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GENERATORS OF THE GAUSS–PICARD MODULAR GROUP IN THREE COMPLEX DIMENSIONS

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In this paper, we will describe a method to obtain the generators system of Gauss–Picard modular group PU(3, 1; $\mathbb{Z}[i]$). More precisely, we will show that PU(3, 1; $\mathbb{Z}[i]$) can be generated by five given transformations: two Heisenberg translations, two Heisenberg rotations and one involution. Indeed, the same method works for the other higher-dimensional Euclidean Picard modular groups.

1. Introduction

There are some natural algebraic generalizations of the classical modular group PSL(2, $\mathbb{Z}$). For example, a Bianchi group is a group of the form PSL(2, $\mathbb{O}_d$), where d is a positive square-free integer. Here, PSL denotes the projective special linear group and $\mathbb{O}_d$ is the ring of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$. These groups were first studied by Bianchi [1892] as a natural class of discrete subgroups of PSL(2, $\mathbb{C}$). A general method to determine finite presentation for each PSL(2, $\mathbb{O}_d$) was developed by Swan [1971] based on the geometrical work of Bianchi, while a separate purely algebraic method was given by Cohn [1968]. As another generalization of the modular group, the construction was generalized by Picard [1883; 1884]. Suppose H is a Hermitian matrix of signature (2, 1) with entries in $\mathbb{O}_d$, and let SU(H; $\mathbb{O}_d$) denote the subgroup of SU(H) consisting of those matrices whose entries lie in $\mathbb{O}_d$. Picard studied the group PU(H; $\mathbb{O}_d$) acting on the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$. Now, Picard modular groups PU(H; $\mathbb{O}_d$) have attracted a great deal of attention both for their intrinsic interest as discrete groups and also for their applications in complex hyperbolic geometry.

One can view the modular group or a Bianchi group acting discontinuously on hyperbolic spaces. Then Poincaré's polyhedra theorem provides a geometric method to obtain their generators from their fundamental polyhedra. But [Mostow 1980] told us that the explicit construction of fundamental domains for lattices in complex hyperbolic spaces was particularly difficult. Until recently, the geometry of SU(H; $\mathbb{O}_3$) had been studied by Falbel and Parker [2006], while the geometry of

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$\mathrm{SU}(H; \mathbb{O}_1)$ had been studied by Francsics and Lax [2005a; 2005b; 2006] and Falbel, Francsics and Parker [Falbel et al. 2011b]. By applying similar ideas to those of [Falbel and Parker 2006; Falbel et al. 2011b], Zhao [2012] obtained generators of the Euclidean Picard groups $\mathrm{PU}(2, 1; \mathbb{O}_d)$ for $d = 2, 7, 11$.

There are some simple algorithms to obtain the generators of the modular group or some Picard modular groups. For example, the continued fraction algorithm may be applied to any element of the modular group $\mathrm{PSL}(2, \mathbb{Z})$. This shows that $S(z) = -1/z$ and $T(z) = z + 1$ generate $\mathrm{PSL}(2, \mathbb{Z})$. This algorithm was extended to $\mathrm{PU}(2, 1; \mathbb{O}_1)$ in [Falbel et al. 2011a], which provided a different system of generators from those obtained via a fundamental domain in [Falbel et al. 2011b]. In [Wang et al. 2011], the authors applied the continued fraction algorithm to $\mathrm{PU}(2, 1; \mathbb{O}_3)$ and produced a different system of generators from that obtained in [Falbel and Parker 2006].

There is an obvious generalization of Picard modular groups to higher complex dimensions. We observe that very little is known about the geometry and algebraic properties, e.g., explicit fundamental domain or generating system of the higher-dimensional Picard modular groups $\mathrm{PU}(n, 1; \mathbb{O}_d)$. In [Xie et al. 2013], the continued fraction algorithm was generalized to Picard modular groups in higher complex dimensions. It contained the first generalization that we were aware of to a group of 4×4 matrices. However, it seems very difficult to extend the continued fraction algorithm to other higher-dimensional Picard modular groups. Using a combination of the ideas from [Falbel et al. 2011a; Xie et al. 2013] and [Falbel and Parker 2006; Zhao 2012], we will present a method to obtain the generating system of the Gauss–Picard modular group $\mathrm{PU}(3, 1; \mathbb{Z}[i])$. We first get the generators of the stabilizer of infinity of $\mathrm{PU}(3, 1; \mathbb{Z}[i])$ by applying a similar argument as in our previous paper [Xie et al. 2013]. Then we will construct a subset in the boundary of complex hyperbolic space which contains the fundamental domain for the stabilizers of infinity in $\mathrm{PU}(3, 1; \mathbb{Z}[i])$. Finally, we will show the boundaries of some isometric spheres that contain this subset. This method works for the other higher-dimensional Euclidean Picard modular groups.

2. Preliminaries

2.1. ***The Siegel domain.*** We recall some basic notions of complex hyperbolic geometry. For more details we refer the reader to [Goldman 1999; Parker 2010].

