above by

$$\begin{aligned}&C\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\sum_{j\in\mathbb{Z}}2^{2js}\sum_{j'}2^{-2\epsilon_2|j-j'|}\|P_{j'}g_{k,\ell}\|^2_{L^2(\mathbb{R}^n)}\sum_{j'}2^{-2\epsilon_1|j-j'|}\\ &\qquad\leqslant C\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\sum_{j'}2^{2j's}\sum_{j\in\mathbb{Z}}2^{2js}2^{-2\epsilon_2|j|}\|P_{j'}g_{k,\ell}\|^2_{L^2(\mathbb{R}^n)}\\ &\qquad\leqslant C\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\sum_{j'}2^{2j's}\|P_{j'}g_{k,\ell}\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

By the definition of $P_{j'}$, Lemma 2.3 and (2-17), we have

$$\begin{aligned}\text{LHS of (2-19)} &\leqslant C\sum_{j'} 2^{2j's}\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\Big\|\chi\Big(\frac{\rho}{2^{j'}}\Big)[\mathcal{H}_{\mu(k)}a_{k,\ell}](\rho)\rho^{\frac{n-1}{2}}\Big\|^2_{L^2(\mathbb{R}^+)}\\ &= C\sum_{j'} 2^{2j's}\Big\|\chi\Big(\frac{\rho}{2^{j'}}\Big)\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} 2\pi i^k[\mathcal{H}_{\mu(k)}a_{k,\ell}](\rho)Y_{k,\ell}(\omega)\Big\|^2_{L^2(\mathbb{R}^n)}\\ &= C\sum_{j'} 2^{2j's}\Big\|\chi\Big(\frac{\rho}{2^{j'}}\Big)\hat{f}\Big\|^2_{L^2(\mathbb{R}^n)} \sim \|f\|^2_{\dot{H}^s}.\end{aligned}$$

We can use a similar argument to prove

$$\text{LHS of (2-19)} \geqslant c\|f\|^2_{\dot{H}^s}.$$

This concludes the proof of Lemma 2.4. □

3. Proof of the main theorems

In this section, we first use the spherical harmonic expansion to write the solution as a linear combination of products of the Hankel transform of radial functions and spherical harmonics. We prove the main theorems by analyzing a property of the Hankel transform. The key ingredients are to use the stationary phase argument and to exploit the asymptotic behavior of the Bessel function.

The expression of the solution. Consider the following Cauchy problem:

$$\begin{cases} i\partial_t u - \Delta u + \dfrac{a}{|x|^2}u = 0,\\ u(x,0) = u_0(x).\end{cases} \tag{3-1}$$

We use the spherical harmonic expansion to write

$$u_0(x) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} a^0_{k,\ell}(r)Y_{k,\ell}(\theta). \tag{3-2}$$

Let us consider (3-1) in polar coordinates. Write $v(t,r,\theta) = u(t,r\theta)$ and $g(r,\theta) = u_0(r\theta)$. Then $v(t,r,\theta)$ satisfies

$$\begin{cases} i\partial_t v - \partial_{rr}v - \dfrac{n-1}{r}\partial_r v - \dfrac{1}{r^2}\Delta_\theta v + \dfrac{a}{r^2}v = 0,\\ v(0,r,\theta) = g(r,\theta).\end{cases} \tag{3-3}$$

By (3-2), we have

$$g(r,\theta) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} a^0_{k,\ell}(r)Y_{k,\ell}(\theta).$$

Using separation of variables, we can write v as a linear combination of products of radial functions and spherical harmonics:

$$v(t,r,\theta)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} v_{k,\ell}(t,r)Y_{k,\ell}(\theta), \tag{3-4}$$

where $v_{k,\ell}$ is given by

$$\begin{cases} i\partial_t v_{k,\ell}-\partial_{rr}v_{k,\ell}-\dfrac{n-1}{r}\partial_r v_{k,\ell}+\dfrac{k(k+n-2)+a}{r^2}v_{k,\ell}=0,\\ v_{k,\ell}(0,r)=a^0_{k,\ell}(r)\end{cases}$$

for each $k,\ell\in\mathbb{N}$, $1\leqslant \ell\leqslant d(k)$. Then we can rewrite the above by the definition of $A_{\nu(k)}$ as

$$\begin{cases} i\partial_t v_{k,\ell}+A_{\nu(k)}v_{k,\ell}=0,\\ v_{k,\ell}(0,r)=a^0_{k,\ell}(r).\end{cases} \tag{3-5}$$

