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SPANNING TREES AND RANDOM WALKS ON WEIGHTED GRAPHS

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Using two graph invariants arising from Chung and Yau's discrete Green's function, we derive explicit formulas and new estimates of hitting times of random walks on weighted graphs through the enumeration of spanning trees.

1. Introduction

Every reversible Markov chain can be viewed as a random walk on a weighted undirected graph $G = (V, E)$ with edge weights w_{xy}. We may assume that G has no multiedges but may have a loop of weight w_{xx} at each vertex x. The weighted degree d_x of x is the sum of all w_{xy}, $y \in V$. The *volume* of a graph is $\mathrm{vol}(G) = \sum_{v \in V} d_v$.

The *Laplacian* of G is the matrix $L = D - A$, where D is the diagonal matrix whose entries are d_x, $x \in V$ and A is the adjacency matrix of G. *Chung's normalized Laplacian*, $\mathcal{L} = D^{-1/2} L D^{-1/2}$, is

$$\mathcal{L}(x, y) = \begin{cases} 1 - w_{xx}/d_x & \text{if } x = y, \\ -w_{xy}/\sqrt{d_x d_y} & \text{if } x \sim y, \\ 0 & \text{otherwise,} \end{cases}$$

where $x \sim y$ denotes that x and y are adjacent. Here we assume that $d_x \neq 0$ for all $x \in V$, since for a random walk it is natural to impose that G is connected and $w_{xy} > 0$ for all $xy \in E$.

If G is connected, denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$ the eigenvalues of $\mathcal{L}$ with the corresponding orthonormal basis of eigenvectors $\phi_0, \phi_1, \ldots, \phi_{n-1}$, as $n \times 1$ column vectors. Obviously $\phi_0(x) = \sqrt{d_x / \mathrm{vol}(G)}$.

Chung and Yau [2000] defined the *discrete Green's function* $\mathcal{G}$ by

$$\mathcal{G} = \sum_{j=1}^{n-1} \frac{1}{\lambda_j} \phi_j \phi_j^*,$$

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which is uniquely determined by the relations $\mathscr{G}\mathscr{L} = \mathscr{L}\mathscr{G} = I - P_0$ and $\mathscr{G}P_0 = 0$, where $P_0 = \phi_0\phi_0^t$ is an $n \times n$ matrix.

A random walk on G is a Markov chain on V with transition probability matrix $(p_{xy})_{x,y\in V}$, where

$$p_{xy} = \begin{cases} w_{xy}/d_x & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

The *hitting time* $H(x, y)$ is the expected number of steps to reach vertex y when started from vertex x. By using a result in [Aldous and Fill 2014], Chung and Yau proved an expression for $H(x, y)$ in terms of the discrete Green's function:

Theorem 1.1 [Chung and Yau 2000]. *On a connected graph G, the hitting time $H(x, y)$ and Green's function $\mathscr{G}(x, y)$ satisfy*

$$H(x, y) = \mathrm{vol}(G)\left(\frac{\mathscr{G}(y, y)}{d_y} - \frac{\mathscr{G}(x, y)}{\sqrt{d_x d_y}}\right). \tag{1}$$

For a weighted graph G, we denote by $\Omega(G)$ the set of spanning trees of G. For $T \in \Omega(G)$, define the weight $w(T)$ of T to be $\prod_{e\in T} w_e$. Let $\tau(G)$ be the weighted counting of spanning trees:

$$\tau(G) = \sum_{T\in\Omega(G)} w(T). \tag{2}$$

Below is a typical identity expressing hitting times in terms of spanning trees (see Theorem 2.11) arising from our study of Chung and Yau's discrete Green's function.

Theorem 1.2. *Let G be a connected weighted graph and $x, y \in V(G)$. Then*

$$H(x, y) = \frac{1}{\tau(G)} \sum_{u\in V(G)} d_u \sum_{\substack{P\in\mathscr{P}_G(x,u) \\ y\notin P}} \prod_{e\in E(P)} w_e \tau(G/\{P, y\}). \tag{3}$$

The paper is organized as follows. In Section 2, we introduce two graph invariants and use them to derive explicit formulas for Chung and Yau's discrete Green's functions and hitting times of random walks on weighted graphs. In Section 3, we apply our formulas to obtain various estimates of hitting times on weighted graphs. In Section 4, we prove an explicit formula for hitting times of random walks on infinite trees. In Section 5, we apply our work to improve estimates of hitting times under different weight schemes on a given simple finite graph.

2. The hitting time of random walks on weighted graphs

Kirchhoff discovered his matrix-tree theorem in 1847 in his work on electrical networks, and this theorem gives an efficient way to calculate $\tau(G)$ using linear algebra.

