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$\mathbb{Q}$-BASES OF THE NÉRON–SEVERI GROUPS OF CERTAIN ELLIPTIC SURFACES

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P. Stiller computed the rank of the Néron–Severi group (known as the Picard number) for several families of elliptic surfaces. However, he did not give the generators of these groups. In this paper we give $\mathbb{Q}$-bases of these groups explicitly. If these surfaces are rational, we also show that they are $\mathbb{Z}$-bases.

1. Introduction

An elliptic surface is a surface which has a surjective map onto a curve such that the generic fiber is a curve of genus one (see [Kodaira 1963a; 1963b]). The Néron–Severi group is the group of divisors modulo algebraic equivalence (see [Hartshorne 1977, Exercise V.1.7]). This group is known to be a finitely generated abelian group, and its rank is called the Picard number. P. Stiller [1987] computed the Picard numbers of several families of elliptic surfaces by studying the action of certain automorphisms on the cohomology group. However he did not give the generators of these groups. The purpose of this paper is to give explicit $\mathbb{Q}$-bases of the Néron–Severi groups of Stiller's list [1987, Examples 1–5] of elliptic surfaces.

We explain briefly how to construct such $\mathbb{Q}$-bases. Let $\mathcal{E}$ be an elliptic surface. We denote by $\mathrm{NS}(\mathcal{E})$ the Néron–Severi group of $\mathcal{E}$. T. Shioda [1972] proved that $\mathrm{NS}(\mathcal{E})$ is generated by fibral divisors and horizontal divisors. Here we mean by a fibral divisor a sum of irreducible components of fibers, and by a horizontal divisor a sum of images of sections. Let $\mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}}$ $(n \in \mathbb{N})$ be one of the families of elliptic surfaces of [Stiller 1987], and let E_n be the generic fiber of $\mathcal{E}_n$ for each n. The E_n are elliptic curves over the function field $\mathbb{C}(t)$. Computing the Picard number of an elliptic surface is equivalent to determining the Mordell–Weil rank (i.e., the rank of the Mordell–Weil group) of the generic fiber. Stiller [1987, Examples 1–5] proved that for each family $\mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}}$ $(n \in \mathbb{N})$, there exists a finite set Adm_i $(1 \le i \le 5)$ of natural numbers such that the Mordell–Weil rank* of $E_n/\mathbb{C}(t)$ is $r = \sum_{\substack{d\mid n \\ d\in \mathrm{Adm}_i}} \varphi(d)$,

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*The referee pointed out that a related or a similar formula is obtained in [Silverman 2000] for arbitrary elliptic surfaces defined over number fields under the Tate conjecture.

where φ is the Euler totient function. In this paper we shall construct r rational points of E_n in an ad hoc manner, and show the linear independence of the associated divisors in $\mathrm{NS}(\mathcal{E}_n)$. If $\mathcal{E}_n$ is rational, then we further show that they form a $\mathbb{Z}$-basis.

This paper is organized as follows. Section 2 is a quick review of some basic results on the Néron–Severi groups of elliptic surfaces. Section 3 is the heart of this paper. We give a number of $\mathbb{Q}$-bases or $\mathbb{Z}$-bases of the Néron–Severi groups of Stiller's list of elliptic surfaces. In Section 4, we give an alternative proof of Stiller's computations of the Picard numbers.

2. The Néron–Severi group of an elliptic surface

In this paper, we mean by an elliptic surface a surjective morphism $f : \mathcal{E} \to C$ onto a curve with a section (say zero section) such that the generic fiber of f is an elliptic curve. Let $f : \mathcal{E} \to \mathbb{P}^1_{\mathbb{C}}$ be a nonsplit minimal elliptic surface, and let $E/\mathbb{C}(t)$ be the generic fiber. There is a natural group isomorphism between the Mordell–Weil group of $E/\mathbb{C}(t)$, denoted by $E(\mathbb{C}(t))$, and the group of sections of $\mathcal{E}$ over $\mathbb{P}^1_{\mathbb{C}}$, denoted by $\mathcal{E}(\mathbb{P}^1_{\mathbb{C}})$ (see [Silverman 1994, Proposition 3.10(c)]):

$$(1)\qquad \begin{aligned} E(\mathbb{C}(t)) &\xrightarrow{\sim} \mathcal{E}(\mathbb{P}^1_{\mathbb{C}}), \\ P = (x_P, y_P) &\longmapsto (\sigma_P : t \mapsto (x_P(t), y_P(t), t)). \end{aligned}$$

According to the Mordell–Weil theorem, $E(\mathbb{C}(t))$ is a finitely generated group. In the following, for each $P \in E(\mathbb{C}(t))$, we denote by (P) the image in $\mathcal{E}$ of the section corresponding to P. For simplicity, we denote by ∞ the image of the zero section, that is, the section corresponding to zero element $O \in E(\mathbb{C}(t))$.

The singular fibers are classified by Kodaira [1963a; 1963b]. We shall follow Kodaira's notation. Let $\Sigma(\mathcal{E})$ be the finite set of points t in $\mathbb{P}^1_{\mathbb{C}}$ such that $\mathcal{E}_t := f^{-1}(t)$ is a singular fiber. For each $t \in \mathbb{P}^1_{\mathbb{C}}$, let m_t be the number of irreducible components of the fiber $\mathcal{E}_t$, and we denote by $F_{t,a}$ $(0 \le a \le m_t - 1)$ the irreducible components. If $t \in \mathbb{P}^1_{\mathbb{C}} \setminus \Sigma(\mathcal{E})$, then $\mathcal{E}_t = F_{t,0}$ is a smooth fiber, and we have

$$\{t \in \mathbb{P}^1_{\mathbb{C}} \mid m_t \ge 2\} = \{t \in \Sigma(\mathcal{E}) \mid m_t \ge 2\}.$$

We fix a general fiber $C_0 := \mathcal{E}_{t_0}$, $t_0 \in \mathbb{P}^1_{\mathbb{C}} \setminus \Sigma(\mathcal{E})$, and we take $F_{t,0}$ to be the unique component of $\mathcal{E}_t$ intersecting with ∞.

Let $E(\mathbb{C}(t))_{\mathrm{free}}$ denote the quotient group $E(\mathbb{C}(t))/E(\mathbb{C}(t))_{\mathrm{tor}}$, where $E(\mathbb{C}(t))_{\mathrm{tor}}$ is the torsion subgroup. Let r be the Mordell–Weil rank of $E/\mathbb{C}(t)$, that is, the rank of $E(\mathbb{C}(t))$, and we take r generators $P_1, \ldots, P_r$ of $E(\mathbb{C}(t))_{\mathrm{free}}$. We put

$$D_i = (P_i) - \infty \in \mathrm{Div}(\mathcal{E}) \quad (1 \le i \le r),$$

where $\mathrm{Div}(\mathcal{E})$ is the group of divisors on $\mathcal{E}$.

Proposition 2.1 [Shioda 1972, Theorem 1.1]. *The free part of the Néron–Severi group* $\mathrm{NS}(\mathcal{E})$ *of the elliptic surface* $\mathcal{E}$, *denoted by* $\mathrm{NS}(\mathcal{E})_{\mathrm{free}}$, *is generated by the divisors*

$$C_0,\ \infty,\ D_1, \dots, D_r,\ F_{t,a} \quad (t \in \Sigma(\mathcal{E}),\ 1 \le a \le m_t - 1). \tag{2}$$

In particular, the Picard number ρ *of the elliptic surface* $\mathcal{E}$ *is given by*

$$\rho = r + 2 + \sum_{t \in \Sigma(\mathcal{E})} (m_t - 1).$$

Stiller computed the Mordell–Weil rank r, but did not give D_i's explicitly. We will give r linearly independent points of the Mordell–Weil group in Section 3. Note that these points are not always generators of the group. We end this section by introducing a practical way to show the linear independence of the divisors $C_0, \infty, D_1, \dots, D_r, F_{t,a}$ ($t \in \Sigma(\mathcal{E})$, $1 \le a \le m_t - 1$), or equivalently the intersection matrix M of these divisors has a nonzero determinant.

For each $P_i \in E(\mathbb{C}(t))$, we have $D_i \cdot C_0 = ((P_i) - \infty) \cdot C_0 = 1 - 1 = 0$. Then there exists a fibral divisor $\Phi_i \in \mathrm{Div}(\mathcal{E}) \otimes \mathbb{Q}$ such that

$$(D_i + \Phi_i) \cdot F = 0 \ \text{ for all fibral divisors } F \in \mathrm{Div}(\mathcal{E}).$$

More explicitly the divisor Φ_i is obtained in the following way (see [Silverman 1994, Proposition 8.3]). We set $a_{t0}(P_i) = 0$ for all $t \in \mathbb{P}^1_{\mathbb{C}}$. Further, when $m_t \ge 2$ we consider the following system of linear equations:

$$\sum_{k=1}^{m_t - 1} a_{tk}(P_i) F_{t,k} \cdot F_{t,l} = -D_i \cdot F_{t,l} \quad (1 \le l \le m_t - 1).$$

This is a system of $m_t - 1$ equations in the $m_t - 1$ variables $a_{tk}(P_i)$. Since the intersection matrix $(F_{t,i} \cdot F_{t,j})_{1 \le i,j \le m_t - 1}$ has a nonzero determinant, this system of equations has a unique solution in rational numbers $a_{tk}(P_i) \in \mathbb{Q}$. Then the divisor

$$\Phi_i := \sum_{t \in \mathbb{P}^1_{\mathbb{C}}} \sum_{k=0}^{m_t - 1} a_{tk}(P_i) F_{t,k} = \sum_{t \in \{t_1, \dots, t_s\}} \sum_{k=1}^{m_t - 1} a_{tk}(P_i) F_{t,k} \in \mathrm{Div}(\mathcal{E}) \otimes \mathbb{Q}$$

has the desired property, where we set $\{t_1, \dots, t_s\} = \{t \in \Sigma(\mathcal{E}) \mid m_t \ge 2\}$. Note that since $F_{t_\alpha,k} \cdot F_{t_\beta,l} = 0$ ($\alpha \ne \beta$), we have

$$0 = (D_i + \Phi_i) \cdot F_{t_\alpha,j} = \left(D_i + \sum_{k=1}^{m_{t_\alpha} - 1} a_{t_\alpha k}(P_i) F_{t_\alpha,k} \right) \cdot F_{t_\alpha,j}.$$

We fix a uniformizer $u_t \in \mathbb{C}(t)$ at t, that is, $\mathrm{ord}_t(u_t) = 1$. Let $f^* : \mathrm{Div}(\mathbb{P}^1_{\mathbb{C}}) \to \mathrm{Div}(\mathcal{E})$ be a homomorphism defined by extending $(t) \mapsto \sum_{j=0}^{m_t - 1} \mathrm{ord}_{F_{t,j}}(u_t \circ f) F_{t,j}$ linearly.