Applying the Hankel transform to (3-5), by Lemma 2.3(iv), we have

$$\begin{cases} i\partial_t \tilde{v}_{k,\ell}+\rho^2\tilde{v}_{k,\ell}=0,\\ \tilde{v}_{k,\ell}(0,\rho)=b^0_{k,\ell}(\rho),\end{cases} \tag{3-6}$$

where

$$\tilde{v}_{k,\ell}(t,\rho)=(\mathcal{H}_\nu v_{k,\ell})(t,\rho),\quad b^0_{k,\ell}(\rho)=(\mathcal{H}_\nu a^0_{k,\ell})(\rho). \tag{3-7}$$

Solving this ODE and inverting the Hankel transform, we obtain

$$\begin{aligned} v_{k,\ell}(t,r)&=\int_0^\infty (r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)\tilde{v}_{k,\ell}(t,\rho)\rho^{n-1}\,\mathrm{d}\rho\\ &=\int_0^\infty (r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)e^{it\rho^2}b^0_{k,\ell}(\rho)\rho^{n-1}\,\mathrm{d}\rho. \end{aligned}$$

Therefore we get

$$\begin{aligned} u(x,t)&=v(t,r,\theta)\\ &=\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta)\int_0^\infty (r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)e^{it\rho^2}b^0_{k,\ell}(\rho)\rho^{n-1}\,\mathrm{d}\rho\\ &=\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta)\mathcal{H}_{\nu(k)}\big[e^{it\rho^2}b^0_{k,\ell}(\rho)\big](r). \end{aligned} \tag{3-8}$$

Proof of Theorem 1.1. By the Sobolev embedding $\dot H^{\frac12-}(\mathbb{R})\cap \dot H^{\frac12+}(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$, it suffices to show:

Proposition 3.1. *Let $\alpha \geqslant \frac12-\frac{\beta}{4}$ and $\beta=1+$ be such that*

$$2\alpha-1+\frac{\beta}{2}<1+\min\Big\{\frac{n-2}{2},\Big(\frac{(n-2)^2}{4}+a\Big)^{\frac12}\Big\}.$$

There exists a constant C independent of u_0 such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}}|\partial_t^\alpha u(x,t)|^2\frac{\mathrm{d}t\,\mathrm{d}x}{(1+|x|)^\beta}\leqslant C\|u_0\|^2_{\dot H^{2\alpha-1+\frac{\beta}{2}}(\mathbb{R}^n)}. \tag{3-9}$$

Proof. By the Plancherel theorem with respect to time t, we obtain

$$\text{LHS of (3-9)}=\int_{\mathbb{R}^n}\int_{\mathbb{R}}\Big|\tau^\alpha\int_{\mathbb{R}}e^{-it\tau}u(x,t)\,\mathrm{d}t\Big|^2\frac{\mathrm{d}\tau\,\mathrm{d}x}{(1+|x|)^\beta}.$$

Using (3-8), this is bounded above by

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}}\Big|\tau^\alpha\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}Y_{k,\ell}(\theta)\int_{\mathbb{R}}\int_0^\infty(r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)e^{it(\rho^2-\tau)}b^0_{k,\ell}(\rho)\rho^{n-1}\,\mathrm{d}\rho\,\mathrm{d}t\Big|^2\\ &\qquad\qquad\times\frac{\mathrm{d}\tau\,\mathrm{d}x}{(1+|x|)^\beta}\\ &\lesssim\int_{\mathbb{R}^{n+1}}\Big|\tau^\alpha\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}Y_{k,\ell}(\theta)\int_0^\infty(r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)b^0_{k,\ell}(\rho)\rho^{n-1}\delta(\tau-\rho^2)\,\mathrm{d}\rho\Big|^2\\ &\qquad\qquad\times\frac{\mathrm{d}\tau\,\mathrm{d}x}{(1+|x|)^\beta}\\ &\lesssim\int_{\mathbb{R}^n}\int_0^\infty\Big|\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}Y_{k,\ell}(\theta)\rho^\alpha(r\sqrt{\rho})^{-\frac{n-2}{2}}J_{\nu(k)}(r\sqrt{\rho})b^0_{k,\ell}(\sqrt{\rho})\rho^{\frac{n-1}{2}}\rho^{-\frac12}\Big|^2\\ &\qquad\qquad\times\frac{\mathrm{d}\rho\,\mathrm{d}x}{(1+|x|)^\beta}.\end{aligned}$$

