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In the first half of this paper, we introduce a prime zeta function associated with the Ihara zeta function, and study several properties of this function. In the last half, using results of the first half, we present graph-theoretic analogues to Mertens' theorems.

1. Introduction

Throughout this paper, we use the notation of [Stark and Terras 1996; Terras 2011] for graph theory and the theory of (Ihara) zeta functions $Z_X(u)$ of graphs, and the notation of [Hardy and Wright 2008] and [Titchmarsh 1958; 1986] for the theory of functions and the Riemann zeta function $\zeta(s)$.

In the analytic theory of the Riemann zeta function, the following theorems are well-known:

- Mertens' first theorem [1874, Equality (5)] (also see [Hardy and Wright 2008, Theorem 425], [Jameson 2003, Theorem 2.6.3], and [Titchmarsh 1986, Equality (3.14.3)]): as $x \to \infty$,
$$\sum_{p \le x} \frac{\log p}{p} = \log x + O(1).$$
- Mertens' second theorem [1874, Equality (13)] (also see [Hardy and Wright 2008, Theorem 427], [Jameson 2003, Theorem 2.6.4/Exercise 4, p. 191], and [Titchmarsh 1986, Equality (3.14.5)]): as $x \to \infty$,
$$\sum_{p \le x} \frac{1}{p} = \log(\log x) + B_1 + O\Big(\frac{1}{\log^k x}\Big)$$
for each $k \ge 1$, where $B_1 = 0.26149\ldots$ is the Mertens constant.
- Mertens' third theorem [1874, Equality (15)] (also see [Hardy and Wright 2008, Theorem 429], [Jameson 2003, Exercise 1, p. 96], and [Titchmarsh 1986,

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Equality (3.15.2)]): as $x \to \infty$,

$$\prod_{p \le x}\left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x},$$

where $\gamma = 0.57721\ldots$ is the Euler–Mascheroni constant.

- Prime number theorem (proved by de la Vallée Poussin and Hadamard in 1896; see, e.g., [Hardy and Wright 2008, Theorem 6], [Jameson 2003, Theorem 3.4.3], and [Titchmarsh 1986, Equality (3.7.1)]): as $x \to \infty$,

$$\pi(x) \sim \frac{x}{\log x},$$

where $\pi(x)$ denotes the number of rational prime numbers p less than x, that is,

$$\pi(x) := \big|\{p : p \text{ is a rational prime number with } p \le x\}\big|.$$

All proofs of the above formulae are related to the Riemann zeta function

$$\zeta(s) = \prod_{p \in \mathcal{P}}\left(1 - \frac{1}{p^s}\right)^{-1},$$

where $\mathcal{P}$ denotes the set of all rational prime numbers, that is,

$$\mathcal{P} := \{p \in \mathbb{Z} : p \text{ is a rational prime number}\},$$

and to the prime zeta function, defined first by Glaisher [1891],

$$P(s) = \sum_{p \in \mathcal{P}} \frac{1}{p^s}.$$

In graph theory, there exists an analogue of the Riemann zeta function, the so-called (Ihara) zeta function $Z_X(u)$ of a graph X (see [Ihara 1966]). Therefore, studying graph-theoretic analogues of these theorems is very interesting. Indeed, Terras and coworkers gave an analogue of the prime number theorem (see Theorem 2.10 in [Horton et al. 2006], and also Theorem 10.1 in [Terras 2011]):

If Δ_X divides n, then, as $n \to \infty$,

$$\pi_X(n) \sim \frac{\Delta_X}{n \cdot R_X^n},$$

and otherwise $\pi_X(n) \sim 0$. (For the definitions of $\pi_X(n)$ and R_X, see this section, and for that of Δ_X, see Section 3.) This is called the graph-theoretic prime number theorem.

In this paper, we define a prime zeta function of a graph, and investigate several properties of this function. In particular, we show that this has a natural boundary. Moreover, by using this function, we present graph-theoretic analogues of Mertens' theorems.

We shall note a relation between previous works and our works. A zeta function of a graph can be specialized from a dynamical zeta function for a flow (see Chapter 4 in [Terras 2011]), and dynamical-systemic analogues to the above formulae are already known (see, e.g., [Sharp 1991] for Mertens' theorems, and [Parry 1983; Parry and Pollicott 1983] for a prime number theorem). In that sense, our statements for Mertens' theorems are not new (see Remark 17). However, since our proofs are graph-theoretic and elementary, they are completely different from previous proofs.

In this section, we first recall the notation for graph theory and zeta functions of graphs, define a prime zeta function of a graph, and finally state the main theorem.

Now we recall the notation of graph theory. Throughout this paper, we always assume that X is a finite, connected, non-cycle and undirected graph without degree-one vertices. Let X be a graph with vertex set V, with $\nu := |V|$, and edge set E, with $\epsilon := |E|$. Simply, such a graph X is denoted by $X := (V, E)$. Note that ϵ is the number of edges of X.

An oriented edge (or an arc) a from a vertex u to a vertex v is denoted by $a = (u, v)$, and the inverse of a is denoted by $a^{-1} = (v, u)$. The origin and terminus of a are denoted by $o(a)$ and $t(a)$, respectively. We can now orient the edges of X, and label the edges as follows:

$$\vec{E} = \{e_1, e_2, \ldots, e_\epsilon, e_{\epsilon+1} = e_1^{-1}, e_{\epsilon+2} = e_2^{-1}, \ldots, e_{2\epsilon} = e_\epsilon^{-1}\}.$$

A path $C = a_1 \cdots a_s$, where the a_i are oriented edges, is said to have a backtrack (resp. tail) if $a_{j+1} = a_j^{-1}$ for some j (resp. $a_s = a_1^{-1}$), and a path C is called a cycle (or a closed path) if $o(a_1) = t(a_s)$. The length $\ell(C)$ of a path $C = a_1 \cdots a_s$ is defined by $\ell(C) = s$.

A cycle C is called prime (or primitive) if it satisfies the following:

- C does not have backtracks or a tail;
- no cycle D exists such that $C = D^f$ for some $f > 1$.

The equivalence class $[C]$ of a cycle $C = a_1 \cdots a_s$ is defined as the set of cycles

$$[C] := \{a_1 a_2 \cdots a_{s-1} a_s, a_2 \cdots a_{s-1} a_s a_1, \ldots, a_s a_1 a_2 \cdots a_{s-1}\},$$

and an equivalence class $[P]$ of a prime cycle P is called a prime in the graph X. Throughout this paper, we denote a prime by the symbol $[P]$. Two cycles C_1 and C_2 are called equivalent if $C_2 \in [C_1]$. Note that if $[C_1] = [C_2]$, then $\ell(C_1) = \ell(C_2)$, and thus $u^{\ell(C_1)} = u^{\ell(C_2)}$.