Theorem 2.1 (Kirchhoff's matrix-tree theorem). *Let G be a connected weighted graph.*

(i) *If the Laplacian L of G has eigenvalues $0 = \mu_0 < \mu_1 \le \cdots \le \mu_{n-1}$, then*

$$\prod_{k=1}^{n-1} \mu_k = n\tau(G). \tag{4}$$

(ii) *Let L_{ij} be the matrix obtained from L by deleting the i-th row and j-th column. Then all cofactors $(-1)^{i+j}\det(L_{ij})$ of L are equal and*

$$\det L_{ii} = n\tau(G) \quad \textit{for all } 1 \le i \le n. \tag{5}$$

We also need the following version of Kirchhoff's matrix-tree theorem for weighted graphs. A proof, with slight changes, can be found in the cited reference.

Theorem 2.2 [Chung 2011, Theorem 1]. *For a connected weighted graph $G = (V, E)$, we have*

$$\prod_{k=1}^{n-1} \lambda_k = \frac{\mathrm{vol}(G)\tau(G)}{\prod_{v\in V} d_v}, \tag{6}$$

where $0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_{n-1}$ are eigenvalues of $\mathcal{L}$ and $\tau(G)$ is defined in (2).

A weighted graph is a graph G equipped with a function $w : E(G) \to \mathbb{R}_+$ that assigns a positive number to each edge. In the following, we fix a weighted graph (G, w) and introduce two invariants for any induced subgraph S of G. Consider the matrix

$$B(x, y) = \begin{cases} d_x^2 s + d_x - w_{xx} & \text{if } x = y, \\ d_x d_y s - w_{xy} & \text{if } x \sim y, \\ d_x d_y s & \text{otherwise,} \end{cases} \tag{7}$$

where $d_x = w_{xx} + \sum_{y\sim x} w_{xy}$. Denote by B_S the principle submatrix of B on indices corresponding to the vertices of S; we define $R(S)$ and $Z(S)$ by

$$\det B_S = R(S) + Z(S)\cdot s.$$

In particular, $R(\varnothing) = 1$, $Z(\varnothing) = 0$ for the empty subgraph $\varnothing$, and $R(\{x\}) = d_x - w_{xx}$, $Z(\{x\}) = d_x^2$ for any $x \in V(G)$.

In the following four lemmas, S is an arbitrary induced subgraph of a fixed weighted graph (G, w). The proofs of these lemmas are similar to those in [Xu and Yau 2013a, Section 2], where we considered unweighted simple graphs with $w_e \equiv 1$ for all $e \in E(G)$.

Lemma 2.3. *If S has k connected components $S_1, \dots, S_k$, then*

$$R(S) = \prod_{i=1}^{k} R(S_i), \quad Z(S) = \sum_{i=1}^{k} Z(S_i) \prod_{\substack{j=1 \\ j \neq i}}^{k} R(S_j).$$

Lemma 2.4. *For any fixed vertex $x \in V(S)$, we have*

$$R(S) = (d_x - w_{xx}) R(S - \{x\}) - \sum_{\substack{y \in V(S) \\ y \sim x}} w_{xy} \sum_{P \in \mathcal{P}_S(x,y)} \prod_{e \in E(P)} w_e R(S - \{P\}),$$

and

$$\begin{aligned} Z(S) = (d_x - w_{xx}) Z(S - \{x\}) \\ - \sum_{\substack{y \in V(S) \\ y \sim x}} w_{xy} \sum_{P \in \mathcal{P}_S(x,y)} \prod_{e \in E(P)} w_e Z(S - \{P\}) + (d_x - w_{xx})^2 R(S - \{x\}) \\ + \sum_{\substack{u, v \in V(S) \\ u \neq v}} d_u d_v \sum_{\substack{P_1 \in \mathcal{P}_S(x,u) \\ P_2 \in \mathcal{P}_S(x,v) \\ P_1 \cap P_2 = x}} \prod_{e \in E(P_1 \cup P_2)} w_e R(S - \{P_1, P_2\}), \end{aligned}$$

where $\mathcal{P}_S(x, y)$ is the set of all simple paths (with no repeated vertices) connecting x and y in S. We assume that $\mathcal{P}_S(x, x)$ consists of the trivial path $\{x\}$ only. Here $S - \{P\}$ means the graph obtained by removing P together with incident edges.