For each two points $t_1, t_2 \in \mathbb{P}^1_{\mathbb{C}} \setminus \Sigma(\mathcal{E})$ with $t_1 \neq t_2$, since C_0 is algebraically equivalent to $f^*(t_i)$ $(i = 1, 2)$, we have

$$C_0^2 = f^*(t_1) \cdot f^*(t_2) = 0.$$

Now it is not hard to show the following lemma.

Lemma 2.2 [Cox and Zucker 1979]. *Let M be the intersection matrix of divisors $C_0, \infty, D_1, \dots, D_r, F_{t,a}$ $(t \in \Sigma(\mathcal{E}), 1 \le a \le m_t - 1)$, and let M_α $(1 \le \alpha \le s)$ be the intersection matrix of divisors $F_{t_\alpha,1}, \dots, F_{t_\alpha,m_{t_\alpha}-1}$. Put*

$$N = \begin{bmatrix} (D_1 + \Phi_1) \cdot D_1 & \cdots & (D_1 + \Phi_1) \cdot D_r \\ \vdots & \ddots & \vdots \\ (D_r + \Phi_r) \cdot D_1 & \cdots & (D_r + \Phi_r) \cdot D_r \end{bmatrix}.$$

Then we have

$$\det M = -\det N \prod_{\alpha=1}^{s} \det M_\alpha.$$

In particular, $\det M \neq 0$ *if and only if* $\det N \neq 0$ *since M_α has nonzero determinant for each α. Note that each M_α gives one of the root lattices A_n, D_n, E_6, E_7 or E_8, and* $\det M_\alpha$ *equals the number of simple components of the singular fiber $\mathcal{E}_{t_\alpha}$.*

3. Stiller's list of elliptic surfaces

In this section, we give explicit $\mathbb{Q}$-bases of the Néron–Severi groups of the elliptic surfaces in Stiller's Examples 1–5. If these surfaces are rational, then we also show that they are $\mathbb{Z}$-bases. Note that these Néron–Severi groups are torsion-free, by [Cox and Zucker 1979] or [Shioda 1990].

We give a proof in detail in the case of Stiller's Example 4. In the other cases we just give results, because the argument is the same.

Throughout this paper, ζ_n will denote a primitive n-th root of unity for a natural number n.

Stiller's Example 4. This example is the minimal elliptic surface whose generic fiber is the elliptic curve defined by the equation

$$Y^2 = 4X^3 - 3u^{4n}X + u^{5n}(u^n - 2) \quad (u \in \mathbb{P}^1_{\mathbb{C}}, n \in \mathbb{N}) \tag{3}$$

over $\mathbb{C}(u)$. We perform the change of variables

$$X = \frac{2^3(3x+1)}{3^6 t^{2n}}, \quad Y = \frac{-2^5\sqrt{2}\,y}{3^7\sqrt{3}t^{3n}}, \quad u = \sqrt[n]{\tfrac{-4}{27}}\, t^{-1}.$$

Then the defining equation (3) becomes

$$y^2 = x^3 + x^2 + t^n \quad (t \in \mathbb{P}^1_{\mathbb{C}}, n \in \mathbb{N}). \tag{4}$$

Let E_n be the elliptic curve defined by (4) and $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}}$ be the associated elliptic surface. For the later use, we write down the construction of $\mathcal{E}_n$. Put

$$\overline{X}_1 = \{([X : Y : Z], t) \in \mathbb{P}^2_{\mathbb{C}} \times \mathbb{A}^1_t \mid Y^2 Z = X^3 + X^2 Z + t^n Z^3\},$$
$$g_1 : \overline{X}_1 \to \mathbb{A}^1_t; \quad ([X : Y : Z], t) \mapsto t.$$

Let us write $n = 6l + k$ with $1 \le k \le 6$. By putting $\bar{x} = x/t^{2(l+1)}$, $\bar{y} = y/t^{3(l+1)}$, $\bar{t} = 1/t$, we obtain the minimal Weierstrass form

$$\bar{y}^2 = \bar{x}^3 + \bar{t}^{2(l+1)}\bar{x}^2 + \bar{t}^{6-k}.$$

over $t = \infty$. Put

$$\overline{X}_2 = \{([\overline{X} : \overline{Y} : \overline{Z}], \bar{t}) \in \mathbb{P}^2_{\mathbb{C}} \times \mathbb{A}^1_{\bar{t}} \mid \overline{Y}^2 \overline{Z} = \overline{X}^3 + \bar{t}^{2(l+1)} \overline{X}^2 \overline{Z} + \bar{t}^{6-k} \overline{Z}^3\},$$
$$g_2 : \overline{X}_2 \to \mathbb{A}^1_{\bar{t}}; \quad ([\overline{X} : \overline{Y} : \overline{Z}], \bar{t}) \mapsto \bar{t}.$$

By gluing the surfaces $\overline{X}_1$ and $\overline{X}_2$, we obtain a projective surface W together with a surjective morphism $g : W \to \mathbb{P}^1_{\mathbb{C}}$. The surface W has singularities in $g^{-1}(0)$ and $g^{-1}(\infty)$. Taking the minimal resolution of singularities of W, we obtain $\mathcal{E}_n$ with $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}}$.

Using Tate's algorithm (see [Silverman 1994]), one can show that the surface $\mathcal{E}_n$ has singular fibers of type I_n over 0, type I_1 over $\zeta_n^i \sqrt[n]{-4/27}$ $(0 \le i \le n-1)$ and type II^* (resp. IV^*, I_0^*, IV, II, I_0) over ∞ as $n \equiv 1$ (resp. 2, 3, 4, 5, 0) modulo 6. Stiller computed the Mordell–Weil rank $r = \operatorname{rank}(E_n(\mathbb{C}(t)))$ and hence the Picard number $\rho = \operatorname{rank}(\mathrm{NS}(\mathcal{E}_n))$, which is given in Table 1. The result for r can be summarized in the following way. Put $\mathrm{Adm}_4 = \{2, 3, 4, 5\}$. Then

$$r = \sum_{\substack{d \mid n \\ d \in \mathrm{Adm}_4}} \varphi(d), \tag{5}$$

where φ is the Euler function. We now define $\varphi(d)$ rational points of E_d for each $d \in \mathrm{Adm}_4$.

Definition 3.1. For $d \in \mathrm{Adm}_4$ and $j \in (\mathbb{Z}/d\mathbb{Z})^\times$, we define $\mathbb{C}(t)$-rational points $P_{d,j}$ of E_d as follows.

$$P_{2,1} = (0, -t),$$
$$P_{3,j} = (-\zeta_3^j t, -\zeta_3^j t),$$
$$P_{4,1} = \left(\sqrt{2}t + 2t^2, \sqrt{2}t + 3t^2 + 2\sqrt{2}t^3\right),$$
$$P_{4,3} = \left(-2 + 2(-1)^{1/4}t, -2\sqrt{-1} + 4\sqrt{-1}(-1)^{1/4}t + t^2\right),$$
$$P_{5,j} = \left(2^{-2/5}\zeta_5^{2j}t^2, 2^{-2/5}\zeta_5^{2j}t^2 + 2^{-3/5}\zeta_5^{3j}t^3\right).$$

n	r	ρ
$6l+1,\ l\geq 0$	$r=\begin{cases}4 & \text{if } l\equiv 4 \mod 5,\\ 0 & \text{otherwise.}\end{cases}$	$\rho=\begin{cases}n+13\\ n+9\end{cases}$
$6l+2,\ l\geq 0$	$r=\begin{cases}7 & \text{if } 3l+1\equiv 0 \mod 10,\\ 5 & \text{if } 3l+1\equiv 5 \mod 10,\\ 3 & \text{if } 3l+1\equiv 2,4,6,8 \mod 10,\\ 1 & \text{otherwise.}\end{cases}$	$\rho=\begin{cases}n+14\\ n+12\\ n+10\\ n+8\end{cases}$
$6l+3,\ l\geq 0$	$r=\begin{cases}6 & \text{if } l\equiv 2 \mod 5,\\ 2 & \text{otherwise.}\end{cases}$	$\rho=\begin{cases}n+11\\ n+7\end{cases}$
$6l+4,\ l\geq 0$	$r=\begin{cases}7 & \text{if } 3l+2\equiv 0 \mod 10,\\ 5 & \text{if } 3l+2\equiv 5 \mod 10,\\ 3 & \text{if } 3l+2\equiv 2,4,6,8 \mod 10,\\ 1 & \text{otherwise.}\end{cases}$	$\rho=\begin{cases}n+10\\ n+8\\ n+6\\ n+4\end{cases}$
$6l+5,\ l\geq 0$	$r=\begin{cases}4 & \text{if } l\equiv 0 \mod 5,\\ 0 & \text{otherwise.}\end{cases}$	$\rho=\begin{cases}n+5\\ n+1\end{cases}$
$6l+6,\ l\geq 0$	$r=\begin{cases}9 & \text{if } l+1\equiv 0 \mod 10,\\ 7 & \text{if } l+1\equiv 5 \mod 10,\\ 5 & \text{if } l+1\equiv 2,4,6,8 \mod 10,\\ 3 & \text{otherwise.}\end{cases}$	$\rho=\begin{cases}n+10\\ n+8\\ n+6\\ n+4\end{cases}$

Table 1. The Mordell–Weil rank r and the Picard number ρ.

