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Let $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}$ be such that $f(a) = \pm 1$ for $1 \le a < q$, and $f(q) = 0$. Then Erdős conjectured that $\sum_{n\ge 1} f(n)/n \ne 0$. For q even, it is easy to show that the conjecture is true. The case $q \equiv 3 \pmod 4$ was solved by Murty and Saradha. In this paper, we show that this conjecture is true for 82% of the remaining integers $q \equiv 1 \pmod 4$.

1. Introduction

In a written communication with Livingston, Erdős made the following conjecture (see [Livingston 1965]): if f is a periodic arithmetic function with period q and

$$f(n) = \begin{cases} \pm 1 & \text{if } q \nmid n, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$L(1, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n} \ne 0$$

where the L-function $L(s, f)$ associated with f is defined by the series

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \tag{1}$$

In 1973, Baker, Birch and Wirsing, using Baker's theory of linear forms in logarithms, proved the conjecture for q prime [Baker et al. 1973, Theorem 1]. In 1982, Okada [1982] established the conjecture if $2\varphi(q) + 1 > q$. Hence, if q is a prime power or a product of two distinct odd primes, the conjecture is true. In 2002, R. Tijdeman [2002] proved the conjecture is true for periodic completely multiplicative functions f. Saradha and Tijdeman [2003] showed that if f is

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periodic and multiplicative with $|f(p^k)| < p-1$ for every prime divisor p of q and every positive integer k, then the conjecture is true.

It is easy to see that

$$L(1, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n}$$

exists if and only if $\sum_{n=1}^{q} f(n) = 0$. If q is even and f takes values ± 1 with $f(q) = 0$, then $\sum_{n=1}^{q} f(n) \neq 0$. Hence the conjecture holds for even q.

In 2007, Murty and Saradha [2007] proved that if q is odd and f is an odd integer-valued odd periodic function then the conclusion of the conjecture holds. In 2010, they proved that the Erdős conjecture is true if $q \equiv 3 \pmod 4$ [Murty and Saradha 2010, Theorem 7]. Thus the conjecture is open only in cases where $q \equiv 1 \pmod 4$. However, it seems that a novel idea will be needed to deal with these cases. In this paper, we adopt a new density-theoretic approach which is orthogonal to earlier methods. Here is the main consequence of our method:

Theorem 1.1. *Let* $S(X) = |\{q \equiv 1 \pmod 4, q \le X \mid$ *Erdős conjecture is true for* $q\}|$. *Then*

$$\liminf_{X\to\infty} \frac{S(X)}{X/4} \ge 0.82.$$

In other words, the Erdős conjecture is true for at least 82% of the integers $q \equiv 1 \pmod 4$. Our method does not extend to show that the Erdős conjecture is true for 100% of the moduli $q \equiv 1 \pmod 4$. We examine this question briefly at the end of the paper. It seems to us that more ideas are needed to resolve the conjecture fully.

These questions have a long history beginning with Baker, Birch and Wirsing [Baker et al. 1973]. Their work was generalized by Gun, Murty and Rath [Gun et al. 2012] to the setting of algebraic number fields. The paper [Chatterjee and Murty 2014] gives new proofs of some of the background results of this area. We also refer the reader to [Tijdeman 2002] for an expanded survey of the early history.

2. Notations and preliminaries

From now onwards, we denote the field of rationals by $\mathbb{Q}$, the field of algebraic numbers by $\overline{\mathbb{Q}}$, Euler's totient function by φ and Euler's constant by γ. We say a function f is Erdősian modulo q if f is a periodic function with period q and

$$f(n) = \begin{cases} \pm 1 & \text{if } q \nmid n, \\ 0 & \text{otherwise.} \end{cases}$$

Also we will write $f(X) \lesssim g(X)$ to mean

$$\limsup_{X\to\infty} \frac{f(X)}{g(X)} \le 1.$$

Similarly, we write $f(x) \gtrsim g(x)$ to mean

$$\liminf_{X\to\infty} \frac{f(X)}{g(X)} \ge 1.$$

2A. ***Okada's criterion.***

Proposition 2.1. *Let the q-th cyclotomic polynomial Φ_q be irreducible over the field $\mathbb{Q}(f(1), \dots, f(q))$. Let $M(q)$ be the set of positive integers which are composed of prime factors of q. For any integer r and prime p, let $v_p(r)$ be the exponent of p dividing r.*

