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FIXED-POINT RESULTS AND THE HYERS–ULAM STABILITY OF LINEAR EQUATIONS OF HIGHER ORDERS

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We present a general method for investigation of the Hyers–Ulam stability of linear equations (differential, difference, functional, integral) of higher orders. It is shown that in many cases, that kind of stability for such equations is a consequence of a similar property of the corresponding first-order equations. Some particular examples of applications for differential, integral, difference and functional equations are described. The method is based on some fixed-point results that are proved in this paper.

1. Introduction

Sometimes we have to deal with functions that satisfy some equations only approximately. One of the possible ways to treat them is just to replace such functions by suitably corresponding exact solutions to those equations. Therefore it seems to be important to know when, why and to what extent we can do this, and what errors we thus commit. Some tools for evaluation of that issue are offered by the theory of Ulam-type stability.

Some information on that theory and further references concerning it are given in Section 3. The following definition somehow describes the main ideas of that kind of stability ($\mathbb{N}$ stands for the set of positive integers, $\mathbb{R}_+ := [0, \infty)$, and C^D denotes the family of all functions mapping a set $D \neq \varnothing$ into a set $C \neq \varnothing$):

Definition 1.1. Let $n \in \mathbb{N}$, A be a nonempty set, (X, d) be a metric space, $\mathscr{E} \subset \mathscr{C} \subset {\mathbb{R}_+}^{A^n}$ be nonempty, $\mathscr{T}$ be an operator (not necessarily linear) mapping $\mathscr{C}$ into ${\mathbb{R}_+}^A$, and $\mathscr{F}_1$, $\mathscr{F}_2$ be operators (not necessarily linear) mapping a nonempty $\mathscr{D} \subset X^A$ into X^{A^n}. We say that the operator equation

$$(1) \qquad \mathscr{F}_1\varphi(x_1, \ldots, x_n) = \mathscr{F}_2\varphi(x_1, \ldots, x_n)$$

is $(\mathscr{E}, \mathscr{T})$-stable, provided for every $\varepsilon \in \mathscr{E}$ and $\varphi_0 \in \mathscr{D}$ with

$$(2) \quad d(\mathscr{F}_1\varphi_0(x_1, \ldots, x_n), \mathscr{F}_2\varphi_0(x_1, \ldots, x_n)) \le \varepsilon(x_1, \ldots, x_n), \quad x_1, \ldots, x_n \in A,$$

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there exists a solution $\varphi \in \mathcal{D}$ of (1) such that

$$d(\varphi(x), \varphi_0(x)) \le \mathcal{T}\varepsilon(x), \quad x \in A. \tag{3}$$

Roughly speaking, $(\mathcal{E}, \mathcal{T})$-stability of (1) means that every approximate (in the sense of (2)) solution of (1) is always close (in the sense of (3)) to an exact solution to (1).

In the particular case when $\mathcal{E}$ contains only all constant functions, $(\mathcal{E}, \mathcal{T})$-stability is called Hyers–Ulam stability. In this paper we describe (in the terms of fixed points) a general method for investigation of the Hyers–Ulam stability of various higher-order linear (differential, integral, difference or functional) equations in a single variable, that is, for $n = 1$. In this way we show how to generalize and easily extend numerous results given in, e.g., [Takahasi et al. 2002; Miura et al. 2003a; 2003b; 2004; 2012; Jung 2004; 2005; 2006; Popa 2005b; Trif 2006; Wang et al. 2008; Brzdęk et al. 2008; 2010; 2011b; Li and Shen 2009; Brzdęk and Jung 2010].

In what follows, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Also, X is a Banach space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $m \in \mathbb{N}$ is fixed and in general we assume that $m > 1$ (unless explicitly stated otherwise), S is a nonempty set, and $a_0, \dots, a_{m-1} \in \mathbb{K}$. Additionally, $\mathcal{U}$ is a linear subspace of X^S (the linear space over $\mathbb{K}$ of all the functions mapping S into X), $F \in \mathcal{U}$ is fixed, $\mathcal{L} : \mathcal{U} \to X^S$ is a linear operator, $P_m : \mathbb{C} \to \mathbb{C}$ is a polynomial given by $P_m(z) := z^m + \sum_{j=0}^{m-1} a_j z^j$ and $r_1, \dots, r_m \in \mathbb{C}$ are the roots of the equation

$$P_m(z) = 1. \tag{4}$$

Moreover, we write $\mathcal{U}_m := \{f \in \mathcal{U} : \mathcal{L}^i f \in \mathcal{U} \text{ for } i = 1, \dots, m-1\}$ and define a linear operator $P_m(\mathcal{L}) : \mathcal{U}_m \to X^S$ by $P_m(\mathcal{L}) := \mathcal{L}^m + \sum_{j=0}^{m-1} a_j \mathcal{L}^j$, where $\mathcal{L}^0 := \mathcal{I}$ is the identity operator (i.e., $\mathcal{I} f = f$ for $f \in X^S$) and $\mathcal{L}^k := \mathcal{L} \circ \mathcal{L}^{k-1}$ for any $k \in \mathbb{N}$. In the next section we present some fixed-point results for the operator

$$\mathcal{P}_m^F := P_m(\mathcal{L}) + F$$

(i.e., $\mathcal{P}_m^F(\varphi) = P_m(\mathcal{L})(\varphi) + F$ for $\varphi \in \mathcal{U}_m$).

