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We prove that any Riemannian two-sphere having area at most 1 can be continuously mapped onto a tree in such a way that the topology of the fibers is controlled and their length is less than 7.6. This result improves previous estimates and relies on a similar statement for Riemannian two-disks.

1. Introduction

In this article we are interested to describe the possible geometries of Riemannian two-disks and two-spheres in the same way a tailor determines the geometry of a body: by taking some relevant measurements. We denote by $A(\cdot)$ the area functional and $|\cdot|$ the length functional. Our main result deals with measurements of two-disks:

Theorem 1.1. *If D is a Riemannian two-disk, then for any $\epsilon > 0$ we can find a continuous map to a trivalent tree such that the preimage of a terminal vertex is either an interior point or the boundary ∂D, the preimage of an interior point of an edge is homeomorphic to a circle, the preimage of a trivalent vertex is homeomorphic to the θ figure, and fibers have length at most*

$$(1+\epsilon)\max\bigl\{|\partial D| + \sqrt{A(D)},\ (4 + 11\sqrt{3}/4)\sqrt{A(D)}\bigr\}.$$

This theorem should be compared to a result by Y. Liokumovich [2014] which states that any Riemannian two-disk D admits a Morse function $f : D \to \mathbb{R}$ which is constant on the boundary and whose fibers have length at most $52\sqrt{A(D)} + |\partial D|$.

Using Theorem 1.1, we are able to estimate the measurements of two-spheres in terms of their area.

Theorem 1.2. *If M is a Riemannian two-sphere with area less than 1, then it admits a continuous map to a trivalent tree such that the preimage of a terminal vertex is a point, the preimage of an interior point of an edge is homeomorphic to a circle, the preimage of a trivalent vertex is homeomorphic to the θ figure and fibers have length at most $2\sqrt{3} + 33/8 \simeq 7.6$.*

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This improves a previous estimate by Liokumovich [2014] proving such a result with $8\sqrt{3}+12 \simeq 26$ as upper bound on the length of the fibers. Note nevertheless that the main result of Liokumovich is stronger: he proved the existence of a Morse function $f : M \to \mathbb{R}$ whose fibers have length at most 52. Also note that L. Guth [2005] proved the existence of maps such as in Theorem 1.2 with the upper bound $120/(2\sqrt{\pi}) \simeq 34$ on the length of the fibers under the weaker assumption that the 1-hypersphericity is less than $1/(2\sqrt{\pi})$. Finally we point out that the constant 7.6 in Theorem 1.2 is within a factor at most 6 from the optimal one; see Remark 2.8.

The interest in obtaining precise measurements for Riemannian two-spheres is illustrated by the fact that we can derive upper bounds on the shortest length of a closed geodesic, on the shortest length of a simple loop dividing the sphere into two subdisks of area at least $A/3$, and on the maximal length of a shortest pants decomposition for punctured spheres. More precisely, we are first able to recover the result of C. Croke [1988] on the existence of short closed geodesics for Riemannian two-spheres: we will deduce from Theorem 1.2 that any Riemannian two-sphere with unit area carries a closed geodesic of length at most $\simeq 10.1$; see Theorem 2.7. This is not as good as the current best constant, due to R. Rotman [2006] and equal to $4\sqrt{2} \simeq 5.7$, but it is not too far from it. Moreover, using Theorem 1.2, we can also recover Theorem VI of [Alvarez Paiva et al. 2013] on the existence of a short closed geodesic for Finsler (eventually nonreversible) two-spheres. The precise statement is the following.

Theorem 1.3. *Let M be a Finsler (eventually nonreversible) two-sphere with Holmes–Thompson area less than* 1. *Then it carries a closed geodesic of length at most* $2\sqrt{\pi}\,(11\sqrt{3}+16) \simeq 31.1$.

This improves the current best constant, due to Liokumovich [2014, Theorem 4]. We also easily deduce from Theorem 1.2 the following result, which also improves one of Liokumovich [2014, Theorem 1].

Theorem 1.4. *Let M be a Riemannian two-sphere. Then there exists a simple loop of length at most* $(2\sqrt{3}+33/8)\sqrt{A(M)}$ *dividing M into two subdisks of area at least* $A(M)/3$.

Finally, it is straightforward to see that Theorem 1.2 implies the following.

Theorem 1.5. *Let M be a Riemannian punctured two-sphere with area less than* 1. *Then there exists a decomposition of M into* 3*-holed spheres such that each boundary curve has length at most* $2\sqrt{3}+33/8 \simeq 7.6$.

This improves the current best bound, even for hyperbolic metrics; compare with [Balacheff and Parlier 2012].

The paper is organized as follows. The first section presents Besicovich's lemma and some of its useful corollaries: Papasoglu's lemma [2009] and the disk subdivision lemma of Liokumovich, A. Nabutovsky and Rotman [Liokumovich et al. 2014]. We also define an invariant called the θ-width, reformulate Theorem 1.1 in terms of this invariant, and show how to prove Theorems 1.2 and 1.3 from Theorem 1.1. In the second section, we prove Theorem 1.1. Our strategy is inspired by the proof of [Liokumovich et al. 2014, Theorem 1.6] where it is shown that the boundary of Riemannian two-disks with uniformly bounded diameter and area can always be contracted through closed curves of bounded length. We show that it is enough to consider the case where the length of the boundary is short in comparison with the area. This step is performed using Besicovich's lemma. Then we use the disk subdivision lemma to argue by induction on the area.