Let $\mathbb{C}^{n,1}$ denote the vector space $\mathbb{C}^{n+1}$ equipped with the Hermitian form of signature $(n, 1)$ given by the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The Hermitian product of two vectors $\boldsymbol{z}$ and $\boldsymbol{w}$ is given by $\langle \boldsymbol{z}, \boldsymbol{w}\rangle = \boldsymbol{w}^* J \boldsymbol{z}$, where $\boldsymbol{w}^*$ denotes the Hermitian transpose of $\boldsymbol{w}$.

We denote by V_- and V_0 the negative and null cones associated to the Hermitian form, respectively. The complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ is the projectivization of V_-, and its boundary is the projectivization of V_0. The model of $\mathbb{H}^n_{\mathbb{C}}$ associated to the Hermitian form given above is often referred to as the Siegel model of $\mathbb{H}^n_{\mathbb{C}}$.

We define the Siegel domain $\mathfrak{S}$ of the complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ by identifying points of $\mathfrak{S}$ with their horospherical coordinates,

$$z = (\zeta, v, u) \in \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}^+.$$

The boundary of $\mathfrak{S}$ is given by $H_0 \cup \{q_\infty\}$, where q_∞ is a distinguished point at infinity and $H_0 = \mathbb{C}^{n-1} \times \mathbb{R} \times \{0\}$.

2.2. ***Heisenberg group.*** The boundary of a complex hyperbolic space is identified with the one-point compactification of the Heisenberg group. The $(2n-1)$-dimensional Heisenberg group $\mathcal{H}_{2n-1}$ is $\mathbb{C}^{n-1} \times \mathbb{R}$ with the group law

$$(\xi, \nu) \cdot (z, u) = (\xi + z, \nu + u + 2\Im\langle\langle \xi, z\rangle\rangle).$$

Here $\langle\langle \xi, z\rangle\rangle = z^*\xi$ is the standard positive definite Hermitian form on $\mathbb{C}^{n-1}$. In particular, we write $\|\xi\|^2 = \xi^*\xi$.

The Heisenberg group acts on itself by Heisenberg translation. For $(\tau, t) \in \mathcal{H}_{2n-1}$, this translation is

$$N_{(\tau,t)}(\xi, \nu) = (\tau + \xi, t + \nu + 2\Im\langle\langle \tau, \xi\rangle\rangle).$$

The unitary group $U(n-1)$ acts on the Heisenberg group by Heisenberg rotation.

2.3. ***Holomorphic isometries.*** Define a map $\mathfrak{S} \to \mathbb{CP}^n$ by

$$\psi : (\xi, \nu, u) \mapsto \begin{pmatrix} \frac{1}{2}(-\|\xi\|^2 - u + i\nu) \\ \xi \\ 1 \end{pmatrix}, \quad \psi : q_\infty \mapsto \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then ψ maps the set of points $z \in \mathfrak{S}$ homeomorphically to the set of points $\boldsymbol{z} \in \mathbb{CP}^n$ with $\langle \boldsymbol{z}, \boldsymbol{z}\rangle < 0$, and maps the set of points in $\partial\mathfrak{S}$ homeomorphically to the set of points $\boldsymbol{z} \in \mathbb{CP}^n$ with $\langle \boldsymbol{z}, \boldsymbol{z}\rangle = 0$. We write $\psi(z) = \boldsymbol{z}$.

The Bergman metric on $\mathfrak{S}$ is given by the distance formula

$$\cosh^2 \frac{\rho(z, w)}{2} = \frac{\langle \boldsymbol{z}, \boldsymbol{w}\rangle\langle \boldsymbol{w}, \boldsymbol{z}\rangle}{\langle \boldsymbol{z}, \boldsymbol{z}\rangle\langle \boldsymbol{w}, \boldsymbol{w}\rangle}.$$

The holomorphic isometry group of $\mathfrak{S}$ with respect to the Bergman metric is the projective unitary group $\mathrm{PU}(n, 1)$, and it acts on $\mathbb{CP}^n$ by matrix multiplication.

2.4. ***Picard modular groups.*** Let $\mathcal{O}_d$ be the ring of integers in the imaginary quadratic number field $\mathbb{Q}(i\sqrt{d})$, where d is a positive square-free integer. If $d \equiv 1, 2 \bmod 4$, then $\mathcal{O}_d = \mathbb{Z}[\sqrt{d}i]$, and if $d \equiv 3 \bmod 4$, then $\mathcal{O}_d = \mathbb{Z}[(1+\sqrt{d}i)/2]$. The subgroup of $\mathrm{PU}(n, 1)$ with entries in $\mathcal{O}_d$ is called the Picard modular group for $\mathcal{O}_d$ and is written as $\mathrm{PU}(n, 1; \mathcal{O}_d)$. Obviously, if $d = 1$, then the ring $\mathcal{O}_d$ can be written as $\mathbb{Z}[i]$.

