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NICK EDELEN AND BRUCE SOLOMON

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As first noted by Korevaar, Kusner, and Solomon, constant mean curvature implies a homological conservation law for hypersurfaces in ambient spaces with Killing fields. We generalize that law by relaxing the topological restrictions assumed by Korevaar et al., and by allowing a weighted mean curvature functional. We also prove a partial converse, which roughly says that when flux is conserved along a Killing field, a hypersurface splits into two regions: one with constant (weighted) mean curvature, and one preserved by the Killing field. We demonstrate our theory by using it to derive a first integral for helicoidal surfaces of constant mean curvature in \mathbb{R}^3 , i.e., “twizzlers”.

1. Introduction

Constant mean curvature (“CMC”) imposes a homological flux conservation law on hypersurfaces in ambient spaces with nontrivial Killing fields. This was first observed and exploited by Korevaar, Kusner, and Solomon [1989] in their paper on the structure of embedded CMC surfaces in \mathbb{R}^3 (see [Kusner 1991] for an alternative exposition). In [Theorem 3.5](#), we generalize that law by relaxing the topological restrictions assumed by Korevaar et al. [1989], and by allowing a weighted version of the mean curvature functional. We further extend the theory via [Theorem 4.1](#), which gives a partial converse to the conservation law. Roughly, it states that when the appropriate flux is conserved along Killing fields, the hypersurface splits into two regions (though either may be empty): a region with constant (weighted) mean curvature, and a region preserved by the Killing fields.

We apply our results in [Case study 4.5](#) by using them to quickly derive the seemingly *ad hoc* first integral that Perdomo [2012], do Carmo and Dajczer [1982], and others have used to analyze the moduli space of CMC surfaces with helicoidal symmetry, also known as *twizzlers*¹. In general, constancy of weighted mean

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¹Twizzlers have also been studied by Wunderlich [1952] and, more recently, Halldorsson [2013].

curvature is characterized by a nonlinear second-order PDE, and its Noetherian reduction to a first-order condition makes it easier to analyze.

When a CMC hypersurface Σ in a manifold N is preserved by the action of a continuous isometry group \mathcal{G} , one can project it into the orbit space N/\mathcal{G} . The projected hypersurface Σ/\mathcal{G} will then be stationary for the *weighted* functional introduced in Section 3.2. We analyze the weighted functional and the resulting weighted mean-curvature invariant with an eye toward this fact. We suspect that virtually all we do here could be developed in a more general, stratified context encompassing both Riemannian manifolds and their quotients under smooth group actions.

We stick with smooth ambient manifolds here, but the orbit space viewpoint can be helpful, and Case study 4.5 could easily have been carried out in that setting. The approach we demonstrate there can also be adapted to spherical and hyperbolic space forms. The first author's report [Edelen 2011] sketches out one way to do that, but we describe the orbit-space approach to those examples in our final Remark 4.6.2.

2. Preliminaries

Let N denote an n -dimensional oriented Riemannian manifold, and consider a smooth, connected, oriented, properly immersed hypersurface $f : \Sigma^{n-1} \rightarrow N$. We will feel free to write Σ when we mean $f(\Sigma)$ or even $f : \Sigma \rightarrow N$, leaving context to clarify our intentions.

Let ν denote the unit normal that completes the orientation of Σ to that of N . The *mean curvature function* $h : \Sigma \rightarrow \mathbb{R}$ is the trace of the shape operator $\nabla\nu$. Notationally,

$$(2-1) \quad h = \operatorname{div}_{\Sigma}(\nu).$$

Here $\operatorname{div}_{\Sigma}(Y)$ denotes the *intrinsic divergence* of a vector field Y along Σ , that is, the trace of the endomorphism $T\Sigma \rightarrow T\Sigma$ gotten at each $p \in \Sigma$ by projecting the ambient covariant derivative ∇Y onto $T_p\Sigma$. One may compute $\operatorname{div}_{\Sigma}$ locally using any orthonormal basis $\{e_i\}$ for $T_p\Sigma$ via

$$\operatorname{div}_{\Sigma}(Y) := \sum_{i=1}^{n-1} \nabla_{e_i} Y \cdot e_i.$$

2.1. Chains and k -area. The homology of the sequence

$$N \rightarrow (N, \Sigma) \rightarrow \Sigma$$

will play a role below in a way that makes it problematic to work solely with smooth submanifolds. We therefore work with a class of piecewise smooth objects:

Definition 2.1.1. A *smooth r -chain* (or simply *chain*) in a smooth manifold M is a finite union of smoothly immersed oriented r -dimensional simplices. We regard a chain X as a formal homological sum

$$(2-2) \quad X = \sum_{i=1}^m m_i f_i.$$

Here each $f_i : \Delta \rightarrow M$ immerses the standard closed oriented r -simplex Δ (along with its boundary) smoothly into M . The m_i are (for us) always integers.

We denote the *support* of a chain X by $\|X\|$.

We write $\mathcal{S}_r(M)$ for the *group of smooth r -chains in M* , and ∂X for the homological *boundary* of a chain X , while $\mathcal{L}_i(M)$ and $\mathcal{B}_i(M)$ denote the spaces of i -dimensional *cycles* and *boundaries* (kernel and image of ∂) in M , respectively. Likewise, $\mathcal{L}_i(M, A)$ and $\mathcal{B}_i(M, A)$ indicate spaces of cycles and boundaries modulo a subset $A \subset M$.

Integration of an r -form ϕ on M over such a chain is trivial:

$$\int_X \phi := \sum_{i=1}^m m_i \int_{\Delta} f_i^* \phi,$$

where f_i^* denotes the usual pullback.

Given a Riemannian metric on M , one can also integrate *functions* over chains, and most importantly for our purposes, compute weighted volumes.

Definition 2.1.2. Let $\mu : M \rightarrow \mathbb{R}$ be any continuous function. Define the μ -*weighted r -volume* $|X|_{\mu}$ of the r -chain X in (2-2) as

$$|X|_{\mu} := \sup \left\{ \int_X e^{\mu} \phi : \phi \text{ is an } r\text{-form on } M \text{ with } \|\phi\|_{\infty} \leq 1 \right\}.$$

For a single immersed simplex, the usual Riemannian volume integral gives a simpler definition. To allow coincident, oppositely oriented simplices to cancel, however, we need the definition above.²

Finally, note that because Stokes' Theorem holds for immersed r -simplices, it holds for r -chains as well.

