ON A PRIME ZETA FUNCTION OF A GRAPH

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In the first half of this paper, we introduce a prime zeta function associated with the Ihara zeta function, and study several properties of this function. In the last half, using results of the first half, we present graph-theoretic analogues to Mertens’ theorems.

1. Introduction

Throughout this paper, we use the notation of [Stark and Terras 1996; Terras 2011] for graph theory and the theory of (Ihara) zeta functions $Z_X(u)$ of graphs, and the notation of [Hardy and Wright 2008] and [Titchmarsh 1958; 1986] for the theory of functions and the Riemann zeta function $\zeta(s)$.

In the analytic theory of the Riemann zeta function, the following theorems are well-known:

- Mertens’ first theorem [1874, Equality (5)] (also see [Hardy and Wright 2008, Theorem 425], [Jameson 2003, Theorem 2.6.3], and [Titchmarsh 1986, Equality (3.14.3)]): as $x \to \infty$,

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

- Mertens’ second theorem [1874, Equality (13)] (also see [Hardy and Wright 2008, Theorem 427], [Jameson 2003, Theorem 2.6.4/Exercise 4, p. 191], and [Titchmarsh 1986, Equality (3.14.5)]): as $x \to \infty$,

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + B_1 + O\left(\frac{1}{\log^k x}\right)$$

for each $k \geq 1$, where $B_1 = 0.26149 \ldots$ is the Mertens constant.

- Mertens’ third theorem [1874, Equality (15)] (also see [Hardy and Wright 2008, Theorem 429], [Jameson 2003, Exercise 1, p. 96], and [Titchmarsh 1986, Equality (3.14.9)]): as $x \to \infty$,

$$\sum_{p \leq x} \frac{1}{p^s} = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(\frac{x}{n}\right)^s + O\left(\frac{1}{\log^k x}\right)$$

for each $k \geq 1$, where $\mu(n)$ is the Möbius function.

MSC2010: primary 11N45; secondary 05C30, 05C38, 05C50.

Keywords: Ihara zeta functions, primes in graphs, Mertens’ theorem.
Equality (3.15.2)): as $x \to \infty$,

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x},$$

where $\gamma = 0.57721\ldots$ is the Euler–Mascheroni constant.

- Prime number theorem (proved by de la Vallée Poussin and Hadamard in 1896; see, e.g., [Hardy and Wright 2008, Theorem 6], [Jameson 2003, Theorem 3.4.3], and [Titchmarsh 1986, Equality (3.7.1))]: as $x \to \infty$,

$$\pi(x) \sim \frac{x}{\log x},$$

where $\pi(x)$ denotes the number of rational prime numbers $p$ less than $x$, that is,

$$\pi(x) := |\{p : p \text{ is a rational prime number with } p \leq x\}|.$$

All proofs of the above formulae are related to the Riemann zeta function

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

where $\mathbb{P}$ denotes the set of all rational prime numbers, that is,

$$\mathbb{P} := \{p \in \mathbb{Z} : p \text{ is a rational prime number}\},$$

and to the prime zeta function, defined first by Glaisher [1891],

$$P(s) = \sum_{p \in \mathbb{P}} \frac{1}{p^s}.$$

In graph theory, there exists an analogue of the Riemann zeta function, the so-called (Ihara) zeta function $Z_X(u)$ of a graph $X$ (see [Ihara 1966]). Therefore, studying graph-theoretic analogues of these theorems is very interesting. Indeed, Terras and coworkers gave an analogue of the prime number theorem (see Theorem 2.10 in [Horton et al. 2006], and also Theorem 10.1 in [Terras 2011]):

If $\Delta_X$ divides $n$, then, as $n \to \infty$,

$$\pi_X(n) \sim \frac{\Delta_X}{n \cdot R_X^n},$$

and otherwise $\pi_X(n) \sim 0$. (For the definitions of $\pi_X(n)$ and $R_X$, see this section, and for that of $\Delta_X$, see Section 3.) This is called the graph-theoretic prime number theorem.
In this paper, we define a prime zeta function of a graph, and investigate several properties of this function. In particular, we show that this has a natural boundary. Moreover, by using this function, we present graph-theoretic analogues of Mertens’ theorems.

We shall note a relation between previous works and our works. A zeta function of a graph can be specialized from a dynamical zeta function for a flow (see Chapter 4 in [Terras 2011]), and dynamical-systemic analogues to the above formulae are already known (see, e.g., [Sharp 1991] for Mertens’ theorems, and [Parry 1983; Parry and Pollicott 1983] for a prime number theorem). In that sense, our statements for Mertens’ theorems are not new (see Remark 17). However, since our proofs are graph-theoretic and elementary, they are completely different from previous proofs.

In this section, we first recall the notation for graph theory and zeta functions of graphs, define a prime zeta function of a graph, and finally state the main theorem.

Now we recall the notation of graph theory. Throughout this paper, we always assume that \( X \) is a finite, connected, non-cycle and undirected graph without degree-one vertices. Let \( X \) be a graph with vertex set \( V \), with \( \nu := |V| \), and edge set \( E \), with \( \epsilon := |E| \). Simply, such a graph \( X \) is denoted by \( X := (V, E) \). Note that \( \epsilon \) is the number of edges of \( X \).

An oriented edge (or an arc) \( a \) from a vertex \( u \) to a vertex \( v \) is denoted by \( a = (u, v) \), and the inverse of \( a \) is denoted by \( a^{-1} = (v, u) \). The origin and terminus of \( a \) are denoted by \( o(a) \) and \( t(a) \), respectively. We can now orient the edges of \( X \), and label the edges as follows:

\[
\vec{E} = \{e_1, e_2, \ldots, e_\epsilon, e_{\epsilon+1} = e_1^{-1}, e_{\epsilon+2} = e_2^{-1}, \ldots, e_{2\epsilon} = e_\epsilon^{-1}\}.
\]

A path \( C = a_1 \cdots a_s \), where the \( a_i \) are oriented edges, is said to have a backtrack (resp. tail) if \( a_{j+1} = a_j^{-1} \) for some \( j \) (resp. \( a_s = a_1^{-1} \)), and a path \( C \) is called a cycle (or a closed path) if \( o(a_1) = t(a_s) \). The length \( \ell(C) \) of a path \( C = a_1 \cdots a_s \) is defined by \( \ell(C) = s \).

A cycle \( C \) is called prime (or primitive) if it satisfies the following:

- \( C \) does not have backtracks or a tail;
- no cycle \( D \) exists such that \( C = D^f \) for some \( f > 1 \).