2. Fixed-point results

For the sake of simplicity we use the notion $\|f\| := \sup_{x \in S} \|f(x)\|$ for $f \in X^S$, which can be considered as an extension (because it admits an infinite value) of the usual supremum norm $\|\cdot\|_\infty$ defined on the linear space (over $\mathbb{K}$) of all bounded functions from X^S. In this section, we write

$$\mathcal{L}_i^v := \mathcal{L} + (1 - r_i)\mathcal{I} - v, \quad v \in X^S, \ i \in \{1, \dots, m\} \tag{5}$$

(i.e., $\mathcal{L}_i^v(\varphi) := \mathcal{L}(\varphi) + (1-r_i)\varphi - v$ for $\varphi \in \mathcal{U}$), provided $r_1, \dots, r_m \in \mathbb{K}$. The next two fixed-point theorems are the main tools in this paper; we present their proofs at the end of the paper.

Theorem 2.1. *Let $r_1, \dots, r_m \in \mathbb{K}$ and $\xi_i : \mathbb{R}_+ \to \mathbb{R}_+$ for $i = 1, \dots, m$. Suppose that*

$$\delta := \|\mathcal{P}_m^F \varphi_s - \varphi_s\| < \infty \tag{6}$$

for some $\varphi_s \in \mathcal{U}_m$ and that the following fixed-point property holds for $i = 1, \dots, m$:

($\mathfrak{L}_i$) *For every $\psi, v \in \mathcal{U}$ such that $\delta := \|\mathcal{L}_i^v \psi - \psi\| < \infty$, there is a fixed point $\phi \in \mathcal{U}$ of $\mathcal{L}_i^v$ such that $\|\psi - \phi\| \le \xi_i(\delta)$.*

Then there exists a fixed point $\varphi \in \mathcal{U}_m$ of $\mathcal{P}_m^F$ such that

$$\|\varphi_s - \varphi\| \le \xi_m \circ \cdots \circ \xi_1(\delta). \tag{7}$$

Moreover, if $\mathcal{L}(\mathcal{U}) \subset \mathcal{U}$ and there is an $L \in \mathbb{R}_+$ with $|r_i| > L$ for $i = 1, \dots, m$ and $\|\mathcal{L} f\| \le L\|f\|$ for $f \in \mathcal{U}$, then there is exactly one fixed point $\varphi \in \mathcal{U}$ of $\mathcal{P}_m^F$ with

$$\|\varphi_s - \varphi\| < \infty. \tag{8}$$

Remark 2.2. From the proof of Theorem 2.1 (see Section 5), φ is equal to ϕ_m, with ϕ_m obtained, step by step, by the following procedure.

Write $\phi_0 = -F$, $\psi_m = \varphi_s$, and $\psi_j(z) = \mathcal{L}\psi_{j+1} - r_{j+1}\psi_{j+1}$ for $j = 1, \dots, m-1$. Then, for $i = 1, \dots, m$, $\phi_i \in \mathcal{U}$ is a fixed point of the operator $\mathcal{L} + (1-r_i)\mathcal{I} - \phi_{i-1}$ with $\|\psi_i - \phi_i\| \le \xi_i \circ \cdots \circ \xi_1(\delta)$. By ($\mathfrak{L}_i$), such a $\phi_i \in \mathcal{U}$ exists for each $i \in \{1, \dots, m\}$. In many cases such a ϕ_i can be described quite precisely (see, e.g., Remark 2.4).

Concerning operators satisfying ($\mathfrak{L}_i$), some recent results can be found in, e.g., [Brzdęk and Jung 2011, Theorem 5.1; Badora and Brzdęk 2012, Theorem 2.1].

In what follows, we say that $\mathcal{U}$ is *closed in the norm* $\|\cdot\|_\infty$ if $\mathcal{U}$ contains every function $f \in X^S$ for which there is a sequence of functions (f_n) in $\mathcal{U}$ that is uniformly convergent to f (i.e., $\lim_{n\to\infty}\|f - f_n\|_\infty = 0$).

Theorem 2.3. *Let $\mathcal{L}(\mathcal{U}) \subset \mathcal{U}$ and let $\mathcal{U}$ be closed in the norm $\|\cdot\|_\infty$. Suppose that there are $\kappa \in \mathbb{R}_+$ and $\varphi_s \in \mathcal{U}$ such that (6) holds, that*

$$\|\mathcal{L} f\| \le \kappa\|f\|, \quad f \in \mathcal{U}, \tag{9}$$

and that one of the following two conditions is valid:

(α) *$r_i \in \mathbb{K}$ and $|r_i| > \kappa$ for $i = 1, \dots, m$;*

(β) *$|r_i| > 2\kappa$ for $i = 1, \dots, m$.*

Then there is a unique fixed point $\varphi \in \mathcal{U}$ of $\mathcal{P}_m^F$ such that $\|\varphi_s - \varphi\| < \infty$; moreover,

$$\|\varphi_s - \varphi\| \le \frac{\delta}{(|r_1| - \rho\kappa)\cdots(|r_m| - \rho\kappa)}, \tag{10}$$

where

$$\rho := \begin{cases} 1 & \textit{if } (\alpha) \textit{ holds}, \\ 2 & \textit{if } (\beta) \textit{ holds}. \end{cases}$$

Remark 2.4. From the proof of Theorem 2.3 (see Section 6) and Remark 2.2, we can deduce that the function φ can be described analogously to Remark 2.2, with the functions ϕ_i (denoted in $(\mathfrak{L}_i)$ by ϕ) given by $\phi_i(x) := \lim_{n\to\infty} \mathcal{T}_i{}^n \psi_i(x)$ for $i = 1, \dots, m$, $x \in S$, where $\mathcal{T}_i$ is defined by (32).