For any d that divides n, there is the surjective map $\rho : E_n \to E_d$ given by $(x, y, t) \mapsto (x, y, t^{n/d})$, and then the inverse image $\rho^*(P_{d,j})$ defines a $\mathbb{C}(t)$-rational point of E_n. In what follows, we use the same symbol $P_{d,j}$ for $\rho^*(P_{d,j})$ since the context will prevent any confusion.

Theorem 3.2. *Let* $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}}(n \in \mathbb{N})$ *be the elliptic surfaces associated to the elliptic curves* $E_n : y^2 = x^3 + x^2 + t^n (n \in \mathbb{N})$ *over* $\mathbb{P}^1_{\mathbb{C}}$. *Then, for each* $n \in \mathbb{N}$, $\mathrm{NS}(\mathcal{E}_n)$ *has a* $\mathbb{Q}$*-basis* $C_0, \infty, D_{d,j}, F_{t,a}$ $(d \in \mathrm{Adm}_4, d|n, j \in (\mathbb{Z}/d\mathbb{Z})^\times, t \in \Sigma(\mathcal{E}_n), 1 \leq a \leq m_t - 1)$, *where* $D_{d,j} = (P_{d,j}) - \infty$ *and the other notations are the same as Section 2. Moreover if* $\mathcal{E}_n$ *is rational* (*i.e.*, $n \leq 6$), *these divisors form a* $\mathbb{Z}$*-basis.*

Proof. In the cases of $n = 6l+1$ with $l \not\equiv 4 \mod 5$ or $n = 6l+5$ with $l \not\equiv 0 \mod 5$, we have $r = 0$ by Table 1, so the assertion follows immediately from Proposition 2.1. We consider the other cases. All we have to do is to show the linear independence of the divisors in Theorem 3.2. Let $d(n)$ be the least common multiple of all numbers in $\{d \in \mathrm{Adm}_4 : d|n\}$. Note that the set is nonempty in these cases. Since n is divisible by $d(n)$, there exists a rational map

$$\begin{aligned}\mathcal{E}_n &\to \mathcal{E}_{d(n)},\\ (x, y, t) &\mapsto (x, y, t^{n/d(n)}).\end{aligned}$$

This map induces an injection

$$\begin{aligned}E_{d(n)}(\mathbb{C}(t)) &\hookrightarrow E_n(\mathbb{C}(t)),\\ (x(t), y(t)) &\mapsto (x(t^{n/d(n)}), y(t^{n/d(n)})).\end{aligned}$$

We obtain $\operatorname{rank}(E_{d(n)}(\mathbb{C}(t))) = \operatorname{rank}(E_n(\mathbb{C}(t)))$ by (5), hence linearly independent points of $E_{d(n)}(\mathbb{C}(t))$ remain so in $E_n(\mathbb{C}(t))$. There exists an injection

$$\begin{aligned}E_{d(n)}(\mathbb{C}(t)) &\hookrightarrow E_{60}(\mathbb{C}(t)),\\ (x(t), y(t)) &\mapsto \big(x(t^{60/d(n)}), y(t^{60/d(n)})\big).\end{aligned}$$

In particular, points in $E_{d(n)}(\mathbb{C}(t))$ are linearly independent if and only if their images are independent in $E_{60}(\mathbb{C}(t))$. Therefore it is sufficient to show the assertion in the case of $n = 60$. Recall that the surface $\mathcal{E}_{60}$ has singular fibers of type I_{60} over 0, type I_1 over $\zeta_{60}^i \sqrt[60]{-4/27}$ $(0 \le i \le 59)$ and type I_0 over ∞.

We want to show that the divisors C_0, ∞, $D_{2,1}$, $D_{3,1}$, $D_{3,2}$, $D_{4,1}$, $D_{4,3}$, $D_{5,1}$, $D_{5,2}$, $D_{5,3}$, $D_{5,4}$, $F_{0,1}, \dots, F_{0,59}$ are a $\mathbb{Q}$-basis of $\mathrm{NS}(\mathcal{E}_{60})$. Equivalently the matrix

$$N = \begin{bmatrix}(D_{2,1}+\Phi_{2,1})\cdot D_{2,1} & \cdots & (D_{2,1}+\Phi_{2,1})\cdot D_{5,4}\\ \vdots & \ddots & \vdots\\ (D_{5,4}+\Phi_{5,4})\cdot D_{2,1} & \cdots & (D_{5,4}+\Phi_{5,4})\cdot D_{5,4}\end{bmatrix}$$

has a nonzero determinant (see Section 2 for the notations).

Firstly we compute the self intersection numbers ∞^2, $(P_{2,1})^2$, $(P_{3,i})^2$, $(P_{4,j})^2$, $(P_{5,k})^2$.

Lemma 3.3. *Let $n = 6l + k$ with $l \ge 0$, $1 \le k \le 6$. Then the canonical divisor $K_{\mathcal{E}_n}$ of the elliptic surface $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}}$ is $K_{\mathcal{E}_n} \cong f^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(l-1)$, and we have $(P)^2 = \infty^2 = -(l+1)$ for each point $P \in E_n(\mathbb{C}(t))$. In particular, if n equals* 60, *then we have*

$$(P_{2,1})^2 = (P_{3,i})^2 = (P_{4,j})^2 = (P_{5,k})^2 = \infty^2 = -10. \tag{6}$$

Proof. By Kodaira's canonical bundle formula (see [1963a; 1963b]), we get

$$K_{\mathcal{E}_n} \cong f^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d-2), \quad d = \frac{1}{12}\sum_{t\in\Sigma(\mathcal{E}_n)} \varepsilon(t), \tag{7}$$

where $\varepsilon(t)$ is defined as follows:

type	I_n	II	III	IV	I_n^*	II^*	III^*	IV^*
$\varepsilon(t)$	n	2	3	4	$n+6$	10	9	8

By the reduction type of the singular fibers of $\mathcal{E}_n$, we have $d = l + 1$. Thus

$$K_{\mathcal{E}_n} \cong f^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d-2) = f^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(l-1). \tag{8}$$

By (1), each point $P \in E_n(\mathbb{C}(t))$ corresponds to a section $\sigma_P : \mathbb{P}^1_{\mathbb{C}} \to \mathcal{E}_n$. The translation-by-$P$ map on E_n can be uniquely extended to a map $\tau_P : \mathcal{E}_n \to \mathcal{E}_n$ by the minimality of $\mathcal{E}_n$ (see [Silverman 1994, Proposition 9.1]). It follows that $\tau_P^* D_1 \cdot \tau_P^* D_2 = D_1 \cdot D_2$ for any two divisors $D_1, D_2 \in \mathrm{Div}(\mathcal{E}_n)$. Hence $(P)^2 = \tau_P^*(P) \cdot \tau_P^*(P) = \infty^2$.

Since ∞ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, ∞ is of genus zero. Thus, by the adjunction formula, we get

$$\tfrac{1}{2}(\infty^2 + K_{\mathcal{E}_n} \cdot \infty) + 1 = 0, \text{ that is, } \infty^2 = -(K_{\mathcal{E}_n} \cdot \infty + 2).$$

On the other hand, we can compute

$$\begin{aligned} K_{\mathcal{E}_n} \cdot \infty = (f^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(l-1)) \cdot \infty &= (l-1) f^*(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)) \cdot \infty \\ &= (l-1) C_0 \cdot \infty = l - 1 \end{aligned}$$

by (8). Therefore we get $(P)^2 = \infty^2 = -(l+1)$ for all $P \in E_n(\mathbb{C}(t))$. □

Next we compute the intersection numbers of the divisors ∞, $(P_{2,1})$, $(P_{3,i})$, $(P_{4,j})$, $(P_{5,k})$, $F_{0,a}$ $(1 \le a \le 59)$ in $\mathcal{E}_{60}$. In the affine surface $X_1 : y^2 = x^3 + x^2 + t^{60}$, the divisors $(P_{2,1})$, $(P_{3,i})$, $(P_{4,j})$ and $(P_{5,k})$ are given by

$$\begin{aligned} &(P_{2,1}) = \big(x = 0,\ y = -t^{30}\big), \\ &(P_{3,i}) = \big(x = -\zeta_3^i t^{20},\ y = -\zeta_3^i t^{20}\big), \\ &(P_{4,1}) = \big(x = \sqrt{2} t^{15} + 2t^{30},\ y = \sqrt{2} t^{15} + 3t^{30} + 2\sqrt{2} t^{45}\big), \\ &(P_{4,3}) = \big(x = -2 + 2(-1)^{1/4} t^{15},\ y = -2\sqrt{-1} + 4\sqrt{-1}(-1)^{1/4} t^{15} + t^{30}\big), \\ &(P_{5,k}) = \big(x = 2^{-2/5} \zeta_5^{2k} t^{24},\ y = 2^{-2/5} \zeta_5^{2k} t^{24} + 2^{-3/5} \zeta_5^{3k} t^{36}\big). \end{aligned}$$