2. Preliminaries

As we deal only with surfaces, we will use the terms of disk and sphere for two-disk and two-sphere. We denote by $A(\,\cdot\,)$ the area functional and $|\cdot|$ the length functional.

Besicovich's lemma and consequences. In order to prove our results, we will use the following fundamental result of metric geometry as well as some of its consequences.

Lemma 2.1 [Besicovitch 1952]. *Let $\mathcal{S}$ be a Riemannian square. Then there exists a simple geodesic path connecting two opposites sides of length at most $\sqrt{A(\mathcal{S})}$.*

In particular, any Riemannian disk D whose boundary satisfies $|\partial D| > 4\sqrt{A(D)}$ can be subdivided into two subdisks of smaller perimeters (divide its boundary into four equal parts and apply Besicovich's lemma).

P. Papasoglu used Besicovich's lemma to derive the following estimate.

Lemma 2.2 [Papasoglu 2009]. *Let M be a Riemannian two-sphere. For any $\delta > 0$ there exists a simple closed curve of length at most $2\sqrt{3}\sqrt{A(M)}+\delta$ and subdividing M into two disks of area at least $A(M)/4$.*

Liokumovich, Nabutovsky and Rotman [2014, Proposition 3.2] apply Papasoglu's result to cut Riemannian disks into two parts of sufficiently big area by a curve of controlled length. We reformulate their result as follows.

Lemma 2.3 (disk subdivision). *Let D be a Riemannian two-disk. For any $\lambda < 1/4$ and $\delta > 0$ there exists a subdisk $D' \subset D$ such that $\lambda A(D) \leq A(D') \leq (1-\lambda)A(D)$ and $|\partial D' \setminus \partial D| \leq 2\sqrt{3}\sqrt{A(D)}+\delta$.*

Technical width. We now introduce our main tool, the θ-width, reformulate Theorem 1.1 in terms of this invariant, and show how to derive Theorems 1.2 and 1.3.

Definition 2.4 (θ-width). Let M be a compact Riemannian surface (possibly with nonempty boundary). We define the θ-width, denoted by $W_\theta(M)$, as the infimum of the $L > 0$ such that there exists a continuous map f from M to a trivalent tree T satisfying the following conditions:

(W1) $f(\partial M) \subset \partial T$ and the preimage of a terminal vertex is either an interior point or a connected component of ∂M.

(W2) The preimage of an interior point of an edge is homeomorphic to a circle.

(W3) The preimage of a trivalent vertex is homeomorphic to the letter θ.

(W4) The preimage of any point has length at most L.

Observe that in particular $W_\theta(M) \geq |\partial M|$.

Our results are consequences of the following estimate.

Theorem 2.5. *Let D be a Riemannian two-disk. Then*

$$W_\theta(D) \leq \max\bigl\{|\partial D| + \sqrt{A(D)},\ (4 + 11\sqrt{3}/4)\sqrt{A(D)}\bigr\}.$$

We will prove this theorem in Section 3. It is straightforward to check that it implies Theorem 1.1. Observe that it also implies the following statement, of which Theorem 1.2 is a direct consequence.

Corollary 2.6. *Let M be a Riemannian two-sphere. Then*

$$W_\theta(M) \leq (2\sqrt{3} + 33/8)\sqrt{A(M)}.$$

Proof of Corollary 2.6. Let M be a Riemannian two-sphere. First divide M into two disks D_1 and D_2 of area at least $A(M)/4$ by a simple closed curve of length at most $3\sqrt{3}\sqrt{A(M)}$ by choosing $\delta = \sqrt{3}\sqrt{A(M)}$ in Papasoglu's result (Lemma 2.2). Observe that choosing a better constant than $3\sqrt{3}$ does not lead to any improvement in our final estimate. Now for each subdisk we have the following bound according to Theorem 2.5:

$$\begin{aligned} W_\theta(D_i) &\leq \max\bigl\{|\partial D_i| + \sqrt{A(D_i)},\ (4 + 11\sqrt{3}/4)\sqrt{A(D_i)}\bigr\} \\ &\leq \bigl(2\sqrt{3} + 33/8\bigr)\sqrt{A(M)} \end{aligned}$$

as $A(D_i) \leq (3/4)A(M)$ for $i = 1, 2$ and $|\partial D_1| = |\partial D_2| \leq 3\sqrt{3}\sqrt{A(M)}$. It is straightforward to check that

$$W_\theta(M) \leq \max\bigl\{W_\theta(D_1),\ W_\theta(D_2)\bigr\} \leq (2\sqrt{3} + 33/8)\sqrt{A(M)}. \qquad \square$$

Existence of short closed geodesics. It is classic to derive for spheres the existence of short closed geodesics from bounds on the θ-width. In particular:

Theorem 2.7. (1) *A Riemannian two-sphere with area* 1 *carries a closed geodesic of length at most* $8/\sqrt{3}+11/2 \simeq 10.1$.

(2) *A Finsler reversible two-sphere with Holmes–Thompson area* 1 *carries a closed geodesic of length at most* $\sqrt{\pi/2}\,(8/\sqrt{3}+11/2) \simeq 12.7$.

(3) *A Finsler possibly nonreversible two-sphere with Holmes–Thompson area* 1 *carries a closed geodesic of length at most* $\sqrt{3\pi}\,(8/\sqrt{3}+11/2) \simeq 31.1$.