The equivalence class \([C]\) of a cycle \( C = a_1 \cdots a_s \) is defined as the set of cycles

\[
[C] := \{a_1a_2 \cdots a_{s-1}a_s, a_2 \cdots a_{s-1}a_s a_1, \ldots, a_s a_1 a_2 \cdots a_{s-1}\},
\]

and an equivalence class \([P]\) of a prime cycle \( P \) is called a prime in the graph \( X \). Throughout this paper, we denote a prime by the symbol \([P]\). Two cycles \( C_1 \) and \( C_2 \) are called equivalent if \( C_2 \in [C_1] \). Note that if \( [C_1] = [C_2] \), then \( \ell(C_1) = \ell(C_2) \), and thus \( u^{\ell(C_1)} = u^{\ell(C_2)} \).
Next, we recall the zeta function of a graph $X = (V = \{v_1, \ldots, v_\nu\}, E)$, and we define a prime zeta function associated with it. Let $u$ be a complex variable, and let $f_X(u)$ denote

$$f_X(u) := \det(I_\nu - Au + Qu^2),$$

where $I_\nu$ is the $\nu \times \nu$ identity matrix, $A$ is the adjacency matrix of $X$ (see Definition 2.1 in [Terras 2011]), and

$$Q = \text{diag}(\deg(v_1) - 1, \ldots, \deg(v_\nu) - 1).$$

Let $\pi_X(n)$ denote

$$\pi(n) = \pi_X(n) := \left| \{ [P] : [P] \text{ is a prime in } X \text{ with } \ell(P) = n \} \right|.$$

Throughout this paper, we fix an arbitrary real number $t > 1$ (that is, $\log t > 0$), and we set $u = t^{-s}$. The (Ihara) zeta function of $X$ (see Definition 2.2 and Theorem 2.5 in [Terras 2011]) and the prime zeta function of $X$ are defined as follows:

$$Z_X(u) := \prod_{[P]} (1 - u^{\ell(P)})^{-1} = \frac{1}{(1 - u^2)^{\nu} f_X(u)}, \quad \mathcal{Z}_X(s) := Z_X(t^{-s}),$$

$$P_X(u) := \sum_{[P]} u^{\ell(P)} = \sum_{n=1}^{\infty} \pi_X(n)u^n, \quad \mathcal{P}_X(s) := P_X(t^{-s}),$$

with $|u|$ sufficiently small, where $[P]$ runs through all primes in $X$. In this paper, we do not distinguish between the two functions $Z_X(u)$ and $\mathcal{Z}_X(s)$, or between $P_X(u)$ and $\mathcal{P}_X(s)$. The right-hand side of the first equality is called the Ihara–Bass formula (see [Bass 1992]). Note that, owing to our assumption for $X$, the zeta function $Z_X(u)$ is expressible like that.

Note that, for two finite connected graphs $X_1$ and $X_2$ without degree-one vertices, $P_{X_1}(u) = P_{X_2}(u)$ if and only if $Z_{X_1}(u) = Z_{X_2}(u)$ (see Proposition 7 in [Storm 2010]).

Let

$$T := \bigcup_{n=1}^{\infty} T_n \quad \text{and} \quad T_n := \{ u \in \mathbb{C} : f_X(u^n) = 0 \}$$

be the zeroes of the $f_X(u^n)$. Note that the elements of $T_n$ are poles of $Z_X(u^n)$. The radius of convergence of $Z_X(u)$ is denoted by $R_X$. Note that $0 < R_X < 1$ since $X$ is a non-cycle graph (see, e.g., [Terras 2011, p. 197]). It follows from the graph-theoretic prime number theorem (see Theorem 10.1 in [Terras 2011]) that the radius of convergence of the other function $P_X(u)$ is also equal to $R_X$. Note that the point $u = R_X$ is a singularity of $P_X(u)$, and that

$$P_X(u) \sim -\log(R_X - u)$$
as \( u \uparrow R_X \), which is similar to

\[ P(s) \sim -\log(s - 1) \]

as \( s \downarrow 1 \) (see, e.g., [Fröberg 1968]), where \( P(s) = \sum_p 1/p^s \) denotes the prime zeta function associated with the Riemann zeta function.

Euclid proved that the number of primes \( p \) is infinite. Euler showed that the prime zeta function \( \sum_p 1/p \) diverges, and as an application he proved the infinitude of primes. In graph theory, it is also well known that the number of primes \([P]\) in \( X \) is infinite. We can give another proof “à la Euler” for this fact since \( u = R_X \) is a singularity of \( P_X(u) \).

Our main theorem is:

**Main Theorem.** Suppose that \( X = (V, E) \) is a finite, connected and non-cycle graph without degree-one vertices.

1. Let \( \mu(n) \) denote the Möbius function. If \(|u| < R_X\), then

\[ P_X(u) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log Z_X(u^n). \]

Furthermore, the right-hand side is absolutely convergent for \( u \) satisfying \(|u| < 1 \) and \( u \notin T \), and so \( P_X(u) \) has an analytic extension to the region \( \{u \in \mathbb{C} : |u| < 1\} \setminus T \).

2. The imaginary axis \( \text{Re}(s) = 0 \) is a natural boundary for the function \( \mathcal{P}_X(s) \), that is, every point on this line can be realized as a limit point of singularities of \( \mathcal{P}_X(s) \).

3. (Graph-theoretic Mertens’ first theorem) As \( N \to \infty \),

\[ \sum_{n \leq N} n \cdot \pi_X(n)R_X^n = N + O(1). \]

4. (Graph-theoretic Mertens’ second theorem) There exists a constant \( B_X \) such that, as \( N \to \infty \),

\[ \sum_{n \leq N} \pi_X(n)R_X^n = \log N + B_X + O\left(\frac{1}{N}\right). \]

5. (Graph-theoretic Mertens’ third theorem) Let \( \gamma = 0.57721\ldots \) denote the Euler–Mascheroni constant. As \( N \to \infty \),

\[ \prod_{\ell \leq N} (1 - R_X^{\ell^{(P)}}) \sim \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N}, \]
where
\[ C_X = -\frac{1}{(1 - R_X^2)^{\epsilon - \nu} R_X f'_X(R_X)} \]

(for the definition, in detail, see Section 3 in this paper).

The contents of this paper are as follows. In the next section, we prove the first two claims in the main theorem, that is, several properties of \( P_X(u) \). In Section 3, we prove the remaining claims in the main theorem, namely, the graph-theoretic Mertens theorems.

### 2. Prime zeta function of a graph

In this section, we give a proof of parts (1) and (2) of the Main Theorem introduced in Section 1.

The following facts about \( Z_X(u) \), etc., are known, and are often used in this paper.

**Facts 1.** (1) (Basic facts) For an arbitrary real number \( t > 1 \), set \( u = t^{-s} \). Then the function \( \Xi_X(s) \) is absolutely convergent and holomorphic for all \( s \) satisfying \( \text{Re}(s) > -\log R_X / \log t \ (\geq 0) \).

Since the function \( Z_X(u) \) is the reciprocal of a polynomial by the Ihara–Bass formula, the function \( Z_X(u) \) is meromorphic for all \( u \in \mathbb{C} \), and therefore \( \Xi_X(s) \) is also meromorphic for all \( s \in \mathbb{C} \).

(2) [Kotani and Sunada 2000, Theorem 1.3(1)] Let \( q + 1 \) and \( p + 1 \) be the maximum and minimum degrees of a graph \( X \), respectively. Then \( 1/q \leq R_X \leq 1/p \), the point \( u = R_X \) is a simple pole of \( Z_X(u) \), and every pole of \( Z_X(u) \) satisfies \( R_X \leq |u| \leq 1 \).

(3) [Terras 2011, p. 197] Suppose that \( X \) is a finite connected graph without degree-one vertices. Then \( R_X = 1 \) if and only if \( X \) is a cycle graph. This follows from the equation \( p = q = 1 \).