3. Hyers–Ulam stability

Let $b_0, \dots, b_m \in \mathbb{K}$, with $b_m \neq 0$, let $Q_m : \mathbb{C} \to \mathbb{C}$ be given by $Q_m(z) := \sum_{j=0}^{m} b_j z^j$, and let $q_1, \dots, q_m \in \mathbb{C}$ be the roots of the equation

$$Q_m(z) = 0. \tag{11}$$

We define a linear operator $Q_m(\mathcal{L}) : \mathcal{U}_m \to X^S$ by $Q_m(\mathcal{L}) := \sum_{j=0}^{m} b_j \mathcal{L}^j$. In this section, we describe some direct consequences of Theorems 2.1 and 2.3 concerning the Hyers–Ulam stability of the operator equation

$$Q_m(\mathcal{L})\varphi = G \tag{12}$$

(for functions $\varphi \in \mathcal{U}_m$ and with a fixed $G \in X^S$), under the assumption

($\mathcal{G}$) $G \in \mathcal{U}$ or (12) has a solution $\hat{\varphi} \in \mathcal{U}_m$.

Let us mention that Hyers–Ulam stability is related to the notions of shadowing and controlled chaos (see, e.g., [Pilyugin 1999; Palmer 2000; Hayes and Jackson 2005; Stević 2008]) as well as the theories of perturbation (see, e.g., [Chang and Howes 1984; Lin and Zhou 1995]) and optimization. At the moment it is a very popular subject of investigation (for more details, references and examples of some recent results, see, e.g., [Hyers 1941; Ulam 1964; Forti 1995; 2007; Hyers et al. 1998; Jung 2001; 2011; Agarwal et al. 2003; Popa 2005a; Jabłoński and Reich 2006; Bahyrycz 2007; Jung and Rassias 2007; 2008; Moszner 2009; Paneah 2009; Ciepliński 2010; 2011; 2012b; Sikorska 2010; Forti and Sikorska 2011; Piszczek 2013a; 2013b]).

Under suitable assumptions, we have the following natural examples of (12):

- The linear differential equation

$$b_m\varphi^{(m)}(z) + b_{m-1}\varphi^{(m-1)}(z) + \cdots + b_1\varphi'(z) + b_0\varphi(z) = G(z). \tag{13}$$

- The linear recurrence (or difference) equation

$$b_m\varphi(n+m) + b_{m-1}\varphi(n+m-1) + \cdots + b_1\varphi(n+1) + b_0\varphi(n) = G(n). \tag{14}$$

- The well-known linear functional equation

$$b_m\varphi(f^m(z)) + b_{m-1}\varphi(f^{m-1}(z)) + \cdots + b_1\varphi(f(z)) + b_0\varphi(z) = G(z). \tag{15}$$

For results on the Hyers–Ulam stability of (13) see [Miura et al. 2003b] (with $G(z) \equiv 0$). Equation (14) is a discrete case of (15); its Hyers–Ulam stability was discussed in [Popa 2005a; 2005b; Brzdęk et al. 2006; 2010]. Equation (15) is one of the most important functional equations, and many results on its solutions can be found in [Kuczma 1968; Kuczma et al. 1990] (see also the references therein); its Hyers–Ulam stability was discussed, e.g., in [Kim 2000; Trif 2002] for $m = 1$ and in [Trif 2006; Brzdęk et al. 2008; 2011b; Brzdęk and Jung 2010] for $m > 1$.

The fixed-point approach has been already applied in the investigation of the Hyers–Ulam stability [Baker 1991; Jung and Chang 2005; Jung and Kim 2006; Mirzavaziri and Moslehian 2006; Jung 2007; Brzdęk et al. 2011a; Brzdęk and Ciepliński 2011; Ciepliński 2012a]. In this section we continue this direction and present two corollaries on such stability, obtained from Theorems 2.1 and 2.3. The first one corresponds to the results in [Brzdęk et al. 2008]. Namely, it states that, in some cases, the Hyers–Ulam stability of (12) can be derived from the analogous properties of the corresponding first-order operator equations, which we express in the form of the following hypothesis:

($\mathcal{H}_i$) $\rho_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that, for every $\varphi_s, \eta \in \mathcal{U}$ and $\delta \in \mathbb{R}_+$ with $\|\mathcal{L}\varphi_s - q_i\varphi_s - \eta\| \le \delta$, there is $\varphi \in \mathcal{U}$ such that $\|\varphi_s - \varphi\| \le \rho_i(\delta)$ and

$$\mathcal{L}\varphi = q_i\varphi + \eta. \tag{16}$$

For examples of operators satisfying ($\mathcal{H}_i$) see [Brzdęk et al. 2010; 2011b] and Section 4.

Corollary 3.1. *Suppose that $G \in X^S$, ($\mathcal{G}$) and ($\mathcal{H}_i$) hold for $i = 1, \dots, m$, $\delta \in \mathbb{R}_+$, and $\varphi_s \in \mathcal{U}_m$ satisfies*

$$\|Q_m(\mathcal{L})\varphi_s - G\| \le \delta. \tag{17}$$

Then there exists a solution $\varphi \in \mathcal{U}_m$ of (12) such that

$$\|\varphi_s - \varphi\| \le \rho_m \circ \cdots \circ \rho_1\left(\frac{\delta}{|b_m|}\right). \tag{18}$$