Proof. It follows from [Alvarez Paiva et al. 2013, Section 4.4] that:

- If any Riemannian sphere M with unit area satisfies $W_\theta(M) \le C$, then any reversible Finsler sphere M' with unit Holmes–Thompson area satisfies $W_\theta(M') \le \sqrt{\pi/2}\,C$.
- If any reversible Finsler sphere M with unit Holmes–Thompson area satisfies $W_\theta(M) \le C$, then any Finsler sphere M'' with unit Holmes–Thompson area satisfies $W_\theta(M') \le \sqrt{6}\,C$.

Now fix a Finsler sphere M. We denote by $\mathrm{sys}(M)$ the systole of M, defined as the length of a shortest closed geodesic. By Corollary 2.6 it remains to prove that

$$\mathrm{sys}(M) \le \tfrac{4}{3}\, W_\theta(M). \tag{2-1}$$

The existence of a closed geodesic on M can be proved through a minimax argument on the one-cycle space $\mathscr{Z}_1(M;\mathbb{Z})$. We refer the reader to [Balacheff and Sabourau 2010] and the references therein for additional information. Loosely speaking, this space arising from geometric measure theory is made of multiple curves (unions of oriented loops) endowed with some special topology. This space allows us to define a minimax process on the Finsler sphere M using F. Almgren's isomorphism between the relative fundamental group $\pi_1(\mathscr{Z}_1(M;\mathbb{Z}),\{0\})$ and the second homology group $H_2(M;\mathbb{Z}) \simeq \mathbb{Z}$. From a result of J. Pitts, the minimax quantity

$$\inf_{(z_t)} \sup_{0\le t\le 1} |z_t|,$$

where (z_t) runs over the families of one-cycles inducing a nontrivial element of $\pi_1(\mathscr{Z}_1(M;\mathbb{Z}),\{0\})$, bounds from above the systole.

We argue by contradiction. Suppose that $\mathrm{sys}(M) > \frac{4}{3}\, W_\theta(M)$. Fix $\epsilon > 0$ such that $\mathrm{sys}(M) > \frac{4}{3}\, W_\theta(M)+\epsilon$. By definition there exists a continuous map f from M to a trivalent tree T satisfying (W1)–(W4) with length $L = W_\theta(M)+\epsilon$.

Let v be a trivalent vertex. Its preimage, denoted by $\theta(v)$, is made of three disjoint oriented arcs α_1, α_2, and α_3 with the same endpoints, ordered such that $|\alpha_1| \le |\alpha_2| \le |\alpha_3|$. Denote by β_{ij} for $1 \le i < j \le 3$ the concatenation of the oriented arcs α_i with $-\alpha_j$. As $\operatorname{sys}(M) > |\beta_{ij}|$ we can continuously contract each of the β_{ij} to a point curve through a length decreasing homotopy by using a Birkhoff process; see [Croke 1988, pp. 4–5]. We denote by $\{\beta_{ij}^t\}_{t\in[0,1]}$ this homotopy with the convention that $\beta_{ij}^0 = \beta_{ij}$. We define an element of $\pi_1(\mathcal{Z}_1(M;\mathbb{Z}), \{0\})$ by

$$f_v(t) = \begin{cases} -\beta_{12}^{1-2t} + \beta_{13}^{1-2t} & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \beta_{23}^{2t-1} & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

This gives rise to an element $[f_v] \in H_2(M, \mathbb{Z})$ such that $|f_v(t)| \leq \frac{4}{3} W_\theta(M) + \epsilon$ for any $t \in [0, 1]$.

Now fix an edge $e = [v_0, v_1] \simeq [0, 1]$ which is not terminal. We denote by α_t the preimage of an interior point of e corresponding to the parameter $t \in]0, 1[$ and orient it in a coherent way. For $i = 0, 1$, denote by α_i the oriented curve obtained as the limit of the curves α_t when $t \to i$. The curve α_i is a simple closed curve contained in $\theta(v_i)$. As before we can contract α_i to a point through a homotopy $\{\alpha_i^t\}_{t\in[0,1]}$. We define an element of $\pi_1(\mathcal{Z}_1(M;\mathbb{Z}), \{0\})$ by

$$f_e(t) = \begin{cases} \alpha_0^{1-3t} & \text{for } t \in \left[0, \frac{1}{3}\right], \\ \alpha_{3t-1} & \text{for } t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ \alpha_1^{3t-2} & \text{for } t \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

This gives rise to an element $[f_e] \in H_2(M, \mathbb{Z})$ such that $|f_e(t)| \leq W_\theta(M) + \epsilon$ for any $t \in [0, 1]$.

Finally, fix a terminal edge $e = [v_0, v_1] \simeq [0, 1]$, with the terminal vertex corresponding to 0. With the same notation as above, the curve α_0 is reduced to a point curve. We define an element of $\pi_1(\mathcal{Z}_1(M;\mathbb{Z}), \{0\})$ by

$$f_e(t) = \begin{cases} \alpha_{2t} & \text{for } t \in [0, \frac{1}{2}], \\ \alpha_1^{2t-1} & \text{for } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

This gives rise to an element $[f_e] \in H_2(M, \mathbb{Z})$ such that $|f_e(t)| \leq W_\theta(M) + \epsilon$ for any $t \in [0, 1]$.