Lemma 2.5. *We have*

$$Z(S) = \sum_{x, y \in V(S)} d_x d_y \sum_{P \in \mathcal{P}_S(x,y)} \prod_{e \in E(P)} w_e R(S - \{P\}). \tag{8}$$

Lemma 2.6. *Regarding G as a subgraph of itself, we have $R(G) = 0$ and $Z(G) =$* $\mathrm{vol}(G)^2 \tau(G)$. *For any $x, y \in V(G)$, we have*

$$R(G - \{x\}) = \sum_{P \in \mathcal{P}_G(x,y)} \prod_{e \in E(P)} w_e R(G - \{P\}) = \tau(G). \tag{9}$$

Now we come to an explicit formula for the Green's function expressed in terms of the above two invariants.

Theorem 2.7. *For a connected graph G and $x, y \in V(G)$, the value of the Green's function $\mathcal{G}(x, y)$ is equal to*

$$\frac{\sqrt{d_x d_y}}{\operatorname{vol}(G)^2 \tau(G)} \Bigg(\sum_{P \in \mathcal{P}_G(x,y)} \prod_{e \in E(P)} w_e \big(R(G - \{P\}) + Z(G - \{P\})\big) - \sum_{\substack{u,v \in V(G)\\ u \neq v}} d_u d_v \sum_{\substack{P_1 \in \mathcal{P}_G(x,u)\\ P_2 \in \mathcal{P}_G(y,v)\\ P_1 \cap P_2 = \varnothing}} \prod_{e \in E(P_1 \cup P_2)} w_e R(G - \{P_1, P_2\}) \Bigg) - \frac{\sqrt{d_x d_y}}{\operatorname{vol}(G)^2}.$$

In particular, *when $x = y$*,

$$\mathcal{G}(y, y) = \frac{d_y}{\operatorname{vol}(G)^2 \tau(G)} \big(R(G - \{y\}) + Z(G - \{y\})\big) - \frac{d_y}{\operatorname{vol}(G)^2}.$$

Proof. The proof is almost the same as that of [Xu and Yau 2013a, Theorem 2.9]. □

Theorem 2.8. *Given a connected graph G and $x, y \in V(G)$, the expected hitting time $H(x, y)$ satisfies*

$$H(x, y) = \frac{1}{\operatorname{vol}(G)\tau(G)} \Bigg(Z(G - \{y\}) - \sum_{P \in \mathcal{P}_G(x,y)} \prod_{e \in E(P)} w_e Z(G - \{P\}) + \sum_{\substack{u,v \in V(G)\\ u \neq v}} d_u d_v \sum_{\substack{P_1 \in \mathcal{P}_G(x,u)\\ P_2 \in \mathcal{P}_G(y,v)\\ P_1 \cap P_2 = \varnothing}} \prod_{e \in E(P_1 \cup P_2)} w_e R(G - \{P_1, P_2\}) \Bigg). \tag{10}$$

Proof. This follows from Theorems 1.1 and 2.7 and Lemma 2.6. □

Corollary 2.9. *On a connected weighted graph G, $H(x, y) = H(y, x)$ for any $x, y \in V(G)$ if and only if $Z(G - \{x\})$ is independent of $x \in V(G)$.*

Proof. By (10), we have

$$H(x, y) - H(y, x) = \frac{1}{\operatorname{vol}(G)\tau(G)} \big(Z(G - \{y\}) - Z(G - \{x\})\big),$$

which implies the corollary. □

Corollary 2.10. *On a connected weighted graph G, $H(x, y) = H(y, x)$ for any $x, y \in V(G)$ if and only if* $\det B_{G-\{x\}}|_{s=1}$ *is independent of $x \in V(G)$.*

Proof. From $\det B_{G-\{x\}}|_{s=1} = R(G - \{x\}) + Z(G - \{x\})$, the conclusion follows from Lemma 2.6 and the previous corollary. □

Theorem 2.11. *Let G be a connected weighted graph and $x, y \in V(G)$. Then*

$$H(x, y) = \frac{1}{\tau(G)} \sum_{u \in V(G)} d_u \sum_{\substack{P \in \mathcal{P}_G(x,u) \\ y \notin P}} \prod_{e \in E(P)} w_e R(G - \{P, y\}). \tag{11}$$

In fact, $R(G - \{P, y\}) = \tau(G/\{P, y\})$, where $G/\{P, y\}$ is obtained from G by contracting $\{P, y\}$ to a point.

