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Let $k[X] = k[x_{i,j} : i = 1, \dots, m;\ j = 1, \dots, n]$ be the polynomial ring in mn variables $x_{i,j}$ over a field k of arbitrary characteristic. Denote by $I_2(X)$ the ideal generated by the 2×2 minors of the generic $m \times n$ matrix $[x_{i,j}]$. We give a closed polynomial formulation for the dimensions of the k-vector space $k[X]/(I_2(X) + (x_{1,1}^q, \dots, x_{m,n}^q))$ as q varies over all positive integers, i.e., we give a closed polynomial form for the generalized Hilbert–Kunz function of the determinantal ring $k[X]/I_2(X)$. We also give a closed formulation of dimensions of other related quotients of $k[X]/I_2(X)$. In the process we establish a formula for the numbers of some compositions (ordered partitions of integers), and we give a proof of a new binomial identity.

1. Introduction

Throughout, let m, n, q be nonnegative integers, and let k, $k[X]$, and $I_2(X)$ be as in the abstract. We write $\mathbb{N}$ for the set of nonnegative integers.

The generalized Hilbert–Kunz function of $R = k[X]/I_2(X)$ is the function $\mathrm{HK}_{R,X} : \mathbb{N} \to \mathbb{N}$ given by

$$\mathrm{HK}_{R,X}(q) = \left(\frac{k[X]}{I_2(X) + (x_{1,1}^q, \dots, x_{m,n}^q)} \right).$$

Namely, $k[X]/(I_2(X) + (x_{1,1}^q, \dots, x_{m,n}^q))$ is a finite-dimensional k-vector space, and length measures that dimension. The standard Hilbert–Kunz function is only defined when k has positive prime characteristic p and when q varies over powers of p, whereas the generalized Hilbert–Kunz function is defined for arbitrary field k, regardless of the characteristic. While the Hilbert–Kunz function is not necessarily a polynomial function, it has a well-defined normalized leading coefficient. The normalized leading coefficient of the generalized Hilbert–Kunz function has been studied for example in [Conca 1996; Eto 2002; Eto and Yoshida 2003], while [Miller and Swanson 2013] studied the whole generalized Hilbert–Kunz function. Miller and Swanson gave a recursive formulation for $\mathrm{HK}_{R,X}$ and proved that it is a

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polynomial function. They gave closed formulations in the case $m \le 2$. This paper is an extension of [Miller and Swanson 2013].

The main result of this paper, Theorem 3.3, is the closed formulation of $\mathrm{HK}_{R,X}$ for arbitrary positive integers m, n. We also give, in Theorem 3.1, an explicit formula for the length of

$$\frac{k[X]}{I_2(X) + (x_{i,j}^q : i, j) + \sum_{j=1}^{n}(x_{1,j}, \ldots, x_{m,j})^q}.$$

In Lemma 2.5 and Corollary 3.4 we give some explicit formulas for the number of tuples of specific length of nonnegative integers that sum up to at most a fixed number and whose first few entries are at most another fixed number. (In other words, we give formulas for the numbers of some specific compositions of integers.)

2. Set-up

Our proofs are based on the following result:

Theorem 2.1 [Miller and Swanson 2013, Theorem 2.4]. *The quotient ring*

$$\frac{k[X]}{I_2(X) + (x_{1,1}^q, \ldots, x_{m,n}^q)}$$

has a k-vector space basis consisting precisely of monomials $\prod_{i,j} x_{i,j}^{p_{i,j}}$ *with the following properties*:

(1) *Whenever* $p_{i,j} > 0$ *and* $i' < i$, $j < j'$, *we have* $p_{i',j'} = 0$. (Monomials satisfying this property will be called *staircase monomials*. The name comes from the southwest-northeast staircase-like shape of the nonzero entries $p_{i,j}$ in the $m \times n$ matrix of all the $p_{i,j}$.)

