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UNIFORM BOUNDEDNESS OF S-UNITS IN ARITHMETIC DYNAMICS

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Let K be a number field and let S be a finite set of places of K which contains all the archimedean places. For any $\phi(z) \in K(z)$ of degree $d \geq 2$ which is not a d-th power in $\bar{K}(z)$, Siegel's theorem implies that the image set $\phi(K)$ contains only finitely many S-units. We conjecture that the number of such S-units is bounded by a function of $|S|$ and d (independently of K, S and ϕ). We prove this conjecture for several classes of rational functions, and show that the full conjecture follows from the Bombieri–Lang conjecture.

1. Introduction

Let K be a number field, let S be a finite set of places of K which contains the set S_∞ of archimedean places of K, and write $\mathfrak{o}_S$ for the ring of S-integers of K and $\mathfrak{o}_S^*$ for the group of S-units of K. The genus-0 case of Siegel's theorem asserts that, for any $\phi(z) \in K(z)$ which has at least three poles in $\mathbb{P}^1(\bar{K})$, the image set $\phi(K)$ contains only finitely many S-integers. However, the number of S-integers in $\phi(K)$ cannot be bounded independently of $\phi(z)$, even if we restrict to functions $\phi(z)$ having a fixed degree, since $\psi(z) := \beta\phi(z)$ satisfies $\psi(K) = \beta\phi(K)$ for any $\beta \in K^*$.

Although the number of S-integers in $\phi(K)$ cannot be bounded in terms of only K, S, and $\deg \phi$, such a bound may be possible for the number of S-units in $\phi(K)$. In fact we conjecture that there is a bound depending only on $|S|$ and $\deg \phi$ (and not on K):

Conjecture 1.1. *For any integers $s \geq 1$ and $d \geq 2$, there is a constant $C = C(s, d)$ such that for any*

- *number field K,*

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- *s-element set S of places of K with $S \supseteq S_\infty$,*
- *degree-d rational function $\phi(z) \in K(z)$ which is not a d-th power in $\overline{K}(z)$,*

we have

$$|\phi(K) \cap \mathfrak{o}_S^*| \le C.$$

We will prove Conjecture 1.1 in case $\phi(z)$ is restricted to certain classes of rational functions, and we will also prove that the full conjecture is a consequence of a variant of the Caporaso–Harris–Mazur conjecture on uniform boundedness of rational points on curves of fixed genus.

We also consider a variant of Conjecture 1.1, which addresses S-units in an orbit of ϕ rather than in the image set $\phi(K)$. Here, for any $\alpha \in \mathbb{P}^1(K)$, the *orbit* of α under $\phi(z)$ is the set

$$\mathcal{O}_\phi(\alpha) := \{\phi^n(\alpha) : n \ge 1\},$$

where $\phi^n = \phi \circ \cdots \circ \phi$ denotes the n-fold composition of ϕ with itself. For any $\phi(z) \in K(z)$ of degree at least 2 such that $\phi^2(z) \notin K[z]$, Silverman [1993] showed that $\mathcal{O}_\phi(\alpha) \cap \mathfrak{o}_S$ is finite. However, for any $\beta \in K^*$ the function $\psi(z) := \beta\phi(z/\beta)$ satisfies $\mathcal{O}_\psi(\alpha\beta) = \beta\mathcal{O}_\phi(\alpha)$, so the size of $\mathcal{O}_\phi(\alpha) \cap \mathfrak{o}_S$ cannot be bounded independently of $\phi(z)$. We conjecture that there is a uniform bound on the number of S-units in an orbit:

Conjecture 1.2. *For any integers $s \ge 1$ and $d \ge 2$, there is a constant $C = C(s, d)$ such that for any*

