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ON CURVES AND POLYGONS WITH THE EQUIANGULAR CHORD PROPERTY

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To the memory of Eugene Gutkin

Let C be a smooth, convex curve on either the sphere $\mathbb{S}^2$, the hyperbolic plane $\mathbb{H}^2$ or the Euclidean plane $\mathbb{E}^2$ with the following property: there exists α and parametrizations $x(t)$ and $y(t)$ of C such that, for each t, the angle between the chord connecting $x(t)$ to $y(t)$ and C is α at both ends.

Assuming that C is not a circle, E. Gutkin completely characterized the angles α for which such a curve exists in the Euclidean case. We study the infinitesimal version of this problem in the context of the other two constant curvature geometries, and in particular, we provide a complete characterization of the angles α for which there exists a nontrivial infinitesimal deformation of a circle through such curves with corresponding angle α. We also consider a discrete version of this property for Euclidean polygons, and in this case, we give a complete description of all nontrivial solutions.

1. Introduction

Given a smooth, convex oriented closed curve C in the Euclidean plane $\mathbb{E}^2$ and $x, y \in C$, $x \neq y$, let $|xy|$ denote the oriented chord connecting x to y. Motivated by his study of mathematical billiards, E. Gutkin [1993] asked the following:

Question 1. Assume the existence of parametrizations $x(t)$ and $y(t)$ of C such that, for each t,

(1) $x'(t), y'(t) \neq 0$;

(2) $x(t) \neq y(t)$;

(3) there exists $\alpha \in (0, \pi]$ such that both angles between C and $|x(t)y(t)|$ equal α.

Then if C is not a circle, what are all possible values of α?

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Gutkin provides a complete answer to Question 1 by establishing the following necessary and sufficient condition for α: there exists an integer $k \geq 2$ such that

$$k \tan \alpha = \tan(k\alpha); \tag{1-1}$$

see [Gutkin 1993; 2012; Tabachnikov 1995]. In particular, only a countable number of values of the angle α are possible.

In terms of billiards, the billiard ball map on the interior of C has a horizontal invariant circle given by the condition that the angle made by the trajectories with the boundary of the table is equal to α. This statement can also be interpreted in terms of capillary floating with zero gravity in neutral equilibrium; see [Finn 2009; Finn and Sloss 2009].

We call a curve satisfying this equiangular chord property a *Gutkin curve*; we will refer to the corresponding angle α as the *contact angle*.

We generalize Gutkin's theorem in two directions: to curves in the standard 2-sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$ and to polygons in $\mathbb{E}^2$ via a discretized version of Question 1. For $\mathbb{S}^2$ and $\mathbb{H}^2$, we consider the following infinitesimal version of Gutkin's question:

Question 2. In either $\mathbb{H}^2$ or $\mathbb{S}^2$, for which angles α are there nontrivial infinitesimal deformations of a radius-R circle through Gutkin curves with contact angle α?

Here, a *nontrivial deformation* of a circle is a deformation that does not correspond to a circle solution (of a different radius).

Our first result yields an answer to Question 2:

Theorem 1.1. *Assume that a circle of radius R in $\mathbb{S}^2$ or in $\mathbb{H}^2$ admits a nontrivial infinitesimal deformation through Gutkin curves with contact angle α. Define angles c via*

$$\cot c = \cos R \cot \alpha$$

in the spherical case and

$$\cot c = \cosh R \cot \alpha$$

in the hyperbolic case. Then there exists $k \in \mathbb{N}$, $k \geq 2$, such that

$$k \tan c = \tan kc.$$

Thus, as in the Euclidean case, only a countable number of values of the contact angle α are possible for a given radius R.

Note that, in the Euclidean plane, Gutkin curves with contact angle $\alpha = \pi/2$ are precisely the curves of constant width; the same holds in the spherical and hyperbolic settings; see [Leichtweiss 2005] for curves of constant width in non-Euclidean geometries.

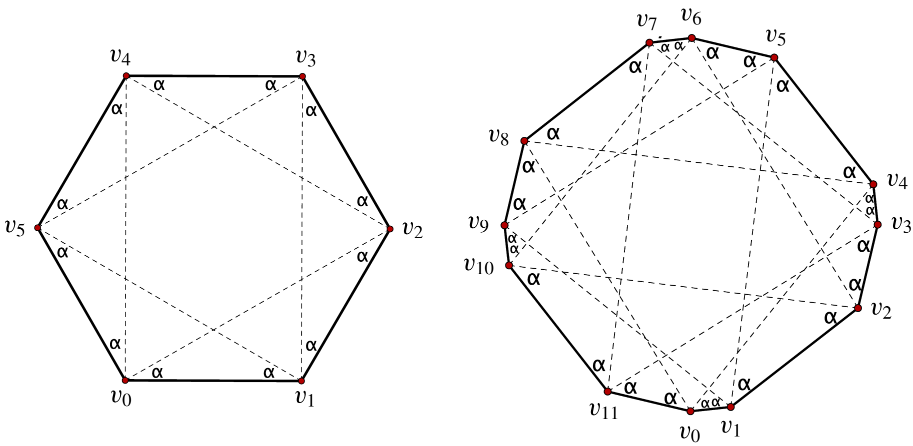

Figure 1. Gutkin (6, 2)-gon and (12, 4)-gon.

In Section 4, we consider the following analog of Gutkin's theorem for polygons in $\mathbb{E}^2$. Let P be a convex n-gon with vertices $\{v_0, \dots, v_{n-1}\}$ in their cyclic order. For $k \in \mathbb{N}$, $2 \le k \le n/2$, a k-diagonal is a straight line segment connecting vertices of P whose indices differ by k modulo n. Then P is a *nontrivial Gutkin* (n, k)*-gon* if P is not regular and there exists α such that, for any k-diagonal D, both contact angles between D and P equal α (see Figure 1 for examples). That is, for each i,

$$\angle v_{i+1} v_i v_{i+k} = \angle v_{i+k-1} v_{i+k} v_i = \alpha,$$

where $\angle v_{i+1} v_i v_{i+k}$ denotes the angle between the edge $|v_{i+1} v_i|$ and the k-diagonal $|v_i v_{i+k}|$.

