THE VIRTUAL FIRST BETTI NUMBER
OF SOLUBLE GROUPS

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We show that if a group $G$ is finitely presented and nilpotent-by-abelian-by-finite, then there is an upper bound on $\dim_{\mathbb{Q}} H_1(M, \mathbb{Q})$, where $M$ runs through all subgroups of finite index in $G$.

1. Introduction

The virtual first betti number of a finitely generated group $G$ is defined as

$$\text{vb}_1(G) = \sup \{ \dim H_1(S, \mathbb{Q}) \mid S \leq G \text{ of finite index} \}.$$ 

A group is said to be large if it has a subgroup of finite index that maps onto a nonabelian free group. If $G$ is large then $\text{vb}_1(G) = \infty$. It is easy to find finitely generated groups $G$ that are not large but have $\text{vb}_1(G) = \infty$. For example, in the metabelian group $\mathbb{Z} \wr \mathbb{Z} = \langle a, t \mid [a, t^{-n}at^n] = 1 \text{ for all } n \rangle$, the subgroup $S_m < \mathbb{Z} \wr \mathbb{Z}$ generated by $t^m$ and the conjugates of $a$ has index $m$ and $H_1(S_m, \mathbb{Z}) = \mathbb{Z}^{m+1}$. In contrast, no example is known of a finitely presented group that is not large but has $\text{vb}_1(G) = \infty$ (see [Button 2010; Lackenby 2010]). Since amenable groups do not contain nonabelian free subgroups, one might hope to resolve this issue by finding a finitely presented amenable group with $\text{vb}_1(G) = \infty$, but this seems to be a nontrivial matter.

We shall prove in this paper that for large classes of finitely presented soluble groups $\text{vb}_1(G)$ is always finite. One would like to prove that the same is true for all finitely presented soluble groups, but here one faces the profound difficulty of deciding which soluble groups admit finite presentations; this is unknown even for abelian-by-polycyclic and nilpotent-by-abelian groups.

In the case of metabelian groups, finite presentability is completely understood in terms of the Bieri–Strebel invariant [Bieri and Strebel 1980]. Some sufficient conditions for finite presentability of nilpotent-by-abelian groups were considered by McIsaac [1984] and later Groves [1991]. In the case of $S$-arithmetic nilpotent-by-abelian groups $G$ one knows more thanks to the work of Abels [1987]: if $G$ is an extension of a nilpotent group $N$ by an abelian group $Q$ then $G$ is finitely

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presented if and only if it is of type FP₂, which it is if and only if $H_2(N, \mathbb{Z})$ is finitely generated as a $\mathbb{Z}Q$-module (where the $Q$ action is induced by conjugation) and $G/N'$ is finitely presented as a group. The first of these conditions is an easy consequence of the fact that $\mathbb{Z}Q$ is a Noetherian ring, and the second is a corollary of a result in [Bieri and Strebel 1980] that every metabelian quotient of a group of type FP₂ that does not contain noncyclic free subgroups is finitely presented.

The case where $G$ is an extension of an abelian normal subgroup $A$ by a polycyclic group $Q$ was approached by Brookes and Groves who studied modules over crossed products of a division ring by a free abelian group; see [Brookes and Groves 1995; 2000; 2002].

Given this background, the natural place to begin our investigation into the virtual first betti number of finitely presented soluble groups is in the setting of metabelian groups. Using methods from commutative algebra, we prove (Theorem 4.3) that if $G$ is finitely presented and metabelian, then $vb_1(G)$ is finite. (The hypothesis that one actually needs to impose on $G$ is somewhat weaker than finite presentability; see Remark 6.5.) The metabelian case is used in the proof of our main theorem, which is the following.

**Theorem A.** Let $G$ be a finitely presented group. If $G$ is nilpotent-by-abelian-by-finite, then $vb_1(G)$ is finite.

Our proof of this theorem relies on the fact that all metabelian quotients of soluble groups of type FP₂ are finitely presented [Bieri and Strebel 1980, Theorem 5.5], as well as a technical result concerning the homology of subgroups of finite index (Proposition 6.2). Groves, Kochloukova and Rodrigues [Groves et al. 2008, Theorem A] proved that if an abelian-by-polycyclic group $G$ is of type FP₃ then it is nilpotent-by-abelian-by-finite, in which case $vb_1(G)$ is finite by Theorem A. The same is true of all soluble groups of type FPᵢ, because they are constructible [Kropholler 1986], hence nilpotent-by-abelian-by-finite, but in this case stronger finiteness results were already known: constructible soluble groups are obtained from the trivial group by finite sequences of ascending HNN extensions and finite extensions, from which it follows that they have finite Prüfer rank (i.e., there is an upper bound on the number of generators for the finitely generated subgroups).

It is natural to wonder if Theorem A might remain true when the field of rationals $\mathbb{Q}$ in the definition of virtual betti number is replaced with other coefficient fields, such as the field with $p$ elements $F_p$. We shall see in Section 5 that it does not.