- *number field K,*
- *s-element set S of places of K with $S \supseteq S_\infty$,*
- *degree-d rational function $\phi(z) \in K(z)$ which is not of the form $\beta z^{\pm d}$ with $\beta \in K^*$,*
- *$\alpha \in \mathbb{P}^1(K)$,*

we have

$$|\mathcal{O}_\phi(\alpha) \cap \mathfrak{o}_S^*| \le C.$$

It turns out that this conjecture is a consequence of Conjecture 1.1:

Proposition 1.3. *If Conjecture 1.1 is true then Conjecture 1.2 is true.*

Remark 1.4. The hypotheses of Conjectures 1.1 and 1.2 imply that $[K : \mathbb{Q}] \le 2s$, since $S_\infty \subseteq S$.

In Section 3 we prove the following preliminary results, which show that Conjectures 1.1 and 1.2 would be true if we allowed the constants C in those conjectures to depend on K, S, and ϕ, rather than just on s and d. We note that in the case of Conjecture 1.1 this simply says that $\phi(K) \cap \mathfrak{o}_S^*$ is finite. These results also indicate the special behavior of the functions excluded in the statements of these conjectures.

Proposition 1.5. *Let K be a number field, let S be a finite set of places of K with $S \supseteq S_\infty$, and let $\phi(z) \in K(z)$ be any rational function.*

(a) *If $|\phi^{-1}(\{0, \infty\})| \neq 2$ then $\phi(K) \cap \mathfrak{o}_S^*$ is finite.*

(b) *If $|\phi^{-1}(\{0, \infty\})| = 2$ then there is a finite set $S' \supseteq S$ for which $\phi(K) \cap \mathfrak{o}_{S'}^*$ is infinite.*

Proposition 1.6. *Let K be a number field, let S be a finite set of places of K with $S \supseteq S_\infty$, and let $\phi(z) \in K(z)$ have degree $d \geq 2$.*

(a) *If $\phi(z)$ does not have the form $\beta z^{\pm d}$ with $\beta \in K^*$, then there is a constant $C(K, S, \phi)$ such that every $\alpha \in \mathbb{P}^1(K)$ satisfies $|\mathcal{O}_\phi(\alpha) \cap \mathfrak{o}_S^*| \leq C(K, S, \phi)$.*

(b) *If $\phi(z) = \beta z^{\pm d}$ with $\beta \in K^*$, then there exist $\alpha \in \mathbb{P}^1(K)$ and a finite set $S' \supseteq S$ for which $\mathcal{O}_\phi(\alpha) \cap \mathfrak{o}_{S'}^*$ is infinite.*

We note that part (a) of each of these propositions follows from Siegel's theorem. For, if $|\phi^{-1}(\{0, \infty\})| > 2$ then $\psi(z) := \phi(z) + 1/\phi(z)$ has at least three poles so that $\psi(K) \cap \mathfrak{o}_S$ is finite; but $\psi(\beta)$ is in $\mathfrak{o}_S$ whenever $\phi(\beta)$ is in $\mathfrak{o}_S^*$, so also $\phi(K) \cap \mathfrak{o}_S^*$ is finite. Next, if $\phi^{-1}(\{0, \infty\})$ is a two-element set other than $\{0, \infty\}$, then Lemma 3.2 implies that $|\phi^{-2}(\{0, \infty\})| > 2$, so that $\phi^2(K) \cap \mathfrak{o}_S^*$ has size $N < \infty$, whence $|\mathcal{O}_\phi(\alpha) \cap \mathfrak{o}_S^*| \leq N + 1 = C(K, S, \phi)$.

In Section 2 we prove Conjectures 1.1 and 1.2 for some families of polynomial maps. The first family consists of monic polynomials in $\mathfrak{o}_S[z]$:

Theorem 1.7. *Let $s \geq 1$ and $d \geq 2$ be integers. There is a constant $C = C(s, d)$ such that for any*

- *number field K,*
- *s-element set S of places of K with $S \supseteq S_\infty$,*
- *degree-d monic polynomial $\phi(z) \in \mathfrak{o}_S[z]$ which does not equal $(z - \beta)^d$ for any $\beta \in K$,*

we have

$$|\phi(K) \cap \mathfrak{o}_S^*| \leq C.$$

Theorem 1.7 proves Conjecture 1.1 for monic polynomials in $\mathfrak{o}_S[z]$; for such polynomials, Conjecture 1.2 follows by applying Theorem 1.7 to $\phi^2(z)$.

