DETERMINANT RANK OF $C^*$-ALGEBRAS

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Dedicated to George A. Elliott on his seventieth birthday

Let $A$ be a unital $C^*$-algebra and let $U_0(A)$ be the group of unitaries of $A$ which are path-connected to the identity. Denote by $CU(A)$ the closure of the commutator subgroup of $U_0(A)$. Let $i_A^{(1,n)} : U_0(A)/CU(A) \to U_0(M_n(A))/CU(M_n(A))$ be the homomorphism defined by sending $u$ to $\text{diag}(u, 1_{n-1})$. We study the problem of when the map $i_A^{(1,n)}$ is an isomorphism for all $n$. We show that it is always surjective and that it is injective when $A$ has stable rank one. It is also injective when $A$ is a unital $C^*$-algebra of real rank zero, or $A$ has no tracial state. We prove that the map is an isomorphism when $A$ is Villadsen’s simple AH-algebra of stable rank $k > 1$. We also prove that the map is an isomorphism for all Blackadar’s unital projectionless separable simple $C^*$-algebras. Let $A = M_n(C(X))$, where $X$ is any compact metric space. We note that the map $i_A^{(1,n)}$ is an isomorphism for all $n$. As a consequence, the map $i_A^{(1,n)}$ is always an isomorphism for any unital $C^*$-algebra $A$ that is an inductive limit of the finite direct sum of $C^*$-algebras of the form $M_n(C(X))$ as above. Nevertheless we show that there is a unital $C^*$-algebra $A$ such that $i_A^{(1,2)}$ is not an isomorphism.

1. Introduction

Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group. Denote by $U_0(A)$ the normal subgroup which is the connected component of $U(A)$ containing the identity of $A$. Denote by $DU(A)$ the commutator subgroup of $U_0(A)$ and by $CU(A)$ the closure of $DU(A)$. We will study the group $U_0(A)/CU(A)$. Recently this group has become an important invariant for the structure of $C^*$-algebras. It plays an important role in the classification of $C^*$-algebras (see [Elliott and Gong 1996; Nielsen and Thomsen 1996; Elliott 1997; Thomsen 1997; Gong 2002; Elliott et al. 2007; Lin 2007; 2011; Gong et al. 2015], for example). It was shown in [Lin 2007] that the map $U_0(A)/CU(A) \to U_0(M_n(A))/CU(M_n(A))$ is an isomorphism for all $n \geq 1$ if $A$ is a unital simple $C^*$-algebra of tracial rank at most one (see also [Lin

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MSC2010: primary 46L06, 46L35; secondary 46L80.

Keywords: determinant rank for $C^*$-algebras.
In general, when $A$ has stable rank $k$, it was shown by Rieffel [1987] that the map $U(M_k(A))/U_0(M_k(A)) \to U(M_{k+m}(A))/U_0(M_{k+m}(A))$ is an isomorphism for all integers $m \geq 1$. In this case $U(M_k(A))/U_0(M_k(A)) = K_1(A)$. This fact plays an important role in the study of the structure of $C^*$-algebras, in particular those $C^*$-algebras of stable rank one, since it simplifies computations when $K$-theory involved. Therefore it seems natural to ask when the map $i_A^{(1,n)} : U_0(A)/CU(A) \to U_0(M_n(A))/CU(M_n(A))$ is an isomorphism for all integers $n \geq 1$. The main tool to study $U_0(M_n(A))/CU(M_n(A))$ is the de la Harpe–Skandalis determinant, studied early by K. Thomsen [1995] (henceforth abbreviated [Th]), which involves the tracial state space $T(A)$ of $A$. On the other hand, we observe that when $T(A) = \emptyset$, $U_0(A)/CU(A) = \{0\}$. So we focus our attention on the case $T(A) \neq \emptyset$. One of the authors was asked repeatedly if the map $i_A^{(1,n)}$ is an isomorphism when $A$ has stable rank one.

It turns out that it is easy to see that the map $i_A^{(1,n)}$ is always surjective for all $n$. Therefore the issue is when $i_A^{(1,n)}$ is injective.

**Definition 1.1.** Let $A$ be a unital $C^*$-algebra. Consider the homomorphism

$$i_A^{(m,n)} : U_0(M_m(A))/CU(M_m(A)) \to U_0(M_n(A))/CU(M_n(A))$$

(induced by $u \mapsto \text{diag}(u, 1_{n-m})$) for integers $n \geq m \geq 1$. The determinant rank of $A$ is defined to be

$$\text{Dur} A = \min \{m \in \mathbb{N} \mid i_A^{(m,n)} \text{ is isomorphism for all } n > m\}.$$

If no such integer exists, we set $\text{Dur} A = \infty$.

We show that if $A = \lim_{n \to \infty} A_n$, then $\text{Dur} A \leq \sup_{n \geq 1} \{\text{Dur} A_n\}$. We prove that $\text{Dur} A = 1$ for all $C^*$-algebras of stable rank one, which answers the question mentioned above. We also show that $\text{Dur} A = 1$ for any unital $C^*$-algebra $A$ with real rank zero. A closely related and repeatedly used fact is that the map $u \to u + (1 - e)$ is an isomorphism from $U(eAe)/CU(eAe)$ onto $U(A)/CU(A)$ when $A$ is a unital simple $C^*$-algebra of tracial rank at most one and $e \in A$ is a projection (see [Lin 2007, Theorem 6.7; 2010b, Theorem 3.4]). We show in this note that this holds for any simple $C^*$-algebra of stable rank one.

Given Rieffel’s early result mentioned above, one might be led to think that, when $A$ has higher stable rank, or at least when $A = C(X)$ for higher-dimensional finite CW complexes, $\text{Dur} A$ is perhaps large. On the other hand it was suggested (see [Th, Section 3]) that $\text{Dur} A = 1$ may hold for most unital simple separable $C^*$-algebras. We found out, somewhat surprisingly, that the determinant rank of $M_n(C(X))$ is always 1 for any compact metric space $X$ and for any integer $n \geq 1$. This, together with previous mentioned result, shows that if $A = \lim_{n \to \infty} A_n$, where $A_n$ is a finite
direct sum of $C^*$-algebras of the form $M_n(C(X))$, then $\text{Dur} A = 1$. Furthermore, we found out that $\text{Dur} A = 1$ for all of Villadsen’s examples of unital simple AH-algebras $A$ with higher stable rank. This research suggests that when $A$ has an abundant amount of projections then $\text{Dur} A$ is likely to be 1 (see Proposition 3.6(3)). In fact, we prove that if $A$ is a unital simple AH-algebra with property (SP), then $\text{Dur} A = 1$. On the other hand, however, we show that if $A$ is one of Blackadar’s examples of unital projectionless simple $C^*$-algebras with infinite many extremal tracial states, then $\text{Dur} A = 1$. Indeed, it seems that it is difficult to find any example of unital separable simple $C^*$-algebras for which $\text{Dur} A$ is larger than 1. Nevertheless Proposition 3.12 below provides a necessary condition for $\text{Dur} A = 1$. In fact, we find that a certain unital separable $C^*$-algebra $A$ violates this condition, which, in turn, provides an example of a unital separable $C^*$-algebra $A$ such that $\text{Dur} A > 1$.

2. Preliminaries

In this section, we list some notation and basic known facts for convenience, many of which are taken from [Th] and other sources.

**Definition 2.1.** Let $A$ be a $C^*$-algebra. Denote by $M_n(A)$ the $n \times n$ matrix algebra of over $A$. If $A$ is not unital, we will use $\tilde{A}$, the unitization of $A$, so suppose that $A$ is unital. For $u$ in $U(A)$, let $[u]$ be the class of $u$ in $U_0(A)/CU(A)$.

We view $A^n$ as the set of all $n \times 1$ matrices over $A$. Set

$$S_n(A) = \left\{ (a_1, \ldots, a_n)^T \in A^n \mid \sum_{i=1}^n a_i^* a_i = 1 \right\},$$

$$Lg_n(A) = \left\{ (a_1, \ldots, a_n)^T \in A^n \mid \sum_{i=1}^n b_i a_i = 1 \text{ for some } b_1, \ldots, b_n \in A \right\}.$$

According to [Rieffel 1983; 1987], the topological stable rank and the connected stable rank of $A$ are defined as

$$\text{tsr} A = \min\{n \in \mathbb{N} \mid Lg_m(A) \text{ is dense in } A^m \text{ for all } m \geq n\},$$

$$\text{csr} A = \min\{n \in \mathbb{N} \mid U_0(M_m(A)) \text{ acts transitively on } S_m(A) \text{ for all } m \geq n\}.$$

If no such integer exists, we set $\text{tsr} A = \infty$ and $\text{csr} A = \infty$. These notions are very useful tools in computing $K$-groups of $C^*$-algebras (see, e.g., [Rieffel 1987; Xue 2000; 2001; 2010]).

**Definition 2.2.** Let $A$ be a $C^*$-algebra. Denote by $A_{sa}$ (resp. $A_+$) the set of all self-adjoint (resp. positive) elements in $A$. Denote by $T(A)$ the tracial state space of $A$. Let $\tau \in T(A)$. We will also use the notation $\tau$ for the unnormalized trace
\( \tau \otimes \text{Tr}_n \) on \( M_n(A) \), where \( \text{Tr}_n \) is the standard trace for \( M_n(\mathbb{C}) \). Every tracial state on \( M_n(A) \) has the form \( (1/n)\tau \).

**Definition 2.3.** For \( a, b \in A \), set \([a, b] = ab - ba\). Furthermore, set
\[
[A, A] = \left\{ \sum_{j=1}^{n} [a_j, b_j] \mid a_j, b_j \in A, \ j = 1, \ldots, n, \ n \geq 1 \right\}.
\]

Now, let \( A_0 \) denote the subset of \( A_{sa} \) consisting of elements of the form \( x - y \) for \( x, y \in A_{sa} \) with \( x = \sum_{j=1}^{\infty} c_j c_j^* \) and \( y = \sum_{j=1}^{\infty} c_j^* c_j \) (convergent in norm) for some sequence \( \{c_j\} \) in \( A \). By [Cuntz and Pedersen 1979], \( A_0 \) is a closed subspace of \( A_{sa} \).

**Proposition 2.4** [Cuntz and Pedersen 1979; Thomsen 1995, Section 3]. Let \( A \) be a \( C^* \)-algebra with unit 1. The following statements are equivalent:

1. \( A_0 = A_{sa} \).
2. \( 1 \in A_0 \).
3. \( \mathcal{T}(A) = \emptyset \).
5. \( A_{sa} = \text{span}\{[a^*, a] \mid a \in A\} \).

**Proof.** (1) \( \Rightarrow \) (2) is obvious.

(2) \( \Rightarrow \) (3). If \( \mathcal{T}(A) \neq \emptyset \), then there is a tracial state \( \tau \) on \( A \). Since \( 1 \in A_0 \), it follows that there is a sequence \( \{a_j\} \) in \( A \) such that \( b = \sum_{j=1}^{\infty} a_j^* a_j \) and \( c = \sum_{j=1}^{\infty} a_j a_j^* \) are convergent in \( A \) and \( 1 = b - c \). Thus, \( \tau(b) = \sum_{j=1}^{\infty} \tau(a_j^* a_j) = \tau(c) \) and \( \tau(1) = \tau(b - c) = 0 \), a contradiction since \( \tau(1) = 1 \).

(3) \( \Rightarrow \) (1). This follows from the proof of [Th, Lemma 3.1].

(4) \( \iff \) (5). Let \( a, b \in A \) and write \( a = a_1 + ia_2 \) and \( b = b_1 + ib_2 \), where \( a_1, a_2, b_1, b_2 \in A_{sa} \). Then
\[
[a, b] = [a_1, b_1] - [a_2, b_2] + i[a_2, b_1] + i[a_1, b_2].
\]

Put \( c_1 = a_1 + ib_1, c_2 = a_2 + ib_2, c_3 = a_2 + ib_1 \) and \( c_4 = a_1 + ib_2 \). Then, from (2-1), we get that
\[
[a, b] = \frac{1}{2i}[c_1^*, c_1] - \frac{1}{2i}[c_2^*, c_2] + \frac{1}{2}[c_3^*, c_3] + \frac{1}{2}[c_4^*, c_4].
\]

So, by (2-2), (4) and (5) are equivalent.

(5) \( \Rightarrow \) (1). Let \( x \in \text{span}\{[a^*, a] \mid a \in A\} \). Then there are elements \( a_1, \ldots, a_k \in A \) and positive numbers \( \lambda_1, \ldots, \lambda_k \) such that \( x = \sum_{i=1}^{j-1} \lambda_i [a_i^*, a_i] - \sum_{i=j+1}^{k} \lambda_i [a_i^*, a_i] \) for some \( j \in \{1, \ldots, k\} \). Put \( c_i = \sqrt{\lambda_i} a_i, i = 1, \ldots, j \) and \( c_i^* = \sqrt{\lambda_i} a_i^* \) when
\[ i = j + 1, \ldots, k. \] Then \( x = \sum_{i=1}^{k} c_i^* c_i - \sum_{i=1}^{k} c_i c_i^* \in A_0. \) Since \( A_0 \) is closed, we get that
\[ A_{sa} = \text{span}\{[a^*, a] \mid a \in A\} \subset A_0 = A_0 \subset A_{sa}. \]

(1) \( \Rightarrow \) (5). According to the definition of \( A_0 \), every element \( x \in A_0 \) has the form \( x = x_1 - x_2 \), where \( x_1 = \sum_{i=1}^{\infty} z_i^* z_i \) and \( x_2 = \sum_{i=1}^{\infty} z_i z_i^* \). Thus, \( x \in \text{span}\{[a^*, a] \mid a \in A\} \) and hence \( A_{sa} = \text{span}\{[a^*, a] \mid a \in A\}. \]

Combining Proposition 2.4 with Definition 2.2, we have:

**Corollary 2.5.** Let \( A \) be a unital \( C^* \)-algebra with \( A_0 = A_{sa} \). Then \( (M_n(A))_0 = (M_n(A))_{sa} \).