**Conjecture.** If $G$ is finitely presented and soluble, then $vb_1(G)$ is finite.

It is difficult to construct finitely presented soluble groups that are not nilpotent-by-abelian-by-finite. The examples provided by the constructions of Robinson and Strebel [1982] all satisfy the conjecture.

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¹Throughout this article, $H'$ denotes the commutator subgroup of a group $H$. 
While editing the final version of this work, we learnt that Andrei Jaikin-Zapirain has, in unpublished work, also proved Theorem A in the metabelian case. Higher dimensional analogues of Theorem A are considered in the forthcoming PhD thesis of Fatemeh Mokari.

2. Preliminary results

2A. Preliminaries on finitely presented metabelian groups. We fix a short exact sequence of groups $A \rightarrow G \rightarrow Q$, where $A$ and $Q$ are abelian and $G$ is finitely generated. The action of $G$ on $A$ by conjugation induces an action of $Q$, which enables us to regard $A$ as a right $\mathbb{Z}Q$-module. Because $G$ is finitely generated and $Q$ is finitely presented, $A$ is finitely generated as a $\mathbb{Z}Q$-module.

Associated to a nonzero real character $\chi : Q \rightarrow \mathbb{R}$ one has the monoid $Q_\chi = \{ g \in Q | \chi(g) \geq 0 \}$.

The character sphere $S(Q)$ is the set of equivalence classes in $\text{Hom}(Q, \mathbb{R}) \setminus \{0\}$ under the relation that identifies $\chi_1 \sim \chi_2$ if $\chi_1 = \lambda \chi_2$ for some $\lambda > 0$. We write $[\chi]$ for the class of $\chi$. Following [Bieri and Strebel 1980], let

$$\Sigma_A(Q) = \{ [\chi] | A \text{ is finitely generated as a } \mathbb{Z}Q_\chi\text{-module} \}.$$ 

By definition, the $\mathbb{Z}Q$-module $A$ is 2-tame if $\Sigma_A(Q)^c = S(Q) \setminus \Sigma_A(Q)$ contains no pair of antipodal points. According to [op. cit., Theorem 5.4], $G$ is finitely presented if and only if $A$ is a 2-tame $\mathbb{Z}Q$-module, and this happens precisely when $G$ is of homological type FP$_2$. We refer the reader to [Bieri 1981] for general results concerning groups of type FP$_m$. If $A_1 \rightarrow A_2 \rightarrow A_3$ is an exact sequence of finitely generated $\mathbb{Z}Q$-modules, then $\Sigma_{A_2}(Q)^c = \Sigma_{A_1}(Q)^c \cup \Sigma_{A_3}(Q)^c$ (see [Bieri and Strebel 1980, Proposition 2.2]), hence every quotient of a 2-tame $\mathbb{Z}Q$-module is 2-tame.

2B. Tensor products and finite presentability. Let $R$ be a noetherian commutative ring with unit 1 and let $W$ be a finitely generated $RQ$-module. As above, we have a Sigma invariant $\Sigma_W(Q) = \{ [\chi] | W \text{ is finitely generated as an } RQ_\chi\text{-module} \}$, and $W$ is defined to be 2-tame as an $RQ$-module if $\Sigma_W^c(Q) = S(Q) \setminus \Sigma_W(Q)$ has no pair of antipodal points.

The question of when the tensor square $W \otimes_R W$ is finitely generated as an $RQ$-module (with $Q$ acting diagonally) is addressed in [Bieri and Groves 1985], where it is shown that $[\chi]$ lies in $\Sigma_W^c(Q)$ if and only if the ring $S = RQ/\text{ann}_{RQ}(W)$ admits a real valuation $v : S \rightarrow \mathbb{R} \cup \{\infty\}$ (in the sense of Bourbaki) that extends $\chi$ and is such that the restriction $v_0$ of $v$ to the image $\bar{R}$ of $R$ in $S$ is nonnegative and discrete. By [loc. cit.], $W \otimes_R W$ is finitely generated as an $RQ$-module if and only if there is no pair of antipodal elements $[\chi], [-\chi] \in \Sigma_W^c(Q)$ that can be
lifted to valuations of $S$ that have the same restriction $v_0$ to $\bar{R}$, with $v_0$ discrete and nonnegative. (These last conditions on $v_0$ are automatic if $\bar{R}$ is $\mathbb{Z}$.)

Returning to the context of Section 2A, we apply these general considerations with $W = A \otimes \mathbb{Q}$ and $R = \mathbb{Q}$, in which case $W \otimes_R W \cong (A \otimes_{\mathbb{Z}} A) \otimes_{\mathbb{Z}} \mathbb{Q}$. We deduce that if there exists a group extension $A \hookrightarrow G \twoheadrightarrow Q$, with $G$ finitely presented, then $W = A \otimes \mathbb{Q}$ is 2-tame as a $\mathbb{Q}Q$-module, and $W \otimes_R W \cong (A \otimes_{\mathbb{Z}} A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finitely generated as a $\mathbb{Q}Q$-module via the diagonal $Q$-action.

