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**-1-PHENOMENA FOR THE PLURI χ_y -GENUS
AND ELLIPTIC GENUS**

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Several independent articles have observed that the Hirzebruch χ_y -genus has an important feature, which we call –1-phenomenon and which tells us that the coefficients of the Taylor expansion of the χ_y -genus at $y = -1$ have explicit expressions. Hirzebruch’s original χ_y -genus can be extended towards two directions: the pluri-case and the case of elliptic genus. This paper contains two parts, in which we investigate the –1-phenomena in these two generalized cases and show that in each case there exists a –1-phenomenon in a suitable sense. Our main results in the first part have an application, which states that all characteristic numbers (Chern numbers and Pontrjagin numbers) on manifolds can be expressed, in a very explicit way, in terms of some rational linear combination of indices of some elliptic operators. This gives an analytic interpretation of characteristic numbers and affirmatively answers a question posed by the author several years ago. The second part contains our attempt to generalize this –1-phenomenon to the elliptic genus, a modern version of the χ_y -genus. We first extend the elliptic genus of an almost-complex manifold to a twisted version where an extra complex vector bundle is involved, and show that it is a weak Jacobi form under some assumptions. A suitable manipulation on the theory of Jacobi forms will produce new modular forms from this weak Jacobi form, and thus much arithmetic information related to the underlying manifold can be obtained, in which the –1-phenomenon of the original χ_y -genus is hidden.

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1. Introduction

The Hirzebruch χ_y -genus and its -1 -phenomenon. In his highly influential book, Hirzebruch [1966] defined a polynomial with integral coefficients $\chi_y(M)$ given a projective manifold M , which encodes the information of indices of Dolbeault complexes and is now called the *Hirzebruch χ_y -genus*. After the discovery of the general index theorem due to Atiyah and Singer, we know that $\chi_y(\cdot)$ can be defined on compact almost-complex manifolds and computed in terms of Chern numbers as follows.

Suppose (M^{2d}, J) is a compact connected almost-complex manifold with an almost-complex structure J . The choice of an almost Hermitian metric on M enables us to define the Hodge star operator $*$ and the formal adjoint $\bar{\partial}^* = - * \bar{\partial} *$ of the $\bar{\partial}$ -operator. For each pair $0 \leq p, q \leq d$, we denote by

$$\Omega^{p,q}(M) := \Gamma(\Lambda^p T^*M \otimes \Lambda^q \overline{T^*M})$$

the complex vector space which consists of smooth complex-valued (p, q) -forms. Here T^*M is the dual of the holomorphic tangent bundle TM in the sense of J . Then for each $0 \leq p \leq d$, we have the Dolbeault-type elliptic differential operator

$$\bigoplus_{q \text{ even}} \Omega^{p,q}(M) \xrightarrow{(\bar{\partial} + \bar{\partial}^*)|_p} \bigoplus_{q \text{ odd}} \Omega^{p,q}(M),$$

whose index is denoted by $\chi^p(M)$ in the notation of [Hirzebruch 1966]. Then the Hirzebruch χ_y -genus of M is nothing but the generating function of the indices $\chi^p(M)$ ($0 \leq p \leq d$):

$$\chi_y(M) := \sum_{p=0}^d \chi^p(M) \cdot y^p.$$

Let us denote by x_1, \dots, x_d the formal Chern roots of TM . This means that the i -th elementary symmetric polynomial of x_1, \dots, x_d represents the i -th Chern class c_i of TM . Then the general form of the Hirzebruch–Riemann–Roch theorem (first proved by Hirzebruch [1966] for projective manifolds, and in the general case by Atiyah and Singer [1968]) tells us that

$$(1-1) \quad \chi_y(M) = \int_M \prod_{i=1}^d \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}}.$$

Among other things, the Hirzebruch χ_y -genus has an important feature, which we call the “ -1 -phenomenon” and has been noticed, implicitly or explicitly, in several independent articles [Narasimhan and Ramanan 1975; Libgober and Wood 1990; Salamon 1996]. This -1 -phenomenon says that at $y = -1$, the coefficients of the

Taylor expansion of $\chi_y(M)$ have explicit expressions. To be more precise, if we write

$$(1-2) \quad \chi_y(M) =: \sum_{i=0}^d a_i(M) \cdot (y + 1)^i,$$

then these $a_i(M)$ can be given explicit expressions in terms of Chern numbers of (M^{2d}, J) as

$$(1-3) \quad \begin{aligned} a_0(M) &= c_d, \\ a_1(M) &= -\frac{1}{2}dc_d, \\ a_2(M) &= -\frac{1}{12}\left[\frac{1}{2}d(3d-5)c_d + c_1c_{d-1}\right], \\ a_3(M) &= -\frac{1}{24}\left[\frac{1}{2}d(d-2)(d-3)c_d + (d-2)c_1c_{d-1}\right], \\ &\vdots \end{aligned}$$

By definition, these $a_i(M)$ are integers. Thus, immediate consequences of their expressions include divisibility properties of Chern numbers. The derivation of these expressions is direct, i.e., by expanding the right-hand side of (1-1) at $y = -1$ and expressing the coefficients in terms of elementary symmetric polynomials of x_1, \dots, x_d . The calculations of a_0 and a_1 are quite easy. The calculation of a_2 appears implicitly in [Narasimhan and Ramanan 1975, p. 18] and explicitly in [Libgober and Wood 1990, p. 141–143]. Narasimhan and Ramanan used a_2 to give a topological restriction on some moduli spaces of stable vector bundles on smooth projective varieties. Libgober and Wood used a_2 to prove the uniqueness of the complex structure on Kähler manifolds of certain homotopy types. Inspired by [Narasimhan and Ramanan 1975], Salamon applied a_2 [1996, Corollary 3.4] to obtain a restriction on the Betti numbers of hyper-Kähler manifolds [ibid., Theorem 4.1]. The expressions of a_3 and a_4 are also included in [ibid., p. 145]. Hirzebruch [1999] used a_1, a_2 and a_3 to obtain a divisibility result on the Euler characteristic of those almost-complex manifolds where $c_1c_{d-1} = 0$. In particular, those almost-complex manifolds with $c_1 = 0$ satisfy this property.

Pluri- χ_y -genus. Some acquaintance with index theory will lead to the observation that $\chi_y(M)$ is the index of the Todd operator (whose index is the Todd genus)

$$(1-4) \quad \Omega^{0,\text{even}}(M) \xrightarrow{(\bar{\partial} + \bar{\partial}^*)|_0} \Omega^{0,\text{odd}}(M)$$

twisted by $\Omega_y(M)$, with

$$\Omega_y(M) := \sum_{p=0}^d \Lambda^p(T^*M) \cdot y^p \in K(M)[y],$$

where $\Lambda^p(\cdot)$ and $K(\cdot)$ denote the p -th exterior power and K -group. Therefore $\chi_y(M)$ can be rewritten as

$$\chi_y(M) = \text{Ind}((\bar{\partial} + \bar{\partial}^*)|_0 \otimes \Omega_y(M)) =: \chi(M, \Omega_y(M)).$$

Here, for simplicity we denote by the standard notation $\chi(M, (\cdot))$ the index of the Todd operator (1-4) twisted by an element $(\cdot) \in K(M)$.