Moreover, if $\mathcal{L}(\mathcal{U}) \subset \mathcal{U}$, and there is $L \in \mathbb{R}_+$ with $\|\mathcal{L}f - \mathcal{L}g\| \le L\|f - g\|$ for $f, g \in \mathcal{U}$ and $|q_i| > L$ for $i = 1, \dots, m$, then there is exactly one solution $\varphi \in \mathcal{U}$ of (12) with

$$\|\varphi_s - \varphi\| < \infty. \tag{19}$$

Proof. Assume first that $G \in \mathcal{U}$. Let

$$F = -\frac{1}{b_m}G, \quad a_0 = \frac{b_0}{b_m} + 1, \quad \text{and} \quad a_i = \frac{b_i}{b_m} \quad \text{for } i = 1, \dots, m-1.$$

Then (17) implies that $\|P_m^F(\mathcal{L})\varphi_s - \varphi_s\| \le \delta/|b_m|$. Further, it is easily seen that $q_1, \dots, q_m$ are the roots of (4) and $(\mathcal{H}_i)$ yields $(\mathfrak{L}_i)$ for $i = 1, \dots, m$ with $\xi_i = \rho_i$. So, by Theorem 2.1, there is a fixed point $\varphi \in \mathcal{U}_m$ of $P_m^F(\mathcal{L})$ such that (18) holds. Clearly φ is a solution to (12).

Now consider the case where (12) has a solution $\hat{\varphi} \in \mathcal{U}_m$. Let $\zeta_s := \varphi_s - \hat{\varphi}$. Then $\zeta_s \in \mathcal{U}_m$ and $\|Q_m(\mathcal{L})\zeta_s - G_0\| = \|Q_m(\mathcal{L})\zeta_s\| \le \delta$, where $G_0 \in X^S$ and $G_0(x) \equiv 0$. Clearly $G_0 \in \mathcal{U}$. Hence, by the first part of the proof, there exists a solution $\zeta \in \mathcal{U}_m$ of (12) (with $G = G_0$) such that

$$\|\zeta_s - \zeta\| \le \rho_m \circ \cdots \circ \rho_1\left(\frac{\delta}{|b_m|}\right).$$

Now, it is easily seen that $\varphi := \zeta + \hat{\varphi}$ is a solution of (12), and (18) holds.

It remains to show the statement concerning the uniqueness of φ. So, suppose that there is an $L \in \mathbb{R}_+$ such that $\|\mathcal{L}f\| \le L\|f\|$ for $f \in \mathcal{U}$ and $|q_i| > L$ for $i = 1, \dots, m$. Then such a φ is the unique fixed point of $\mathcal{P}_m^F$ satisfying (19), and therefore it is also the unique solution of (12) such that (19) is valid. □

It is easily seen that [Brzdęk et al. 2008, Theorem 1] is a particular case of our Corollary 3.1.

Remark 3.2. In the case where $|q_i| < L$ for some $i \in \{1, \dots, m\}$, it follows from [Brzdęk et al. 2010, Theorem 3(c)] that in the general situation we may not have uniqueness of φ in Corollary 3.1.

Corollary 3.3. *Let $\mathcal{L}(\mathcal{U}) \subset \mathcal{U}$, $G \in X^S$, $(\mathcal{G})$ be valid, $\mathcal{U}$ be closed in the supremum norm $\|\cdot\|_\infty$, $\delta, \kappa \in \mathbb{R}_+$, $\varphi_s \in \mathcal{U}$, (17) hold, and*

$$\|\mathcal{L}f - \mathcal{L}g\| \le \kappa\|f - g\|, \quad f, g \in \mathcal{U}. \tag{20}$$

Assume that one of the following two conditions is valid:

(α) *$q_i \in \mathbb{K}$ and $|q_i| > \kappa$ for $i = 1, \dots, m$;*

(β) *$|q_i| > 2\kappa$ for $i = 1, \dots, m$.*

Then there is a unique solution $\varphi \in \mathcal{U}$ of (12) with $\|\varphi_s - \varphi\| < \infty$; moreover,

$$\|\varphi_s - \varphi\| \le \frac{\delta}{|b_m|(|q_1| - \rho\kappa)\cdots(|q_m| - \rho\kappa)}, \tag{21}$$

where

$$\rho := \begin{cases} 1 & \text{if } (\alpha) \text{ holds}, \\ 2 & \text{if } (\beta) \text{ holds}. \end{cases}$$

Proof. Arguing analogously as in the proof of Corollary 3.1, we deduce the statement from Theorem 2.3. □

If $\sigma : S \to \mathbb{K}$ is bounded, $f : S \to S$, and $\mathscr{L}g := \sigma g \circ f$ for $g \in \mathcal{U}$, then (20) holds with $\kappa := \sup_{t\in S} |\sigma(t)|$. So, Corollary 3.3 yields the following result, which complements (and generalizes to a certain extent) some results in [Trif 2006; Brzdęk et al. 2008; 2011b]:

Corollary 3.4. *Let one of the conditions* (α), (β) *of* Corollary 3.3 *hold, and let* $\sigma : S \to \mathbb{K}$, $G \in X^S$, $f : S \to S$, $\delta \in \mathbb{R}_+$, $\varphi_s : S \to X$, $\kappa := \sup_{t\in S} |\sigma(t)| < |q_j|$ *for* $j = 1, \dots, m$, *and*

$$\sup_{t\in S} \left\| \sum_{j=0}^{m} b_j \sigma_j(t) \varphi_s(f^j(t)) - G(t) \right\| \le \delta, \tag{22}$$

where $\sigma_0(t) = 1$ *and* $\sigma_j(t) = \sigma_{j-1}(t)\sigma(f^{j-1}(t))$ *for* $t \in S$, $j = 1, \dots, m$. *Then there is a unique solution* $\varphi : S \to X$ *of the functional equation*

$$\sum_{j=0}^{m} b_j \sigma_j(t) \varphi(f^j(t)) = G(t) \tag{23}$$

such that (21) *holds. Moreover, if* S *is endowed with a topology and* $\sigma_1, \dots, \sigma_m$ *and* f *are continuous, then the following two statements are valid*:

(i) *If* φ_s *and* G *are continuous, then* φ *is continuous.*

(ii) *If* $X = \mathbb{K}$ *and* φ_s *and* G *are Borel measurable, then* φ *is Borel measurable.*

Proof. It is enough to take $\mathscr{L}\xi = \sigma\, \xi \circ f$ for $\xi \in X^S$ in Corollary 3.3. Moreover, if S is endowed with a topology and $\sigma_1, \dots, \sigma_m$ and f are continuous, then taking $\mathcal{U} := \{\xi \in X^S : \xi$ is continuous$\}$ or $\mathcal{U} := \{\xi \in X^S : \xi$ is Borel measurable$\}$ we obtain that φ is continuous or Borel measurable, respectively. □

Remark 3.5. The form of σ_j seems to be a bit complicated for greater m, but for instance with $m = 2$, (23) has the simple and quite general form

$$b_2\sigma(t)\sigma(f(t))\varphi(f^2(t)) + b_1\sigma(t)\varphi(f(t)) + b_0\varphi(t) = G(t).$$

4. Some further consequences

Let I be an open real interval, let $C^1(I, X)$ denote the space of strongly differentiable functions mapping I into X, and let $\mathcal{U} = C^1(I, X)$ and $\mathscr{L} = d/dt$. In the next remark we show that, in view of [Miura et al. 2004, Remark 1, Corollaries 2, 3] (see also [Takahasi et al. 2002]), hypothesis ($\mathcal{H}_i$) holds for each $i \in \{1, \dots, m\}$ such that $\Re q_i \neq 0$, where $\Re z$ denotes the real part of the complex number z for $\mathbb{K} = \mathbb{C}$ and $\Re z = z$ for $\mathbb{K} = \mathbb{R}$, with

$$\rho_i(\delta) := \frac{\delta}{|\Re q_i|}, \quad \delta \in \mathbb{R}_+. \tag{24}$$

Moreover, in the case where I is finite and $\Re q_i = 0$, $(\mathcal{H}_i)$ holds for each $i \in \{1, \dots, m\}$, with

$$\rho_i(\delta) := d(I)\delta, \quad \delta \in \mathbb{R}_+, \tag{25}$$

where $d(I)$ denotes the diameter of I; see [Miura et al. 2004, Remark 1, Corollary 4].

Remark 4.1. For $i \in \{1, \dots, m\}$ and $\delta \in \mathbb{R}_+$, we write

$$\rho_i(\delta) = \begin{cases} d(I)\delta & \text{if } \Re q_i = 0 \text{ and } d(I) < \infty, \\ \delta/|\Re q_i| & \text{if } \Re q_i \neq 0. \end{cases} \tag{26}$$

Let $\mathcal{L} = d/dt$ and $\eta \in \mathcal{U}$. Take $\varphi_s \in \mathcal{U}$, $i \in \{1, \dots, m\}$, and $\delta \in \mathbb{R}_+$ with $\|\mathcal{L}\varphi_s - q_i\varphi_s - \eta\| \le \delta$. There is a solution $\varphi_0 \in \mathcal{U}$ of the equation $\mathcal{L}\varphi_0 = q_i\varphi_0 + \eta$. Write $\varphi_1 = \varphi_s - \varphi_0$. Then $\|\mathcal{L}\varphi_1 - q_i\varphi_1\| = \|\mathcal{L}\varphi_s - q_i\varphi_s - \eta\| \le \delta$. Hence, according to the results in [Miura et al. 2004], there is $\hat{\varphi} \in \mathcal{U}$ such that $\|\varphi_1 - \hat{\varphi}\| \le \rho_i(\delta)$ and $\mathcal{L}\hat{\varphi} = q_i\hat{\varphi}$. Now, it is easily seen that $\varphi := \hat{\varphi} + \varphi_0$ satisfies (16) and that $\|\varphi_s - \varphi\| = \|\varphi_1 - \hat{\varphi}\| \le \rho_i(\delta)$.

If $d(I) = \infty$, then $(\mathcal{H}_i)$ may not be valid for $i \in \{1, \dots, m\}$ with $\Re q_i = 0$ (see, e.g., [Takahasi et al. 2002, Theorem 2.1(iii)]).

In view of Remark 4.1, from Theorem 2.1 we can deduce the following corollary, which generalizes the main result obtained in [Miura et al. 2003b], although it was proved in that paper by a different method.

Corollary 4.2. *Let I be an open real interval, and let $G \in C^0(I, X)$, $q_i \in \mathbb{K}$ for $i = 1, \dots, m$, $\delta \in \mathbb{R}_+$ and $\varphi_s \in C^m(I, X)$ satisfy*

$$\|b_m\varphi_s^{(m)} + \cdots + b_1\varphi_s' + b_0\varphi_s - G\| \le \delta. \tag{27}$$

Suppose that $d(I) < \infty$ or $\Re q_i \neq 0$ for $i = 1, \dots, m$. Then there exists a solution $\varphi \in C^m(I, X)$ of (13) such that

$$\|\varphi_s - \varphi\| \le \frac{\delta}{|b_m|} \prod_{i=1}^{m} D_i, \tag{28}$$

where

$$D_i = \begin{cases} d(I) & \text{if } \Re q_i = 0 \text{ and } d(I) < \infty, \\ 1/|\Re q_i| & \text{if } \Re q_i \neq 0. \end{cases} \tag{29}$$