(4) [Kotani and Sunada 2000, p. 8] The leading coefficient of the polynomial \( f_X \) is given by
\[
c = \prod_{v \in V} (\deg(v) - 1),
\]
and therefore that of the polynomial \( 1/Z_X \) is equal to \( c_{2\epsilon} = (-1)^{\epsilon - \nu} c \).

In this section, the following lemma is important.

**Key Lemma 2.** Let
\[
\phi(u) = 1 + \sum_{i=1}^{d} c_i u^i \in \mathbb{Z}[u]
\]
be a polynomial function of degree $d \geq 0$, and let

$$T = \{ u \in \mathbb{C} : \text{there exists } n \geq 1 \text{ such that } \phi(u^n) = 0 \}$$

denote the zeroes of the $\phi(u^n)$. Suppose that $r$ is an arbitrary real number, and assume that $\Phi(u)$ is a series defined by

$$\Phi(u) = \sum_{n=1}^{\infty} \frac{1}{n^r} \log \phi(u^n).$$

Then $\Phi(u)$ is absolutely convergent for $u$ satisfying $|u| < 1$ and $u \notin T$.

**Proof.** First, we suppose that $d = 0$. Then the $\phi(u^n) = 1$ are constant, and therefore $\Phi(u) = 0$ is also constant. Hence, the claim is trivial. From now on, we assume that $d \geq 1$. Set $c := \max\{|c_i| : 1 \leq i \leq d\}$, choose a number $C_0$ with $C_0 \geq cd + 1$ ($\geq 2$), and fix it.

Let $r_n$ ($n \geq 3$) be a number defined by

$$r_n := \left( \frac{1 - \exp(-1/n^{2-r})}{C_0} \right)^{1/n}.$$ 

Note that $r_n < (1/C_0)^{1/n}$, the sequence $\{r_n\}_{n \geq 3}$ is increasing, and $\lim_{n \to \infty} r_n = 1$.

Take $u$ satisfying $|u| < 1$ and $u \notin T$, and fix it. Then there exists a number $N$ such that $|u| \leq r_N$, and thus $|u| < r_n$ for all $n \geq N + 1$. Now we fix such numbers $N$ and $n$.

Since $|u| < (1/C_0)^{1/n}$ and $|u^n| \leq |u| < 1$, we obtain, by the triangle inequality,

$$0 < 1 - C_0 |u^n| \leq |\phi(u^n)|, \quad \text{and so} \quad -\log |\phi(u^n)| \leq -\log(1 - C_0 |u^n|).$$

On the other hand, since $|u| < r_n$, then $C_0 |u^n| < 1 - \exp(-1/n^{2-r})$, so we obtain the inequality $-\log(1 - C_0 |u^n|) < 1/n^{2-r}$. Combining this result with (1), we obtain

$$\text{Re}(-\log \phi(u^n)) = -\log |\phi(u^n)| < \frac{1}{n^{2-r}}.$$

The first inequality in (1) also shows that the function $\log \phi(u^n)$ is holomorphic in the closed disk $|u| \leq r_{N+1}$. By applying the Borel–Carathéodory theorem (see, e.g., [Titchmarsh 1958, §5.5]) to the function $\log \phi(u^n)$ and the two circles $|u| = r_{N+1}$, $|u| = r_N$, we obtain

$$|\log \phi(u^n)| \leq \max_{|u|=r_N} |\log \phi(u^n)| \leq K \max_{|u|=r_{N+1}} \text{Re}(-\log \phi(u^n)) \leq K \frac{1}{n^{2-r}},$$

where $K := 2r_N/(r_{N+1} - r_N)$. Therefore, it follows that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^r} |\log \phi(u^n)| \leq K \sum_{n=N+1}^{\infty} \frac{1}{n^z} < K \cdot \zeta(2) < \infty.$$
Hence, for \( u \) satisfying \( |u| < 1 \) and \( u \notin T \), the series \( \Phi(u) \) converges absolutely. □

Using this lemma, we can prove the following proposition.

**Proposition 3.** Let \( \mu(n) \) denote the Möbius function. If \( |u| < R_X \), then

\[
P_X(u) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log Z_X(u^n).
\]

Moreover, the right-hand side of (3) is absolutely convergent for \( u \) satisfying \( |u| < 1 \) and \( u \notin T \), and therefore \( P_X(u) \) extends analytically to the region \( \{ u \in \mathbb{C} : |u| < 1 \} \setminus T \).

Equivalently, if \( \text{Re}(s) > -\log R_X / \log t \), then

\[
P_X(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log Z_X(ns).
\]

The right-hand side of (4) is absolutely convergent for \( s \) satisfying \( \text{Re}(s) > 0 \) and \( t^{-s} \notin T \), and so (4) gives the analytic continuation of \( P_X(s) \) to the region.

**Proof.** Note that \( R_X \leq 1 \) (from Fact 1(2)) and \( \exp(z) = \prod_{n=1}^{\infty} (1 - z)^{-\mu(n)/n} \) for \( |z| < 1 \). Suppose that \( |u| < R_X \). Since \( |u^{\ell(P)}| \leq |u| < 1 \), we obtain the equality

\[
\exp(P_X(u)) = \prod_{[P]} \exp(u^{\ell(P)}) = \prod_{[P]} \prod_{n=1}^{\infty} (1 - u^{n\ell(P)})^{-\mu(n)/n} = \prod_{n=1}^{\infty} Z_X(u^n)^{\mu(n)/n},
\]

and therefore (3) holds for \( u \) satisfying \( |u| < R_X \).

Set

\[
1/Z_X(u) = (1 - u^2)^{-\nu} f_X(u) = 1 + c_1 u + \cdots + c_{2\epsilon} u^{2\epsilon} \in \mathbb{Z}[x],
\]

\( c = \max\{|c_i| : 1 \leq i \leq 2\epsilon\} \) and \( C_0 = 2\epsilon c \geq 2 \). By applying Key Lemma 2 to \( \phi(u) = 1/Z_X(u) \) and \( r = 1 \), it follows that, for \( u \) satisfying \( |u| < 1 \) and \( u \notin T \), the series \( \sum_{n=1}^{\infty} \log Z_X(u^n)/n \) is absolutely convergent, and so the right-hand side of (3) is absolutely convergent.

Moreover, for a Ramanujan graph, we can prove the following.

**Corollary 4.** Suppose that \( X \) is a finite connected Ramanujan graph with degree \( q + 1 \), that is, \( Z_X(u) \) satisfies the Riemann hypothesis (see Theorem 7.4 in [Terras 2011]). Then the function \( P_X(u) \) is absolutely convergent for \( u \) satisfying \( |u| < 1 \) and \( u \notin (1/q)^{1/n} \) for all \( n \).

Equivalently, the function \( P_X(s) \) is absolutely convergent for \( s \) such that \( \text{Re}(s) > 0 \) and \( \text{Re}(s) \neq \log q / \log t^n \) for all \( n \).

**Proof.** Since \( X \) is a Ramanujan graph, by Theorem 1.3 in [Kotani and Sunada 2000], every real (resp. nonreal) zero of \( f_X(u) \) satisfies \( |u| = 1 \) or \( 1/q \) (resp. \( |u| = 1/\sqrt{q} \)). Thus, every point \( |u| \neq (1/q)^{1/n} \) is not zero of \( f_X(u^n) \). Hence, the proof of the assertion follows from Proposition 3. □
We can completely interchange the roles of the functions \( P_X(u) \) and \( \log Z_X(u) \).