Then $L(1, f) = 0$ if and only if the following conditions are satisfied:

$$\sum_{m\in M(q)} \frac{f(am)}{m} = 0 \qquad \text{for every } a \text{ with } 1 \le a < q \text{ and } (a, q) = 1, \text{ and}$$

$$\sum_{\substack{r=1\\(r,q)>1}}^{q} f(r)\epsilon(r, p) = 0 \quad \text{for every prime divisor } p \text{ of } q,$$

where

$$\epsilon(r, p) = \begin{cases} v_p(r) & \text{if } v_p(r) < v_p(q), \\ v_p(q) + \dfrac{1}{p-1} & \text{otherwise.} \end{cases}$$

This proposition is a modification, due to Saradha and Tijdeman [2003], of a result of Okada [1986]. Note that Okada deduced the sufficient condition $2\varphi(q) + 1 > q$ stated in the introduction from his original version of this criterion.

2B. ***Wirsing's theorem.*** The following proposition is due to Wirsing [1961].

Proposition 2.2. *Let f be a nonnegative multiplicative arithmetic function*, *satisfying*

$$|f(p)| \le G \text{ for all primes } p,$$

$$\sum_{p\le X} p^{-1} f(p) \log p \sim \tau \log X,$$

with some constants $G > 0$, $\tau > 0$ and

$$\sum_{p} \sum_{k\ge 2} p^{-k} |f(p^k)| < \infty;$$

if $0 < \tau \le 1$, *then*, *in addition*, *the condition*

$$\sum_p \sum_{\substack{k\ge 2\\ p^k \le X}} |f(p^k)| = O(X/\log X)$$

is assumed to hold. Then

$$\sum_{n\le X} f(n) = (1+o(1))\frac{X}{\log X}\frac{e^{-\gamma\tau}}{\Gamma(\tau)}\prod_{p\le X}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\right).$$

2C. *Mertens' theorem.* We also need a classical theorem of Mertens in a later section. We record the theorem here (see, for example, [Murty 2008, page 130]):

Proposition 2.3.
$$\lim_{X\to\infty} \log X \prod_{p\le X}\left(1-\frac{1}{p}\right) = e^{-\gamma}.$$

3. Exceptions to the conjecture of Erdős

We say that the Erdős conjecture is false modulo q, if there is an Erdősian function f for which $L(1, f) = 0$. The following proposition plays a fundamental role in our approach.

Proposition 3.1. *If the Erdős conjecture is false modulo* q *with* q *odd*, *then*

$$1 \le \sum_{\substack{d|q\\ d\ge 3}} \frac{1}{\varphi(d)}.$$

Proof. By the hypothesis, there is an Erdősian function $f \pmod q$ for which, we have $L(1, f) = 0$. Applying Okada's criterion, we get

$$\sum_{b\in M(q)} \frac{f(b)}{b} = 0. \tag{2}$$

Let $d = (b, q)$, so that $b = db_1$ with $(b_1, q/d) = 1$. Then (2) can be written as

$$-f(1) = \sum_{\substack{d|q\\ d\ge 3}} \frac{1}{d} \sum_{\substack{b_1\in M(q)\\ (b_1,q/d)=1}} \frac{f(db_1)}{b_1}.$$

Taking absolute value of both sides, we get

$$1 \le \sum_{\substack{d|q\\ d\ge 3}} \frac{1}{d} \sum_{b_1\in M(d)} \frac{1}{b_1}. \tag{3}$$

Notice that the inner sum can be written as

$$\sum_{b_1 \in M(d)} \frac{1}{b_1} = \prod_{p|d} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p|d} \left(1 - \frac{1}{p}\right)^{-1} = \frac{d}{\varphi(d)}.$$

Hence from (3), we get

$$1 \le \sum_{\substack{d|q \\ d \ge 3}} \frac{1}{\varphi(d)}. \qquad \square$$

Corollary 3.2. *If q is a prime power or a product of two distinct odd primes, then the Erdős conjecture is true modulo q.*

Proof. This is a pleasant elementary exercise. $\square$

Hence we have recovered the two basic cases of the conjecture which were given in the introduction, of course, also as a consequence of Okada's criterion.