Proof. The proof is almost identical to that of [Xu and Yau 2013b, Theorem 2.7]. $R(G - \{P, y\}) = \tau(G/\{P, y\})$ follows from Theorem 2.1(ii). □

Corollary 2.12. *For any connected weighted graph G, we have*

$$Z(G - \{x\}) = \tau(G) \sum_{y \in V(G)} d_y H(y, x). \tag{12}$$

Proof. By (8) and (11), we have

$$\begin{aligned} Z(G - \{x\}, d_G) &= \sum_{u, y \in V(G - \{x\})} d_u d_y \sum_{P \in \mathcal{P}_{G-\{x\}}(u,y)} \prod_{e \in E(P)} w_e R(G - \{P, x\}) \\ &= \sum_{y, u \in V(G)} d_y d_u \sum_{\substack{P \in \mathcal{P}_G(y,u) \\ x \notin P}} \prod_{e \in E(P)} w_e R(G - \{P, x\}) \\ &= \tau(G) \sum_{y \in V(G)} d_y H(y, x). \end{aligned}$$

□

3. Identities and estimates of hitting times

It is natural to regard a weighted graph as an electrical network, where an edge xy has conductance w_{xy} and hence resistance $1/w_{xy}$. Chandra et al. [1989] proved that the commute time $\kappa(x, y) := H(x, y) + H(y, x)$ can be expressed in terms of the effective resistance R_{xy} between x, y,

$$\kappa(x, y) = \operatorname{vol}(G) R_{xy}. \tag{13}$$

The effective resistance R_{xy} can be expressed in terms of spanning trees (cf. Theorem 3.2):

$$R_{xy} = \frac{\tau(G/\{x, y\})}{\tau(G)}. \tag{14}$$

Tetali's formula [1991] expresses $H(x, y)$ in terms of effective resistances:

$$H(x, y) = \frac{1}{2} \sum_{z \in V(G)} d_z (R_{xy} + R_{yz} - R_{xz}). \tag{15}$$

We have the following well-known upper bound of R_{xy}.

Theorem 3.1. *Given a connected graph G and $x, y \in V(G)$, we have $R_{xy} \le d(x, y)$, where the distance $d(x, y)$ between x, y is defined by*

$$d(x, y) = \min\left\{ \sum_{e \in E(P)} \frac{1}{w_e} \;\middle|\; P \in \mathcal{P}_G(x, y) \right\}.$$

As remarked in [Lovász 1996, Corollary 4.2], the following formula could be proved by the method of electric networks. Here we give a proof by using (11).

Theorem 3.2. *Let $x, y \in V(G)$ be two distinct vertices of a connected weighted graph G. Then we have*

$$H(x, y) + H(y, x) = \mathrm{vol}(G)\frac{\tau(G/\{x, y\})}{\tau(G)}, \tag{16}$$

where $G/\{x, y\}$ is obtained from G by contracting $\{x, y\}$ to a point.

Proof. Define a graph G' by

$$G' = \begin{cases} G & \text{if } x \sim y, \\ G \cup \{xy\} & \text{otherwise.} \end{cases}$$

Namely, we modify G by adding an edge xy if x, y are not adjacent.

If $u \neq x, y$, define

$$\Omega_1 = \{T \in \Omega(G') \mid T \text{ contains } xy \text{ and a path from } u \text{ to } x \text{ not containing } y\},$$
$$\Omega_2 = \{T \in \Omega(G') \mid T \text{ contains } xy \text{ and a path from } u \text{ to } y \text{ not containing } x\},$$
$$\Omega_3 = \{T \in \Omega(G') \mid T \text{ contains } xy\}.$$

It is not difficult to see that $\Omega_1 \cup \Omega_2 = \Omega_3 = \Omega(G/\{x, y\})$. (More precisely, Ω_3 is in one-to-one correspondence with $\Omega(G/\{x, y\})$.) Then, by (11),

$$\begin{aligned} H(x, y) + H(y, x) &= \frac{1}{\tau(G)} \sum_{u \in V(G)} d_u \Bigg(\sum_{\substack{P \in \mathcal{P}_G(x,u) \\ y \notin P}} \prod_{e \in E(P)} w_e \tau(G/\{P, y\}) \\ &\qquad\qquad + \sum_{\substack{P \in \mathcal{P}_G(y,u) \\ x \notin P}} \prod_{e \in E(P)} w_e \tau(G/\{P, x\}) \Bigg) \\ &= \frac{1}{\tau(G)} \sum_{u \in V(G)} d_u \Bigg(\sum_{T \in \Omega_1} \prod_{e \in T} w_e + \sum_{T \in \Omega_2} \prod_{e \in T} w_e \Bigg) \\ &= \frac{1}{\tau(G)} \sum_{u \in V(G)} d_u \tau(G/\{x, y\}). \end{aligned}$$

The term in parenthesis on the third line is equal to $\tau(G/\{x, y\})$, independently of $u \in V(G)$. $\square$

In the rest of this section, we apply Theorem 2.11 to prove some estimates for the hitting time on weighted graphs. It is interesting to see the role of edge weights in these estimates.