We can also consider, for an arbitrarily fixed positive integer g , the *pluri χ_y -genus* $\chi_{\underline{y}}(M)$ by using sufficiently many forms of the type

$$\begin{aligned} (1-5) \quad \Omega_{\underline{y}}(M) &:= \sum_{0 \leq p_1, \dots, p_g \leq d} \Lambda^{p_1}(T^*M) \otimes \dots \otimes \Lambda^{p_g}(T^*M) \cdot y_1^{p_1} \dots y_g^{p_g} \\ &= \Omega_{y_1}(M) \otimes \dots \otimes \Omega_{y_g}(M) \in K(M)[y_1, \dots, y_g] \end{aligned}$$

to twist $(\bar{\partial} + \bar{\partial}^*)|_0$, i.e.,

$$\chi_{\underline{y}}(M) := \text{Ind}((\bar{\partial} + \bar{\partial}^*)|_0 \otimes \Omega_{\underline{y}}(M)) = \chi(M, \Omega_{\underline{y}}(M)),$$

which specializes to Hirzebruch’s original χ_y -genus when $g = 1$.

Inspired by the above-mentioned -1 -phenomenon of the χ_y -genus, we may ask what the coefficients look like if we expand $\chi_{\underline{y}}(M)$ at $y_1 = \dots = y_g = -1$. Our first main observation in this article is that the coefficients of $(y + 1)^{p_1} \dots (y + 1)^{p_g}$ in $\chi_{\underline{y}}(M)$ can be divided into three parts, which is our main result in Section 3 (Theorem 2.2). Moreover, we can do a similar manipulation for signature operator on closed smooth oriented manifolds, and their coefficients also have a similar feature (Theorem 2.3). A direct corollary of these two theorems is that any Chern number of (M^{2d}, J) or any Pontrjagin number of a closed smooth oriented manifold can be written explicitly as a rational linear combination of indices of some elliptic operators, which provides an analytic interpretation of characteristic numbers and answers [Li 2011, Question 1.1] affirmatively.

Elliptic genus. Elliptic genera of oriented differentiable manifolds and almost-complex manifolds were first constructed by Ochanine, Landweber, Stong and Hirzebruch in a topological way; Witten gave it a geometric interpretation, in which they can be viewed as the loop space analogues of the Hirzebruch L -genus and χ_y -genus (see [Landweber 1988] and the references therein). The most remarkable property of elliptic genera is their rigidity for spin manifolds and almost-complex Calabi–Yau manifolds (in the very weak sense that c_1 vanishes up to torsion, i.e., $c_1 = 0 \in H^2(M, \mathbb{R})$), which was conjectured by Witten and generalizes the famous rigidity property of the original L -genus, \hat{A} -genus [Atiyah and Hirzebruch 1970] and χ_y -genus [Lusztig 1971]. The first rigorous proof was presented in [Bott and Taubes 1989; Taubes 1989]. A quite simple, unified and enlightening proof was discovered by Liu [1996], in which modular invariance of the four classical

Jacobi theta functions and their various transformation laws play key roles. Later on, this modular invariance property, its various remarkable extensions and relation with vertex operator algebra were established by Liu and his coauthors from various perspectives [Liu 1995a; 1995b; Liu and Ma 2000; Liu et al. 2001; 2003; Han and Zhang 2004; Dong et al. 2005; Chen and Han 2009; Chen et al. 2011; Han et al. 2012; Han and Liu 2014].

We are concerned in this paper with the elliptic genus of almost-complex manifolds. The elliptic genus of a compact, almost-complex manifold (M^{2d}, J) , which we denote by $\text{Ell}(M, \tau, z)$, is defined as a function of two variables $(\tau, z) \in \mathbb{H} \times \mathbb{C}$, where \mathbb{H} is the upper half plane. To be more precise, $\text{Ell}(M, \tau, z)$ is defined to be the index of the Todd operator (1-4) twisted by

$$y^{-d/2} \bigotimes_{n \geq 1} (\Lambda_{-yq^{n-1}} T^* \otimes \Lambda_{-y^{-1}q^n} T \otimes S_{q^n} T^* \otimes S_{q^n} T) =: E_{q,y},$$

i.e., $\text{Ell}(M, \tau, z) := \chi(M, E_{q,y})$, where $q = e^{2\pi\sqrt{-1}\tau}$, $y = e^{2\pi\sqrt{-1}z}$ and T (resp. T^*) is the holomorphic (resp. dual of the holomorphic) tangent bundle of M in the sense of J . Here, for any complex vector bundle W ,

$$\Lambda_t(W) := \bigoplus_{i \geq 0} \Lambda^i(W) \quad \text{and} \quad S_t(W) := \bigoplus_{i \geq 0} S^i(W)$$

denote the generating series of the exterior and symmetric powers of W , respectively.

According to the Atiyah–Singer index theorem, we have

$$\begin{aligned} \text{Ell}(M, \tau, z) &= \int_M \text{td}(M) \cdot \text{ch}(E_{q,y}) \\ &= y^{-\frac{1}{2}d} \chi_{-y}(M) + q \cdot [y^{-\frac{1}{2}d} \chi_{-y}(M, T^*(1-y) + T(1-y^{-1}))] + q^2 \cdot (\dots), \end{aligned}$$

where

$$\text{td}(M) := \prod_{i=1}^d \frac{x_i}{1 - e^{-x_i}}$$

is the Todd class of M and $\text{ch}(\cdot)$ is the Chern character.

Thus, the elliptic genus $\text{Ell}(M, \tau, z)$ can be viewed as a generalization of the Hirzebruch χ_y -genus, in the sense that the q^0 -term of the Fourier expansion of $\text{Ell}(M, \tau, z)$ is essentially $\chi_y(M)$. If (M^{2d}, J) is Calabi–Yau, the coefficients of q -expansion of $\text{Ell}(M, \tau, z)$ are rigid for arbitrary y [Liu 1996, Theorem B]. Moreover, in this case, $\text{Ell}(M, \tau, z)$ itself is a weak Jacobi form of weight 0 and index $\frac{1}{2}d$ [Gritsenko 1999b, Proposition 1.2; Borisov and Libgober 2000, Theorem 2.2].