By orthogonality, therefore, the LHS of (3-9) is

$$\lesssim\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\int_0^\infty\int_0^\infty\big|\rho^{2\alpha+\frac12}(r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)b^0_{k,\ell}(\rho)\rho^{n-2}\big|^2\frac{\mathrm{d}\rho\,r^{n-1}\,\mathrm{d}r}{(1+r)^\beta}.$$

Let χ be a smoothing function equals 1 in $\big[1,\frac32\big]$ and vanishes outside $\big[\frac12,2\big]$. For our purpose, we make a dyadic decomposition to obtain

LHS of (3-9)

$$\lesssim \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\sum_{M\in 2^{\mathbb{Z}}}\int_0^\infty\int_0^\infty\Big|\rho^{2\alpha+\frac12}(r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)b^0_{k,\ell}(\rho)\rho^{n-2}\chi\Big(\frac{\rho}{M}\Big)\Big|^2 \times\frac{r^{n-1}\,\mathrm{d}r\,\mathrm{d}\rho}{(1+r)^\beta}$$

$$\lesssim \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\sum_{M\in 2^{\mathbb{Z}}} M^{2(n-2+2\alpha+\frac12)+1-n} \times\int_0^\infty\int_0^\infty\Big|(r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)b^0_{k,\ell}(M\rho)\chi(\rho)\Big|^2\frac{r^{n-1}\,\mathrm{d}r\,\mathrm{d}\rho}{(1+\frac{r}{M})^\beta}$$

$$\lesssim \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\sum_{M\in 2^{\mathbb{Z}}}\sum_{R\in 2^{\mathbb{Z}}} M^{n-2+4\alpha}R^{n-1} \times\int_R^{2R}\int_0^\infty\Big|(r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)b^0_{k,\ell}(M\rho)\chi(\rho)\Big|^2\frac{\mathrm{d}r\,\mathrm{d}\rho}{(1+\frac{r}{M})^\beta}.$$

Define

$$G_{k,\ell}(R,M)=\int_R^{2R}\int_0^\infty\big|(r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)b^0_{k,\ell}(M\rho)\chi(\rho)\big|^2\frac{\mathrm{d}r\,\mathrm{d}\rho}{(1+\frac{r}{M})^\beta}.$$

Proposition 3.2. (1) *If* $R\lesssim 1$, *then*

$$G_{k,\ell}(R,M)\lesssim R^{2\nu(k)-n+3}M^{-n}\min\Big\{1,\Big(\frac{M}{R}\Big)^\beta\Big\}\Big\|b^0_{k,\ell}(\rho)\chi\Big(\frac{\rho}{M}\Big)\rho^{\frac{n-1}{2}}\Big\|^2_{L^2}.$$

(2) *If* $R\gg 1$, *then*

$$G_{k,\ell}(R,M)\lesssim \min\Big\{1,\Big(\frac{M}{R}\Big)^\beta\Big\}R^{-(n-2)}M^{-n}\Big\|b^0_{k,\ell}(\rho)\chi\Big(\frac{\rho}{M}\Big)\rho^{\frac{n-1}{2}}\Big\|^2_{L^2}.$$

Proof. (1) Since $\rho\sim 1$, we have $r\rho\lesssim 1$. By the property (2-5) of the Bessel function, we obtain

$$\begin{aligned}G_{k,\ell}(R,M)&\lesssim\int_R^{2R}\int_0^\infty\bigg|\frac{(r\rho)^{\nu(k)}(r\rho)^{-\frac{n-2}{2}}}{2^{\nu(k)}\Gamma(\nu(k)+\frac12)\Gamma(\frac12)}b^0_{k,\ell}(M\rho)\chi(\rho)\bigg|^2\mathrm{d}\rho\frac{\mathrm{d}r}{(1+\frac{r}{M})^\beta}\\&\lesssim R^{2\nu(k)-n+3}M^{-n}\min\Big\{1,\Big(\frac{M}{R}\Big)^\beta\Big\}\|b^0_{k,\ell}(\rho)\chi(\tfrac{\rho}{M})\rho^{\frac{n-1}{2}}\|^2_{L^2}.\end{aligned}$$