Proof. There is a function $\varphi_0 \in C^m(I, X)$ satisfying the equation

$$b_m \varphi_0^{(m)}(x) + b_{m-1}\varphi_0^{(m-1)}(x) + \cdots + b_1 \varphi_0'(x) + b_0 \varphi_0(x) = G(x).$$

Let $\zeta_s := \varphi_s - \varphi_0$. Then, by (27),

$$\|b_m \zeta_s^{(m)} + b_{m-1}\zeta_s^{(m-1)} + \cdots + b_1 \zeta_s' + b_0 \zeta_s\| \le \delta,$$

i.e., (17) is valid with $G(x) \equiv 0$. As we have already observed in Remark 4.1, hypothesis $(\mathscr{H}_i)$ holds for $i = 1, \ldots, m$ with $\mathscr{U} = C^1(I, X)$, $\mathscr{L} = d/dt$ and ρ_i given by (26). So, by Corollary 3.1, there is a solution $\zeta \in C^m(I, X)$ of the equation

$$b_m \zeta^{(m)}(x) + b_{m-1}\zeta^{(m-1)}(x) + \cdots + b_1 \zeta'(x) + b_0 \zeta(x) = 0$$

such that

$$\|\zeta_s - \zeta\| \le \frac{\delta}{|b_m|} \prod_{i=1}^{m} D_i,$$

where D_i is given by (29). Now, it is easily seen that $\varphi := \zeta + \varphi_0$ is a solution of (13), and (28) holds. □

Similar results for integral equations can be derived from [Miura et al. 2012] in analogous ways. It seems that so far that no paper has been published containing stability results (of the type discussed in this paper) for linear integral equations of higher orders.

5. Proof of Theorem 2.1

This proof proceeds via induction with respect to m. The case $m = 1$ is a consequence of $(\mathfrak{L}_1)$ with $v = -F$. Assume that the theorem is true for $m = k$. Let $\varphi_s \in \mathscr{U}_m$ satisfy (6) with $m = k + 1$, which in view of the Viète formulas can be written in the form

$$\delta = \|\mathscr{P}_{k+1}^F \varphi_s - \varphi_s\| = \left\| \mathscr{L}^{k+1}\varphi_s + \sum_{j=0}^{k} a_j \mathscr{L}^j \varphi_s + F - \varphi_s \right\|$$
$$= \left\| \mathscr{L}^{k+1}\varphi_s + (-1)\left(\sum_{j=1}^{k+1} r_j\right)\mathscr{L}^k \varphi_s + \cdots + (-1)^{k+1} r_1 \cdot \cdots \cdot r_{k+1}\varphi_s + F \right\|.$$

Let $\psi_s := \mathscr{L}\varphi_s - r_{k+1}\varphi_s$. Since $\mathscr{L} : \mathscr{U} \to X^S$ is a linear operator, we have

$$\mathscr{L}^p \psi_s = \mathscr{L}^{p+1}\varphi_s - r_{k+1}\mathscr{L}^p \varphi_s$$

for $p = 1, \dots, k$, whence

$$\begin{aligned}
&\left\| \mathscr{L}^k \psi_s + (-1)\Big(\sum_{j=1}^{k} r_j\Big)\mathscr{L}^{k-1}\psi_s + \cdots + [(-1)^k r_1 \cdots r_k + 1]\psi_s + F - \psi_s \right\| \\
&\quad = \left\| \mathscr{L}^k \psi_s + (-1)\Big(\sum_{j=1}^{k} r_j\Big)\mathscr{L}^{k-1}\psi_s + \cdots + (-1)^k r_1 \cdots r_k \psi_s + F \right\| \\
&\quad = \bigg\| \mathscr{L}^{k+1}\varphi_s - r_{k+1}\mathscr{L}^k\varphi_s + (-1)\Big(\sum_{j=1}^{k} r_j\Big)(\mathscr{L}^k\varphi_s - r_{k+1}\mathscr{L}^{k-1}\varphi_s) \\
&\qquad\qquad + \cdots + (-1)^k r_1 \cdots r_k(\mathscr{L}\varphi_s - r_{k+1}\varphi_s) + F \bigg\| \\
&\quad = \left\| \mathscr{L}^{k+1}\varphi_s + (-1)\Big(\sum_{j=1}^{k+1} r_j\Big)\mathscr{L}^k\varphi_s + \cdots + (-1)^{k+1} r_1 \cdots r_{k+1}\varphi_s + F \right\| \le \delta.
\end{aligned}$$

Since, by the Viète formulas, $r_1, \dots, r_k$ are the roots of the equation

$$\begin{aligned}
1 &= (z - r_1)(z - r_2)\cdots(z - r_k) + 1 \\
&= z^k + (-1)\Big(\sum_{j=1}^{k} r_j\Big)z^{k-1} + \cdots + [(-1)^k r_1 \cdots r_k + 1]z^0,
\end{aligned}$$

by the inductive assumption, there is $\psi \in \mathscr{U}_k$ such that

$$\psi = \mathscr{L}^k\psi + (-1)\Big(\sum_{j=1}^{k} r_j\Big)\mathscr{L}^{k-1}\psi + \cdots + [(-1)^k r_1 \cdots r_k + 1]\psi + F \tag{30}$$

and $\|\mathscr{L}^{\psi}_{k+1}\varphi_s - \varphi_s\| = \|\mathscr{L}\varphi_s - r_{k+1}\varphi_s - \psi\| = \|\psi_s - \psi\| \le \xi_k \circ \cdots \circ \xi_1(\delta)$. Hence, in view of $(\mathfrak{L}_{k+1})$, there is a fixed point $\varphi \in \mathscr{U}$ of $\mathscr{L}^{\psi}_{k+1}$ with

$$\|\varphi_s - \varphi\| \le \xi_{k+1}(\xi_k \circ \cdots \circ \xi_1(\delta)).$$