We shall also need a refinement of this observation that involves the annihilator $\text{ann}_{\mathbb{Z}Q}(A)$ of $A$ in $\mathbb{Z}Q$, which we denote $I$. Bieri and Strebel [1981, (1.3)] prove that

$$\Sigma_A(Q) = \Sigma_{\mathbb{Z}Q/I}(Q).$$

Thus if $A$ is 2-tame as a $\mathbb{Z}Q$-module, then so is $\mathbb{Z}Q/I$.

**Lemma 2.1.** If there exists a group extension $A \hookrightarrow G \twoheadrightarrow Q$ with $A$ and $Q$ abelian and $G$ finitely presented, and $I = \text{ann}_{\mathbb{Z}Q}(A)$, then $(\mathbb{Z}Q/I) \otimes_{\mathbb{Z}} (\mathbb{Z}Q/I) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finitely generated as a $\mathbb{Q}Q$-module via the diagonal $Q$-action.

### 2C. Preliminaries on commutative algebra

We will need the following basic facts from commutative algebra; for details see, for example, [Bourbaki 1961–1965; Atiyah and Macdonald 1969; Eisenbud 1995]. Let $Q$ be a finitely generated abelian group and recall that the Krull dimension of a commutative ring is the supremum of the lengths of all chains of prime ideals in the ring.

1. The radical $\sqrt{J}$ of each ideal $J \triangleleft \mathbb{Q}Q$ is the intersection of the finitely many prime ideals that contain $J$ and are minimal subject to this condition.
2. Finite dimensional $\mathbb{Q}$-algebras are Artinian and thus have Krull dimension 0.

Throughout, if $R$ is a commutative ring and $m$ a positive integer, then $R^m$ will denote the subring generated by $m$-th powers, except that $\mathbb{Z}^m$ and $\mathbb{Q}^m$ will denote Cartesian powers. Where no ring is specified, tensor products are assumed to be taken over $\mathbb{Z}$.

### 3. A finiteness result in commutative algebra

Lemma 2.1 assures us that the following theorem applies to the modules that arise from short exact sequences $N \hookrightarrow G \twoheadrightarrow \mathbb{Z}^m$ associated to finitely presented metabelian groups.

**Theorem 3.1.** Let $Q \cong \mathbb{Z}^n$ be a group and let $S = \mathbb{Z}Q/I$ be a commutative ring such that $(S \otimes_{\mathbb{Z}} S) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finitely generated as a $\mathbb{Q}Q$-module via the diagonal $Q$-action. Then,

$$\sup_m \dim_{\mathbb{Q}}(S \otimes_{\mathbb{Z}} \mathbb{Q}^m) < \infty.$$
Proof. Let $B = S \otimes \mathbb{Q} = \mathbb{Q}Q/J$ and for each positive integer $m$ define $J_m \triangleleft \mathbb{Q}$ to be $(J, Q^m - 1)$ and

$$B_m := B \otimes_{\mathbb{Q}Q^n} \mathbb{Q} = \mathbb{Q}Q/J_m \cong S \otimes_{\mathbb{Z}} \mathbb{Q}.$$

As $\mathbb{Q}Q/(Q^m - 1)$ is finite dimensional over $\mathbb{Q}$, so is $B_m = \mathbb{Q}Q/J_m$. Hence $B_m$ has Krull dimension 0; i.e., every prime ideal in $B_m$ is a maximal one. Therefore, the finite collection of prime ideals $P_{m,t}$ whose intersection is $\sqrt{B_m}$ are the only prime ideals in $\mathbb{Q}Q$ above $J_m$, and each of the quotients $\mathbb{Q}Q/P_{m,t}$ is a field.

We shall establish the theorem by proving the following:

**Claim 1.** There exist only finitely many fields $F$ such that for some $m \geq 1$ (depending on $F$) the field $F$ is a quotient of $B_m$.

Claim 1 provides an integer $m_0$ such that if a field $F$ is a quotient of $B_m$ then the natural map $\mathbb{Q}Q \rightarrow F$ factors through $\mathbb{Q}Q/(Q^{m_0} - 1)$.

**Claim 2.** If $m_0$ divides $m$ then $J_m = J_{mr}$ for every $r \in \mathbb{N}$.

To see that the theorem follows from these claims, note that for an arbitrary positive integer $m$ we have $J_m \supseteq J_{mm_0} = J_{m_0}$, whence

$$\dim_{\mathbb{Q}}(\mathbb{Q}Q/J_m) \leq \dim_{\mathbb{Q}}(\mathbb{Q}Q/J_{m_0}) \leq \dim_{\mathbb{Q}}(\mathbb{Q}Q/(Q^{m_0} - 1)) = \dim_{\mathbb{Q}}[\mathbb{Q}/Q^{m_0}] = m_0^n.$$