Let \( a, b \in A_{sa}. \) Then, for any \( n \geq 1,
\[ \exp(i a) \exp(i b) \left( \exp\left(-i \frac{a}{n}\right) \exp\left(-i \frac{b}{n}\right) \right)^n \in DU(A) \]
and
\[ \exp(-i(a + b)) = \lim_{n \to \infty} (\exp(-i a/n) \exp(-i b/n))^n \]
by the Trotter product formula [Masani 1981, Theorem 2.2]. So \( \exp(i a) \exp(i b) \exp(-i(a + b)) \in CU(A). \) Consequently,
\[ \text{(2-3)} \quad [\exp(i a)] [\exp(i b)] = [\exp(i(a + b))] \quad \text{in} \quad U_0(A)/CU(A). \]

The following is taken from the proof of [Th, Lemma 3.1].

**Lemma 2.6.** Let \( a \in A_{sa}. \)

(1) If \( a \in A_0 \), then \( [\exp(i a)] = 0 \) in \( U_0(A)/CU(A); \)

(2) If \( T(A) \neq \emptyset \) and \( \tau(a) = \tau(b) \) for all \( \tau \in T(A) \), then \( a - b \in A_0 \) and \( [\exp(i a)] = [\exp(i b)] \) in \( U_0(A)/CU(A). \)

Combining Lemma 2.6(1) with Corollary 2.5, we have

**Corollary 2.7.** If \( T(A) = \emptyset \), then \( U_0(M_n(A)) = CU(M_n(A)) \) for \( n \geq 1. \)

**Definition 2.8.** Let \( A \) be a unital \( C^* \)-algebra with \( T(A) \neq \emptyset \). Let \( PU^n_0(A) \) denote the set of all piecewise smooth maps \( \xi : [0, 1] \to U_0(M_n(A)) \) with \( \xi(0) = 1_n \), where \( 1_n \) is the unit of \( M_n(A). \) For \( \tau \in T(A) \), the de la Harpe–Skandalis function \( \Delta^\tau_n \) on \( PU^n_0(A) \) is given by
\[ \Delta^\tau_n(\xi(t)) = \frac{1}{2\pi i} \int_0^1 \tau(\xi'(t)) (\xi(t))^* \, dt \quad \text{for all} \quad \xi \in PU^n_0(A). \]

Note that we use an unnormalized trace \( \tau = \tau \otimes \text{Tr}_n \) on \( M_n(A). \) This gives a homomorphism \( \Delta^n : PU^n_0(A) \to \text{Aff}(T(A)) \), the space of all real affine continuous functions on \( T(A). \)

We list some properties of \( \Delta^\tau_n(\cdot): \)
Lemma 2.9 [de la Harpe and Skandalis 1984, Lemmas 1 and 3]. Let $A$ be a unital $C^*$-algebra with $T(A) \neq \emptyset$. Let $\xi_1, \xi_2, \xi \in PU^n_0(A)$. Then:

1. $\Delta^n_\tau(\xi_1(t)) = \Delta^n_\tau(\xi_2(t))$ for all $\tau \in T(A)$, if $\xi_1(1) = \xi_2(1)$ and $\xi_1 \xi_2 \in U_0((C_0(S^1, M_n(A))))$.

2. There are $y_1, \ldots, y_k \in M_n(A)_{sa}$ such that $\Delta^n_\tau(\xi(t)) = \sum_{j=1}^k \tau(y_j)$ for all $\tau \in T(A)$ and $\xi(1) = \exp(i2\pi y_1) \cdots \exp(i2\pi y_k)$.

Definition 2.10. Let $A$ be a $C^*$-algebra with $T(A) \neq \emptyset$. Let $\text{Aff}(T(A))$ be the set of all real continuous affine functions on $T(A)$. Define $\rho_A : K_0(A) \to \text{Aff}(T(A))$ by

$$\rho_A([p])(\tau) = \tau(p) \quad \text{for all } \tau \in T(A),$$

where $p \in M_n(A)$ is a projection.

Define $P^n_0(A)$ to be the subgroup of $K_0(A)$ generated by projections in $M_n(A)$. Denote by $\rho_A^n(K_0(A))$ the subgroup $\rho_A(P^n_0(A))$ of $\rho_A(K_0(A))$. In particular, $\rho_A^n(K_0(A))$ is the subgroup of $\rho_A(K_0(A))$ generated by the images of projections in $A$ under the map $\rho_A$.

Definition 2.11. Let $A$ be a unital $C^*$-algebra. Denote by $LU^n_0(A)$ the set of piecewise smooth loops in $U(C_0(S^1, M_n(A)))$.

Then, by Bott periodicity, $\Delta^n(LU^n_0(A)) \subset \rho_A(K_0(A))$. Denote by

$$q^n : \text{Aff}(T(A)) \to \text{Aff}(T(A))/\Delta^n(LU^n_0(A))$$

the quotient map. Put $\overline{\Delta^n} = q^n \circ \Delta^n$. Since $\overline{\Delta^n}$ vanishes on $LU^n_0(A)$, we also use $\overline{\Delta^n}$ for the homomorphism from $U_0(M_n(A))$ into $\text{Aff}(T(A))/\Delta^n(LU^n_0(A))$. An important fact that we will repeatedly use is that the kernel of $\overline{\Delta^n}$ is exactly $CU(M_n(A))$, by [Th, Lemma 3.1]. In other words, if $u \in U_0(M_n(A))$ and $\overline{\Delta^n}(u) = 0$, then $u \in CU(M_n(A))$.

Corollary 2.12. Let $A$ be a unital $C^*$-algebra and let $u \in U_0(M_n(A))$ for $n \geq 1$. Then there are an $a \in A_{sa}$ and a $v \in CU(M_n(A))$ such that

$$u = \text{diag}(\exp(i2\pi a), 1_{n-1})v$$

(in the case $n = 1$, we define $\text{diag}(\exp(i2\pi a), 1_{n-1}) = \exp(i2\pi a)$).

Moreover, if there is a $u \in PU^n_0(A)$ with $u(1) = u$, we can choose a self-adjoint element $a$ so that $\hat{a} = \Delta^n(u(t))$, where $\hat{a}(\tau) = \tau(a)$ for all $\tau \in T(A)$. 
Proof. Fix a piecewise smooth path \( u(t) \in PU^n_0(A) \) with \( u(0) = 1 \) and \( u(1) = u \). By Lemma 2.9(2), there are \( a_1, a_2, \ldots, a_m \in M_n(A)_{sa} \) such that

\[
u = \prod_{j=1}^{m} \exp(i2\pi a_j) \quad \text{and} \quad \Delta^n_\tau(u(t)) = \tau \sum_{j=1}^{m} a_j \quad \text{for all} \quad \tau \in T(A).
\]

Put \( a_0 = \sum_{j=1}^{n} a_j \). Write \( a_0 = (b_{i,j})_{n \times n} \). Define \( a = \sum_{i=1}^{n} b_{i,i} \). Then \( a \in A_{sa} \). Moreover,

\[
\Delta^n(a) = \exp(-i2\pi a), \quad \text{and} \quad \nabla_n(a) = 0.
\]

Thus, by [Th, Lemma 3.1], \( \exp(-i2\pi a), \) for all \( \tau \in T(A) \). Put \( v = \exp(-i2\pi a), \) for all \( \tau \in T(A) \). Then \( u = \exp(i2\pi a), \) for all \( \tau \in T(A) \). \( \square \)

3. Determinant rank

Let \( A \) be a unital \( C^* \)-algebra. Consider the homomorphism

\[
i_{A}^{(m,n)} : U_0(M_m(A)) / CU(M_m(A)) \rightarrow U_0(M_n(A)) / CU(M_n(A))
\]

for integers \( n \geq m \geq 1 \).

Proposition 3.1. Let \( A \) be a unital \( C^* \)-algebra with \( T(A) \neq \emptyset \). Then

\[
i_{A}^{(m,n)} : U_0(M_m(A)) / CU(M_m(A)) \rightarrow U_0(M_n(A)) / CU(M_n(A))
\]

is surjective for \( n \geq m \geq 1 \).

Proof. It suffices to show that \( i_{A}^{(1,n)} \) is surjective. Let \( u \in U_0(M_n(A)) \). It follows from Corollary 2.12 that \( u = \exp(i2\pi a), 1_{n-1}u \) for some \( a \in A_{sa} \) and \( v \in CU(M_n(A)) \). Then \( i_{A}^{(1,n)}([\exp(i2\pi a)]) = [u] \). \( \square \)

Lemma 3.2. Let \( A \) be a unital \( C^* \)-algebra with \( T(A) \neq \emptyset \). Assume \( u \in U_0(M_m(A)) \).

(1) If \( \Delta^n(\exp(u(t), 1_{n-m}u) \in \Delta^n(LU^n_0(A)) \) for some \( n > m \), where \( \{u(t) : t \in [0, 1]\} \) is a piecewise smooth path with \( u(0) = 1_m \) and \( u(1) = u \), then, for any \( \epsilon > 0 \), there exist \( a \in M_m(A)_{sa} \) with \( \|a\| < \epsilon, b \in M_m(A)_{sa} \), \( v \in CU(M_m(A)) \) and \( w \in LU^n_0(A) \) such that

\[
u = \exp(i2\pi a) \exp(i2\pi b)v \quad \text{and} \quad \tau(b) = \Delta^n_\tau(w(t)) \quad \text{for all} \quad \tau \in T(A).
\]

(2) If \( \Delta^n(u(t)) \in \rho_A(K_0(A)) \) for some \( u \in PU^n_0(A) \) with \( u(1) = u \), then, for any \( \epsilon > 0 \), there exist \( a \in M_m(A)_{sa} \) with \( \|a\| < \epsilon, b \in M_m(A)_{sa} \) and \( v \in CU(M_m(A)) \) such that

\[
u = \exp(i2\pi a) \exp(i2\pi b)v \quad \text{and} \quad \hat{b} \in \rho_A(K_0(A)),
\]

where \( \hat{b}(\tau) = \tau(b) \) for all \( \tau \in T(A) \).
Proof. Let $\epsilon > 0$. For (1), there is a $w \in LU^n_0(A)$ such that
\begin{equation}
\sup\{ |\Delta^n_\tau(u(t)) - \Delta^n_\tau(w(t))| : \tau \in T(A) \} < \epsilon/3\pi.
\end{equation}
There is an $a_1 \in M_m(A)_{sa}$ by Corollary 2.12 such that
\begin{equation}
t(\tau(a_1)) = \Delta^n_\tau(u(t)) - \Delta^n_\tau(w(t)) \quad \text{for all } \tau \in T(A).
\end{equation}
Combining (3-3) with [Cuntz and Pedersen 1979] and the proof of [Th, Lemma 3.1], we can find $a \in M_m(A)_{sa}$ such that $|a| < \epsilon/2\pi$. There is also a $b \in A_{sa}$ such that $\tau(b) = -\Delta^n_\tau(w(t))$ for all $\tau \in T(A)$. Put
\begin{equation}
v(t) = \exp(-i2\pi bt) \exp(-i2\pi at)u(t) \quad \text{for } t \in [0, 1]
\end{equation}
and $v = v(1)$. Then $\Delta^n(v(t)) = 0$. It follows from [Th, Lemma 3.1] that $v \in CU(A)$. Then $u = \exp(i2\pi a) \exp(i2\pi b)v$.

For (2), there are an integer $n \geq m$ and projections $p, q \in M_n(A)$ such that (for a piecewise smooth path $\{u(t) : t \in [0, 1]\}$ with $u(0) = 1_n$ and $u(1) = u$)
\begin{equation}
||\Delta^n_\tau(u(t)) - \tau(p) + \tau(q)|| < \epsilon \quad \text{for all } \tau \in T(A).
\end{equation}
Let $b \in M_m(A)_{sa}$ such that $\tau(b) = \tau(p) - \tau(q)$ for all $\tau \in T(A)$ (see the proof above); there is an $a \in M_m(A)_{sa}$ with $|a| < \epsilon$ such that
\begin{equation}
t(\tau(a)) = \Delta^n_\tau(u(t)) - \tau(p) + \tau(q) \quad \text{for all } \tau \in T(A).
\end{equation}
Let $v = u \exp(-i2\pi a) \exp(-i2\pi b)$ and $v(t) = u(t) \exp(-i2\pi at) \exp(-i2\pi bt)$. Then $\Delta^n(v(t)) = 0$. It follows from [Th, Lemma 3.1] that $v \in CU(M_m(A))$. \qed

Let $A$ be a unital $C^*$-algebra. Let $Dur A$ be defined as in Definition 1.1. It follows from Corollary 2.7 that if $T(A) = \emptyset$ then $Dur A = 1$.