Our second result is a complete characterization of the pairs (n, k) for which a nontrivial Gutkin (n, k)-gon exists:

Theorem 1.2. *A nontrivial Gutkin* (n, k)*-gon in the Euclidean plane exists if and only if* n *and* $k-1$ *are not coprime.*

Interestingly, the main ingredient of our proof is the Diophantine equation

$$\tan \frac{kr\pi}{n} \tan \frac{\pi}{n} = \tan \frac{k\pi}{n} \tan \frac{r\pi}{n},$$

which is a discrete version of (1-1). This equation also appeared in [Tabachnikov 2006], and it was solved in [Connelly and Csikós 2009].

2. A proof of Gutkin's theorem in E^2

Although the existing proofs of Gutkin's theorem in $\mathbb{E}^2$ [Gutkin 1993; 2012; Tabachnikov 1995] are very clear and simple, our goal in this paper is to study the situations in $\mathbb{S}^2$ and $\mathbb{H}^2$. Therefore, in this section, we reprove (the necessary part of) Gutkin's

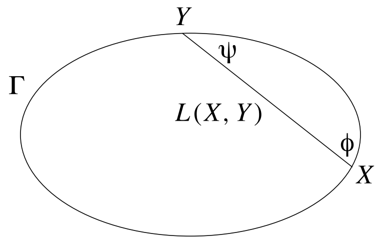

Figure 2. Curve Γ with chord xy.

theorem using methods that can be applied to the other constant-curvature settings. This proof is motivated by the study of integrable billiards by M. Bialy [1993; 2013].

Let $\tilde{\gamma}:\mathbb{R}\to\mathbb{R}^2$ be a periodic unit-speed parametrization of a smooth strictly convex curve Γ. For $x, y \in \mathbb{R}$, let X and Y be the points $\tilde{\gamma}(x)$ and $\tilde{\gamma}(y)$, ϕ and ψ the angles made by the chord XY with Γ, and $L = |XY|$ the length of the chord, the generating function of the billiard ball map. See Figure 2.

We have

$$L_x = -\cos\phi, \quad L_y = \cos\psi,$$
$$L_{xy} = \frac{\sin\phi\sin\psi}{L},\ L_{xx} = \frac{\sin^2\phi}{L} - \kappa(x)\sin\phi,\ L_{yy} = \frac{\sin^2\psi}{L} - \kappa(y)\sin\psi, \tag{2-1}$$

where κ is the curvature of the curve and subscripts denote partial differentiation; see, e.g., [Bialy 1993].

We interpret $L(x, y)$ as a function on the torus $\Gamma\times\Gamma$. If Γ is a Gutkin curve with contact angle α, then there exists a curve s on this torus where both angles, ϕ and ψ, have the same constant value α.

We seek a reparametrization $\gamma(t(x)) = \tilde{\gamma}(x)$ so that the values $t(x)$ and $t(y)$ of the new parameter at the points X and Y differ by a constant: $2c = t(y) - t(x)$. Denote d/dt by a prime.

Proposition 2.1. *The parameter t is determined by the condition $x' = a/\kappa(x)$, where a is a constant.*

Proof. Since α is constant as a function of t,

$$0 = L_{xt} = L_{xx}x' + L_{xy}y' \quad \text{and} \quad 0 = L_{yt} = L_{xy}x' + L_{yy}y'. \tag{2-2}$$

This implies that $L_{xx}L_{yy} = L_{xy}^2$ along our curve, and substituting from (2-1), we have

$$\frac{\sin\alpha}{\kappa(x)} + \frac{\sin\alpha}{\kappa(y)} = L. \tag{2-3}$$

We compute y'/x' from (2-1)–(2-3),

$$\frac{y'}{x'} = -\frac{L_{xy}}{L_{yy}} = \frac{\sin\alpha}{\kappa(y)L - \sin\alpha} = \frac{\kappa(x)}{\kappa(y)},$$

which implies the claim. □

Since the curvature is the rate of turning of the direction of the curve, Proposition 2.1 defines (up to a multiplicative coefficient) the angular parameter along the curve. Note that $0 \le x \le L(\gamma)$ and $0 \le t \le T$, where T is the upper bound of t and $L(\gamma)$ is the length of γ. It follows that

$$T = \int_0^T dt = \frac{1}{a}\int_0^{L(\gamma)} \kappa(x)\,dx.$$

Choose $a = 1$ to make $T = 2\pi$, which agrees with the angle. Then $c = \alpha$.

In view of Proposition 2.1, we set

$$f_1 := f(t-\alpha) = \frac{\sin\alpha}{\kappa(x)} \quad\text{and}\quad f_2 := f(t+\alpha) = \frac{\sin\alpha}{\kappa(y)}.$$

From (2-3), we have

$$L = \frac{\kappa(x)\sin\alpha + \kappa(y)\sin\alpha}{\kappa(x)\kappa(y)} = f_1 + f_2.$$

It follows that $L' = f_1' + f_2'$. By the chain rule, we have

$$L_x x' + L_y y' = \cot\alpha(f_2 - f_1) = f_1' + f_2',$$

and therefore,

$$f'(t+\alpha) + f'(t-\alpha) = \cot\alpha(f(t+\alpha) - f(t-\alpha)). \tag{2-4}$$

Since $f(t)$ is a function with period 2π, using the Fourier expansion, we obtain $f(t) = \sum b_k e^{ikt}$, where $b_k \in \mathbb{C}$ and $b_{-k} = \overline{b_k}$. Thus,

$$f(t\pm\alpha) = \sum b_k e^{\pm ik\alpha} e^{ikt} \quad\text{and}\quad f'(t\pm\alpha) = \sum b_k i k e^{\pm ik\alpha} e^{ikt}.$$

Let LHS be the left-hand side of (2-4) and RHS the right-hand side. It follows that

$$\text{LHS} = \sum b_k ik(e^{ik\alpha} + e^{-ik\alpha})e^{ikt} \quad\text{and}\quad \text{RHS} = \cot\alpha \sum b_k(e^{ik\alpha} - e^{-ik\alpha})e^{ikt}.$$

Equating both sides, we have

$$b_k(k\cos k\alpha - \cot\alpha\sin k\alpha) = 0.$$

For $k = 1$, this automatically holds, and if $b_k \neq 0$ for some $k \ge 2$, then

$$k\tan\alpha = \tan k\alpha.$$

If the curve is a circle, then $f(t)$ is constant and all $b_k = 0$, and if the curve is not a circle, then $b_k \neq 0$ for some $k \geq 1$. It remains to show that $b_1 = 0$.