**Corollary 5.** If \(|u| < 1\) and \( u \notin T \), then

\[
(5) \quad \log Z_X(u) = \sum_{n=1}^{\infty} \frac{1}{n} P_X(u^n).
\]

Equivalently, if \( \text{Re}(s) > 0 \) and \( t^{-s} \notin T \), then

\[
(6) \quad \log \mathcal{E}_X(s) = \sum_{n=1}^{\infty} \frac{1}{n} \mathcal{P}_X(ns).
\]

**Proof.** By applying the Möbius inversion formula (see, e.g., Theorem 270 in [Hardy and Wright 2008], or Theorem 2.2.8 in [Jameson 2003]) to the equality (3) for \(|u| < 1\), we obtain the equality (5).

\[ \square \]

**Remark 6.** The equalities (4) and (6) indicate that \( \mathcal{P}_X(s) \) is a graph-theoretic analogue to the prime zeta function \( \mathcal{P}(s) \) for the Riemann zeta function \( \zeta(s) \). The relations between \( \mathcal{P}(s) \) and \( \zeta(s) \) are given as follows (see [Glaisher 1891], and also [Fröberg 1968] and Equality (1.6.1) in [Titchmarsh 1986]):

For \( \text{Re}(s) > 1 \),

\[
P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns) \quad \text{and} \quad \log \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n} P(ns).
\]

We can orient the edges of \( X \), and label the edges as follows:

\[ \vec{E} = \{a_1, a_2, \ldots, a_\epsilon, a_{\epsilon+1} = a_1^{-1}, a_{\epsilon+2} = a_2^{-1}, \ldots, a_{2\epsilon} = a_\epsilon^{-1}\}. \]

Let \( W = W_X := (w_{ij}) \) denote the edge adjacency matrix of a graph \( X \), that is, a \( 2\epsilon \times 2\epsilon \) matrix defined by

\[
w_{ij} := \begin{cases} 1 & \text{if } t(a_i) = o(a_j) \text{ and } a_j \neq a_i^{-1} \text{ for } a_i, a_j \in \vec{E}, \\ 0 & \text{otherwise} \end{cases}
\]

(see p. 28 in [Terras 2011]). Let \( \lambda_1, \ldots, \lambda_k \) be the distinct eigenvalues of \( W \), and let \( e_1, \ldots, e_k \) be their multiplicities. Note that \( \sum_{i=1}^{k} e_i = 2\epsilon \). Let \( e := \sum_{i=1, \lambda_i \neq \pm 1}^{k} e_i \).

By the determinant formula given by Hashimoto [1989] and Bass [1992], the polynomial \( 1/Z_X(u) \) can be written as

\[
1/Z_X(u) = \det(I_{2\epsilon} - Wu) = \prod_{i=1}^{k} (1 - \lambda_i u)^{e_i}.
\]

Note that \( f_X(1) = 0 \). We now define a polynomial \( g_X(u) \) by

\[
g_X(u) := f_X(u)/(1 - u).
\]
Note that since \( f'_X(1) = 2(\epsilon - \nu)\kappa \) by [Northshield 1998, Theorem],
\[
g_X(1) = -f'_X(1) = -2(\epsilon - \nu)\kappa,
\]
where \( \kappa \) is the complexity of \( X \), that is, the number of spanning trees in \( X \). Since \( X \) is a non-cycle graph, that is, \( \epsilon \neq \nu \), the polynomial \( g_X(u) \) can be also written as
\[
(7) \quad g_X(u) = \frac{1/Z_X(u)}{(1-u^2)^{\epsilon-\nu}(1-u)} = (1+u)^{2\nu-1-e} \prod_{\lambda_i \neq \pm 1} (1-\lambda_i u)^{e_i}.
\]

We can show that the function \( P_X(s) \) has a natural boundary.

**Proposition 7.** Let \( X = (V, E) \) be a finite, connected and non-cycle graph without degree-one vertices.

1. There exists an eigenvalue \( \lambda \) of \( W \) such that \( |\lambda| > 1 \).
2. The imaginary axis \( \text{Re}(s) = 0 \) is a natural boundary for the function \( P_X(s) \), that is, every point on this line can be realized as a limit point of singularities of \( P_X(s) \).

**Proof.**

1. The leading coefficient \( c_{2\epsilon} \) of the polynomial \( 1/Z_X(u) \) is given by
\[
(-1)^{\epsilon-\nu} \prod_{v \in V} (\deg(v) - 1) = c_{2\epsilon} = \prod_{i=1}^k \lambda_i^{e_i}
\]
(from Fact 1(4)). By our assumption for \( X \), the graph \( X \) is not a 2-regular graph. Thus \( |c_{2\epsilon}| > 1 \) and so there exists an eigenvalue \( \lambda_i \) with \( |\lambda_i| \neq 1 \). Note that every pole \( 1/\lambda_i \) of \( Z_X(u) \) satisfies \( |1/\lambda_i| \leq 1 \) by Fact 1(2). So there exists an eigenvalue \( \lambda_i \) with \( |\lambda_i| > 1 \).

2. Note that \( \exp(z) = \prod_{n=1}^\infty (1-z^n)^{-\mu(n)/n} \) for \( |z| < 1 \). If \( |u| < 1 \) and \( u \notin T \), then
\[
\exp(P_X(u)) = \prod_{n=1}^\infty Z_X(u^n)^{\mu(n)/n}
= \left( \prod_{n=1}^\infty (1-u^{2n})^{-\mu(n)/n} \right)^{\epsilon-\nu} \left( \prod_{n=1}^\infty (1-u^n)^{-\mu(n)/n} \right) \prod_{n=1}^\infty g_X(u^n)^{-\mu(n)/n}
= \exp((\epsilon - \nu)u^2 + u) \prod_{n=1}^\infty g_X(u^n)^{-\mu(n)/n},
\]
and therefore the equality
\[
P_X(u) = (\epsilon - \nu)u^2 + u - \sum_{n=1}^\infty \frac{\mu(n)}{n} \log g_X(u^n)
\]
holds.
Note that \( u = t^{-s} \). By using the equalities (7) and 2, the function \( \mathcal{P}_X(s) \) can be written as
\[
\mathcal{P}_X(s) = (\epsilon - \nu)t^{-2s} + t^{-s} - \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left( (2\nu - 1 - e) \log(1 + t^{-ns}) + \sum_{\lambda_i \neq \pm 1} k \log(1 - \lambda_it^{-ns}) \right)
\]
for all \( s \) satisfying \( \text{Re}(s) > 0 \). By part (1), there exists \( \lambda \) such that \( |\lambda| > 1 \) among the eigenvalues \( \lambda_1, \ldots, \lambda_k \) of \( W \). Note that \( 1 - \lambda t^{-ns} = 0 \) if and only if \( s = r(\lambda, n, m) \), where
\[
r(\lambda, n, m) := \frac{\log|\lambda|}{n \log t} + i \frac{\text{Arg}(\lambda) + 2\pi m}{n \log t},
\]
and \( \text{Arg}(\lambda) \) is the argument of \( \lambda \) with \( -\pi \leq \text{Arg}(\lambda) < \pi \). Note that
\[
\epsilon_n := \frac{\log|\lambda|}{n \log t} \to 0
\]
as \( n \to \infty \). We now fix an arbitrary point \( \alpha = ia \) on the imaginary axis \( \text{Re}(s) = 0 \). Then, we can arrange a sequence of integers \( \{m_n\} \) for each integer \( n \) so that
\[
\frac{\text{Arg}(\lambda) + 2\pi m_n}{n \log t} \to a
\]
as \( n \to \infty \). Hence, each point \( \alpha \) on the boundary is a limit point of singularities of \( \mathcal{P}_X(s) \). Since \( \epsilon_n > 0 \) for all \( n \), we cannot continue \( \mathcal{P}_X(s) \) beyond the boundary at \( \text{Re}(s) = 0 \). \( \square \)

**Remark 8.** Proposition 7(2) is an analogue of the fact that the imaginary axis \( \text{Re}(s) = 0 \) is a natural boundary for the prime zeta function \( P(s) \) of the Riemann zeta function \( \zeta(s) \) (see [Landau and Walfisz 1920]).