As we have mentioned above, the elliptic genus $\text{Ell}(M, \tau, z)$ can be viewed as a generalization of $\chi_y(M)$, and also has a rigidity property when M is Calabi–Yau. So we may ask in the Calabi–Yau case whether $\text{Ell}(M, \tau, z)$ has some kind of arithmetic phenomenon which extends the original -1 -phenomenon of $\chi_y(M)$. Note that, strictly speaking, $\text{Ell}(M, \tau, z)$ is a generalization of $\chi_{-y}(M)$ rather than $\chi_y(M)$, as the q^0 -term of $\text{Ell}(M, \tau, z)$ is $y^{-d/2}\chi_{-y}(M)$. So if there exists some kind of phenomenon which extends the original -1 -phenomenon of $\chi_y(M)$, the parameter $y = e^{2\pi\sqrt{-1}z}$ should correspond to 1 rather than -1 . Thus the variable z should correspond to 0. Indeed, there does exist such a kind of generalization, which depends on some arithmetic properties of Jacobi forms and has been implicitly used by Gritsenko [1999b]. Our aim in Section 3 is twofold. On the one hand, given a compact almost-complex manifold (M^{2d}, J) and a rank- l complex vector bundle W over it, we construct a generalized elliptic genus $\text{Ell}(M, W, \tau, z)$, which is defined to be the index of the Todd operator (1-4) twisted by

$$\left[\prod_{i=1}^{\infty} (1 - q^i) \right]^{2(d-l)} \cdot y^{-l/2} \bigotimes_{n \geq 1} (\Lambda_{-yq^{n-1}} W^* \otimes \Lambda_{-y^{-1}q^n} W \otimes S_{q^n} T^* \otimes S_{q^n} T),$$

and show that it is a weak Jacobi form of weight $d - l$ and index $\frac{1}{2}l$ if the first Pontrjagin classes $p_1(M)$ equals $p_1(W)$ and the first Chern class $c_1(W)$ is 0 in $H^*(M, \mathbb{R})$. On the other hand, we highlight a well-known manipulation in Jacobi forms to obtain modular forms from $\text{Ell}(M, W, \tau, z)$, whose arithmetic information will in turn give geometric results on M and W . Some examples are given to illustrate this observation.

2. -1 -phenomenon of the pluri- χ_y -genus

Statements of the main results related to the pluri- χ_y -genus. Let (M^{2n}, J) (resp. X^{2n}) be a compact almost-complex manifold of complex dimension n (resp. smooth, closed oriented manifold of real dimension $2n$). As before, we use $(\bar{\partial} + \bar{\partial}^*)|_0$ to denote the Todd operator on (M^{2n}, J) , whose index is the Todd genus of M . We denote by D the signature operator on X , whose index is the signature of X^{2n} [Atiyah and Singer 1968, Section 6]. By definition $\text{Ind}(D)$ is zero unless n is even.

Let W be a complex vector bundle over M or X . By means of a connection on W , the elliptic operator $(\bar{\partial} + \bar{\partial}^*)|_0$ and D can be extended to a new elliptic operator $((\bar{\partial} + \bar{\partial}^*)|_0) \otimes W$ and $D \otimes W$, whose indices via the Atiyah–Singer index theorem are

$$\begin{aligned} \chi(M, W) &= \text{Ind}(((\bar{\partial} + \bar{\partial}^*)|_0) \otimes W) = \int_M [\text{td}(M) \cdot \text{ch}(W)] \\ &= \int_M \left[\prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}} \cdot \text{ch}(W) \right] \end{aligned}$$

and

$$\text{Ind}((D \otimes W)) = \int_X \left[\left(\prod_{i=1}^n \frac{x_i}{\tanh(x_i/2)} \right) \cdot \text{ch}(W) \right]$$

respectively. Here we use the i -th elementary symmetric polynomial of x_1, \dots, x_n (resp. x_1^2, \dots, x_n^2) to denote the i -th Chern class (resp. Pontrjagin class) of (M^{2n}, J) (resp. X^{2n}).

Definition 2.1. For an arbitrary fixed positive integer g , we define

$$\begin{aligned} \Omega_{\underline{y}}(M) &:= \sum_{0 \leq p_1, \dots, p_g \leq n} \Lambda^{p_1}(T^*M) \otimes \dots \otimes \Lambda^{p_g}(T^*M) \cdot y_1^{p_1} \dots y_g^{p_g} \\ &= \Omega_{y_1}(M) \otimes \dots \otimes \Omega_{y_g}(M) \in K(M)[y_1, \dots, y_g], \\ \Omega_{\underline{y}}^{\mathbb{R}}(X) &:= \sum_{0 \leq p_1, \dots, p_g \leq 2n} \Lambda^{p_1}(T_{\mathbb{C}}^*X) \otimes \dots \otimes \Lambda^{p_g}(T_{\mathbb{C}}^*X) \cdot y_1^{p_1} \dots y_g^{p_g} \\ &= \Omega_{y_1}^{\mathbb{R}}(X) \otimes \dots \otimes \Omega_{y_g}^{\mathbb{R}}(X) \in (KO(X) \otimes \mathbb{C})[y_1, \dots, y_g], \end{aligned}$$

where

$$\Omega_{y_p}^{\mathbb{R}}(X) := \sum_{p=0}^{2n} \Lambda^p(T_{\mathbb{C}}^*X) \cdot y^p$$

and $T_{\mathbb{C}}^*X$ is the dual of the complexified tangent bundle of X , and

$$\begin{aligned} \chi_{\underline{y}}(M) &:= \sum_{0 \leq p_1, \dots, p_g \leq n} \text{Ind}[(\bar{\partial} + \bar{\partial}^*)|_0 \otimes (\Lambda^{p_1}(T^*M) \otimes \dots \otimes \Lambda^{p_g}(T^*M))] \\ &\quad \cdot y_1^{p_1} \dots y_g^{p_g} \\ &= \int_M \left[\prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}} \cdot \text{ch}(\Omega_{\underline{y}}(M)) \right] \\ D_{\underline{y}}(X) &:= \sum_{0 \leq p_1, \dots, p_g \leq 2n} \text{Ind}[D \otimes (\Lambda^{p_1}(T_{\mathbb{C}}^*X) \otimes \dots \otimes \Lambda^{p_g}(T_{\mathbb{C}}^*X))] \cdot y_1^{p_1} \dots y_g^{p_g} \\ &= \int_X \left[\left(\prod_{i=1}^n \frac{x_i}{\tanh(x_i/2)} \right) \cdot \text{ch}(\Omega_{\underline{y}}^{\mathbb{R}}(X)) \right]. \end{aligned}$$

Our main result in this section is:

Theorem 2.2. *The coefficient of $(1 + y_1)^{n-q_1} \dots (1 + y_g)^{n-q_g}$ in $\chi_{\underline{y}}(M)$ is equal to*

$$\begin{cases} 0 & \text{if } \sum_{i=1}^g q_i > n, \\ \int_M \prod_{i=1}^g c_{q_i}(M) & \text{if } \sum_{i=1}^g q_i = n, \\ \text{a rational linear combination of Chern numbers of } M & \text{if } \sum_{i=1}^g q_i < n. \end{cases}$$

We have a similar result for smooth manifolds.