Corollary 3.3. *Let G be a connected weighted graph with n vertices and $x, y \in V(G)$. Then*

$$H(x, y) \le (n-1)^2 \frac{d_{\max}}{w_{\min}}, \tag{17}$$

where $d_{\max} = \max\{d_v \mid v \in V(G)\}$ *and* $w_{\min} = \min\{w_e \mid e \in E(G)\}$.

Proof. Fix $x, y, u \in V(G)$ with $y \ne u$. Given a spanning tree $T \in \Omega(G)$ and an edge $e \in E(T)$, denote by $T(e)$ the subgraph of G' obtained from T by removing e and adding an edge uy if $uy \notin E(T)$, namely,

$$T(e) = \begin{cases} T & \text{if } uy \in T, \\ T \cup \{uy\} - \{e\} & \text{if } uy \notin T. \end{cases}$$

Define a subset S of $\Omega(G) \times E(G)$ by

$$S = \{(T, e) \mid T \in \Omega(G), e \in E(T), T(e) \in \Omega(G')\}$$

and let $S' = \{T \in \Omega(G') \mid T \text{ contains } uy\}$. Then the map $(T, e) \to T(e)$ is a surjective map from S to S'. Since

$$\bigcup_{\substack{P \in \mathcal{P}_G(x,u) \\ y \notin P}} \Omega(G/\{P, y\}) = \{T \in \Omega(G') \mid T \text{ contains } uy \text{ and a path from } u \text{ to } x \text{ not passing through } y\}$$

is a subset of S' and the left-hand side is a disjoint union over $P \in \mathcal{P}_G(x, u)$, $y \notin P$, we have

$$\sum_{\substack{P \in \mathcal{P}_G(x,u) \\ y \notin P}} \prod_{e \in E(P)} w_e \tau(G/\{P, y\}) \le (n-1)\tau(G) \frac{1}{w_{\min}}.$$

Then (17) follows from (11). □

Corollary 3.4. *Let G be a connected weighted graph and $xy \in E(G)$. Then*

$$H(x, y) \le \frac{\operatorname{vol}(G) - d_y}{w_{xy}}.$$

Proof. Fix $x, y, u \in V(G)$ with $y \ne u$. It is not difficult to see that

$$\bigcup_{\substack{P \in \mathcal{P}_G(x,u) \\ y \notin P}} \Omega(G/\{P, y\}) = \{T \in \Omega(G) \mid T \text{ contains } xy \text{ and a path from } u \text{ to } x \text{ not passing through } y\}$$

is a subset of $\Omega(G)$ and the left-hand side is a disjoint union over $P \in \mathcal{P}_G(x,u)$, $y \notin P$. Thus (11) implies that

$$H(x,y) \le \frac{1}{\tau(G)} \sum_{\substack{u\in V(G)\\ u\neq y}} d_u \frac{\tau(G)}{w_{xy}} = \frac{\mathrm{vol}(G)-d_y}{w_{xy}},$$

as claimed. □

Corollary 3.5. *Let G be a connected weighted graph. Then for any two distinct vertices $x, y \in V(G)$, we have*

$$H(x,y) \le \max\left\{\frac{d_u}{w_{yu}} \;\middle|\; u \in \mathcal{S}\right\}, \tag{18}$$

where $\mathcal{S} = \{u \in V(G) \mid$ there is a path from x to u not passing through $y\}$.

Proof. Fix two distinct vertices $x, y \in V(G)$. We may assume that $w_{yu} > 0$ for all $u \in \mathcal{S}$, i.e., there is an edge connecting y and u. Otherwise the right-hand side of (18) is infinite. Define $\Omega_{xy} = \{T \in \Omega(G) \mid xy \in T\}$ and

$$V_T = \{u \in V(G) \mid T \text{ contains a path from } x \text{ to } u \text{ not passing through } y\}.$$

Let $S = \{(T,u) \mid T \in \Omega_{xy}, u \in V_T\}$. Define a map $f : S \to \Omega(G)$ by

$$f(T,u) = \begin{cases} T & \text{if } u = x,\\ \{T - xy\} \cup \{uy\} & \text{if } u \neq x,\end{cases}$$

where we used the fact that $d_y = n - 1$. It is not difficult to see that f is injective. Thus, we have

$$\begin{aligned}\sum_{u\in V(G)} d_u \sum_{\substack{P\in\mathcal{P}_G(x,u)\\ y\notin P}} \prod_{e\in E(P)} w_e \tau(G/\{P,y\}) &= \sum_{u\in V(G)} \frac{d_u}{w_{yu}} \sum_{(T,u)\in S} w(f(T,u))\\ &\le \max\left\{\frac{d_u}{w_{yu}} \;\middle|\; u\in\mathcal{S}\right\}\tau(G).\end{aligned}$$

Therefore (11) implies (19). □

There is a direct probabilistic proof of Corollary 3.5 (see [Xu and Yau 2013b, Remark 2.14]). In fact, all the above three corollaries may be obtained from the following more refined estimates (see also [Chang and Xu ≥ 2015]):

Theorem 3.6. *Let G be a connected weighted graph. Then*

$$H(x,y) \le \max\{d_u \mid u \in \Gamma(y), u \neq y\} + \sum_{\substack{u\in V(G), u\neq y\\ u\notin\Gamma(y)}} d_u \min\{d(x,y), d(u,y)\}, \tag{19}$$

where $\Gamma(y)$ is the set of vertices adjacent to y and $d(x,y)$ is defined in Theorem 3.1.