Remark 1. The matrices corresponding to the generators obtained in this paper belong to the group $U(3, 1; \mathbb{Z}[i])$. In relation to complex hyperbolic isometries, the relevant group is $\mathrm{PU}(3, 1; \mathbb{Z}[i]) = \mathrm{SU}(3, 1; \mathbb{Z}[i])/\mathbb{Z}_4$. The center of $\mathrm{SU}(3, 1)$ is isomorphic to $\mathbb{Z}_4$, the group of fourth roots of unity. By abuse of notation, we will denote the Gauss–Picard modular group in three complex dimensions by $U(3, 1; \mathbb{Z}[i])$.

2.5. ***Heisenberg automorphism groups.*** The action of Heisenberg isometries extends to the Siegel domain, fixing q_∞. Some examples of Heisenberg isometries are as follows: for $U \in U(n-1)$ and $(\tau, t) \in \mathcal{H}_{2n-1}$, the Heisenberg rotation and Heisenberg translation correspond to the matrices

$$M_U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad N_{(\tau,t)} = \begin{pmatrix} 1 & -\tau^* & \frac{1}{2}(-\|\tau\|^2 + it) \\ 0 & I_{n-1} & \tau \\ 0 & 0 & 1 \end{pmatrix}$$

in $\mathrm{SU}(n, 1)$, respectively. The Heisenberg dilation by r fixing q_∞ and 0 corresponds to the matrix $A_r \in \mathrm{SU}(n, 1)$, where

$$A_r = \begin{pmatrix} r & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & 1/r \end{pmatrix}.$$

Finally, the Heisenberg inversion interchanging q_∞ and 0 corresponds to the matrix $R \in \mathrm{SU}(n, 1)$, where

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

2.6. ***Isometric spheres.*** Given an element $G \in \mathrm{PU}(3, 1)$ such that $G(q_\infty) \neq q_\infty$, we define the isometric sphere of G to be the hypersurface

$$\{z \in \mathbb{H}^3_{\mathbb{C}} : |\langle z, q_\infty\rangle| = |\langle z, G^{-1}(q_\infty)\rangle|\}.$$

For example, the isometric sphere of

$$R=\begin{pmatrix}0&0&0&1\\0&-1&0&0\\0&0&-1&0\\1&0&0&0\end{pmatrix}$$

is

$$\mathcal{B}_0=\left\{(\zeta_1,\zeta_2,t,u)\in\mathfrak{S}:\left||\zeta_1|^2+|\zeta_2|^2+u+it\right|=2\right\}$$

in horospherical coordinates.

All other isometric spheres are images of $\mathcal{B}_0$ by Heisenberg dilations, rotations and translations. Thus, the isometric sphere with radius r and center $(\zeta_1^0,\zeta_2^0,t^0,0)$ is given by

$$\left\{(\zeta_1,\zeta_2,t,u):\left||\zeta_1-\zeta_1^0|^2+|\zeta_2-\zeta_2^0|^2+u+it-it^0+2i\Im(\zeta_1\overline{\zeta_1^0}+\zeta_2\overline{\zeta_2^0})\right|=r^2\right\}.$$

If G has the matrix form

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{pmatrix},$$

then $G(q_\infty)\neq q_\infty$ if and only if $a_{41}\neq 0$. The isometric sphere of G has radius $r=\sqrt{2/|a_{41}|}$ and center $G^{-1}(q_\infty)$, which in horospherical coordinates is

$$(\zeta_1^0,\zeta_2^0,t^0,0)=(\overline{a_{42}}/\overline{a_{41}},\overline{a_{43}}/\overline{a_{41}},2\Im(\overline{a_{44}}/\overline{a_{41}}),0).$$

3. The generators of the stabilizer

Let Γ_∞ be the stabilizer subgroup of q_∞ in $\mathrm{PU}(n,1)$. That is,

$$\Gamma_\infty\equiv\{g\in\mathrm{PU}(n,1):g(q_\infty)=q_\infty\}.$$

We recall from [Falbel et al. 2011a; Francsics and Lax 2005a; 2005b; Xie et al. 2013] that the Langlands decomposition can be used to parametrize a transformation in the stabilizer subgroup of q_∞.

Lemma 2 (Langlands decomposition). *Any element $P\in\Gamma_\infty$ can be decomposed as a product of a Heisenberg translation, dilation, and a rotation*:

$$P=N_{(\tau,t)}A_rM_U=\begin{pmatrix}r&-\tau^*U&(-\|\tau\|^2+it)/2r\\0&U&\tau/r\\0&0&1/r\end{pmatrix}.$$

The parameters satisfy the corresponding conditions. That is, $U\in U(n-1)$, $r\in\mathbb{R}^+$ and $(\tau,t)\in\mathcal{H}_{2n-1}$.

First, we describe the Heisenberg rotations in the Gauss–Picard modular group $U(3,1;\mathbb{Z}[i])$. Let $U(2;\mathbb{Z}[i])$ be the unitary group $U(2)$ over the ring $\mathbb{Z}[i]$. Then we have the following result.