Theorem 2.3. *If n is even, the coefficient of*

$$(1 + y_1)^{2(n-q_1)} \dots (1 + y_g)^{2(n-q_g)}$$

in $D_y(X)$ is equal to

$$\begin{cases} 0 & \text{if } \sum_{i=1}^g q_i > \frac{1}{2}n, \\ (-1)^{n/2} \cdot 2^n \cdot \int_X \prod_{i=1}^g p_{q_i}(X) & \text{if } \sum_{i=1}^g q_i = \frac{1}{2}n, \\ \text{a rational linear combination of Pontrjagin numbers of } X & \text{if } \sum_{i=1}^g q_i < \frac{1}{2}n, \end{cases}$$

where $p_i(X)$ is the i -th Pontrjagin class of X .

Clearly, a direct corollary of this theorem is the following result, which gives an affirmative answer to [Li 2011, Question 1.1].

Corollary 2.4. *Any Chern number (resp. Pontrjagin number) on a compact almost-complex manifold (resp. compact smooth manifold) can be expressed in an explicit way in terms of the indices of some elliptic differential operators over this manifold.*

Proofs of Theorems 2.2 and 2.3. Abusing notation, we use $c_q(\dots)$ to denote both the q -th Chern class of an almost-complex manifold and the q -th elementary symmetric polynomial of the variables in the bracket.

The proofs of Theorems 2.2 and 2.3 depend on the following lemma:

Lemma 2.5. *If we assign each x_i ($1 \leq i \leq n$) the same degree, then we have:*

(1) *the coefficient of $(1 + y)^{n-q}$ ($0 \leq q \leq n$) in $\prod_{i=1}^n (1 + ye^{-x_i})$ is*

$$c_q(x_1, \dots, x_n) + \text{higher-degree terms};$$

(2) *the coefficient of $(1 + y)^{2(n-q)}$ ($0 \leq q \leq n$) in $\prod_{i=1}^n (1 + ye^{-x_i})(1 + ye^{x_i})$ is*

$$(-1)^q c_q(x_1^2, \dots, x_n^2) + \text{higher-degree terms}.$$

Proof. We have

$$\prod_{i=1}^n (1 + ye^{-x_i}) = \prod_{i=1}^n [(1 - e^{-x_i}) + (1 + y)e^{-x_i}] = e^{-c_1} \prod_{i=1}^n [(e^{x_i} - 1) + (1 + y)].$$

Thus the coefficient of $(1 + y)^{n-q}$ in $\prod_{i=1}^n (1 + ye^{-x_i})$ is

$$e^{-c_1} \cdot c_q(e^{x_1} - 1, \dots, e^{x_n} - 1) = c_q(x_1, \dots, x_n) + \text{higher-degree terms}.$$

Similarly,

$$\prod_{i=1}^n (1 + ye^{-x_i})(1 + ye^{x_i}) = \prod_{i=1}^n [(e^{x_i} - 1) + (1 + y)][(e^{-x_i} - 1) + (1 + y)],$$

and the coefficient of $(1 + y)^{2n-q}$ is

$$\begin{aligned} c_q(e^{x_1} - 1, \dots, e^{x_n} - 1, e^{-x_1} - 1, \dots, e^{-x_n} - 1) \\ = c_q(x_1, \dots, x_n, -x_1, \dots, -x_n) + \text{higher-degree terms.} \end{aligned}$$

Note that

$$c_q(x_1, \dots, x_n, -x_1, \dots, -x_n) = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ (-1)^{q/2} c_{q/2}(x_1^2, \dots, x_n^2) & \text{if } q \text{ is even.} \end{cases}$$

This gives the desired property. □

Now we can prove Theorems 2.2 and 2.3.

Proof. If we use x_1, \dots, x_n (resp. $x_1, \dots, x_n, -x_1, \dots, -x_n$) to denote the formal Chern roots of TM (resp. $T_{\mathbb{C}}X$), then we have (see [Hirzebruch et al. 1992, p. 11])

$$\text{ch}(\Omega_{\underline{y}}(M)) = \prod_{j=1}^g \left[\prod_{i=1}^n (1 + y_j e^{-x_i}) \right]$$

and

$$\text{ch}(\Omega_{\underline{y}}^{\mathbb{R}}(X)) = \prod_{j=1}^g \left[\prod_{i=1}^n (1 + y_j e^{-x_i})(1 + y_j e^{x_i}) \right].$$

Thus,

$$\begin{aligned} \chi_{\underline{y}}(M) &= \int_M \left[\left(\prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}} \right) \cdot \text{ch}(\Omega_{\underline{y}}(M)) \right] \\ &= \int_M \left\{ \left(\prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}} \right) \cdot \prod_{j=1}^g \left[\prod_{i=1}^n (1 + y_j e^{-x_i}) \right] \right\} \end{aligned}$$

and

$$\begin{aligned} \text{Ind}(D_{\underline{y}}^{\mathbb{R}}(X)) &= \int_X \left[\left(\prod_{i=1}^n \frac{x_i}{\tanh(x_i/2)} \right) \cdot \text{ch}(\Omega_{\underline{y}}^{\mathbb{R}}(X)) \right] \\ &= \int_X \left\{ \left(\prod_{i=1}^n \frac{x_i}{\tanh(x_i/2)} \right) \cdot \prod_{j=1}^g \left[\prod_{i=1}^n (1 + y_j e^{-x_i})(1 + y_j e^{x_i}) \right] \right\}. \end{aligned}$$

Note that the constant terms of

$$\frac{x_i}{1 - e^{-x_i}} = 1 + \dots \quad \text{and} \quad \frac{x_i}{\tanh(x_i/2)} = \frac{x_i(1 + e^{-x_i})}{1 - e^{-x_i}} = 2 + \dots$$

are 1 and 2 respectively. So by Lemma 2.5, when considering the Taylor expansions of $\text{Ind}(D_{\underline{y}}(M))$ and $\text{Ind}(D_{\underline{y}}^{\mathbb{R}}(X))$ at $y_1 = \dots = y_g = -1$, the coefficients before the terms $(1 + y_1)^{n-q_1} \dots (1 + y_g)^{n-q_g}$ and $(1 + y_1)^{2(n-q_1)} \dots (1 + y_g)^{2(n-q_g)}$ are

$$\begin{aligned} \int_M \left\{ (1 + \text{higher-degree terms}) \cdot \prod_{j=1}^g [c_{q_j}(x_1, \dots, x_n) + \text{higher-degree terms}] \right\} \\ = \int_M \prod_{i=1}^g c_{q_i}(M) + \int_M (\text{higher-degree terms}) \end{aligned}$$