Proof. We split the right-hand side of (11) into two terms:

$$H(x,y)=\frac{1}{\tau(G)}\sum_{u\in\Gamma(y)} d_u \sum_{\substack{P\in\mathcal{P}_G(x,u)\\ y\notin P}}\prod_{e\in E(P)} w_e\tau(G/\{P,y\}) + \frac{1}{\tau(G)}\sum_{\substack{u\in V(G)\\ u\notin\Gamma(y)}} d_u \sum_{\substack{P\in\mathcal{P}_G(x,u)\\ y\notin P}}\prod_{e\in E(P)} w_e\tau(G/\{P,y\}). \tag{20}$$

If $u\in\Gamma(y)$, then define

$\Omega_{uy}=\{T\in\Omega(G)\mid T$ contains uy and a path from x to u not passing through $y\}$.

Since $\bigcup_{u\in\Gamma(y)}\Omega_{uy}$ is a disjoint union in $\Omega(G)$ and

$$\sum_{\substack{P\in\mathcal{P}_G(x,u)\\ y\notin P}}\prod_{e\in E(P)} w_e\tau(G/\{P,y\})=\sum_{T\in\Omega_{uy}} w(T)\quad\text{for all } u\in\Gamma(G),$$

the first summand in the right-hand side of (20) satisfies

$$\frac{1}{\tau(G)}\sum_{u\in\Gamma(y)} d_u \sum_{\substack{P\in\mathcal{P}_G(x,u)\\ y\notin P}}\prod_{e\in E(P)} w_e\tau(G/\{P,y\})\le\max\{d_u\mid u\in\Gamma(y),\,u\neq y\}. \tag{21}$$

By using (14) and the inequality

$$\sum_{\substack{P\in\mathcal{P}_G(x,u)\\ y\notin P}}\prod_{e\in E(P)} w_e\tau(G/\{P,y\})\le\min\{\tau(G/\{x,y\}),\tau(G/\{u,y\})\}, \tag{22}$$

the second summand in the right-hand of (20) satisfies

$$\frac{1}{\tau(G)}\sum_{\substack{u\in V(G)\\ u\notin\Gamma(y)}} d_u \sum_{\substack{P\in\mathcal{P}_G(x,u)\\ y\notin P}}\prod_{e\in E(P)} w_e\tau(G/\{P,y\}) \le \sum_{\substack{u\in V(G)\\ u\notin\Gamma(y)}} d_u\min\{R_{xy},R_{uy}\}\le\sum_{\substack{u\in V(G)\\ u\notin\Gamma(y)}} d_u\min\{d(x,y),d(u,y)\}. \tag{23}$$

The last inequality follows from Theorem 3.1. So (19) follows from (21) and (23). □

Corollary 3.7. *Let G be a connected weighted graph. Then*

$$H(x,y)\le\sum_{\substack{u\in V(G)\\ u\neq y}} d_u\min\{R_{xy},R_{uy}\}\le\sum_{\substack{u\in V(G)\\ u\neq y}} d_u\min\{d(x,y),d(u,y)\}. \tag{24}$$

Proof. The first inequality follows from (14), (22) and Theorem 2.11. The second inequality follows from Theorem 3.1. □

4. Some examples

First we consider infinite but locally finite connected graphs. Since an infinite (locally finite) graph can be considered as a limit of a sequence of finite graphs, the hitting time formula (11) is still valid as long as the limit exists. A weighted tree T is a locally finite tree (possibly with loops) whose edges are assigned positive weights.