It is straightforward to see (compare with [Balacheff 2003–2004, Section 1.3]):

$$[S^2] = \sum_{e\in E(T)} \varepsilon_e \cdot [f_e] + \sum_{v\in V(T)} \varepsilon_v \cdot [f_v]$$

for some choice of coefficients ε_v and ε_e in $\{-1, 1\}$. Here $E(T)$ and $V(T)$ denote the set of edges and the set of vertices of T, respectively. This implies that there exists an edge e such that $[f_e] \neq 0$ or a vertex v such that $[f_v] \neq 0$. According to the minimax principle on the one-cycle space, we conclude that

$$\operatorname{sys}(M) \leq \tfrac{4}{3} W_\theta(M) + \epsilon,$$

which is a contradiction. $\square$

Remark 2.8. Using the estimate (2-1), we observe that the flat metric with three conical singularities of angle $2\pi/3$ on the two-sphere obtained by gluing two flat equilateral triangles of side 1 along their boundary satisfies

$$\frac{W_\theta}{\sqrt{A}} \geq \tfrac{3}{4} \cdot 2^{\frac{1}{2}} 3^{\frac{1}{4}} \geq 1.39.$$

This proves that the constant in Theorem 1.2 is within a factor at most 6 from the optimal one.

3. The θ-width of a Riemannian disk

In this section we prove Theorem 2.5. For this we adapt the strategy of the proof of [Liokumovich et al. 2014, Theorem 1.6] to control our invariant W_θ.

Reduction to the short boundary case.

Lemma 3.1. *Let D be a Riemannian two-disk and $C \geq 0$. Suppose that there exists $\eta > 0$ such that for any subdisk $D' \subset D$ for which*

$$|\partial D'| < (4+\eta)\sqrt{A(D')},$$

we have

$$W_\theta(D') \leq (1+\eta)\ \max\bigl\{|\partial D'| + \sqrt{A(D')},\ C\sqrt{A(D')}\bigr\}.$$

Then, for any subdisk $D' \subset D$,

$$W_\theta(D') \leq (1+\eta)\ \max\bigl\{|\partial D'| + \sqrt{A(D')},\ C\sqrt{A(D')}\bigr\}.$$

In the sequel, we will use this lemma with constant $C = 0$ (small area case) and

$$C = C_{\lambda,\eta} := 4 + 2\eta + 2\sqrt{3} + \frac{1-\lambda}{\sqrt{3}(1-2\eta)} + \sqrt{1-\lambda}$$

for $0 < \lambda < \frac{1}{4}$ and $\eta > 0$ (general case).

Proof. For any subdisk $D' \subset D$ we define $n(D')$ to be the smallest integer n such that

$$|\partial D'| < \bigl(4 + \eta\left(\tfrac{4}{3}\right)^n\bigr)\sqrt{A(D')}.$$

Let D' be a subdisk such that $n(D') = 0$. Equivalently, we have that $|\partial D'| < (4+\eta)\sqrt{A(D')}$, and so we are done by assumption.

Now fix an integer n and suppose that for any subdisk $D' \subset D$ such that $n(D') \leq n-1$, we have proven that

$$W_\theta(D') \leq (1+\eta)\ \max\bigl\{|\partial D'| + \sqrt{A(D')},\ C\sqrt{A(D')}\bigr\}.$$

Let $D' \subset D$ be a subdisk with $n(D') = n$. In particular we have $|\partial D'| > 4\sqrt{A(D')}$, so we can subdivide D' into two subdisks D'_1 and D'_2 of smaller perimeters using a Besicovich cut α of length $\sqrt{A(D')}$ (Lemma 2.1). More precisely,

$$\begin{aligned}|\partial D_i'| &\le \tfrac{3}{4}|\partial D'| + \sqrt{A(D')}\\ &< \tfrac{3}{4}\left(\eta\left(\tfrac{4}{3}\right)^n + 4\right)\sqrt{A(D')} + \sqrt{A(D')}\\ &< \left(\eta\left(\tfrac{4}{3}\right)^{n-1} + 4\right)\sqrt{A(D')}\end{aligned}$$

so $n(D_i') \le n-1$.

Let $\epsilon > 0$ be small enough so that all points of D' at a distance at least ϵ from $\partial D'$ form a subdisk denoted by $D'' \subset D'$. The subdisk D'' is itself subdivided by the Besicovich's cut α into two subdisks $D_i'' \subset D_i'$ for $i = 1, 2$. By considering ϵ smaller if necessary, we can suppose that $\partial D_i''$ is sufficiently close to $\partial D_i'$ so that $|\partial D_i''| < |\partial D'|$ and $n(D_i'') \le n-1$. In particular,

$$W_\theta(D_i'') \le (1+\eta)\max\left\{|\partial D_i''| + \sqrt{A(D_i'')},\ C\sqrt{A(D_i'')}\right\}$$

for $i = 1, 2$ by the induction assumption, which implies that

$$W_\theta(D_i'') \le (1+\eta)\max\left\{|\partial D'| + \sqrt{A(D')},\ C\sqrt{A(D')}\right\}.$$

Claim 3.2. $W_\theta(D') \le \max\left\{|\partial D'| + \sqrt{A(D')} + o(\epsilon),\ W_\theta(D_1''),\ W_\theta(D_2'')\right\}$.

Proof of Claim 3.2. Indeed for any $\delta > 0$ and $i = 1, 2$, let $f_i : D_i'' \to T_i$ be a continuous map to a trivalent tree T_i satisfying conditions (W1)–(W4) with length strictly less than $W_\theta(D_i'') + \delta$. Denote by v_i the terminal vertex of T_i corresponding to the boundary $\partial D_i''$. Consider a new edge $e \simeq [0, 1]$ and define a new trivalent tree T obtained from T_1, T_2, and e by identifying v_1, v_2, and the vertex of e corresponding to $\{1\}$ into the same vertex denoted by v. The trees T_1 and T_2 can be thought as subgraphs of T. Denote by $\{\gamma_t\}_{t\in[0,1]}$ a monotone isotopy from $\partial D'$ to $\partial D''$ formed by level sets of the distance function to $\partial D'$. It satisfies $|\gamma_t| \le |\partial D'| + o(\epsilon)$.