Recall that x is arc length and t is the angular parameter on the curve γ. Then $\gamma_x = (\cos t, \sin t)$ and $dt/dx = \kappa$. Therefore,

$$\gamma_t = \frac{1}{\kappa}(\cos t, \sin t) \quad \text{and} \quad \int_0^{2\pi} \gamma_t \, dt = 0.$$

Hence,

$$\int_0^{2\pi} \frac{\cos t}{\kappa} \, dt = \int_0^{2\pi} \frac{\sin t}{\kappa} \, dt = 0;$$

that is, the function f is L^2-orthogonal to the first harmonics. Hence, f has no first harmonics in the Fourier expansion; that is, $b_1 = 0$.

3. Infinitesimal analogs of Gutkin's theorem in $\mathbb{S}^2$ and $\mathbb{H}^2$

We prove Theorem 1.1 in detail for $\mathbb{S}^2$. The hyperbolic case being analogous, we only indicate the necessary changes.

Let γ be a Gutkin curve, and as before, let x and y be arc length parameters. Then ϕ and ψ should have constant value, namely, the contact angle α. By [Bialy 2013], we have the following formulas for the first and second partials of L (valid along the curve $s \subset \Gamma \times \Gamma$):

$$L_x = -\cos\alpha, \quad L_y = \cos\alpha,$$

$$L_{xy} = \frac{\sin^2\alpha}{\sin L}, \quad L_{xx} = \frac{\sin^2\alpha}{\tan L} - \kappa(x)\sin\alpha, \quad L_{yy} = \frac{\sin^2\alpha}{\tan L} - \kappa(y)\sin\alpha. \tag{3-1}$$

(The function κ is the geodesic curvature of the curve.) Once again, we seek a parametrization on the curve such that the values of the parameter at points x and y differ by a constant: $t(y) = t(x) + 2c$.

Proposition 3.1. *The desired parametrization $\gamma(t)$ is given by the equation*

$$x' = \frac{a}{\sqrt{\kappa^2(x) + \sin^2\alpha}},$$

where a is a constant.

Proof. Equation (2-2) holds along our curve as before, so $L_{xx}L_{yy} = L_{xy}^2$. Substitute from (3-1) to obtain the equation

$$\left(\kappa(x) - \frac{\sin\alpha}{\tan L}\right)\left(\kappa(y) - \frac{\sin\alpha}{\tan L}\right) = \frac{\sin^2\alpha}{\sin^2 L}. \tag{3-2}$$

Then we can compute y'/x' from (2-2),

$$\frac{y'}{x'} = -\frac{L_{xx}}{L_{xy}} = \left(\kappa(x) - \frac{\sin\alpha}{\tan L}\right)\frac{\sin L}{\sin\alpha} = \frac{\sqrt{\kappa(x) - \dfrac{\sin\alpha}{\tan L}}}{\sqrt{\kappa(y) - \dfrac{\sin\alpha}{\tan L}}}, \tag{3-3}$$

with the last equality due to (3-2). Next, we claim that

$$\frac{\sqrt{\kappa(x) - \dfrac{\sin\alpha}{\tan L}}}{\sqrt{\kappa(y) - \dfrac{\sin\alpha}{\tan L}}} = \frac{\sqrt{\kappa^2(x) + \sin^2\alpha}}{\sqrt{\kappa^2(y) + \sin^2\alpha}}, \tag{3-4}$$

which, along with (3-3), implies the statement of the proposition.

It remains to prove (3-4). Rewrite (3-2) as

$$\kappa(x)\kappa(y) - \frac{\sin\alpha}{\tan L}(\kappa(x) + \kappa(y)) - \sin^2\alpha = 0,$$

and multiply by $\kappa(y) - \kappa(x)$ to obtain

$$\kappa(x)\kappa^2(y) - \frac{\sin\alpha}{\tan L}\kappa^2(y) + \kappa(x)\sin^2\alpha = \kappa^2(x)\kappa(y) - \frac{\sin\alpha}{\tan L}\kappa^2(x) + \kappa(y)\sin^2\alpha,$$

or

$$\left(\kappa(x) - \frac{\sin\alpha}{\tan L}\right)(\kappa^2(y) + \sin^2\alpha) = \left(\kappa(y) - \frac{\sin\alpha}{\tan L}\right)(\kappa^2(x) + \sin^2\alpha).$$

This implies (3-4). □

We choose a in such a way that

$$T = \frac{1}{a}\int_0^{L(\gamma)} \sqrt{\kappa^2(x) + \sin^2\alpha}\, dx = 2\pi \tag{3-5}$$

in order to make Fourier expansion more convenient.

Define a function f on the curve by

$$\cot f = \frac{\kappa}{\sin\alpha}. \tag{3-6}$$

Remark 3.2. The meaning of the function f is illustrated in Figure 3. Let O be the center of the osculating circle at point $x \in \gamma$, and let R be its radius. Then $\cot R = \kappa(x)$. Drop the perpendicular from O to the segment xy. Then we have a right triangle PxO with an angle $\pi/2 - \alpha$. Solving a right spherical triangle yields $\cot|Px|\sin\alpha = \cot R$. Hence, $f = |Px|$.

Denote by f_1 and f_2 the values of this function at points y and x.

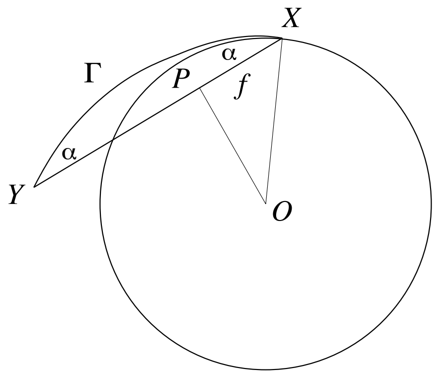

Figure 3. Geometric interpretation of the function f.