Note that $\mathscr{L}\varphi = \psi + r_{k+1}\varphi$, whence $\mathscr{L}\varphi \in \mathscr{U}$, which means that $\varphi \in \mathscr{U}_2$. Analogously, step by step, finally we get $\varphi \in \mathscr{U}_{k+1}$. Consequently, (30) yields

$$\begin{aligned}
0 &= \mathscr{L}^{k+1}\varphi - r_{k+1}\mathscr{L}^k\varphi + (-1)\Big(\sum_{j=1}^{k} r_j\Big)(\mathscr{L}^k\varphi - r_{k+1}\mathscr{L}^{k-1}\varphi) \\
&\qquad\qquad + \cdots + (-1)^k r_1 \cdots r_k(\mathscr{L}\varphi - r_{k+1}\varphi) + F \\
&= \mathscr{L}^{k+1}\varphi + (-1)\Big(\sum_{j=1}^{k+1} r_j\Big)\mathscr{L}^k\varphi + \cdots + (-1)^{k+1} r_1 \cdots r_{k+1}\varphi + F \\
&= P_m(\mathscr{L})\varphi + F - \varphi.
\end{aligned}$$

It remains to prove the statement of the uniqueness of φ. Notice that if $\varphi_1, \varphi_2 \in \mathcal{U}$ are both fixed points of $\mathcal{P}_m^F$ with $\|\varphi_s - \varphi_i\| < \infty$ for $i = 1, 2$, then $\|\varphi_1 - \varphi_2\| < \infty$. So, it suffices to prove that $\varphi_1 = \varphi_2$ if $\varphi_1, \varphi_2 \in \mathcal{U}$ are fixed points of $\mathcal{P}_m^F$ such that

$$M := \|\varphi_1 - \varphi_2\| < \infty. \tag{31}$$

The proof of uniqueness proceeds via induction with respect to m. For $m = 1$ we have $r_1 = 1 - a_0$ and, therefore, for arbitrary fixed points $\varphi_1, \varphi_2 \in \mathcal{U}$ of $\mathcal{P}_m^F$ satisfying (31), we have $|r_1|^n \|\varphi_1 - \varphi_2\| = \|\mathcal{L}^n \varphi_1 - \mathcal{L}^n \varphi_2\| \le L^n M$ for $n \in \mathbb{N}$, whence $\varphi_1 = \varphi_2$, because $|r_1| > L$. We further assume that the fact is true for $m = k$. Consider fixed points $\varphi_1, \varphi_2 \in \mathcal{U}$ of $\mathcal{P}_{k+1}^F$ satisfying (31), and write $\phi_i := \mathcal{L}\varphi_i - r_{k+1}\varphi_i$ for $i = 1, 2$. Then, arguing analogously as before for ψ_s, we see that $\phi_1, \phi_2 \in \mathcal{U}$ are fixed points of $\mathcal{P}_m^F$ with $m = k$ and appropriate (possibly different) $a_0, \dots, a_{k-1}$. Moreover, $\|\phi_1 - \phi_2\| \le (L + |r_{k+1}|)M$. Hence, according to the inductive assumption, $\phi_1 = \phi_2$ and, analogously to the case $m = 1$, finally we obtain that $\varphi_1 = \varphi_2$. □

6. Proof of Theorem 2.3

First, consider the case of (α). In view of Theorem 2.1, it is enough to show that $(\mathfrak{L}_i)$ holds for $i = 1, \dots, m$. Fix $i \in \{1, \dots, m\}$, $v \in \mathcal{U}$ and $\psi \in \mathcal{U}$ and assume that $\delta_0 := \|\mathcal{L}_i^v \psi - \psi\| < \infty$. Write

$$\mathcal{T}_i := \frac{1}{r_i}(\mathcal{L} - v). \tag{32}$$

In view of (5), $\|\mathcal{T}_i \psi - \psi\| \le \delta_0 / |r_i|$, and, for every $f, g \in \mathcal{U}$,

$$\|\mathcal{T}_i f - \mathcal{T}_i g\| = \left\| \frac{1}{r_i}\mathcal{L} f - \frac{1}{r_i}\mathcal{L} g \right\| \le \frac{\kappa}{|r_i|}\|f - g\|.$$

Define a generalized metric (i.e., admitting an infinite value) d in X^S by $d(f, g) = \|f - g\|$ for $f, g \in X^S$ (see [Luxemburg 1958]). Applying the fixed-point alternative of J. B. Diaz and B. Margolis [1968, p. 306–307], we see that (for the generalized metric d) the limit $\phi := \lim_{n\to\infty} {\mathcal{T}_i}^n \psi$ exists in X^S and ϕ is the unique fixed point of $\mathcal{T}_i$ with

$$\|\psi - \phi\| \le \frac{\delta_0}{|r_i|} \frac{1}{1 - \kappa/|r_i|} = \frac{\delta_0}{|r_i| - \kappa}.$$

Since the sequence $({\mathcal{T}_i}^n \psi)$ converges to ϕ uniformly and $\mathcal{U}$ is closed in the norm $\|\cdot\|_\infty$, ϕ belongs to $\mathcal{U}$. Next, $\mathcal{L}_i^v \phi = \mathcal{L}\phi + (1 - r_i)\phi - v = r_i \mathcal{T}_i \phi + v + (1 - r_i)\phi - v = \phi$, implying that $(\mathfrak{L}_i)$ is valid, which completes the proof in the case of (α).