(2) Since $\rho\sim 1$, we have $r\rho\gg 1$. We estimate

$$G_{k,\ell}(R,M)\lesssim R^{-(n-2)}\int_0^\infty\big|b^0_{k,\ell}(M\rho)\chi(\rho)\big|^2\int_R^{2R}\big|J_{\nu(k)}(r\rho)\big|^2\frac{\mathrm{d}r}{(1+\frac{r}{M})^\beta}\,\mathrm{d}\rho. \tag{3-10}$$

Subcase (i): $R\lesssim M$. Noting that $\rho\sim 1$, we obtain by Lemma 2.2

$$\int_R^{2R} |J_{\nu(k)}(r\rho)|^2 \frac{\mathrm{d}r}{(1+\frac{r}{M})^\beta} \lesssim \int_R^{2R} |J_{\nu(k)}(r\rho)|^2 \,\mathrm{d}r \lesssim 1. \tag{3-11}$$

Subcase (ii): $R \gg M$. Noticing that $\rho \sim 1$ again, we obtain by Lemma 2.2

$$\int_R^{2R} |J_{\nu(k)}(r\rho)|^2 \frac{\mathrm{d}r}{(1+\frac{r}{M})^\beta} \lesssim \Big(\frac{M}{R}\Big)^\beta \int_R^{2R} |J_{\nu(k)}(r\rho)|^2 \,\mathrm{d}r \lesssim \Big(\frac{M}{R}\Big)^\beta. \tag{3-12}$$

Putting (3-11) and (3-12) into (3-10), we have

$$\begin{aligned} G_{k,\ell}(R,M) &\lesssim \min\Big\{1, \Big(\frac{M}{R}\Big)^\beta\Big\} R^{-(n-2)} \int_0^\infty \Big|b^0_{k,\ell}(M\rho)\chi(\rho)\Big|^2 \mathrm{d}\rho \\ &\lesssim \min\Big\{1, \Big(\frac{M}{R}\Big)^\beta\Big\} R^{-(n-2)} M^{-n} \Big\| b^0_{k,\ell}(\rho)\chi\Big(\frac{\rho}{M}\Big)\rho^{\frac{n-1}{2}}\Big\|^2_{L^2}. \end{aligned}$$

Thus we have proved Proposition 3.2. □

Now we return to proving Proposition 3.1. By Proposition 3.2, we show

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}} |\partial_t^\alpha u(x,t)|^2 \frac{\mathrm{d}t\,\mathrm{d}x}{(1+|x|)^\beta} \\ &\lesssim \sum_{k=0}^\infty \sum_{\ell=1}^{d(k)} \sum_{M\in 2^{\mathbb{Z}}} \sum_{\{R\in 2^{\mathbb{Z}}:R\lesssim 1\}} \bigg(M^{4\alpha-2} R^{2(\nu(k)+1)} \min\Big\{1, \Big(\frac{M}{R}\Big)^\beta\Big\} \\ &\qquad\qquad\qquad\qquad \times \big\| b^0_{k,\ell}(\rho)\chi\big(\tfrac{\rho}{M}\big)\rho^{\frac{n-1}{2}}\big\|^2_{L^2}\bigg) \\ &\qquad + \sum_{k=0}^\infty \sum_{\ell=1}^{d(k)} \sum_{M\in 2^{\mathbb{Z}}} \sum_{\{R\in 2^{\mathbb{Z}}:R\gg 1\}} M^{4\alpha-2+\beta} R^{1-\beta} \big\| b^0_{k,\ell}(\rho)\chi\big(\tfrac{\rho}{M}\big)\rho^{\frac{n-1}{2}}\big\|^2_{L^2}. \end{aligned}$$