Proposition 3.3. *One has*

$$a \cot\alpha(\sin f_1 - \sin f_2) = f_1' + f_2'. \tag{3-7}$$

Proof. First, note that Proposition 3.1 and (3-6) imply that

$$x' = \frac{a \sin f}{\sin\alpha}. \tag{3-8}$$

Next, as before, $L_{xx}L_{yy} = L_{xy}^2$, and substituting from (3-1), we obtain

$$\cot L = \frac{\kappa(x)\kappa(y) - \sin^2\alpha}{\kappa(x)\sin\alpha + \kappa(y)\sin\alpha}.$$

Substituting $\kappa(x)$ and $\kappa(y)$ from (3-6) yields

$$\cot L = \frac{\cot f_1 \cot f_2 - 1}{\cot f_1 + \cot f_2} = \cot(f_1 + f_2).$$

Thus, $L = f_1 + f_2$, and hence, $L' = f_1' + f_2'$. By the chain rule,

$$L' = L_x x' L_y y' = \frac{a\cos\alpha}{\sin\alpha}(\sin f_1 - \sin f_2),$$

where the last equality is due to (3-1) and (3-8). This implies the statement. □

Remark 3.4. Equation (3-7) appeared in [Tabachnikov 2006] in a study of a different rigidity problem also related to a flotation problem (Ulam's problem on bodies that float in equilibrium in all positions) and to a problem of bicycle kinematics.

Equation (3-7) is an analog of (2-4), but unlike the Euclidean case, it is nonlinear, and we do not know how to solve it. Thus, we resort to linearization of the problem, that is, start from a circle γ_0 of radius R and then deform it to find infinitesimal solutions.

Write $f_1(t) = f(t+c)$ and $f_2(t) = f(t-c)$, where the constant c depends on the Gutkin curve and the contact angle (in the Euclidean case, $c = \alpha$). For a circle on S^2, we compute the relation between R, α and c and the value of a.

Lemma 3.5. *One has*

$$\cos\alpha = \frac{\cos c}{\sqrt{\sin^2 R\cos^2 c + \cos^2 R}} \quad \textit{or equivalently} \quad \cot c = \cos R\cot\alpha,$$

and

$$a = \sqrt{\cos^2 R + \sin^2\alpha\sin^2 R}.$$

Proof. The circle of radius R is parametrized as

$$\gamma_0(t) = (\sin R\cos t, \sin R\sin t, \cos R),$$

where $t \in [0, 2\pi]$. We need to find the angle α made by the geodesic segment $[\gamma_0(-c), \gamma_0(c)]$ with this circle.

The great circle through points $\gamma_0(-c)$ and $\gamma_0(c)$ is the parametric curve

$$\Gamma(s) = \frac{\cos c}{\sqrt{\sin^2 R\cos^2 c + \cos^2 R}}(\sin R\cos c, 0, \cos R) + \sin s(0, 1, 0),$$

and $\Gamma(s_0) = \gamma_0(c)$ for $\sin s_0 = \sin R\sin c$. It remains to compute the velocity vectors $d\Gamma(s)/ds$ and $d\gamma_0(t)/dt$, evaluate them at $s = s_0$ and $t = c$, respectively, and compute the angle between these vectors. This straightforward computation yields the first formula of the lemma. A calculation using trigonometric identities yields the simpler, equivalent, formula.

To obtain the formula for a, note that the length and the geodesic curvature of the circle γ_0 are equal to $2\pi\sin R$ and $\cot R$, respectively. Then (3-5) yields the result. □

Remark 3.6. A referee pointed out that this lemma can be proved, in a simpler way, by applying formulas of spherical trigonometry to the spherical triangle XPO in Figure 3.

Now we are ready for the proof of Theorem 1.1 in the spherical case. Let γ_0 be a circle of radius R. Then the function f is a constant satisfying $\cot f = \cot R/\sin\alpha$ (see (3-6)), and the constants c and a are as in Lemma 3.5. Consider an infinitesimal deformation of the curve in the class of Gutkin curves with the contact angle α. Then f, c and a deform as

$$f \mapsto f + \varepsilon g(t), \quad c \mapsto c + \varepsilon\delta, \quad a \mapsto a + \varepsilon\beta,$$

where $g(t)$ is a 2π-periodic function and all the previous relations hold. Substitute into (3-7):

$$(a+\varepsilon\beta)\cot\alpha\big(\sin(f+\varepsilon g(t+c+\varepsilon\delta))-\sin(f+\varepsilon g(t-c-\varepsilon\delta))\big)$$
$$=\varepsilon(g'(t+c+\varepsilon\delta)+g'(t-c-\varepsilon\delta)).$$

Computing modulo ε^2 yields

$$a\cot\alpha\cos f(g(t+c)-g(t-c))=g'(t+c)+g'(t-c).$$

As before, this implies that, if $g(t)$ is not a constant (which would correspond to a trivial deformation to a circle of possibly different radius), then

$$k\cos kc=a\cot\alpha\cos f\sin kc$$

for each k for which the Fourier coefficient $b_k\neq 0$. Substituting the values of the constants f and a and eliminating α using Lemma 3.5 yields, after a straightforward, albeit tedious, computation,

$$k\cos kc=\cot c\sin kc \quad\text{or}\quad k\tan c=\tan kc.$$

For $k=1$, this formula holds for all c, and it remains to explain the condition $k\geq 2$ in the formulation of the theorem. The next proposition shows that the first Fourier coefficient b_1 vanishes.

Proposition 3.7. *The function $g(t)$ is L^2-orthogonal to the first harmonics; that is, its Fourier expansion does not contain* $\cos t$ *and* $\sin t$.

Proof. Let φ and θ be the spherical coordinates. Recall that the spherical metric is $\sin^2\theta\, d\varphi^2+d\theta^2$. The unperturbed curve $\gamma_0(t)$, the circle of latitude of radius R, has the coordinates (t, R). Consider its infinitesimal deformation

$$\gamma_\varepsilon(t)=(t+\varepsilon\bar f(t),\, R+\varepsilon\bar g(t)),$$

where $\bar f$ and $\bar g$ are 2π-periodic functions. The curvature of γ_0 is $\cot R$. Let $\cot R+\varepsilon k(t)$ be the curvature of γ_ε. Here and below, all computations are modulo ε^2.