**Proof of Claim 1.** Our hypothesis on $S$ implies that $B \otimes \mathbb{Q} B$ is finitely generated as $\mathbb{Q}Q$-module via the diagonal $Q$-action, by $d$ elements say. Let $F$ be a field quotient of $B_m$ and let $\theta : \mathbb{Q}Q \rightarrow F$ be the canonical projection; so $Q^m - 1 \subseteq \ker(\theta)$. Then, $\theta(Q)$ is a finitely generated multiplicative subgroup of $F^*$ that has finite exponent and $F$, being finite dimensional over $\mathbb{Q}$, embeds in $\mathbb{C}$. Hence $\theta(Q)$ is a finite cyclic group, generated by a root of unity, $\epsilon$ of order $s$, say. Thus we obtain a subgroup $H < Q$ such that $Q/H$ is cyclic of order $s$ and $H - 1 \subseteq \ker(\theta)$. Now, $F \cong \mathbb{Q}[x]/(f)$, where $f$ is the minimal polynomial of $\epsilon$ over $\mathbb{Q}$. And $f$ is an irreducible factor of $x^s - 1$ in $\mathbb{Q}[x]$, whose zeroes are distinct roots of unity with order precisely $s$. Thus $\dim_{\mathbb{Q}} F = \deg(f) = \varphi(s)$, where $\varphi$ is Euler’s totient function. On the other hand, $F \otimes_{\mathbb{Q}} B$ is an epimorphic image of the $\mathbb{Q}Q$-module $B \otimes_{\mathbb{Q}} B$ and the action of $Q$ on $F \otimes_{\mathbb{Q}} F$ factors through the action of $Q/H$, so $F \otimes_{\mathbb{Q}} F$ is generated as a $\mathbb{Q}[Q/H]$-module by $d$ elements. Hence

$$\varphi(s)^2 = (\dim_{\mathbb{Q}} F)^2 = \dim_{\mathbb{Q}}(F \otimes_{\mathbb{Q}} F) \leq d \dim_{\mathbb{Q}}[\mathbb{Q}/Q/H] = ds.$$

An elementary calculation shows that $\varphi(n)/\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$, so for fixed $d$ there are only finitely many possible values of $s$ and $\epsilon$. Let $b$ be a natural number such that the order of $\epsilon$ is at most $b$. Then, the order of $\epsilon$ is a divisor of $m_0 = b!$ and $F$ is a quotient of $\mathbb{Q}Q/(Q^{m_0} - 1)$. 


Since \( Q/(Q^{m_0} - 1) \) is finite dimensional over \( Q \) it has Krull dimension 0, so has only finitely many prime ideals and finitely many field quotients. This completes the proof of Claim 1.

**Proof of Claim 2.** Since \( m_0 \) divides \( m \) we have \( J_m \subseteq J_{m_0} \), so the prime ideals containing \( J_{m_0} \) also contain \( J_m \). On the other hand, we saw earlier that for each of the prime ideals \( P_m \), containing \( J_{m_0} \) contains \( J_{m_0} = (J, Q^{m_0} - 1) \). The radical of \( J_m \) is the intersection of the prime ideals containing it, so

\[
\sqrt{J_m} = \sqrt{J_{m_0}}.
\]

Arguing by induction on \( r \), Claim 2 will follow if we can prove that for every prime number \( p \) we have \( J_m = J_{mp} \), which is equivalent to the assertion that \( q^m - 1 \in J_{mp} \) for all \( q \in Q \).

We now fix \( q \in Q \). From the preceding argument, \( \sqrt{J_m} = \sqrt{J_{mp}} \). In particular, \( Q^m - 1 \subseteq J_m \subseteq \sqrt{J_m} = \sqrt{J_{mp}} \), so there is a natural number \( s \) (over which we have no control) such that

\[
(q^m - 1)^s \in J_{mp}.
\]

As \( Q^{mp} - 1 \subseteq J_{mp} \), we also have

\[
q^{mp} - 1 \in J_{mp}.
\]

Let \( g(x) \) be the greatest common divisor of \( x^{pm} - 1 \) and \( (x^m - 1)^s \) in \( \mathbb{Q}[x] \). In characteristic zero, the polynomial \( x^{pm} - 1 \) has no repeated roots, so neither does \( g(x) \). Since \( g(x) \) divides \( (x^m - 1)^s \), it must actually divide \( x^m - 1 \), so in fact \( g(x) = x^m - 1 \). From (3-1), (3-2) and Bézout’s lemma, we have \( g(q) \in J_{pm} \). Since \( q \in Q \) is arbitrary, this implies that \( J_{mp} = J_m \). □

4. The main theorem for metabelian groups

In this section we prove that all finitely presented metabelian groups have finite virtual first betti number. The proof relies on the finiteness theorem proved in the previous section and two technical lemmas, the first of which is a simple observation about commensurable groups.

**Lemma 4.1.** Let \( G \) be a group. If \( G_0 < G \) is a subgroup of finite index, then \( \text{vb}_1(G) = \text{vb}_1(G_0) \).