Let $d(n)$ be the divisor function, that is, $d(n)$ is the number of divisors of n.

Corollary 3.3. *If the smallest prime factor of q is at least $d(q)$, then the Erdős conjecture is true for q.*

Proof. Let l be the smallest prime factor of q. From the above proposition, if the Erdős conjecture is false modulo q, then we have

$$\begin{aligned} 1 &\le \sum_{\substack{d|q \\ d \ge 3}} \frac{1}{\varphi(d)} \\ &< \frac{1}{\varphi(l)} \sum_{\substack{d|q \\ d \ge 3}} 1 = \frac{d(q) - 2}{l - 1}, \end{aligned}$$

the strict inequality in the penultimate step coming from the fact that q has at least two prime divisors. Thus, $l < d(q)$. Hence if $l \ge d(q)$, then the Erdős conjecture is true modulo q. $\square$

Note that, Corollary 3.3 was not known previously. It implies that the conjecture is true for any squarefree number q with k prime factors, provided the smallest prime factor of q is greater than 2^k. Proposition 3.1 opens the door for a new approach to the study of Erdős's conjecture. Let us consider the following:

$$S_1(X) = \left|\{q \equiv 1 \pmod 4, q \le X \mid \text{Erdős conjecture is false modulo } q\}\right|.$$

Then, we have

$$\begin{aligned}
S_1(X) &\le \sum_{\substack{q\le X\\ q\equiv 1 \pmod 4}} \sum_{\substack{d\mid q\\ d\ge 3}} \frac{1}{\varphi(d)} \le \sum_{\substack{3\le d\le X\\ d \text{ odd}}} \frac{1}{\varphi(d)} \sum_{\substack{q\le X\\ q\equiv 1 \pmod 4\\ d\mid q}} 1 \\
&\le \sum_{\substack{3\le d\le X\\ d \text{ odd}}} \frac{1}{\varphi(d)}\left(\frac{X}{4d}+O(1)\right) \le \sum_{\substack{3\le d\le X\\ d \text{ odd}}} \frac{1}{\varphi(d)}\frac{X}{4d} + O\biggl(\sum_{3\le d\le X}\frac{1}{\varphi(d)}\biggr) \\
&\le \sum_{\substack{3\le d\le X\\ d \text{ odd}}} \frac{1}{\varphi(d)}\frac{X}{4d} + O(\log X),
\end{aligned}$$

where we have used the well-known fact that (see, for example, [Murty 2008, page 67])

$$\sum_{d\le X}\frac{1}{\varphi(d)} = O(\log X).$$

Hence, we get

$$\begin{aligned}
S_1(X) &\lesssim \frac{X}{4}\sum_{\substack{3\le d\\ d \text{ odd}}} \frac{1}{d\varphi(d)} \\
&\lesssim \frac{X}{4}\biggl(\prod_{p \text{ odd}}\Bigl(1+\frac{1}{p\varphi(p)}+\frac{1}{p^2\varphi(p^2)}+\cdots\Bigr)-1\biggr) \\
&\lesssim \frac{X}{4}\biggl(\prod_{p \text{ odd}}\Bigl(1+\frac{1}{p(p-1)}+\frac{1}{p^3(p-1)}+\cdots\Bigr)-1\biggr) \\
&\lesssim \frac{X}{4}\biggl(\prod_{p \text{ odd}}\Bigl(1+\frac{1}{p(p-1)}\Bigl(1+\frac{1}{p^2}+\frac{1}{p^4}+\cdots\Bigr)\Bigr)-1\biggr) \\
&\lesssim \frac{X}{4}\biggl(\prod_{p \text{ odd}}\Bigl(1+\frac{p}{(p-1)(p^2-1)}\Bigr)-1\biggr).
\end{aligned}$$

The product is easily computed numerically and we have $S_1(X) \lesssim 0.33(X/4)$. The following is an immediate corollary.