2.2. Symmetry. Our work here is vacuous unless the ambient space N has nontrivial Killing fields.

Write \mathcal{I} and $L(\mathcal{I})$ respectively for the isometry group of N and its Lie algebra. Identify $L(\mathcal{I})$ with the linear space of Killing fields on N in the usual way, associating each $Y \in L(\mathcal{I})$ with the Killing field (also called Y) we get by differentiating

²Definition 2.1.2 amounts to a weighted version of the *mass* of X as a *current*, in the sense of geometric measure theory [Federer 1969, p. 358].

the flow that sends $p \in N$ along the path $t \mapsto \exp(tY)p$. We write Y_p for the value of Y at p .

One often studies CMC hypersurfaces (like surfaces of revolution and twizzlers in \mathbb{R}^3) in relation to the action of a closed, connected subgroup $\mathcal{L} \subset \mathcal{F}$. Though it complicates our exposition to some extent, the presence of such a subgroup \mathcal{L} —like the density function e^μ —lets us broaden our theory. Even when $\mu \equiv 0$ and \mathcal{L} is the full isometry group of N , however, our results go beyond those of [Korevaar et al. 1989].

In Theorem 4.1 (a converse to our conservation law) we must consider the possibility that all Killing fields associated with \mathcal{L} lie tangent to an open subset S of our hypersurface $\Sigma \subset N$. The following lemma (and its corollary) then lets us deduce \mathcal{L} -invariance of S .

Lemma 2.3. *Suppose $S \subset N$ is a hypersurface, and that for some $Y \in L(\mathcal{L})$, we have $Y_p \in T_p S$ for every $p \in S$. Then for each $p \in S$, there exist a compact neighborhood $\mathbb{O}_p \subset S$ and an $\varepsilon > 0$ such that $e^{tY}q \in S$ whenever $|t| < \varepsilon$ and $q \in \mathbb{O}_p$.*

Proof. Since S is a submanifold, some open set $W \subset N$ contains S , but no point of $\bar{S} \setminus S$ (\bar{S} = closure of S). Let $\Theta : S \times \mathbb{R} \rightarrow N$ denote the flow of Y , so that $\Theta(q, t) := \exp(tY)q$. Then $\Theta^{-1}(W)$ is an open neighborhood of $S \times \{0\}$.

Now $\Theta(q, t)$ parametrizes the integral curve of Y with initial velocity Y_q . But $Y_p \in T_p S$ for all $p \in S$, and first-order ODE's have unique solutions, so this curve must stay in S for all $(q, t) \in W$. It follows that $\Theta^{-1}(W) = \Theta^{-1}(S)$.

For any compact neighborhood \mathbb{O}_p of $p \in S$, there now exists an $\varepsilon > 0$ such that

$$\mathbb{O}_p \times (-\varepsilon, \varepsilon) \subset \Theta^{-1}(S),$$

and the lemma consequently holds with this choice of \mathbb{O}_p and ε . □

2.4. Flux. Korevaar et al. [1989] showed that when a hypersurface $\Sigma \subset N^n$ has constant mean curvature $h \equiv H$ and the homology groups $H_{n-1}(N)$ and $H_{n-2}(N)$ are both trivial (over \mathbb{Z} —all homology groups in this paper have integer coefficients), there exists a *flux homomorphism*

$$\phi : H_{n-2}(\Sigma) \otimes L(\mathcal{F}) \rightarrow \mathbb{R}$$

defined by assigning, to any Killing field Y and any class $\mathbf{k} \in H_{n-2}(\Sigma)$, the *flux* $\phi(\mathbf{k}, Y)$ of Y across \mathbf{k} , where

$$(2-3) \quad \phi(\mathbf{k}, Y) := \int_{\Gamma} \eta \cdot Y + H \int_K \nu \cdot Y.$$

Here,

- Γ can be an $(n - 2)$ -cycle representing \mathbf{k} ;

- $K \subset N$ can be any $(n - 1)$ -chain bounded by Γ ;
- η is the orienting unit conormal to Γ in Σ ; and
- ν is the orienting unit normal to K in N .

To ensure that $\phi(\mathbf{k}, Y)$ is well-defined by (2-3), [Korevaar et al. 1989] makes two topological assumptions: namely, that $H_{n-1}(N)$ and $H_{n-2}(N)$ both vanish. The vanishing of $H_{n-2}(N)$ ensures Γ will bound *some* chain K , while that of $H_{n-1}(N)$ means any competing chain K' with $\partial K' = \partial K$ can be written $K' = K + \partial U$ for some n -chain U . Since Killing fields are divergence-free, the divergence theorem then makes the second integral in (2-3) independent of the choice of K .

Here, we extend this flux theory in [Korevaar et al. 1989] in several ways.

First, in Section 3.2, we broaden the mean curvature functional by allowing μ -weighted area and volume as in Definition 2.1.2. This is a minor tweak of the standard theory, but it does *not* correspond to a mere conformal change of metric, since n - and $(n - 1)$ -dimensional volumes scale differently under conformal change. We do this with a geometric application in mind: the μ -weighted theory relates the geometry of \mathcal{L} -invariant CMC hypersurfaces in N to that of hypersurfaces in the orbit space N/\mathcal{L} (see Remark 3.3.2 below).

Second, and more importantly, we eliminate the homological triviality assumptions mentioned above. Though we follow the same variational strategy as in [Korevaar et al. 1989], we show the flux invariant lives more naturally in a certain *relative* homology group. Instead of focusing the invariant on $(n - 2)$ -cycles in the surface Σ , we realize the flux as an invariant on certain $(n - 1)$ -dimensional relative cycles we shall call *caps*.

The homological restriction can be naively avoided by defining the flux on $H_{n-1}(N, \Sigma)$. When $H_{n-2}(\Sigma) \neq 0$, however, one gets a more sensitive invariant by designating a set of “reference cycles”. We call this set a *spine*. It not only gives better invariants, but it tends to make flux calculations more tractable.

The new viewpoint reproduces the invariant in [Korevaar et al. 1989] when $H_{n-1}(N) = H_{n-2}(N) = 0$. In that case, the reference cycle is trivial, and the long exact sequence for the pair (N, Σ) , namely

$$0 = H_{n-1}(N) \rightarrow H_{n-1}(N, \Sigma) \xrightarrow{\partial} H_{n-2}(\Sigma) \rightarrow H_{n-2}(N) = 0,$$

shows that $H_{n-1}(N, \Sigma) \cong H_{n-2}(\Sigma)$.

We derive our generalized conservation law in Section 3, and then, in Section 4, develop a partial converse. Before proceeding to these extensions, however, we present a motivating example that we can review later as an illustration of our theory.