(2) *Either* $\sum_j p_{i,j} < q$ *for all* $i = 1, \ldots, m$ *or* $\sum_i p_{i,j} < q$ *for all* $j = 1, \ldots, n$. □

Thus, to compute the Hilbert–Kunz function, we need to be able to count such monomials. The recursive formulations for this function in [Miller and Swanson 2013], as well as the explicit formulations below, require counting related sets of monomials:

Definition 2.2 [Miller and Swanson 2013, Section 3]. Let $r_1, \ldots, r_m, c_1, \ldots, c_n \in \mathbb{N} \cup \{\infty\}$. (In general we think of the r_i as the row sums and the c_j as the column sums.) Define $N_q(m, n; r_1, \ldots, r_m; c_1, \ldots, c_n)$ to be the number of monomials $\prod_{i,j} x_{i,j}^{p_{i,j}}$ with the following properties:

(1) $\prod_{i,j} x_{i,j}^{p_{i,j}}$ is a staircase monomial, i.e., whenever $p_{i,j} > 0$ and $i' < i$, $j < j'$, we have $p_{i',j'} = 0$.

(2) $\sum_j p_{i,j} \le r_i$ for all $i \in \{1,\dots,m\}$ and $\sum_i p_{i,j} \le c_j$ for all $j \in \{1,\dots,n\}$.

(3) Either $\sum_j p_{i,j} < q$ for all $i \in \{1,\dots,m\}$ or $\sum_i p_{i,j} < q$ for all $j \in \{1,\dots,n\}$.

For ease of notation, for any $c \in \mathbb{N}\cup\{\infty\}$ we let $\overline{c}$ denote a repetition of cs, where the number of occurrences depends on the context. For example, $N_q(m,n;\overline{\infty},\overline{\infty})$ stands for $N_q(m,n;\infty,\dots,\infty;\infty,\dots,\infty)$, with m occurrences of ∞ in the first instance and n in the second. By convention, $N_q(0,n;\,;c_1,\dots,c_n)=1$.

It was proved in [Miller and Swanson 2013, Section 3] that

$$N_q(m,n;r_1,\dots,r_m;c_1,\dots,c_n) = \text{ length of } \left(\frac{k[X]}{I_2(X)+(x_{i,j}^q : i,j)+\sum_{i=1}^m (x_{i,1},\dots,x_{i,n})^{r_i+1}+\sum_{j=1}^n (x_{1,j},\dots,x_{m,j})^{c_j+1}}\right),$$

where for an ideal I, we set I^∞ to be the 0 ideal. Thus, in particular,

$$N_q(m,n;\overline{\infty};\overline{\infty}) = \mathrm{HK}_{K[X]/I_2(X),X}(q).$$

Our main result in this paper relies on the count of the following monomials as well:

Definition 2.3. Let $r_1,\dots,r_m,c_1,\dots,c_n \in \mathbb{N}\cup\{\infty\}$. (We think of r_i as the i-th row sum, and of c_j as the j-th column sum.) Define $M_q(m,n;r_1,\dots,r_m;c_1,\dots,c_n)$ to be the number of monomials $\prod_{i,j} x_{i,j}^{p_{i,j}}$ such that:

(1) $\prod_{i,j} x_{i,j}^{p_{i,j}}$ is a staircase monomial, i.e., whenever $p_{i,j}>0$ and $i'<i$, $j<j'$, we have $p_{i',j'}=0$.

(2) $\sum_j p_{i,j} \le \min\{r_i, q-1\}$ for all $i \in \{1,\dots,m\}$.

(3) There exists $j \in \{1,\dots,n\}$ such that $\sum_i p_{i,j} > c_j$.

The following lemma says that mn exponents $p_{i,j}$ of a staircase monomial can be identified by $m+n$ or even $m+n-1$ numbers:

Lemma 2.4. *Suppose that $r_1,\dots,r_m,c_1,\dots,c_n$ are nonnegative integers and that $\sum_i r_i = \sum_j c_j$. Then there exists a unique staircase monomial $\prod_{i,j} x_{i,j}^{p_{i,j}}$ such that $r_i = \sum_j p_{i,j}$ for all $i = 1,\dots m$ and $c_j = \sum_i p_{i,j}$ for all $j = 1,\dots,n$.*

Proof. If $m=1$, then clearly $p_{1,j}=c_j$, which is uniquely determined. If $n=1$, necessarily $p_{i,1}=r_i$.