### 3. Graph-theoretic Mertens’ theorem

In this section, we prove parts (3)–(5) of the Main Theorem introduced in Section 1.

Throughout this section, we always assume that \( X = (V, E) \) is a finite, connected, non-cycle graph without degree-one vertices. Note in particular that \( \nu \neq \epsilon \) and \( 0 < R_X < 1 \).

First, we define the constants \( H_X, C_X \) and \( \gamma_X \), and study their properties, which play important roles in this section. Let \( u \) be a complex variable. We define a function by
\[
H_X(u) := \log Z_X(u) - P_X(u) = \sum_{n \geq 2} \frac{1}{n} P_X(u^n) = \sum_{[P]} \sum_{n \geq 2} \frac{1}{n} u^{n\ell(P)}.
\]
Note that the point \( u = R_X \) is a common pole of \( Z_X(u) \) and \( P_X(u) \) by Fact 1(2), and that the series \( H_X(u) \) is absolutely convergent for \( u \) satisfying \( |u| < 1 \) and \( u \not\in T \), from Corollary 5.

Since \( u = R_X \) is a simple pole of \( Z_X(u) \), we can define constants \( c_X \) and \( C_X \) by

\[
c_X := - \text{Res}_{u=R_X} Z_X(u) = \lim_{u \uparrow R_X} (R_X - u) Z_X(u) = \frac{-1}{(1 - R_X^2)^{\varepsilon - v} f_X'(R_X)}
\]

and \( C_X := c_X / R_X \).

**Lemma 9.**

(1) The value \( H_X := H_X(R_X) \) is finite.

(2) The constants \( c_X \) and \( C_X \) are positive.

**Proof.** (1) Since \( R_X^n < R_X < 1 \) \((n \geq 2)\), the function \( P_X(u) \) is holomorphic at \( u = R_X^n \), and therefore \( P_X(u^n) \) is holomorphic at \( u = R_X \). We have

\[
H_X(R_X) = \sum_{[P]} \sum_{n \geq 2} \frac{1}{n} R_X^{n\ell(P)} \leq \sum_{[P]} \sum_{n \geq 2} R_X^{n\ell(P)}
\]

\[
= \sum_{[P]} \frac{R_X^{2\ell(P)}}{1 - R_X^{\ell(P)}} \leq \frac{1}{1 - R_X} \sum_{[P]} R_X^{2\ell(P)} = \frac{P_X(R_X^2)}{1 - R_X} < +\infty,
\]

and the assertion follows.

(2) Note that the leading coefficient of the polynomial \( f_X \) is given by

\[
c = \prod_{v \in \mathcal{V}} (\deg(v) - 1) > 0
\]

by Fact 1(4). Then \( f_X \) factors as the product of irreducible polynomials such that

\[
f_X(u) = c \prod_{i=1}^{m_1} (u - \alpha_i) \cdot \prod_{j=1}^{m_2} f_j(u),
\]

where the \( f_j \) are monic of \( \deg f_j = 2 \), and \( \deg f_X = 2\nu = m_1 + 2m_2 \). Note that \( m_1 \) is even. Since \( u = R_X \) is a simple pole of \( Z_X(u) \), it is a simple zero of \( f_X \). We may assume that \( \alpha_1 = R_X \). Since \( \alpha_i > R_X \) \((2 \leq i \leq m_1)\) and the discriminants of the \( f_j \) are negative, the sign of

\[
f_X'(R_X) = c \prod_{i=2}^{m_1} (R_X - \alpha_i) \prod_{j=1}^{m_2} f_j(R_X)
\]

is equal to \((-1)^{m_1-1} = -1\), i.e., \( f_X'(R_X) < 0 \), so \( c_X > 0 \) and \( C_X = c_X / R_X > 0 \). \( \square \)
Since the function $Z_X(u) - c_X/(R_X - u)$ is holomorphic at $u = R_X$, we can define a constant $\gamma_X$ by

$$\gamma_X := \lim_{u \uparrow R_X} \left( Z_X(u) - \frac{c_X}{R_X - u} \right),$$

which is an analogue of the Euler–Mascheroni constant $\gamma = \lim_{s \downarrow 1} (\zeta(s) - 1/(s-1))$ for $\zeta(s)$.

In a neighborhood of $u = R_X$, the function $Z_X(u)$ can be expanded as

$$Z_X(u) = \frac{c_X}{R_X - u} + \gamma_X + O(R_X - u),$$

and so

$$\log Z_X(u) = \log \frac{c_X}{R_X - u} + O(R_X - u).$$

Similarly, in a neighborhood of $u = R_X$, the function $P_X(u)$ can be expanded as

$$P_X(u) = \log \frac{c_X}{R_X - u} - H_X(u) + O(R_X - u) = \log \frac{c_X}{R_X - u} - H_X(R_X) + O(R_X - u).$$

In this section, the following facts are used.

**Facts 10.** (1) (See, for example, Theorem 18.1 in [Korevaar 2002].) Let $x$ be a complex variable and let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with $a_n \geq 0$ that converges for $|x| < 1$. Suppose that

$$F(x) - \frac{C}{1-x} = O(1)$$

as $x \to 1$. Then the partial sum $A(N) = \sum_{n \leq N} a_n$ satisfies

$$A(N) = C \cdot N + O(\log N)$$

as $N \to \infty$.

(2) (See, for example, Exercises 9-6 in [Apostol 1974], and Theorem 1.3.6 in [Jameson 2003], the Abel partial summation formula). Let $\{a_n\}$ be real numbers, and let $f(t)$ be a (real- or complex-valued) function with a continuous derivative in the interval $[1, N]$. Then

$$\sum_{n \leq N} a_n f(n) = A(N) f(N) - \int_{1}^{N} A(t) f'(t) \, dt.$$

By using Fact 10, we can prove the following proposition.