Example 2.4.1. *Twizzlers* are “helical” CMC surfaces invariant under a 1-parameter group of screw motions in \mathbb{R}^3 . Any such surface can be gotten by applying

a screw motion to a curve γ in a plane perpendicular to the screw-axis. The resulting helicoidal surface will then have mean curvature $h \equiv H$ if and only if γ satisfies an easily derived second-order ODE. However, as others have noted [do Carmo and Dajczer 1982; Wunderlich 1952; Perdomo 2012; Halldorsson 2013], the second-order ODE has a useful first integral. We show how to derive it from flux conservation below.

The conservation law formulated in [Korevaar et al. 1989], however, yields nothing for twizzlers, since the typical CMC twizzler is generated by a nonperiodic curve γ in the transverse plane, and thus lacks homology. To remedy that, one can mod out the translational period of the helicoidal motion, realizing the twizzler as an immersion of a *cylinder* in $N := \mathbb{R}^2 \times \mathbf{S}^1$. Cylinders *do* have nontrivial loops, but those loops don't bound in N , and hence can't be capped off as required by [Korevaar et al. 1989].

Our approach evades that obstruction; see [Example 3.1.1](#) and [Case study 4.5](#).

3. Conservation

Like Korevaar et al. [1989], we derive flux conservation using a constrained first-variation formula. We make two notable modifications, however.

First, we weight both the areas of hypersurfaces and the volumes of domains by an \mathcal{L} -invariant *density function*

$$(3-1) \quad e^\mu : N \rightarrow (0, \infty).$$

Here μ can be any smooth function fixed by \mathcal{L} . The formula in [Korevaar et al. 1989] effectively takes $\mu \equiv 0$, as will become clear in [Section 3.2](#) below.

Secondly, we encode the homology of our immersion $f : \Sigma \rightarrow N$ into a set of reference cycles B . Let f_* denote the induced homomorphism

$$f_* : H_{n-2}(\Sigma) \rightarrow H_{n-2}(N).$$

Definition 3.0.1 (Spine). We call a subgroup $B \subset \mathcal{X}_{n-2}(N)$ a *spine* for the pair (N, Σ) if:

- (a) $B \cap \mathcal{X}_{n-2}(\Sigma) = 0$;
- (b) B generates $f_* H_{n-2}(\Sigma)$;
- (c) the composition $B \rightarrow \mathcal{X}_{n-2}(N) \rightarrow H_{n-2}(N)$ is injective.

We won't always draw an explicit distinction between the subgroup B and a set of generating cycles for B .

A nontrivial spine lets us assign fluxes to classes in $H_{n-2}(\Sigma)$ that don't bound in N .

Note that a spine for (N, Σ) always exists. Indeed, any independent set of cycles that generate $f_*H_{n-2}(\Sigma)$ in $H_{n-2}(N)$ will satisfy conditions (b) and (c) of [Definition 3.0.1](#), and one can always perturb slightly, if needed, to realize (a). That condition is really an artifact of the language we use to define the flux invariants; we want the sum $\mathcal{L}_{n-2}(\Sigma) + B$ to be direct. The assumption could be omitted in favor of more precision in distinguishing “caps with nontrivial spines” and “caps without spines”.

Definition 3.0.2 (Cap). A *cap* K is any chain in $\mathcal{S}_{n-1}(N)$ such that

$$\partial K \in \mathcal{L}_{n-2}(\Sigma) \oplus B.$$

As the kernel of the composition

$$\mathcal{S}_{n-1}(N) \xrightarrow{\partial} \mathcal{S}_{n-2}(N) \rightarrow \mathcal{S}_{n-2}(N)/(\mathcal{L}_{n-2}(\Sigma) \oplus B),$$

the set of all caps forms a group, which we denote by $\mathcal{X}(N, \Sigma, B)$.

A *reduced cap* is a class belonging to the quotient

$$\mathcal{K}(N, \Sigma, B) = \mathcal{X}(N, \Sigma, B)/\mathcal{B}_{n-1}(N, \Sigma).$$

We call two caps K, K' *homologous*, written $K \sim K'$, if they represent the same reduced cap in $\mathcal{K}(N, \Sigma, B)$.

In spirit, a reduced cap is a class in $H_{n-1}(N, \Sigma \cup \|B\|)$, where $\|B\|$ denotes the support of B . Indeed, when $\|B\|$ is disjoint from Σ , we have $\mathcal{K}(N, \Sigma, B) = H_{n-1}(N, \Sigma \cup \|B\|)$. When $\|B\|$ does meet Σ , however, ambiguity can arise as to which part of ∂K to take as β in [Observation 3.1](#) below. The need to remove that ambiguity motivated [Definition 3.0.2](#). The direct sum decomposition of $\mathcal{X}(N, \Sigma, B)$ there immediately yields the fact we need:

Observation 3.1. *For any cap K , there exists a unique $\beta \in B$ with $\|\partial K - \beta\| \subset \Sigma$.*

To make the notions of *spine* and *cap* more concrete, we illustrate using twizzlers:

Example 3.1.1. As explained in [Example 2.4.1](#), we may regard a twizzler as a cylinder $\Sigma = \mathbb{R} \times \mathcal{S}^1$ immersed in $N = \mathbb{C} \times \mathcal{S}^1$ and preserved by the helical \mathcal{S}^1 -action

$$(3-2) \quad [e^{i\theta}](z, e^{it}) = (e^{i\theta}z, e^{i(t+\theta)}).$$

The length of the \mathcal{S}^1 -factor is geometrically significant, but we can take it to be the usual 2π for purposes of this example.

We call the orbits of the screw-action *helices*. By construction, both N and the twizzler $f(\Sigma)$ are foliated by such helices, any one of which generates $f_*H_1(\Sigma) = H_1(N)$. It follows that any helix, viewed as a 1-cycle in N , qualifies as a spine for (N, Σ) . We take the shortest one, namely $\mathbf{0} \times \mathcal{S}^1 \subset N$, as our spine B .

Suppose a twizzler is generated by a particular curve $\gamma : \mathbb{R} \rightarrow \mathbb{C}$, so that we can immerse it in N via

$$f(t, e^{i\theta}) = (e^{i\theta} \gamma(t), e^{i\theta}).$$

For each fixed $t \in \mathbb{R}$, the helix $\Gamma_t := f(t, \mathbf{S}^1)$ forms a nontrivial cycle in $H_1(\Sigma)$. Any oriented surface that realizes the homology between Γ_t and the compatibly oriented cycle $\beta \in B$ is then a *cap* for Γ_t .

For instance, the line segment (or any arc) joining $\mathbf{0}$ to $\gamma(t)$ in \mathbb{C} will, under the \mathbf{S}^1 -action (3-2), sweep out a cap, and all arcs give rise to the same reduced cap in this way. Such caps are also preserved by the \mathbf{S}^1 -action, a useful property that many other caps lack.