In general, for arbitrary m and n, knowing c_1 and r_m is enough information to uniquely determine $p_{m,1}$: if $p_{m,1}<\min\{c_1,r_m\}$, then the m-th row has a nonzero number beyond the first entry and the first column has a nonzero number in the first $m-1$ rows, which then makes the corresponding monomial nonstaircase and is not allowed. So necessarily $p_{m,1}=\min\{c_1,r_m\}$. If $p_{m,1}=c_1$, then no more nonzero exponents appear in the first column, and it remains to fill in the remaining $m\times(n-1)$

matrix of $p_{i,j}$ with the remaining numbers $r_1, \dots, r_{m-1}, r_m - c_1, c_2, \dots, c_n$. If instead $p_{m,1} = r_m$, then no more nonzero exponents appear in the last row, and it remains to fill in the remaining $(m-1) \times n$ matrix of $p_{i,j}$ with the remaining numbers $r_1, \dots, r_{m-1}, c_1 - r_m, c_2, \dots, c_n$. $\square$

Lemma 2.5. *Let a, b, w, z be integers with $a \le b$. The number of b-tuples of nonnegative integers that sum up to at most w and for which the first a entries are strictly smaller than z equals*

$$\sum_{i=0}^{a} (-1)^i \binom{a}{i} \binom{w-iz+b}{b}.$$

Proof. (This proof was suggested by the referee.) Let E be the set of all b-tuples $(v_1, \dots, v_b)$ of nonnegative integers that sum up to at most w. It is well known that $|E| = \binom{w+b}{b}$. Let E_j be the subset E of those tuples for which $v_i \ge z$. Then $|E_j| = \binom{w-z+b}{b}$, and more generally, $|E_{j_1} \cap \cdots \cap E_{j_i}| = \binom{w-iz+b}{b}$. The desired cardinality is $|E \setminus (E_1 \cup \cdots \cup E_a)|$, which, by the inclusion-exclusion principle, equals

$$\begin{aligned} |E| - \sum_{i=1}^{a} (-1)^{i-1} \sum_{1 \le j_1 < \cdots < j_i \le a} |E_{j_1} \cap \cdots \cap E_{j_i}| & \\ &= \binom{w-iz+b}{b} - \sum_{i=1}^{a} (-1)^{i-1} \binom{a}{i} \binom{w-iz+b}{b} \\ &= \sum_{i=0}^{a} (-1)^i \binom{a}{i} \binom{w-iz+b}{b}. \end{aligned}$$ $\square$

Lemma 2.6. *Let a, b, w, z be integers with $a < b$. The following numbers are the same*:

(1) *The number of b-tuples of nonnegative integers that sum to at most $a(z-1) - w$ and for which the first a entries are strictly smaller than z.*

(2) $$\sum_{i=0}^{a} (-1)^i \binom{a}{i} \binom{a(z-1)-w-iz+b}{b}.$$

(3) *The number of b-tuples of nonnegative integers for which the first a entries are strictly smaller than z and the sum of the first a entries is greater than or equal to w plus the sum of the remaining entries.*

Proof. The first two numbers are the same by Lemma 2.5.

Let

$$E = \left\{ (v_1, \dots, v_b) \in \mathbb{N}^b : v_1, \dots, v_a < z, \sum_i v_i \le a(z-1) - w \right\},$$

$$F = \left\{ (u_1, \dots, u_b) \in \mathbb{N}^b : u_1, \dots, u_a < z, \sum_{i=1}^{a} u_i \ge \sum_{i>a} u_i + w \right\},$$

so $|E|$ is the number in (1) and $|F|$ is the number in (3). Define $\varphi : E \to F$ by

$$\varphi(v_1, \dots, v_b) = (z-1-v_1, \dots, z-1-v_a, v_{a+1}, \dots, v_b).$$

Certainly this image is in $\mathbb{N}^b$, each of the first a entries is strictly smaller than z, and the sum of the first a entries is $a(z-1) - \sum_{i=1}^{a} v_i = a(z-1) - \sum_{i=1}^{b} v_i + \sum_{i=a+1}^{b} v_i \ge w + \sum_{i=a+1}^{b} v_i$, so that the range of φ is in F. The proof of surjectivity is similar, and injectivity is clear. Thus φ is bijective, which proves that the numbers in (1) and (3) are the same. □

3. Main theorems

In this section we give explicit (nonrecursive) formulas for $N_q(m, n; \overline{\infty}; \overline{q-1})$, $M_q(m, n; \overline{q-1}; \overline{q-1})$, and $N_q(m, n; \overline{\infty}; \overline{\infty})$ for arbitrary positive integers m, n.