From $\beta = 1+$, one has

$$\begin{aligned} &\sum_{M\in 2^{\mathbb{Z}}} \sum_{\{R\in 2^{\mathbb{Z}}:R\lesssim 1\}} M^{4\alpha-2} R^{2(\nu(k)+1)} \min\Big\{1, \Big(\frac{M}{R}\Big)^\beta\Big\} \big\| b^0_{k,\ell}(\rho)\chi\big(\tfrac{\rho}{M}\big)\rho^{\frac{n-1}{2}}\big\|^2_{L^2} \\ &\qquad\qquad \lesssim \sum_{M\in 2^{\mathbb{Z}}} M^{4\alpha-2+\beta} \big\| b^0_{k,\ell}(\rho)\chi\big(\tfrac{\rho}{M}\big)\rho^{\frac{n-1}{2}}\big\|^2_{L^2}. \end{aligned}$$

Since $\alpha \geqslant \frac{1}{2} - \frac{\beta}{4}$, we have by Lemma 2.5

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}} |\partial_t^\alpha u(x,t)|^2 \frac{\mathrm{d}t\,\mathrm{d}x}{(1+|x|)^\beta} &\lesssim \sum_{k=0}^\infty \sum_{\ell=1}^{d(k)} \sum_{M\in 2^{\mathbb{Z}}} M^{4\alpha-2+\beta} \Big\| b^0_{k,\ell}(\rho)\chi\Big(\frac{\rho}{M}\Big)\rho^{\frac{n-1}{2}}\Big\|^2_{L^2} \\ &\leqslant C\|u_0\|^2_{\dot H^{2\alpha-1+\frac{\beta}{2}}(\mathbb{R}^n)}. \end{aligned}$$

□

Finally, we apply Proposition 3.1 with $\alpha = \frac{1}{2}+$ and $\alpha = \frac{1}{2}-$ to prove Theorem 1.1.

Proof of Theorem 1.2. We now construct an example to show Theorem 1.2. The main idea is the stationary phase argument. By (3-8), we recall

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}Y_{k,\ell}(\theta)\int_0^\infty (r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)e^{it\rho^2}b^0_{k,\ell}(\rho)\rho^{n-1}\,\mathrm{d}\rho, \tag{3-13}$$

where

$$b^0_{k,\ell}(\rho)=(\mathcal{H}_\nu a^0_{k,\ell})(\rho),\quad u_0(x)=u_0(r\theta)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}a^0_{k,\ell}(r)Y_{k,\ell}(\theta).$$

In particular we choose $u_0(x)$ to be a radial function such that $(\mathcal{H}_{\nu(0)}u_0)(\xi)=\chi_N(|\xi|)$, where χ_N is a smooth positive function supported in J_N (to be chosen later) and $N\gg 1$. Then

$$u(x,t)=\int_0^\infty (r\rho)^{-\frac{n-2}{2}}J_{\nu(0)}(r\rho)e^{it\rho^2}\chi_N(\rho)\rho^{n-1}\,\mathrm{d}\rho. \tag{3-14}$$

Recalling the asymptotic expansion of the Bessel function,

$$J_\nu(r)=r^{-\frac{1}{2}}\sqrt{\frac{2}{\pi}}\cos\big(r-\frac{\nu\pi}{2}-\frac{\pi}{4}\big)+O_\nu(r^{-\frac{3}{2}})\quad\text{as } r\to\infty,$$

with a constant depending on ν (see [Stein and Weiss 1971]), we can write

$$\begin{aligned}u(x,t)=C_\nu\int_0^\infty (r\rho)^{-\frac{n-1}{2}}\big(e^{i(r\rho-\frac{\nu\pi}{2}-\frac{\pi}{4})}-e^{-i(r\rho-\frac{\nu\pi}{2}-\frac{\pi}{4})}\big)e^{it\rho^2}\chi_N(\rho)\rho^{n-1}\,\mathrm{d}\rho\\ +C_\nu\int_0^\infty (r\rho)^{-\frac{n-2}{2}}O_\nu\big((r\rho)^{-\frac{3}{2}}\big)e^{it\rho^2}\chi_N(\rho)\rho^{n-1}\,\mathrm{d}\rho.\end{aligned}$$