3.2. First variation. To prepare for our first variation formula, fix a spine B for (N, Σ) , and suppose we have homologous caps K, K' in $\mathfrak{L}(N, \Sigma, B)$. There then exists an n -chain U satisfying

$$(3-3) \quad \partial U = S + K - K'$$

for some $S \in \mathcal{G}_{n-1}(\Sigma)$. Applying the boundary operator to (3-3), we then get

$$(3-4) \quad \partial K - \partial K' = -\partial S.$$

In particular, $\partial K - \partial K'$ is a cycle in Σ , and by the definition of a *cap*, there now exist unique $\beta, \beta' \in B$ such that

$$\partial K - \beta, \partial K' - \beta' \in \mathfrak{L}_{n-2}(\Sigma)$$

and (3-4) forces $\beta = \beta'$. This proves:

Proposition 3.3. *If two caps $K, K' \in \mathfrak{L}(N, \Sigma, B)$ are homologous, there exists a unique $\beta \in B$ such that both $\partial K - \beta$ and $\partial K' - \beta$ are supported in Σ .*

Definition 3.3.1. The proposition above lets us define the *spine of a reduced cap* $\mathbf{k} \in \mathfrak{K}(N, \Sigma, B)$ as the unique $\beta \in B$ with $\partial K - \beta \in \mathfrak{L}_{n-2}(\Sigma)$ for any representative K .

In the situation just described, and in the presence of a density function e^μ , we now consider the n - and $(n-1)$ -dimensional μ -weighted volumes $|U|_\mu$ and $|S|_\mu$ of the chains U and S respectively (Definition 2.1.2) as we deform along the flow of a smooth vector field Y . Fix a scalar H , and consider the initial derivative of $|S|_\mu - H |U|_\mu$ with respect to this flow, written

$$(3-5) \quad \delta_Y(|S|_\mu - H |U|_\mu).$$

Calling this the (μ -weighted) *volume-constrained first variation* of S , we obtain our conservation law for hypersurfaces with constant μ -mean curvature $h_\mu \equiv H$, as defined in (3-8) below, by evaluating (3-5) on Killing vector fields of N . To simplify the task, we analyze $\delta_Y |U|_\mu$ and $\delta_Y |S|_\mu$ separately before combining results.

A familiar derivation shows $\delta_Y |U|_\mu$ to equal the integral of $\operatorname{div}_N(Y)$ over U when $\mu \equiv 0$. A routine modification of that calculation shows that for general μ ,

$$\delta_Y |U|_\mu = \int_U \operatorname{div}_N(e^\mu Y) = \int_{\partial U} e^\mu Y \cdot \nu,$$

where ν denotes the orienting unit normal along ∂U . By (3-3), we can rewrite this as

$$(3-6) \quad \delta_Y |U|_\mu = \int_S e^\mu Y \cdot \nu + \int_{K-K'} e^\mu Y \cdot \nu.$$

A similar modification of the $\mu \equiv 0$ case, as analyzed in [Simon 1983, pp. 46–51], computes the μ -weighted first variation of $|S|_\mu$ along Y :

$$(3-7) \quad \delta_Y |S|_\mu = \int_S e^\mu d\mu(\nu) \nu \cdot Y + \operatorname{div}_\Sigma(e^\mu Y^\top) + \operatorname{div}_\Sigma(e^\mu Y^\perp).$$

Here Y^\top and Y^\perp signify the tangential and normal components, respectively, of Y along S .

Recall that for vector fields *tangent* to Σ , the divergence theorem applies in its usual form: given an $(n-1)$ -chain S in Σ with oriented unit conormal η along its boundary, we have

$$\int_S \operatorname{div}_\Sigma(X) = \int_{\partial S} X \cdot \eta \quad (X \text{ tangent to } \Sigma).$$

For vector fields *normal* to Σ , on the other hand, the divergence operator invokes the mean curvature of Σ , due to (2-1). When $Z = (Z \cdot \nu)\nu$ is purely normal, then, the Leibniz rule yields

$$\int_S \operatorname{div}_\Sigma(Z) = \int_S (Z \cdot \nu)h \quad (Z \text{ normal to } \Sigma).$$

Accordingly, we define the μ -mean curvature h_μ along Σ as

$$(3-8) \quad h_\mu := h + d\mu(\nu).$$

Using this notation, the facts above reduce (3-7) to

$$(3-9) \quad \delta_Y |S|_\mu = \int_{\partial S} e^\mu Y \cdot \eta + \int_S e^\mu h_\mu Y \cdot \nu.$$

Finally, using (3-6), (3-9), and (3-4), we can put our volume-constrained first-variation formula (3-5) into the form we need:

$$(3-10) \quad \delta_Y (|S|_\mu - H |U|_\mu) = - \int_{\partial K - \partial K'} e^\mu \eta \cdot Y - H \int_{K-K'} e^\mu \nu \cdot Y + \int_S e^\mu (h_\mu - H) \nu \cdot Y.$$

Remark 3.3.2. The μ -mean curvature h_μ arises naturally in the context of Riemannian submersions, which we encounter here whenever a compact Lie group \mathcal{G} of dimension $k > 0$ acts isometrically on a Riemannian manifold X . In that situation, the principal orbits (roughly speaking, the orbits of highest dimension) foliate a dense open subset $X' \subset X$, and the submersion $X' \rightarrow X'/\mathcal{G}$ becomes Riemannian, given the right metric on X'/\mathcal{G} (see [Hsiang and Lawson 1971]).

In any case, every Riemannian submersion $\pi : P \rightarrow N$ induces a *fiber volume function*

$$e^\mu : N \rightarrow (0, \infty), \quad e^\mu(p) := |\pi^{-1}(p)|,$$

where $|\pi^{-1}(p)|$ is the k -dimensional volume of the fiber over p . A standard first-variation calculation then shows:

Observation 3.4. *The μ -mean curvature h_μ of a hypersurface $\Sigma \subset N$ gives the classical mean curvature h of its preimage $\pi^{-1}(\Sigma) \subset P$.*

In the context of an isometric \mathcal{G} -action as discussed above, one may then study G -invariant hypersurfaces of constant (classical) mean curvature $h \equiv H$ in X by considering, instead, hypersurfaces of constant μ -mean curvature $h_\mu \equiv H$ in the orbit space X/\mathcal{G} . This can be especially fruitful when X/\mathcal{G} is just two- or three-dimensional. We consider examples involving twizzlers at the end of the paper.

In any case, the constrained first-variation formula (3-10) lets us extend the conservation law presented in [Korevaar et al. 1989]. As before, $\mathcal{L} \subset \mathcal{F}$ denotes a μ -preserving group of isometries on N , and the Killing fields that generate its identity component correspond to $L(\mathcal{L})$.