**Proposition 3.3.** Let $A$ be a unital $C^*$-algebra. Then, for any integer $n \geq 1$,
\[
Dur(M_n(A)) \leq \left\lfloor \frac{Dur A - 1}{n} \right\rfloor + 1,
\]
where $\lfloor x \rfloor$ is the integer part of $x$.

**Proof.** Note that $n(\lfloor (Dur A - 1)/n \rfloor + 1) \geq Dur A$. \qed

**Theorem 3.4.** Let $A$ be a unital $C^*$-algebra, and $I \subset A$ a closed ideal of $A$ such that the quotient map $\pi : A \to A/I$ induces the surjective map from $K_0(A)$ onto $K_0(A/I)$. Then $Dur(A/I) \leq Dur A$.

**Proof.** Let $m = Dur A$ and $n > m$. Let $u \in U_0(M_m(A/I))$ be a unitary such that $\text{diag}(u, 1_{n-m}) \in CU(M_n(A/I))$. We will show that $u \in CU(M_m(A/I))$. 
Let \( \epsilon > 0 \). By Lemma 3.2, without loss of generality we may assume that there are \( a_1, b_1 \in (M_m(A/I))_{sa} \) such that

\[
(3-8) \quad u = \exp(i2\pi a_1)\exp(i2\pi b_1)v,
\]

where \( q_1, q_2 \in M_K(A/I) \) are projections for some large \( K \geq m \), for all \( \tau \in T(A/I) \).

By the assumption, without loss of generality we may assume further that there are projections \( p_1, p_2 \in M_K(A) \) such that \( \pi_*([p_1 - [p_2]] = [q_1] - [q_2], \) where \( \pi_* : K_0(A) \to K_0(A/I) \) is induced by \( \pi \). Let \( b_2 \in (M_m(A))_{sa} \) such that \( \tau(b_2) = \tau(p_1) - \tau(p_2) \) for all \( \tau \in T(A) \). There exists an \( a \in (M_m(A))_{sa} \) such that \( \pi_m(a) = a_1 \), where \( \pi_m : M_m(A) \to M_m(A/I) \) is the map induced by \( \pi \). Then, by (3-8),

\[
(3-9) \quad \pi_m(\exp(i2\pi a))\pi_m(\exp(i2\pi b_2))u^* \in CU(M_m(A/I)).
\]

Put \( u_1 = \pi_m(\exp(i2\pi a))\pi_m(\exp(i2\pi b_2)) \). Let \( w = \exp(i2\pi b_2) \). Then \( \Delta(w) = 0 \). Since \( m = \text{Dur } A \), this implies that \( w \in CU(M_m(A)) \). It follows that \( \pi_m(w) \in CU(M_m(A/I)) \), which implies by (3-9) that \( \text{dist}(u, CU(M_m(A/I))) < \epsilon \). \( \square \)

**Theorem 3.5.** Let \( A = \lim_{n \to \infty} (A_n, \phi_n) \) be a unital \( C^* \)-algebra, where each \( A_n \) is unital. Suppose that \( \text{Dur } A_n \leq r \) for all \( n \). Then \( \text{Dur } A \leq r \).

**Proof.** We write \( \phi_{n_1,n_2} : A_{n_1} \to A_{n_2} \) for \( \phi_{n_2} \circ \phi_{n_2-1} \circ \cdots \circ \phi_1 \) and \( \phi_{n_1} : A_{n_1} \to A \) for the map induced by the inductive limit system. Let \( u \in U_0(M_r(A)) \) such that \( u_1 = \text{diag}(u, 1_{n-r}) \in CU(M_n(A)) \) for some \( n > r \). Let \( \epsilon > 0 \). There is a \( v \in DU(M_n(A)) \) such that

\[
(3-10) \quad \|u_1 - v\| < \frac{\epsilon}{8n}.
\]

Write \( v = \prod_{j=1}^K v_j \), where \( v_j = x_jy_jx_j^*y_j \) and \( x_j, y_j \in U_0(M_n(A)) \) for \( j = 1, 2, \ldots, K \). Choose a large \( N \geq 1 \) such that there are \( v' \in U_0(M_r(A_N)) \) and \( x_j', y_j' \in U_0(M_n(A_N)) \) such that

\[
(3-11) \quad \|u - \phi_{N,\infty}(u')\| < \frac{\epsilon}{8nK} \quad \text{and} \quad \|\phi_{N,\infty}(x_j') - x_j\| < \frac{\epsilon}{8nK}
\]

for \( j = 1, 2, \ldots, K \). Then we have by (3-10) and (3-11)

\[
(3-12) \quad \|\phi_{N,\infty}(u_1') - \prod_{j=1}^K \phi_{N,\infty}(v_j')\| < \frac{\epsilon}{4n},
\]

for \( j = 1, 2, \ldots, K \), where \( u_1' = \text{diag}(u', 1_{n-r}) \) and \( v_j' = x_j'y_j'(x_j')^*(y_j')^* \). Then (3-12) implies that there is an \( N_1 > N \) such that

\[
(3-13) \quad \|\phi_{N,N_1}(u_1') - \prod_{j=1}^K \phi_{N,N_1}(v_j')\| < \frac{\epsilon}{2n}.
\]
Put $U = \phi_{N, N_1}(u')$, $U_1 = \text{diag}(U, 1_{n-r})$ and $w_j = \phi_{N, N_1}(v'_j)$, $j = 1, 2, \ldots, K$. Note that $\phi_{N_1, \infty}(U) = \phi_{N, \infty}(u')$. There is an $a \in (M_n(A_{N_1}))_{sa}$ (by (3-13)) such that

$$U_1 = \exp(i 2\pi a) \prod_{j=1}^{K} w_j \quad \text{and} \quad \|a\| < 2 \arcsin \frac{\epsilon}{8n}.$$  

(3-14) There is a $b \in (M_r(A_{N_1}))_{sa}$ such that

$$\tau(b) = \tau(a) \quad \text{for all} \quad \tau \in T(A) \quad \text{and} \quad \|b\| < 2n \arcsin \frac{\epsilon}{8n}.$$  

(3-15) Put $W = \text{diag}(U \exp(-i 2\pi b), 1_{n-r})$; then $W \in CU(M_n(A_{N_1}))$. Since $\text{Dur} A_{N_1} \leq r$, we conclude that $U \exp(-i 2\pi b) \in CU(M_r(A_{N_1}))$. It follows that

$$\phi_{N_1, \infty}(U \exp(-i 2\pi b)) \in CU(M_r(A)).$$

However, by (3-10), (3-11), (3-15),

$$\|u - \phi_{N_1, \infty}(U \exp(-i 2\pi b))\| \\
\leq \|u - \phi_{N_1, \infty}(u')\| + \|\phi_{N_1, \infty}(U) - \phi_{N_1, \infty}(U \exp(-i 2\pi b))\| \\
< \frac{\epsilon}{8nK} + \|1 - \exp(-i 2\pi \phi_{N_1, \infty}(b))\| < \frac{\epsilon}{8nK} + \epsilon/4 < \epsilon.$$  

Therefore, $\text{Dur} A \leq r$. \hfill \Box

**Proposition 3.6.** Let $A$ be a unital $C^*$-algebra with $T(A) \neq \emptyset$. Let $a \in A_{sa}$ and put $\hat{a}(\tau) = \tau(a)$ for all $\tau \in T(A)$.

1. If $\exp(2\pi i a) \in CU(A)$, then $\hat{a} \in \rho_{\hat{a}}(K_0(A))$.

2. If $u \in U_0(A)$ and for some piecewise smooth path $\{u(t) : t \in [0, 1]\}$ with $u(0) = 1$ and $u(1) = u$, $\Delta^1(u(t)) \in \rho_{\hat{a}}^k(K_0(A))$ for some $k \geq 1$, then $\text{diag}(u, 1_{k-1}) \in CU(M_k(A))$.  

3. If $\rho_{\hat{a}}^1(K_0(A)) = \rho_{\hat{a}}(K_0(A))$, then $\text{Dur} A = 1$.

**Proof.** Part (1) follows from [Th].

(2) By applying Corollary 2.12, there exists a $v \in CU(A)$ such that

$$u = \exp(i 2\pi a)v \quad \text{and} \quad \tau(a) = \Delta^1_{\tau}(u(t)) \quad \text{for all} \quad \tau \in T(A).$$

So for any $\epsilon \in (0, 1)$, there are projections $p_1, \ldots, p_{m_1}, q_1, \ldots, q_{m_2} \in M_k(A)$ such that

$$\sup \left\{ \sum_{j=1}^{m_1} \tau(p_j) - \sum_{j=1}^{m_2} \tau(q_j) - \tau(a) : \tau \in T(A) \right\} < \frac{\arcsin(\epsilon/4)}{\pi}.$$  

(3-16)
Set \( b = \sum_{j=1}^{m_1} p_j - \sum_{j=1}^{m_2} q_j \) and \( a_0 = \text{diag}(a, 0, 0, \ldots, 0) \). Then \( a_0, b \in M_k(A)_{sa} \) and
\[
|\tau(a_0) - \tau(b)| < \frac{\arcsin(\epsilon/4)}{k\pi} \quad \text{for all} \quad \tau \in T(M_k(A))
\]
by (3-16). Thus, by the proof of [Th, Lemma 3.1], we have
\[
\inf\{\|a_0 - b - x\| \mid x \in (M_k(A))_0\} = \sup\{|\tau(a_0 - b)| \mid \tau \in T(M_k(A))\} \leq \frac{\arcsin(\epsilon/4)}{k\pi}.
\]
Choose \( x_0 \in (M_k(A))_0 \) such that \( \|a_0 - b - x_0\| < 2 \arcsin(\epsilon/4)/k\pi \). Put \( y_0 = a_0 - b - x_0 \). Then \( \|y_0\| \leq 2 \arcsin(\epsilon/4)/k\pi \). Put \( u_1 = \text{diag}(u, 1_{k-1}) \exp(-i2\pi y_0) \). Define
\[
w(t) = \text{diag}(u(t), 1_{k-1}) \exp(-i2\pi y_0 t) \prod_{j=1}^{m_1} \exp(-i2\pi p_j t) \prod_{j=1}^{m_2} \exp(i2\pi q_j t)
\]
for \( t \in [0, 1] \). Then \( w(0) = 1, w(1) = u(1) \exp(-i2\pi y_0) = u_1 \) and, moreover,
\[
\Delta^k_t(w(t)) = \tau(a) - \tau(y_0) - \left(\sum_{j=1}^{m_1} \tau(p_j) - \sum_{j=1}^{m_2} \tau(q_j)\right)
\]
\[
= \tau(a) - \tau(a_0) + \tau(b) - \tau(x_0) - \tau(b)
\]
\[
= \tau(a) - \tau(a_0) = 0 \quad \text{for all} \quad \tau \in T(A).
\]
It follows that \( w(1) = u_1 \in CU(M_k(A)) \). Then
\[
\|\text{diag}(u, 1_{k-1}) - u_1\| = \|\exp(i2\pi y_0) - 1_k\| < \epsilon.
\]
(3) Let \( u \in U_0(A) \) such that \( \text{diag}(u, 1_{n-1}) \in CU(M_n(A)) \). Let \( u(t) \) be a piecewise smooth path with \( u(0) = 1 \) and \( u(1) = u \). Then
\[
\Delta^1(u(t)) \in \rho_A(K_0(A)) = \rho^1_A(K_0(A)).
\]
By Part (2), \( u \in CU(A) \). This implies that \( \text{Dur} A = 1 \). □

**Proposition 3.7.** Let \( X \) be a compact metric space. Then \( \text{Dur}(M_n(C(X))) = 1 \) for all \( n \geq 1 \).

**Proof.** By Proposition 3.3, it suffices to consider the case \( A = C(X) \). One has
\[
\rho^1_A(K_0(A)) = C(X, \mathbb{Z}) = \rho_A(K_0(A)).
\]
It follows from Proposition 3.6(3) that \( \text{Dur} A = 1 \). □

Combining Theorem 3.5 with Proposition 3.7, we have:
Corollary 3.8. Let $A = \lim_{n \to \infty} (A_n, \phi_n)$, where $A_m = \bigoplus_{j=1}^{m(n)} \mathbf{M}_{k(n,j)}(X_{n,j})$ and each $X_{n,j}$ is a compact metric space. Then $\text{Dur} A = 1$.

Theorem 3.9. Let $A$ be a unital $C^*$-algebra with real rank zero. Then $\rho_A^1(K_0(A)) = \rho_A(K_0(A))$ and $\text{Dur} A = 1$.

Proof. By Corollary 2.7, we may assume that $T(A) \neq \emptyset$. Since $A$ is of real rank zero, by [Zhang 1990, Theorem 3.3], for any $n \geq 2$ and any nonzero projection $p \in \mathbf{M}_n(A)$, there are projections $p_1, \ldots, p_n \in A$ such that $p \sim \text{diag}(p_1, \ldots, p_n)$ in $\mathbf{M}_n(A)$. Thus, $\tau(p) = \sum_{j=1}^{n} \tau(p_j)$ for all $\tau \in T(A)$ and, consequently, $\rho_A^1(K_0(A)) = \rho_A(K_0(A))$. It follows from Proposition 3.6(3) that $\text{Dur} A = 1$. \hfill \Box

Theorem 3.10. Let $A$ be a unital $C^*$-algebra with $T(A) \neq \emptyset$. If $\text{csr}(C(S^1, A)) \leq n + 1$ for some $n \geq 1$, then $\text{Dur} A \leq n$.