Let us define

$$I_1(r)=C_\nu e^{i(\frac{\nu\pi}{2}+\frac{\pi}{4})}\int_0^\infty (r\rho)^{-\frac{n-1}{2}}e^{i(-r\rho+t\rho^2)}\chi_N(\rho)\rho^{n-1}\,\mathrm{d}\rho, \tag{3-15}$$

$$I_2(r)=C_\nu e^{-i(\frac{\nu\pi}{2}+\frac{\pi}{4})}\int_0^\infty (r\rho)^{-\frac{n-1}{2}}e^{i(r\rho+t\rho^2)}\chi_N(\rho)\rho^{n-1}\,\mathrm{d}\rho, \tag{3-16}$$

$$I_3(r)=C_\nu\int_0^\infty (r\rho)^{-\frac{n-2}{2}}O_\nu\big((r\rho)^{-\frac{3}{2}}\big)e^{it\rho^2}\chi_N(\rho)\rho^{n-1}\,\mathrm{d}\rho. \tag{3-17}$$

Let $\phi_r(\rho)=t\rho^2-r\rho$. The fundamental idea is to choose sets J_N and $E\subset B^n$, in which $t(r)$ can be chosen, so that $\partial_\rho\phi_r(\rho)=2t(r)\rho-r$ almost vanishes for all $\rho\in J_N$ and $r\in\{|x|:x\in E\}$. To this end, we choose

$$E=\{x:\tfrac{1}{100}\leqslant|x|\leqslant\tfrac{1}{8}\}\quad\text{and}\quad J_N=[N,N+2N^{\frac{1}{2}}].$$

Choose $t(r) = r/(2(N+\sqrt{N}))$; then $\partial_\rho \phi_r(N+N^{\frac{1}{2}}) = 0$. Then

$$I_1(r) = C_\nu e^{i(\frac{\nu\pi}{2}+\frac{\pi}{4})} e^{i\phi_r(N+\sqrt{N})} \int_0^\infty (r\rho)^{-\frac{n-1}{2}} \exp \frac{ir[\rho-(N+\sqrt{N})]^2}{2(N+\sqrt{N})} \chi_N(\rho)\rho^{n-1}\,\mathrm{d}\rho.$$

Observe that

$$|I_1(r)| \geqslant c_\nu \int_0^\infty (r\rho)^{-\frac{n-1}{2}} \cos \frac{r[\rho-(N+\sqrt{N})]^2}{2(N+\sqrt{N})} \chi_N(\rho)\rho^{n-1}\,\mathrm{d}\rho. \tag{3-18}$$

Moreover, there exists a small constant $c > 0$ such that

$$\cos \frac{r[\rho-(N+\sqrt{N})]^2}{2(N+\sqrt{N})} \geqslant c,$$

since $\dfrac{r[\rho-(N+\sqrt{N})]^2}{2(N+\sqrt{N})} \leqslant \dfrac{\pi}{4}$ for all $r \in \left[\frac{1}{100}, \frac{1}{8}\right]$, $N \gg 1$ and $\rho \in J_N$. Therefore,

$$|I_1(r)| \geqslant c_\nu r^{-\frac{n-1}{2}} \int_0^\infty \chi_N(\rho)\rho^{\frac{n-1}{2}}\,\mathrm{d}\rho \geqslant c_\nu r^{-\frac{n-1}{2}} N^{\frac{n}{2}}. \tag{3-19}$$

On the other hand, let $\varphi_r(\rho) = t\rho^2 + r\rho$, $t = t(r)$ as before; then $\partial_\rho \varphi_r(\rho) = 2t(r)\rho + r \geqslant \frac{1}{200}$ when $\rho \in J_N$ and $r \in \left[\frac{1}{100}, \frac{1}{8}\right]$. Integrating by parts, we obtain

$$|I_2(r)| \leqslant C_\nu r^{-\frac{n}{2}} N^{\frac{n-2}{2}}. \tag{3-20}$$

Obviously, we have

$$|I_3(r)| \leqslant C_\nu r^{-\frac{n}{2}} N^{\frac{n-2}{2}}. \tag{3-21}$$

Combining (3-19)–(3-21), we get for $N \gg 1$ and $r \in \left[\frac{1}{100}, \frac{1}{8}\right]$

$$u^*(x) \geqslant cN^{\frac{n}{2}}. \tag{3-22}$$

On the other hand, let $j_0 = \log_2 N$; we obtain by the definitions of P_j and $\tilde{P}_j$

$$\|u_0(x)\|_{H^s}^2 = \sum_j 2^{2js}\|P_j u_0\|_{L^2}^2 = \sum_j 2^{2js}\|P_j \tilde{P}_{j_0} u_0\|_{L^2}^2.$$