We also prove Conjecture 1.2 for monic polynomials in $K[z]$ in which the coefficients of all but one term are in $\mathfrak{o}_S$, so long as this exceptional term does not have degree $d - 1$. We deduce this from the following more general result in v-adic dynamics.

Theorem 1.8. *Let K be a field with a nonarchimedean valuation v, and let*

$$\phi(z) = a_d z^d + \cdots + a_1 z + a_0 \in K[z]$$

be a polynomial satisfying

- $v(a_d) = 0$,
- *there is exactly one integer i for which $v(a_i) < 0$, and this exceptional i satisfies $i \neq d-1$.*

Then for each $\alpha \in K$, the set $\{n \geq 1 \mid v(\phi^n(\alpha)) = 0\}$ contains at most one element.

As an immediate corollary, we have the stated case of Conjecture 1.2:

Corollary 1.9. *Let K be a number field, and let S be a finite set of places of K with $S \supseteq S_\infty$. For any monic $\phi_0(z) \in \mathfrak{o}_S[z]$, any $\alpha, \beta \in K$ with $\beta \notin \mathfrak{o}_S$, and any integer i with $0 \leq i < \deg \phi_0 - 1$, the polynomial $\phi(z) := \phi_0(z) + \beta z^i$ satisfies*

$$|\mathcal{O}_\phi(\alpha) \cap \mathfrak{o}_S^*| \leq 1.$$

Remark 1.10. Conjecture 1.2 also follows from Theorem 2 of [Levin 2012] for rational functions of the form

$$\phi(z) := \frac{z^d + \beta_{d-1} z^{d-1} + \cdots + \beta_1 z}{\gamma_{d-1} z^{d-1} + \gamma_{d-2} z^{d-2} + \cdots + \gamma_1 z + 1}$$

with $\beta_1, \ldots, \beta_{d-1}, \gamma_1, \ldots, \gamma_{d-1} \in \mathfrak{o}_S$ and $\phi(z) \neq z^d$. For that theorem gives a uniform bound on the number of elements of K in the backwards orbit of any element of $\mathfrak{o}_S^*$. This also bounds the number of S-units in $\mathcal{O}_\phi(\alpha)$ for any $\alpha \in K$, since if $\phi^n(\alpha) \in \mathfrak{o}_S^*$ then $\alpha, \phi(\alpha), \ldots, \phi^{n-1}(\alpha)$ are elements of K in the backwards orbit of $\phi^n(\alpha)$.

We prove our conjectures for some further classes of rational functions in Section 4.

In Section 3 we show that our conjectures are consequences of the following variant of the deep conjecture of Caporaso, Harris and Mazur [Caporaso et al. 1997] concerning rational points on curves of a fixed genus.

Conjecture 1.11. *Fix integers $g \geq 2$ and $D \geq 1$. There is a constant $N = N(D, g)$ such that $|X(K)| \leq N$ for every smooth, projective, geometrically irreducible genus-g curve X defined over a degree-D number field K.*

Theorem 1.12. *If Conjecture 1.11 is true, then Conjecture 1.1 and Conjecture 1.2 are true.*

Remark 1.13. Conjecture 1.11 follows from the Bombieri–Lang conjecture [Pacelli 1997].