Proof. Let $u \in U_0(\mathbf{M}_n(A))$ such that $\text{diag}(u, 1_k) \in CU(\mathbf{M}_{n+k}(A))$ for some integer $k \geq 1$. Let $\{u(t) : t \in [0, 1]\}$ be a piecewise smooth path with $u(0) = 1_n$ and $u(1) = u$. By [Th], $\Delta^{n+k}(\text{diag}(u(t), 1_k)) \in \Delta^{n+k}(LU_{0}^{n+k}(A))$. It follows from Lemma 3.2(1) that, for any $\epsilon > 0$, there are $a, b \in \mathbf{M}_n(A)_{sa}$ and $v \in CU(\mathbf{M}_n(A))$ with $\|a\| < 2 \arcsin(\epsilon/4)/\pi$ such that

$$
(3-17) \ u = \exp(i2\pi a) \exp(i2\pi b)v \quad \text{and} \quad \tau(b) = \Delta^{n+k}_\tau(w(t)) \quad \text{for all} \ \tau \in T(A),
$$

where $w \in LU_{0}^{n+k}(A)$. Since $\text{csr}(C(S^1, A)) \leq n + 1$, by Proposition 2.6 of [Rieffel 1987] there is a $w_1 \in LU_{0}^{n}(A)$ such that $\text{diag}(w_1, 1_{n+k})$ is homotopy to $w$. In particular, $\Delta^{n}_\tau(w_1(t)) = \Delta^{n+k}_\tau(w(t))$ for all $\tau \in T(A)$. Consider the piecewise smooth path

$$
U(t) = \exp(-i2\pi at) \exp(i2\pi bt)\overline{w}_1(t), \quad t \in [0, 1].
$$

Then $U(0) = 1_n$ and $U(1) = \exp(i2\pi b)$. We compute that $\Delta^{n}_\tau(U(t)) = 0$ for all $\tau \in T(A)$. It follows by [Th, Lemma 3.1] that $\exp(i2\pi b) \in CU(\mathbf{M}_n(A))$. By (3-17),

$$
[u] = [\exp(i2\pi a)] \quad \text{in} \quad U_0(\mathbf{M}_n(A))/CU(\mathbf{M}_n(A)),
$$

Therefore $\text{dist}(u, CU(\mathbf{M}_n(A))) \leq \|\exp(i2\pi a) - 1_n\| < \epsilon$. \hfill \Box

Corollary 3.11. Let $A$ be a unital $C^*$-algebra of stable rank one. Then $\text{Dur} A = 1$.

Proof. This follows from the inequality $\text{csr}(C(S^1, A)) \leq \text{tsr} A + 1$ (see [Rieffel 1983, Corollary 8.6]) and Theorem 3.10. \hfill \Box

We end this section with the following:

Proposition 3.12. Let $A$ be a unital $C^*$-algebra. Suppose that there is a projection $p \in \mathbf{M}_2(A)$ such that, for any $x \in K_0(A)$ with $\rho_A(x) = \rho_A([p])$, no unitary in $U(\mathbf{C})$ represents $x$, where $C = C_0((0, 1), A)$. Then $\text{Dur} A > 1$. 

**Proof.** There exists an $a \in A_+$ such that $\tau(a) = \rho_A([p])\tau$ for all $\tau \in T(A)$. Put $u = \exp(i2\pi a)$ and $v = \text{diag}(u, 1)$. Then it follows from Proposition 3.6(2) that $v \in CU(M_2(A))$. This implies that $i_A^{(1,2)}([u]) = 0$. Now we show that $u \notin CU(A)$. Let

$$w(t) = \exp(2i(1-t)\pi a) \quad \text{for all} \ t \in [0, 1].$$

Then $w(0) = u$ and $w(1) = 1_A$. If $u \in CU(A)$, then, by [Th, Lemma 3.1], there is a continuous and piecewise smooth path of unitaries $\xi(t) \in \hat{C}$, where $C = C_0((0, 1), A)$, such that

$$(3-18) \quad \Delta_\tau(\xi(t)) = \tau(p) \quad \text{for all} \ \tau \in T(A).$$

The Bott map shows that the unitary $\xi$ is homotopic to a projection loop which corresponds to some $x \in K_0(A)$ with $\rho_A(x) = \rho_A([p])$, which contradicts the assumption. \hfill $\square$

### 4. Simple C*-algebras

Let us begin with the following:

**Theorem 4.1.** Let $A$ be a unital infinite-dimensional simple C*-algebra of real rank zero with $T(A) \neq \emptyset$. Then

$$\rho_A^1(K_0(A)) = \text{Aff}(T(A)) \quad \text{and} \quad U_0(A) = CU(A).$$

**Proof.** Let $p \in A$ be a nonzero projection, let $\lambda = n/m$ with $n, m \in \mathbb{N}$ and let $\epsilon > 0$. Then by Zhang’s half theorem (see [Lin 2010a, Lemma 9.4]), there is a projection $e \in A$ such that $\max_{\tau \in T(A)} |\tau(p) - n \tau(e)| < n \epsilon / m$. Thus,

$$\max_{\tau \in T(A)} |\lambda \tau(p) - m \tau(e)| < \epsilon,$$

and consequently $r \rho_A(p) \in \rho_A^1(K_0(A))$ for all $r \in \mathbb{R}$.

Let $a \in A_{sa}$. Since $A$ has real rank zero, $a$ is a limit of the form $\sum_{j=1}^k \lambda_j p_j$, where $p_1, p_2, \ldots, p_k$ are mutually orthogonal projections in $A$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$. Therefore $\hat{a} \in \rho_A^1(K_0(A))$ by the above argument, where $\hat{a}(\tau) = \tau(a)$ for all $\tau \in T(A)$. Since $\text{Aff}(T(A)) = \{\hat{a} \mid a \in A_{sa}\}$ by [Lin 2007, Theorem 9.3], it follows from Theorem 3.9 that

$$\text{Aff}(T(A)) \subseteq \rho_A^1(K_0(A)) = \rho_A(K_0(A)) \subseteq \text{Aff}(T(A)),$$

that is, $\text{Aff}(T(A)) = \rho_A^1(K_0(A))$.

Note that

$$\rho_A^1(K_0(A)) \subseteq \Delta^1(LU_0^1(A)) \subseteq \rho_A(K_0(A)) = \rho_A^1(K_0(A)).$$
So $\Delta^1(LU^1_k(A)) = \rho^1_A(K_0(A)) = \text{Aff}(T(A))$. Thus, $\overline{\Delta^1} = 0$ (see Definition 2.11), and the assertion follows. \hfill \Box

For unital simple $C^*$-algebras, we have:

**Theorem 4.2.** Let $A$ be a unital infinite-dimensional simple $C^*$-algebra. Then $\text{Dur} A = 1$ if one of the following holds:

1. $A$ is not stably finite.
2. $A$ has stable rank one.
3. $A$ has real rank zero.
4. $A$ is projectionless and $\rho_A(K_0(A)) = \mathbb{Z}$ (with $\rho_A([1_A]) = 1$).
5. $A$ has property (SP) and has a unique tracial state.

**Proof.** (1) In this case, there is a nonunitary isometry $u \in M_k(A)$ for some $k \geq 2$. Since $M_k(A)$ is also simple, every tracial state on $M_k(A)$ is faithful if $T(A) \neq \emptyset$. This implies that $T(A) = \emptyset$. The assertion follows from Corollary 2.7.

(2) This follows from Corollary 3.11.

(3) This follows from Theorem 4.1 or Theorem 3.9.

(4) By the assumption, we have $\rho^1_A(K_0(A)) = \rho_A(K_0(A)) = \mathbb{Z}$. By Proposition 3.6, $\text{Dur} A = 1$.

(5) Let $\epsilon > 0$ and let $\tau \in T(A)$ be the unique tracial state. Let $k \geq 1$ be an integer and $p \in M_k(A)$ a projection. Since $A$ has (SP), there is a nonzero projection $q \in A$ such that $0 < \tau(q) < \frac{1}{2}\epsilon$ (see, for example, [Lin 2001, Lemma 3.5.7]). Then, there is an integer $m \geq 1$ such that $|m\tau(q) - \tau(p)| < \epsilon$. This implies that $\rho^1_A(K_0(A)) = \rho_A(K_0(A))$. Therefore, by Proposition 3.6, $\text{Dur} A = 1$. \hfill \Box

For a unital simple $C^*$-algebra $A$, Theorem 4.2 indicates that the only case when $\text{Dur} A$ might not be 1 is when $A$ is stably finite and has stable rank greater than 1. The only example of this that we know so far is given by Villadsen [1999].

However, we have the following:

**Theorem 4.3.** For each integer $n \geq 1$, there is a unital simple AH-algebra $A$ with $\text{tsr} A = n$ such that $\text{Dur} A = 1$.

**Proof.** Fix an integer $n > 1$. Let $A = \lim_{k \to \infty} (A_k, \phi_k)$ be the unital simple AH-algebra with $\text{tsr} A = n$ constructed by Villadsen [1999]. Then $A_1 = C(D^n)$. The connecting maps $\phi_k$ are “diagonal” maps. More precisely, $\phi_k(f) = \sum_{j=1}^{n(k)} f(y_{k,j}) \otimes p_{k,j}$ for all $f \in A_k$, where $p_{k,1}$ is a trivial rank-1 projection, $A_{k+1} = \phi_k(id_{A_k})M(r(k))(C(X_k))\phi_k(id_{A_k})$ (for some large $r(n)$) for some spaces $X_k$, and $\gamma_{k,j} : X_{k+1} \to X_k$ is a continuous map (these are $\pi_{i+1}$ and some point evaluations as denoted in [Villadsen 1999, p. 1092]). Clearly $A_1$ contains a rank-1 projection. Suppose that $A_k$, as a unital hereditary $C^*$-subalgebra of
$M_{r(k)}(C(X_k))$, contains a rank-1 projection $e_k$ (of $M_{r(k)}(C(X_k))$). Then, since $(\text{id}_{A_k} \circ \gamma_{k,1}) \otimes p_{k,1} \leq \phi_k(\text{id}_{A_k})$, we have $(\text{id}_{A_k} \circ \gamma_{k,1}) \otimes p_{k,1} \in A_{k+1}$. Then $e_k \circ \gamma_{k,1} \otimes p_{k,1} \in A_{k+1}$, which is a rank-1 projection.

The above shows every $A_k$ contains a rank-1 projection.

Now let $p \in M_m(A)$ be a projection. We may assume that there is a projection $q \in M_m(A_{k+1})$ such that $\phi_{k+1,\infty}(q) = p$. Let $e_{k+1} \in A_{k+1}$ be a rank-1 projection. Then there is an integer $L \geq 1$ such that $L \tau(e_{k+1}) = \tau(q)$ for all $\tau \in T(A_{k+1})$. It follows that

$$L \tau(\phi_{k+1,\infty}(e_{k+1})) = \tau(p) \quad \text{for all} \quad \tau \in T(A).$$

So $\rho_A^1(K_0(A)) = \rho_A(K_0(A))$ and hence $\text{Dur} A = 1$ by Proposition 3.6.

**Theorem 4.4.** Let $A$ be a unital simple AH-algebra with (SP) property. Then $\text{Dur} A = 1$.

**Proof.** By Proposition 3.1, it suffices to show that $i_A^{1,n}$ is injective, and by Proposition 3.6 it suffices to show that $\rho_A^1(K_0(A)) = \rho_A(K_0(A))$.