**Proposition 11.** Suppose that $X$ is a finite, connected and non-cycle graph without degree-one vertices. In a neighborhood of $u = R_X$, expand $Z_X(u)$ into the
power series

\[ Z_X(u) = \sum_{n=0}^{\infty} a'_n u^n. \]

Then, as \( N \to \infty \),

\[ \sum_{n \leq N} a'_n R_X^n = C_X \cdot N + O(\log N). \]

**Proof.** First, for simplicity of arguments, we normalize the function \( Z_X(u) \):

\[ F(x) = Z_X(R_X x) = \sum_{n=0}^{\infty} a'_n R_X^n x^n = \sum_{n=0}^{\infty} a_n x^n, \]

where \( a_n = a'_n R_X^n \). Note that the normalized function \( F(x) \) converges for \( |x| < 1 \). Since all coefficients \( a'_n \) are nonnegative (by page 13 in [Terras 2011]), all coefficients \( a_n \) are also nonnegative. Since \( X \) is a non-cycle graph, the point \( x = 1 \) is a simple pole of \( F(x) \). Hence, we obtain

\[ F(x) - \frac{C_X}{1-x} = O(1) \]

as \( x \to 1 \). By applying Fact 10(1) to this equality, as \( N \to \infty \),

\[ \sum_{n \leq N} a_n = C_X \cdot N + O(\log N), \quad \text{and so} \quad \sum_{n \leq N} a'_n R_X^n = C_X \cdot N + O(\log N) \]

holds, and the assertion follows. \( \square \)

Now, we compute the following example.

**Example 12** [Terras 2011, Example 2.8, p. 18]. Consider the graph \( X = K_4 - \{ \text{one edge} \} \). Then

\[ f_X(u) = (1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3) \quad \text{and} \quad Z_X(u)^{-1} = (1-u^2) f_X(u). \]

Since the radius of convergence \( R_X \) of \( Z_X(u) \) is the smallest positive real zero of \( f_X(u) \),

\[ R_X = \frac{1}{6} (\alpha - 1 + \alpha^{-1}) = 0.6572981 \ldots, \quad \alpha = (53 + 6\sqrt{78})^{1/3}. \]

Then \( C_X \) is computed as \( C_X = 0.5540954 \ldots \). For example, if \( N = 50000 \), then

\[ \frac{1}{N} \sum_{n \leq N} a'_n R_X^n = 0.5540867 \ldots \approx C_X. \]
Let $X = (V, E)$ be a graph, and set $|V| = \nu$ and $|E| = \epsilon$. Let $W = W_X$ be the edge adjacency matrix of $X$ (see page 28 in [Terras 2011], or Section 2 in this paper), and let $\text{Spec}(W)$ denote the spectrum of $W$, that is, the list of its eigenvalues together with their multiplicities. Note that $|\text{Spec}(W)| = 2\epsilon$. The polynomial $1/Z_X(u)$ has an expression different from that in Section 2. In fact, this can be written as

$$1/Z_X(u) = \det(I_{2\epsilon} - Wu) = \prod_{\lambda \in \text{Spec}(W)} (1 - \lambda u) = \prod_{i=1}^{k}(1 - \lambda_i u)^{e_i}.$$ 

Since the points $u = 1/\lambda$ are the poles of $Z_X(u)$, we obtain $1 \leq |\lambda| \leq 1/R_X$ by Fact 1(2).

The following lemma is used in the proof of Theorem 14 in this section.

**Key Lemma 13.** Suppose that $X$ is a finite, connected and non-cycle graph without degree-one vertices.

1. As $N \to \infty$, we have
   $$\sum_{n=1}^{N} \sum_{\lambda \in \text{Spec}(W)} (\lambda R_X)^n = N + O(1).$$

2. Let $0 < \alpha < \frac{1}{2}$ be a fixed real number. Then there exists a natural number $N_0$ such that, for any $n \geq N_0$,
   $$\left| n \cdot \pi(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right| < 2\epsilon \left( \frac{1}{R_X} \right)^{(1-\alpha)n}.$$ 

**Proof.** (1) Let $\Delta_X$ denote

$$\Delta = \Delta_X := \gcd\{\ell(P) : [P] \text{ is a prime in } X\}$$

(see Definition 2.12 in [Terras 2011]). It follows from Theorem 1.4 in [Kotani and Sunada 2000] that the poles of $Z_X(u)$ on the circle $|u| = R_X$ have the form $u = R_X e^{2\pi i a/\Delta}$ ($1 \leq a \leq \Delta$). It is well known that

$$\sum_{a=1}^{\Delta} e^{2\pi i an/\Delta} = \begin{cases} \Delta & \text{if } \Delta \mid n, \\ 0 & \text{otherwise} \end{cases}$$

(see, e.g., Exercise 10.1 in [Terras 2011]). Then we obtain

$$\left| N - \sum_{|\lambda|=1/R_X} \sum_{n=1}^{N} (\lambda R_X)^n \right| = \left| N - \sum_{n=1}^{\Delta} \sum_{a=1}^{\Delta} e^{2\pi i an/\Delta} \right| = N - \left\lceil \frac{N}{\Delta} \right\rceil \Delta < \Delta,$$
where \([r]\) denotes the integer part of the real number \(r\). On the other hand, we obtain
\[
\left| \sum_{|\lambda| < 1/R_X} \sum_{n=1}^{N} (\lambda R_X)^n \right| < 2\epsilon \sum_{n \geq 1} (\rho R_X)^n = \frac{2\epsilon \rho R_X}{1 - \rho R_X},
\]
where
\[
\rho := \max\{|\lambda| : \lambda \in \text{Spec}(W), |\lambda| < 1/R_X\}.
\]
Combining these inequalities, by the triangle inequality we obtain
\[
\left| N - \sum_{n=1}^{N} \sum_{\lambda \in \text{Spec}(W)} (\lambda R_X)^n \right| < \Delta + \frac{2\epsilon \rho R_X}{1 - \rho R_X}
\]
as \(N \to \infty\), and the assertion follows.

(2) Let \(\mu(n)\) denote the Möbius function. Note that \(\sum_{d|n} |\mu(d)| \leq n\). It is known that
\[
\pi(n) = \frac{1}{n} \sum_{d|n} \mu(d)N_{n/d} \quad \text{and} \quad N_n = \sum_{\lambda \in \text{Spec}(W)} \lambda^n
\]
(see (10.3) and (10.4) in [Terras 2011]). Combining these equalities, we obtain
\[
n \cdot \pi(n) = \sum_{\lambda \in \text{Spec}(W)} \sum_{d|n} \mu(d)\lambda^{n/d},
\]
and thus
\[
\left| n \cdot \pi(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right| = \left| \sum_{\lambda \in \text{Spec}(W)} \sum_{d|n} \mu(d)\lambda^{n/d} \right|
\]
\[
\leq \sum_{\lambda \in \text{Spec}(W)} \sum_{d|n} |\mu(d)| \cdot |\lambda|^{n/d} \leq \sum_{\lambda \in \text{Spec}(W)} \sum_{d|n} |\mu(d)| \cdot |\lambda|^{n/2}
\]
\[
\leq n \sum_{\lambda \in \text{Spec}(W)} \left( \frac{1}{R_X} \right)^{n/2} \leq 2\epsilon n \left( \frac{1}{R_X} \right)^{n/2}.
\]
On the other hand, since \(R_X < 1\) and \(0 < \alpha < \frac{1}{2}\) by our assumptions, there exists a natural number \(N_0\) such that, for any \(n \geq N_0\),
\[
n \leq \left( \frac{1}{R_X} \right)^{(1/2-\alpha)n}, \quad \text{and so} \quad n \left( \frac{1}{R_X} \right)^{n/2} \leq \left( \frac{1}{R_X} \right)^{(1-\alpha)n}.
\]
Hence, for any \(n \geq N_0\),
\[
\left| n \cdot \pi(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right| \leq 2\epsilon \left( \frac{1}{R_X} \right)^{(1-\alpha)n},
\]
and the assertion follows.