The referee provided the following geometric explanation of the difference between the questions of S-integers and S-units in the image set $\phi(K)$ of a rational function ϕ, indicating possible directions for future work. Writing $\phi(x/y) = \frac{f(x,y)}{g(x,y)}$ as the ratio of two coprime homogeneous polynomials, we see that the S-integral points of $\phi(K)$ correspond to the S-integral points of the quasi-affine variety cut out by

$$zg(x, y) = f(x, y) \quad \text{in} \quad \mathbb{P}^1 \times \mathbb{A}^1.$$

Similarly, the S-unit points in $\phi(K)$ correspond to the S-integral points of the variety defined by

$$zg(x, y) = wf(x, y) \quad \text{and} \quad zw = 1 \quad \text{in} \quad \mathbb{P}^1 \times \mathbb{A}^2.$$

It would be interesting to seek generalizations of Conjecture 1.1 by considering more generally what sorts of families of varieties are likely to satisfy uniform boundedness statements for their S-integral points.

2. Special classes of rational functions

In this section we prove Theorems 1.7 and 1.8.

Proof of Theorem 1.7. Let K be a number field, let S be a finite set of places of K with $S \supseteq S_\infty$, and let $\phi(z) \in \mathfrak{o}_S[z]$ be monic of degree $d \geq 2$ with $\phi(z) \neq (z - \beta)^d$ for any $\beta \in K$. Then $\phi(z)$ has at least two distinct roots δ_1, δ_2 in $\overline{K}$. Let $K' = K(\delta_1, \delta_2)$ and let S' be the set of places of K' which lie over places in S, so that $|S'| \leq [K' : K]|S| \leq d(d-1)|S|$ and $\delta_i \in \mathfrak{o}_{S'}$. Then we can write

$$\phi(z) = (z - \delta_1)(z - \delta_2)\psi(z),$$

where $\psi(z)$ is a monic polynomial in $\mathfrak{o}_{S'}[z]$. Let $\gamma \in K$ satisfy $\phi(\gamma) \in \mathfrak{o}_S^*$. Then we must have $\gamma \in \mathfrak{o}_S$, so that both $u_i := \gamma - \delta_i$ and $\psi(\gamma)$ are in $\mathfrak{o}_{S'}$. Since $u_1 u_2 \psi(\gamma) = \phi(\gamma)$ is in $\mathfrak{o}_S^*$, it follows that $u_1, u_2 \in \mathfrak{o}_{S'}^*$. In addition we have

$$\frac{1}{\delta_2 - \delta_1} u_1 - \frac{1}{\delta_2 - \delta_1} u_2 = 1. \tag{2-1}$$

Moreover, γ is uniquely determined by u_1, so the number of elements $\gamma \in \mathfrak{o}_S$ for which $\phi(\gamma) \in \mathfrak{o}_S^*$ is at most the number of solutions to (2-1) in elements $u_1, u_2 \in \mathfrak{o}_{S'}^*$. Finally, by [Evertse 1984], the number of such solutions is at most $C_1 C_2^{|S'|-1}$ for some absolute constants C_1, C_2 (in fact, we can take $C_1 = C_2 = 256$ [Beukers and Schlickewei 1996]). Therefore $|\phi(K) \cap \mathfrak{o}_S^*|$ is bounded by a function of $|S'|$, and hence by a function of $|S|$ and d. □

Proof of Theorem 1.8. Suppose that $\mathcal{O}_\phi(\alpha)$ contains a unit of the valuation ring, and let m be the least positive integer for which $v(\phi^m(\alpha)) = 0$. Writing $\gamma := \phi^m(\alpha)$, we will show by induction that $|\phi^n(\gamma)|_v = |a_i|_v^{d^{n-1}}$ for every $n \geq 1$. The strong triangle inequality implies that $|\phi(\gamma)|_v = |a_i|_v$, proving the base case $n = 1$. If $\delta := \phi^n(\gamma)$ satisfies $|\delta|_v = |a_i|_v^{d^{n-1}}$ for some $n \geq 1$, then $|a_i \delta^i|_v = |a_i|_v^{1+id^{n-1}}$ and $|a_j \delta^j|_v \leq |a_i|_v^{jd^{n-1}}$ for $j \neq i$, with equality when $j = d$. Our hypothesis $i < d - 1$

implies that $d^n > 1 + id^{n-1}$, so that $|\phi^{n+1}(\gamma)|_v = |a_i|_v^{d^n}$, which completes the induction. It follows that $v(\phi^n(\gamma)) < 0$ for every $n > 0$, so that $\mathcal{O}_\phi(\alpha)$ contains exactly one unit, which concludes the proof. □