Next, we recall the zeta function of a graph $X = (V = \{v_1, \dots, v_\nu\}, E)$, and we define a prime zeta function associated with it. Let u be a complex variable, and let $f_X(u)$ denote

$$f_X(u) := \det(I_\nu - Au + Qu^2),$$

where I_ν is the $\nu \times \nu$ identity matrix, A is the adjacency matrix of X (see Definition 2.1 in [Terras 2011]), and

$$Q = \operatorname{diag}(\deg(v_1) - 1, \dots, \deg(v_\nu) - 1).$$

Let $\pi_X(n)$ denote

$$\pi(n) = \pi_X(n) := \big|\{[P] : [P] \text{ is a prime in } X \text{ with } \ell(P) = n\}\big|.$$

Throughout this paper, we fix an arbitrary real number $t > 1$ (that is, $\log t > 0$), and we set $u = t^{-s}$. The (Ihara) zeta function of X (see Definition 2.2 and Theorem 2.5 in [Terras 2011]) and the prime zeta function of X are defined as follows:

$$Z_X(u) := \prod_{[P]} (1 - u^{\ell(P)})^{-1} = \frac{1}{(1 - u^2)^{\epsilon - \nu} f_X(u)}, \quad \mathscr{Z}_X(s) := Z_X(t^{-s}),$$

$$P_X(u) := \sum_{[P]} u^{\ell(P)} = \sum_{n=1}^{\infty} \pi_X(n) u^n, \qquad \mathscr{P}_X(s) := P_X(t^{-s}),$$

with $|u|$ sufficiently small, where $[P]$ runs through all primes in X. In this paper, we do not distinguish between the two functions $Z_X(u)$ and $\mathscr{Z}_X(s)$, or between $P_X(u)$ and $\mathscr{P}_X(s)$. The right-hand side of the first equality is called the Ihara–Bass formula (see [Bass 1992]). Note that, owing to our assumption for X, the zeta function $Z_X(u)$ is expressible like that.

Note that, for two finite connected graphs X_1 and X_2 without degree-one vertices, $P_{X_1}(u) = P_{X_2}(u)$ if and only if $Z_{X_1}(u) = Z_{X_2}(u)$ (see Proposition 7 in [Storm 2010]).

Let

$$T := \bigcup_{n=1}^{\infty} T_n \quad \text{and} \quad T_n := \{u \in \mathbb{C} : f_X(u^n) = 0\}$$

be the zeroes of the $f_X(u^n)$. Note that the elements of T_n are poles of $Z_X(u^n)$. The radius of convergence of $Z_X(u)$ is denoted by R_X. Note that $0 < R_X < 1$ since X is a non-cycle graph (see, e.g., [Terras 2011, p. 197]). It follows from the graph-theoretic prime number theorem (see Theorem 10.1 in [Terras 2011]) that the radius of convergence of the other function $P_X(u)$ is also equal to R_X. Note that the point $u = R_X$ is a singularity of $P_X(u)$, and that

$$P_X(u) \sim -\log(R_X - u)$$

as $u \uparrow R_X$, which is similar to

$$P(s) \sim -\log(s-1)$$

as $s \downarrow 1$ (see, e.g., [Fröberg 1968]), where $P(s) = \sum_p 1/p^s$ denotes the prime zeta function associated with the Riemann zeta function.

Euclid proved that the number of primes p is infinite. Euler showed that the prime zeta function $\sum_p 1/p$ diverges, and as an application he proved the infinitude of primes. In graph theory, it is also well known that the number of primes $[P]$ in X is infinite. We can give another proof "à la Euler" for this fact since $u = R_X$ is a singularity of $P_X(u)$.

Our main theorem is:

Main Theorem. *Suppose that $X = (V, E)$ is a finite, connected and non-cycle graph without degree-one vertices.*

(1) *Let $\mu(n)$ denote the Möbius function. If $|u| < R_X$, then*

$$P_X(u) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log Z_X(u^n).$$

Furthermore, the right-hand side is absolutely convergent for u satisfying $|u| < 1$ and $u \notin T$, and so $P_X(u)$ has an analytic extension to the region $\{u \in \mathbb{C} : |u| < 1\} \setminus T$.

(2) *The imaginary axis* $\mathrm{Re}(s) = 0$ *is a natural boundary for the function $\mathcal{P}_X(s)$, that is, every point on this line can be realized as a limit point of singularities of $\mathcal{P}_X(s)$.*

(3) (*Graph-theoretic Mertens' first theorem*) *As $N \to \infty$,*

$$\sum_{n \le N} n \cdot \pi_X(n) R_X^n = N + O(1).$$

(4) (*Graph-theoretic Mertens' second theorem*) *There exists a constant B_X such that, as $N \to \infty$,*

$$\sum_{n \le N} \pi_X(n) R_X^n = \log N + B_X + O\Big(\frac{1}{N}\Big).$$

(5) (*Graph-theoretic Mertens' third theorem*) *Let $\gamma = 0.57721\ldots$ denote the Euler–Mascheroni constant. As $N \to \infty$,*

$$\prod_{\ell(P) \le N} (1 - R_X^{\ell(P)}) \sim \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N},$$

where

$$C_X = -\frac{1}{(1-R_X^2)^{\epsilon-\nu} R_X f_X'(R_X)}$$

(*for the definition*, *in detail*, *see Section 3 in this paper*).

The contents of this paper are as follows. In the next section, we prove the first two claims in the main theorem, that is, several properties of $P_X(u)$. In Section 3, we prove the remaining claims in the main theorem, namely, the graph-theoretic Mertens theorems.

2. Prime zeta function of a graph

In this section, we give a proof of parts (1) and (2) of the Main Theorem introduced in Section 1.

The following facts about $Z_X(u)$, etc., are known, and are often used in this paper.

Facts 1. (1) (Basic facts) *For an arbitrary real number $t > 1$, set $u = t^{-s}$. Then the function $\mathfrak{X}_X(s)$ is absolutely convergent and holomorphic for all s satisfying* $\mathrm{Re}(s) > -\log R_X/\log t$ (≥ 0).

Since the function $Z_X(u)$ is the reciprocal of a polynomial by the Ihara–Bass formula, *the function $Z_X(u)$ is meromorphic for all $u \in \mathbb{C}$*, *and therefore $\mathfrak{X}_X(s)$ is also meromorphic for all $s \in \mathbb{C}$.*

(2) [Kotani and Sunada 2000, Theorem 1.3(1)] *Let $q+1$ and $p+1$ be the maximum and minimum degrees of a graph X, respectively. Then $1/q \leq R_X \leq 1/p$, the point $u = R_X$ is a simple pole of $Z_X(u)$, and every pole of $Z_X(u)$ satisfies $R_X \leq |u| \leq 1$.*

(3) [Terras 2011, p. 197] *Suppose that X is a finite connected graph without degree-one vertices. Then $R_X = 1$ if and only if X is a cycle graph. This follows from the equation $p = q = 1$.*

(4) [Kotani and Sunada 2000, p. 8] *The leading coefficient of the polynomial f_X is given by*

$$c = \prod_{v \in V} (\deg(v) - 1),$$

and therefore that of the polynomial $1/Z_X$ is equal to $c_{2\epsilon} = (-1)^{\epsilon-\nu} c$.

In this section, the following lemma is important.