Corollary 3.4. $|\{q \equiv 1 \pmod 4,\, q \le X \mid$ *Erdős conjecture is true for* $q\}| \gtrsim 0.67\frac{X}{4}$.

3A. *Refinement using the second moment.* By considering higher moments, we can improve the lower bound in the above corollary. We begin with the second moment. We include these estimates since they are of independent interest and self contained.

Proposition 3.5. $|\{q \equiv 1 \pmod 4,\, q \le X \mid$ *Erdős conjecture is true for* $q\}| \gtrsim 0.78\frac{X}{4}$.

Proof. Let us first consider the following inequality:

$$
\begin{aligned}
S_1(X) &\le \sum_{\substack{q\le X\\ q\equiv 1 \pmod 4}} \Bigg(\sum_{\substack{d|q\\ d\ge 3}} \frac{1}{\varphi(d)}\Bigg)^2\\
&\le \sum_{\substack{q\le X\\ q\equiv 1 \pmod 4}} \sum_{\substack{d_1|q, d_2|q\\ 3\le d_1, d_2<q}} \frac{1}{\varphi(d_1)\varphi(d_2)}\\
&\le \sum_{\substack{3\le d_1, d_2\le X\\ d_1, d_2 \text{ odd}}} \frac{1}{\varphi(d_1)\varphi(d_2)} \sum_{\substack{q\le X\\ q\equiv 1 \pmod 4\\ d_1|q, d_2|q}} 1\\
&\le \sum_{\substack{3\le d_1, d_2\le X\\ d_1, d_2 \text{ odd}}} \frac{1}{\varphi(d_1)\varphi(d_2)} \sum_{\substack{q\le X\\ q\equiv 1 \pmod 4\\ [d_1, d_2]|q}} 1\\
&\le \sum_{\substack{3\le d_1, d_2\le X\\ d_1, d_2 \text{ odd}}} \frac{1}{\varphi(d_1)\varphi(d_2)} \left(\frac{X}{4[d_1, d_2]} + O(1)\right).
\end{aligned}
$$

Hence, we have

$$
S_1(X) \le \frac{X}{4} \sum_{\substack{3\le d_1, d_2\le X\\ d_1, d_2 \text{ odd}}} \frac{1}{\varphi(d_1)\varphi(d_2)[d_1, d_2]} + O(\log^2 X).
$$

By a simple numerical calculation, we deduce that

$$
S_1(X) \lesssim 0.22\frac{X}{4}.
$$

Hence the conjecture holds for at least 78% of the positive integers congruent to $1 \pmod 4$. □

Similarly one can compute higher fractional moments to get an optimal result. For any $r > 1$, we have

$$
S_1(X) \le \sum_{\substack{q\le X\\ q\equiv 1 \pmod 4}} \Bigg(\sum_{\substack{d|q\\ d\ge 3}} \frac{1}{\varphi(d)}\Bigg)^r.
$$

We study this as a function of r. Using Maple we computed that the minimal value occurs at $r \sim 3.85$[1] and we get

$$
S_1(X) \lesssim 0.18\frac{X}{4}.
$$

[1]Code available at www.mast.queensu.ca/∼murty/maplecode.pdf.

Thus, we get $|\{q \equiv 1 \pmod 4, q \leq X \mid \text{Erdős conjecture is true for } q\}| \gtrsim 0.82\frac{X}{4}$, that is,

$$\liminf_{X\to\infty} \frac{S(X)}{X/4} \geq 0.82.$$

Hence, we have shown Theorem 1.1: the conjecture holds for at least 82% of the numbers congruent to $1 \pmod 4$.