Due to (3-6),

$$\sin\alpha\cot(f+\varepsilon g(t))=\cot R+\varepsilon k(t);$$

hence, up to a constant multiplier, $g=k$. We shall compute $k(t)$ and show that it is free from first harmonics.

We shall use Liouville's formula for curvature of a curve in an orthogonal coordinate system (u, v); see, e.g., [do Carmo 1976]. Recall this formula. Let ψ be the angle made by the curve with the curves $v=\text{const}$, let K_u and K_v be the geodesic curvatures of the coordinate curves $v = \text{const}$ and $u = \text{const}$ and let x be the arc length parameter on the curve. Then the curvature of the curve is

$$\frac{d\psi}{dx} + K_u \cos\psi + K_v \sin\psi. \tag{3-9}$$

Here u and v are the longitude and latitude, so $K_v = 0$ and $K_u(\varphi, \theta) = \cot\theta$.

Since

$$x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi, \quad z = \cos\theta,$$

one has

$$\gamma_\varepsilon = \big(\sin R\cos t + \varepsilon(\bar g(t)\cos R\cos t - \bar f(t)\sin R\sin t), \\ \sin R\sin t + \varepsilon(\bar g(t)\cos R\sin t + \bar f(t)\sin R\cos t), \cos R - \varepsilon\bar g(t)\sin R\big).$$

Then

$$\gamma_\varepsilon' = \big(-\sin R\sin t + \varepsilon(-\bar g\cos R\sin t + \bar g'\cos R\cos t - \bar f\sin R\cos t - \bar f'\sin R\sin t), \\ \sin R\cos t + \varepsilon(\bar g\cos R\cos t + \bar g'\cos R\sin t - \bar f\sin R\sin t + \bar f'\sin R\cos t), \\ -\varepsilon\bar g'\sin R\big).$$

It follows that

$$|\gamma_\varepsilon'| = \sin R + \varepsilon(\bar g\cos R + \bar f'\sin R).$$

The angle ψ between γ_ε' and the circles of latitude is infinitesimal. Therefore, $\cos\psi = 1$ (modulo ε^2). Using the formula for γ_ε', one computes this angle:

$$\psi = -\varepsilon\frac{\bar g'(t)}{\sin R}.$$

(The minus sign is due to the fact that increasing $\bar g$ pushes the curve down to the equator.) Hence,

$$\frac{d\psi}{dx} = \frac{\psi'}{x'} = \frac{\psi'}{|\gamma_\varepsilon'|} = -\varepsilon\frac{\bar g''(t)}{\sin^2 R}.$$

Finally,

$$\cot\theta = \cot(R + \varepsilon\bar g(t)) = \cot R - \varepsilon\frac{\bar g(t)}{\sin^2 R}.$$

Now (3-9) implies that, up to a constant factor, $k(t) = \bar g(t) + \bar g''(t)$. Since the differential operator $d^2/dx^2 + 1$ "kills" the first harmonics, the result follows. □

This concludes the proof in the spherical case.

For the case of $\mathbb{H}^2$, we apply a similar method, so we briefly describe the differences. The formulas for the partials of L read [Bialy 2013]

$$L_x = -\cos\alpha, \quad L_y = \cos\alpha,$$
$$L_{xy} = \frac{\sin^2\alpha}{\sinh L}, \quad L_{xx} = \frac{\sin^2\alpha}{\tanh L} - \kappa(x)\sin\alpha, \quad L_{yy} = \frac{\sin^2\alpha}{\tanh L} - \kappa(y)\sin\alpha.$$

The parametrization of a Gutkin curve is given by $x_t = a/\sqrt{\kappa(x)^2 - \sin^2\alpha}$, where the constant a is normalized so that the parameter t takes values in $[0, 2\pi]$. One defines the function $f(t)$ by $\coth f = \kappa/\sin\alpha$, and as before, one obtains a difference-differential equation

$$a\cot\alpha(\sinh f_1 - \sinh f_2) = f_1' + f_2'.$$

Analogs of Lemma 3.5 hold:

$$\cos\alpha = \frac{\cos c}{\sqrt{\cosh^2 R - \sinh^2 R\cos^2 c}} \quad \text{or equivalently} \quad \cot c = \cosh R\cot\alpha$$

and

$$a = \sqrt{\cosh^2 R - \sin^2\alpha\sinh^2 R}.$$

The computations in Euclidean space $\mathbb{R}^3$ involving the unit sphere are replaced by similar computations in the Minkowski space $\mathbb{R}^{1,2}$ involving a hyperboloid of two sheets, used as a model of $\mathbb{H}^2$.

4. Gutkin polygons

Refer to the introduction for the definition of a Gutkin (n, k)-gon. Let $G(n, k)$ denote the set of all Gutkin (n, k)-gons. Given $P \in G(n, k)$, it will be convenient to think of P as being embedded in the complex plane $\mathbb{C}$. Let l_i denote the side length, $|v_{i+1} - v_i|$.

Notice that if $n = 2k$, for every index i, one has $i - k = i + k$. Therefore, in this case, each vertex is the end point of exactly one diagonal. If $n \neq 2k$, then $i - k \neq i + k$, so each vertex is the endpoint of two diagonals. In this case, for each v_i, we call the angle between the two diagonals β_i; i.e., $\beta_i = \angle v_{i-k} v_i v_{i+k}$.

The first two propositions in this section will establish basic geometric properties of a Gutkin (n, k)-gon.