At last, we can prove the main theorem in this section.

**Theorem 14.** Suppose that $X$ is a finite, connected and non-cycle graph without degree-one vertices. Let $\gamma = 0.57721 \ldots$ be the Euler–Mascheroni constant, and let $H_X = H_X(R_X)$ and $C_X$ be the constants.

1. *(Graph-theoretic Mertens’ first theorem)* As $N \to \infty$,
   \[
   \sum_{n \leq N} n \cdot \pi(n) R_X^n = N + O(1).
   \]

2. *(Graph-theoretic Mertens’ second theorem)* There exists a constant $B_X$ such that, as $N \to \infty$,
   \[
   \sum_{n \leq N} \pi(n) R_X^n = \log N + B_X + O\left(\frac{1}{N}\right).
   \]

3. The equality $B_X = \gamma + \log C_X - H_X$ holds. Equivalently,
   \[
   B_X = \gamma + \log C_X - \sum_{\lambda \in \text{Spec}(W)} (\log(1 - R_X^\lambda)) + R_X^\lambda.
   \]

4. *(Graph-theoretic Mertens’ third theorem)* As $N \to \infty$,
   \[
   \prod_{\ell(P) \leq N} (1 - R_X^{\ell(P)}) = \prod_{n \leq N} (1 - R_X^n)^{\pi(n)} \sim \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N}.
   \]

**Proof.** (1) Let $N_0$ be a number as in the proof of Key Lemma 13(2), and let $K$ denote the constant
   \[
   K := \left| \sum_{n=1}^{N_0-1} n \cdot \pi(n) R_X^n - \sum_{n=1}^{N_0-1} \sum_{\lambda \in \text{Spec}(W)} (\lambda R_X)^n \right|.
   \]

Assume that $N$ is sufficiently large. Then it follows from Key Lemma 13(2) that
   \[
   \left| \sum_{n=1}^{N} n \cdot \pi(n) R_X^n - \sum_{n=1}^{N} \sum_{\lambda \in \text{Spec}(W)} (\lambda R_X)^n \right| \leq K + \left| \sum_{n=N_0}^{N} R_X^n \left( n \cdot \pi(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right) \right| \leq K + 2\varepsilon \sum_{n=N_0}^{N} R_X^{\alpha n} < K + \frac{2\varepsilon}{1 - R_X^{\alpha}}.
   \]
and therefore by Key Lemma 13(1) we have

\[ \sum_{n=1}^{N} n \cdot \pi(n) R_X^n = \sum_{n=1}^{N} \sum_{\lambda \in \text{Spec}(W)} (\lambda R_X)^n + O(1) = N + O(1) \quad \text{as } N \to \infty. \]

(2) We set \( a_n = n \cdot \pi(n) R_X^n \). By part (1), we obtain \( A(t) = t + O(1) \). By applying Fact 10(2) with \( f(t) = 1/t \), we get

\[
\sum_{n \leq N} \pi(n) R_X^n = \frac{A(N)}{N} + \frac{1}{1} \int_{1}^{N} A(t) \frac{dt}{t^2} = \frac{N + O(1)}{N} + \int_{1}^{N} t + O(1) dt
\]

\[
= 1 + O\left(\frac{1}{N}\right) + \int_{1}^{N} \left( \frac{1}{t} + O\left(\frac{1}{t^2}\right) \right) dt
\]

\[
= 1 + O\left(\frac{1}{N}\right) + \left[ \log t + O\left(\frac{1}{t}\right) \right]_{1}^{N}
\]

\[
= 1 + O\left(\frac{1}{N}\right) + \log N + O\left(\frac{1}{N}\right) + O(1) = \log N + O(1) + O\left(\frac{1}{N}\right),
\]

and the assertion follows.

(3) Fix an arbitrary \( x \) satisfying \( 0 < x < 1 \). By applying Fact 10(2) with \( a_n = \pi(n) R_X^n \) and \( f(t) = x^t \),

\[
\sum_{n \leq N} \pi(n) R_X^n x^n = A(N)x^N - \log x \int_{1}^{N} x^t A(t) dt
\]

holds. It follows from part (2) that

\[
\sum_{n \leq N} \pi(n) R_X^n x^n = \left( \log N + B_X + O\left(\frac{1}{N}\right) \right) x^N - \log x \int_{1}^{N} x^t \left( \log t + B_X + O\left(\frac{1}{t}\right) \right) dt,
\]

and, moreover, as \( N \to \infty \),

\[
P_X(R_X x) = -\log x \int_{1}^{\infty} x^t \left( \log t + B_X + O\left(\frac{1}{t}\right) \right) dt.
\]

In order to calculate the right-hand side of this equality, for simplicity of arguments, we define the functions \( I_n = I_n(x) \):

\[
-\log x \int_{1}^{\infty} x^t \left( \log t + B_X + O\left(\frac{1}{t}\right) \right) dt = I_1 + I_2 + O(I_3),
\]
where
\[ I_1 = -\log x \int_1^\infty x^t \log t \, dt, \]
\[ I_2 = -B_X \cdot \log x \int_1^\infty x^t \, dt = B_X \cdot x, \quad \text{and} \]
\[ I_3 = -\log x \int_1^\infty \frac{x^t}{t} \, dt. \]

First, we compute the function \( I_1 \):
\[
I_1 = -\int_1^\infty (x^t)' \log t \, dt = \int_1^\infty \frac{x^t}{t} \, dt.
\]
Now we take \( r = -t \log x \). Note that \( \log x < 0 \). Then we obtain
\[
I_1 = \int_{-\log x}^{\infty} \frac{e^{-r}}{r} \, dr = -\text{Ei}(\log x),
\]
where \( \text{Ei}(z) \) \( (z \in \mathbb{C} \text{ and } |\text{Arg}(-z)| < \pi) \) is the exponential integral
\[
-\text{Ei}(-z) = \int_{-z}^{\infty} \frac{e^{-r}}{r} \, dr
\]
(see, e.g., Equality (3.1.3) in [Lebedev 1972]). Since the function \( \text{Ei}(z) \) expands as
\[
\text{Ei}(z) = \gamma + \log(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!}
\]
(see Equality (3.1.6) in [ibid.]),
\[
I_1 = -\gamma - \log(-\log x) + O(\log x) = -\gamma - \log(-\log x) + O(1 - x).
\]

Next we calculate the function \( I_3 \). It follows from the above result that
\[
I_3 = -\log x \int_1^\infty \frac{x^t}{t} \, dt = (-\log x)I_1 = O(1 - x)
\]
as \( x \uparrow 1 \).