Thus $(P_{4,3})$ does not pass through the singular point $(0, 0, 0)$ of X_1, however $(P_{2,1})$, $(P_{3,i})$, $(P_{4,1})$ and $(P_{5,k})$ pass through this point. Since this singular point is of type A_{59}, we can resolve it blowing up 30 times. We denote by $(x_{(m)}, y_{(m)}, t_{(m)})$ the coordinates in the neighborhood of the singular point after the m-th blowing-up $(1 \le m \le 29)$, and denote by $(P_{2,1})^{(m)}$, $(P_{3,i})^{(m)}$, $(P_{4,1})^{(m)}$, $(P_{5,k})^{(m)}$ the m-th blowing-up of $(P_{2,1})$, $(P_{3,i})$, $(P_{4,1})$, $(P_{5,k})$, respectively. These curves are given by

$$\begin{aligned} &(P_{2,1})^{(m)} = \big(x_{(m)} = 0,\ y_{(m)} = -t_{(m)}^{30-m}\big), \\ &(P_{3,i})^{(m)} = \big(x_{(m)} = -\zeta_3^i t_{(m)}^{20-m},\ y_{(m)} = \zeta_3^i t_{(m)}^{20-m}\big), \\ &(P_{4,1})^{(m)} = \big(x_{(m)} = \sqrt{2} t^{15-m} + 2t^{30-m},\ y_{(m)} = \sqrt{2} t^{15-m} + 3t^{30-m} + 2\sqrt{2} t^{45-m}\big), \\ &(P_{5,k})^{(m)} = \big(x_{(m)} = 2^{-2/5} \zeta_5^{2k} t_{(m)}^{24-m},\ y_{(m)} = 2^{-2/5} \zeta_5^{2k} t_{(m)}^{24-m} + 2^{-3/5} \zeta_5^{3k} t_{(m)}^{36-m}\big). \end{aligned}$$

In particular, in $\mathcal{E}_{60}$ the divisors $(P_{3,i})$ (resp. $(P_{4,1})$, $(P_{5,k})$) intersect with either of two $\mathbb{P}^1_{\mathbb{C}}$ which appear by the 20-th (resp. 15, 24-th) blowing-up, and the divisors $(P_{2,1})$ intersect with unique $\mathbb{P}^1_{\mathbb{C}}$ which appears by the 30-th blowing-up. Hence we may assume that $(P_{2,1})$ (resp. $(P_{3,i})$, $(P_{4,1})$, $(P_{4,3})$, $(P_{5,k})$) intersects with $F_{0,30}$ (resp. $F_{0,20}$, $F_{0,15}$, $F_{0,0}$, $F_{0,24}$). In addition, in $X_1 \cap (t \neq 0)$,

$$(P_{2,1}) \cap (P_{3,i}) = \varnothing,$$

$$(P_{2,1}) \cap (P_{4,1}) = \bigl\{\bigl(0, -\tfrac{1}{2}, t\bigr) \mid t^{15} = -\tfrac{1}{\sqrt{2}}\bigr\},$$

$$(P_{2,1}) \cap (P_{4,3}) = \bigl\{(0, -\sqrt{-1}, t) \mid t^{15} = (-1)^{3/4}\bigr\},$$

$$(P_{2,1}) \cap (P_{5,k}) = \varnothing,$$

$$(P_{3,1}) \cap (P_{3,2}) = \varnothing,$$

$$(P_{3,i}) \cap (P_{4,1}) = \bigl\{\bigl(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, t\bigr) \mid t^5 = -\zeta_3^{2i}\tfrac{1}{\sqrt{2}}\bigr\},$$

$$(P_{3,i}) \cap (P_{4,3}) = \bigl\{(x(t), y(t), t) \mid \zeta_3^{2i}t^{10} - (1+\sqrt{-1})((-1)^{1/4}\zeta_3^i t^5 - 1) = 0\bigr\},$$

$$(P_{3,i}) \cap (P_{5,k}) = \varnothing,$$

$$(P_{4,1}) \cap (P_{4,3}) = \varnothing,$$

$$(P_{4,1}) \cap (P_{5,k}) = \bigl\{(x(t), y(t), t) \mid 2^{7/10}\zeta_5^{3k}t^6 + \zeta_5^{4k}t^3 + 2^{3/10} = 0\bigr\},$$

$$\begin{aligned}(P_{4,3}) \cap (P_{5,k}) = \bigl\{(x(t), y(t), t) \mid \zeta_5^{3k}t^6\bigl(\zeta_5^{3k}t^6 - 2^{4/5}(-1)^{1/4}\zeta_5^{4k}t^3 + 2^{3/5}\sqrt{-1}\,\bigr)\\ - 2^{1/5}(1+\sqrt{-1})\bigl(2^{1/5}(-1)^{1/4}\zeta_5^{4k}t^3 - \sqrt{-1}\,\bigr) = 0\bigr\},\end{aligned}$$

$$(P_{5,k_1}) \cap (P_{5,k_2}) = \varnothing \quad (k_1 \neq k_2).$$

and the local intersection numbers of the divisors $(P_{2,1})$, $(P_{3,i})$, $(P_{4,j})$, $(P_{5,k})$ at these intersection points are all one.

On the other hand, in the ∞-model X_2 (the surface obtained by the variable transformation $\bar{x} = x/t^{20}$, $\bar{y} = y/t^{30}$, $\bar{t} = 1/t$) or its projection $\overline{X}_2$, the divisors $(P_{2,1})$, $(P_{3,i})$, $(P_{4,j})$ and $(P_{5,k})$ are given by

$$(P_{2,1}) = (\bar{x} = 0, \bar{y} = -1),$$

$$(P_{3,i}) = (\bar{x} = -\zeta_3^i, \bar{y} = -\zeta_3^i\bar{t}^{10}),$$

$$(P_{4,1}) = \bigl\{([\sqrt{2}\bar{t}^{20} + 2\bar{t}^5 : \sqrt{2}\bar{t}^{30} + 3\bar{t}^{15} + 2^{2/3} : \bar{t}^{15}], \bar{t}\,) \mid \bar{t} \in \mathbb{A}^1\bigr\},$$

$$(P_{4,3}) = \bigl(\bar{x} = -2\bar{t}^{20} + 2(-1)^{1/4}\bar{t}^5, \bar{y} = -2\sqrt{-1}\bar{t}^{30} + 4\sqrt{-1}(-1)^{1/4}\bar{t}^{15} + 1\bigr),$$

$$(P_{5,k}) = \bigl\{([2^{-2/5}\zeta_5^{2k}\bar{t}^2 : 2^{-2/5}\zeta_5^{2k}\bar{t}^{12} + 2^{-3/5}\zeta_5^{3k} : \bar{t}^6], \bar{t}\,) \mid \bar{t} \in \mathbb{A}^1\bigr\}.$$

Thus when $\bar{t}$ equals 0 (t equals ∞), the divisors $(P_{4,1})$, $(P_{5,k})$ and ∞ intersect at $([0:1:0], 0) \in \overline{X}_2$ and the other pairs of $(P_{2,1})$, $(P_{3,i})$, $(P_{4,j})$, $(P_{5,k})$ and ∞ do not intersect. Moreover the local intersection number of $(P_{4,1})$ and ∞ at this point is five, and the numbers of the other pairs of $(P_{4,1})$, $(P_{5,k})$ and ∞ are two. From the above and (6), we obtain

$$(P_{2,1})\cdot(P_{d,l}) = \begin{cases} -10 & (d=2), \\ 0 & (d=3,5), \\ 15 & (d=4), \end{cases}$$

$$(P_{3,i})\cdot(P_{d,l}) = \begin{cases} -10 & (d=3, i=l), \\ 0 & (d=3, i\neq l), \\ 5 & (d=4, j=1), \\ 10 & (d=4, j=3), \\ 0 & (d=5), \end{cases}$$

$$(P_{4,j})\cdot(P_{d,l}) = \begin{cases} -10 & (d=4, j=l), \\ 0 & (d=4, j\neq l), \\ 8 & (d=5, j=1), \\ 12 & (d=5, j=3), \end{cases}$$

$$(P_{5,k})\cdot(P_{5,l}) = \begin{cases} -10 & (k=l), \\ 2 & (k\neq l). \end{cases}$$

Finally we give $\Phi_{d,l}$ for $d \in \mathrm{Adm}_4$, $l \in (\mathbb{Z}/d\mathbb{Z})^\times$ by the method mentioned in Section 2 and compute $(D_{d,l}+\Phi_{d,l})\cdot D_{d',l'}$ where d, $d' \in \mathrm{Adm}_4$, $l \in (\mathbb{Z}/d\mathbb{Z})^\times$, $l' \in (\mathbb{Z}/d'\mathbb{Z})^\times$. Recall that $\Phi_{d,l}$ is defined by $\Phi_{d,l} = \sum_{i=1}^{59} a_i(P_{d,l})F_{0,i}$ and

$$(a_1(P_{d,l}),\ldots,a_{59}(P_{d,l})) = -(D_{d,l}\cdot F_{0,1},\ldots,D_{d,l}\cdot F_{0,59})(F_{0,i}\cdot F_{0,j})^{-1}.$$