Lemma 3. $U(2;\mathbb{Z}[i])$ *can be generated by the two unitary matrices*

$$U_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad \textit{and}\quad U_2=\begin{pmatrix}i&0\\0&1\end{pmatrix}.$$

Remark 4. A similar lemma was proved for $U(2;\mathbb{Z}[(1+\sqrt{-3})/2])$ in [Xie et al. 2013].

Next, we characterize the elements of the stabilizer subgroup Γ_∞ of infinity in the Picard modular group $U(3,1;\mathbb{Z}[i])$. We denote this stabilizer by $\Gamma_\infty(3,1;\mathbb{Z}[i])$.

Lemma 5. *An element $P\in U(3,1;\mathbb{Z}[i])$ lies in $\Gamma_\infty(3,1;\mathbb{Z}[i])$ if and only if the parameters in the Langlands decomposition of P satisfy the conditions*

$$r=1,\quad t\in 2\mathbb{Z},\quad \tau=(\tau_1,\tau_2)^T\in\mathbb{Z}[i]^2,\quad U\in U(2;\mathbb{Z}[i]),\quad \|\tau\|\in 2\mathbb{Z}.$$

Proof. The proof of this lemma follows from the Langlands decomposition form of $P\in\Gamma_\infty(3,1;\mathbb{Z}[i])$. □

We are now in a position to determine the generators of the stabilizer subgroup of q_∞.

Proposition 6. *Let $\Gamma_\infty(3,1;\mathbb{Z}[i])$ be stated as above. Then $\Gamma_\infty(3,1;\mathbb{Z}[i])$ is generated by the Heisenberg translations $N_{((1,1)^T,0)}$, $N_{((0,0)^T,2)}$ and the Heisenberg rotations M_{U_i} $(i=1,2)$.*

Proof. Our proof starts with the observation that there is no dilation component of $P\in\Gamma_\infty(3,1;\mathbb{Z}[i])$ in its Langlands decomposition. That is, P must have the form

$$P=N_{(\tau,t)}M_U=\begin{pmatrix}1&-\tau^*&(-||\tau||^2+it)/2\\0&I_2&\tau\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&U&0\\0&0&1\end{pmatrix}.$$

Since the unitary matrix U lies in $U(2;\mathbb{Z}[i])$, the rotation component of P in the Langlands decomposition is generated by M_{U_i} $(i=1,2)$ by Lemma 3.

What is left is to consider the Heisenberg translation part $N_{(\tau,t)}$ of P. Let

$$\tau=(m_1+n_1i,m_2+n_2i)^T,$$

where $m_1,n_1,m_2,n_2\in\mathbb{Z}$. Since $|\tau|^2=m_1^2+n_1^2+m_2^2+n_2^2\in 2\mathbb{Z}$, there are two cases:

(1) $m_1^2+n_1^2\in 2\mathbb{Z}$ and $m_2^2+n_2^2\in 2\mathbb{Z}$;

(2) $m_1^2+n_1^2\in 2\mathbb{Z}+1$ and $m_2^2+n_2^2\in 2\mathbb{Z}+1$.

We first consider the case (1). We can write τ as

$$\tau = \big(k_1(1+i)+l_1(1-i),\, k_2(1+i)+l_2(1-i)\big),$$

where $k_1, l_1, k_2, l_2 \in \mathbb{Z}$. $N_{(\tau,t)}$ splits as

$$N_{(\tau,t)} = N_{((0,0)^T,t)} \circ N_{(\tau,0)}.$$

Since $t = 2k \in 2\mathbb{Z}$, $N_{((0,0)^T,t)} = N^k_{((0,0)^T,2)}$. We also have

$$N_{(\tau,0)} = N^{k_1}_{((1+i,0)^T,0)} \circ N^{l_1}_{((i-1,0)^T,0)} \circ N^{2k_1 l_1}_{((0,0)^T,2)} \circ N^{k_2}_{((0,1+i)^T,0)} \circ N^{l_2}_{((0,1+i)^T,0)} \circ N^{-2k_2 l_2}_{((0,0)^T,2)}.$$

We observe that

$$\begin{aligned}
N_{((1+i,0)^T,0)} &= N_{((1,1)^T,0)} \circ N_{((i,-1)^T,0)} \circ N_{((0,0)^T,2)},\\
N_{((i-1,0)^T,0)} &= N_{((i,1)^T,0)} \circ N^{-1}_{((1,1)^T,0)} \circ N^{-1}_{((0,0)^T,2)},\\
N_{((0,1+i)^T,0)} &= N_{((1,1)^T,0)} \circ N_{((-1,i)^T,0)} \circ N_{((0,0)^T,2)},\\
N_{((0,i-1)^T,0)} &= N_{((1,i)^T,0)} \circ N^{-1}_{((1,1)^T,0)} \circ N^{-1}_{((0,0)^T,2)}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
N_{((i,1)^T,0)} &= M_{U_2} N_{((1,1)^T,0)} M^{-1}_{U_2},\\
N_{((i,-1)^T,0)} &= M_{U_1} M^2_{U_2} M_{U_1} M_{U_2} N_{((1,1)^T,0)} M^3_{U_2} (M_{U_1} M_{U_2} M_{U_1})^2,\\
N_{((-1,i)^T,0)} &= M^2_{U_2} M_{U_1} M_{U_2} M_{U_1} N_{((1,1)^T,0)} M^2_{U_2} (M_{U_1} M_{U_2} M_{U_1})^3,\\
N_{((1,i)^T,0)} &= M_{U_1} M_{U_2} M_{U_1} N_{((1,1)^T,0)} (M_{U_1} M_{U_2} M_{U_1})^3.
\end{aligned}$$