We define a new map $f : D' \to T$ as follows:

$$f(x) = \begin{cases} f_i(x) & \text{if } x \in D_i'' \setminus \partial D_i'', \\ v & \text{if } x \in \partial D_1'' \cup \partial D_2'', \\ t & \text{if } x \in \gamma_t \text{ for } t \in [0, 1]. \end{cases}$$

By construction we have that the length of the preimages is always strictly less than

$$\max\left\{|\partial D'| + \sqrt{A(D')} + o(\epsilon),\ W_\theta(D_1'') + \delta,\ W_\theta(D_2'') + \delta\right\}.$$

It is easy to check that $f : D' \to T$ satisfies conditions (W1)–(W3), which yields the claim if we let $\delta \to 0$. □

Now Claim 3.2 implies

$$W_\theta(D') \le (1+\eta)\max\left\{|\partial D'| + \sqrt{A(D')},\ C\sqrt{A(D')}\right\}$$

by letting $\epsilon \to 0$, and we are done by induction. □

The small area case.

Lemma 3.3. *Let D be a Riemannian two-disk and $\eta > 0$. There exists $\epsilon > 0$ such that any subdisk $D' \subset D$ with $A(D') \leq \epsilon$ satisfies*

$$W_\theta(D') \leq (1+\eta)\left(|\partial D'| + \sqrt{A(D')}\right).$$

Proof. According to Lemma 3.1 with $C = 0$, it is enough to prove the lemma for subdisks D' with

$$|\partial D'| < (4+\eta)\sqrt{\epsilon}.$$

As observed in the proof of [Liokumovich et al. 2014, Lemma 2.3], for r small enough, every ball of radius r is $(1 + O(r))$-bilipschitz homeomorphic to a convex subset of $\mathbb{R}^2$. Hence for ϵ small enough the condition $|\partial D'| < (4+\eta)\sqrt{\epsilon}$ ensures that D' is $(1+\eta)$-bilipschitz to a subset $U \subset \mathbb{R}^2$ with analytic boundary. It is easy to continuously contract the boundary of U into a point through a continuous one-parameter family of closed multicurves — that is, the union of a finite number of closed curves — of U with decreasing length. For this, consider a supporting line ℓ of U. We linearly translate this line in the inner orthogonal direction until we sweep out U and denote by $\{\ell_t\}_{t\in[0,1]}$ this family of translated lines (with the convention that $\ell_0 = \ell$). For each $t \in [0, 1]$ the intersection $\ell_t \cap U$ consists of a finite number of disjoint segments. By transversality we can assume that this number of disjoint segments changes at each step by at most 1, and because the boundary is analytic the number of such steps is finite.

Consider the family of closed multicurves γ_t defined as the boundary of the union $\bigcup_{s\in[t,1]} U \cap \ell_t$. This is a continuous one-parameter family of closed multicurves of U with decreasing length that contracts ∂U to a point. The multicurves involved in this family are not disjoint, but it can be done by slightly perturbing the family in the neighborhood of ∂U without significantly increasing their length. Finally, it is classic to derive from this family a map $f : U \to T$ with T a trivalent tree and satisfying conditions (W1)–(W4), with L as close as wanted from $|\partial U|$; compare with [Gromov 1983, p. 128]. In particular $W_\theta(U) \leq |\partial U|$ which in turn implies that $W_\theta(D') \leq (1+\eta)|\partial D'|$. □

The general case.

Let D be a Riemannian disk. Fix $\eta > 0$ and $0 < \lambda < \frac{1}{4}$ and define

$$C_{\lambda,\eta} = 4 + 2\eta + 2\sqrt{3} + \frac{1-\lambda}{\sqrt{3}(1-2\eta)} + \sqrt{1-\lambda}.$$

We will argue by induction and prove that for any subdisk $D' \subset D$,

$$W_\theta(D') \leq (1+\eta)\max\left\{|\partial D'| + \sqrt{A(D')},\ C_{\lambda,\eta}\sqrt{A(D')}\right\}.$$

This implies the conclusion of Theorem 2.5 by letting $\eta \to 0$ and $\lambda \to \frac{1}{4}$. According to Lemma 3.1 with $C = C_{\lambda,\eta}$, it is enough to estimate the θ-width of D' under the stronger assumption that $|\partial D'| < (4+\eta)\sqrt{A(D')}$.

Let $\epsilon > 0$ such that the conclusion of Lemma 3.3 holds. For any subdisk $D' \subset D$ we define $m(D')$ to be the smallest integer m such that

$$A(D') \leq \epsilon \left(\frac{1}{1-\lambda}\right)^m.$$

Let D' be a subdisk such that $m(D') = 0$. Equivalently, $A(D') \leq \epsilon$ and we are done according to Lemma 3.3.

Now fix a positive integer m and suppose that for any subdisk $D' \subset D$ with $m(D') \leq m-1$ we have proven that

$$W_\theta(D') \leq (1+\eta)\,\max\bigl\{|\partial D'| + \sqrt{A(D')},\ C_{\lambda,\eta}\sqrt{A(D')}\bigr\}.$$

Let $D' \subset D$ be a subdisk with $m(D') = m$.