and

$$\begin{aligned} \int_X \left\{ (2^n + \text{higher-degree terms}) \cdot \prod_{j=1}^g [(-1)^{q_j} c_{q_j}(x_1^2, \dots, x_n^2) + \text{higher degree terms}] \right\} \\ = 2^n \cdot (-1)^{\sum_{j=1}^g q_j} \int_X \prod_{j=1}^g p_{q_j}(X) + \int_X (\text{higher degree terms}), \end{aligned}$$

respectively, which give the desired proofs of Theorems 2.2 and 2.3. □

3. The generalized elliptic genus and its -1 -phenomenon

The generalized elliptic genus of almost-complex manifolds. In this subsection, we extend the original definition of the elliptic genus of almost-complex manifolds by considering an extra complex vector bundle and showing that it is a weak Jacobi form. As before, let (M^{2d}, J) be a compact almost-complex manifold and W a rank- l complex vector bundle over it.

Definition 3.1. The generalized elliptic genus of (M^{2d}, J) with respect to W , which we denote by $\text{Ell}(M, W, \tau, z)$, is defined to be the index of the Todd operator

$$\Omega^{0, \text{even}}(M) \xrightarrow{(\bar{\partial} + \bar{\partial}^*)|_0} \Omega^{0, \text{odd}}(M)$$

twisted by

$$c^{2(d-l)} \cdot y^{-l/2} \bigotimes_{n \geq 1} (\Lambda_{-yq^{n-1}} W^* \otimes \Lambda_{-y^{-1}q^n} W \otimes S_{q^n} T^* \otimes S_{q^n} T) =: E(W, q, y),$$

where

$$q = e^{2\pi\sqrt{-1}\tau}, \quad y = e^{2\pi\sqrt{-1}z},$$

and for simplicity $c := \prod_{i=1}^{\infty} (1 - q^i)$. If $W = T$, this definition degenerates to the original elliptic genus.

Our first observation in this section is the following, which extends [Gritsenko 1999b, Proposition 1.2; Borisov and Libgober 2000, Theorem 2.2], in which $W = T$.

Theorem 3.2. *The generalized elliptic genus $\text{Ell}(M, W, \tau, z)$ is a weak Jacobi form of weight $d - l$ and index $\frac{1}{2}l$ provided that the first Pontrjagin classes $p_1(M)$ equals $p_1(W)$ and the first Chern class $c_1(W)$ is 0 in $H^*(M, \mathbb{R})$.*

Remark 3.3. (1) A two-variable function $\varphi(\tau, z)$ for $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ is called a *weak Jacobi form of weight k and index m* for $k \in \mathbb{Z}$ and $m \in \mathbb{Z}/2$ if it is a holomorphic function with respect to the two variables τ and z , has no negative powers of q in its Fourier expansion in terms of $q^i y^j$ and satisfies some transformation laws involving k and m ; the precise definition can be found in [Eichler and Zagier 1985, p. 9, p. 104]. There, only the integral indices are considered. However, with minor modifications of inserting a character, this notion can be easily extended to the case where the index is allowed to be a half-integer (see [Gritsenko 1999b, p. 102]).

(2) Motivated by his ingenious proof of the rigidity theorem, Liu constructed a two-variable function for (M, J) and W and showed that it is a weak Jacobi form under some assumptions, and the original Witten theorem exactly corresponds to the case where the index is equal to zero [Liu 1995b, Theorem 3, Corollary 3.1]. This construction later was generalized to the family case by Liu and Ma [2000, Theorem 3.1]. So our theorem has a similar flavor to their work.

(3) Gritsenko [1999a, Theorem 1.2] further extended the original elliptic genus to another case where an extra complex bundle is involved. But his construction is different from ours as it is still of weight zero.

The Atiyah–Singer index theorem tells us that

$$\text{Ell}(M, W, \tau, z) = \int_M \text{td}(M) \cdot \text{ch}(E(W, q, y)).$$

In particular, if J is integrable, $\text{Ell}(M, W, \tau, z)$ is the holomorphic Euler characteristic of the (virtual) bundle $E(W, q, y)$.

Let us recall one of the Jacobi-theta series [Chandrasekharan 1985, Chapter 5]:

$$\begin{aligned} \theta(\tau, z) &:= \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+1/2)^2/2} y^{n+1/2} \\ &= 2cq^{1/8} \sin(\pi z) \prod_{n=1}^{\infty} (1 - q^n y)(1 - q^n y^{-1}) \\ &= 2cq^{1/8} \sinh(\pi \sqrt{-1}z) \prod_{n=1}^{\infty} (1 - q^n y)(1 - q^n y^{-1}) \\ &= 2cq^{1/8} \sinh(\pi \sqrt{-1}z) \prod_{n=1}^{\infty} (1 - q^n e^{2\pi \sqrt{-1}z})(1 - q^n e^{-2\pi \sqrt{-1}z}). \end{aligned}$$

The following lemma says that $\text{Ell}(M, W, \tau, z)$ can be expressed in terms of $\theta(\tau, z)$.

Lemma 3.4. *If we denote by $2\pi\sqrt{-1}x_i$ ($1 \leq i \leq d$) and $2\pi\sqrt{-1}w_i$ ($1 \leq i \leq l$) the Chern roots of TM and W , respectively, then we have*

$$\begin{aligned} &\text{Ell}(M, W, \tau, z) \\ &= \int_M \left[\exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \cdot (\eta(\tau))^{3(d-l)} \cdot \prod_{i=1}^d \frac{2\pi\sqrt{-1}x_i}{\theta(\tau, x_i)} \cdot \prod_{j=1}^l \theta(\tau, w_j - z) \right], \end{aligned}$$

where

$$\eta(\tau) := q^{1/24} \cdot c = q^{1/24} \prod_{i=1}^{\infty} (1 - q^i)$$

is the famous Dedekind eta function. In particular, $\text{Ell}(M, W, \tau, z)$ is a holomorphic function with respect to the two variables τ and z and has no negative powers of q in its Fourier expansion.