Theorem 4.1. *Let x, y be two distinct vertices of a weighted tree T, and denote by P_{xy} the path $[x = v_0, v_1, \ldots, v_{k-1}, v_k = y]$ connecting x to y. For any $v_i \in V(P_{xy})$, we denote by T_i the component of $T - E(P_{xy})$ that contains v_i, and denote by $w_{i-1,i}$ the weight of the edge $v_{i-1}v_i$. Then the hitting time $H(x, y)$ is given by*

$$H(x, y) = \sum_{j=0}^{k-1} \Big(\sum_{u \in T_j} d_u \Big) \Big(\sum_{i=j+1}^{k} \frac{1}{w_{i-1,i}} \Big). \tag{25}$$

Proof. First we define induced subtrees $T(N)$ of T for $N \in \mathbb{N}$. The vertices of $T(N)$ are those vertices whose distances from x are within N. We may apply (11) to get $H(x, y)$ on $T(N)$, which increases as N increases. Then (25) follows easily. We omit the details. □

Corollary 4.2. *Let x, y be two distinct vertices of a weighted tree T. Then $H(x, y) < \infty$ if and only if*

$$\sum_{u \in \mathcal{S}} d_u < \infty,$$

where $\mathcal{S} = \{u \in V(T) \mid \text{there is a path from } x \text{ to } u \text{ not passing through } y\}$.

Corollary 4.3. *On the weighted one-dimensional lattice $\mathbb{Z}$,*

$$H(j, j+1) = \frac{\sum_{i \le j} d_i}{w_{j,j+1}}. \tag{26}$$

Both corollaries follow easily from (25). For unweighted trees, formula (25) was obtained in [Haiyan and Fuji 2004] (see also [Moon 1973]). Formula (26) can be found in [Palacios and Tetali 1996], where it was used to study hitting times for birth and death chains.

Now let G be a locally finite connected weighted graph and $xy \in E(G)$. Then the inequality of Corollary 3.4 still holds: $H(x, y) \le (\mathrm{vol}(G) - d_y)/w_{xy}$. Next we show that the equality essentially holds when xy is a cut edge of G.

Let $\mathcal{S} = \{u \in V(G) \mid \text{there is a path from } x \text{ to } u \text{ not passing through } y\}$. Let G' be the subgraph obtained by removing all vertices in $V(G)/\{\mathcal{S} \cup y\}$ from G. If $xy \in E(G)$ is a cut edge of G, note that $H(x, y)$ is the same for random walks on either G or G'. Moreover, for each spanning tree T of G' and $u \in \mathcal{S}$, there exists a path from x to u. Therefore,

$$H(x, y) = \frac{1}{\tau(G')} \sum_{u \in \mathcal{S}} d_u \frac{1}{w_{xy}} \tau(G') = \frac{1}{w_{xy}} \sum_{u \in \mathcal{S}} d_u = \frac{\mathrm{vol}(G') - d'_y}{w_{xy}},$$

where d'_y is the degree of y in G'.

Following [Georgakopoulos 2012], we call a weighted graph G *reversible* if $H(x, y) = H(y, x)$ holds for any $x, y \in V(G)$. For simplicity, we assume that G has no loops, i.e., $w_{xx} = 0$ for all $x \in V(G)$, and all edge weights of G are positive. It is interesting to study restrictions on edge weights for a reversible graph G.

Conjecture 4.4. *Let G be a weighted cycle on n vertices. Assume all edge weights of G are positive. Denote $w_{n,n+1} = w_{n,1}$.*

(i) *If n is odd, then G is reversible if and only if there exists some $a > 0$ such that $w_{i,i+1} = a$ for all $1 \leq i \leq n$.*

(ii) *If n is even, then G is reversible if and only if there exist $a, b > 0$ such that $w_{1,2} = w_{3,4} = \cdots = w_{n-1,n} = a$ and $w_{2,3} = w_{4,5} = \cdots = w_{n,1} = b$.*

The sufficiency in (ii) follows from Corollary 2.10.

5. Weight schemes on graphs

Given a simple, connected, undirected graph G with n vertices, we obtain a weighted graph by assigning a positive number w_e to each edge $e \in E(G)$. The hitting and cover times of a simple random walk on G (i.e., $w_e = 1$, for all $e \in E(G)$) have order $O(n^3)$. The work of [Ikeda et al. 2009; Abdullah 2011] showed that if a token knows not only the degree of the current vertex that it is on, but also the degrees of neighboring vertices, we can guarantee $O(n^2)$ hitting times.

In this section, we will denote by $d(u)$ the number of edges adjacent to a vertex u in G and assume that G has no loops.