Now, consider the case when (β) is valid and $\mathbb{K} = \mathbb{R}$. As is well-known (see, e.g., [Fabian et al. 2001, p. 39; Ferrera and Muñoz 2003] or [Kadison and Ringrose 1997, p. 66, Exercise 1.9.6]), X^2 is a complex Banach space with linear structure defined by the operations $(x, y) + (z, w) := (x + z, y + w)$ and

$(\alpha + i\beta)(x, y) := (\alpha x - \beta y, \beta x + \alpha y)$ and the Taylor norm $\|\cdot\|_T$ given by $\|(x, y)\|_T := \sup_{0\le\theta\le 2\pi} \|(\cos\theta)x + (\sin\theta)y\|$ for $x, y, z, w \in X$, $\alpha, \beta \in \mathbb{R}$. Clearly,

$$\max\{\|x\|, \|y\|\} \le \|(x, y)\|_T \le \|x\| + \|y\|, \quad x, y \in X. \tag{33}$$

We write $p_i(w_1, w_2) := w_i$ for $w_1, w_2 \in X$, $i = 1, 2$ and $\|\mu\|_T := \sup_{x\in S} \|\mu(x)\|_T$ for $\mu \in (X^2)^S$. Let $\mathcal{U}_0 := \{\mu : S \to X^2 : p_i \circ \mu \in \mathcal{U}, i \in \{1, 2\}\}$ and $\mathcal{L}_0\mu(x) := \big(\mathcal{L}(p_1 \circ \mu)(x), \mathcal{L}(p_2 \circ \mu)(x)\big)$ for $\mu \in \mathcal{U}_0$, $x \in S$. Since $\mathcal{L}$ is linear and $\mathcal{U}$ is a linear subspace of X over $\mathbb{K} = \mathbb{R}$, we see that $\mathcal{U}_0$ is a linear subspace of X^2 over $\mathbb{C}$ and $\mathcal{L}_0$ is a linear operator (also over $\mathbb{C}$) such that $\mathcal{L}_0(\mathcal{U}_0) \subset \mathcal{U}_0$.

Choose $\mu \in (X^2)^S$ and consider a sequence (μ_n) in $\mathcal{U}_0$ which is uniformly convergent to μ (in the Taylor norm). Then, by (33),

$$\max\{\|p_1 \circ \mu_n - p_1 \circ \mu\|, \|p_2 \circ \mu_n - p_2 \circ \mu\|\} \le \|\mu_n - \mu\|_T, \quad n \in \mathbb{N},$$

which means that $p_i \circ \mu_n$ is uniformly convergent to $p_i \circ \mu$ for $i = 1, 2$. Consequently, $p_1 \circ \mu$, $p_2 \circ \mu \in \mathcal{U}$, whence $\mu \in \mathcal{U}_0$. Thus, $\mathcal{U}_0$ is closed in the supremum norm connected with the norm $\|\cdot\|_T$. Further, according to (9) and (33), we have

$$\begin{aligned}\|\mathcal{L}_0\mu\|_T &= \|(\mathcal{L}(p_1 \circ \mu), \mathcal{L}(p_2 \circ \mu))\|_T \le \|\mathcal{L}(p_1 \circ \mu)\| + \|\mathcal{L}(p_2 \circ \mu)\| \\ &\le \kappa\|p_1 \circ \mu\| + \kappa\|p_2 \circ \mu\| \le 2\kappa \max\{\|p_1 \circ \mu\|, \|p_2 \circ \mu\|\} \le 2\kappa\|\mu\|_T\end{aligned}$$

for every $\mu \in \mathcal{U}_0$. We write $\chi := (\varphi_s, 0)$ and $v_0 = (v, 0)$ for $v \in \mathcal{U}$. Then we have $\|\mathcal{L}_0\chi + (1 - r_i)\chi - v_0 - \chi\|_T = \|\mathcal{L}\varphi_s + (1 - r_i)\varphi_s - v - \varphi_s\| = \delta < \infty$, because $p_2 \circ \chi(x) = 0$ for $x \in S$. So, we have again the case of (α), where $\mathcal{L}$, κ, φ_s, F, $\mathcal{U}$, $\mathcal{P}_m^F$ are replaced with $\mathcal{L}_0$, 2κ, χ, $F_0 := (F, 0)$, $\mathcal{U}_0$ and $\widehat{\mathcal{P}_m^F} := P_m(\mathcal{L}_0) + F_0$, respectively. So, by the first part of the proof, there is a fixed point $H \in \mathcal{U}_0$ of $\widehat{\mathcal{P}_m^F}$ with

$$\|\chi - H\|_T \le \frac{\delta}{(|r_1| - 2\kappa)\cdots(|r_m| - 2\kappa)}. \tag{34}$$

Observe that $\varphi := p_1 \circ H$ is a fixed point of $\mathcal{P}_m^F$. Moreover, by (34), (10) holds with $\rho = 2$.

It remains to prove the statement of the uniqueness of φ. Let $\varphi_0 \in \mathcal{U}$ be a fixed point of $\mathcal{P}_m^F$ such that $\|\varphi_s - \varphi_0\| < \infty$. Write $H_0(x) := (\varphi_0(x), 0)$ for $x \in S$. Note that $H_0 \in \mathcal{U}_0$ is a fixed point of $\widehat{\mathcal{P}_m^F}$. Moreover, $\|\chi - H_0\|_T = \|\varphi_s - \varphi\| < \infty$. By Theorem 2.1 (with $L = 2\kappa$), we deduce that $H_0 = H$, whence $\varphi_0 = p_1 \circ H_0 = p_1 \circ H = \varphi$. □

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