Proof. We have

$$\begin{aligned} &\text{ch}(E(W, q, y)) \\ &= c^{2(d-l)} y^{-l/2} \prod_{j=1}^l (1 - ye^{-2\pi\sqrt{-1}w_j}) \\ &\quad \times \prod_{n=1}^{\infty} \frac{\prod_{j=1}^l (1 - yq^n e^{-2\pi\sqrt{-1}w_j})(1 - y^{-1}q^n e^{2\pi\sqrt{-1}w_j})}{\prod_{i=1}^d (1 - q^n e^{-2\pi\sqrt{-1}x_i})(1 - q^n e^{2\pi\sqrt{-1}x_i})} \\ &= c^{2(d-l)} y^{-l/2} \prod_{j=1}^l (1 - ye^{-2\pi\sqrt{-1}w_j}) \\ &\quad \times \prod_{j=1}^l \frac{\theta(\tau, w_j - z)}{2cq^{1/8} \sinh(\pi\sqrt{-1}(w_j - z))} \prod_{i=1}^d \frac{2cq^{1/8} \sinh(\pi\sqrt{-1}x_i)}{\theta(\tau, x_i)} \\ &= \exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \cdot (\eta(\tau))^{3(d-l)} \cdot \prod_{i=1}^d \frac{1 - e^{-2\pi\sqrt{-1}x_i}}{\theta(\tau, x_i)} \cdot \prod_{j=1}^l \theta(\tau, w_j - z). \end{aligned}$$

The last equality is due to the fact that

$$c_1(M) = \sum_{i=1}^d 2\pi\sqrt{-1}x_i \quad \text{and} \quad c_1(W) = \sum_{j=1}^l 2\pi\sqrt{-1}w_j.$$

Therefore,

$$\begin{aligned} \text{Ell}(M, W, \tau, z) &= \int_M \text{td}(M) \cdot \text{ch}(E(W, q, y)) = \int_M \prod_{i=1}^d \frac{2\pi\sqrt{-1}x_i}{1 - e^{-2\pi\sqrt{-1}x_i}} \cdot \text{ch}(E(W, q, y)) \\ &= \int_M \left[\exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \cdot (\eta(\tau))^{3(d-l)} \cdot \prod_{i=1}^d \frac{2\pi\sqrt{-1}x_i}{\theta(\tau, x_i)} \cdot \prod_{j=1}^l \theta(\tau, w_j - z) \right]. \end{aligned}$$

The holomorphicity of $\text{Ell}(M, W, \tau, z)$ is now clear from this expression, as the Jacobi theta function $\theta(\tau, z)$ only has zeroes of order 1 along $z = m_1 + m_2\tau$ ($m_1, m_2 \in \mathbb{Z}$) [Chandrasekharan 1985, p. 59]. Also it is obvious from this expression that $\text{Ell}(M, W, \tau, z)$ has no negative powers of q in its Fourier expansion. \square

Proof of Theorem 3.2. $\text{SL}_2(\mathbb{Z})$ is generated by the two matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

To verify that $\text{Ell}(M, W, \tau, z)$ satisfies the required transformation laws, it suffices to show the four identities

- (3-1) $\text{Ell}(M, W, \tau + 1, z) = \text{Ell}(M, W, \tau, z),$
- (3-2) $\text{Ell}(M, W, \tau, z + 1) = (-1)^l \text{Ell}(M, W, \tau, z),$
- (3-3) $\text{Ell}(M, W, \tau, z + \tau) = (-1)^l \exp(-\pi\sqrt{-1}l(\tau + 2z)) \text{Ell}(M, W, \tau, z),$
- (3-4) $\text{Ell}(M, W, -1/\tau, z/\tau) = \tau^{d-l} \exp(\pi\sqrt{-1}lz^2/\tau) \text{Ell}(M, W, \tau, z).$

For Dedekind eta function $\eta(\tau)$ and Jacobi theta function $\theta(\tau, z)$ we have transformation laws [Chandrasekharan 1985]:

$$\begin{aligned} \eta^3\left(-\frac{1}{\tau}\right) &= \left(\frac{\tau}{\sqrt{-1}}\right)^{3/2} \eta^3(\tau), \\ \eta^3(\tau + 1) &= \exp\left(\frac{\pi\sqrt{-1}}{4}\right) \eta^3(\tau), \\ \theta(\tau, z + 1) &= -\theta(\tau, z), \\ \theta(\tau, z + \tau) &= -q^{-1/2} \exp(-2\pi\sqrt{-1}z)\theta(\tau, z), \\ \theta(\tau + 1, z) &= \exp\left(\frac{\pi\sqrt{-1}}{4}\right) \theta(\tau, z), \\ \theta\left(-\frac{1}{\tau}, z\right) &= -\sqrt{-1} \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} \exp(\pi\sqrt{-1}\tau z^2)\theta(\tau, \tau z). \end{aligned}$$

The first three identities (3-1)–(3-3) are easy to verify by using the transformation laws above. Here we only need to check (3-4) carefully. Indeed,

$$\begin{aligned}
 (3-5) \quad \prod_{i=1}^d \theta\left(-\frac{1}{\tau}, x_i\right) &= \prod_{i=1}^d -\sqrt{-1} \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} \exp(\pi\sqrt{-1}\tau x_i^2) \theta(\tau, \tau x_i) \\
 &= \exp\left(\frac{\tau p_1(M)}{4\pi\sqrt{-1}}\right) \prod_{i=1}^d -\sqrt{-1} \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} \theta(\tau, \tau x_i).
 \end{aligned}$$

Here, we use the assumption that

$$p_1(M) = \sum_{i=1}^d (2\pi\sqrt{-1}x_i)^2.$$

Similarly,

$$\begin{aligned}
 (3-6) \quad \prod_{j=1}^l \theta\left(-\frac{1}{\tau}, w_j - \frac{z}{\tau}\right) \\
 &= \prod_{j=1}^l -\sqrt{-1} \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} \exp\left(\pi\sqrt{-1}\tau \left(w_j - \frac{z}{\tau}\right)^2\right) \theta(\tau, \tau w_j - z) \\
 &= \exp\left(\frac{\tau p_1(W)}{4\pi\sqrt{-1}} + \frac{\pi\sqrt{-1}lz^2}{\tau}\right) \prod_{j=1}^l -\sqrt{-1} \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} \theta(\tau, \tau w_j - z).
 \end{aligned}$$

In the last equality, we used the assumption that

$$c_1(W) = \sum_{j=1}^l 2\pi\sqrt{-1}w_j = 0.$$

Combining the transformation law of $\eta(\tau)$, (3-5), (3-6) and the fact that $p_1(M) = p_1(W)$ leads to