3B. ***An alternative approach.*** In this subsection, we discuss an alternative approach to this problem. It leads to a slightly weaker result. However this method is of independent interest, so we record it here. We begin with a further refinement of Proposition 3.1 by considering fractional moments there. From Proposition 3.1, if the Erdős conjecture is false for odd q, then

$$1 \leq \sum_{\substack{d\mid q\\ d\geq 3}} \frac{1}{\varphi(d)}.$$

Adding 1 to both sides of the above inequality, we get

$$2 \leq \sum_{d\mid q} \frac{1}{\varphi(d)},$$

which can be rewritten as

$$1 \leq \frac{1}{2}\sum_{d\mid q} \frac{1}{\varphi(d)}.$$

Hence for any $\alpha > 0$, Proposition 3.1 can be rewritten as follows.

Proposition 3.6. *If Erdős conjecture is false for odd q, then*

$$1 \leq \frac{1}{2^\alpha}\left(\sum_{d\mid q} \frac{1}{\varphi(d)}\right)^\alpha.$$

As before, $S_1(X) = |\{q \equiv 1 \pmod 4, q \leq X \mid \text{Erdős conjecture is false for } q\}|$. Then from the above proposition, we get

$$S_1(X) \leq \frac{1}{2^\alpha} \sum_{\substack{q\leq X\\ q\equiv 1 \pmod 4}} \left(\sum_{d\mid q} \frac{1}{\varphi(d)}\right)^\alpha.$$

Let $f_\alpha(q) = \left(\sum_{d\mid q} 1/\varphi(d)\right)^\alpha$ and χ be the nontrivial Dirichlet character mod 4. Then the above inequality becomes

$$S_1(X) \leq \frac{1}{2^{\alpha+1}}\left(\sum_{\substack{q\leq X\\ q \text{ odd}}} f_\alpha(q) + \sum_{\substack{q\leq X\\ q \text{ odd}}} \chi(q) f_\alpha(q)\right). \tag{4}$$

Again, note that $f_\alpha(q)$ is a multiplicative arithmetic function. One can check that it also satisfies all the other hypotheses of Wirsing's theorem (Proposition 2.2) with $G = 2^\alpha$ and $\tau = 1$. So in light of Wirsing's theorem, we get

$$\sum_{\substack{q \le X \\ q \text{ odd}}} f_\alpha(q) \sim X \frac{e^{-\gamma}}{\log X} \prod_{\substack{p \le X \\ p \ne 2}} \left(1 + \frac{f_\alpha(p)}{p} + \frac{f_\alpha(p^2)}{p^2} + \cdots\right)$$

and

$$\sum_{\substack{q \le X \\ q \text{ odd}}} \chi(q) f_\alpha(q) \sim X \frac{e^{-\gamma}}{\log X} \prod_{\substack{p \le X \\ p \ne 2}} \left(1 + \frac{\chi(p) f_\alpha(p)}{p} + \frac{\chi(p^2) f_\alpha(p^2)}{p^2} + \cdots\right).$$

Again, from Mertens theorem we know that

$$\prod_{p \le X} (1 - 1/p) \sim \frac{e^{-\gamma}}{\log X}.$$

Hence we have

$$\begin{aligned} \sum_{\substack{q \le X \\ q \text{ odd}}} f_\alpha(q) &\sim \frac{X}{2} \prod_{\substack{p \le X \\ p \ne 2}} (1 - 1/p) \left(1 + \frac{f_\alpha(p)}{p} + \frac{f_\alpha(p^2)}{p^2} + \cdots\right) \\ &\sim \frac{X}{2} p_1 \text{ (say)} \end{aligned}$$

and

$$\begin{aligned} \sum_{\substack{q \le X \\ q \text{ odd}}} \chi(q) f_\alpha(q) &\sim \frac{X}{2} \prod_{\substack{p \le X \\ p \ne 2}} (1 - 1/p) \left(1 + \frac{\chi(p) f_\alpha(p)}{p} + \frac{\chi(p^2) f_\alpha(p^2)}{p^2} + \cdots\right) \\ &\sim \frac{X}{2} p_2 \text{ (say).} \end{aligned}$$

Now using the above two inequalities, (4) becomes

$$S_1(X) \lesssim \frac{X}{2^{\alpha+2}} (p_1 + p_2).$$

Finally, using Maple[2] we find that the quantity on the right hand side is minimized at $\alpha \sim 8.11$ and we get

$$S_1(X) \lesssim 0.20 \frac{X}{4}.$$

[2]Code available at www.mast.queensu.ca/∼murty/maplecode.pdf.