By Lemma 2.3 there exists a subdisk $D'_0 \subset D'$ such that

$$\lambda A(D') \leq A(D'_0) \leq (1-\lambda)A(D') \quad \text{and} \quad |\partial D'_0 \setminus \partial D'| \leq (2\sqrt{3}+\eta)\sqrt{A(D')}.$$

First case. If $\partial D'_0 \cap \partial D' \neq \varnothing$, then D' decomposes into an union of subdisks $D'_0, \dots, D'_k$ with disjoint interiors such that $A(D'_i) \leq (1-\lambda)A(D')$ for $i = 0, \dots, k$. In particular for each i,

$$W_\theta(D'_i) \leq (1+\eta)\,\max\bigl\{|\partial D'_i| + \sqrt{A(D'_i)},\ C_{\lambda,\eta}\sqrt{A(D'_i)}\bigr\}$$

as $m(D'_i) \leq m-1$, and the inductive assumption applies.

Using a similar argument to that of Claim 3.2, it is straightforward to check that

$$W_\theta(D') \leq (1+\eta)\,\max\bigl\{|\partial D'| + |\partial D'_0 \setminus \partial D'| + \sqrt{1-\lambda}\sqrt{A(D')},\ C_{\lambda,\eta}\sqrt{1-\lambda}\sqrt{A(D')}\bigr\}$$

as $|\partial D'_i| \leq |\partial D'| + |\partial D'_0 \setminus \partial D'|$ and $A(D'_i) \leq (1-\lambda)A(D')$ for $i = 0, \dots, k$.

Combined with the fact that $|\partial D'| < (4+\eta)\sqrt{A(D')}$, this implies that

$$\begin{aligned} W_\theta(D') &\leq (1+\eta)\,\max\bigl\{(4+2\eta+2\sqrt{3}+\sqrt{1-\lambda})\sqrt{A(D')},\ C_{\lambda,\eta}\sqrt{A(D')}\bigr\} \\ &\leq (1+\eta)\,\max\bigl\{|\partial D'| + \sqrt{A(D')},\ C_{\lambda,\eta}\sqrt{A(D')}\bigr\}, \end{aligned}$$

as claimed.

Second case. If $\partial D'_0 \cap \partial D' = \varnothing$, then D' decomposes into the union of the disk D'_0 and an annulus $\mathscr{A}$. Recall that

$$\begin{aligned} |\partial D'| &< (4+\eta)\sqrt{A(D')}, & A(\mathscr{A}) &\leq (1-\lambda)A(D'), \\ |\partial D'_0| &\leq (2\sqrt{3}+\eta)\sqrt{A(D')}, & A(D'_0) &\leq (1-\lambda)A(D'). \end{aligned}$$

Thus $m(D'_0) \leq m-1$, so that, by the inductive assumption,

$$W_\theta(D'_0) \leq \max\bigl\{|\partial D'_0| + \sqrt{A(D'_0)},\ C_{\lambda,\eta}\sqrt{A(D')}\bigr\} \leq C_{\lambda,\eta}\sqrt{A(D')}.$$

Denote by $h(\mathscr{A})$ the height of the annulus, that is, the distance between its two boundary curves. We say that $\mathscr{A}$ *decomposes into a stack of annuli* if there exist a finite number of annuli $\mathscr{A}_1, \ldots, \mathscr{A}_k$ with disjoint interiors such that $\mathscr{A} = \bigcup_{i=1}^k \mathscr{A}_i$ and $\mathscr{A}_i \cap \mathscr{A}_{i+1} = \beta_i$ is a common boundary simple closed curve for $i = 1, \ldots, k-1$. The following lemma will help us to estimate the θ-width of D'.

Lemma 3.4. *The Riemannian annulus $\mathscr{A}$ decomposes into a stack of annuli $\mathscr{A}_1$, $\mathscr{A}_2, \ldots, \mathscr{A}_k$ such that*

$$h(\mathscr{A}_i) \leq \frac{\sqrt{1-\lambda}}{2\sqrt{3}(1-2\eta)}\sqrt{A(\mathscr{A})} \quad \textit{for } i = 1, \ldots, k,$$

$$|\beta_i| \leq \frac{2\sqrt{3}+\eta}{\sqrt{1-\lambda}}\sqrt{A(\mathscr{A})} \qquad \textit{for } i = 1, \ldots, k-1.$$

Proof. Suppose that

$$h(\mathscr{A}) > \frac{\sqrt{1-\lambda}}{2\sqrt{3}(1-2\eta)}\sqrt{A(\mathscr{A})}.$$

Consider for every $0 < t < h(\mathscr{A})$ the 1-cycle c_t formed by points of $\mathscr{A}$ at distance t of β_0. By the coarea formula

$$A\bigl(\{c_t \mid t \in [\eta h(\mathscr{A}), (1-\eta)h(\mathscr{A})]\}\bigr) = \int_{\eta h(\mathscr{A})}^{(1-\eta)h(\mathscr{A})} |c_t|\, dt \leq A(\mathscr{A}),$$

so that there exists some $t \in [\eta h(\mathscr{A}), (1-\eta)h(\mathscr{A})]$ such that

$$|c_t| \leq \frac{2\sqrt{3}}{\sqrt{1-\lambda}}\sqrt{A(\mathscr{A})};$$

otherwise, we derive a contradiction. The cycle c_t can be approximated by a union of smooth closed simple curves with total length at most

$$\frac{2\sqrt{3}+\eta}{\sqrt{1-\lambda}}\sqrt{A(\mathscr{A})}.$$

So $\mathscr{A}$ decomposes into a stack of two annuli $\mathscr{A}_1$ and $\mathscr{A}_2$ such that $\mathscr{A}_1 \cap \mathscr{A}_2$ is a simple closed curve of length at most

$$\frac{2\sqrt{3}+\eta}{\sqrt{1-\lambda}}\sqrt{A(\mathscr{A})}$$

and such that $h(\mathscr{A}_i) \leq (1-\eta)h(\mathscr{A})$ for $i = 1, 2$. By iterating this process, we derive the lemma. □