Theorem 3.1. *For all nonnegative integers m, n, q, the k-length of the quotient ring $k[X]/\big(I_2(X) + (x_{i,j}^q : i, j) + \sum_{j=1}^{n} (x_{1,j}, \dots, x_{m,j})^q\big)$ equals*

$$N_q(m, n; \overline{\infty}; \overline{q-1}) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{iq+m-1}{m+n-1},$$

and furthermore, this number equals the number of $(m+n-1)$-tuples of nonnegative integers that sum up to at most $n(q-1)$ and for which the first n entries are strictly smaller than q.

Proof. Let $T_{m,n,q}$ be the set of all staircase monomials $\prod_{i,j} x_{i,j}^{p_{i,j}}$ such that $\sum_i p_{i,j} < q$ for all $j = 1, \dots, n$. By [Miller and Swanson 2013, Section 3],

$$\begin{aligned} |T_{m,n,q}| &= N_q(m, n; \overline{\infty}; \overline{q-1}) \\ &= \left(\frac{k[X]}{I_2(X) + (x_{i,j}^q : i, j) + \sum_{j=1}^{n} (x_{1,j}, \dots, x_{m,j})^q} \right). \end{aligned}$$

Let W be the set of $(m+n-1)$-tuples of nonnegative integers such that the first n entries are strictly smaller than q, and the sum of the first n entries is greater than or equal to the sum of the remaining entries. To each element $\prod_{i,j} x_{i,j}^{p_{i,j}}$ in $T_{m,n,q}$ we associate the $(m+n-1)$-tuple $\big(\sum_i p_{i1}, \dots, \sum_i p_{in}, \sum_j p_{2j}, \dots, \sum_j p_{mj}\big)$ of nonnegative integers. This is an element of W. For any $(c_1, \dots, c_n, r_2, \dots, r_m) \in W$, set $r_1 = \sum_j c_j - \sum_{i=2}^m r_m \in \mathbb{N}$. By Lemma 2.4, there is a unique element of $T_{m,n,q}$ that corresponds to $(c_1, \dots, c_n, r_2, \dots, r_m)W$. Thus $|T_{m,n,q}| = |W|$, and by Lemma 2.6 applied with $w = 0$,

$$\begin{aligned} N_q(m,n;\overline{\infty};\overline{q-1}) &= |W| \\ &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n(q-1)-iq+m+n-1}{m+n-1} \\ &= \sum_{i=0}^{n} (-1)^i \binom{n}{n-i} \binom{(n-i)q+m-1}{m+n-1} \\ &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{iq+m-1}{m+n-1}. \end{aligned}$$

□

Theorem 3.2. *For all positive integers m, n,*

$$M_q(m,n;\overline{q-1};\overline{q-1}) = \sum_{i=1}^{n} \sum_{j=0}^{m} (-1)^{m-j+i-1} \binom{n}{i} \binom{m}{j} \binom{jq-iq+n-1}{m+n-1}.$$

Proof. Let S be the set of all $(m+n)$-tuples $(r_1, \dots, r_m, c_1, \dots, c_n)$ of nonnegative integers such that $\sum_i r_i = \sum_j c_j$, $r_1, \dots, r_m < q$, and there exists j such that $c_j \geq q$. By Lemma 2.4, each such tuple uniquely determines a staircase monomial, and by definition of M_q, the number of these monomials is precisely $M_q(m,n;\overline{q-1};\overline{q-1})$. We will count these monomials via the tuples.

We define the function $f : S \to 2^{\{1,\dots,n\}}$ as $f(r_1, \dots, r_m, c_1, \dots, c_n) = \{j : c_j \geq q\}$. By S_k we denote the set of all those $x \in S$ for which $|f(x)| = k$. Consider the set A of all (L, x) for which $x \in S$ and L is a nonempty subset of $f(x)$. For each $l = 1, \dots, n$, the number of $(L, x) \in A$ with $|L| = l$ equals

$$\sum_{k=l}^{n} \binom{k}{l} |S_k|.$$

In other words, (L, x) only arises if $L \subseteq f(x)$, and each $x \in S_k$, generates $\binom{k}{l}$ distinct elements (L, x) in A with $|L| = l$.