Hence, we get

$$\liminf_{X\to\infty} \frac{S(X)}{X/4} \geq 0.80.$$

Remarks. One cannot hope to obtain 100% by these methods. In fact, one can show that there is a positive density (albeit small) of q for which the inequality of Proposition 3.1 holds. Indeed, since

$$\sum_{d|q} \frac{1}{\varphi(d)} \geq \prod_{p|q}\left(1+\frac{1}{p-1}\right)$$

we can make the product (and hence the sum) arbitrarily large by ensuring that q is divisible by all the primes in an initial segment. We can even ensure that these primes are congruent to $1 \pmod 4$. We then take numbers which are divisible by this q and congruent to $1 \pmod 4$ and deduce that for all these numbers, the inequality in the proposition holds. Since the product on the right diverges slowly to infinity as we go through such numbers q, we obtain in this way a small density of numbers for which the inequality holds.

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References

[Baker et al. 1973] A. Baker, B. J. Birch, and E. A. Wirsing, "On a problem of Chowla", *J. Number Theory* **5** (1973), 224–236. MR 49 #4959 Zbl 0267.10065

[Chatterjee and Murty 2014] T. Chatterjee and M. R. Murty, "Non-vanishing of Dirichlet series with periodic coefficients", *J. Number Theory* **145** (2014), 1–21. MR 3253290

[Gun et al. 2012] S. Gun, M. R. Murty, and P. Rath, "Linear independence of Hurwitz zeta values and a theorem of Baker-Birch-Wirsing over number fields", *Acta Arith.* **155**:3 (2012), 297–309. MR 2983455

[Livingston 1965] A. E. Livingston, "The series $\sum_1^\infty f(n)/n$ for periodic f", *Canad. Math. Bull.* **8** (1965), 413–432. MR 32 #4104 Zbl 0129.02801

[Murty 2008] M. R. Murty, *Problems in analytic number theory*, Second ed., Graduate Texts in Mathematics **206**, Springer, New York, 2008. Readings in Mathematics. MR 2008j:11001 Zbl 1190.11001

[Murty and Saradha 2007] M. R. Murty and N. Saradha, "Transcendental values of the digamma function", *J. Number Theory* **125**:2 (2007), 298–318. MR 2008g:11123 Zbl 1222.11097

[Murty and Saradha 2010] M. R. Murty and N. Saradha, "Euler-Lehmer constants and a conjecture of Erdös", *J. Number Theory* **130**:12 (2010), 2671–2682. MR 2011h:11078

[Okada 1982] T. Okada, "On a certain infinite series for a periodic arithmetical function", *Acta Arith.* **40**:2 (1982), 143–153. MR 83h:10099 Zbl 0402.10035

[Okada 1986] T. Okada, "Dirichlet series with periodic algebraic coefficients", *J. London Math. Soc.* (2) **33**:1 (1986), 13–21. MR 87i:11087 Zbl 0589.10034

[Saradha and Tijdeman 2003] N. Saradha and R. Tijdeman, "On the transcendence of infinite sums of values of rational functions", *J. London Math. Soc.* (2) **67**:3 (2003), 580–592. MR 2004b:11103 Zbl 1045.11051

[Tijdeman 2002] R. Tijdeman, "Some applications of Diophantine approximation", pp. 261–284 in *Number theory for the millennium, III (Urbana, IL, 2000)*, A K Peters, Natick, MA, 2002. MR 2003j:11076 Zbl 1045.11022

[Wirsing 1961] E. Wirsing, "Das asymptotische Verhalten von Summen über multiplikative Funktionen", *Math. Ann.* **143** (1961), 75–102. MR 24 #A1241 Zbl 0104.04201

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