Key Lemma 2. *Let*

$$\phi(u) = 1 + \sum_{i=1}^{d} c_i u^i \in \mathbb{Z}[u]$$

be a polynomial function of degree $d \geq 0$, *and let*

$$T = \{u \in \mathbb{C} : \text{there exists } n \geq 1 \text{ such that } \phi(u^n) = 0\}$$

denote the zeroes of the $\phi(u^n)$. *Suppose that* r *is an arbitrary real number, and assume that* $\Phi(u)$ *is a series defined by*

$$\Phi(u) = \sum_{n=1}^{\infty} \frac{1}{n^r} \log \phi(u^n).$$

Then $\Phi(u)$ *is absolutely convergent for* u *satisfying* $|u| < 1$ *and* $u \notin T$.

Proof. First, we suppose that $d = 0$. Then the $\phi(u^n) = 1$ are constant, and therefore $\Phi(u) = 0$ is also constant. Hence, the claim is trivial. From now on, we assume that $d \geq 1$. Set $c := \max\{|c_i| : 1 \leq i \leq d\}$, choose a number C_0 with $C_0 \geq cd + 1$ (≥ 2), and fix it.

Let r_n $(n \geq 3)$ be a number defined by

$$r_n := \left(\frac{1 - \exp(-1/n^{2-r})}{C_0} \right)^{1/n}.$$

Note that $r_n < (1/C_0)^{1/n}$, the sequence $\{r_n\}_{n \geq 3}$ is increasing, and $\lim_{n\to\infty} r_n = 1$.

Take u satisfying $|u| < 1$ and $u \notin T$, and fix it. Then there exists a number N such that $|u| \leq r_N$, and thus $|u| < r_n$ for all $n \geq N + 1$. Now we fix such numbers N and n.

Since $|u| < (1/C_0)^{1/n}$ and $|u^n| \leq |u| < 1$, we obtain, by the triangle inequality,

$$0 < 1 - C_0|u^n| \leq |\phi(u^n)|, \quad \text{and so} \quad -\log|\phi(u^n)| \leq -\log(1 - C_0|u^n|). \tag{1}$$

On the other hand, since $|u| < r_n$, then $C_0|u^n| < 1 - \exp(-1/n^{2-r})$, so we obtain the inequality $-\log(1 - C_0|u^n|) < 1/n^{2-r}$. Combining this result with (1), we obtain

$$\operatorname{Re}(-\log \phi(u^n)) = -\log|\phi(u^n)| < \frac{1}{n^{2-r}}. \tag{2}$$

The first inequality in (1) also shows that the function $\log \phi(u^n)$ is holomorphic in the closed disk $|u| \leq r_{N+1}$. By applying the Borel–Carathéodory theorem (see, e.g., [Titchmarsh 1958, §5.5]) to the function $\log \phi(u^n)$ and the two circles $|u| = r_{N+1}$, $|u| = r_N$, we obtain

$$|\log \phi(u^n)| \leq \max_{|u|=r_N} |\log \phi(u^n)| \leq K \max_{|u|=r_{N+1}} \operatorname{Re}(-\log \phi(u^n)) \leq K \frac{1}{n^{2-r}},$$

where $K := 2r_N/(r_{N+1} - r_N)$. Therefore, it follows that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^r} |\log \phi(u^n)| \leq K \sum_{n=N+1}^{\infty} \frac{1}{n^2} < K \cdot \zeta(2) < \infty.$$

Hence, for u satisfying $|u| < 1$ and $u \notin T$, the series $\Phi(u)$ converges absolutely. $\square$

Using this lemma, we can prove the following proposition.

Proposition 3. *Let $\mu(n)$ denote the Möbius function. If $|u| < R_X$, then*

$$P_X(u) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log Z_X(u^n). \tag{3}$$

Moreover, the right-hand side of (3) is absolutely convergent for u satisfying $|u| < 1$ and $u \notin T$, and therefore $P_X(u)$ extends analytically to the region $\{u \in \mathbb{C} : |u| < 1\} \setminus T$.

Equivalently, if $\mathrm{Re}(s) > -\log R_X / \log t$, *then*

$$\mathcal{P}_X(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \mathcal{Z}_X(ns). \tag{4}$$

The right-hand side of (4) is absolutely convergent for s satisfying $\mathrm{Re}(s) > 0$ *and $t^{-s} \notin T$, and so (4) gives the analytic continuation of $\mathcal{P}_X(s)$ to the region.*

Proof. Note that $R_X \leq 1$ (from Fact 1(2)) and $\exp(z) = \prod_{n=1}^{\infty} (1 - z^n)^{-\mu(n)/n}$ for $|z| < 1$. Suppose that $|u| < R_X$. Since $|u^{\ell(P)}| \leq |u| < 1$, we obtain the equality

$$\exp(P_X(u)) = \prod_{[P]} \exp(u^{\ell(P)}) = \prod_{[P]} \prod_{n=1}^{\infty} (1 - u^{n\ell(P)})^{-\mu(n)/n} = \prod_{n=1}^{\infty} Z_X(u^n)^{\mu(n)/n},$$

and therefore (3) holds for u satisfying $|u| < R_X$.

Set

$$1/Z_X(u) = (1 - u^2)^{\epsilon - \nu} f_X(u) = 1 + c_1 u + \cdots + c_{2\epsilon} u^{2\epsilon} \in \mathbb{Z}[x],$$

$c = \max\{|c_i| : 1 \leq i \leq 2\epsilon\}$ and $C_0 = 2\epsilon c \geq 2$. By applying Key Lemma 2 to $\phi(u) = 1/Z_X(u)$ and $r = 1$, it follows that, for u satisfying $|u| < 1$ and $u \notin T$, the series $\sum_{n=1}^{\infty} \log Z_X(u^n)/n$ is absolutely convergent, and so the right-hand side of (3) is absolutely convergent. $\square$

Moreover, for a Ramanujan graph, we can prove the following.

Corollary 4. *Suppose that X is a finite connected Ramanujan graph with degree $q + 1$, that is, $Z_X(u)$ satisfies the Riemann hypothesis* (see Theorem 7.4 in [Terras 2011]). *Then the function $P_X(u)$ is absolutely convergent for u satisfying $|u| < 1$ and $|u| \neq (1/q)^{1/n}$ for all n.*

Equivalently, the function $\mathcal{P}_X(s)$ is absolutely convergent for s such that $\mathrm{Re}(s) > 0$ *and* $\mathrm{Re}(s) \neq \log q / \log t^n$ *for all n.*

Proof. Since X is a Ramanujan graph, by Theorem 1.3 in [Kotani and Sunada 2000], every real (resp. nonreal) zero of $f_X(u)$ satisfies $|u| = 1$ or $1/q$ (resp. $|u| = 1/\sqrt{q}$). Thus, every point $|u| \neq (1/q)^{1/n}$ is not zero of $f_X(u^n)$. Hence, the proof of the assertion follows from Proposition 3. $\square$

We can completely interchange the roles of the functions $P_X(u)$ and $\log Z_X(u)$.