Theorem 3.5 (conservation law). *Suppose $\Sigma \subset N$ is an oriented hypersurface with $h_\mu \equiv H$, and B is a spine for the pair (N, Σ) . Then the formula*

$$(3-11) \quad \phi_B[\mathbf{k}](Y) := \int_{\partial K - \beta} e^\mu \eta \cdot Y + H \int_K e^\mu \nu \cdot Y$$

yields a well-defined homomorphism

$$\phi_B : \mathcal{K}(N, \Sigma, B) \otimes L(\mathcal{L}) \rightarrow \mathbb{R}.$$

Here Y is any Killing field in $L(\mathcal{L})$, K is any cap in \mathbf{k} , and $\beta \in B$ is the spine of \mathbf{k} given by Definition 3.3.1.

Proof. The basic linearity properties of the integral make ϕ_B a homomorphism once we establish well-definition: that $\phi_B[\mathbf{k}](Y)$ doesn't depend on which cap $K \in \mathbf{k}$ we use to compute it. We thus need to show, for all $Y \in L(\mathcal{L})$ and all $K, K' \in \mathbf{k}$, that

$$(3-12) \quad \int_{\partial K - \beta} e^\mu \eta \cdot Y + H \int_K e^\mu \nu \cdot Y = \int_{\partial K' - \beta} e^\mu \eta \cdot Y + H \int_{K'} e^\mu \nu \cdot Y$$

for any other $K' \in \mathbf{k}$. This follows easily from the constrained first-variation formula (3-10), however.

For μ is \mathcal{L} -invariant, and Y generates a flow that leaves both $|S|_\mu$ and $|U|_\mu$ unchanged, and hence the left-hand side of (3-10) must vanish. The integral over S on the right of (3-10) vanishes too, because $h_\mu \equiv H$. So (3-10) reduces to

$$0 = \int_{\partial K - \partial K'} e^\mu \eta \cdot Y + H \int_{K - K'} e^\mu \nu \cdot Y.$$

This is clearly equivalent to (3-12), since the integrals over β there cancel. \square

Remark 3.5.1. The simplest case of Theorem 3.5, where $\mu \equiv 0$ and \mathcal{L} is the full isometry group of N (so that $L(\mathcal{L})$ includes all Killing fields), already improves on the conservation law in [Korevaar et al. 1989] by eliminating the triviality assumptions there on $H_{n-1}(N)$ and $H_{n-2}(N)$.

Remark 3.5.2. The particular choice of spine B in Theorem 3.5 is of no real consequence. For when B and B' are both spines for (N, Σ) , the well-definition of ϕ_B on a class in $\mathcal{K}(N, \Sigma, B)$ implies that of $\phi_{B'}$ on a corresponding class in $\mathcal{K}(N, \Sigma, B')$.

To see this, suppose ϕ_B is well-defined on a class \mathbf{k} containing a cap K with boundary $\Gamma + \beta$, where $\beta \in B$ and Γ is supported in Σ . Then there exists a cycle $\beta' \in B'$ homologous to β , and hence an $(n-1)$ -chain P with

$$\partial P = \beta' - \beta.$$

We claim $\phi_{B'}$ will now be well-defined on the class \mathbf{k}' represented by $K + P$ in $\mathcal{K}(N, \Sigma, B')$.

Indeed, take any cap \tilde{K} homologous to $K + P$ in the latter group. Then $\tilde{K} - P \in \mathbf{k} \in \mathcal{K}(N, \Sigma, B)$, and if ϕ_B is well-defined there for some $Y \in L(\mathcal{G})$, we have, on the one hand,

$$\phi_B(\tilde{K} - P, Y) = \phi_B(K, Y).$$

On the other hand, we have

$$\begin{aligned} \phi_B(\tilde{K} - P, Y) &= \int_{\Gamma'} e^\mu \eta \cdot Y + H \int_{\tilde{K} - P} e^\mu \nu \cdot Y \\ &= \int_{\Gamma'} e^\mu \eta \cdot Y + H \int_{\tilde{K}} e^\mu \nu \cdot Y + H \int_P e^\mu \nu \cdot Y \\ &= \phi_{B'}(\tilde{K}, Y) + H \int_P e^\mu \nu \cdot Y. \end{aligned}$$

Together, these facts yield

$$\phi_{B'}(\tilde{K}, Y) = \phi_B(K, Y) - H \int_P e^\mu \nu \cdot Y.$$

Since \tilde{K} was arbitrary in k' , while P is fixed, we see that $\phi_{B'}$ is well-defined on $k' \in \mathcal{K}(N, \Sigma, B')$, as claimed.

4. Partial converse

Suppose the isometry group \mathcal{I} of our ambient manifold N contains a closed, connected group \mathcal{L} preserving a density function e^μ as above. Consider an immersed hypersurface $f : \Sigma \rightarrow N$, together with a spine B for the pair (N, Σ) .

Above, we assumed constancy of μ -mean curvature on Σ , and deduced conservation of flux. We now seek a *converse* conservation law to the effect that well-definition of the flux functional ϕ_B implies constancy of μ -mean curvature. Well-definition of ϕ_B , however, means nothing without Killing fields on which to pose it, so the strength of any such converse must correlate with the abundance of Killing fields.

Similarly, one shouldn't need to assume well-definition of ϕ_B on *all* Killing fields to get a conservation law. We could restrict ϕ_B to a nonempty subset of $L(\mathcal{L})$ (even a singleton) and ask whether well-definition of ϕ_B there influences geometry.

Dually, we needn't assume constancy of ϕ_B on all caps. We have in mind the case where Σ is preserved by a closed, connected subgroup $\mathcal{G} \subset \mathcal{L}$ and ϕ_B takes a fixed value on a sufficiently "crowded" set of homologous \mathcal{G} -invariant caps.

Definition 4.0.1 (\mathcal{G} -crowded). A set of trivial caps $\mathcal{C} \subset \mathcal{B}_{n-1}(N, \Sigma)$ is a *\mathcal{G} -crowded set of boundaries* if, for every \mathcal{G} -orbit λ and every $\epsilon > 0$, we can find a cap $K \in \mathcal{C}$ satisfying

$$K = \partial U - S.$$

Here U is an $(n+1)$ -chain in N , and S is an n -chain in Σ , which, as an n -current, is represented by some constant multiple of a submanifold-with-boundary within distance ϵ of the orbit λ . In other words, for some constant c and any integrable function f , we have

$$\int_S f = c \int_{\text{spt } S} f.$$

We say that a set \mathcal{C} of *nonbounding* caps in $\mathcal{L}(N, \Sigma, B)$ is *\mathcal{G} -crowded* if the difference set $\{K - K' : K, K' \in \mathcal{C}\}$ forms a \mathcal{G} -crowded set of boundaries. Note that in this case, each $K \in \mathcal{C}$ represents the same reduced cap in $\mathcal{K}(N, \Sigma, B)$.