**Proof.** By definition, \( \text{vb}_1(G) = \sup_M \dim H_1(M, \mathbb{Q}) \), where the supremum is taken over finite-index subgroups of \( G \). If \( M \) has finite index in \( G_0 \), then it also has finite index in \( G \), so \( \text{vb}_1(G) \geq \text{vb}_1(G_0) \). Conversely, if \( S \) has finite index in \( G \), then
$S_0 = G_0 \cap S$ has finite index in $G_0$, and since it also has finite index in $S$, we have
$\dim H_1(S_0, \mathbb{Q}) \geq \dim H_1(S, \mathbb{Q})$, so $\text{vb}_1(G_0) \geq \text{vb}_1(G)$. \qed

**Lemma 4.2.** Let $A \hookrightarrow G \twoheadrightarrow Q$ be a short exact sequence of groups with $A$ and $Q$ abelian and let $n$ be the torsion-free rank of $Q$. Then:
(a) Writing $[G, A] = \{(g, a) = g^{-1}a^{-1}ga \mid g \in G, a \in A\}$, we have
$$\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) \leq \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + n.$$  

In the split case, $G = A \rtimes Q$, we have $H_1(G, \mathbb{Q}) \cong (G/[G, A]) \otimes_{\mathbb{Z}} \mathbb{Q}$, and
$$\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) = \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + n.$$

(b) If $G_m$ is a subgroup of finite index in $G$ and $Q_m$ is the image of $G_m$ in $Q$, then
$$\dim_{\mathbb{Q}} H_1(G_m, \mathbb{Q}) \leq \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) + n.$$  

In the split case, $G_m = (A \cap G_m) \rtimes Q_m$, equality is attained:
$$\dim_{\mathbb{Q}} H_1(G_m, \mathbb{Q}) = \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) + n.$$

(c) If $G = A \rtimes Q$ and $\mathcal{B}$ denotes the set of subgroups of finite index in $Q$, then
$$\text{vb}_1(G) = \sup_{S \in \mathcal{B}} \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}S} \mathbb{Q}) + n.$$

**Proof.** (a) As $[G, A] \subseteq [G, G]$, we see that $H_1(G, \mathbb{Z}) = G/[G, G]$ is a quotient of $G/[G, A]$. So from the central extension $A/[G, A] \to G/[G, A] \to Q$, we get
$$\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) \leq \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + \dim_{\mathbb{Q}}(Q \otimes \mathbb{Q}) = \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + n.$$

If $G = A \rtimes Q$ then, using that $A, Q$ are abelian and $A$ is normal in $G$, we get $[G, G] = [AQ, AQ] = [Q, A] \subseteq [G, A] \subseteq [G, G]$; hence $[G, G] = [G, A]$ and $A/[G, A] \to G/[G, G] \to Q$ is an exact sequence of abelian groups.

(b) We consider the short exact sequence $A_m \to G_m \to Q_m$, where $A_m = A \cap G_m$. From part (a) we have

$$\text{dim}_\mathbb{Q} H_1(G_m, \mathbb{Q}) \leq \text{dim}_\mathbb{Q}(A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) + n,$$

with equality if the sequence splits. Furthermore, since $A/A_m$ is finite we have
$$0 = \text{Tor}_1^{\mathbb{Z}Q_m}(A/A_m, \mathbb{Q}) \quad \text{and} \quad (A/A_m) \otimes_{\mathbb{Z}Q_m} \mathbb{Q} = 0.$$  

Thus there is an exact sequence (part of the long exact sequence in Tor associated to $A \cap G_m \to A \to A/(A \cap G_m)$)
$$0 = \text{Tor}_1^{\mathbb{Z}Q_m}(A/A_m, \mathbb{Q}) \to A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q} \to A \otimes_{\mathbb{Z}Q_m} \mathbb{Q} \to (A/A_m) \otimes_{\mathbb{Z}Q_m} \mathbb{Q} = 0,$$

whence $A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q} \cong A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}$. Thus, we may replace $A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q}$ in (4-1) by $A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}$, and (b) is proved.
(c) From the first part of (b) we have
\[ \text{vb}_1(G) \leq \sup_{S \in \mathcal{H}} \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} S \mathbb{Q}) + n, \]
and to obtain the reverse inequality, we use the second part of (b)
\[ \sup_{S \in \mathcal{H}} \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} S \mathbb{Q}) + n = \sup_{S \in \mathcal{H}} H_1(A \rtimes S, \mathbb{Q}), \]
noting that \( A \rtimes S \) has finite index in \( G \).

\[ \square \]

**Theorem 4.3.** Let \( A \hookrightarrow G \twoheadrightarrow Q \) be a short exact sequence of groups with \( A \) and \( Q \) abelian. If \( G \) is finitely presented then its virtual first betti number \( \text{vb}_1(G) \) is finite.

**Proof.** By passing to a subgroup of finite index in \( Q \) and replacing \( G \) by the inverse image of this subgroup, we may assume that \( Q \) is free abelian. Lemma 4.1 assures us that it is enough to consider this case, and Lemma 4.2(b) tells us that we will be done if we can establish an upper bound on \( \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} Q_m \mathbb{Q}) \) as \( Q_m \) ranges over the subgroups of finite index in \( Q \).