Corollary 5. *If* $|u| < 1$ *and* $u \notin T$, *then*

(5) $$\log Z_X(u) = \sum_{n=1}^{\infty} \frac{1}{n} P_X(u^n).$$

Equivalently, *if* $\mathrm{Re}(s) > 0$ *and* $t^{-s} \notin T$, *then*

(6) $$\log \mathscr{Z}_X(s) = \sum_{n=1}^{\infty} \frac{1}{n} \mathscr{P}_X(ns).$$

Proof. By applying the Möbius inversion formula (see, e.g., Theorem 270 in [Hardy and Wright 2008], or Theorem 2.2.8 in [Jameson 2003]) to the equality (3) for $|u| < 1$, we obtain the equality (5). □

Remark 6. The equalities (4) and (6) indicate that $\mathscr{P}_X(s)$ is a graph-theoretic analogue to the prime zeta function $P(s)$ for the Riemann zeta function $\zeta(s)$. The relations between $P(s)$ and $\zeta(s)$ are given as follows (see [Glaisher 1891], and also [Fröberg 1968] and Equality (1.6.1) in [Titchmarsh 1986]):

For $\mathrm{Re}(s) > 1$,

$$P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns) \quad \text{and} \quad \log \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n} P(ns).$$

We can orient the edges of X, and label the edges as follows:

$$\vec{E} = \{a_1, a_2, \ldots, a_\epsilon, a_{\epsilon+1} = a_1^{-1}, a_{\epsilon+2} = a_2^{-1}, \ldots, a_{2\epsilon} = a_\epsilon^{-1}\}.$$

Let $W = W_X := (w_{ij})$ denote the edge adjacency matrix of a graph X, that is, a $2\epsilon \times 2\epsilon$ matrix defined by

$$w_{ij} := \begin{cases} 1 & \text{if } t(a_i) = o(a_j) \text{ and } a_j \neq a_i^{-1} \text{ for } a_i, a_j \in \vec{E}, \\ 0 & \text{otherwise} \end{cases}$$

(see p. 28 in [Terras 2011]). Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of W, and let $e_1, \ldots, e_k$ be their multiplicities. Note that $\sum_{i=1}^{k} e_i = 2\epsilon$. Let $e := \sum_{i=1, \lambda_i \neq \pm 1}^{k} e_i$. By the determinant formula given by Hashimoto [1989] and Bass [1992], the polynomial $1/Z_X(u)$ can be written as

$$1/Z_X(u) = \det(I_{2\epsilon} - Wu) = \prod_{i=1}^{k} (1 - \lambda_i u)^{e_i}.$$

Note that $f_X(1) = 0$. We now define a polynomial $g_X(u)$ by

$$g_X(u) := f_X(u)/(1-u).$$

Note that since $f'_X(1) = 2(\epsilon - \nu)\kappa$ by [Northshield 1998, Theorem],

$$g_X(1) = -f'_X(1) = -2(\epsilon - \nu)\kappa,$$

where κ is the complexity of X, that is, the number of spanning trees in X. Since X is a non-cycle graph, that is, $\epsilon \neq \nu$, the polynomial $g_X(u)$ can be also written as

$$g_X(u) = \frac{1/Z_X(u)}{(1-u^2)^{\epsilon-\nu}(1-u)} = (1+u)^{2\nu-1-e} \prod_{\substack{i=1 \\ \lambda_i \neq \pm 1}}^{k} (1-\lambda_i u)^{e_i}. \tag{7}$$

We can show that the function $\mathcal{P}_X(s)$ has a natural boundary.

Proposition 7. *Let $X = (V, E)$ be a finite, connected and non-cycle graph without degree-one vertices.*

(1) *There exists an eigenvalue λ of W such that $|\lambda| > 1$.*

(2) *The imaginary axis* $\mathrm{Re}(s) = 0$ *is a natural boundary for the function $\mathcal{P}_X(s)$, that is, every point on this line can be realized as a limit point of singularities of $\mathcal{P}_X(s)$.*

Proof. (1) The leading coefficient $c_{2\epsilon}$ of the polynomial $1/Z_X(u)$ is given by

$$(-1)^{\epsilon-\nu} \prod_{v \in V} (\deg(v) - 1) = c_{2\epsilon} = \prod_{i=1}^{k} \lambda_i^{e_i}$$

(from Fact 1(4)). By our assumption for X, the graph X is not a 2-regular graph. Thus $|c_{2\epsilon}| > 1$ and so there exists an eigenvalue λ_i with $|\lambda_i| \neq 1$. Note that every pole $1/\lambda_i$ of $Z_X(u)$ satisfies $|1/\lambda_i| \le 1$ by Fact 1(2). So there exists an eigenvalue λ_i with $|\lambda_i| > 1$.

(2) Note that $\exp(z) = \prod_{n=1}^{\infty} (1-z^n)^{-\mu(n)/n}$ for $|z| < 1$. If $|u| < 1$ and $u \notin T$, then

$$\begin{aligned}
\exp(P_X(u)) &= \prod_{n=1}^{\infty} Z_X(u^n)^{\mu(n)/n} \\
&= \left(\prod_{n=1}^{\infty} (1-u^{2n})^{-\mu(n)/n}\right)^{\epsilon-\nu} \left(\prod_{n=1}^{\infty} (1-u^n)^{-\mu(n)/n}\right) \prod_{n=1}^{\infty} g_X(u^n)^{-\mu(n)/n} \\
&= \exp((\epsilon-\nu)u^2 + u) \prod_{n=1}^{\infty} g_X(u^n)^{-\mu(n)/n},
\end{aligned}$$

and therefore the equality

$$P_X(u) = (\epsilon - \nu)u^2 + u - \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log g_X(u^n)$$

holds.

Note that $u = t^{-s}$. By using the equalities (7) and 2, the function $\mathcal{P}_X(s)$ can be written as

$$\mathcal{P}_X(s) = (\epsilon - \nu)t^{-2s} + t^{-s} - \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\Biggl((2\nu - 1 - e)\log(1 + t^{-ns}) + \sum_{\substack{i=1 \\ \lambda_i \neq \pm 1}}^{k} e_i \log(1 - \lambda_i t^{-ns})\Biggr)$$

for all s satisfying $\mathrm{Re}(s) > 0$. By part (1), there exists λ such that $|\lambda| > 1$ among the eigenvalues $\lambda_1, \dots, \lambda_k$ of W. Note that $1 - \lambda t^{-ns} = 0$ if and only if $s = r(\lambda, n, m)$, where

$$r(\lambda, n, m) := \frac{\log|\lambda|}{n \log t} + i\frac{\mathrm{Arg}(\lambda) + 2\pi m}{n \log t},$$

and $\mathrm{Arg}(\lambda)$ is the argument of λ with $-\pi \le \mathrm{Arg}(\lambda) < \pi$. Note that

$$\varepsilon_n := \frac{\log|\lambda|}{n \log t} \to 0$$

as $n \to \infty$. We now fix an arbitrary point $\alpha = ia$ on the imaginary axis $\mathrm{Re}(s) = 0$. Then, we can arrange a sequence of integers $\{m_n\}$ for each integer n so that

$$\frac{\mathrm{Arg}(\lambda) + 2\pi m_n}{n \log t} \to a$$

as $n \to \infty$. Hence, each point α on the boundary is a limit point of singularities of $\mathcal{P}_X(s)$. Since $\varepsilon_n > 0$ for all n, we cannot continue $\mathcal{P}_X(s)$ beyond the boundary at $\mathrm{Re}(s) = 0$. $\square$

Remark 8. Proposition 7(2) is an analogue of the fact that the imaginary axis $\mathrm{Re}(s) = 0$ is a natural boundary for the prime zeta function $P(s)$ of the Riemann zeta function $\zeta(s)$ (see [Landau and Walfisz 1920]).