Let $p$ be a projection in $M_n(A)$. Since $A$ is simple, $\inf \{\tau(p) \mid \tau \in T(A)\} = d > 0$. Given a positive number $\epsilon < \min\{\frac{1}{2}, \frac{1}{2}d\}$. Choose an integer $K \geq 1$ such that $1/K < \frac{1}{2} \epsilon$. Since $A$ is a simple unital $C^*$-algebra with (SP), it follows from [Lin 2001, Lemma 3.5.7] that there are mutually orthogonal and mutually equivalent nonzero projections $p_1, p_2, \ldots, p_K \in A$ such that $\sum_{j=1}^K p_j \leq p$. We compute that

$$(4-1) \quad \tau(p_1) < \epsilon/2 \quad \text{and} \quad \tau(p_1) < d/K \quad \text{for all} \quad \tau \in T(A).$$

Since $A$ is simple and unital, there are $x_1, x_2, \ldots, x_N \in A$ such that

$$\sum_{j=1}^N x_j^* p_1 x_j = 1_A.$$

Let $A = \lim(A_m, \phi_m)$, where $A_m = \bigoplus_{i=1}^{r(m)} M_{p_{m,j}}(C(X_{m,j})) P_{n,j}$ for each $m$, $X_{n,j}$ is a connected finite CW-complex and $P_{m,j} \in M_{p_{m,j}}(C(X_{m,j}))$ is a projection. Without loss of generality, we may assume that, there are projections $p_1' \in A_m, p' \in M_n(A_m)$ and elements $y_1, y_2, \ldots, y_N \in A_m$ such that $\phi_{m,\infty}(p_1') = p_1, \phi_{m,\infty}(y_j) = x_j, (\phi_{m,\infty} \otimes \text{id}_{M_n})(p') = p$ and

$$(4-2) \quad \left\| \sum_{j=1}^N y_j^* p_1' y_j - 1_A \right\| < 1.$$

Write $p_1'$ and $p'$ as

$$p_1' = p_{1,1} + p_{1,2} + \cdots + p_{1,r(m)} \quad \text{and} \quad p' = q_1 + q_2 + \cdots + q_{r(m)},$$
where, for each \( j = 1, \ldots, r(m) \), \( p'_{1,j} \in P_{m,j}M_{R(m,j)}(C(X_{m,j}))P_{m,j} \) and \( q_j \in M_n(P_{m,j}M_{R(m,j)}(C(X_{m,j}))P_{m,j}) \) are projections. Note that (4-2) implies that \( p'_{1,j} \neq 0 \) for \( j = 1, 2, \ldots, r(m) \). Define

\[
r_{1,j} = \text{rank} \, p'_{1,j} \quad \text{and} \quad r_j = \text{rank} \, q_j \quad \text{for} \quad j = 1, 2, \ldots, r(m).
\]

Then \( r_j = l_j r_{1,j} + s_j \), where \( l_j, s_j \geq 0 \) are integers and \( s_j < r_{1,j} \). It follows that

\[
(4-3) \quad \left| t(p') - \sum_{j=1}^{r(m)} l_j t(p'_{1,j}) \right| < t(p'_1) \quad \text{for all} \quad t \in T(A_m).
\]

Define \( q_{1,j} = \phi_{m,\infty}(p'_{1,j}) \) for \( j = 1, \ldots, r(m) \). Then each \( q_{1,j} \) is a projection in \( A \). Note that for each \( \tau \in T(A) \), \( \tau \circ \phi_{m,\infty} \) is a tracial state on \( A_m \). So, by (4-3),

\[
\left| \tau(p) - \sum_{j=1}^{r(m)} l_j \tau(q_{1,j}) \right| < \tau(p_1) < \epsilon \quad \text{for all} \quad \tau \in T(A).
\]

This implies that \( \rho^1_A(K_0(A)) = \rho_A(K_0(A)) \). \qed

**Lemma 4.5.** Let \( A \) be a unital simple C*-algebra with \( T(A) \neq \emptyset \), and let \( a \in A_+ \setminus \{0\} \). Then, for any \( b \in A_{sa} \), there is a \( c \in \text{Her} \, a \) such that \( b - c \in A_0 \).

**Proof.** Since \( A \) is simple and unital, there are \( x_1, x_2, \ldots, x_m \in A \) such that \( \sum_{j=1}^m x_j^* a x_j = 1_A \). Set \( c = \sum_{j=1}^m a^{1/2} x_j b x_j^* a^{1/2} \). Then \( c \in \text{Her} \, a \) and

\[
\tau(c) = \sum_{j=1}^m \tau(a^{1/2} x_j b x_j^* a^{1/2}) = \sum_{j=1}^m \tau(b x_j^* a x_j) = \tau(b) \quad \text{for all} \quad \tau \in T(A).
\]

It follows from Lemma 2.6(2) that \( b - c \in A_0 \). \qed

A special case of the following can be found in [Lin 2010b, Theorem 3.4]:

**Theorem 4.6.** Let \( A \) be a unital simple C*-algebra and let \( e \in A \) be a nonzero projection. Consider the map \( U_0(eAe)/CU(eAe) \to U_0(A)/CU(A) \) given by \( i_e([u]) = [u+(1-e)] \). This map is always surjective, and is also injective if \( \text{tsr} \, A = 1 \).

**Proof.** To see that \( i_e \) is surjective, let \( u \in U_0(A) \). Write \( u = \prod_{k=1}^n \exp(i a_k) \) for \( a_k \in A_{sa} \), \( k = 1, 2, \ldots, n \). By Lemma 4.5, there are \( b_1, \ldots, b_n \in eAe \) such that \( b_k - a_k \in A_0 \). Put \( w = e \prod_{k=1}^n \exp(i b_k) \). Then \( w \in U_0(eAe) \). Set \( v = w + (1-e) \).

Then \( v = \prod_{k=1}^n \exp(i b_k) \). Thus, by Lemma 2.6(1),

\[
i_e([w]) = [v] = \sum_{k=1}^n [\exp(i b_k)] = \sum_{k=1}^n [\exp(i a_k)] = [u] \quad \text{in} \quad U_0(A)/CU(A),
\]

that is, \( i_e \) is surjective.
To see that $i_e$ is injective when $A$ has stable rank one, let $w \in U_0(eAe)$ such that $w + (1 - e) \in CU(A)$. Since $A$ is simple, there are $z_1, \ldots, z_n \in A$ such that $1 - e = \sum_{j=1}^n z_j^* ez_j$. Set
\[
X = \begin{bmatrix}
e z_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
e z_n & 0 & \cdots & 0
\end{bmatrix} \in M_n(A).
\]
Then
\[(4-4) \quad \text{diag}(1 - e, 0, \ldots, 0) = X^*X, \quad XX^* \leq \text{diag}(e, e, \ldots, e).
\]
Equation (4-4) indicates that $[1 - e] \leq n[e]$ in $K_0(A)$. Since $tsr A = 1$, we can find a projection $p \in M_s(A)$ for some $s \geq n$ and a unitary $U \in M_{s+1}(A)$ such that
\[(4-5) \quad \text{diag}(e, \ldots, e, 0, \ldots, 0) = U \text{diag}(1 - e, p)U^*,
\]
where $r = s - n + 1$. Write $v = w + (1 - e)$ as $v = \begin{bmatrix} w & 1 - e \end{bmatrix}$, and set
\[
W = \begin{bmatrix} e \\ U \end{bmatrix} \quad \text{and} \quad Q = \text{diag}(e, \ldots, e, 0, \ldots, 0).
\]
Then $W \text{diag}(e, 1 - e, p)M_{s+2}(A) \text{diag}(e, 1 - e, p)W^* \subset M_{n+1}(eAe) \oplus 0$ and
\[(4-6) \quad W \begin{bmatrix} v \\ p \end{bmatrix} W^* = \begin{bmatrix} w \\ U \text{diag}(1 - e, p)U^* \end{bmatrix} = \text{diag}(w, Q),
\]
by (4-5). Note that $\text{diag}(v, p) \in CU(\text{diag}(e, 1 - e, p)M_{s+2}(A) \text{diag}(e, 1 - e, p))$. So, by (4-6),
\[
\text{diag}(w, e, \ldots, e) \in CU(M_{n+1}(eAe)).
\]
Since $tsr(eAe) = 1$, it follows from Theorem 4.2(2) that $w \in CU(eAe)$.

\[\square\]

**Lemma 4.7.** Let $C$ be a nonunital $C^*$-algebra and $B = \overline{C}$. Assume $u_1, u_2, \ldots, u_n \in U(M_k(B))$ for some $k \geq 2$. Then, there are unitaries $u_1', u_2', \ldots, u_n' \in M_k(\overline{C})$ with $\pi_k(u_j') = 1_k$ and $w, z_j, \tilde{u}_j \in U(M_k(\mathbb{C}))$ for $j = 1, \ldots, n$ such that
\[
\prod_{j=1}^n u_j = \left( \prod_{j=1}^n u_j' \right) w, \quad \text{with} \quad u_j' = z_j^* u_j \tilde{u}_j z_j \quad \text{for} \quad j = 1, \ldots, n,
\]
\[
w = \pi_k \prod_{j=1}^n u_j.
\]
where \( \pi(x + \lambda) = \lambda \) for all \( x \in \mathbb{C} \) and \( \lambda \in \mathbb{C} \) and \( \pi_k \) is the induced homomorphism of \( \pi \) on \( M_k(B) \).

Moreover, if \( u_j \in U_0(M_k(B)) \), then we may assume that each \( u'_j \in U_0(M_k(\mathbb{C})) \) for \( j = 1, \ldots, n \).

**Proof.** Put \( \tilde{u}_j = \pi_k(u_j) \in U(M_k(\mathbb{C})) \). If \( n = 2 \), then

\[
\begin{align*}
\pi_k(u'_1) &= 1_k, \\
\pi_k(u'_2) &= \pi_k(\tilde{u}_1(u_2u^*_s)u^*_1) = 1_k, \\
\pi_k(w_1) &= 1_k, \\
\pi_k(w_2) &= 1_k, \\
\pi_k(w_3) &= 1_k, \\
\pi_k(w_4) &= 1_k.
\end{align*}
\]

Thus the lemma holds if \( n = 2 \). Suppose that the lemma holds for \( s \). Then

\[
\begin{align*}
\pi_k(u'_1) &= 1_k, \\
\pi_k(u'_2) &= \pi_k(\tilde{u}_1(u_2u^*_s)u^*_1) = 1_k, \\
\pi_k(w_1) &= 1_k, \\
\pi_k(w_2) &= 1_k, \\
\pi_k(w_3) &= 1_k, \\
\pi_k(w_4) &= 1_k,
\end{align*}
\]

The first part of the lemma follows.

To see the second part, we first assume that \( u_j = \exp(ia_j) \) for some \( a_j \in (M_k(B))_{sa} \). Note that \( \tilde{u}_j = \exp(i\tilde{a}_j) \), where \( \tilde{a}_j = \pi_k(a_j) \in (M_k(\mathbb{C}))_{sa}, j = 1, \ldots, n \). Consider the path \( u'_j(t) = \exp(it\tilde{a}_j) \exp(-it\tilde{a}_j) \) for \( t \in [0, 1] \). Note that, for each \( t \in [0, 1] \) and \( j = 1, \ldots, n \),

\[
\pi_k(\exp(it\tilde{a}_j) \exp(-it\tilde{a}_j)) = \exp(it\pi_k(a_j)) \exp(-it\pi_k(a_j)) = 1_k.
\]

It follows that \( u'_j(t) \in \widehat{M_k(\mathbb{C})} \) for all \( t \in [0, 1] \) and \( j = 1, \ldots, n \). The case that \( u_j = \exp(\prod_{k=1}^m(ia_k)) \) follows from this and what has been proved. \( \Box \)
Lemma 4.8. Let $C$ be a nonunital $C^*$-algebra and $B = \widetilde{C}$. Suppose that $z = aba^*b^*$, where $a, b \in U_0(M_k(B))$. Then $z = yw$, where $y \in CU(M_k(C))$ with $\pi_k(y) = 1_k$ and $w \in CU(M_k(C))$. Moreover, if $u = \prod_{j=1}^n z_j$, where each $z_j \in CU(M_k(B))$, then $u = yv$, where $y \in CU(M_k(C))$ with $\pi_k(y) = 1_k$ and $v \in CU(M_k(C))$.

**Proof.** Let $\tilde{a} = \pi_k(a)$ and $\tilde{b} = \pi_k(b)$. Then $\tilde{a}, \tilde{b} \in U(M_k(C))$. It follows from Lemma 4.7 that for $j = 1, 2$ there are $a_j, b_j \in U_0(M_k(C))$ with $\pi_k(a_j) = \pi_k(b_j) = 1_k$ and $z_j \in U(M_k(C))$ such that

$$ab = a_1b_1w_1, \quad a_1 = a\tilde{a}^*, \quad b_1 = z^*b\tilde{b}^*z_1, \quad w_1 = \tilde{a}\tilde{b},$$

(4-7)

$$ba = b_2a_2w_2, \quad b_2 = \tilde{b}\tilde{a}^*, \quad a_2 = z^*a\tilde{a}^*z_2, \quad w_2 = z^*\tilde{a}\tilde{b}.\tag{4-8}$$

Set $x_1 = w_1w^*_2z_2$ and $x_2 = w_1w^*_2z_1$. Then $x_1, x_2 \in U_0(M_k(C))$ and

$$aba^*b^* = a_1b_1(w_1w^*_2z_2(a\tilde{a}^*)z_2w_2u^*)(w_1w^*_2(b\tilde{b}^*)w_2w^*_1))w_1w^*_2$$

by (4-7) and (4-8).