For integers $1 \le m \le 59$, since $\infty \cdot F_{0,m} = 0$, we have, in Kronecker delta notation,

$$D_{2,1}\cdot F_{0,m} = (P_{2,1})\cdot F_{0,m} = \delta_{m,30},$$

$$D_{3,i}\cdot F_{0,m} = (P_{3,i})\cdot F_{0,m} = \delta_{m,20},$$

$$D_{4,1}\cdot F_{0,m} = (P_{4,1})\cdot F_{0,m} = \delta_{m,15},$$

$$D_{4,3}\cdot F_{0,m} = (P_{4,3})\cdot F_{0,m} = 0,$$

$$D_{5,k}\cdot F_{0,m} = (P_{5,k})\cdot F_{0,m} = \delta_{m,24}.$$

Since the reduction type of $(\mathcal{E}_{60})_0$ is I_{60}, we have the intersection matrix

$$(F_{0,i}\cdot F_{0,j})_{1\le i,j\le 59} = \begin{bmatrix} -2 & 1 & \cdots & 0 \\ 1 & -2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & -2 \end{bmatrix}.$$

It is an easy exercise to show that the j-th row of the inverse of this matrix is $\frac{-1}{60}[60-j, 2\cdot(60-j), \ldots, j\cdot(60-j), j\cdot(59-j), \ldots j\cdot 2, j]$. Therefore we obtain

$a_1(P_{4,3}) = \cdots = a_{59}(P_{4,3}) = 0$ and

$$\begin{aligned}
(a_1(P_{2,1}), \ldots, a_{59}(P_{2,1})) &= \tfrac{1}{60}[30, 2\cdot 30, \ldots, 30\cdot 30, \ldots, 30\cdot 2, 30],\\
(a_1(P_{3,i}), \ldots, a_{59}(P_{3,i})) &= \tfrac{1}{60}[40, 2\cdot 40, \ldots, 20\cdot 40, \ldots, 20\cdot 2, 20],\\
(a_1(P_{4,1}), \ldots, a_{59}(P_{4,1})) &= \tfrac{1}{60}[45, 2\cdot 45, \ldots, 15\cdot 45, \ldots, 15\cdot 2, 15],\\
(a_1(P_{5,k}), \ldots, a_{59}(P_{5,k})) &= \tfrac{1}{60}[36, 2\cdot 36, \ldots, 24\cdot 36, \ldots, 24\cdot 2, 24].
\end{aligned}$$

In particular, $\Phi_{4,3} = 0$ and $\Phi_{3,i}$, $\Phi_{5,k}$ do not depend on i, k, so we put $\Phi_2 = \Phi_{2,1}$, $\Phi_3 = \Phi_{3,i}$, $\Phi_4 = \Phi_{4,1}$, $\Phi_5 = \Phi_{5,k}$. We can compute $(D_{d,l} + \Phi_d)\cdot D_{d',l'}$, where $d, d' \in \mathrm{Adm}_4$, $l \in (\mathbb{Z}/d\mathbb{Z})^\times$, $l' \in (\mathbb{Z}/d'\mathbb{Z})^\times$, as follows:

$$(D_{2,1} + \Phi_2)\cdot D_{d,l} = \begin{cases} -5 & (d=2),\\ 0 & (d=3),\\ \frac{15}{2} & (d=4, l=1),\\ 5 & (d=4, l=3),\\ 0 & (d=5),\end{cases}$$

$$(D_{3,i} + \Phi_3)\cdot D_{d,l} = \begin{cases} -\frac{20}{3} & (d=3, i=l),\\ \frac{10}{3} & (d=3, i\neq l),\\ 0 & (d=4,5),\end{cases}$$

$$(D_{4,j} + \Phi_4)\cdot D_{d,l} = \begin{cases} -\frac{75}{4} & (d=4,\ j=l=1),\\ -20 & (d=4,\ j=l=3),\\ -15 & (d=4,\ j\neq l),\\ 0 & (d=5),\end{cases}$$

$$(D_{5,k} + \Phi_5)\cdot D_{d,l} = \begin{cases} -\frac{48}{5} & (d=5, k=l),\\ \frac{12}{5} & (d=5, k\neq l).\end{cases}$$

Since $(D_{d,l} + \Phi_d)\cdot F = 0$ for all fibral divisors F, we have $(D_{d,l} + \Phi_d)\cdot D_{d',l'} = (D_{d',l'} + \Phi_{d'})\cdot D_{d,l}$. Thus we obtain

$$N = \begin{bmatrix}
-5 & 0 & 0 & \frac{15}{2} & 5 & 0 & 0 & 0 & 0\\
0 & -\frac{20}{3} & \frac{10}{3} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{10}{3} & -\frac{20}{3} & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{15}{2} & 0 & 0 & -\frac{75}{4} & -15 & 0 & 0 & 0 & 0\\
5 & 0 & 0 & -15 & -20 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -\frac{48}{5} & \frac{12}{5} & \frac{12}{5} & \frac{12}{5}\\
0 & 0 & 0 & 0 & 0 & \frac{12}{5} & -\frac{48}{5} & \frac{12}{5} & \frac{12}{5}\\
0 & 0 & 0 & 0 & 0 & \frac{12}{5} & \frac{12}{5} & -\frac{48}{5} & \frac{12}{5}\\
0 & 0 & 0 & 0 & 0 & \frac{12}{5} & \frac{12}{5} & \frac{12}{5} & -\frac{48}{5}
\end{bmatrix},$$

and $\det N = -2^8 3^5 5^4 \neq 0$. Therefore the divisors C_0, ∞, $D_{2,1}$, $D_{3,i}$, $D_{4,j}$, $D_{5,k}$, $F_{0,1}, \ldots, F_{0,59}$ form a $\mathbb{Q}$-basis of $\mathrm{NS}(\mathcal{E}_{60})$.

Similarly in the cases of $n \leq 6$, we can compute $\det M = \pm 1$, where M is the intersection matrix of the divisors in Theorem 3.2 (see Lemma 2.2). In particular, the divisors form a $\mathbb{Z}$-basis of $\mathrm{NS}(\mathcal{E}_n)$. □

Stiller's Example 1. This example is the minimal elliptic surface whose generic fiber is the elliptic curve defined by

$$Y^2 = 4X^3 - 3u^{3n}X - u^{5n} \quad (u \in \mathbb{P}^1_{\mathbb{C}}, n \in \mathbb{N})$$

over $\mathbb{C}(u)$. By changing the variables suitably, the defining equation becomes

$$y^2 = x^3 + t^n x + t^n \quad (t \in \mathbb{P}^1_{\mathbb{C}}). \tag{9}$$

We denote by E_n the elliptic curve defined by (9) and by $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}}$ the associated elliptic surface. Stiller [1987] proved that the Mordell–Weil rank $r = \mathrm{rank}(E_n(\mathbb{C}(t)))$ is given by

$$r = \sum_{\substack{d \mid n \\ d \in \mathrm{Adm}_1}} \varphi(d), \tag{10}$$

where φ is the Euler function and $\mathrm{Adm}_1 = \{1, 2, 3, 7, 8, 10, 12, 15, 18, 20, 42\}$.

Remark 3.4. For use in Section 4, we note that $d \in \mathrm{Adm}_1$ if and only if each $j \in \{j \in \mathbb{N} : 9d \leq 12j \leq 10d\}$ is not relatively prime to d. Such d's are called *admissible* in [Stiller 1987].

Definition 3.5. For $d \in \mathrm{Adm}_1$ and $j \in (\mathbb{Z}/d\mathbb{Z})^\times$, we define $\mathbb{C}(t)$-rational points $P_{d,j}$ of E_d as follows.

$$\begin{aligned}
P_{1,1} &= \bigl(-1, \sqrt{-1}\bigr),\\
P_{2,1} &= \bigl(\sqrt{-1}\, t, -t\bigr),\\
P_{3,j} &= \bigl(-\zeta_3^j t, \sqrt{-1}\zeta_3^{2j} t^2\bigr),\\
P_{7,j} &= \bigl(-\zeta_7^{2j} t^2 - \zeta_7^{3j} t^3, \sqrt{-1}(\zeta_7^{3j} t^3 + \zeta_7^{4j} t^4 + \zeta_7^{5j} t^5)\bigr),\\
P_{8,j} &= \left(\sum_{i=0}^{2} a_i(8, j) t^{i+2}, \sum_{i=0}^{3} b_i(8, j) t^{i+3}\right),\\
P_{10,j} &= \bigl(2^{2/5}\zeta_{10}^{4j} t^4, -\zeta_{10}^{5j} t^5 - 2^{1/5}\zeta_{10}^{7j} t^7\bigr),\\
P_{12,j} &= \left(\sum_{i=0}^{2} a_i(12, j) t^{4+i}, \sum_{i=0}^{3} b_0(12, j) t^{6+i}\right),
\end{aligned}$$

$$P_{15,j} = \big(-\zeta_{15}^{5j} t^5 - 3^{1/5}\zeta_{15}^{6j} t^6 - 3^{2/5}\zeta_{15}^{7j} t^7,$$
$$\sqrt{-1}(3^{3/5}\zeta_{15}^{8j} t^8 + 3^{4/5}\zeta_{15}^{9j} t^9 + 2\zeta_{15}^{10j} t^{10} + 3^{1/5}\zeta_{15}^{11j} t^{11})\big),$$
$$P_{18,j} = \left(\sum_{i=0}^{2} a_i(18, j)(\zeta_{18}^{j} t)^{6+2i}, \sum_{i=0}^{3} b_i(18, j)(\zeta_{18}^{j} t)^{9+2i}\right),$$
$$P_{20,j} = \left(\sum_{i=0}^{2} a_i(20, j)(\zeta_{20}^{j} t)^{6+2i}, \sum_{i=0}^{3} b_i(20, j)(\zeta_{20}^{j} t)^{9+2i}\right),$$
$$P_{42,j} = \left(\sum_{i=1}^{5} a_i(\zeta_{42}^{j} t)^{12+2i}, \sum_{i=1}^{7} b_i(\zeta_{42}^{j} t)^{19+2i}\right),$$