Proposition 4.1. *Given n and k, the associated contact angle is equal to $\pi(k-1)/n$ for any Gutkin (n, k)-gon.*

Proof. Let $P \in G(2k, k)$ for some $k \geq 2$. For each i, $\angle v_{i+k} v_i v_{i+1} = \angle v_{i+k} v_i v_{i-1} = \alpha$. Then all interior angles of P are equal to 2α. Since the sum of the interior angles

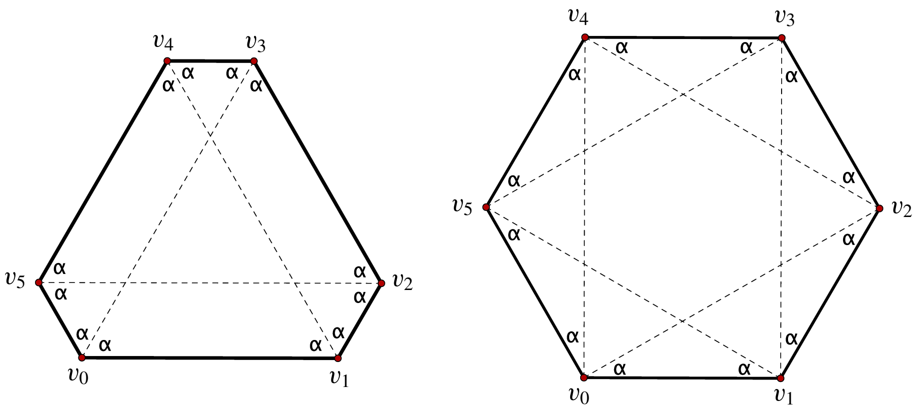

Figure 4. Two Gutkin polygons with angles labeled. Left: Gutkin (6, 3)-gon. Right: Gutkin (6, 2)-gon.

of any n-gon is equal to $\pi(n-2)$, we have $\alpha = \pi(n-2)/(2n)$, which is equal to $\pi(k-1)/n$.

Now assume that $n \neq 2k$. First, note that the sum of the interior angles of the Gutkin polygon equals $(n-2)\pi$ and also equals

$$\sum_{i=0}^{n-1} \beta_i + 2n\alpha;$$

see Figure 4. Therefore,

$$\alpha = \frac{\pi(n-2) - \sum_{i=0}^{n-1} \beta_i}{2n}. \tag{4-1}$$

For fixed n and k, let $P \in G(n,k)$. For $1 \le j \le \gcd(n,k)$, define the polygon

$$Q_j = \overline{v_j v_{j+k} v_{j+2k} \cdots v_{j+(nk/\gcd(n,k))-1}}.$$

Two examples of the Q_j are shown in Figure 5. Note that the sides of Q_j are the diagonals of P. The vertices of all Q_j form a disjoint partition of $\{v_0, v_1, \dots, v_{n-1}\}$ into $\gcd(n,k)$ subsets of equal size. Thus, the sum of the interior angles of all Q_j is $\sum_{i=0}^{n-1} \beta_i$.

Each Q_j is a star polygon with the number of vertices $N = n/\gcd(n,k)$ and the turning number $W = k/\gcd(n,k)$. The sum of the interior angles of such a polygon equals $\pi(N-2W)$, that is, $\pi(n-2k)/\gcd(n,k)$. One has $\gcd(n,k)$ polygons Q_j; hence, the total sum of their exterior angles is $\pi(n-2k)$. Substituting into (4-1) yields the result. □

Proposition 4.2. *In a Gutkin (n,k)-gon, the interior angles associated to vertices v_i and v_{i+k-1} are equal for all i.*

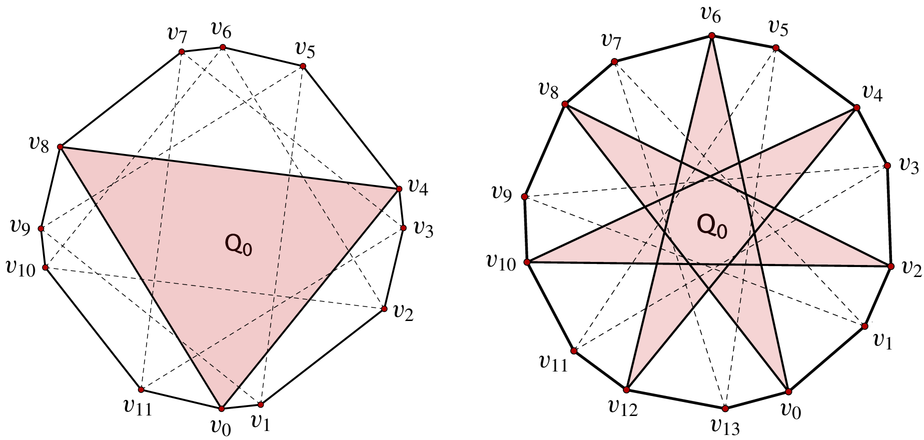

Figure 5. Polygons Q_0 on two Gutkin polygons. Left: Q_0 for a Gutkin (12, 4)-gon. Right: Q_0 for a Gutkin (14, 6)-gon.

Proof. Consider the self-intersecting quadrilateral $B_i = v_i v_{i+k} v_{i+k+1} v_{i+1}$; see Figure 6. Let w_i denote the intersection point of the two diagonals, $\overline{v_i v_{i+k}}$ and $\overline{v_{i+1} v_{i+k+1}}$. Notice that B_i is comprised of two triangles meeting at w_i. The opposite angles at w_i are equal, and the angle at v_i and v_{i+k+1} is equal to α. Therefore, the angles at v_{i+1} and v_{i+k} are equal, which are also equal to $\alpha + \beta_{i+1}$ and $\alpha + \beta_{i+k}$, respectively. Then $\beta_{i+1} = \beta_{i+k}$. Since the interior angle associated to any v_j is equal to $2\alpha + \beta_j$, the desired result follows. □

Corollary 4.3. *If n and $k - 1$ are coprime, then any $P \in G(n, k)$ is equiangular.*

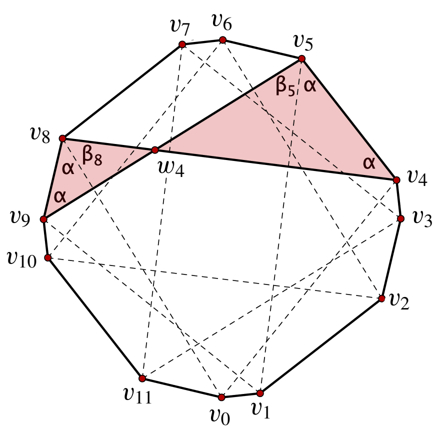

Figure 6. A Gutkin (12, 4)-gon. The shaded region is B_4.