Recall that \( A \) is finitely generated as a \( \mathbb{Z} Q \)-module, say by \( d \) elements. Thus, denoting the annihilator \( \text{ann}_{\mathbb{Z} Q}(A) = \{ \lambda \in \mathbb{Z} Q \mid A\lambda = 0 \} \) by \( I \), we have an epimorphism of \( \mathbb{Z} Q \)-modules
\[ (\mathbb{Z} Q/I)^{[d]} = \mathbb{Z} Q/I \oplus \cdots \oplus \mathbb{Z} Q/I \rightarrow A \]
that induces an epimorphism of \( \mathbb{Q} \)-vector spaces
\[ ((\mathbb{Z} Q/I) \otimes_{\mathbb{Z} Q_m} \mathbb{Q})^{[d]} = (\mathbb{Z} Q/I)^{[d]} \otimes_{\mathbb{Z} Q_m} \mathbb{Q} \rightarrow A \otimes_{\mathbb{Z} Q_m} \mathbb{Q}. \]
Thus,
\[ \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z} Q_m} \mathbb{Q}) \leq d \dim_{\mathbb{Q}}((\mathbb{Z} Q/I) \otimes_{\mathbb{Z} Q_m} \mathbb{Q}) \]
and it suffices to show that
\[ \sup_m \dim_{\mathbb{Q}}((\mathbb{Z} Q/I) \otimes_{\mathbb{Z} Q_m} \mathbb{Q}) < \infty. \]

For every \( m \) there is a natural number \( \alpha_m \) such that \( Q^{\alpha_m} \leq Q_m \), and \( \mathbb{Z} Q/I \otimes_{\mathbb{Z} Q_m} \mathbb{Q} \) is a quotient of \( \mathbb{Z} Q/I \otimes_{\mathbb{Z} Q^{\alpha_m}} \mathbb{Q} \). Thus,
\[ \dim_{\mathbb{Q}}((\mathbb{Z} Q/I) \otimes_{\mathbb{Z} Q_m} \mathbb{Q}) \leq \dim_{\mathbb{Q}}((\mathbb{Z} Q/I) \otimes_{\mathbb{Z} Q^{\alpha_m}} \mathbb{Q}), \]
and we have reduced to showing that
\[ \sup_s \dim_{\mathbb{Q}}((\mathbb{Z} Q/I) \otimes_{\mathbb{Z} Q_s} \mathbb{Q}) < \infty. \]

The theorem now follows from Lemma 2.1 and Theorem 3.1. \( \square \)
5. Characteristic $p$ case

In this section we shall construct examples which show that the restriction to fields of characteristic 0 in Theorem A is essential, even in the metabelian case.\footnote{John Wilson [1998] proved that the dimension of $H_1(S, F_p)$ can grow at most like the square root of the index $[G : S]$. Jack Button [2010] exhibited a finitely presented soluble group that exhibits this growth for all $p$.} To this end, we consider the mod $p$ virtual first betti number of a finitely generated group $G$,

$$
\text{vb}_1^{(p)}(G) = \sup\{ \dim H_1(S, F_p) \mid S < G \text{ of finite index} \}.
$$

**Proposition 5.1.** For every prime $p$ there exist finitely presented metabelian groups $\Gamma$ such that $\text{vb}_1^{(p)}(\Gamma)$ is infinite.

**Proof.** Let $Q$ be a free abelian group with generators $x$ and $y$ and let $A = F_p Q/I$, where $I$ is the ideal of $F_p Q$ generated by $y - x^2 + x - 1$. Then,

$$
A \cong F_p \left[ x, x^{-1}, \frac{1}{x^2 - x + 1} \right].
$$

Consider

$$
A_m = A \otimes_{\mathbb{Z} Q_p} F_p \cong F_p Q/(I, Q^{p^m} - 1).
$$

Since $(x^2 - x + 1)^{p^m} - 1 = x^{2^{p^m}} - x^{p^m} + 1 - 1 = x^{p^m}(x^{p^m} - 1)$, we have

$$
A_m = F_p \left[ x, x^{-1}, \frac{1}{x^2 - x + 1} \right] / (x^{p^m} - 1, (x^2 - x + 1)^{p^m} - 1)
$$

$$
= F_p \left[ x, x^{-1}, \frac{1}{x^2 - x + 1} \right] / (x^{p^m} - 1)
$$

is the localisation

$$
B_m S^{-1}
$$

where $B_m = F_p[x, x^{-1}]/(x^{p^m} - 1)$ and $S$ is the image of $\{(x^2 - x + 1)^j\}_{j \geq 1}$ in $B_m$. Note that $x^{p^m} - 1$ and $x^2 - x + 1$ do not have a common root in any finite field extension of $F_p$, for if $z$ were a common root we would have $1 = z^{2^{p^m}} = (z - 1)^{p^m} = z^{p^m} - 1 = 0$, which is a contradiction. Thus the polynomials $x^{p^m} - 1$ and $(x^2 - x + 1)^j$ are coprime in $F_p[x, x^{-1}]$; i.e., they generate the whole ring as an ideal, and so the elements of $S$ are invertible in $B_m$. Therefore $B_m S^{-1} = B_m$ and

$$
\dim_{F_p} A_m = \dim_{F_p} B_m S^{-1} = \dim_{F_p} B_m = p^m.
$$

Now define

$$
\Gamma = A \rtimes Q \quad \text{and} \quad \Gamma_m = A \rtimes Q^{p^m}.
$$

Then, as in the split case of Lemma 4.2(b) (with coefficients in $F_p$ in place of $Q$),

$$
\dim_{F_p} H_1(\Gamma_m, F_p) = \dim_{F_p} A_m + 2 = p^m + 2,
$$

$$
\text{vb}_1^{(p)}(\Gamma) = \sup\{ \dim H_1(S, F_p) \mid S < \text{finite metabelian} \} = \infty.
$$
which tends to infinity with $m$.