In case (2), similar considerations apply to the translation $N_{(\tau,0)} \circ N_{((1,1)^T,0)}$, where $N_{(\tau,0)}$ belongs to case (1). □

4. Fundamental domain for the stabilizer in PU(2, 1; $\mathbb{Z}[i]$)

In [Falbel et al. 2011b], the authors described a method to find the fundamental domain for the stabilizer of q_∞ in the Gauss–Picard modular group PU(2, 1; $\mathbb{Z}[i]$) in two complex dimensions. We review it now.

Let Γ be PU(2, 1; $\mathbb{Z}[i]$) and Γ_∞ be the stabilizer of q_∞. Every element of Γ_∞ is upper triangular, and its diagonal entries are units in $\mathbb{Z}[i]$. Recall that the units of $\mathbb{Z}[i]$ are $\pm 1, \pm i$. Therefore Γ_∞ contains no dilations and fits into the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_\infty \xrightarrow{\Pi_*} \Delta \longrightarrow 1,$$

where $\Delta \subset \operatorname{Isom} \mathbb{Z}[i]$ is of index 2 and Π is the vertical projection defined by $\Pi : (z,t) \in \mathscr{H} \mapsto z \in \mathbb{C}$.

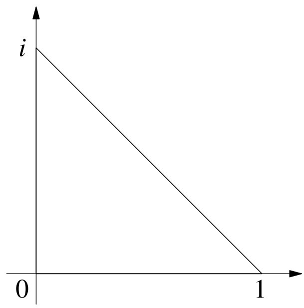

Figure 1. A fundamental domain for the index two subgroup $\Delta \subset \operatorname{Isom}\mathbb{Z}[i]$ is a triangle $\triangle$ with vertices $0, 1, i$.

As the first step toward the construction of a fundamental domain for the action of Γ_∞ on $\mathscr{H}$, one should construct a fundamental domain in $\mathbb{C}$ of $\Delta \subseteq \operatorname{Isom}\mathbb{Z}[i]$. From the generators of Δ, one finds that a fundamental domain for $\Delta \subseteq \operatorname{Isom}\mathbb{Z}[i]$ is the triangle $\triangle$ with vertices $0, 1, i$; see Figure 1.

In order to produce a fundamental domain for Γ_∞, we look at all the preimages of the triangle (that is, a fundamental domain of $\Pi_*(\Gamma_\infty)$) under the vertical projection Π and we intersect this with a fundamental domain for $\ker(\Pi_*)$. The inverse image of the triangle under Π is an infinite prism. The kernel of Π_* is the infinite cyclic group generated by T, the vertical translation by $(0, 2)$. Hence, a fundamental domain for Γ_∞ is the prism in $\mathscr{H}$ with vertices $(0, \pm 1)$, $(1, \pm 1)$, $(i, \pm 1)$.

5. Statement of the results

In this section, we recall the geometric method used in [Falbel and Parker 2006; Falbel et al. 2011b] to determine the generators of the Euclidean Picard groups, and then state our method and results.

The geometric method is based on the special feature that the Euclidean Picard modular orbifold has only one cusp for $d = 1, 2, 3, 7, 11$. The basic idea of the proof can be described easily. Analogously to Theorem 3.5 of [Falbel and Parker 2006], it can be proved that $\langle \Gamma_\infty, R\rangle$ has only one cusp. The fact that $\mathrm{PU}(2, 1; \mathbb{O}_d)$ has the same cusp and the stabilizer of infinity as the group generated by $\langle \Gamma_\infty, R\rangle$ shows that they are the same. The key step is to find a union of isometric spheres so that a fundamental domain for Γ_∞ is contained in the intersection of their exteriors and a fundamental domain for the stabilizer, which implies that the group $\langle \Gamma_\infty, R\rangle$ has only one cusp. In other words, one should show that the union of the boundaries of these isometric spheres in the Heisenberg group contains a fundamental domain for the stabilizer of infinity.

We will prove our result by using a similar idea. The main observation is that there is no need to know the exact fundamental domain for the stabilizer of infinity.

We will construct a set in the Heisenberg group which contains a fundamental set for the stabilizer of infinity as a subset. Then we show that the union of the boundaries of some isometric spheres in the Heisenberg group covers this set. This also show that the group $\langle \Gamma_\infty, R\rangle$ has only one cusp.