3. Graph-theoretic Mertens' theorem

In this section, we prove parts (3)–(5) of the Main Theorem introduced in Section 1.

Throughout this section, we always assume that $X = (V, E)$ is a finite, connected, non-cycle graph without degree-one vertices. Note in particular that $\nu \neq \epsilon$ and $0 < R_X < 1$.

First, we define the constants H_X, C_X and γ_X, and study their properties, which play important roles in this section. Let u be a complex variable. We define a function by

$$H_X(u) := \log Z_X(u) - P_X(u) = \sum_{n \ge 2} \frac{1}{n} P_X(u^n) = \sum_{[P]} \sum_{n \ge 2} \frac{1}{n} u^{n\ell(P)}.$$

Note that the point $u = R_X$ is a common pole of $Z_X(u)$ and $P_X(u)$ by Fact 1(2), and that the series $H_X(u)$ is absolutely convergent for u satisfying $|u| < 1$ and $u \notin T$, from Corollary 5.

Since $u = R_X$ is a simple pole of $Z_X(u)$, we can define constants c_X and C_X by

$$c_X := -\operatorname*{Res}_{u=R_X} Z_X(u) = \lim_{u \uparrow R_X} (R_X - u) Z_X(u) = \frac{-1}{(1 - R_X^2)^{\epsilon - \nu} f_X'(R_X)}$$

and $C_X := c_X / R_X$.

Lemma 9. (1) *The value $H_X := H_X(R_X)$ is finite.*

(2) *The constants c_X and C_X are positive.*

Proof. (1) Since $R_X^n < R_X < 1$ $(n \geq 2)$, the function $P_X(u)$ is holomorphic at $u = R_X^n$, and therefore $P_X(u^n)$ is holomorphic at $u = R_X$. We have

$$\begin{aligned} H_X(R_X) &= \sum_{[P]} \sum_{n \geq 2} \frac{1}{n} R_X^{n\ell(P)} \leq \sum_{[P]} \sum_{n \geq 2} R_X^{n\ell(P)} \\ &= \sum_{[P]} \frac{R_X^{2\ell(P)}}{1 - R_X^{\ell(P)}} \leq \frac{1}{1 - R_X} \sum_{[P]} R_X^{2\ell(P)} = \frac{P_X(R_X^2)}{1 - R_X} < +\infty, \end{aligned}$$

and the assertion follows.

(2) Note that the leading coefficient of the polynomial f_X is given by

$$c = \prod_{v \in V} (\deg(v) - 1) > 0$$

by Fact 1(4). Then f_X factors as the product of irreducible polynomials such that

$$f_X(u) = c \prod_{i=1}^{m_1} (u - \alpha_i) \cdot \prod_{j=1}^{m_2} f_j(u),$$

where the f_j are monic of $\deg f_j = 2$, and $\deg f_X = 2\nu = m_1 + 2m_2$. Note that m_1 is even. Since $u = R_X$ is a simple pole of $Z_X(u)$, it is a simple zero of f_X. We may assume that $\alpha_1 = R_X$. Since $\alpha_i > R_X$ $(2 \leq i \leq m_1)$ and the discriminants of the f_j are negative, the sign of

$$f_X'(R_X) = c \prod_{i=2}^{m_1} (R_X - \alpha_i) \prod_{j=1}^{m_2} f_j(R_X)$$

is equal to $(-1)^{m_1 - 1} = -1$, i.e., $f_X'(R_X) < 0$, so $c_X > 0$ and $C_X = c_X / R_X > 0$. $\square$

Since the function $Z_X(u) - c_X/(R_X - u)$ is holomorphic at $u = R_X$, we can define a constant γ_X by

$$\gamma_X := \lim_{u \uparrow R_X} \left(Z_X(u) - \frac{c_X}{R_X - u} \right),$$

which is an analogue of the Euler–Mascheroni constant $\gamma = \lim_{s \downarrow 1} (\zeta(s) - 1/(s-1))$ for $\zeta(s)$.

In a neighborhood of $u = R_X$, the function $Z_X(u)$ can be expanded as

$$Z_X(u) = \frac{c_X}{R_X - u} + \gamma_X + O(R_X - u),$$

and so

$$\log Z_X(u) = \log \frac{c_X}{R_X - u} + O(R_X - u). \tag{8}$$

Similarly, in a neighborhood of $u = R_X$, the function $P_X(u)$ can be expanded as

$$P_X(u) = \log \frac{c_X}{R_X - u} - H_X(u) + O(R_X - u) = \log \frac{c_X}{R_X - u} - H_X(R_X) + O(R_X - u).$$

In this section, the following facts are used.

Facts 10. (1) (See, for example, Theorem 18.1 in [Korevaar 2002].) *Let x be a complex variable and let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with $a_n \geq 0$ that converges for $|x| < 1$. Suppose that*

$$F(x) - \frac{C}{1-x} = O(1)$$

as $x \to 1$. Then the partial sum $A(N) = \sum_{n \leq N} a_n$ satisfies

$$A(N) = C \cdot N + O(\log N)$$

as $N \to \infty$.

(2) (See, for example, Exercises 9-6 in [Apostol 1974], and Theorem 1.3.6 in [Jameson 2003], the Abel partial summation formula). *Let $\{a_n\}$ be real numbers, and let $f(t)$ be a (real- or complex-valued) function with a continuous derivative in the interval $[1, N]$. Then*

$$\sum_{n \leq N} a_n f(n) = A(N) f(N) - \int_1^N A(t) f'(t)\, dt.$$

By using Fact 10, we can prove the following proposition.

Proposition 11. *Suppose that X is a finite, connected and non-cycle graph without degree-one vertices. In a neighborhood of $u = R_X$, expand $Z_X(u)$ into the* *power series*

$$Z_X(u) = \sum_{n=0}^{\infty} a'_n u^n.$$

Then, as $N \to \infty$,

$$\sum_{n \le N} a'_n R_X^n = C_X \cdot N + O(\log N).$$

Proof. First, for simplicity of arguments, we normalize the function $Z_X(u)$:

$$F(x) = Z_X(R_X x) = \sum_{n=0}^{\infty} a'_n R_X^n x^n = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_n = a'_n R_X^n$. Note that the normalized function $F(x)$ converges for $|x| < 1$. Since all coefficients a'_n are nonnegative (by page 13 in [Terras 2011]), all coefficients a_n are also nonnegative. Since X is a non-cycle graph, the point $x = 1$ is a simple pole of $F(x)$. Hence, we obtain

$$F(x) - \frac{C_X}{1-x} = O(1)$$

as $x \to 1$. By applying Fact 10(1) to this equality, as $N \to \infty$,

$$\sum_{n \le N} a_n = C_X \cdot N + O(\log N), \quad \text{and so} \quad \sum_{n \le N} a'_n R_X^n = C_X \cdot N + O(\log N)$$

holds, and the assertion follows. □

Now, we compute the following example.