By combining the above results, the equality (9) is written as follows:
\[
P_X(R_X x) = -\gamma - \log(-\log x) + B_X x + O(1 - x),
\]
and, moreover, as \( x \uparrow 1 \),
\[
P_X(R_X x) + \log(-\log x) \to B_X - \gamma.
\]

On the other hand, since
\[
\log Z_X(R_X x) = \log \frac{1}{1-x} + \log C_X + O(1 - x)
\]
from the equality (8), as \( x \uparrow 1 \),

\[
\log Z_X(R_X x) + \log(-\log x) = \log\left(\frac{-\log x}{1-x}\right) + \log C_X \to \log C_X.
\]

Combining (10) with (11), we obtain

\[
H_X = \lim_{x \uparrow 1} H_X(R_X x) = \lim_{x \uparrow 1} \left(\log Z_X(R_X x) - P_X(R_X x)\right)
\]
\[
= \lim_{x \uparrow 1} \left(\log Z_X(R_X x) + \log(-\log x)\right) - \left(P_X(R_X x) + \log(-\log x)\right)
\]
\[
= \log C_X + \gamma - B_X.
\]

(4) Fix an arbitrary positive real number \( N \). We define the following functions:

\[
H_X^{\leq N} = \sum_{n \leq N} \pi(n) \sum_{m=2}^{\infty} \frac{1}{m} R_X^{mn}
\]
\[
H_X^{> N} = \sum_{n > N} \pi(n) \sum_{m=2}^{\infty} \frac{1}{m} R_X^{mn}.
\]

Note that \( H_X = H_X^{\leq N} + H_X^{> N} \). From parts (2) and (3), we obtain

\[
\sum_{n \leq N} \pi(n) R_X^n + H_X^{\leq N} = \log N + \gamma + \log C_X - H_X^{> N} + O\left(\frac{1}{N}\right).
\]

Since the left-hand side of this equality is equal to

\[
\sum_{n \leq N} \pi(n) R_X^n + H_X^{\leq N} = \sum_{n \leq N} \pi(n) \sum_{m=1}^{\infty} \frac{1}{m} R_X^{mn}
\]
\[
= -\sum_{n \leq N} \pi(n) \log(1 - R_X^n) = -\log \left(\prod_{n \leq N} (1 - R_X^n)^{\pi(n)}\right),
\]

we obtain

\[
\prod_{n \leq N} (1 - R_X^n)^{\pi(n)} = \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N} \exp\left(H_X^{> N} + O\left(\frac{1}{N}\right)\right).
\]

Since \( H_X^{> N} \to 0 \) and \( 1/N \to 0 \) as \( N \to \infty \), the assertion follows.

Last, we compute the following example.

**Example 15** (continued from Example 12). Consider the graph \( X = K_4 - \{\text{one edge}\} \). Then

\[
H_X = 0.25613 \ldots, \quad B_X = \gamma + \log C_X - H_X = -0.26933 \ldots.
\]
For example, if \( N = 550 \), then
\[
\sum_{n \leq N} \pi(n) R_X^n - \log N = -0.26842 \ldots \approx B_X,
\]
\[
\prod_{n \leq N} (1 - R_X^n) \pi(n) = 0.18447 \ldots \approx \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N} = 0.18457 \ldots .
\]

**Remark 16.** (See [Mertens 1874, Equation (17)], or [Hardy and Wright 2008, Theorem 428].) A number-theoretic analogue to part (3) in the preceding theorem is
\[
B_1 = \gamma - H = \gamma + \sum_p \left( \log \left( 1 - \frac{1}{p} \right) + \frac{1}{p} \right),
\]
where \( H = \sum_{n \geq 2} P(n)/n \) is a constant, and \( P(s) \) is the prime zeta function.

**Remark 17.** We now compare parts (2)–(4) of our Theorem 14 with Theorem 1 in [Sharp 1991]. We define
\[
h_X := -\log R_X, \quad N(P) = e^{h_X \ell(P)} \quad \text{and} \quad x = e^{h_X N}.
\]
The quantity \( h_X \) is called the topological entropy of a flow in ergodic theory (see [Sharp 1991]), which is a constant in our setting. Note that \( \ell(P) \leq N \) if and only if \( N(P) \leq x \). Note that \( R_X^{\ell(P)} = 1/N(P) \). Then our Mertens’ second theorem can be rewritten as
\[
\sum_{N(P) \leq x} \frac{1}{N(P)} = \log(\log x) + B + O\left( \frac{1}{\log x} \right),
\]
where \( B := -\log h_X + B_X \), and, similarly, our Mertens’ third theorem becomes
\[
\prod_{N(P) \leq x} \left( 1 - \frac{1}{N(P)} \right) \sim \frac{1}{C_X/h_X} \cdot \frac{e^{-\gamma}}{\log x}.
\]
In Theorem 1 in [Sharp 1991], our constant \( C_X/h_X \), which is equal to a residue (up to sign) of the Ihara zeta function, corresponds with that of a dynamical zeta function for a flow.

Moreover, our Theorem 14(3) becomes
\[
B = \gamma + \log(C_X/h_X) + \sum_{[P]} \left( \log \left( 1 - \frac{1}{N(P)} \right) + \frac{1}{N(P)} \right).
\]

**Remark 18.** Let \( X = (V, E) \) be a finite, connected, non-cycle graph without degree-one vertices, and let \( S = (V', E') \) be its \( k \)-subdivision (that is, let \( S \) be the graph obtained from \( X \) by adding \( k \) new vertices to each edge of \( X \)) (see Examples 6.4 and 8.5 in [Terras 2011]). Then
\[
H_X = H_S, \quad C_X = (k + 1)C_S, \quad \text{and} \quad B_X = B_S + \log(k + 1).
\]
This is proved as follows: note that $\Delta_S = (k+1)\Delta_X$, $R_S^{k+1} = R_X$, and

$$\pi_S(n) = \begin{cases} \pi_X(n/(k+1)) & \text{if } (k+1) \mid n, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore,

$$H_S = \sum_{m \geq 2} \frac{1}{m} P_S(R_S^m) = \sum_{m \geq 2} \frac{1}{m} \sum_{n=1}^{\infty} \pi_S(n) R_S^{mn} = \sum_{m \geq 2} \frac{1}{m} \sum_{n=1}^{\infty} \pi_X(n) R_S^{(k+1)mn}$$

$$= \sum_{m \geq 2} \frac{1}{m} \sum_{n=1}^{\infty} \pi_X(n) R_X^{mn} = \sum_{m \geq 2} \frac{1}{m} P_X(R_X^m) = H_X.$$ 

Note that $\nu' = \nu + k\epsilon$, $\epsilon' = (k+1)\epsilon$, and $Z_S(u) = Z_X(u^{k+1})$, and so

$$(1-u^2)^{-\nu} f_S(u) = (1-u^{2(k+1)})^{-\nu} f_X(u^{k+1}),$$

$$(1-R_S^2)^{-\nu} R_S f'_S(R_S) = (k+1)(1-R_X^2)^{-\nu} R_X f'_X(R_X).$$

Therefore,

$$(k+1)C_S = \frac{-(k+1)}{(1-R_S^2)^{-\nu'} R_S f'_S(R_S)} = C_X,$$

and so

$$B_X = \gamma + \log C_X - H_X = \gamma + \log C_S - H_S + \log(k+1) = B_S + \log(k+1).$$

Acknowledgements

Saito would like to thank Professor Toyokazu Hiramatsu for his encouragement and thoughtful suggestions. The authors thank the referee for his or her valuable comments and careful review of this paper.

References


TAKEHIRO HASEGAWA
SHIGA UNIVERSITY
OTSU
SHIGA 520-0862
JAPAN
thasegawa3141592@yahoo.co.jp

SEIKEN SAITO
WASEDA UNIVERSITY
SHINJUKU
TOKYO 169-8050
JAPAN
seiken.saitou@gmail.com
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