We count the elements of A another way. Put $(L, x) = (L, r_1, \dots, r_m, c_1, \dots, c_n)$ with $|L| = l$. Since $\sum_i r_i = \sum_j c_j$, one of the c_j is redundant, and we remove c_s, where $s = \min L$. Furthermore, we lose no information if we subtract q from each c_l with $l \in L$. Set $c'_j = c_j - q$ if $j \in L$ and $c'_j = c_j$ otherwise. Thus, to count all (L, x), it suffices to count all $(L, r_1, \dots, r_m, c'_1, \dots, c'_{s-1}, c'_{s+1}, \dots, c'_n)$. But the set of all such $(m+n)$-tuples equals $P_l(n) \times W$, where $P_l(n)$ is the set of all l-element subsets of $\{1, \dots, n\}$, and W is the set of all $(m+n-1)$-tuples of nonnegative integers whose first m entries are strictly smaller than q, and the sum of the first

m entries is greater than or equal to lq plus the sum of the remaining entries. By Lemma 2.6, W has cardinality

$$\begin{aligned}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{m(q-1)-lq-iq+m+n-1}{m+n-1}\\ &= \sum_{i=0}^{m}(-1)^i\binom{m}{m-i}\binom{(m-i)q-lq+n-1}{m+n-1}\\ &= \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\binom{iq-lq+n-1}{m+n-1}.\end{aligned}$$

Using $|P_l(n)| = \binom{n}{l}$, this gives a system of n linear equations with matrix form:

$$\begin{bmatrix} \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{n}{1} \\ \binom{1}{2} & \binom{2}{2} & \binom{3}{2} & \cdots & \binom{n}{2} \\ \binom{1}{3} & \binom{2}{3} & \binom{3}{3} & \cdots & \binom{n}{3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{1}{n} & \binom{2}{n} & \binom{3}{n} & \cdots & \binom{n}{n} \end{bmatrix} \begin{bmatrix} |S_1| \\ |S_2| \\ |S_2| \\ \vdots \\ |S_n| \end{bmatrix} = \begin{bmatrix} \binom{n}{1}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\binom{iq-1q+n-1}{m+n-1} \\ \binom{n}{2}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\binom{iq-2q+n-1}{m+n-1} \\ \binom{n}{3}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\binom{iq-3q+n-1}{m+n-1} \\ \vdots \\ \binom{n}{n}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\binom{iq-nq+n-1}{m+n-1} \end{bmatrix}.$$

Note that the matrix $\left[\binom{j}{i}\right]_{i,j}$ is upper triangular with determinant 1. Its inverse is the upper triangular matrix $\left[(-1)^{i+j}\binom{j}{i}\right]_{i,j}$, as for all $i \le j$,

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{i+k}\binom{k}{i}\binom{j}{k} &= \sum_{k=i}^{j}(-1)^{i+k}\frac{k!}{i!(k-i)!}\frac{j!}{k!(j-k)!}\\ &= \sum_{k=i}^{j}(-1)^{i+k}\frac{j!}{i!(j-i)!}\frac{(j-i)!}{(k-i)!(j-k)!}\\ &= \binom{j}{i}\sum_{k=i}^{j}(-1)^{i+k}\binom{j-i}{k-i} = \binom{j}{i}\sum_{k=0}^{j-i}(-1)^k\binom{j-i}{k},\end{aligned}$$

which is 0 if $j > i$ and is 0 if $j = i$. Thus, by Cramer's rule,

$$\begin{aligned}
M_q(m,n;\overline{q-1};\overline{q-1}) &= \sum_{k=1}^{n} |S_k| \\
&= \sum_{k=1}^{n}\sum_{j=1}^{n}(-1)^{k+j}\binom{j}{k}\binom{n}{j}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\binom{iq-jq+n-1}{m+n-1} \\
&= \sum_{j=1}^{n}(-1)^{j}\binom{n}{j}\sum_{k=1}^{n}(-1)^{k}\binom{j}{k}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\binom{iq-jq+n-1}{m+n-1} \\
&= \sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\binom{iq-jq+n-1}{m+n-1} \\
&= \sum_{j=1}^{n}\sum_{i=0}^{m}(-1)^{m-i+j-1}\binom{n}{j}\binom{m}{i}\binom{iq-jq+n-1}{m+n-1} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}(-1)^{m-i+j-1}\binom{n}{j}\binom{m}{i}\binom{iq-jq+n-1}{m+n-1}. \qquad \square
\end{aligned}$$