Example 12 [Terras 2011, Example 2.8, p. 18]. Consider the graph $X = K_4 - \{\text{one edge}\}$. Then

$$f_X(u) = (1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3) \quad \text{and} \quad Z_X(u)^{-1} = (1-u^2) f_X(u).$$

Since the radius of convergence R_X of $Z_X(u)$ is the smallest positive real zero of $f_X(u)$,

$$R_X = \tfrac{1}{6}(\alpha - 1 + \alpha^{-1}) = 0.6572981\ldots, \qquad \alpha = (53 + 6\sqrt{78})^{1/3}.$$

Then C_X is computed as $C_X = 0.5540954\ldots$. For example, if $N = 50000$, then

$$\frac{1}{N} \sum_{n \le N} a'_n R_X^n = 0.5540867\ldots \approx C_X.$$

Let $X=(V,E)$ be a graph, and set $|V|=\nu$ and $|E|=\epsilon$. Let $W=W_X$ be the edge adjacency matrix of X (see page 28 in [Terras 2011], or Section 2 in this paper), and let $\mathrm{Spec}(W)$ denote the spectrum of W, that is, the list of its eigenvalues together with their multiplicities. Note that $|\mathrm{Spec}(W)|=2\epsilon$. The polynomial $1/Z_X(u)$ has an expression different from that in Section 2. In fact, this can be written as

$$1/Z_X(u)=\det(I_{2\epsilon}-Wu)=\prod_{\lambda\in\mathrm{Spec}(W)}(1-\lambda u)\quad\left(=\prod_{i=1}^{k}(1-\lambda_i u)^{e_i}\right).$$

Since the points $u=1/\lambda$ are the poles of $Z_X(u)$, we obtain $1\le|\lambda|\le 1/R_X$ by Fact 1(2).

The following lemma is used in the proof of Theorem 14 in this section.

Key Lemma 13. *Suppose that X is a finite, connected and non-cycle graph without degree-one vertices.*

(1) *As $N\to\infty$, we have*

$$\sum_{n=1}^{N}\sum_{\lambda\in\mathrm{Spec}(W)}(\lambda R_X)^n=N+O(1).$$

(2) *Let $0<\alpha<\frac{1}{2}$ be a fixed real number. Then there exists a natural number N_0 such that, for any $n\ge N_0$,*

$$\left|n\cdot\pi(n)-\sum_{\lambda\in\mathrm{Spec}(W)}\lambda^n\right|<2\epsilon\left(\frac{1}{R_X}\right)^{(1-\alpha)n}.$$

Proof. (1) Let Δ_X denote

$$\Delta=\Delta_X:=\gcd\{\ell(P):[P]\text{ is a prime in }X\}$$

(see Definition 2.12 in [Terras 2011]). It follows from Theorem 1.4 in [Kotani and Sunada 2000] that the poles of $Z_X(u)$ on the circle $|u|=R_X$ have the form $u=R_Xe^{2\pi ia/\Delta}$ $(1\le a\le\Delta)$. It is well known that

$$\sum_{a=1}^{\Delta}e^{2\pi ian/\Delta}=\begin{cases}\Delta & \text{if }\Delta\mid n,\\ 0 & \text{otherwise}\end{cases}$$

(see, e.g., Exercise 10.1 in [Terras 2011]). Then we obtain

$$\left|N-\sum_{|\lambda|=1/R_X}\sum_{n=1}^{N}(\lambda R_X)^n\right|=\left|N-\sum_{n=1}^{N}\sum_{a=1}^{\Delta}e^{2\pi ian/\Delta}\right|=N-\left[\frac{N}{\Delta}\right]\Delta<\Delta,$$

where $[r]$ denotes the integer part of the real number r. On the other hand, we obtain

$$\left|\sum_{|\lambda|<1/R_X}\sum_{n=1}^{N}(\lambda R_X)^n\right| < 2\epsilon\sum_{n\geq 1}(\rho R_X)^n = \frac{2\epsilon\rho R_X}{1-\rho R_X},$$

where

$$\rho := \max\{|\lambda| : \lambda \in \operatorname{Spec}(W),\ |\lambda| < 1/R_X\}.$$

Combining these inequalities, by the triangle inequality we obtain

$$\left|N - \sum_{n=1}^{N}\sum_{\lambda\in\operatorname{Spec}(W)}(\lambda R_X)^n\right| < \Delta + \frac{2\epsilon\rho R_X}{1-\rho R_X}$$

as $N \to \infty$, and the assertion follows.

(2) Let $\mu(n)$ denote the Möbius function. Note that $\sum_{d|n}|\mu(d)| \leq n$. It is known that

$$\pi(n) = \frac{1}{n}\sum_{d|n}\mu(d)N_{n/d} \quad\text{and}\quad N_n = \sum_{\lambda\in\operatorname{Spec}(W)}\lambda^n$$

(see (10.3) and (10.4) in [Terras 2011]). Combining these equalities, we obtain

$$n\cdot\pi(n) = \sum_{\lambda\in\operatorname{Spec}(W)}\sum_{d|n}\mu(d)\lambda^{n/d},$$

and thus

$$\begin{aligned}\left|n\cdot\pi(n) - \sum_{\lambda\in\operatorname{Spec}(W)}\lambda^n\right| &= \left|\sum_{\lambda\in\operatorname{Spec}(W)}\sum_{\substack{d|n\\ d\geq 2}}\mu(d)\lambda^{n/d}\right| \\ &\leq \sum_{\lambda\in\operatorname{Spec}(W)}\sum_{\substack{d|n\\ d\geq 2}}|\mu(d)|\cdot|\lambda|^{n/d} \leq \sum_{\lambda\in\operatorname{Spec}(W)}\sum_{\substack{d|n\\ d\geq 2}}|\mu(d)|\cdot|\lambda|^{n/2} \\ &\leq n\sum_{\lambda\in\operatorname{Spec}(W)}\left(\frac{1}{R_X}\right)^{n/2} \leq 2\epsilon n\left(\frac{1}{R_X}\right)^{n/2}.\end{aligned}$$

On the other hand, since $R_X < 1$ and $0 < \alpha < \frac{1}{2}$ by our assumptions, there exists a natural number N_0 such that, for any $n \geq N_0$,

$$n \leq \left(\frac{1}{R_X}\right)^{(1/2-\alpha)n}, \quad\text{and so}\quad n\left(\frac{1}{R_X}\right)^{n/2} \leq \left(\frac{1}{R_X}\right)^{(1-\alpha)n}.$$

Hence, for any $n \geq N_0$,

$$\left|n\cdot\pi(n) - \sum_{\lambda\in\operatorname{Spec}(W)}\lambda^n\right| \leq 2\epsilon\left(\frac{1}{R_X}\right)^{(1-\alpha)n},$$

and the assertion follows. □

At last, we can prove the main theorem in this section.