Lemma 5.1. *Let G be connected graph with n vertices and $u_0 = x, u_1, \ldots, u_l = y$ a shortest path (achieving minimum l) connecting any two distinct vertices x and y. Then $\sum_{i=0}^{l} d(u_i) \leq 3n - 4$. More precisely,*

$$\sum_{i=0}^{l} d(u_i) \leq \begin{cases} 2n-2 & \text{if } l = 1, \\ 3n - l - 3 & \text{if } l \geq 2. \end{cases}$$

Proof. The proof is due to [Ikeda et al. 2009, Theorem 2]. Each vertex of $V(G)$ not lying on the path can be connected to at most 3 vertices of the path, due to its minimality, which also implies that u_i, u_j are adjacent if and only if $|i - j| = 1$. The asserted inequalities follow easily. □

Next we will apply Theorem 3.6 to estimate hitting times under three different weight schemes: $w_{uv} = 1/\sqrt{d(u)d(v)}$, $1/\min\{d(u), d(v)\}$ or $1/\max\{d(u), d(v)\}$. The leading terms of the bounds in Theorems 5.2 and 5.3 were obtained in [Ikeda et al. 2009, Theorem 2] and [Abdullah 2011, Theorem 68] respectively.

Theorem 5.2. *Let G be a graph with assigned weights $w_{uv} = 1/\sqrt{d(u)d(v)}$ for each edge uv. Then the hitting time satisfies $H(x, y) \le 3n^2 - 9n + \frac{15}{2}$.*

Proof. For the two terms in the right-hand side of (19), we have the estimates

$$d_u = \sum_{v\in\Gamma(u)} \frac{1}{\sqrt{d(u)d(v)}} \le \frac{1}{2}\sum_{v\in\Gamma(u)}\left(\frac{1}{d(u)} + \frac{1}{d(v)}\right) \le \frac{1}{2} + \frac{d(u)}{2} \le \frac{n}{2}. \tag{27}$$

and, by using $\sum_{u\in V(G)}\sum_{v\in\Gamma(u)} \frac{1}{2}(1/d(u) + 1/d(v)) = n$,

$$\sum_{\substack{u\in V(G),u\neq y\\ u\notin\Gamma(y)}} d_u \le \sum_{\substack{u\in V(G),u\neq y\\ u\notin\Gamma(y)}}\sum_{v\in\Gamma(u)} \frac{1}{2}\left(\frac{1}{d(u)} + \frac{1}{d(v)}\right) \le n-1-\frac{d(y)}{2} \le n-\frac{3}{2}. \tag{28}$$

Let $u_0 = x, u_1, \ldots, u_l = y$ be a shortest path (achieving minimum l) connecting x and y. Then

$$d(x, y) \le \sum_{i=0}^{l-1} \sqrt{d(u_i)d(u_{i+1})} \le \sum_{i=0}^{l-1} \frac{d(u_i) + d(u_{i+1})}{2} \le 3n - 5. \tag{29}$$

The last inequality follows from Lemma 5.1. By (19), we have

$$H(x, y) \le (3n - 5)(n - \tfrac{3}{2}) + \tfrac{1}{2}n = 3n^2 - 9n + \tfrac{15}{2},$$

as claimed. □

Theorem 5.3. *Let G be a graph with assigned weights $w_{uv} = 1/\min\{d(u), d(v)\}$ for each edge uv. Then the hitting time satisfies $H(x, y) \le 6n^2 - 18n + 14$.*

Proof. For the two terms in the right-hand side of (19), we have

$$d_u = \sum_{v\in\Gamma(u)} \frac{1}{\min\{d(u), d(v)\}} \le d(u) \le n - 1 \tag{30}$$

and, similarly to (28),

$$\sum_{\substack{u\in V(G),u\neq y\\ u\notin\Gamma(y)}} d_u \le \sum_{\substack{u\in V(G),u\neq y\\ u\notin\Gamma(y)}}\sum_{v\in\Gamma(u)}\left(\frac{1}{d(u)} + \frac{1}{d(v)}\right) \le 2n - 3. \tag{31}$$

Similarly to (29), we have

$$d(x, y) \le \sum_{i=0}^{l-1} \min\{d(u_i), d(u_{i+1})\} \le \sum_{i=0}^{l-1} d(u_i) \le 3n - 5.$$

The desired upper bound of $H(x, y)$ follows from (19). □

Theorem 5.4. *Let G be a graph with assigned weights $w_{uv} = 1/\max\{d(u), d(v)\}$ for each edge uv. Then the hitting time satisfies $H(x, y) \le 6n^2 - 23n + 23$.*

Proof. For the two terms in the right-hand side of (19), we have

$$d_u = \sum_{v \in \Gamma(u)} \frac{1}{\max\{d(u), d(v)\}} \le 1 \quad \text{and} \quad \sum_{\substack{u \in V(G), u \ne y \\ u \notin \Gamma(y)}} d_u \le n - 2.$$

Similar to (29), we have

$$d(x, y) \le \sum_{i=0}^{l-1} \max\{d(u_i), d(u_{i+1})\} \le \sum_{i=0}^{l-1} (d(u_i) + d(u_{i+1}) - 1) \le 6n - 11.$$

The desired upper bound of $H(x, y)$ follows from (19). □

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