More precisely, let Σ be the set

$$\{(\xi_1, \xi_2, t) : \xi_i \in \triangle, -1 \leq t \leq 1\}.$$

Here $\triangle$ is the fundamental domain of $\Delta \subset \operatorname{Isom} \mathbb{Z}[i]$.

Note that Σ is not a fundamental domain for the stabilizer of infinity because this set is preserved by some Heisenberg rotations.

Proposition 7. Σ *contains a fundamental domain for the stabilizer of infinity.*

Proof. The restriction of the action of the stabilizer of infinity on each copy of $\mathbb{C}$ has the same fundamental domain $\triangle$ as $\Delta \subset \operatorname{Isom} \mathbb{Z}[i]$. Then Σ is the preimage of $\triangle \times \triangle$ under vertical projection intersected with a fundamental domain for the vertical translation by $((0, 0)^T, 2)$. It is clear that Σ is preserved by the Heisenberg rotations M_{U_1}. Hence, a fundamental domain for Γ_∞ lies inside Σ. $\square$

In next section we will prove our main theorem. Our main step is to show that Σ lies inside the boundaries of some isometric spheres in the Heisenberg group. It is obvious that the geodesic cone from q_∞ over Σ contains a fundamental domain for the Gauss–Picard modular group $U(3, 1; \mathbb{Z}[i])$.

Theorem 8. *The Picard modular group* $U(3, 1; \mathbb{Z}[i])$ *is generated by the Heisenberg translations*

$$N_{((1,1)^T,0)} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad N_{((0,0)^T,2)} = \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the Heisenberg rotations

$$M_{U_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{U_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the involution

$$R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

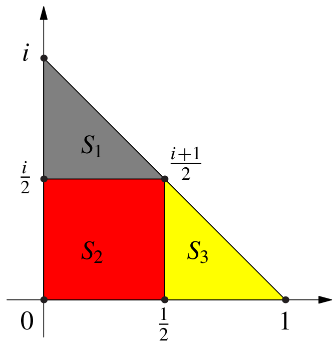

Figure 2. The decomposition of the fundamental domain $\triangle$ for $\Delta \subset \operatorname{Isom} \mathbb{Z}[i]$ into three parts.

6. Proof of Theorem 8

In this section we will prove that the generators of Picard modular groups consist of the generators of the stabilizer and the involution.

Recall that the Cygan sphere $\mathcal{B}_0$ is the isometric sphere of R. The boundary $\mathcal{S}_0$ of $\mathcal{B}_0$ is called the Heisenberg sphere in the Heisenberg group. $\mathcal{S}_0$ is defined by

$$\mathcal{S}_0 = \left\{ \left| |\xi_1|^2 + |\xi_2|^2 + ti \right| = 2 \right\}.$$

Indeed, we only need to consider the boundaries of isometric spheres in the Heisenberg group because two isometric spheres have a nonempty interior intersection if and only if the boundaries have a nonempty interior intersection.

It is not hard to see that parts of Σ lie outside $\mathcal{S}_0$. Therefore we need to find more isometric spheres whose boundaries together with $\mathcal{S}_0$ contain the set Σ.

Note that Σ has the form

$$\Sigma = \{(\xi_1, \xi_2, t) : \xi_1 \in \triangle, \xi_2 \in \triangle, -1 \le t \le 1\}.$$

First, we decompose $\triangle$ into three parts. We write $\triangle = S_1 \cup S_2 \cup S_3$, where S_1 is a triangle with vertices i, $\frac{i}{2}$, $\frac{1}{2}(1+i)$, S_2 is a square with vertices 0, $\frac{i}{2}$, $\frac{1}{2}$, $\frac{1}{2}(1+i)$, and S_3 is a triangle with vertices 0, 1, $\frac{1}{2}(1+i)$; see Figure 2.

Therefore, Σ will be decomposed into nine subsets:

- $\Sigma_1 = \{(\xi_1, \xi_2, t) : \xi_1 \in S_1, \xi_2 \in S_1, -1 \le t \le 1\}$,
- $\Sigma_2 = \{(\xi_1, \xi_2, t) : \xi_1 \in S_1, \xi_2 \in S_2, -1 \le t \le 1\}$,
- $\Sigma_3 = \{(\xi_1, \xi_2, t) : \xi_1 \in S_1, \xi_2 \in S_3, -1 \le t \le 1\}$,
- $\Sigma_4 = \{(\xi_1, \xi_2, t) : \xi_1 \in S_2, \xi_2 \in S_1, -1 \le t \le 1\}$,
- $\Sigma_5 = \{(\xi_1, \xi_2, t) : \xi_1 \in S_2, \xi_2 \in S_2, -1 \le t \le 1\}$,
- $\Sigma_6 = \{(\xi_1, \xi_2, t) : \xi_1 \in S_2, \xi_2 \in S_3, -1 \le t \le 1\}$,

- $\Sigma_7 = \{(\xi_1, \xi_2, t) : \xi_1 \in S_3, \xi_2 \in S_1, -1 \le t \le 1\}$,
- $\Sigma_8 = \{(\xi_1, \xi_2, t) : \xi_1 \in S_3, \xi_2 \in S_2, -1 \le t \le 1\}$,
- $\Sigma_9 = \{(\xi_1, \xi_2, t) : \xi_1 \in S_3, \xi_2 \in S_3, -1 \le t \le 1\}$.