Theorem 14. *Suppose that X is a finite, connected and non-cycle graph without degree-one vertices. Let $\gamma = 0.57721\ldots$ be the Euler–Mascheroni constant, and let $H_X = H_X(R_X)$ and C_X be the constants.*

(1) (*Graph-theoretic Mertens' first theorem*) *As $N \to \infty$,*

$$\sum_{n \le N} n \cdot \pi(n) R_X^n = N + O(1).$$

(2) (*Graph-theoretic Mertens' second theorem*) *There exists a constant B_X such that, as $N \to \infty$,*

$$\sum_{n \le N} \pi(n) R_X^n = \log N + B_X + O\Big(\frac{1}{N}\Big).$$

(3) *The equality $B_X = \gamma + \log C_X - H_X$ holds. Equivalently,*

$$\begin{aligned} B_X &= \gamma + \log C_X - \sum_{[P]} \sum_{n \ge 2} \frac{1}{n} R_X^{n\ell(P)} \\ &= \gamma + \log C_X + \prod_{[P]} \big(\log(1 - R_X^{\ell(P)}) + R_X^{\ell(P)}\big). \end{aligned}$$

(4) (*Graph-theoretic Mertens' third theorem*) *As $N \to \infty$,*

$$\prod_{\ell(P) \le N} \big(1 - R_X^{\ell(P)}\big) = \prod_{n \le N} (1 - R_X^n)^{\pi(n)} \sim \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N}.$$

Proof. (1) Let N_0 be a number as in the proof of Key Lemma 13(2), and let K denote the constant

$$K := \Bigg| \sum_{n=1}^{N_0 - 1} n \cdot \pi(n) R_X^n - \sum_{n=1}^{N_0-1} \sum_{\lambda \in \mathrm{Spec}(W)} (\lambda R_X)^n \Bigg|.$$

Assume that N is sufficiently large. Then it follows from Key Lemma 13(2) that

$$\begin{aligned} \Bigg| \sum_{n=1}^{N} n \cdot \pi(n) R_X^n - \sum_{n=1}^{N} \sum_{\lambda \in \mathrm{Spec}(W)} (\lambda R_X)^n \Bigg| &\le K + \Bigg| \sum_{n=N_0}^{N} R_X^n \Bigg(n \cdot \pi(n) - \sum_{\lambda \in \mathrm{Spec}(W)} \lambda^n \Bigg) \Bigg| \\ &\le K + 2\epsilon \sum_{n=N_0}^{N} R_X^{\alpha n} < K + \frac{2\epsilon}{1 - R_X^\alpha}, \end{aligned}$$

and therefore by Key Lemma 13(1) we have

$$\sum_{n=1}^{N} n \cdot \pi(n) R_X^n = \sum_{n=1}^{N} \sum_{\lambda \in \mathrm{Spec}(W)} (\lambda R_X)^n + O(1) = N + O(1) \quad \text{as } N \to \infty.$$

(2) We set $a_n = n \cdot \pi(n) R_X^n$. By part (1), we obtain $A(t) = t + O(1)$. By applying Fact 10(2) with $f(t) = 1/t$, we get

$$\begin{aligned}\sum_{n \le N} \pi(n) R_X^n &= \frac{A(N)}{N} + \int_1^N \frac{A(t)}{t^2}\, dt = \frac{N + O(1)}{N} + \int_1^N \frac{t + O(1)}{t^2}\, dt \\ &= 1 + O\Big(\frac{1}{N}\Big) + \int_1^N \Big(\frac{1}{t} + O\Big(\frac{1}{t^2}\Big)\Big)\, dt \\ &= 1 + O\Big(\frac{1}{N}\Big) + \Big[\log t + O\Big(\frac{1}{t}\Big)\Big]_1^N \\ &= 1 + O\Big(\frac{1}{N}\Big) + \log N + O\Big(\frac{1}{N}\Big) + O(1) = \log N + O(1) + O\Big(\frac{1}{N}\Big),\end{aligned}$$

and the assertion follows.

(3) Fix an arbitrary x satisfying $0 < x < 1$. By applying Fact 10(2) with $a_n = \pi(n) R_X^n$ and $f(t) = x^t$,

$$\sum_{n \le N} \pi(n) R_X^n x^n = A(N) x^N - \log x \int_1^N x^t A(t)\, dt$$

holds. It follows from part (2) that

$$\sum_{n \le N} \pi(n) R_X^n x^n = \Big(\log N + B_X + O\Big(\frac{1}{N}\Big)\Big) x^N - \log x \int_1^N x^t \Big(\log t + B_X + O\Big(\frac{1}{t}\Big)\Big)\, dt,$$

and, moreover, as $N \to \infty$,

$$P_X(R_X x) = -\log x \int_1^\infty x^t \Big(\log t + B_X + O\Big(\frac{1}{t}\Big)\Big)\, dt. \tag{9}$$

In order to calculate the right-hand side of this equality, for simplicity of arguments, we define the functions $I_n = I_n(x)$:

$$-\log x \int_1^\infty x^t \Big(\log t + B_X + O\Big(\frac{1}{t}\Big)\Big)\, dt = I_1 + I_2 + O(I_3),$$

where

$$I_1 = -\log x \int_1^{\infty} x^t \log t \, dt,$$

$$I_2 = -B_X \cdot \log x \int_1^{\infty} x^t \, dt = B_X \cdot x, \quad \text{and}$$

$$I_3 = -\log x \int_1^{\infty} \frac{x^t}{t} \, dt.$$

First, we compute the function I_1:

$$I_1 = -\int_1^{\infty} (x^t)' \log t \, dt = \int_1^{\infty} \frac{x^t}{t} \, dt.$$

Now we take $r = -t \log x$. Note that $\log x < 0$. Then we obtain

$$I_1 = \int_{-\log x}^{\infty} \frac{e^{-r}}{r} \, dr = -\mathrm{Ei}(\log x),$$

where $\mathrm{Ei}(z)$ ($z \in \mathbb{C}$ and $|\mathrm{Arg}(-z)| < \pi$) is the exponential integral

$$-\mathrm{Ei}(-z) = \int_z^{\infty} \frac{e^{-r}}{r} \, dr$$

(see, e.g., Equality (3.1.3) in [Lebedev 1972]). Since the function $\mathrm{Ei}(z)$ expands as

$$\mathrm{Ei}(z) = \gamma + \log(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!}$$

(see Equality (3.1.6) in [ibid.]),

$$I_1 = -\gamma - \log(-\log x) + O(\log x) = -\gamma - \log(-\log x) + O(1 - x).$$

Next we calculate the function I_3. It follows from the above result that

$$I_3 = -\log x \int_1^{\infty} \frac{x^t}{t} \, dt = (-\log x) I_1 = O(1 - x)$$

as $x \uparrow 1$.