where the coefficients $a_k(d, j), b_k(d, j)$ are given by Table 2 and the set of complex numbers $(a_1, \dots, a_5, b_1, \dots, b_7)$ are solutions of a system of equations

$$\begin{aligned}
b_7^2 &= a_5^3,\\
2b_6b_7 &= 3a_4a_5^2 + a_5,\\
2b_5b_7 + b_6^2 &= 3a_3a_5^2 + 3a_4^2a_5 + a_4,\\
2b_4b_7 + 2b_5b_6 &= 3a_2a_5^2 + 6a_3a_4a_5 + a_4^3 + a_3,\\
2b_3b_7 + 2b_4b_6 + b_5^2 &= 3a_1a_5^2 + (6a_2a_4 + 3a_3^2)a_5 + 3a_3a_4^2 + a_2,\\
2b_2b_7 + 2b_3b_6 + 2b_4b_5 &= (6a_1a_4 + 6a_2a_3)a_5 + 3a_2a_4^2 + 3a_3^2a_4 + a_1,\\
2b_1b_7 + 2b_2b_6 + 2b_3b_5 + b_4^2 &= (6a_1a_3 + 3a_2^2)a_5 + 3a_1a_4^2 + 6a_2a_3a_4 + a_3^3,\\
2b_1b_6 + 2b_2b_5 + 2b_3b_4 &= 6a_1a_2a_5 + (6a_1a_3 + 3a_2^2)a_4 + 3a_2a_3^2,\\
2b_1b_5 + 2b_2b_4 + b_3^2 &= 3a_1^2a_5 + 6a_1a_2a_4 + 3a_1a_3^2 + 3a_2^2a_3,\\
2b_1b_4 + 2b_2b_3 &= 3a_1^2a_4 + 6a_1a_2a_3 + a_2^3,\\
2b_1b_3 + b_2^2 &= 3a_1^2a_3 + 3a_1a_2^2,\\
2b_1b_2 &= 3a_1^2a_2,\\
b_1^2 &= a_1^3 + 1.
\end{aligned}$$

The system is given by comparing the coefficients of $(b_1t^{21} + b_2t^{23} + \cdots + b_7t^{33})^2$ with those of $(a_1t^{14} + a_2t^{16} + \cdots + a_5t^{22})^3 + t^{42}(a_1t^{14} + a_2t^{16} + \cdots + a_5t^{22}) + t^{42}$.

Theorem 3.6. *Let $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}} (n \in \mathbb{N})$ be the elliptic surfaces associated to the elliptic curves $E_n : y^2 = x^3 + t^n x + t^n$ $(n \in \mathbb{N})$ over $\mathbb{P}^1_{\mathbb{C}}$. Then, for each $n \in \mathbb{N}$, $\mathrm{NS}(\mathcal{E}_n)$ has a $\mathbb{Q}$-basis $C_0, \infty, D_{d,j}, F_{t,a}(d \in \mathrm{Adm}_1, d|n, j \in (\mathbb{Z}/d\mathbb{Z})^\times, t \in \Sigma(\mathcal{E}_n), 1 \le a \le m_t - 1)$. Moreover if $\mathcal{E}_n$ is rational* (*i.e.*, $n = 1, 2, 3, 4, 6, 7, 8$ *or* 12), *then these divisors form a $\mathbb{Z}$-basis.*

The argument of the proof is the same as the previous one, though the computation is very complicated.

j	1	3	5	7
$a_0(8, j)$	$2^{-1/2}$	$-\sqrt{2}$	$\sqrt{-1}+(-1)^{1/4}+1$	0
$a_1(8, j)$	0	$2^{3/4}$	0	$a_2(8, 7)^{1/4}(2^{5/2}+4)$
$a_2(8, j)$	0	-1	$\sqrt{-1}$	$\sqrt{-2\sqrt{2}-3}$
$b_0(8, j)$	$2^{-3/4}$	$-2^{3/4}\sqrt{-1}$	$(\sqrt{-1}+(-1)^{1/4}+1)^{3/2}$	0
$b_1(8, j)$	0	$3\sqrt{-1}$	0	1
$b_2(8, j)$	$2^{-1/4}$	$-2^{5/4}\sqrt{-1}$	0	$a_2(8, 7)^{3/4}\sqrt{2\sqrt{2}+2}(\sqrt{2}+1)^{1/4}$
$b_3(8, j)$	0	$\sqrt{-2}$	$\big(3(-1)^{1/4}(\sqrt{-1}-1)-4\big) \big/ \big(\sqrt{-1}+(-1)^{1/4}+1\big)^{3/2}$	$a_2(8, 7)^{3/2}\sqrt{2\sqrt{2}+2}(\sqrt{2}-1)^{1/4}$

j	1	5	7	11
$a_0(12, j)$	$-\frac{1}{12}a_1(12, 1)^2a_2(12, 1)(-33+5a_2(12, 1)^2)$	-1	$(1+\sqrt{-3})/2$	$\frac{1}{4}a_1(12, 11)^2a_2(12, 11)(5+3a_2(12, 11)^2)$
$a_1(12, j)$	a root of $z^6+180a_2(12, 1)+388a_2(12, 1)^3=0$	0	0	a root of $z^6-12a_2(12, 11)+36a_2(12, 11)^3=0$
$a_2(12, j)$	$\sqrt{2\sqrt{3}+3}$	$\sqrt{(2/3)\sqrt{3}-1}$	$\sqrt{(2/3)\sqrt{3}-1}$	$\sqrt{(2/3)\sqrt{3}-1}$

j	1, 5, 7, 11
$b_0(12, j)$	$\big(4a_0(12, j)a_1(12, j)a_2(12, j)(a_2(12, j)^2+1)(3a_2(12, j)^4+6a_2(12, j)^2-1) - a_1(12, j)^3(a_2(12, j)^2-1)(a_2(12, j)^4+6a_2(12, j)^2+1)\big)/16b_3(12, j)^5$
$b_1(12, j)$	$\big(4a_0(12, j)a_2(12, j)(a_2(12, j)^2+1)(3a_2(12, j)^2+1)+a_1(12, j)^2(3a_2(12, j)^4+6a_2(12, j)^2-1)\big)/8b_3(12, j)^3$
$b_2(12, j)$	$\big(a_1(12, j)(3a_2(12, j)^2+1)\big)/b_3(12, j)$
$b_3(12, j)$	$\sqrt{a_2(12, j)^3+a_2(12, j)}$

j	1, 11	5, 13
$a_0(18, j)$	0	$2b_1(18, j)b_2(18, j)-b_2(18, j)^6$
$a_1(18, j)$	$-2^{2/9}3^{-1/3}$	$b_2(18, j)^2$
$a_2(18, j)$	$2^{-2/9}3^{-2/3}$	0

j	1, 11	5, 13
$b_0(18, j)$	1	a root of $z^3 - 9z^2 - 9z + 9 = 0$
$b_1(18, j)$	0	a root of $8919936 - 8011872 b_0(18, j) - 9735552 b_0(18, j)^2 + z^9 = 0$
$b_2(18, j)$	0	$\frac{1}{36} b_1(18, j)^2(-45 - 15 b_0(18, j) + 2 b_0(18, j)^2)$
$b_3(18, j)$	$-2^{-1/3}3^{-1}$	0

j	7, 17
$a_0(18, j)$	$-4^{1/3}$
$a_1(18, j)$	a root of $4 + 3a_0(18, j)^2 z^3 + 6a_0(18, j) z^6 + z^9 = 0$
$a_2(18, j)$	$-\frac{1}{12} a_1(18, j)^2 (5a_0(18, j)^2 + 6a_0(18, j) a_1(18, j)^3 + a_1(18, j)^6)$
$b_0(18, j)$	$\big((12a_1(18, j)a_2(18, j)^4 - 4a_2(18, j)^3)a_0(18, j) - a_1(18, j)^3 a_2(18, j)^3 + 3a_1(18, j)^2 a_2(18, j)^2 + 5a_1(18, j)a_2(18, j) + 1\big)/16a_2(18, j)^{9/2}$
$b_1(18, j)$	$\big(12a_0(18, j)a_2(18, j)^3 + 3a_1(18, j)^2 a_2(18, j)^2 - 2a_1(18, j)a_2(18, j) - 1\big)/8a_2(18, j)^{5/2}$
$b_2(18, j)$	$\big(3a_1(18, j)a_2(18, j)^2 + a_2(18, j)\big)/2a_2(18, j)^{3/2}$
$b_3(18, j)$	$a_2(18, j)^{3/2}$

j	1, 3, 7, 9
$a_0(20, j)$	$\frac{1}{80} a_1(20, j)^2 a_2(20, j)(172 - 220a_2(20, j)^2 + 117a_1(20, j)^5 + 205a_1(20, j)^5 a_2(20, j)^2)$
$a_1(20, j)$	a root of $56 - 328a_2(20, j)^2 - 2500a_2(20, j)^2 z^5 + 625z^{10} = 0$
$a_2(20, j)$	a root of $5z^4 + 2z^2 + 1 = 0$

j	11, 13, 17, 19
$a_0(20, j)$	$-\frac{1}{16} a_1(20, j)^2 a_2(20, j)(-191 - 188a_2(20, j)^2 + 69a_2(20, j)^4 + 6a_2(20, j)^6)$
$a_1(20, j)$	a root of $z^5 + 1 - 45a_2(20, j)^2 - 15a_2(20, j)^4 + 15a_2(20, j)^6 = 0$
$a_2(20, j)$	a root of $z^8 + 12z^6 - 26z^4 - 52z^2 + 1 = 0$

j	1, 3, 7, 9, 11, 13, 17, 19
$b_0(20, j)$	$a_0(20, j)^{3/2}$
$b_1(20, j)$	$\big(3a_0(20, j)^2 a_1(20, j) + 1\big)/a_0(20, j)^{3/2}$
$b_2(20, j)$	$\big(12a_0(20, j)^5 a_2(20, j) + 3a_0(20, j)^4 a_1(20, j)^2 - 6a_0(20, j)^2 a_1(20, j) - 1\big)/8a_0(20, j)^{9/2}$
$b_3(20, j)$	$\big(12(a_0(20, j)^7 a_1(20, j) - a_0(20, j)^5)a_2(20, j) - a_0(20, j)^6 a_1(20, j)^3 + 15a_0(20, j)^4 a_1(20, j)^2 + 9a_0(20, j)^2 a_1(20, j) + 1\big)/16a_0(20, j)^{15/2}$