We first prove that $\mathcal{S}_0$ covers the subsets Σ_2, Σ_4, Σ_5, Σ_6, Σ_8.

If $(\xi_1, \xi_2, t) \in \Sigma_5$, then

$$|\xi_1|^2 + |\xi_2|^2 \le \left(\tfrac{\sqrt{2}}{2}\right)^2 + \left(\tfrac{\sqrt{2}}{2}\right)^2 = 1,$$

so

$$\left||\xi_1|^2 + |\xi_2|^2 + it\right| \le \sqrt{1+1} = \sqrt{2} < 2.$$

Hence $\Sigma_5 \subset \mathcal{S}_0$.

If $(\xi_1, \xi_2, t) \in \Sigma_2$, then

$$|\xi_1|^2 + |\xi_2|^2 \le 1 + \tfrac{\sqrt{2}}{2} = \tfrac{3}{2},$$

so

$$\left||\xi_1|^2 + |\xi_2|^2 + it\right| \le \sqrt{\left(\tfrac{3}{2}\right)^2 + 1} = \sqrt{\tfrac{13}{4}} < 2.$$

Therefore $\Sigma_2 \subset \mathcal{S}_0$.

Similarly, we can show that Σ_4, Σ_6, Σ_8 are included in $\mathcal{S}_0$.

In order to prove this theorem, it is sufficient to prove that the remaining four subsets are covered by some Heisenberg spheres.

For the set Σ_9, we consider the map $N_{((1,1)^T,0)} R N^{-1}_{((1,1)^T,0)}$. The isometric sphere $\mathcal{B}_1$ of this map is the Cygan sphere centered at the point $((1, 1)^T, 0, 0)$ (in horospherical coordinates) with radius 1. The boundary of $\mathcal{B}_1$ is a Heisenberg sphere given by

$$\mathcal{S}_1 = \left\{\left||\xi_1 - 1|^2 + |\xi_2 - 1|^2 + i(t + 2\Im(\xi_1 + \xi_2))\right| = 2\right\}.$$

If $(\xi_1, \xi_2, t) \in \Sigma_9$, then $\xi_1 \in S_3$, $\xi_2 \in S_3$, $-1 \le t \le 1$. We get that

$$0 \le \Im\xi_i \le \tfrac{1}{2}, \quad |\xi_i - 1|^2 \le \tfrac{1}{2},$$

so

$$-1 \le t + 2\Im(\xi_1 + \xi_2) \le 3.$$

Let $T = N_{((0,0)^T,2)}$. It is easy to see that the subset

$$\{(\xi_1, \xi_2, t) : \xi_1 \in S_3, \xi_2 \in S_3, -1 \le t + 2\Im(\xi_1 + \xi_2) \le 1\}$$

lies inside $\mathcal{S}_1$ and the set

$$\{(\xi_1, \xi_2, t) : \xi_1 \in S_3, \xi_2 \in S_3, 1 \le t + 2\Im(\xi_1 + \xi_2) \le 3\}$$

lies inside

$$T^{-1}(\mathscr{S}_1)=\big\{\big||\xi_1-1|^2+|\xi_2-1|^2+i(t-2+2\Im(\xi_1+\xi_2))\big|=2\big\}.$$

Therefore, $\mathscr{S}_1$ and $T^{-1}(\mathscr{S}_1)$ cover the set Σ_9.

For the set Σ_7, we consider the map $N_{((1,i)^T,0)}RN^{-1}_{((1,i)^T,0)}$. The isometric sphere $\mathscr{B}_2$ of this map is the Cygan sphere centered at the point $((1,i)^T,0,0)$. The boundary of $\mathscr{B}_2$ is given by

$$\mathscr{S}_2=\big\{\big||\xi_1-1|^2+|\xi_2-i|^2+i(t+2\Im(\xi_1)+2\Re(\xi_2))\big|=2\big\}.$$

If $(\xi_1,\xi_2,t)\in\Sigma_7$, then $\xi_1\in S_3$, $\xi_2\in S_1$, $-1\le t\le 1$. We get that

$$0\le\Im\xi_1\le\tfrac{1}{2},\quad 0\le\Re\xi_2\le\tfrac{1}{2},\quad |\xi_1-1|^2\le\tfrac{1}{2},\quad |\xi_2-i|^2\le\tfrac{1}{2},$$

so

$$-2\le t+2\Im(\xi_1+\xi_2)\le 2.$$

If $-1\le t+2\Im(\xi_1+\xi_2)\le 1$, then the subset

$$\{(\xi_1,\xi_2,t):\xi_1\in S_3,\ \xi_2\in S_1,\ -1\le t+2\Im(\xi_1+\xi_2)\le 1\}$$

lies inside $\mathscr{S}_2$.