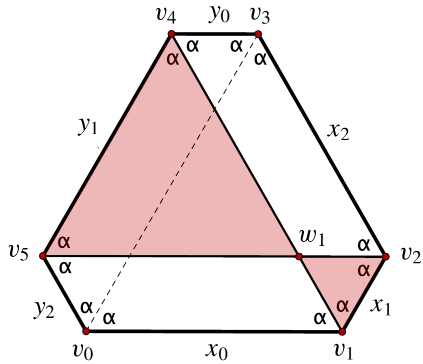

Figure 7. A Gutkin $(6, 3)$-gon with side lengths labeled. The shaded region is B_1.

The case $n = 2k$ is special in that Gutkin polygons abound (in the continuous case, this corresponds to the contact angle $\pi/2$, that is, when Gutkin curves are curves of constant width). Let $\mathbb{R}^n_+$ be the positive orthant.

Proposition 4.4. *The dimension of the space of Gutkin* $(2k, k)$*-gons, considered modulo similarities, equals* $k - 2$*. This quotient space is the intersection of a* $(k-2)$*-dimensional affine subspace with an open cube in* $\mathbb{R}^k$.

Proof. Let P be a Gutkin $(2k, k)$-gon. Consider the diagonals $\overline{v_i v_{i+k}}$ and $\overline{v_{i+1} v_{i+k+1}}$ of $G(2k, k)$; see Figure 7. Let w_i denote the intersection of these two diagonals, and let B_i be the bow-tie-shaped polygon $\overline{v_i v_{i+1} v_{i+k+1} v_{i+k}}$. Notice that $\triangle v_i v_{i+1} w_i$ and $\triangle v_{i+k+1} v_{i+k} w_i$ are both isosceles triangles and are similar.

Thus, $v_i w_i = v_{i+1} w_i$ and $v_{i+k} w_i = v_{i+k+1} w_i$. Hence, the diagonals $v_i v_{i+k}$ and $v_{i+1} v_{i+k+1}$ have equal length. Since i is arbitrary and the indices are circular, all diagonals have the same length, say, h. Since h is just a scaling factor, we set $h = 1$ for the remainder of the proof.

Notice that P is comprised of k polygons B_i. Let x_i denote the length of $\overline{v_i v_{i+1}}$ for $0 \le i \le k-1$, and let y_i denote the length of $\overline{v_{i+k} v_{i+k+1}}$, where $0 \le i \le k-1$. Note that x_i and y_i denote the lengths of the nonintersecting sides of B_i.

Assume that v_0 is at the origin and v_1 lies on the positive x axis, and recall that the vertices are labeled in counterclockwise order. This factors out the action of the isometry group of the plane. We shall show that $x_0, \dots, x_{k-1}$ uniquely determine $y_0, \dots, y_{k-1}$ and study the condition that these sides form a closed polygon.

Since the diagonals have fixed length equal to 1, one has $y_i = 2\cos\alpha - x_i$. Also, v_k is at the point $(\cos\alpha, \sin\alpha)$. Viewing the sides of $G(2k, k)$ as vectors, the i-th side is $x_i(\cos i\theta, \sin i\theta)$, where $\theta = \pi - 2\alpha = \pi/k$, and the sum of these vectors must be equal to v_k. Thus,

$$\sum_{i=0}^{k-1} x_i(\cos i\theta, \sin i\theta) = (\cos\alpha, \sin\alpha). \tag{4-2}$$

If the side lengths $x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}$ form a closed polygon, then the sides with lengths y_i must start at v_k and end at v_0. In other words, the side lengths satisfy

$$v_k + \sum_{i=0}^{k-1} y_i(\cos(\pi + i\theta), \sin(\pi + i\theta)) = v_0. \tag{4-3}$$

Simplifying the left-hand side yields

$$\begin{aligned}
(\cos\alpha, \sin\alpha) &+ \sum_{i=0}^{k-1} y_i(-\cos i\theta, -\sin i\theta)\\
&= (\cos\alpha, \sin\alpha) - \sum_{i=0}^{k-1}(2\cos\alpha - x_i)(\cos i\theta, \sin i\theta)\\
&= (\cos\alpha, \sin\alpha) - 2\cos\alpha \sum_{i=0}^{k-1}(\cos i\theta, \sin i\theta) + \sum_{i=0}^{k-1} x_i(\cos i\theta, \sin i\theta)\\
&= (\cos\alpha, \sin\alpha) - 2\cos\alpha(1, \tan\alpha) + (\cos\alpha, \sin\alpha) = (0, 0) = v_0.
\end{aligned}$$

Thus, (4-2) implies (4-3).

Hence, $G(2k, k)$ is determined by the k-tuple $x_0, \ldots, x_{k-1}$ satisfying the two linear equations (4-2). In addition, $0 < x_i < 2\cos\alpha$ for all i. This implies the result. □

Next we consider other equiangular cases.

Proposition 4.5. *The quotient space of the space of equiangular Gutkin (n, k)-gons by the group of similarities is identified with the intersection of an M-dimensional affine subspace with $\mathbb{R}^n_+$, where M is equal to the number of positive integers $2 \le r \le n-2$ satisfying the equation*

$$\tan\frac{kr\pi}{n}\tan\frac{\pi}{n} = \tan\frac{k\pi}{n}\tan\frac{r\pi}{n}. \tag{4-4}$$

Proof. Let $P \in G(n, k)$ be embedded in the complex plane with $v_0 = 0$ and v_1 on the positive real axis. Let $x_i = |v_{i+1} - v_i|$ for $0 \le i \le k-1$ be the side lengths of P. Let $\omega = \exp(2\pi/n)$. Notice that $v_{i+1} - v_i = x_i\omega^i$, and a diagonal can be represented as

$$v_{i+k} - v_i = a_i\omega^{i+m}, \tag{4-5}$$

where $a_i \in \mathbb{R}$, $a_i > 0$ and $m = (k-1)/2$. Notice that in this representation,

$$\begin{aligned}\arg(v_{i+1} - v_i) &= (2\pi i)/n,\\ \arg(v_{i+k} - v_{i+k-1}) &= 2\pi(i+k-1)/n,\\ \arg(v_{i+k} - v_i) &= \pi(2i+k-1)/n.\end{aligned}$$

Then

$$\angle v_{i+1} v_i v_{i+k} = \angle v_{i+k-1} v_{i+k} v_i = \pi(k-1)/n = \alpha.$$