3. Connection with rational points on curves

In this section we prove Theorem 1.12 and Propositions 1.3, 1.5, and 1.6. We begin by relating S-units in the image set $\phi(K)$ of a rational function to rational points on certain curves.

Lemma 3.1. *Let K be a number field, let S be a finite set of places of K with $S \supseteq S_\infty$, and let $\phi(z) \in K(z)$ be a nonconstant rational function. For any prime p with $p >$ deg ϕ, there are elements $\gamma_1, \dots, \gamma_t \in \mathfrak{o}_S^*$, where $t \le p^{|S|}$, with the following properties*:

- *For each i, the affine curve X_i defined by $y^p = \gamma_i \phi(z)$ is geometrically irreducible.*
- *We have $|\phi(K) \cap \mathfrak{o}_S^*| \le \sum_{i=1}^t N_i$ where N_i is the number of points in $X_i(K)$ having nonzero y-coordinate.*

Proof. First note that $y^p = \gamma\phi(z)$ is geometrically irreducible for any $\gamma \in K^*$, since $\gamma\phi(z)$ is not a p-th power in $\overline{K}(z)$. Dirichlet's S-unit theorem asserts that $\mathfrak{o}_S^* \cong \mu_K \times \mathbb{Z}^{|S|-1}$, where μ_K denotes the group of roots of unity in K. Since μ_K is cyclic, it follows that $\mathfrak{o}_S^* / (\mathfrak{o}_S^*)^p \cong (\mathbb{Z}/p\mathbb{Z})^r$ where $r \in \{|S| - 1, |S|\}$. Let Γ be a set of p^r elements in $\mathfrak{o}_S^*$ whose images in $\mathfrak{o}_S^* / (\mathfrak{o}_S^*)^p$ are pairwise distinct. For any $\beta \in K$ such that $\phi(\beta) \in \mathfrak{o}_S^*$, we can write $\phi(\beta) = \gamma^{-1}\delta^p$ for some $\gamma \in \Gamma$ and $\delta \in \mathfrak{o}_S^*$. Then (δ, β) is a K-rational point on the curve $y^p = \gamma\phi(z)$ whose y-coordinate is nonzero. Since the z-coordinate of this point is β, the result follows. □

Proof of Theorem 1.12. By Proposition 1.3, it suffices to show that Conjecture 1.11 implies Conjecture 1.1. Let K be a number field, let S be a finite set of places of K with $S \supseteq S_\infty$, and let $\phi(z) \in K(z)$ have degree $d \ge 2$. Assume that $\phi(z)$ is not a d-th power in $\overline{K}(z)$, so that $m := |\phi^{-1}(\{0, \infty\})|$ is at least 3. Let p be the smallest prime for which $p > d$ and $(p - 1)(m - 2) > 2$. Then $p = 5$ if $d = 2$ and $m = 3$, and in all other cases $p < 2d$ by Bertrand's postulate. Let $\gamma_1, \dots, \gamma_t$ satisfy the conclusion of Lemma 3.1, so that $\gamma_i \in K^*$ and $t \le p^{|S|}$. Writing X_i for the curve $y^p = \gamma_i\phi(z)$, and N_i for the number of points in $X_i(K)$ having nonzero y-coordinate, it follows that $|\phi(K) \cap \mathfrak{o}_S^*| \le \sum_{i=1}^t N_i$. Since every point on X_i having nonzero y-coordinate is nonsingular, we see that N_i is bounded above by the number of K-rational points on the unique smooth projective curve Y_i over K which is birational to X_i. Since $p > d$, the classical genus formula for Kummer covers [Stichtenoth 2009, Proposition III.7.3] implies that the genus g of Y_i is