Write $a_1 = \prod_{j=1}^{m_1} \exp(iy_{1j})$ and $b_1 = \prod_{k=1}^{m_2} \exp(iy_{2k})$, where $y_{1j}, y_{2k} \in (M_k(C))_{sa}$, $j = 1, \ldots, m_1$, $k = 1, \ldots, m_2$. Let

$$y_{1j} = y^+_{1j} - y^-_{1j} \quad \text{and} \quad y_{2k} = y^+_{2k} - y^-_{2k},$$

with $y^+_{1j}, y^-_{1j}, y^+_{2k}, y^-_{2k} \in (M_k(C))_+$ for $j = 1, \ldots, m_1$ and $k = 1, \ldots, m_2$. Set

$$c_1 = \sum_{j=1}^{m_1} (y^+_{1j} + x_1y_{1j}x^*_1) + \sum_{k=1}^{m_2} (y^+_{2k} + x_2y_{2k}x^*_2),$$

$$c_2 = \sum_{j=1}^{m_1} (y^-_{1j} + x_1y^+_{1j}x^*_1) + \sum_{k=1}^{m_2} (y^-_{2k} + x_2y^+_{2k}x^*_2),$$

$$d_1 = \sum_{j=1}^{m_1} (y^+_{1j} + y^-_{1j}) + \sum_{k=1}^{m_2} (y^+_{2k} + y^-_{2k}),$$

$$d_2 = \sum_{j=1}^{m_1} (y^-_{1j} + y^+_{1j}) + \sum_{k=1}^{m_2} (y^-_{2k} + y^+_{2k}).$$

Then $c_1, c_2, d_1, d_2 \in (M_2(C))_+$ and clearly $c_1 - d_1, c_2 - d_2 \in (M_k(C))_0$. Therefore, $(c_1 - c_2) - (d_1 - d_2) \in (M_k(C))_0$. Put $y = a_1b_1(x_1a_1^*x_1^*)(x_2^*b_1^*x_2)$ and $w = w_1w^*_2$. Then $y \in U_0(M_k(C))$ with $\pi_k(y) = 1_k$ and $w = \tilde{a}\tilde{b}\tilde{a}^*\tilde{b}^* \in DU_k(\mathbb{C})$. Moreover, in $U_0(M_k(C))/CU(M_k(C))$, 

$$[y] = [\exp(i(c_1 - c_2))] = [\exp(i(d_1 - d_2))] = [a_1][b_1][a_1^*][b_1^*] = 0.$$
This proves the first part of the lemma. The second part follows. \hfill \Box

**Theorem 4.9.** Let $A$ be an infinite-dimensional unital simple $C^*$-algebra with $T(A) \neq \emptyset$ such that there is an $m \geq 1$, for every hereditary $C^*$-subalgebra $C$, with $\text{Dur} \tilde{C} \leq m$. Then $\text{Dur} A = 1$.

**Proof.** Let $n \geq 1$. By Proposition 3.1, it suffices to show that $i_A^{(1,n)}$ is injective. Let $u \in U_0(A)$ with $\text{diag}(u, 1_{n-1}) \in CU(M_n(A))$. Since $A$ is simple and infinite-dimensional, we can find nonzero mutually orthogonal positive elements $c_1, \ldots, c_m \in A$ and $x_1, \ldots, x_m \in A$ such that

\[ x_j^*x_j = c_1 \quad \text{and} \quad x_jx_j^* = c_j, \quad j = 2, 3, \ldots, m. \]

Put $\text{Her} c_1 = C$ and $B = \tilde{C}$. Then $\text{Her}(c_1 + c_2 + \cdots + c_m) \cong M_m(C)$. Note that $M_m(B)$ is not isomorphic to a subalgebra of $M_m(A)$.

By Lemma 4.5, we may assume, without loss of generality, that $u = \exp(2\pi i b)$ for some $b \in C_{sa}$. Then, by Proposition 3.6(1), $\hat{b} \in \rho_A(K_0(A))$.

Since $A$ is simple and $C$ is $\sigma$-unital, it follows from [Brown 1977, Theorem 2.8] that there is a unitary element $W$ in $M(A \otimes \mathcal{K})$ (the multiplier algebra of $A \otimes \mathcal{K}$) such that $W^* (C \otimes \mathcal{K}) W = A \otimes \mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra consisting of all compact operators on $l^2$. Note that since $A$ is a unital simple $C^*$-algebra, every tracial state $\tau$ on $C$ is the normalization of a tracial state restricted on $C$. Therefore

\[ \hat{b} \in \rho_A(K_0(A)) = \rho_B(K_0(C)) \subset \rho_B(K_0(B)). \]

Viewing $b$ in $B_{s,a}$, consider $v = \exp(i 2\pi b) \in U_0(B)$ and $v(t) = \exp(i 2\pi tb)$, $t \in [0, 1]$. Then (4-9) implies that $A^1(v(t)) \in \rho_B(K_0(B))$. By Lemma 3.2(2), for any $\epsilon > 0$, there are $a \in B_{sa}$ with $\|a\| < \epsilon$, $d \in B_{sa}$ with $\tilde{d} \in \rho_B(K_0(B))$ and $v_0 \in CU(B)$ such that

\[ v = \exp(i 2\pi a) \exp(i 2\pi d)v_0. \]

Choose projections $p, q \in M_n(B)$ for some $n > m$ such that for all $\tau \in T(B)$,

\[ \tau(\text{diag}(d, 0_{(n-1) \times (n-1)})) = \tau(p) - \tau(q). \]

So $\text{diag}(\exp(i 2\pi d), 1_{n-1}) \in CU(M_n(B))$ by Lemma 2.6(2). By assumption, $i_B^{(m,k)}$ is injective for all $k > m$. Therefore, we have $\text{diag}(v, 1_{m-1}) \in CU(M_m(B))$ by (4-10).

Let $\epsilon > 0$. Then there is a $v_1 \in DU(M_m(B))$ such that $\|\text{diag}(v, 1_{m-1}) - v_1\| < \frac{1}{2} \epsilon$. We may write $v_1 = \prod_{j=1}^r z_j$, where $z_j \in M_m(B)$ is a commutator. It follows from Lemma 4.8 that there are $y \in CU(M_m(C))$ with $\pi_m(y) = 1_m$ and $w \in DU(M_m(C))$ such that $v_1 = yw$. Noting that $w = \pi_m(w) = \pi_m(v_1)$ and $\pi(v) = 1$, we have $\|1_m - w\| < \frac{1}{2} \epsilon$. Thus $\|\text{diag}(v, 1_{m-1}) - y\| < \epsilon$. Set $v_0 = v - 1$ and $y_0 = y - 1_m$. Then

\[ \text{diag}(v_0, 0_{(m-1) \times (m-1)}), y_0 \in M_m(C), \]

\[ \|\text{diag}(v_0, 0_{(m-1) \times (m-1)}) - y_0\| < \epsilon. \]
By identifying $1_m + M_m(C)$ with a unital $C^*$-subalgebra $1_A + \text{Her}(c_1 + c_2 + \cdots + c_m)$ of $A$, we get that $\| \exp(i2\pi b) - y \| < \epsilon$ by (4-11). Since $y \in CU(M_m(C)) \subset CU(A)$ and hence $u \in CU(A)$, we have $\text{Dur} A = 1$.

**Corollary 4.10.** Let $A$ be a unital simple $C^*$-algebra. Suppose that there is an integer $K \geq 1$ such that $\text{csr}(C(S^1, C)) \leq K$ for every hereditary $C^*$-subalgebra $C$. Then $\text{Dur} A = 1$.

**Proof.** It follows from Theorem 3.10 that $\text{Dur} \tilde{C} \leq \max\{K - 1, 1\}$. Theorem 4.9 then applies. □

**Definition 4.11.** Let $A$ be a $C^*$-algebra with $T(A) \neq \emptyset$. Define

$$D_1 = \sup \{ \text{dist}(x, K_0(A)) : x \in \overline{\rho_A(K_0(A))} \}$$

**Theorem 4.12.** Let $A$ be a unital simple $C^*$-algebra with $T(A) \neq \emptyset$ such that there is an $M > 0$ with $D_1 < M$ for all nonzero hereditary $C^*$-subalgebras $C$ of $A$. Then $\text{Dur} A = 1$.

**Proof.** Let $u \in U_0(A)$ such that $\text{diag}(u, 1_{n-1}) \in CU(M_n(A))$. By Corollary 2.12, we may assume that $u = \exp(i2\pi a)$ for some $a \in A_{sa}$. Then $\hat{a} \in \rho_A(K_0(A))$ by Proposition 3.6(1).

Given $\epsilon > 0$, choose an integer $N \geq 1$ such that $M/N < \epsilon/2\pi$. There are mutually orthogonal nonzero positive elements $c_1, c_2, \ldots, c_N$ in $A$ and elements $x_1, x_2, \ldots, x_N \in A$ such that

$$(4-12) \quad x_j^* x_j = c_1 \quad \text{and} \quad x_j x_j^* = c_j, \quad j = 2, 3, \ldots, N.$$ 

Let $C = \text{Her} c_1$ and $B = \tilde{C}$. It follows from Lemma 4.5 that there is a $b \in C_{sa}$ such that $a - b$ is in $A_0$, i.e., $\tau(a) = \tau(b)$ for all $\tau \in T(A)$. Therefore $[\exp(i2\pi a)] = [\exp(i2\pi b)]$ in $U_0(A)/CU(A)$ by Lemma 2.6(2).

Since $A$ is a unital simple $C^*$-algebra and $C$ is $\sigma$-unital, it follows from the proof of Theorem 4.9 that $\rho_C(b) \in \rho_C(K_0(C))$. Therefore, by assumption, there are projections $p_1, p_2, \ldots, p_{k_1}, q_1, q_2, \ldots, q_{k_2} \in C$ such that

$$\sup_{\tau \in T(C)} \left| \tau(b) - \left( \sum_{i=1}^{k_1} \tau(p_i) - \sum_{j=1}^{k_2} \tau(q_j) \right) \right| < M.$$ 

Put $d = \sum_{i=1}^{k_1} p_i - \sum_{j=1}^{k_2} q_j$ and $f = b - d$. Then $\exp(i2\pi d) \in CU(A)$ by (2-3) and $[\exp(i2\pi f)] = [\exp(i2\pi b)] \in U_0(A)/CU(A)$. Moreover, from

$$\inf \{ \| f - x \| : x \in C_0 \} = \sup \{ |\tau(f)| : \tau \in T(C) \} < M.$$
(see the proof of [Th, Lemma 3.1]), there are \( f_0 \in C_0 \) and \( f_1 \in C_{sa} \) with \( \| f_1 \| < M \) such that \( f = f_1 + f_0 \). By Lemma 2.6(1), \( \exp(i2\pi f_0) \in CU(A) \). Since \( f_1 \in C_{sa} \), by (4-12), for \( i = 1, 2, \ldots, N \) there are \( g_i \in \text{Her} \ c_i \) with
\[
(4-13) \quad \| g_i \| \leq \| f_1 \|/N \quad \text{and} \quad \tau(g_i) = \tau(f_1/N) \quad \text{for all} \ \tau \in T(A).
\]
Set \( g = \sum_{i=1}^{n} g_i \in A \). Then, by (4-13),
\[
(4-14) \quad \| \exp(i2\pi g) - 1_A \| < M/N < \epsilon \quad \text{and} \quad \Delta \left( \exp(i2\pi f) \exp(-i2\pi g) \right) = 0.
\]
So \( \exp(i2\pi f) \exp(-i2\pi g) \in CU(A) \) and consequently \( \text{dist}(e^{i2\pi a}, CU(A)) < \epsilon \). \( \square \)

Bruce Blackadar [1981] constructed three examples of unital simple separable nuclear C*-algebras \( A, A_\Delta, A_H \) with no nontrivial projections. By [Blackadar 1981, Theorem 4.9], \( K_0(A) = \mathbb{Z} \) with a unique tracial state. It follows from Theorem 4.2(4) that \( \text{Dur} \ A = 1 \). We turn to his examples \( A_\Delta \) and \( A_H \), which may have rich tracial spaces. It should be also noted that, as Blackadar showed, when \( \Delta \) is not trivial (for example), \( M_2(A_\Delta) \) has a projection \( p \) with \( \tau(p) = 1 \) for all \( \tau \in T(A_\Delta) \). In particular, this implies that
\[
\bar{\rho}^1_{A_\Delta}(K_0(A_\Delta)) \neq \bar{\rho}_{A_\Delta}(K_0(A_\Delta)).
\]
However, \( \text{Dur} \ A_\Delta = 1 \) as shown below. It follows that there is a unitary \( u \in \tilde{C} \), where \( C = C_0((0, 1), A) \), which represents a projection \( q \) with \( \tau(q) = 1 \) for all \( \tau \in T(A_\Delta) \).

**Proposition 4.13.** Let \( B \) be a unital AF-algebra and \( \sigma \) an automorphism of \( B \). Put \( M_\sigma = \{ f \in C([0, 1], B) \mid f(1) = \sigma(f(0)) \} \). Then \( \text{Dur} M_\sigma = 1 \).

**Proof.** Clearly, \( T(M_\sigma) \neq \emptyset \). From the exact sequence of C*-algebras
\[
0 \to C_0((0, 1), B) \to M_\sigma \to B \to 0,
\]
we obtain the exact sequence of C*-algebras
\[
(4-15) \quad 0 \to C_0((0, 1) \times S^1, B) \to C(S^1, M_\sigma) \to C(S^1, B) \to 0.
\]
Since \( B \) is an AF-algebra, it follows from [Nistor 1986, Corollary 2.11] that
\[
\text{csr}(C(S^1, B)) = \text{csr}(C(S^1)) = 2,
\]
\[
\text{csr}(C_0((0, 1) \times S^1, B)) = \text{csr}(C_0((0, 1) \times S^1)) = 2,
\]
and consequently, applying [Nagy 1987, Lemma 2] to (4-15), we get
\[
\text{csr}(C(S^1, M_\sigma)) \leq \max\{\text{csr}(C(S^1, B)), \text{csr}(C_0((0, 1) \times S^1, B))\} \leq 2.
\]
Therefore \( \text{Dur} A = 1 \) by Theorem 3.10. \( \square \)

**Corollary 4.14.** \( \text{Dur} A_\Delta = 1 \) and \( \text{Dur} A_H = 1 \).
Proof. Both $C^*$-algebras are of the form $\lim_{n \to \infty} A_n$, where each $A_n \cong M_\sigma$, where $M_\sigma$ is as in Proposition 4.13, and thus $\text{Dur} A_n = 1$. By Theorem 3.5, $\text{Dur} A_\Delta = 1$ and $\text{Dur} A_H = 1$.