The main theorem on the generalized Hilbert–Kunz function now follows:

Theorem 3.3. *For all positive integers m, n, the Hilbert function $\mathrm{HK}_{R,X}(q)$ of $k[X]/I_2(X)$ at q, i.e., the length of $k[X]/\big(I_2(X) + (x_{1,1}^q, \dots, x_{m,n}^q)\big)$, equals*

$$\begin{aligned}
N_q(m,n;\overline{\infty};\overline{\infty}) &= N_q(m,n;\overline{\infty};\overline{q-1}) + M_q(m,n;\overline{q-1};\overline{q-1}) \\
&= \sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}\binom{iq+m-1}{m+n-1} \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{m}(-1)^{m-j+i-1}\binom{n}{i}\binom{m}{j}\binom{jq-iq+n-1}{m+n-1}.
\end{aligned}$$

Proof. By Definition 2.2, $N_q(m,n;\overline{\infty};\overline{\infty})$ counts all the staircase monomials $\prod_{i,j} x_{i,j}^{p_{i,j}}$ with the property that either $\sum_i p_{i,j} < q$ for all j or $\sum_j p_{i,j} < q$ for all i.

The number $N_q(m,n;\overline{\infty};\overline{q-1})$ counts those monomials in the previous paragraph for which $\sum_i p_{i,j} < q$ for all j, and $M_q(m,n;\overline{q-1};\overline{q-1})$ counts those monomials for which $\sum_i p_{i,j} \ge q$ for some j. Thus

$$N_q(m,n;\overline{\infty};\overline{\infty}) = N_q(m,n;\overline{\infty};\overline{q-1}) + M_q(m,n;\overline{q-1};\overline{q-1}),$$

and by Theorems 3.1 and 3.2, this is equal to the claimed sums of binomial coefficients. $\square$

In particular, comparison with Theorem 4.4 in [Miller and Swanson 2013] when $m = 2$ gives:

Corollary 3.4. *The number of $(n+1)$-tuples of nonnegative integers that sum up to at most $n(q-1)$ and for which the first n entries are strictly smaller than q equals*

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}\binom{iq+1}{n+1}=\frac{nq^{n+1}-(n-2)q^n}{2}.$$

Proof. According Theorem 4.4 in to [Miller and Swanson 2013],

$$N_q(2,n;\overline{\infty};\overline{\infty})=\frac{nq^{n+1}-(n-2)q^n}{2}+n\binom{q+n-1}{n+1},$$

and by Theorem 3.3,

$$\begin{aligned}N_q(2,n;\overline{\infty};\overline{\infty})&=\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}\binom{iq+1}{n+1}+2\sum_{i=1}^{n}(-1)^{i}\binom{n}{i}\binom{q-iq+n-1}{n+1}\\&\qquad+\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}\binom{2q-iq+n-1}{n+1}\\&=\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}\binom{iq+1}{n+1}+\binom{n}{1}\binom{2q-1q+n-1}{n+1}\\&=\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}\binom{iq+1}{n+1}+n\binom{q+n-1}{n+1}.\end{aligned}$$

Thus $\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}\binom{iq+1}{n+1}=\left(nq^{n+1}-(n-2)q^n\right)/2$. By Theorem 3.1, this number is the number of $(n+1)$-tuples of nonnegative integers that sum up to at most $n(q-1)$ and for which the first n entries are strictly smaller than q. □

We remark here that we know of no other proof of the equality in the last corollary. Natural first attempts would be induction and Gosper's algorithm, and neither of these is successful, as for one thing, the summands depend not only on the summing index i but also on n. The challenge remains to establish a closed-form expression for $N_q(m,n;\overline{\infty};\overline{\infty})$ and $N_q(m,n;\overline{\infty};\overline{q-1})$ for higher m.

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