Moreover,

$$\begin{aligned} v_{i+k} - v_i &= (v_{i+k} - v_{i+k-1}) + (v_{i+k-1} - v_{i+k-2}) + \cdots + (v_{i+1} - v_i)\\ &= \omega^{i+k-1} x_{i+k-1} + \omega^{i+k-2} x_{i+k-2} + \cdots + \omega^i x_i\\ &= \omega^i x_i + \omega^{i+1} x_{i+1} + \cdots + \omega^{i+k-1} x_{i+k-1}.\end{aligned}$$

From (4-5), $v_{i+k} - v_i$ is also equal to $a_i \omega^{i+m}$. Thus,

$$\begin{aligned} a_i \omega^{i+m} &= \omega^i x_i + \omega^{i+1} x_{i+1} + \cdots + \omega^{i+k-1} x_{i+k-1}\\ a_i &= \omega^{-m} x_i + \omega^{1-m} x_{i+1} + \cdots + \omega^{k-1-m} x_{k-1}.\end{aligned}$$

Using $a_i - \overline{a_i} = 0$, one has

$$(\omega^{-m} - \omega^m) x_i + (\omega^{1-m} - \omega^{m-1}) x_{i+1} + \cdots + (\omega^{k-1-m} - \omega^{m-k+1}) x_{k-1} = 0.$$

This gives a system of n linear equations on variables x_i. The coefficient matrix, A, is a circulant matrix where the first row is equal to

$$\begin{pmatrix}\omega^{-m} - \omega^m & \omega^{1-m} - \omega^{m-1} & \cdots & \omega^{k-1-m} - \omega^{m-k+1} & 0 & 0 & \cdots & 0\end{pmatrix}.$$

Then the eigenvalues of A are

$$\lambda_r = \sum_{\nu=0}^{k-1} (\omega^{\nu-m} - \omega^{m-\nu}) \omega^{\nu r}; \tag{4-6}$$

see [Davis 1979].

We expect one of the eigenvalues to be equal to zero because we have not factorized by scaling yet. If no other eigenvalue equals zero, then only trivial solutions exist. Now, we compute λ_r in three cases: $r = 0$, $r = 1$ or $r = n-1$, and $2 \le r \le n-2$.

For $r = 0$, we have

$$\lambda_0 = \omega^{-m} \sum_{\nu=0}^{k-1} \omega^\nu - \omega^m.$$

Let h be equal to $|\omega^i + \cdots + \omega^{i+k}|$. By rotational symmetry, h does not vary with i. Now evaluating the above equation,

$$\lambda_0 = h\omega^{-m}\omega^m - h\omega^m\omega^{-m} = 0.$$

Thus, for $r = 0$, A has eigenvalue λ_0 equal to zero.

Assume that λ_r is equal to zero for some other r. Set (4-6) to zero and simplify:

$$\sum_{\nu=0}^{k+1} \omega^{(r+1)\nu} = \omega^{k-1} \sum_{\nu=0}^{k-1} \omega^{(r-1)\nu}. \tag{4-7}$$

For $r = 1$, (4-7) can be written as $k\omega^{k-1} = \sum_{\nu=0}^{k-1} \omega^{2\nu}$. Then $k = \left|\sum_{\nu=0}^{k-1} \omega^{2\nu}\right|$. This is true only if the $\omega^{2\nu}$ are collinear, which is clearly not the case. Thus, $\lambda_1 \neq 0$ and likewise for $r = n-1$.

For $2 \le r \le n-2$, using geometric series, we can rewrite (4-7) as

$$\frac{\omega^{k(r+1)} - 1}{\omega^{r+1} - 1} = \omega^{k-1} \frac{\omega^{k(r-1)} - 1}{\omega^{r-1} - 1}. \tag{4-8}$$

After expanding this equation in terms of sines and cosines and using trigonometric identities, one rewrites it as (4-4). For any solution r, one obtains $\lambda_r = 0$. This implies the claim. □

We are ready to prove Theorem 1.2.

If n and $k-1$ are coprime, then a Gutkin polygon is equiangular by Corollary 4.3. Connelly and Csikós [2009] show that a solution to (4-4) for integer values $1 < k$ and $r < n/2$ must satisfy $k + r = n/2$ and $n \mid (k-1)(r-1)$. Since n and $k-1$ are coprime, there are no solutions. Note also that, if r is a solution, so is $n-r$. Thus, by Proposition 4.5, the matrix A has corank 1 and the Gutkin polygon must be regular.

It remains to construct a nontrivial Gutkin polygon for noncoprime n and $k-1$. Let $p = \gcd(n, k-1)$ and $q = n/p$. Choose angles $\theta_1, \ldots, \theta_p$ such that $\theta_1 + \cdots + \theta_p = 2\pi/q$. Divide a unit circle into q equal parts, and divide each of these equal arcs into p arcs of lengths $\theta_1, \ldots, \theta_p$ in this order. One obtains an inscribed n-gon. See Figure 8 for $n = 8$ and $k = 3$.

Lemma 4.6. *The constructed n-gon is a Gutkin polygon.*

Proof. The angular measure of an inscribed angle is half that of the subtended arc. It follows that

$$\angle v_{i+1} v_i v_{i+k} = \angle v_{i+k-1} v_{i+k} v_i = \frac{\theta_1 + \cdots + \theta_p}{2} = \frac{\pi}{q}. \qquad \square$$

Since the choice of the angles $\theta_1, \ldots, \theta_p$ was arbitrary, we obtain a $(p-1)$-parameter family of pairwise nonsimilar Gutkin polygons.

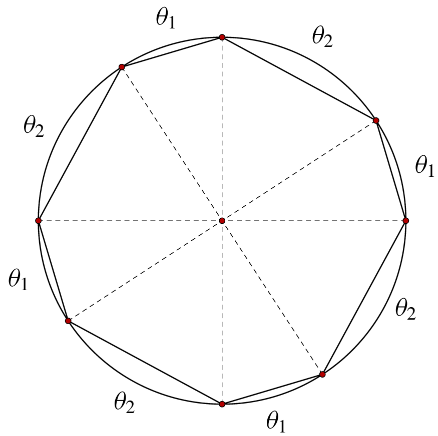

Figure 8. Constructing a nontrivial Gutkin polygon.

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