5. $C^*$-algebras with $\text{Dur} A > 1$

In this section, we will present a unital $C^*$-algebra $C$ such that $\text{Dur} C = 2$. In particular, we will show that there are $C^*$-algebras which satisfy the condition described in Proposition 3.12.

5.1. We first list some standard facts from elementary topology. We will give a brief proof of each fact for the reader’s convenience.

Fact 1. Let

$$B_d(0) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \leq d\}.$$  

Let $f : B_d(0) \times S^1 \to S^3 = \text{SU}(2)$ be a continuous map which is not surjective. Then there is a homotopy

$$F : B_d(0) \times S^1 \times [0, 1] \to S^3 = \text{SU}(2)$$

such that $F(x, e^{i\theta}, 0) = f(x, e^{i\theta})$, $F(x, e^{i\theta}, s) = f(x, e^{i\theta})$ if $\|x\| = d$ (i.e., if $x \in \partial B_d(0)$) and $g(x, e^{i\theta}) = F(x, e^{i\theta}, 1)$ satisfies

$$g(0, e^{i\theta}) = F(0, e^{i\theta}, 1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \text{SU}(2) = S^3.$$  

Proof. Assume that $f$ misses a point $z \in S^3 = \text{SU}(2)$ and that $z \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \text{SU}(2)$. Then $S^3 \setminus \{z\}$ is homeomorphic to $D^3 = \{(x, y, z) \mid x^2 + y^2 + z^2 < 1\}$, with the identity matrix mapping to $(0, 0, 0)$. Without loss of generality, we can assume that $f$ is a map from $B_d(0) \times S^1$ to $D^3$. Let $F : B_d(0) \times S^1 \times [0, 1] \to D^3$ be defined by

$$F(x, e^{i\theta}, s) = f(x, e^{i\theta}) \max\{1 - s, \|x\|/d\},$$

which satisfies the condition.

Fact 2. Let $f, g : S^4 \times S^1 \to \text{SU}(n) \subset U(n) = U_n(\mathbb{C})$ (where $n \geq 2$) be continuous maps. If $f$ is homotopic to $g$ in $U(n)$, then they are also homotopic in $\text{SU}(n)$.

Proof. This follows from the fact that there is a continuous map $\pi : U(n) \to \text{SU}(n)$ with $\pi \circ i = \text{id}_{\text{SU}(n)}$, where $i : \text{SU}(n) \to U(n)$ is inclusion.
**Fact 3.** Let \( \xi \in S^4 \) be the north pole. Suppose that \( f, g : S^4 \times S^1 \to \text{SU}(n) \) are two continuous maps such that
\[
f(\xi, e^{i\theta}) = 1_n = g(\xi, e^{i\theta})
\]
for all \( e^{i\theta} \in S^1 \). If \( f \) and \( g \) are homotopic in \( \text{SU}(n) \), then there is a homotopy
\[
F : S^4 \times S^1 \times [0, 1] \to \text{SU}(n)
\]
such that \( F(x, e^{i\theta}, 0) = f(x, e^{i\theta}) \), \( F(x, e^{i\theta}, 1) = g(x, e^{i\theta}) \) for all \( x \in S^4 \), \( e^{i\theta} \in S^1 \) and \( F(\xi, e^{i\theta}, t) = 1_n \) for all \( e^{i\theta} \in S^1 \).

**Proof.** Let \( G : S^4 \times S^1 \times [0, 1] \to \text{SU}(n) \) be a homotopy between \( f \) and \( g \). That is, \( G(\cdot, \cdot, 0) = f \) and \( G(\cdot, \cdot, 1) = g \). Let \( F : S^4 \times S^1 \times [0, 1] \to \text{SU}(n) \) be defined by
\[
F(x, e^{i\theta}, t) = G(x, e^{i\theta}, t)(G(\xi, e^{i\theta}, t))^*.
\]
Then \( F \) satisfies the condition. \( \square \)

**5.2.** We will describe the projection \( P \in M_4(C(S^4)) \) of rank two which represents the class of \((2, 1) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_0(C(S^4))\) as follows: One can regard \( S^4 \) as the quotient space \( D^4/\partial D^4 \), where
\[
D^4 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 \leq 1\}.
\]
It is standard to construct a unitary
\[
\alpha : D^4 \to U_4(\mathbb{C}) = U(\mathbb{M}_4(\mathbb{C}))
\]
such that \( \alpha(0) = 1_4 \) and such that, for any \((z, w) \in \partial D^4\) (i.e., \(|z|^2 + |w|^2 = 1\)),
\[
\alpha(z, w) :=
\begin{bmatrix}
z & w & 0 & 0 \\
-\bar{w} & \bar{z} & 0 & 0 \\
0 & 0 & \bar{z} & -w \\
0 & 0 & \bar{w} & z
\end{bmatrix}
\triangleq
\begin{bmatrix}
\beta(z, w) & 0 \\
0 & \beta(z, w)^*
\end{bmatrix},
\]
where \( \beta(z, w) = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \), for \((z, w) \in \partial D^4 = S^3\), represents the generator of \( K_1(C(S^3)) \). Define \( P : S^4 \to U_4(\mathbb{C}) \) by
\[
P(z, w) \triangleq \alpha(z, w) \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \alpha^*(z, w).
\]
Note that \( \alpha \) is not defined as a function from \( S^4 = D^4/\partial D^4 \) to \( U(4) \), but \( P \) is, since
\[
P(z, w) = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix}
\]
for all \((z, w) \in \partial D^4\)
and \( \partial D^4 \) is identified with the north pole \( \xi \in S^4 \). Hence \( P(\xi) = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \).
5.3. In the rest of the paper, for a compact metric space $X$ with a given base point and a $C^*$-algebra $A$, by $C_0(X, A)$ we mean the $C^*$-algebra of the continuous functions from $X$ to $A$ which vanish at the base point (and $C_0(X, C)$ will be denoted by $C_0(X)$). (Most spaces we used here have an obvious base point, which we will not mention afterward.) Let $A = C_0(S^1, PM_4(S^4)P)$. Let $\tilde{A}$ be the unitization of $A$. Let $B = C_0(S^1, C(S^4))$. Since $A$ is a corner of $M_4(B)$ and $B$ is a corner of $M_2(A)$ (note that a trivial projection of rank 1 is equivalent to a subprojection of $P \oplus P$), $A$ is stably isomorphic to $B$. Let $\tilde{B}$ be a unitization of $B$. Then $\tilde{B} = C(S^4 \times S^1)$ and

$$K_1(\tilde{A}) \cong K_1(A) \cong K_1(B) \cong K_1(\tilde{B}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$ 

5.4. For a unitary $u \in M_4(C(S^4 \times S^1))$, in the identification of $[u] \in K_1(C(S^4 \times S^1))$ with $\mathbb{Z} \oplus \mathbb{Z}$, the first component corresponds to the winding number of

$$S^1 \hookrightarrow S^4 \times S^1 \xrightarrow{\text{det} u} S^1 \subset \mathbb{C},$$

that is, the winding number of the map

$$e^{i\theta} \rightarrow \text{det} u(\xi, e^{i\theta}),$$

where $\xi$ is the north pole of $S^4$. Hence, if $u : S^4 \times S^1 \rightarrow SU(n)$, then the first component of $[u] \in K_1(C(S^4 \times S^1)) \cong \mathbb{Z} \oplus \mathbb{Z}$ is automatically zero.

**Lemma 5.5.** Let $u : S^4 \times S^1 \rightarrow SU(2)$. Then $u \in M_2(C(S^4 \times S^1))$ represents the zero element in $K_1(C(S^4 \times S^1))$. In other words, if $u \in SU_n(S^4 \times S^1)$ represents a nonzero element in $K$-theory, then $n \geq 3$.

**Proof.** Let $f : S^4 \times S^1 \rightarrow S^5$ be the standard quotient map sending $\{\xi\} \times S^1 \cup S^4 \times \{1\}$ to a single point. Consider $u : S^4 \times S^1 \rightarrow SU(2)$. Without loss of generality, assume $u(\xi, 1) = 1_2 \in SU(2)$. Then $u|_{S^4 \times \{1\}} : S^4 \rightarrow SU(2) = S^3$ represents an element in $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore $u^2|_{S^4 \times \{1\}} : S^4 \rightarrow SU(2) = S^3$ is homotopically trivial, with $(\xi, 1) \in S^4 \times S^1$ as a fixed point. Evidently, $u^2|_{\{\xi\} \times S^1} : S^1 \rightarrow S^3 = SU(2)$ is homotopically trivial with $(\xi, 1) \in S^4 \times S^1$ as a fixed point. Consequently

$$u^2|_{S^4 \times \{1\} \cup \{\xi\} \times S^1} : S^4 \times \{1\} \cup \{\xi\} \times S^1 \rightarrow S^3$$

is homotopically trivial with $(\xi, 1) \in S^4 \times S^1$ as a fixed base point. There is a homotopy

$$F : (S^4 \times \{1\} \cup \{\xi\} \times S^1) \times [0, 1] \rightarrow S^3$$

with $F(\cdot, 0) = u^2|_{S^4 \times \{1\} \cup \{\xi\} \times S^1}$ and

$$F(x, 1) = 1_2 \text{ for all } x \in S^4 \times \{1\} \cup \{\xi\} \times S^1.$$
The following is a well-known easy fact: For any relative CW complex \((X, Y)\) \((Y \subset X)\), any continuous map \(Y \times I \cup X \times \{0\} \to Z\) (where \(Z\) is any other CW complex) can be extended to a continuous map \(X \times I \to Z\).

Hence, there is a homotopy \(G : (S^4 \times S^1) \times [0, 1] \to S^3\) with \(G(\cdot, 0) = u^2\), and \(G|_{S^4 \times \{1\} \cup \xi_1 \times S^1 \times [0, 1]} = F\). Let \(v : S^4 \times S^1 \to SU(2)\) be defined by \(v(x) = G(x, 1)\); then \([v] = [u^2] \in K_1(C(S^4 \times S^1))\) and \(v\) maps \(S^4 \times \{1\} \cup \xi_1 \times S^1\) to \(1_2 \in SU(2)\). Consequently, \(v\) passes to a map

\[
v_1 : S^5 \defeq S^4 \times S^1 / S^4 \times \{1\} \cup \{\xi_1\} \times S^1 \to S^3 = SU(2)
\]

and represents an element in \(\pi_5(S^3) = \mathbb{Z}/2\mathbb{Z}\). Hence \(v_1^2 : S^5 \to S^3\) is homotopically trivial, and therefore \(v_1^2\) is as well. So we have

\[4[u] = 2[u^2^2] = 2[v] = [v^2^2] = 0 \in K_1(C(S^4 \times S^1)),\]

which implies \([u] = 0 \in K_1(C(S^4 \times S^1))\). \(\square\)

**Remark 5.6.** In the proof of Lemma 5.5, we in fact proved the following fact: For any \(u : S^4 \times S^1 \to SU(2)\), the map \(u^4 : S^4 \times S^1 \to SU(2)\) is homotopically trivial.

5.7. Note that \(P \in M_4(C(S^4))\) can be regarded as a projection in \(M_4(C(S^4 \times S^1))\), still denoted by \(P\), i.e., for fixed \(x \in S^4\), \(P(x, \cdot)\) is a constant projection along the \(S^1\) direction. Then

\[(5-1) \quad K_1(A) \cong K_1(\overline{A}) \cong K_1(C(S^4 \times S^1)) \cong K_1(PM_4(C(S^4 \times S^1))) P,\]

where \(A = C_0(S^1), PM_4(C(S^4)) P\) is defined in Section 5.2. Let

\[E = \{(\xi, u) : \xi \in S^4 \times S^1, u \in M_4(\mathbb{C}) \text{ with } P(x)uP(x) = u, uu^* = uu^* = P(x)\}, \]

\[SE = \{(\xi, u) \in E : \det(P(x)uP(x) + (1_4 - P(x)) = 1)\}.\]

Then \(E \to S^4 \times S^1\) and \(SE \to S^4 \times S^1\) are fiber bundles with fibers \(U(2)\) and \(SU(2)\), respectively. Also the unitaries in \(PM_4(C(S^4 \times S^1)) P\) correspond bijectively to the cross-sections of a bundle \(E \to S^4 \times S^1\). For this reason, we will call a unitary (of \(PM_4(C(S^4 \times S^1)) P\)) with determinant 1 everywhere a cross-section of a bundle \(SE \to S^4 \times S^1\).

**Theorem 5.8.** If \(u \in PM_4(C(S^4 \times S^1)) P\) has determinant 1 everywhere, i.e., if \(u\) is a cross-section of \(SE \to S^4 \times S^1\), then \([u] = 0 \in K_1(PM_4(C(S^4 \times S^1)) P)\).