Example 4.0.2. For any point $p \in \Sigma$, and $\epsilon > 0$, let $V_\epsilon(p)$ be the \mathcal{G} -orbit of the ball $B_\epsilon(p)$. When ϵ is sufficiently small, Σ will separate V_ϵ into two open sets V_ϵ^\pm . Then $K = \partial V_\epsilon^+ - \Sigma \cap V_\epsilon$ will be a trivial cap, and the collection of all these K for $p \in \Sigma$ and $\epsilon > 0$ small will form a \mathcal{G} -crowded set.

Using this definition, we can state and prove our partial converse, which says (roughly) that when our hypersurface Σ and the density e^μ are preserved by a

closed, connected subgroup $\mathcal{G} \subset \mathcal{F}$, and the flux is constant on a \mathcal{G} -crowded set of caps — with respect to Killing fields that commute with \mathcal{G} — we can split Σ into two nice subsets: one with constant μ -mean curvature, and one preserved by the flows of those Killing fields. These subsets may overlap, and either can be empty, as seen in [Examples 4.4.1](#) below.

Theorem 4.1. *Let $\Sigma \subset N$ be a complete oriented \mathcal{G} -invariant hypersurface, and B a spine for the pair (N, Σ) . Suppose $\mathcal{C} \subset \mathcal{X}(N, \Sigma, B)$ is a \mathcal{G} -crowded set of caps, and $\beta \in B$ is the spine of the reduced cap containing \mathcal{C} .*

If \mathcal{G} preserves a Killing field Y , and the μ -weighted flux functional

$$\int_{\partial K - \beta} e^\mu \eta \cdot Y + H \int_K e^\mu \nu \cdot Y$$

is constant on \mathcal{C} , then the set

$$\Sigma' := \Sigma \setminus h_\mu^{-1}(H)$$

is preserved by the flow of Y .

Proof. [Definition 4.0.1](#) and the form of the flux functional immediately show that constancy of flux on any \mathcal{G} -crowded set of caps in $\mathcal{X}(N, \Sigma, B)$ forces *vanishing* of flux on a \mathcal{G} -crowded set of *boundaries*. So without losing generality, we may assume $\mathcal{C} \subset \mathcal{B}_{n-1}(N, \Sigma)$.

The heart of our argument then lies with the following:

Claim. *If $p \in \Sigma'$, then $Y_p \in T_p \Sigma$.*

The definition makes Σ' relatively open in Σ . Since \mathcal{G} preserves Σ and μ , it preserves h_μ and hence Σ' . The \mathcal{G} -crowdedness of \mathcal{C} ensures the existence of a cap

$$K = \partial U - S \in \mathcal{C},$$

with $S \subset \Sigma'$ supported within an arbitrarily small distance to the \mathcal{G} -orbit of p .

We use the volume-constrained first-variation formula with K as above, and $K' = 0$ since K bounds modulo $\Sigma' \subset \Sigma$. The first two integrals in [\(3-10\)](#) now vanish on our Killing field Y , since together they compute the flux of Y across a trivial cap.

This reduces the constrained first variation to a single integral:

$$\delta_Y(|S|_\mu - H|U|_\mu) = \int_S e^\mu (h_\mu - H) Y \cdot \nu.$$

Finally, since Y preserves μ , the left side of this equation must *vanish*, leaving the identity

$$(4-1) \quad \int_{\text{spt } S} e^\mu (h_\mu - H) Y \cdot \nu = 0.$$

If $Y_p \notin T_p S$, then by assumption the integrand $(h_\mu - H)Y \cdot \nu$ is not 0 at p . Since all quantities are continuous and preserved by \mathcal{G} , it follows that $(h_\mu - H)Y \cdot \nu$ is strictly positive (or negative) in a neighborhood of the \mathcal{G} -orbit of p . Since the \mathcal{G} -crowdedness of \mathcal{C} lets us confine the support of S to such a neighborhood, we can contradict (4-1), thereby proving the claim.

To finish proving the theorem, it suffices to show that whenever $p \in \Sigma'$, the entire Y -streamline with initial velocity Y_p lies in Σ' .

Let $T > 0$ be the maximal time such that $\Theta(p, t) \subset \Sigma'$ for all $t < T$. By Lemma 2.3 (with $q := p$), some such T exists. Since Y generates a μ -preserving isometric flow, we have $h_\mu(\Theta(p, t)) \equiv H'$ with H' constant for all $t \in [0, T)$. Moreover, $H' \neq H$, as we are in Σ' . We now claim $T = \infty$. For otherwise, the continuity of h_μ and the completeness of the larger hypersurface Σ immediately yields both $\Theta(p, T) \in \Sigma$ and $h_\mu(\Theta(p, T)) = H' \neq H$, so that $\Theta(p, T) \in \Sigma'$. But then Lemma 2.3 (with $q := \Theta(p, T)$) contradicts the maximality of T . In short, $\Theta(p, t) \in \Sigma'$ for all $t \geq 0$. Since the same reasoning shows that $\Theta(p, t) \in \Sigma'$ for all $t \leq 0$ too, the proof is complete. \square

Remark 4.1.1. We emphasize again that our converse remains interesting even when \mathcal{G} is trivial. Theorem 4.1 then implies, for instance, that when the flux across every sufficiently small trivial cap vanishes on the generators of a subgroup $\mathcal{L} \subset \mathcal{F}$, the part of Σ that does *not* have constant μ -mean curvature $h_\mu = H$ must be \mathcal{L} -invariant.

Corollary 4.2. *If, as in Theorem 4.1, the μ -weighted flux functional is constant on one \mathcal{G} -crowded set of caps, it actually extends as a well-defined conserved quantity to all of $\mathcal{K}(N, \Sigma, B)$.*

Proof. While the theorem assumes constancy of ϕ_B only on a \mathcal{G} -crowded set of caps, the proof then deduces that at every point $p \in \Sigma$, either $h_\mu = H$ or Y belongs to $T_p \Sigma$. In this case, the last integral in the volume-constrained first-variation formula (3-5) clearly vanishes on any $(n - 1)$ -chain S in Σ , so that $\phi_B(K, Y) = \phi_B(K', Y)$ for any two homologous caps $K, K' \in \mathcal{L}(N, \Sigma, B)$. \square

Let us henceforth agree that when \mathcal{G} is trivial, we call a \mathcal{G} -crowded set of caps simply *crowded*.