If $-2\le t+2\Im(\xi_1+\xi_2)\le -1$, then the subset

$$\{(\xi_1,\xi_2,t):\xi_1\in S_3,\ \xi_2\in S_1,\ -2\le t+2\Im(\xi_1+\xi_2)\le -1\}$$

lies inside $T(\mathscr{S}_2)$.

If $1\le t+2\Im(\xi_1+\xi_2)\le 2$, then the subset

$$\{(\xi_1,\xi_2,t):\xi_1\in S_3,\ \xi_2\in S_1,\ 1\le t+2\Im(\xi_1+\xi_2)\le 2\}$$

lies inside $T^{-1}(\mathscr{S}_2)$.

For the set Σ_1, we consider the map $N_{((i,i)^T,0)}RN^{-1}_{((i,i)^T,0)}$. The isometric sphere $\mathscr{B}_3$ of this map is the Cygan sphere centered at the point $((i,i)^T,0,0)$. The boundary of $\mathscr{B}_3$ is a Heisenberg sphere given by

$$\mathscr{S}_3=\big\{\big||\xi_1-1|^2+|\xi_2-1|^2+i(t+2\Im(\xi_1+\xi_2))\big|=2\big\}.$$

If $(\xi_1,\xi_2,t)\in\Sigma_1$, then $\xi_1\in S_1$, $\xi_2\in S_1$, $-1\le t\le 1$. We get that

$$0\le\Re\xi_i\le\tfrac{1}{2},\ |\xi_i-1|^2\le\tfrac{1}{2},$$

so

$$-3\le t+2\Im(\xi_1+\xi_2)\le 1.$$

As before, we can see that Σ_1 is covered by the Heisenberg spheres corresponding to the maps

$$N_{((i,i)^T,0)}RN^{-1}_{((i,i)^T,0)}\quad\text{and}\quad TN_{((i,i)^T,0)}RN^{-1}_{((i,i)^T,0)}T^{-1}.$$

It remains to consider the set Σ_3. We consider the map $N_{((i,1)^T,0)} R N^{-1}_{((i,1)^T,0)}$. The isometric sphere $\mathcal{B}_4$ of this map is the Cygan sphere centered at the point $((i,1)^T,0,0)$. The boundary of $\mathcal{B}_4$ is a Heisenberg sphere given by

$$\mathcal{S}_4 = \big\{\big||\xi_1 - i|^2 + |\xi_2 - 1|^2 + i(t - 2\Re(\xi_1) + 2\Im(\xi_2))\big| = 2\big\}.$$

If $(\xi_1, \xi_2, t) \in \Sigma_1$, then $\xi_1 \in S_1$, $\xi_2 \in S_3$, $-1 \le t \le 1$. We get that

$$0 \le \Re\xi_1 \le \tfrac{1}{2}, \quad 0 \le \Im\xi_1 \le \tfrac{1}{2}, \quad |\xi_1 - i|^2 \le \tfrac{1}{2}, \quad |\xi_2 - 1|^2 \le \tfrac{1}{2},$$

so

$$-2 \le t - 2\Re(\xi_1) + 2\Im(\xi_2) \le 2.$$

If $-1 \le t - 2\Re(\xi_1) + 2\Im(\xi_2) \le 1$, then the subset

$$\{(\xi_1, \xi_2, t) : \xi_1 \in S_3, \xi_2 \in S_1, -1 \le t - 2\Re(\xi_1) + 2\Im(\xi_2) \le 1\}$$

lies inside $\mathcal{S}_4$.

If $-2 \le t - 2\Re(\xi_1) + 2\Im(\xi_2) \le -1$, then the subset

$$\{(\xi_1, \xi_2, t) : \xi_1 \in S_3, \xi_2 \in S_1, -2 \le t - 2\Re(\xi_1) + 2\Im(\xi_2) \le -1\}$$

lies inside $T(\mathcal{S}_4)$.

If $1 \le t - 2\Re(\xi_1) + 2\Im(\xi_2) \le 2$, then the subset

$$\{(\xi_1, \xi_2, t) : \xi_1 \in S_3, \xi_2 \in S_1, 1 \le t - 2\Re(\xi_1) + 2\Im(\xi_2) \le 2\}$$

lies inside $T^{-1}(\mathcal{S}_4)$. Thus Σ_3 is covered by the Heisenberg spheres corresponding to the maps

$$N_{((i,1)^T,0)} R N^{-1}_{((i,1)^T,0)}, \quad T N_{((i,1)^T,0)} R N^{-1}_{((i,1)^T,0)} T^{-1},$$

and

$$T^{-1} N_{((i,1)^T,0)} R N^{-1}_{((i,1)^T,0)} T.$$

Remark 9. This method works for the other higher-dimensional Euclidean Picard modular groups $\mathrm{PU}(n, 1; \mathcal{O}_d)$ for $d = 2, 7, 11$ and $n \ge 3$. But the calculation will be more complicated. For example, the set Σ will be decomposed into smaller parts. Then one needs more Heisenberg spheres to cover the set Σ which contains the fundamental set.

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