Table 2. Coefficients $a_k(d, j), b_k(d, j)$

Stiller's Example 2. This example is the minimal elliptic surface whose generic fiber is the elliptic curve defined by

$$Y^2 = 4X^3 - 3u^n X - u^{2n} \quad (u \in \mathbb{P}^1_{\mathbb{C}}, n \in \mathbb{N})$$

over $\mathbb{C}(u)$. By putting $X = -9x/4$, $Y = 27\sqrt{-1}y/4$, $u = \sqrt[n]{-27/4}t$, the defining equation becomes

$$y^2 = x^3 + t^n x + t^{2n} \quad (t \in \mathbb{P}^1_{\mathbb{C}}). \tag{11}$$

We denote by E^2_n the elliptic curve defined by (11) and by $f : \mathcal{E}^2_n \to \mathbb{P}^1_{\mathbb{C}}$ the associated elliptic surface. Similarly to Stiller's Example 1, the Mordell–Weil rank $r = \operatorname{rank}(E_n(\mathbb{C}(t)))$ is given by

$$r = \sum_{\substack{d \mid n \\ d \in \mathrm{Adm}_2}} \varphi(d),$$

where φ is the Euler function and $\mathrm{Adm}_2 = \{1, 2, 5, 6, 8, 9, 12, 14, 20, 21, 30\}$.

We now denote by $\mathcal{E}^1_n$ the elliptic surface of Stiller's Example 1 and by E^1_n the associated elliptic curve. Recall that

$$E^1_n : y^2 = x^3 + t^n x + t^n.$$

We assume that n is even and write $n = 2m$. By putting $\bar{x} = x/t^{2m}$, $\bar{y} = y/t^{3m}$, $\bar{t} = 1/t$, we obtain

$$\bar{y}^2 = \bar{x}^3 + \bar{t}^{2m}\bar{x} + \bar{t}^{4m}.$$

Since this equation is the defining equation of E^2_{2m}, we obtain an isomorphism

$$\begin{aligned} \mathcal{E}^1_{2m} &\xrightarrow{\sim} \mathcal{E}^2_{2m}, \\ (x, y, t) &\longmapsto (x/t^{2m}, y/t^{3m}, 1/t). \end{aligned} \tag{12}$$

Therefore we define the $\varphi(d)$ rational points of E^2_d for each $d (\geq 2) \in \mathrm{Adm}_2$ as the images of the points of Definition 3.5 via this isomorphism.

Definition 3.7. For $d \in \mathrm{Adm}_2$ and $j \in (\mathbb{Z}/d\mathbb{Z})^\times$, we define $\mathbb{C}(t)$-rational points $P_{d,j}$ of E^2_d as follows.

$$\begin{aligned}
P_{1,1} &= (0, -t), \\
P_{2,1} &= \left(\sqrt{-1}\, t, -t^2\right), \\
P_{5,j} &= \left(2^{2/5}\zeta_5^{3j} t^3, -2^{1/5}\zeta_5^{4j} t^4 - t^5\right), \\
P_{6,j} &= \left(-\zeta_6^{4j} t^4, \sqrt{-1}\zeta_6^{5j} t^5\right), \\
P_{8,j} &= \left(\sum_{i=0}^{2} a_i(8, j) t^{6-i}, \sum_{i=0}^{3} b_i(8, j) t^{9-i}\right),
\end{aligned}$$

$$
\begin{aligned}
P_{9,j} &= \Bigl(\sum_{i=0}^{2} a_i(18,\tilde{j})(\zeta_9^j t)^{6-i},\ \sum_{i=0}^{3} b_i(18,\tilde{j})(\zeta_9^j t)^{9-i}\Bigr),\\
P_{12,j} &= \Bigl(\sum_{i=0}^{2} a_i(12,j)t^{8-i},\ \sum_{i=0}^{3} b_i(12,j)t^{12-i}\Bigr),\\
P_{14,j} &= \bigl(-\zeta_{14}^{8j}t^8-\zeta_{14}^{10j}t^{10},\ \sqrt{-1}(\zeta_{14}^{11j}t^{11}+\zeta_{14}^{13j}t^{13}+\zeta_{14}^{15j}t^{15})\bigr),\\
P_{20,j} &= \Bigl(\sum_{i=0}^{2} a_i(20,j)(\zeta_{20}^j t)^{14-2i},\ \sum_{i=0}^{3} b_i(20,j)(\zeta_{20}^j t)^{21-2i}\Bigr),\\
P_{21,j} &= \Bigl(\sum_{i=1}^{5} a_i(\zeta_{21}^j t)^{15-i},\ \sum_{i=1}^{7} b_i(\zeta_{21}^j t)^{22-i}\Bigr),\\
P_{30,j} &= \bigl(-3^{2/5}\zeta_{30}^{16j}t^{16}-3^{1/5}\zeta_{30}^{18j}t^{18}-\zeta_{30}^{20j}t^{20},\\
&\qquad \sqrt{-1}(3^{1/5}\zeta_{30}^{23j}t^{23}+2\zeta_{30}^{25j}t^{25}+3^{4/5}\zeta_{30}^{27j}t^{27}+3^{1/5}\zeta_{30}^{29j}t^{29})\bigr),
\end{aligned}
$$

where $\tilde{j}$ equals j if j is odd and equals $j+9$ if j is even, and the coefficients $a_k(d, j)$, $b_k(d, j)$, $a_1, \dots, a_5$, $b_1, \dots, b_7$ are same as them of Definition 3.5.

The following theorem follows from the isomorphism (12) and Theorem 3.6.

Theorem 3.8. *Let $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}} (n \in \mathbb{N})$ be the elliptic surfaces associated to the elliptic curves $E_n : y^2 = x^3 + t^n x + t^{2n} (n \in \mathbb{N})$ over $\mathbb{P}^1_{\mathbb{C}}$. Then, for each $n \in \mathbb{N}$, $\mathrm{NS}(\mathcal{E}_n)$ has a $\mathbb{Q}$-basis $C_0, \infty, D_{d,j}, F_{t,a} (d \in \mathrm{Adm}_2, d|n, j \in (\mathbb{Z}/d\mathbb{Z})^\times, t \in \Sigma(\mathcal{E}_n), 1 \le a \le m_t - 1)$. Moreover if $\mathcal{E}_n$ is rational* (*i.e.*, $n = 1, 2, 3, 4, 5, 6, 8, 9$ *or* 12), *then these divisors form a $\mathbb{Z}$-basis.*

Stiller's Example 3. Here we consider the minimal elliptic surface whose generic fiber is the elliptic curve defined by

$$Y^2 = 4X^3 - 3u^{3n}\bigl(u^n - \tfrac{8}{9}\bigr)X + u^{4n}\bigl(u^{2n} - \tfrac{4}{3}u^n + \tfrac{8}{27}\bigr) \quad (u \in \mathbb{P}^1_{\mathbb{C}}, n \in \mathbb{N})$$

over $\mathbb{C}(u)$. By changing the variables suitably, the defining equation becomes

$$y^2 = x^3 + x^2 + t^n x + \tfrac{1}{4}t^{2n} \quad (t \in \mathbb{P}^1_{\mathbb{C}}). \tag{13}$$

We denote by E_n the elliptic curve defined by (13) and by $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}}$ the associated elliptic surface. By [Stiller 1987, Example 3], the Mordell–Weil rank $r = \mathrm{rank}(E_n(\mathbb{C}(t)))$ equals 1 if n is even and equals 0 if n is odd. Therefore similarly to Examples 1, 2 and 4, we obtain $r = \sum_{\substack{d \in \mathrm{Adm}_3 \\ d|n}} \varphi(d)$, $\mathrm{Adm}_3 = \{2\}$, and we can show:

Theorem 3.9. *Let $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}} (n \in \mathbb{N})$ be the elliptic surfaces associated to the elliptic curves $E_n : y^2 = x^3 + x^2 + t^n x + t^{2n}/4$ $(n \in \mathbb{N})$ over $\mathbb{P}^1_{\mathbb{C}}$. For each $n \in \mathbb{N}$, if n*

is odd, then $\mathrm{NS}(\mathcal{E}_n)$ *has a* $\mathbb{Z}$*-basis* $C_0, \infty, F_{t,a}$ $(t \in \Sigma(\mathcal{E}_n), 1 \le a \le m_t - 1)$, *and if* n *is even, then the group has a* $\mathbb{Q}$*-basis* $C_0, \infty, D_{2,1}, F_{t,a}$ $(t \in \Sigma(\mathcal{E}_n), 1 \le a \le m_t - 1)$, *where* $D_{2,1} = (P_{2,1}) - \infty$ *and* $\mathbb{C}(t)$*-rational point* $P_{2,1}$ *is defined by*

$$P_{2,1} = \left(-\tfrac{1}{2}t^n, \tfrac{1}{4}\sqrt{-2}t^{3n/2}\right).$$

Moreover if $\mathcal{E}_n$ *is rational* (*i.e.*, $n \le 3$), *then these divisors form a* $\mathbb{Z}$*-basis.*