We suppose that the stack decomposition is ordered in such a way that D'_0 and $\mathscr{A}_1$ are adjacent. In the sequel we will denote by β_0 the boundary curve of $\mathscr{A}$ corresponding to $\partial D'_0$ and by β_k the one corresponding to $\partial D'$. Observe in particular that

$$|\beta_i| \le (2\sqrt{3}+\eta)\sqrt{A(D')}$$

for $i = 0, \dots, k-1$ and that

$$|\beta_k| \le (4+\eta)\sqrt{A(D')}.$$

Now it remains to estimate the θ-width of D' using this stack decomposition. For each annulus $\mathscr{A}_i$ of the decomposition, choose a minimizing simple path α_i between its two boundary curves. Cutting then along the curve α_i yields to a disk we denote by D'_i whose boundary consists in the concatenation of β_{i-1}, a copy of α_i, β_i and another copy of α_i. Observe that

$$\begin{aligned}|\partial D'_i| &\le (4+\eta)\sqrt{A(D')} + (2\sqrt{3}+\eta)\sqrt{A(D')} + 2\left(\frac{\sqrt{1-\lambda}}{2\sqrt{3}(1-2\eta)}\right)\sqrt{A(\mathscr{A}_i)}\\ &\le \left(4+2\eta+2\sqrt{3}+\frac{1-\lambda}{\sqrt{3}(1-2\eta)}\right)\sqrt{A(D')}.\end{aligned}$$

Since $A(D'_i) = A(\mathscr{A}_i) \le (1-\lambda)A(D')$, we have $m(D'_i) \le n-1$ for $i = 1, \dots, k$, so that

$$\begin{aligned}W_\theta(D'_i) &\le \max\left\{|\partial D'_i| + \sqrt{A(D'_i)},\, C_{\lambda,\eta}\sqrt{A(D'_i)}\right\}\\ &\le \max\left\{\left(4+2\eta+2\sqrt{3}+\frac{1-\lambda}{\sqrt{3}(1-2\eta)}+\sqrt{1-\lambda}\right)\sqrt{A(D')},\, C_{\lambda,\eta}\sqrt{A(D')}\right\}\\ &\le C_{\lambda,\eta}\sqrt{A(D')}.\end{aligned}$$

by the inductive assumption.

Lemma 3.5. *For $i = 1, \dots, k$,*

$$W_\theta(D'_0 \cup \dots \cup D'_i) \le \max\left\{W_\theta(D'_0 \cup \dots \cup D'_{i-1}),\, W_\theta(D'_i)\right\}.$$

In particular, $W_\theta(D') \le \max\{W_\theta(D'_0), \dots, W_\theta(D'_k)\} \le C_{\lambda,\eta}\sqrt{A(D)}$, which concludes the proof of Theorem 2.5.

Proof. Fix $\delta > 0$ and $i \in [\![1, k]\!]$. Choose a trivalent tree T_i (resp. T'_i) together with a continuous map $f_i : D'_i \to T_i$ (resp. $f'_i : D'_0 \cup \dots \cup D'_{i-1} \to T'_i$) satisfying conditions (W1)–(W4) with associated length strictly less than $W_\theta(D'_i) + \delta$ (resp. $W_\theta(D'_0 \cup \dots \cup D'_{i-1}) + \delta$).

We now fix some notation; see Figure 1. Let v_i denote the terminal vertex of T_i whose preimage is $\partial D'_i$, and v'_i the terminal vertex of T'_i whose preimage is $\beta_{i-1} = \partial(D'_0 \cup \dots \cup D'_{i-1})$. Denote by $U_i \subset T_i$ a small neighborhood of $v_i \in T_i$, by

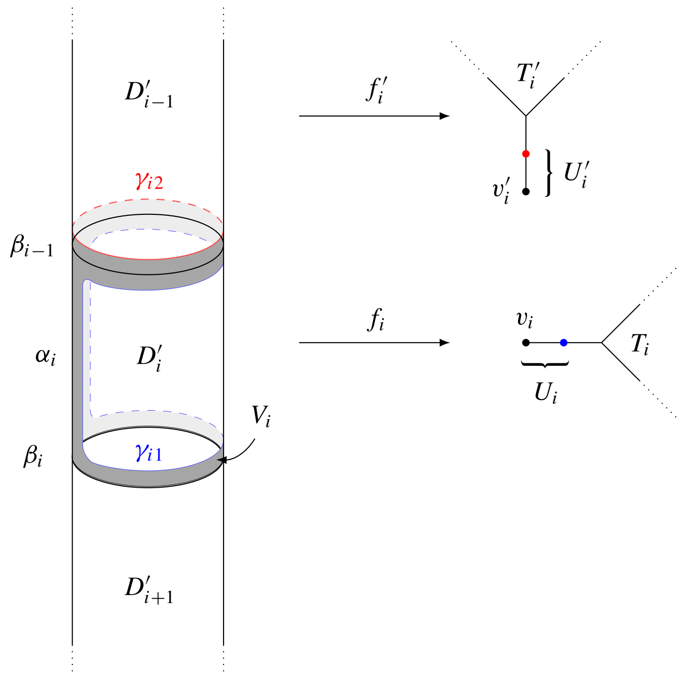

Figure 1. The annulus $\mathscr{A}$ near D'_i.