$$\begin{aligned}
 \text{Ell}\left(M, W, -\frac{1}{\tau}, \frac{z}{\tau}\right) \\
 &= \int_M \left[\exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \left(\eta\left(-\frac{1}{\tau}\right)\right)^{3(d-l)} \right. \\
 &\quad \left. \times \prod_{i=1}^d \frac{2\pi\sqrt{-1}x_i}{\theta(-1/\tau, x_i)} \prod_{j=1}^l \theta\left(-\frac{1}{\tau}, w_j - \frac{z}{\tau}\right) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \tau^{d-l} \exp\left(\frac{\pi \sqrt{-1} l z^2}{\tau}\right) \\
 &\quad \times \int_M \left[\exp\left(\frac{c_1(M) - c_1(W)}{2}\right) (\eta(\tau))^{3(d-l)} \prod_{i=1}^d \frac{2\pi \sqrt{-1} x_i}{\theta(\tau, \tau x_i)} \prod_{j=1}^l \theta(\tau, \tau w_j - z) \right] \\
 &= \tau^{-l} \exp\left(\frac{\pi \sqrt{-1} l z^2}{\tau}\right) \\
 &\quad \times \int_M \left[\exp\left(\frac{c_1(M) - c_1(W)}{2}\right) (\eta(\tau))^{3(d-l)} \prod_{i=1}^d \frac{2\pi \sqrt{-1} (\tau x_i)}{\theta(\tau, \tau x_i)} \prod_{j=1}^l \theta(\tau, \tau w_j - z) \right] \\
 &= \tau^{d-l} \exp\left(\frac{\pi \sqrt{-1} l z^2}{\tau}\right) \\
 &\quad \times \int_M \left[\exp\left(\frac{c_1(M) - c_1(W)}{2}\right) (\eta(\tau))^{3(d-l)} \prod_{i=1}^d \frac{2\pi \sqrt{-1} x_i}{\theta(\tau, x_i)} \prod_{j=1}^l \theta(\tau, w_j - z) \right] \\
 &= \tau^{d-l} \exp\left(\frac{\pi \sqrt{-1} l z^2}{\tau}\right) \text{Ell}(M, W, \tau, z)
 \end{aligned}$$

The penultimate equality is due to the fact that in the integrand we are only concerned with the homogeneous part of degree d ($\deg(x_i) = \deg(w_j) = 1$). This completes the proof of [Theorem 3.2](#). □

Algebraic preliminaries. Before discussing the arithmetic properties of the generalized elliptic genus $\text{Ell}(M, W, \tau, z)$, we need to review a well-known manipulation in algebraic number theory which helps derive modular forms from Jacobi forms.

Recall that the *Eisenstein series* $G_{2k}(\tau)$ are defined to be

$$G_{2k}(\tau) := -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) \cdot q^n,$$

[[Hirzebruch et al. 1992](#), p. 131], where

$$\sigma_k(n) := \sum_{\substack{m>0 \\ m|n}} m^k$$

and the B_{2k} are the Bernoulli numbers.

These $G_{2k}(\tau)$ carry rich arithmetic information. It is well-known that $G_{2k}(\tau)$ ($k \geq 2$) are modular forms of weight $2k$ over the full modular group $\text{SL}_2(\mathbb{Z})$ and the whole graded ring of modular forms over $\text{SL}_2(\mathbb{Z})$ are generated by $G_4(\tau)$ and $G_6(\tau)$. However, $G_2(\tau)$ is *not* a modular form but a *quasimodular form*, as it

transforms as [Hirzebruch et al. 1992, p. 138]

$$(3-7) \quad G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - \frac{c(c\tau + d)}{4\pi\sqrt{-1}} \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}).$$

The next proposition, which is a well-known fact in algebraic number theory and has been used implicitly by Gritsenko in the proof of [1999b, Lemma 1.6], provides us with a method for deriving modular forms from Jacobi forms.

Proposition 3.5. *Suppose a function $\varphi(\tau, z) : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfies*

$$(3-8) \quad \varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \exp\left(\frac{2\pi\sqrt{-1}mcz^2}{c\tau + d}\right) \cdot \varphi(\tau, z),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, i.e., $\varphi(\tau, z)$ transforms like a Jacobi form of weight k and index m .

Then, if we define

$$\Phi(\tau, z) := \exp(-8\pi^2 m G_2(\tau) z^2) \varphi(\tau, z),$$

we have

$$(3-9) \quad \Phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \Phi(\tau, z).$$

This means that if we set

$$\Phi(\tau, z) =: \sum_{n \in \mathbb{Z}} a_n(\tau) \cdot z^n,$$

then

$$a_n\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{k+n} a_n(\tau).$$

In particular, if $\varphi(\tau, z)$ is a weak Jacobi form of weight k and index m , then these $a_n(\tau)$ are modular forms of weight $k + n$ over $\text{SL}_2(\mathbb{Z})$.

Proof. Equation (3-9) can be verified directly by using the assumption (3-8) and the transformation law (3-7). If, moreover, $\varphi(\tau, z)$ is a weak Jacobi form, then $\varphi(\tau, z)$ and thus $\Phi(\tau, z)$ are holomorphic and have no negative powers of q when considering their Fourier expansions in terms of q and y . This implies that these $a_n(\tau)$ are also holomorphic and have no negative powers of q when considering the Fourier expansions of q , which gives the desired proof. \square

With the assumptions in Theorem 3.2 understood, we know that $\text{Ell}(M, W, \tau, z)$ is a weak Jacobi form of weight $d - l$ and index $\frac{1}{2}l$. Then Proposition 3.5 tells us:

Proposition 3.6. *The series $a_n(M, W, \tau)$ determined by*

$$\exp[l \cdot G_2(\tau) \cdot (2\pi\sqrt{-1z})^2] \cdot \text{Ell}(M, W, \tau, z) =: \sum_{n \geq 0} a_n(M, W, \tau) \cdot (2\pi\sqrt{-1z})^n$$

are modular forms of weight $d - l + n$ over $\text{SL}_2(\mathbb{Z})$. Furthermore, the first three series of $a_n(M, W, \tau)$ are of the form

$$a_0(M, W, \tau) = \chi(M, \Lambda_{-1}W^*) + q \cdot \chi(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)) + q^2 \cdot (\dots),$$

$$a_1(M, W, \tau) = \sum_{p=0}^l (-1)^p \left(p - \frac{l}{2}\right) \chi(M, \Lambda^p W^*) + q \cdot (\dots),$$

$$a_2(M, W, \tau) = -\frac{l}{24} \chi(M, \Lambda_{-1}W^*) + \frac{1}{2} \sum_{p=0}^l (-1)^p \left(p - \frac{l}{2}\right)^2 \chi(M, \Lambda^p W^*) + q \cdot (\dots).$$