Corollary 4.3. *If N is homogeneous, μ is constant, and on some crowded set of caps, the flux functional is well-defined for all Killing fields on N , then Σ has mean curvature $h \equiv H$ everywhere.*

Proof. With \mathcal{G} trivial in Theorem 4.1, well-definition on all Killing fields makes Σ' invariant under the entire isometry group \mathcal{F} . But in a homogeneous space, all nonempty \mathcal{F} -invariant sets have top dimension. So Σ' , having codimension one, must be empty, forcing $h \equiv H$ throughout Σ . \square

When $\mathcal{L} \subset \mathcal{F}$ is a subgroup, we say that N has *cohomogeneity* k with respect to \mathcal{L} when the highest-dimensional orbits of \mathcal{L} have codimension k in N . Cohomogeneity zero is the same as *homogeneity*.

Corollary 4.4. *Suppose a real-analytic Riemannian manifold N has cohomogeneity one with respect to a μ -preserving group \mathcal{L} , and on some crowded set of caps, the flux functional is well-defined on all of $L(\mathcal{L})$. Then either $h_\mu \equiv H$, or Σ is an orbit of \mathcal{L} . Either way, h_μ is constant on Σ .*

Proof. In an analytic ambient space, hypersurfaces with constant μ -mean curvature are analytic [Federer 1969, 5.2.16]. Cohomogeneity one means the only connected \mathcal{L} -invariant hypersurfaces are single orbits of \mathcal{L} , which clearly have constant μ -mean curvature. Since Σ is connected, the corollary now follows from Theorem 4.1. □

Examples 4.4.1. Take $N = \mathbb{R}^3$ and let \mathcal{G} be the circular group acting by rotation about the x -axis. The Killing field $Y = (1, 0, 0)$ generates translational flow along that axis, and \mathcal{G} commutes with this flow as required by Theorem 4.1. The noncylindrical Delaunay surfaces — CMC surfaces of revolution about the x -axis analyzed by C. Delaunay in 1841 — show that Theorem 4.1 may obtain with \mathcal{G} -invariant hypersurfaces having $h \equiv H$ and no flow-invariant subset Σ' .

In contrast, if we take Σ to be any cylinder centered about the x -axis with radius *not* equal to $1/H$, we get an example with $\Sigma' = \Sigma$. That is, Σ has mean curvature H nowhere, and yet the flux functional remains well-defined on Y , thanks to the global flow-invariance of Σ .

Of course, the cylinder of radius $1/H$ about the x -axis has *both* $h \equiv H$ and the extra translational symmetry.

All these possibilities arise in the family of twizzlers too, as we shall shortly see.

Case study 4.5 (first integrals for twizzlers). Consider the Riemannian product $N := \mathbb{C} \times \mathcal{S}_R^1$, where the complex plane \mathbb{C} and \mathcal{S}_R^1 (the circle of radius R) have their standard metrics. Take $\mu \equiv 0$, and let $\mathcal{G} \approx \mathcal{S}^1$ act via screw-motion:

$$[e^{it}](z, Re^{i\theta}) = (e^{it}z, Re^{i(t+\theta)}).$$

In this situation, each helical orbit of the \mathcal{G} -action generates $H_1(N) \approx \mathbb{Z}$. Let $\Sigma \subset N$ be any connected \mathcal{G} -invariant surface, and with no loss of generality assume it does not contain the shortest orbit $\beta := \mathbf{0} \times \mathcal{S}_R^1$. Then β clearly generates a spine for (N, Σ) .

We can parametrize Σ by letting \mathcal{G} act on an immersed curve $\gamma : \mathbb{R} \rightarrow \mathbb{C} \times \{1\} \approx \mathbb{C}$ via the map

$$(4-2) \quad X(u, v) = (e^{iv}\gamma(u), Re^{iv}).$$

Assume the orientation of γ makes the natural frame $\{X_u, X_v\}$ *positively* oriented along Σ .

Now fix any point p on the generating curve γ , and join it to the origin in \mathbb{C} by a line segment. This segment sweeps out a helicoidal cap K^p , invariant under the \mathcal{G} -action, and the reduced class of K^p in $\mathcal{H}(N, \Sigma, B)$ is clearly independent of p . One easily sees that as p varies over γ , the resulting caps K^p form a \mathcal{G} -crowded set \mathcal{C} according to [Definition 4.0.1](#).

Now let Y be the circular Killing field generating the purely “horizontal” isometric flow $[e^{is}](z, Re^{i\theta}) = (e^{is}z, Re^{i\theta})$. Note that Y commutes with \mathcal{G} and preserves μ , as required by [Theorem 4.1](#).

Finally, suppose that when we put $K = K^p$ and β as above in the flux formula of [Theorem 3.5](#), the result is independent of p .

Since N has cohomogeneity one with respect to the extension of \mathcal{G} by the flow of Y , [Corollary 4.4](#) dictates that *either* Σ is a CMC twizzler with $h \equiv H$, or Σ is an orbit of the combined action, and thus a circular cylinder with $h \equiv 1/r$ (r giving the radius of the cylinder; typically $1/r \neq H$).

As an application of our theory, we now show that constancy of ϕ_B on the \mathcal{G} -crowded set of caps K^p described above “explains” the first-order ODE known to characterize generating curves of CMC twizzlers, as mentioned in our introduction.

Proposition 4.6. *A noncircular immersed curve γ in \mathbb{C} generates a twizzler in $\mathbb{C} \times \mathbf{S}_R^1$ with $h \equiv H$ if and only if, for some $c \in \mathbb{R}$, it solves*

$$(4-3) \quad \frac{2\pi R^2(\dot{\gamma} \cdot i\gamma)}{\sqrt{R^2|\dot{\gamma}|^2 + (\dot{\gamma} \cdot \gamma)^2}} - \pi RH|\gamma|^2 = c.$$

Proof. Since we assume γ is not circular, [Theorem 3.5](#) and [Corollary 4.4](#), as noted above, tell us that $h \equiv H$ if and only if the flux of the circular vector field

$$Y_{(z,\tau)} = (iz, 0)$$

across K^p is independent of p . That is,

$$(4-4) \quad \phi_B(K^p, Y) \equiv c \quad \text{for all } p \in \gamma.$$

[Equation \(4-3\)](#) merely evaluates this assertion.

To reach [\(4-3\)](#) from [\(4-4\)](#), we temporarily fix a point $p = \gamma(t)$ on the generating curve γ , and specify an orientation on the cap K^p , by declaring the frame field $\{K_u, K_v\}$ associated with the parametrization

$$K(u, v) = (ue^{iv}p, Re^{iv}), \quad (u, v) \in (0, 1) \times (0, 2\pi)$$

to be positively oriented.