$U'_i \subset T'_i$ a small neighborhood of $v'_i \in T'_i$, and by V_i the closure of the union of the preimages $f_i^{-1}(U_i)$ and $f'^{-1}_i(U'_i)$. The set V_i is isomorphic to a sphere with three boundary components. One of these components is β_i; the other two are denoted by γ_{i1} and γ_{i2}, as in Figure 1.

Observe that γ_{i1} is a small deformation of $\partial D'_i$ viewed as a curve in D'_i, while γ_{i2} is a small deformation of $\beta_{i-1} \subset D'_0 \cup \cdots \cup D'_{i-1}$. In particular,

$$|\gamma_{i1}| = |\partial D'_i| + o(\epsilon) \quad \text{and} \quad |\gamma_{i2}| = |\beta_{i-1}| + o(\epsilon).$$

We will define a new map from $D'_0 \cup \cdots \cup D'_i$ to a trivalent tree by using the restriction of the previous maps f_i and f'_i on the complementary regions of V_i, and completing it on V_i using the following map, whose straightforward construction is depicted in Figure 2 on the next page.

Claim 3.6. *There exists a map $f : V_i \to Y$ where Y is a tripod* (*a trivalent tree with only three edges*) *and satisfying conditions* (W1)–(W4) *with associated length* $|\partial D'_i| + o(\epsilon)$. □

We apply the claim as follows. Consider the trivalent tree T''_i obtained from the disjoint union of $T_i \setminus U_i$, $T'_i \setminus U'_i$, and Y after identification of the terminal vertices of $T_i \setminus U_i$ and Y corresponding to γ_{i1} and the one of $T'_i \setminus U'_i$ and Y corresponding

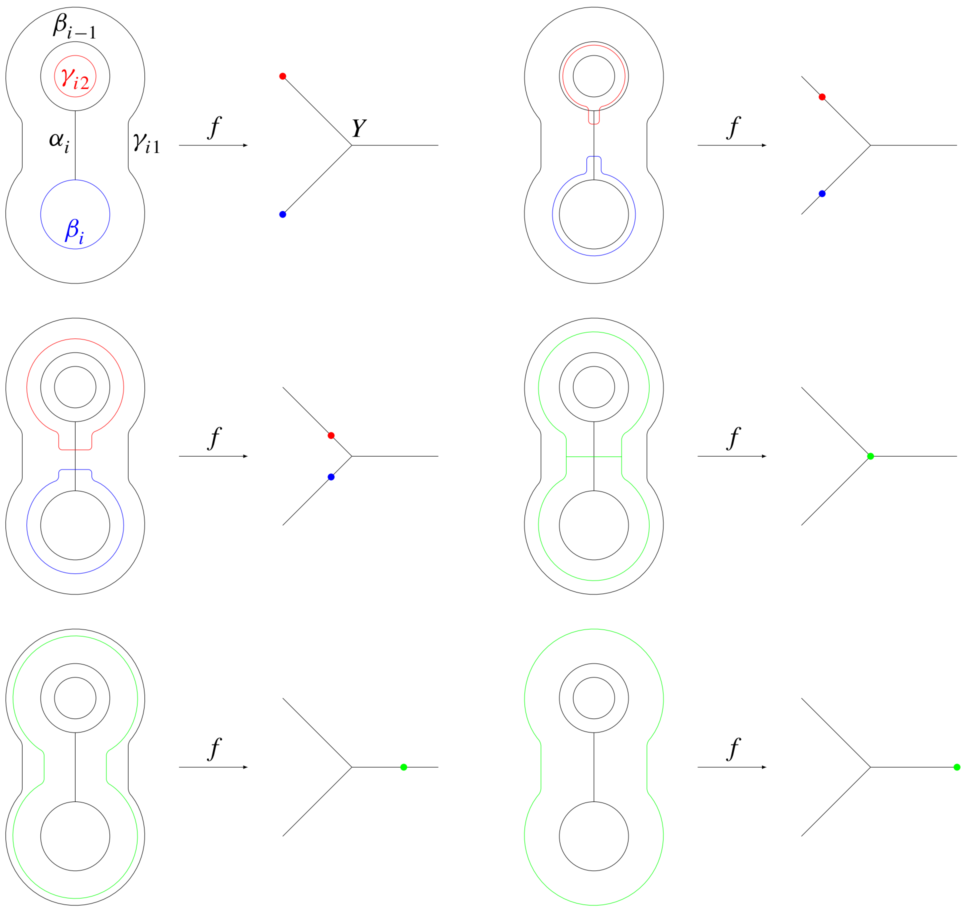

Figure 2. The map $f : V_i \to Y$.

to γ_{i2}. We then define $f_i'' : D_0' \cup \cdots \cup D_i' \to T_i''$ as follows:

$$f_i''(x) = \begin{cases} f_i(x) & \text{if } x \in D_i' \setminus V_i, \\ f_i'(x) & \text{if } x \in D_0' \cup \cdots \cup D_{i-1}' \setminus V_i, \\ f(x) & \text{if } x \in V_i. \end{cases}$$

By construction we have that the length of the preimages is always less than

$$\max\big\{W_\theta(D_0' \cup \cdots \cup D_{i-1}') + \delta,\ W_\theta(D_i') + \delta,\ |\partial D_i'| + o(\epsilon)\big\}.$$

This concludes the proof by letting $\epsilon \to 0$ and $\delta \to 0$ as $W_\theta(M) \geq |\partial M|$ for any Riemannian surface M. $\square$

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