$(p-1)(m-2)/2$. Thus our choice of p ensures that

$$2 \le g \le \tfrac{1}{2}\big(\tfrac{5}{2}d-1\big)\big(2d-2\big).$$

If Conjecture 1.11 is true then $|Y_i(K)|$ is bounded by a constant which depends only on the genus of $Y_i(K)$ and the degree $[K:\mathbb{Q}]$. Since the genus is bounded by a function of d, and the degree $[K:\mathbb{Q}]$ is bounded by a function of $|S|$ (by Remark 1.4), it follows that $|Y_i(K)|$ is bounded by a constant depending on d and $|S|$. Since $t \le p^{|S|} \le (5d/2)^{|S|}$, this proves that Conjecture 1.11 implies Conjecture 1.1. □

Our proof of Proposition 1.3 relies on the following well-known lemma.

Lemma 3.2. *Let $\phi(z) \in \mathbb{C}(z)$ be any rational function of degree $d \ge 2$ which is not of the form $\beta z^{\pm d}$ with $\beta \in \mathbb{C}^*$. Then $|\phi^{-2}(\{0,\infty\})| \ge 3$.*

Proof. Write $m := |\phi^{-2}(\{0,\infty\})|$, so we must show that $m \ge 3$. Plainly $m \ge |\phi^{-1}(\{0,\infty\})| \ge 2$, so the conclusion holds unless $|\phi^{-1}(\{0,\infty\})| = 2$. In this case ϕ is totally ramified over both 0 and ∞, so the Riemann–Hurwitz formula (or writing down $\phi(z)$) implies that ϕ is unramified over all other points. Since $\phi(z)$ does not have the form $\beta z^{\pm d}$, we know that $\phi^{-1}(\{0,\infty\}) \ne \{0,\infty\}$, so that at least one point in $\phi^{-1}(\{0,\infty\})$ has d distinct ϕ-preimages. Since each point has at least one preimage, we conclude that $m \ge d+1 \ge 3$, as desired. □

Proof of Proposition 1.3. If $\phi(z) \ne \beta z^{\pm d}$ then $\phi^2(z)$ has a total of at least three zeroes and poles by Lemma 3.2, and hence is not a d^2-th power in $\overline{K}(z)$. Thus Conjecture 1.1 implies that $|\phi^2(K) \cap \mathfrak{o}_S^*| \le C(s,d)$, so that

$$|\mathcal{O}_\phi(\alpha) \cap \mathfrak{o}_S^*| \le C(s,d) + 1. \qquad \square$$

Part (a) of Proposition 1.5 follows from our proof of Theorem 1.12, by using Faltings' theorem [1983] instead of Conjecture 1.11. We now give a more elementary proof of Proposition 1.5.

Proof of Proposition 1.5. If $|\phi^{-1}(\{0,\infty\})| > 2$, the function $\psi(z) := \phi(z) + 1/\phi(z)$ satisfies $|\psi^{-1}(\{0,\infty\})| \ge 3$, so $\psi(K) \cap \mathfrak{o}_S$ is finite by Siegel's theorem; but $\psi(\beta)$ is in $\mathfrak{o}_S$ whenever $\phi(\beta)$ is in $\mathfrak{o}_S^*$, so it follows that $\phi(K) \cap \mathfrak{o}_S^*$ is finite. Now assume that $|\phi^{-1}(\{0,\infty\})| = 2$, so that $\phi(z) = \gamma\mu(z)^d$ for some $d \ge 1$, some $\gamma \in K^*$, and some degree-one $\mu(z) \in K(z)$. Let S' be a finite set of places of K such that $\gamma \in \mathfrak{o}_{S'}^*$, $S' \supseteq S$, and $|S'| > 1$. Since $\mu(K)$ contains all but at most one element of K, it follows that $\phi(K)$ contains all but at most one element of $\gamma(\mathfrak{o}_{S'}^*)^d$. Since $\gamma \in \mathfrak{o}_{S'}^*$ and $|S'| > 1$, this shows that $\phi(K) \cap \mathfrak{o}_{S'}^*$ is infinite. □