Now consider the second integral in the flux formula (3-11) — the one that pairs Y with the unit normal ν along K^p . The correctly oriented unit normal will be a positive multiple of

$$K_u \wedge K_v = (-Rie^{iv}p, u|p|^2ie^{iv}).$$

The length of $K_u \wedge K_v$ is actually irrelevant: we divide by it to normalize, but then multiply it back in as the Jacobian in the flux integral, namely

$$\int_{K^p} \nu \cdot Y = \int_0^{2\pi} \int_0^1 (K_u \wedge K_v) \cdot Y|_{K(u,v)} du dv.$$

At $K(u, v)$, we have $Y = (ue^{iv}p, 0)$, so the corresponding flux term evaluates easily to

$$(4-5) \quad H \int_{K^p} \nu \cdot Y = -2\pi HR|p|^2 \int_0^1 u du = -\pi HR|p|^2.$$

Now consider the other integral in the flux formula (3-11), the integral over $\Gamma := \partial K - \beta$, where K^p meets Σ . This curve is the helical \mathcal{G} -orbit of p , and one easily computes its length as

$$|\Gamma| = 2\pi\sqrt{R^2 + |p|^2}.$$

Our chosen orientation of K induces an orientation on Γ . Since K_u at Γ is parallel to the outer conormal in K , the velocity Γ' of Γ is equal to a positive multiple of X_v . The outer conormal in Σ along Γ , which we called η , must then give the pair $\{\eta, \Gamma'\}$ positive orientation, so we can obtain η by orthonormalizing X_u along Γ , i.e., by normalizing

$$|X_v|^2 X_u - (X_u \cdot X_v) X_v.$$

Both η and Y are \mathcal{G} -invariant, making $\eta \cdot Y$ constant along Γ , and careful calculation then shows that indeed,

$$\eta \cdot Y \equiv \frac{R^2 \dot{\gamma} \cdot ip}{\sqrt{R^2 + |p|^2} \sqrt{(R^2 + |p|^2) |\dot{\gamma}|^2 - (\gamma \cdot ip)^2}},$$

where we evaluate $\dot{\gamma}$ at p . We can simplify the second square root in the denominator here via the elementary identity

$$(\dot{\gamma} \cdot ip)^2 = |\dot{\gamma}|^2 |p|^2 - (\dot{\gamma} \cdot p)^2.$$

This lets us express the conormal flux integral as

$$(4-6) \quad \int_{\Gamma} \eta \cdot Y = \frac{2\pi R^2 (\dot{\gamma} \cdot ip)}{\sqrt{R^2 |\dot{\gamma}|^2 + (\dot{\gamma} \cdot p)^2}}.$$

Setting $p = \gamma(t)$ and recalling (3-11), we now get $\phi_B(K^p, Y)$ by adding (4-5) to (4-6). \square

Remark 4.6.1. If we parametrize a convex arc of the generating curve γ using its *support function*, namely

$$k(t) := \sup_{\theta} \gamma(t) \cdot e^{i\theta},$$

then

$$\gamma(t) = (k(t) + i\dot{k}(t))e^{it}.$$

It now follows from Proposition 4.6 that when γ generates a pitch- R twizzler with $h \equiv H$, its support function satisfies a simple nonlinear ODE:

$$\frac{2Rk}{\sqrt{R^2 + \dot{k}^2}} - H(k^2 + \dot{k}^2) = C.$$

In other words, the phase portrait of k lies on one of the “heart-shaped” level curves of the function

$$F(x, y) := \frac{2Rx}{\sqrt{R^2 + y^2}} - H(x^2 + y^2).$$

Perdomo [2012; 2013] based his dynamical characterization of twizzler generating curves and his study of their moduli space on this observation.

Remark 4.6.2 (twizzlers in other 3D-space forms). It is natural to see the curve γ in Case study 4.5 as the projection of the hypersurface Σ into the orbit space $N/\mathcal{G} \approx \mathbb{C}$. The length of the orbit above $z \in \mathbb{C}$ is easily computed as $|\Gamma_z| = 2\pi\sqrt{R^2 + |z|^2}$, and if we adopt this as our density function, i.e., $e^{\mu(z)} = |\Gamma_z|$, on the orbit space (see Definition 2.1.2), a simple reworking of Proposition 4.6 reinterprets the first integral there as the condition for γ to have $h_\mu \equiv H$ as a “hypersurface” in the two-dimensional orbit space.

Similarly, one can seek CMC “twizzlers” in the 3-sphere $\mathcal{S}^3 \subset \mathbb{R}^4$ invariant under one of the helical (k, l) “torus knot” circle actions given by

$$[e^{it}](z, w) = (e^{ikt}z, e^{ilt}w).$$

This is the standard Hopf action when $k = l = 1$, in which case the orbit space $\mathcal{S}^3/\mathcal{G}$ is of course the standard 2-sphere \mathcal{S}^2 . More generally, when $\gcd(k, l) = 1$, one can realize the orbit space as an eccentric “football” or “teardrop” shaped surface of revolution in \mathbb{R}^3 , smooth except for conical singularities at one or both ends. The \mathcal{G} -invariant CMC twizzlers in \mathcal{S}^3 then correspond one-to-one with curves having constant μ -mean curvature in the orbit space, where the density function is again given by orbit length: $e^{\mu(p)} = |\pi^{-1}(p)|$ for p in the orbit space. By Theorem 4.1, these $h_\mu \equiv H$ curves are precisely the noncircular curves that conserve flux along the Killing fields that generate the rotational symmetry of the orbit space. It is then

straightforward to use this fact, as in [Proposition 4.6](#), to derive the first integral they satisfy. See [\[Edelen 2011\]](#) for the resulting expression. We should note here that the special case $h_\mu \equiv 0$ (*minimal twizzlers* in S^3) was analyzed using Hamilton–Jacobi theory in [\[Hsiang and Lawson 1971, Chapter IV\]](#).

Analogous helical actions exist in the hyperbolic space form \mathbb{H}^3 , and the resulting CMC twizzlers have a first integral derivable in precisely the same way. The reader may consult [\[Edelen 2011\]](#) for a description of the group action and the resulting first integral in this case as well.

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NICK EDELEN
STANFORD UNIVERSITY
STANFORD, CA 94305
UNITED STATES
nedelen@math.stanford.edu

BRUCE SOLOMON
INDIANA UNIVERSITY
BLOOMINGTON, IN 47405
UNITED STATES
solomon@indiana.edu

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Stanford, CA 94305-2125
finn@math.stanford.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
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chari@math.ucr.edu

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Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

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Princeton University
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