By the calculation [Bieri and Strebel 1981, Theorem 5.2] of $\Sigma_A(Q)$ for $A = F_p Q/I$, where the ideal $I$ is 1-generated, or by the link between $\Sigma_A^c(Q)$ and valuation theory (as described in Section 2B), we have

$$\Sigma_A^c(Q) = \{[\chi_1], [\chi_2], [\chi_3]\},$$

with

$$\chi_1(x) = 0, \quad \chi_2(x) = 1, \quad \chi_3(x) = -1,$$

$$\chi_1(y) = 1, \quad \chi_2(y) = 0, \quad \chi_3(y) = -2.$$ 

Thus, $A$ is 2-tame as a $\mathbb{Z}Q$-module, and by the classification of finitely presented metabelian groups in [Bieri and Strebel 1980], $\Gamma$ is finitely presented. □

**Corollary 5.2.** There exists a finitely presented metabelian group $G$ such that for the class $\mathcal{A}$ of all subgroups of finite index in $G$, 

$$\sup_{M \in \mathcal{A}} d(M) = \infty,$$

where $d(M)$ is the minimal number of generators of $M$.

**Proof.** Immediate, since $d(M) \geq \dim_{F_p} H_1(M, F_p)$. □

It is natural to wonder if the lack of finiteness exhibited in the preceding proposition might be avoided by restricting to subgroups whose index is coprime to $p$. The following refinement shows that this is not the case.

**Proposition 5.3.** Let $p$ be a prime. There exist finitely presented metabelian groups $G$ such that

$$\sup\{\dim_{F_p} H_1(S, F_p) \mid S \in \mathcal{A}_p\} = \infty,$$

where

$$\mathcal{A}_p = \{S \leq G \mid [G : S] \text{ is finite and coprime to } p\}.$$

**Proof.** Let $A = F_p[x, x^{-1}, (x + 1)^{-1}]$ and let $Q$ be a free abelian group of rank 2 whose generators $x_1, x_2$ act on $A$ as multiplication by $x$ and $x + 1$, respectively. We consider the group $G = A \rtimes Q$. As an $F_p[Q]$-module, $A \cong F_p[Q]/I$ where $I$ is the ideal generated by $x_2 - x_1 - 1$, and the argument given in the preceding proposition shows that $\Sigma_A(Q)^c$ consists of precisely 3 points, no pair of which is antipodal. Therefore, $G$ is finitely presented.

Let $F$ be a finite field with $p^r$ elements, $r \geq 2$. Let $w$ be a generator of the multiplicative group $F^* = F \setminus \{0\}$. Let $Q_r$ be the kernel of the homomorphism $Q \to F^*$ defined by $x_1 \mapsto w$ and $x_2 \mapsto w + 1$. Let $G_r = A \rtimes Q_r$ and note that $|G/G_r| = |Q/Q_r| = p^r - 1$ is coprime to $p$.

The ring epimorphism $A \to F$ sending $x$ to $w$ provides an epimorphism of the underlying additive groups which extends to a group epimorphism $A \rtimes Q_r \to F \times \mathbb{Z}^2$. 

[506] MARTIN R. BRIDSON AND DESSISLAVA H. KOCHLOUKOVA
According to [Quillen 1969, Appendix A, Corollary 3.8], for any nilpotent group without changing the kernel of the canonical epimorphism \(G \to H\), there exists a positive integer \(r\) such that\(\dim_{F_p} F = r\), it follows that \(\dim_{F_p} H_1(G_r, F_p) \geq r + 2\). And, \(r \geq 2\) was arbitrary.

6. Beyond the metabelian case

In this section we shall prove Theorem A, but first we present a consequence of Theorem 4.3 that describes what one can deduce about towers of finite-index subgroups above the commutator subgroup in amenable and related groups.

**Proposition 6.1.** Let \(G\) be a group of type \(FP_2\) that does not contain a nonabelian free group and let \((\cdot)\) be the set of finite-index subgroups in \(G\) that contain the commutator subgroup \(G'\). Then, \(\sup_{M \in (\cdot)} \dim_{\mathbb{Q}} H_1(M, \mathbb{Q}) < \infty\).