Stiller's Example 5. We finally consider the minimal elliptic surface whose generic fiber is the elliptic curve defined by the equation†

$$Y^2 = 4X^3 - 3u^{12k+3}\left(u^{4k+1} - \tfrac{3}{4}\right)X - u^{20k+5}\left(u^{4k+1} - \tfrac{9}{8}\right) \quad (u \in \mathbb{P}^1_{\mathbb{C}}, k \in \mathbb{N})$$

over $\mathbb{C}(u)$. By changing the variables suitably, the defining equation becomes

$$y^2 = x^3 + x^2 + t^{4k+1}x \quad (t \in \mathbb{P}^1_{\mathbb{C}}).$$

Here we discuss a slightly more general equation:

$$y^2 = x^3 + x^2 + t^n x \quad (t \in \mathbb{P}^1_{\mathbb{C}}, n \in \mathbb{N}). \tag{14}$$

We denote by E_n the elliptic curve defined by (14) and by $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}}$ the associated elliptic surface. The surface $\mathcal{E}_n$ has singular fibers of type I_{2n} over 0, type I_1 over $\zeta_n^i \sqrt[n]{1/4}$ $(0 \le i \le n-1)$ and III^* (resp. I_0^*, III, I_0) over ∞ as $n \equiv 1$ (resp. 2, 3, 0) modulo 4. Using Stiller's method, one can show that the Mordell–Weil rank $r = \mathrm{rank}(E_n(\mathbb{C}(t)))$ is given by

$$r = \sum_{\substack{d|n \\ d \in \mathrm{Adm}_5}} \varphi(d),$$

where φ is the Euler function and $\mathrm{Adm}_5 = \{2, 3\}$. We obtain the following theorem similarly to the other examples.

Theorem 3.10. *Let* $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}}$ $(n \in \mathbb{N})$ *be the elliptic surfaces associated to the elliptic curves* $E_n : y^2 = x^3 + x^2 + t^n x$ $(n \in \mathbb{N})$ *over* $\mathbb{P}^1_{\mathbb{C}}$. *Then, for each* $n \in \mathbb{N}$, $\mathrm{NS}(\mathcal{E}_n)$ *has a* $\mathbb{Q}$*-basis* $C_0, \infty, D_{d,j}, F_{t,a}$ $(d \in \mathrm{Adm}_5, d|n, j \in (\mathbb{Z}/d\mathbb{Z})^\times, t \in \Sigma(\mathcal{E}_n)$, $1 \le a \le m_t - 1)$, *where* $D_{d,j} = (P_{d,j}) - \infty$ *and* $\mathbb{C}(t)$*-rational points* $P_{2,1}, P_{3,j}$ *are defined by*

$$P_{2,1} = \left(\sqrt{-1}\, t^{n/2}, \sqrt{-1}\, t^{n/2}\right),$$
$$P_{3,j} = \left(2^{2/3}\zeta_3^j t^{n/3}, 2^{2/3}\zeta_3^j t^{n/3} + 2^{1/3}\zeta_3^{2j} t^{2n/3}\right).$$

Moreover if $\mathcal{E}_n$ *is rational* (*i.e.*, $n \le 4$), *then these divisors form a* $\mathbb{Z}$*-basis.*

†The equation in [Stiller 1987, page 188] is incorrect.

4. Alternative proof of Stiller's computations

Each surface in [Stiller 1987, Examples 1–5] has an automorphism such that it acts in multiplicity one (i.e., each eigenspace is one-dimensional) on the second de Rham cohomology modulo zero and fibral divisor classes. Stiller showed this by using the *inhomogeneous* de Rham cohomology, and this is essentially used in his argument on the computation of Picard numbers.

We gave the explicit $\mathbb{Q}$-bases of the Néron–Severi groups in the last section, where we used his results on the Picard numbers. However once one has the divisors as in the last section, one can conclude that they automatically form a $\mathbb{Q}$-basis of the Néron–Severi group. We show it in this section.

Let $f : \mathcal{E}_n \to \mathbb{P}^1_{\mathbb{C}}$ ($n \in \mathbb{N}$) be one of the families of elliptic surfaces of Examples 1–5. Let $\mathrm{NS}(\mathcal{E}_n)'$ be the subgroup of $\mathrm{NS}(\mathcal{E}_n)$ which is generated by all the divisors in Theorem 3.2, 3.6, 3.8, 3.9 or 3.10, as the case may be. Put $H^2_{\mathrm{tr}}(\mathcal{E}_n) = H^2(\mathcal{E}_n, \mathbb{Q})/\mathrm{NS}(\mathcal{E}_n)_{\mathbb{Q}}$, $V(\mathcal{E}_n) = H^2(\mathcal{E}_n, \mathbb{Q})/\mathrm{NS}(\mathcal{E}_n)'_{\mathbb{Q}}$. The goal is to show that $\mathrm{NS}(\mathcal{E}_n)_{\mathbb{Q}} = \mathrm{NS}(\mathcal{E}_n)'_{\mathbb{Q}}$, or equivalently

$$\dim V(\mathcal{E}_n) \le \dim H^2_{\mathrm{tr}}(\mathcal{E}_n). \tag{15}$$

We give a proof of (15) only for Stiller's Example 1 since the same argument works in the other cases. We already know the dimension of $V(\mathcal{E}_n)$. In the case at hand, the result can be written as

$$\dim V(\mathcal{E}_n) = \sum_{d \in S^1_n} \varphi(d), \tag{16}$$

where we put $S^1_n = \{d \in \mathbb{N} : d|n,\ d \notin \mathrm{Adm}_1 \cup \{4, 6\}\}$ and $\varphi(d)$ is the Euler function. In particular, when $n = 1, 2, 3, 4, 6, 7, 8$ or 12, the value of (16) is zero and there is nothing to prove. We assume $n \neq 1, 2, 3, 4, 6, 7, 8$ or 12.

Let $\sigma : \mathcal{E}_n \to \mathcal{E}_n$ be an automorphism which is defined by $(x, y, t) \mapsto (x, y, \zeta_n^{-1} t)$, and let σ^* be the automorphism on $H^2_{\mathrm{tr}}(\mathcal{E}_n)$ induced by σ. We denote by $f(T)$ the minimal polynomial of σ^* over $\mathbb{Q}$. If we have

$$f(\zeta_d) = 0 \ \text{ for each } d \in S^1_n, \tag{17}$$

then d-th cyclotomic polynomial divides into $f(T)$ and hence we have

$$\dim H^2_{\mathrm{tr}}(\mathcal{E}_n) \ge \deg f(T) \ge \sum_{d \in S^1_n} \varphi(d) = \dim V(\mathcal{E}_n)$$

and (15) follows. Let us prove (17).

Lemma 4.1. *Let $n = 12l + k$ with $l \ge 0$, $1 \le k \le 12$. If n equals $1, 2, 3, 4, 6, 7, 8$ or 12, then $H^0(\mathcal{E}_n, \Omega^2_{\mathcal{E}_n}) = 0$. Otherwise, $H^0(\mathcal{E}_n, \Omega^2_{\mathcal{E}_n})$ has a basis*

$$t^{2n-a(n)-3} dt \frac{dx}{y}, \quad \dots, \quad t^{2n-a(n)-b(n)-3} dt \frac{dx}{y},$$

k	1	2	3	4	5	6
$a(n)$	$9l-1$	$9l$	$9l+1$	$9l+2$	$9l+2$	$9l+3$
$b(n)$	$l-1$	$l-1$	$l-1$	$l-1$	l	$l-1$
k	7	8	9	10	11	12
$a(n)$	$9l+4$	$9l+5$	$9l+5$	$9l+6$	$9l+7$	$9l+8$
$b(n)$	$l-1$	$l-1$	l	l	l	$l-1$

Table 3. Definitions of $a(n)$ and $b(n)$ for Lemma 4.1.

where $a(n), b(n)$ are defined in Table 3.

Proof. The proof is left as an exercise (see, for example, [Stiller 1987, Proposition 3.3] for details). □

For an integer i with $0 \le i \le b(n)$, since we have

$$\sigma^*\Big(t^{2n-a(n)-i-3}dt\frac{dx}{y}\Big) = \zeta_n^{a(n)+i+2}\Big(t^{2n-a(n)-i-3}dt\frac{dx}{y}\Big),$$

the automorphism σ^* on $H^2_{\mathrm{tr}}(\mathcal{E}_n)$ over $\mathbb{Q}$ has eigenvalues $\zeta_n^{a(n)+i+2}$, so we have $f(\zeta_n^{a(n)+i+2}) = 0$ $(0 \le i \le b(n))$. On the other hand, since we have

$$J(n) := \{a(n)+2, \dots, a(n)+b(n)+2\} = \{j \in \mathbb{N} \mid 9n < 12j < 10n\},$$

we obtain $\{(a(d)+2)n/d, \dots, (a(d)+b(d)+2)n/d\} \subset J(n)$ for each d which divides into n. Then $J(d) = \varnothing$ if and only if $d = 1, 2, 3, 4, 6, 7, 8$ or 12. In addition, $d \in \mathrm{Adm}_1$ if and only if each $j \in \{j \in \mathbb{N} \mid 9d \le 12j \le 10d\}$ is not relatively prime to d (see Remark 3.4). Therefore for each $d \in S_n^1$, there exists a natural number j which is relatively prime to d such that $jn/d \in J(n)$, and we have $\zeta_n^{jn/d} = \zeta_d^j$. This implies (17).

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