**Proof.** Note that \(SE \to S^4 \times S^1\) is a smooth fiber bundle over the smooth manifold \(S^4 \times S^1\). By a standard result in differential topology, \(u\) is homotopic to a \(C^\infty\)-section. Without loss of generality, we may assume that \(u\) itself is smooth. Identify the north pole \(\xi \in S^4\) with \(0 \in \mathbb{R}^4\) and a neighborhood of \(\xi\) with \(B_\epsilon(0) \subset \mathbb{R}^4\) for \(\epsilon > 0\). Since \(B_\epsilon(0)\) is contractible, \(SE|_{B_\epsilon(0) \times S^1}\) is a trivial bundle. Note that the projection \(P \in M_4(C(S^4 \times S^1))\) is constant along \(S^1\), hence \(SE \cong SE|_{S^4 \times \{1\} \times S^1}\).
and \( SE|_{B_\epsilon(0) \times S^1} \cong SE|_{B_\epsilon(0) \times \{1\}} \times S^1 \); in other words, the fiber is constant along \( S^1 \) and \( SE|_{B_\epsilon(0) \times \{1\}} \) is trivial and isomorphic to \((B_\epsilon(0) \times \{1\}) \times \text{SU}(2)\). There is a smooth bundle isomorphism

\[
\gamma : SE|_{B_\epsilon(0) \times S^1} \rightarrow (B_\epsilon(0) \times S^1) \times \text{SU}(2).
\]

Then

\[
\gamma \circ u|_{B_\epsilon(0) \times S^1} : B_\epsilon(0) \times S^1 \rightarrow (B_\epsilon(0) \times S^1) \times \text{SU}(2)
\]

is a smooth map with

\[
\pi_1 \circ (\gamma \circ u)|_{B_\epsilon(0) \times S^1} = \text{id}_{B_\epsilon(0) \times S^1},
\]

where \( \pi_1 : (B_\epsilon(0) \times S^1) \times \text{SU}(2) \rightarrow B_\epsilon(0) \times S^1 \) is the projection onto the first coordinate. Define \( \phi = \pi_2 \circ (\gamma \circ u)|_{B_\epsilon(0) \times S^1} \), where \( \pi_2 : (B_\epsilon(0) \times S^1) \times \text{SU}(2) \rightarrow \text{SU}(2) \) is the projection onto the second coordinate. Since \( \phi \) is smooth, \( \phi|_{\{\xi\} \times S^1} \) is not onto \( \text{SU}(2) \) (note \( \dim(\text{SU}(2)) = 3 \) and \( \dim(S^1) = 1 \)). Therefore, if \( \epsilon \) is small enough, \( \phi|_{B_\epsilon(0) \times S^1} \) is not onto. By Fact 1 of Section 5.1, \( \phi \) is homotopic to a constant map \( \phi_1 : B_\epsilon(0) \times S^1 \rightarrow \text{SU}(2) \) with

\[
\phi_1(\{\xi\} \times S^1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \phi|_{\partial B_\epsilon(0) \times S^1} = \phi_1|_{\partial B_\epsilon(0) \times S^1},
\]

via a homotopy \( F : (B_\epsilon(0) \times S^1) \times [0, 1] \rightarrow \text{SU}(2) \) with \( F(x, e^{i\theta}, t) \) constant with respect to \( t \) if \( x \in \partial B_\epsilon(0) \).

Let \( u_1 : B_\epsilon(0) \times S^1 \rightarrow SE \) be the cross-section defined by

\[ u_1(x, e^{i\theta}) = \gamma^{-1}((x, e^{i\theta}), (x, e^{i\theta})) \in SE. \]

Then \( u_1(x, e^{i\theta}) = u(x, e^{i\theta}) \) if \( x \in \partial B_\epsilon(0) \). We can extend \( u_1 \) to \( S^4 \times S^1 \) by defining

\[ u_1(x, e^{i\theta}) = u(x, e^{i\theta}) \quad \text{if} \quad (x, e^{i\theta}) \notin B_\epsilon(0) \times S^1. \]

Hence \( u_1 \) is a section of \( SE \) with

\[ u_1(\xi, e^{i\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = P(\xi) \quad \text{for all} \quad e^{i\theta} \in S^1. \]

Moreover, \( u_1 \) is homotopic to \( u \) by a homotopy that is constant on \((S^4 \setminus B_\epsilon(0)) \times S^1 \) (on which \( u_1 = u \)) and that agrees with \( F \) on \( B_\epsilon(0) \times S^1 \). Hence \([u] = [u_1] \in K_1(PM_4(C(S^4 \times S^1))P)\). Recall that \( S^4 \) is obtained from

\[ D^4 = \{(z, w) \in \mathbb{C}^2 | |z|^2 + |w|^2 \leq 1\} \]

by identifying

\[ \partial D^4 = \{(z, w) \in \mathbb{C}^2 | |z|^2 + |w|^2 = 1\}. \]
with the north pole $\xi \in S^4$. Recall that $P \in M_4(C(S^4))$ (viewed as a projection in $M_4(C(S^4 \times S^1))$ constant along the $S^1$ direction) is defined as

$$P(z, w) = \alpha(z, w) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \alpha^*(z, w),$$

where $\alpha(z, w)$ is defined as in Section 5.2.

Define

$$v(z, w, e^{i\theta}) = \alpha^*(z, w)u_1(z, w, e^{i\theta})\alpha(z, w).$$

Then we have that

(i) $v(z, w, e^{i\theta}) = \begin{bmatrix} 1 \\ 2 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$ for all $(z, w) \in \partial D^4$.

and therefore $v$ can be regarded as a map from $S^4 \times S^1$ to $M_4(\mathbb{C})$. Moreover,

(ii) $v(z, w, e^{i\theta}) = \begin{bmatrix} 1 \\ 2 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} v(z, w, e^{i\theta}) \begin{bmatrix} 1 \\ 2 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$ for all $(z, w, e^{i\theta}) \in S^4 \times S^1$.

By considering the upper-left corner of $v$ (still denoted by $v$), we obtain a unitary $v: S^4 \times S^1 \to SU(2)$. By Lemma 5.5 and Remark 5.6, $v^4$ is homotopically trivial. Furthermore, by Fact 3 of Section 5.1, there is a homotopy $F: S^4 \times S^1 \times [0, 1] \to SU(2)$ such that

(iii) $F(z, w, e^{i\theta}, 0) = v^4(z, w, e^{i\theta})$ for all $(z, w, e^{i\theta}) \in S^4 \times S^1$,

(iv) $F(\xi, e^{i\theta}, t) = 1_2$ for all $e^{i\theta} \in S^1$,

(v) $F(z, w, e^{i\theta}, 1) = 1_2$ for all $(z, w) \in S^4 \times S^1$.

Define $G: D^4 \times S^1 \times [0, 1] \to M_4(\mathbb{C})$ by

$$G(z, w, e^{i\theta}, t) = \alpha(z, w) \begin{bmatrix} F(z, w, e^{i\theta}, t) \\ 0 \\ 2 \\ 0 \end{bmatrix} \alpha^*(z, w).$$

Then, by (iv), for $(z, w) \in \partial D^4$ we have

$$G(z, w, e^{i\theta}, t) = \begin{bmatrix} 1 \\ 2 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Hence $G$ defines a map (still denoted by $G$) from $S^4 \times S^1 \times [0, 1] \to M_4(\mathbb{C})$. Furthermore $G(z, w, e^{i\theta}, t) \in P(z, w)M_4(\mathbb{C})P(z, w)$, and

$$G(z, w, e^{i\theta}, 0) = \alpha(z, w) \begin{bmatrix} v^4 \\ 0 \\ 2 \\ 0 \end{bmatrix} \alpha^*(z, w) = u_1^4.$$
That is, $G$ defines a homotopy between $u_1^4$ and the unit $P \in P(M_4(C(S^4 \times S^1)))$. Consequently $[u_1^4] = 0$ and $[u_1] = 0 \in K_1(P(M_4(C(S^4 \times S^1)))P)$. Moreover, $[u] = 0 \in K_1(C(S^4 \times S^1))$, as desired.

\section*{5.9.} We identify $P(M_4(C(S^4 \times S^1)))P$ as a corner of $M_4(C(S^4 \times S^1))$; then $K_1(P(M_4(C(S^4 \times S^1)))P)$ is isomorphic to $K_1(C(S^4 \times S^1)) = \mathbb{Z} \oplus \mathbb{Z}$ naturally. Let $a \in P(M_4(C(S^4 \times S^1)))P$ be defined by

$$a(x,e^{i\theta}) = e^{i\theta} P(x).$$

On the other hand, $a$ could also be regarded as a unitary in $M_4(C(S^4 \times S^1))$ as $a(x,e^{i\theta}) = e^{i\theta} P(x) + (1 - P(x))$. Then $[a] = (2, 1) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_1(C(S^4 \times S^1))$, since $[a]$ is the image of $[P] \in K_0(C(S^4))$ under the exponential map

$$K_1(C(S^4)) \to K_1(C(0(S^1, C(S^4))),$$

and $[P] = (2, 1) \in K_0(C(S^4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

**Theorem 5.10.** No element $(1, k) \in K_1(C(S^4 \times S^1))$ can be realized by a unitary $b \in PM_4(C(S^4 \times S^1))P$.

**Proof.** We argue by contradiction. Assume $b \in PM_4(C(S^4 \times S^1))P$ satisfies $[b] = (1, k) \in K_1(PM_4(C(S^4 \times S^1))P)$. Without loss of generality, we assume that $b(\xi, 1) = P$. Then

$$[b^2a^*] = (0, 2k - 1) \in K_1(PM_4(C(S^4 \times S^1))P).$$

In particular, the map

$$e^{i\theta} \to \det\begin{bmatrix} P(\xi)(b^2a^*)(\xi, e^{i\theta}) P(\xi) & 0 \\ 0 & 14 - P(\xi) \end{bmatrix}_{8 \times 8}$$

has winding number 0. That is, it is homotopically trivial. Hence

$$(x, e^{i\theta}) \to \det\begin{bmatrix} P(\xi)(b^2a^*)(x, e^{i\theta}) P(\xi) & 0 \\ 0 & 14 - P(\xi) \end{bmatrix}_{8 \times 8}$$

defines a map $h : S^4 \times S^1 \to S^1$ such that $h_* : \pi_1(S^4 \times S^1) \to \pi_1(S^1)$ is the zero map. Hence there is a lifting $\tilde{h} : S^4 \times S^1 \to \mathbb{R}$ with $h(x, e^{i\theta}) = \exp(i \tilde{h}(x, e^{i\theta}))$. Define a unitary $b_1 \in PM_4(C(S^4 \times S^1))P$ by $b_1(x, e^{i\theta}) = \exp(i \frac{1}{2} \tilde{h}(x, e^{i\theta})) P(x)$. Then $[b_1] \in \mathbb{Z} \cong K_1(C(S^4 \times S^1))$, and $b^2a^*b_1^* \in U(PM_4C(S^4 \times S^1))P$ has determinant 1 everywhere. By Theorem 5.8, $[b^2a^*b_1^*] = 0 \in K_1(C(S^4 \times S^1))$. On the other hand,

$$[b^2a^*b_1^*] = [b^2a^*] = (0, 2k - 1) \neq 0 \in K_1(C(S^4 \times S^1)),$$

which is a contradiction.

**Remark 5.11.** Similarly, we can show that for any unitary $u \in PM_4(C(S^4 \times S^1))P$, $[u] = l[u] = (2l, l) \in K_1(C(S^4 \times S^1))$ for some $l \in \mathbb{Z}$.  

Corollary 5.12. Let $A = C_0(S^1, PC(S^4))$, and let $\tilde{A}$ be the unitization of $A$. Then there is no unitary $u \in \tilde{A}$ such that $[u] = (1, k) \in K_1(A)$. In particular, no unitary $u$ can correspond to a rank-1 projection in $M_4(C(S^4))$.

Proof. Note that we may view $P$ as a projection in $M_4(C(S^4 \times S^1))$ which is constant along the direction of $S^1$ (Section 5.7). So we may view $\tilde{A}$ as a unital $C^*$-subalgebra of $PM_4(C(S^4 \times S^1))P$. Thus, by the identification (5-1), Theorem 5.10 applies.

Theorem 5.13. Let $A = PM_4(C(S^4))P$. Then $\text{Dur} A = 2$.

Proof. There is a projection $e \in M_2(A)$ which is unitarily equivalent to a rank-1 projection in $M_8(C(S^4))$ corresponding to $(1,0) \in K_0(C(S^4))$. Let $C = C_0((0,1),A)$. By Corollary 5.12, there is no unitary in $\tilde{C}$ which represents a rank-1 projection. It follows from Proposition 3.12 that $\text{Dur} A > 1$.

However, since $\rho_C(K_0(M_2(C))) = \frac{1}{2}\mathbb{Z}$ and $M_2(C)$ contains a rank-1 projection (with trace $\frac{1}{2}$), by Proposition 3.6(3), $\text{Dur}(M_2(C)) = 1$. It follows that $\text{Dur} C = 2$. \qed

Acknowledgements

The majority of this work was done when Lin and Xue were in the Research Center for Operator Algebras in the East China Normal University. They are both partially supported by the center. Lin is also partially supported by a grant from the NSF.

References


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Received May 16, 2014.
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