**Proof.** By [Bieri and Strebel 1980, Theorem 5.5], \(G/G''\) is finitely presented. Since \(M \supseteq G''\), we have \(M' \supseteq G''\) and can replace \(G\) by \(G/G''\) and \(M\) by \(MG''/G''\) without changing \(H_1(M, \mathbb{Q})\). Then we can apply Theorem 4.3.

Our proof of Theorem A relies on the following proposition, which is of interest in its own right.

**Proposition 6.2.** Let \(N \hookrightarrow G \to Q\) be a short exact sequence of groups, where \(N\) is nilpotent, \(Q\) is abelian and \(G\) is finitely generated. Let \(G_n\) be a subgroup of finite index in \(G\) and let \(\overline{G}_n\) be the image of \(G_n\) in the metabelian group \(G/N'\). Then,

\[
\dim_{\mathbb{Q}} H_1(G_n, \mathbb{Q}) = \dim_{\mathbb{Q}} H_1(\overline{G}_n, \mathbb{Q}).
\]

**Proof.** We argue using the Malcev completion \(j_N : N \to N^*\) [Malcev 1949]. According to [Quillen 1969, Appendix A, Corollary 3.8], for any nilpotent group \(N\), the homomorphism \(j_N : N \to N^*\) is characterized up to isomorphism by the following properties:

(a) \(N^*\) is nilpotent and uniquely divisible.

(b) \(\ker j_N\) is the torsion subgroup of \(N\).

(c) For every \(x \in N^*\), there is a positive integer \(n\) such that \(x^n \in N\).

In any nilpotent group, the set \(\sqrt{S}\) of elements that have powers in a fixed subgroup \(S\) is a subgroup. It follows that, for every subgroup \(M < N\), the map \(M \to \sqrt{j_N(M)}\) satisfies properties (a) to (c). Thus we may identify \(M^*\) with \(\sqrt{j_N(M)} < N^*\). If \(M < N\) has finite index, then \(M^* = \sqrt{j_N(M)} = N^*\). And \(((N^*)') = (N')^*\).

With these facts in hand, for all subgroups of finite index \(G_n < G\) we have \((G_n')^* \supseteq ((G_n \cap N')^*) = ((G_n \cap N)^*)' = (N')' = (N')^*\). Thus \((G_n'N')^* = (G_n')^*\), and from (c) we deduce that \(G_n'(N' \cap G_n)/G_n'\) is torsion. As \(G_n'(N' \cap G_n)/G_n'\) is the kernel of the canonical epimorphism \(G_n/G_n' \to G_nN'/G_nN'\), we have

\[
H_1(G_n, \mathbb{Q}) \cong (G_n/G_n') \otimes \mathbb{Q} \cong (G_nN'/G_n'N') \otimes \mathbb{Q} \cong H_1(\overline{G}_n, \mathbb{Q}).
\]
Theorem 6.3. Let $N \hookrightarrow G \twoheadrightarrow Q$ be a short exact sequence of groups. If $N$ is nilpotent, $Q$ is abelian and $G$ is of type $FP_2$, then the virtual first betti number $\text{vb}_1(G)$ is finite.

Proof. In the light of Proposition 6.2, this follows directly from Theorem 4.3 and the fact [Bieri and Strebel 1980, Theorem 5.5] that $G/N'$ is a finitely presented metabelian group.

Corollary 6.4 (Theorem A). If a group $G$ is nilpotent-by-abelian-by-finite and of type $FP_2$, then $\text{vb}_1(G)$ is finite.

Proof. Let $G_0$ be a subgroup of finite index in $G$ such that $G_0$ is nilpotent-by-abelian. Then, $G_0$ has type $FP_2$, so $\text{vb}_1(G_0)$ is finite, by Theorem 6.3, and hence, so is $G$, by Lemma 4.1.

Remark 6.5. We did not use the full force of finite presentability in establishing Theorem A: in fact, it is enough to assume that $G$ has a subgroup of finite index $G_0$ in which there is a nilpotent subgroup $N \triangleleft G_0$ such that $Q = G_0/N$ is free abelian and, writing $A = N/N'$, the $\mathbb{Q}Q$-module $A \otimes A \otimes \mathbb{Q}$, with diagonal action, should be finitely generated. These requirements follow from the finite presentability of $G_0/N'$ but are strictly weaker.

Corollary 6.6. Every soluble group of type $FP_\infty$ has finite virtual first betti number.

Proof. Soluble groups $S$ of type $FP_\infty$ are constructible and hence nilpotent-by-abelian-by-finite [Kropholler 1986].

Corollary 6.7. Every abelian-by-polycyclic group of type $FP_3$ has finite virtual first betti number.

Proof. By the main result of [Groves et al. 2008], abelian-by-polycyclic groups of type $FP_3$ are nilpotent-by-abelian-by-finite.

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ANGELO BIANCHI, TIAGO MACELO and ADRIANO MOURA

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MARTIN PUCHOL and JIALIN ZHU

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GUIHUA GONG, HUAXIN LIN and YIFENG XUE

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DAVID HARTLEY

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GEORGE M. BERGMAN

The virtual first Betti number of soluble groups
MARTIN R. BRIDSON and DESSISLAVA H. KOCHLOUKOVA