By Lemma 2.4, we choose $s < \epsilon < 1 + \min\left\{\frac{n-2}{2}, \left(\frac{(n-2)^2}{4} + a\right)^{\frac{1}{2}}\right\}$ to obtain

$$\begin{aligned} \|u_0(x)\|_{H^s}^2 &\leqslant C \sum_j 2^{2js-2\epsilon|j-j_0|}\|u_0\|_{L^2}^2 \\ &= CN^{2s} \sum_j 2^{2js-2\epsilon|j|}\|\chi_N\|_{L^2}^2 = N^{2s+n-\frac{1}{2}}. \end{aligned} \tag{3-23}$$

Thus, by (1-5) and (3-22), we must have $s \geqslant \frac{1}{4}$.

Proof of Theorem 1.3. Even though there is a loss of the angular regularity in Theorem 1.3, the result implies the sharp result for the radial initial data. The key ingredient here is the following lemma proved in [Gigante and Soria 2008].

Lemma 3.1. *Let $\tilde{J}_\nu(s) = s^{\frac{1}{2}} J_\nu(s)$ with $s \geqslant 0$, and let*

$$T_\nu g(r) = \int_I \frac{e^{it(r)\rho^2} \tilde{J}_\nu(r\rho)}{\rho^{\frac{1}{4}}} g(\rho)\,\mathrm{d}\rho. \tag{3-24}$$

Then

$$\int_0^1 |T_\nu g(r)|^2 \,\mathrm{d}r \leqslant C \int_I |g(\rho)|^2 \,\mathrm{d}\rho, \tag{3-25}$$

where the constant C is independent of $g \in L^2(I)$, of the interval I, of the measurable function $t(r)$ and of the order $\nu \geqslant 0$.

We also can follow the Carleson approach [1980] to linearize our maximal operator, by making t into a function of r, $t(r)$. By the triangle inequality, we have the estimate

$$\begin{aligned}\|u^*(x)\|_{L^2(B^n)} \\ \leqslant C \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) e^{it(r)\rho^2} b^0_{k,\ell}(\rho)\rho^{n-1}\,\mathrm{d}\rho \right\|_{L^2_{r^{n-1}\mathrm{d}r}}.\end{aligned}$$

Let $g(\rho) = b^0_{k,\ell}(\rho)\rho^{\frac{n-1}{2}+\frac{1}{4}}$; then

$$\|u^*(x)\|_{L^2(B^n)} \lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| \int_0^\infty \tilde{J}_{\nu(k)}(r\rho) e^{it(r)\rho^2} \rho^{-\frac{1}{4}} g(\rho)\,\mathrm{d}\rho \right\|_{L^2_r([0,1])}. \tag{3-26}$$

Using Lemma 3.1, we obtain

$$\|u^*(x)\|_{L^2(B^n)} \lesssim C \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \| b^0_{k,\ell}(\rho)\rho^{\frac{n-1}{2}+\frac{1}{4}} \|_{L^2_\rho(\mathbb{R}^+)}. \tag{3-27}$$

Let $\alpha = (n-1)/2 + \epsilon$ with $\epsilon > 0$, we have by the Cauchy–Schwarz inequality

$$\begin{aligned}\|u^*(x)\|_{L^2(B^n)} \\ \leqslant C \bigg(\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1+k)^{-2\alpha} \bigg)^{\frac{1}{2}} \bigg(\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1+k)^{2\alpha} \| b^0_{k,\ell}(\rho)\rho^{\frac{n-1}{2}+\frac{1}{4}} \|^2_{L^2_\rho(\mathbb{R}^+)} \bigg)^{\frac{1}{2}}.\end{aligned}$$

Since $d(k) \simeq \langle k\rangle^{n-2}$, we have by Lemma 2.5

$$\|u^*(x)\|_{L^2(B^n)} \lesssim \Bigg(\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}(1+k)^{2\alpha}\big\|b^0_{k,\ell}(\rho)\rho^{\frac{n-1}{2}+\frac{1}{4}}\big\|^2_{L^2_\rho(\mathbb{R}^+)}\Bigg)^{\frac{1}{2}} \lesssim \|u_0\|_{H_r^{\frac{1}{4}}H_\theta^\alpha}.$$

This completes the proof of Theorem 1.3.

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