By combining the above results, the equality (9) is written as follows:

$$P_X(R_X x) = -\gamma - \log(-\log x) + B_X x + O(1 - x),$$

and, moreover, as $x \uparrow 1$,

$$P_X(R_X x) + \log(-\log x) \to B_X - \gamma. \tag{10}$$

On the other hand, since

$$\log Z_X(R_X x) = \log \frac{1}{1-x} + \log C_X + O(1 - x)$$

from the equality (8), as $x \uparrow 1$,

$$\log Z_X(R_X x) + \log(-\log x) = \log\left(\frac{-\log x}{1-x}\right) + \log C_X \to \log C_X. \tag{11}$$

Combining (10) with (11), we obtain

$$\begin{aligned} H_X &= \lim_{x\uparrow 1} H_X(R_X x) = \lim_{x\uparrow 1}\bigl(\log Z_X(R_X x) - P_X(R_X x)\bigr) \\ &= \lim_{x\uparrow 1}\bigl((\log Z_X(R_X x) + \log(-\log x)) - (P_X(R_X x) + \log(-\log x))\bigr) \\ &= \log C_X + \gamma - B_X. \end{aligned}$$

(4) Fix an arbitrary positive real number N. We define the following functions:

$$H_X^{\le N} = \sum_{n \le N} \pi(n) \sum_{m=2}^{\infty} \frac{1}{m} R_X^{mn} \quad \text{and} \quad H_X^{> N} = \sum_{n > N} \pi(n) \sum_{m=2}^{\infty} \frac{1}{m} R_X^{mn}.$$

Note that $H_X = H_X^{\le N} + H_X^{>N}$. From parts (2) and (3), we obtain

$$\sum_{n \le N} \pi(n) R_X^n + H_X^{\le N} = \log N + \gamma + \log C_X - H_X^{>N} + O\left(\frac{1}{N}\right).$$

Since the left-hand side of this equality is equal to

$$\begin{aligned} \sum_{n \le N} \pi(n) R_X^n + H_X^{\le N} &= \sum_{n \le N} \pi(n) \sum_{m=1}^{\infty} \frac{1}{m} R_X^{mn} \\ &= -\sum_{n \le N} \pi(n) \log(1 - R_X^n) = -\log\left(\prod_{n \le N} (1 - R_X^n)^{\pi(n)}\right), \end{aligned}$$

we obtain

$$\prod_{n \le N} (1 - R_X^n)^{\pi(n)} = \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N} \exp\left(H_X^{>N} + O\left(\frac{1}{N}\right)\right).$$

Since $H_X^{>N} \to 0$ and $1/N \to 0$ as $N \to \infty$, the assertion follows. □

Last, we compute the following example.

Example 15 (continued from Example 12). Consider the graph $X = K_4 - \{\text{one edge}\}$. Then

$$H_X = 0.25613\ldots, \qquad B_X = \gamma + \log C_X - H_X = -0.26933\ldots.$$

For example, if $N = 550$, then

$$\sum_{n \le N} \pi(n) R_X^n - \log N = -0.26842\cdots \approx B_X,$$

$$\prod_{n \le N} (1 - R_X^n)^{\pi(n)} = 0.18447\cdots \approx \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N} = 0.18457\ldots.$$

Remark 16. (See [Mertens 1874, Equation (17)], or [Hardy and Wright 2008, Theorem 428].) A number-theoretic analogue to part (3) in the preceding theorem is

$$B_1 = \gamma - H = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right),$$

where $H = \sum_{n \ge 2} P(n)/n$ is a constant, and $P(s)$ is the prime zeta function.

Remark 17. We now compare parts (2)–(4) of our Theorem 14 with Theorem 1 in [Sharp 1991]. We define

$$h_X := -\log R_X, \quad N(P) = e^{h_X \ell(P)} \quad \text{and} \quad x = e^{h_X N}.$$

The quantity h_X is called the topological entropy of a flow in ergodic theory (see [Sharp 1991]), which is a constant in our setting. Note that $\ell(P) \le N$ if and only if $N(P) \le x$. Note that $R_X^{\ell(P)} = 1/N(P)$. Then our Mertens' second theorem can be rewritten as

$$\sum_{N(P) \le x} \frac{1}{N(P)} = \log(\log x) + B + O\left(\frac{1}{\log x}\right),$$

where $B := -\log h_X + B_X$, and, similarly, our Mertens' third theorem becomes

$$\prod_{N(P) \le x} \left(1 - \frac{1}{N(P)}\right) \sim \frac{1}{C_X/h_X} \cdot \frac{e^{-\gamma}}{\log x}.$$

In Theorem 1 in [Sharp 1991], our constant C_X/h_X, which is equal to a residue (up to sign) of the Ihara zeta function, corresponds with that of a dynamical zeta function for a flow.

Moreover, our Theorem 14(3) becomes

$$B = \gamma + \log(C_X/h_X) + \sum_{[P]} \left(\log\left(1 - \frac{1}{N(P)}\right) + \frac{1}{N(P)}\right).$$

Remark 18. Let $X = (V, E)$ be a finite, connected, non-cycle graph without degree-one vertices, and let $S = (V', E')$ be its k-subdivision (that is, let S be the graph obtained from X by adding k new vertices to each edge of X) (see Examples 6.4 and 8.5 in [Terras 2011]). Then

$$H_X = H_S, \quad C_X = (k+1)C_S, \quad \text{and} \quad B_X = B_S + \log(k+1).$$

This is proved as follows: note that $\Delta_S = (k+1)\Delta_X$, $R_S^{k+1} = R_X$, and

$$\pi_S(n) = \begin{cases} \pi_X(n/(k+1)) & \text{if } (k+1) \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} H_S &= \sum_{m\geq 2} \frac{1}{m} P_S(R_S^m) = \sum_{m\geq 2} \frac{1}{m} \sum_{n=1}^{\infty} \pi_S(n) R_S^{mn} = \sum_{m\geq 2} \frac{1}{m} \sum_{n=1}^{\infty} \pi_X(n) R_S^{(k+1)mn} \\ &= \sum_{m\geq 2} \frac{1}{m} \sum_{n=1}^{\infty} \pi_X(n) R_X^{mn} = \sum_{m\geq 2} \frac{1}{m} P_X(R_X^m) = H_X. \end{aligned}$$

Note that $\nu' = \nu + k\epsilon$, $\epsilon' = (k+1)\epsilon$, and $Z_S(u) = Z_X(u^{k+1})$, and so

$$\begin{aligned} (1-u^2)^{\epsilon-\nu} f_S(u) &= (1-u^{2(k+1)})^{\epsilon-\nu} f_X(u^{k+1}), \\ (1-R_S^2)^{\epsilon-\nu} R_S f_S'(R_S) &= (k+1)(1-R_X^2)^{\epsilon-\nu} R_X f_X'(R_X). \end{aligned}$$

Therefore,

$$(k+1)C_S = \frac{-(k+1)}{(1-R_S^2)^{\epsilon'-\nu'} R_S f_S'(R_S)} = C_X,$$

and so

$$B_X = \gamma + \log C_X - H_X = \gamma + \log C_S - H_S + \log(k+1) = B_S + \log(k+1).$$

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