Proof. The first statement is a direct application of [Proposition 3.5](#) as $\text{Ell}(M, W, \tau, z)$ is a weak Jacobi form of weight $d - l$ and index $\frac{1}{2}l$. For the second one, if we set

$$\exp[lG_2(\tau)(2\pi\sqrt{-1z})^2] =: A_0(y) + A_1(y) \cdot q + (\dots) \cdot q^2$$

and

$$\text{Ell}(M, W, \tau, z) =: B_0(y) + B_1(y) \cdot q + (\dots) \cdot q^2,$$

we can easily deduce from their explicit expressions that

$$A_0(y) = \exp\left[-\frac{l}{24}(2\pi\sqrt{-1z})^2\right] = 1 - \frac{l}{24}(2\pi\sqrt{-1z})^2 + \dots,$$

$$A_1(y) = l(2\pi\sqrt{-1z})^2 - \frac{l^2}{24}(2\pi\sqrt{-1z})^4 + \dots,$$

$$\begin{aligned} B_0(y) &= \sum_{p=0}^l (-1)^p \chi(M, \Lambda^p W^*) y^{p-1/2} \\ &= \sum_{p=0}^l (-1)^p \chi(M, \Lambda^p W^*) \\ &\quad \times \left[1 + \left(p - \frac{l}{2}\right)(2\pi\sqrt{-1z}) + \frac{1}{2} \left(p - \frac{l}{2}\right)^2 (2\pi\sqrt{-1z})^2 + \dots \right], \end{aligned}$$

$$B_1(y) = \chi(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)) + 2\pi\sqrt{-1z}(\dots).$$

Note that

$$\sum_{n \geq 0} a_n(M, W, \tau) (2\pi\sqrt{-1z})^n = A_0(y)B_0(y) + [A_0(y)B_1(y) + A_1(y)B_0(y)]q + \dots;$$

then it is easy to deduce the expressions in [Proposition 3.6](#) in terms of those of $A_0(y)$, $A_1(y)$, $B_0(y)$ and $B_1(y)$. \square

−1-phenomenon of the generalized elliptic genus. Here, using Proposition 3.6, presented in the last subsection, we investigate the arithmetic information of the generalized elliptic genus $\text{Ell}(M, W, \tau, z)$, which can be viewed as an appropriate −1-phenomenon of $\text{Ell}(M, W, \tau, z)$.

We will present one proposition and two examples related to $a_2(M, W, \tau)$, $a_0(M, W, \tau)$ and $a_1(M, W, \tau)$, respectively, to illustrate an appropriate −1-phenomenon of the generalized elliptic genus $\text{Ell}(M, W, \tau, z)$.

Our next proposition related to $a_2(M, W, \tau)$ gives the “reason” why these $a_n(M, W, \tau)$ should be the −1-phenomenon of $\text{Ell}(M, W, \tau, z)$.

Proposition 3.7. $a_2(M, W, \tau)$ is a modular form of weight $d - l + 2$ over $\text{SL}_2(\mathbb{Z})$ provided that $p_1(M) = p_1(W)$ and $c_1(W) = 0$ in $H^*(M, \mathbb{R})$. Consequently, if either (i) $d - l$ is odd, or (ii) $d \leq l$ but $d - l \neq -2$, we have

$$(3-10) \quad \sum_{p=0}^l (-1)^p \left(p - \frac{l}{2}\right)^2 \chi(M, \Lambda^p W^*) = \frac{l}{12} \chi(M, \Lambda_{-1} W^*).$$

Moreover, if $W = T$ and $c_1(M) = 0$ in $H^*(M, \mathbb{R})$, (3-10) is nothing but the original −1-phenomenon of the Hirzebruch χ_y -genus.

Proof. If either (i) $d - l$ is odd or (ii) $d \leq l$ but $d - l \neq -2$, $a_2(M, W, \tau)$ is a modular form over $\text{SL}_2(\mathbb{Z})$ whose weight is either (i) odd or (ii) no more than 2 but not zero. This means $a_2(M, W, \tau) \equiv 0$; then its expression in Proposition 3.6 gives (3-10).

If $W = T$, then the right-hand side of (3-10) is

$$\frac{d}{12} \chi(M, \Lambda_{-1} T^*) = \frac{d}{12} \chi_y(M) \Big|_{y=-1} = \frac{d}{12} c_d(M).$$

However, the left-hand side of (3-10) is

$$\begin{aligned} & \sum_{p=0}^d (-1)^p \left(p - \frac{d}{2}\right)^2 \chi^p(M) \\ &= \sum_{p=0}^d (-1)^p \left[2 \cdot \frac{p(p-1)}{2} + (1-d)p + \frac{d^2}{4} \right] \chi^p(M) \\ &= 2a_2(M) - (1-d)a_1(M) + \frac{d^2}{4} a_0(M) \\ &= \frac{d(3d-5)}{12} c_d(M) + \frac{(1-d)d}{2} c_d(M) + \frac{d^2}{4} c_d(M) \quad (\text{via (1-3) and } c_1(M) = 0) \\ &= \frac{d}{12} c_d(M) = \text{the right-hand side of (3-10)}. \quad \square \end{aligned}$$

The next two examples, related to $a_0(M, W, \tau)$ and $a_1(M, W, \tau)$, give much arithmetic information about M and W .

Example 3.8. By Proposition 3.6, we know that $a_0(M, W, \tau)$ is a modular form of weight $d - l$ over $\mathrm{SL}_2(\mathbb{Z})$ provided that $p_1(M) = p_1(W)$ and $c_1(M) = 0$ in $H^2(M, \mathbb{R})$. Consequently:

(1) If either $d - l$ is odd or $d - l \leq 2$ but is nonzero, we have

$$\chi(M, \Lambda_{-1}W^*) = \chi(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)) = 0.$$

(2) If $d - l = 4$, $a_0(M, W, \tau)$ is proportional to the Eisenstein series

$$G_4(\tau) = -\frac{B_4}{8} + q + \dots = \frac{1}{240} + q + \dots,$$

and so

$$\chi(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)) = 240\chi(M, \Lambda_{-1}W^*).$$

(3) If $d - l = 6$, $a_0(M, W, \tau)$ is proportional to the Eisenstein series

$$G_6(\tau) = -\frac{B_6}{12} + q + \dots = -\frac{1}{504} + q + \dots,$$

and so

$$\chi(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)) = -504\chi(M, \Lambda_{-1}W^*).$$

(4) If $d - l = 8$, $a_0(M, W, \tau)$ is proportional to

$$[G_4(\tau)]^2 = \left[\frac{1}{240} + q + \dots \right]^2 = \frac{1}{240^2} + \frac{1}{120}q + \dots,$$

and so

$$\chi(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)) = 480\chi(M, \Lambda_{-1}W^*).$$

Example 3.9. By Proposition 3.6, we know that $a_1(M, W, \tau)$ is a modular form of weight $d - l + 1$ over $\mathrm{SL}_2(\mathbb{Z})$ provided that $p_1(M) = p_1(W)$ and $c_1(M) = 0$ in $H^2(M, \mathbb{R})$. Consequently, if either $d - l$ is even or $d - l \leq 1$ but $d - l \neq -1$, we have

$$\sum_{p=0}^l (-1)^p \left(p - \frac{l}{2} \right) \chi(M, \Lambda^p W^*) = 0.$$

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PING LI
DEPARTMENT OF MATHEMATICS
TONGJI UNIVERSITY
SHANGHAI, 200092
CHINA
pingli@tongji.edu.cn

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Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

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