Proof of Proposition 1.6. If $\phi(z)$ does not have the form $\beta z^{\pm d}$ then $|\phi^{-2}(\{0,\infty\})| \ge 3$ by Lemma 3.2, so Proposition 1.5 implies that $\phi^2(K) \cap \mathfrak{o}_S^*$ has size $N < \infty$, whence

$$|\mathcal{O}_\phi(\alpha) \cap \mathfrak{o}_S^*| \le N + 1 = C(K,S,\phi).$$

Now consider $\phi(z)=\beta z^{\pm d}$ with $\beta\in K^*$ and $d\geq 2$. Any $\alpha\in K^*$ satisfies $\mathcal{O}_\phi(\alpha)\subseteq\mathfrak{o}_{S'}^*$ where S' is the union of S with the set of places v of K for which $|\alpha|_v\neq 1$ or $|\beta|_v\neq 1$. If $\alpha\in K^*$ is not a root of unity then $\mathcal{O}_\phi(\alpha)$ is infinite, so that $\mathcal{O}_\phi(\alpha)\cap\mathfrak{o}_{S'}^*$ is infinite. □

4. Additional remarks

We make two final remarks. First, the proofs of Theorems 1.7 and 1.8 can be modified to treat some classes of Laurent polynomials. For example, let d and d' be distinct positive integers, and let $\phi(z)=(\gamma_d z^d+\cdots+\gamma_1 z+\gamma_0)/z^{d'}$ where $\gamma_i\in\mathfrak{o}_S$ and $\gamma_d,\gamma_0\in\mathfrak{o}_S^*$. Suppose in addition that the numerator is not a d-th power in $\overline{K}[z]$. Then $|\phi(K)\cap\mathfrak{o}_S^*|\leq C(s,d)$ for any $\alpha\in\mathbb{P}^1(K)$. Indeed, since γ_0 and γ_d are assumed to be units, $\phi(\beta)$ cannot be in $\mathfrak{o}_S^*$ if $|\beta|_v\neq 1$ for some $v\notin S$. Thus we need only consider $\beta\in\mathfrak{o}_S^*$, and now the desired bound follows from the proof of Theorem 1.7.

As another example, consider $\phi(z)=(\gamma_d z^d+\cdots+\gamma_1 z+\gamma_0)/z^{d'}$ where $d>d'$, $\gamma_i\in K$, and there is some $v\notin S$ for which $|\gamma_d|_v>\max(1,|\gamma_i|_v)$ for each $i<d$. Then $|\mathcal{O}_\phi(\alpha)\cap\mathfrak{o}_S^*|\leq 1$ for any $\alpha\in\mathbb{P}^1(K)$, as the orbit of an S-unit cannot contain another S-integer by the proof of Theorem 1.8. Both this class of examples and the previous class are quite special, but they serve as further evidence for Conjectures 1.1 and 1.2.

We conclude this paper by noting that the constant C that appears in Conjectures 1.1 and 1.2 must depend on both s and d. The necessity of dependence on s is clear. Dependence on d is also required, since by Lagrange interpolation one can construct polynomials $\phi(z)\in K[z]$ in which the first several $\phi^i(\alpha)$ take on any prescribed distinct values in K while also $\phi(z)$ has at least two zeroes (and hence is not $\beta z^{\pm d}$).

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