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Using parabolic induction, a global representation of a double cover of the conformal group $\mathrm{SO}(2, n+1)_0$ is constructed. Its space of finite vectors is realized as a direct sum of eigenspaces of the Yamabe operator on $S^1 \times S^n$. The explicit form of the corresponding eigenvalues is obtained. An explicit basis of K-finite eigenvectors is used to study its structure as a representation of the Lie algebra of the conformal group.

1. Introduction

M. Hunziker, M. Sepanski, and R. Stanke [Hunziker et al. 2012] used parabolic induction to construct a representation $I_{m,r}$ of a twofold cover $\widetilde{G}$ of the conformal group $G := \mathrm{SO}(2, n+1)_0$ of $\mathbb{R}^{2,n+1}$ and studied the kernel of a distinguished central element Ω in the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of $\widetilde{G}$. It was shown that this kernel carries the minimal representation of $\widetilde{G}$ as a positive energy representation and that elements in this kernel correspond to solutions of the wave equation. In this article, we study the structure of $I_{m,r}$ and its compact picture $I''_{m,r}$ as $(\mathfrak{g}_{\mathbb{C}}, \widetilde{K})$-modules, where $\widetilde{K}$ denotes the maximal compact subgroup of $\widetilde{G}$. We generalize the results of [loc. cit.] by considering the space of $\widetilde{K}$-finite vectors $\ker(\Omega-\mu)_{\widetilde{K}}$ of $\ker(\Omega-\mu)$ where $\mu \in \mathbb{R}$. We explicitly determine the conditions on μ such that $\ker(\Omega-\mu)_{\widetilde{K}}$ is nonzero. We show that Ω can be realized as the Yamabe operator $\widetilde{\Delta}_{S^1\times S^n}$ acting on $S^1 \times S^n$ embedded in the Minkowski space $\mathbb{R}^{2,n+1}$. Using this realization, we show that the space of $\widetilde{K}$-finite vectors $(I''_{m,r})_{\widetilde{K}}$ of $I''_{m,r}$ is isomorphic to a direct sum of eigenspaces of the Yamabe operator $\widetilde{\Delta}_{S^1\times S^n}$. An explicit basis of eigenvectors vectors for $(I''_{m,r})_{\widetilde{K}}$ in terms of harmonic and Gegenbauer polynomials is constructed. We show that each of these eigenspaces is invariant under the action of $\widetilde{K}$, but not invariant under the action of all of $\mathfrak{g}$ with the exception of the null eigenspace. While $\mathfrak{g}$ does not preserve each individual eigenspace, it preserves the direct sum of them, $(I''_{m,r})_{\widetilde{K}}$. The zero eigenspace is left invariant under the action of $\mathfrak{g}$ and $(I''_{m,r})_{\widetilde{K}}/(\ker\Omega)_{\widetilde{K}}$ decomposes as the direct

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sum of two irreducible representations explicitly identified in Theorem 1. The elements in the zero eigenspace correspond to solutions to the wave equation in the noncompact picture.

1.1. ***Related work.*** This article is motivated by [Hunziker et al. 2012], where the case $\mu = 0$ was studied. This work is similar in spirit to [Franco and Sepanski 2013], which provides a generalization of [Sepanski and Stanke 2010]. Since the minimal representation of conformal groups, the wave equation, and the Yamabe operator are heavily studied topics in mathematics there exists substantial overlap with existing literature. See [Hunziker et al. 2012] for a more extensive set of references. Of particular note, for $n = 3$, some of the results and formulas in this article appear in [Segal et al. 1981] and the references therein. In terms of the minimal representation of the conformal group G, B. Binegar and R. Zierau [1991] realized the minimal representation of $\mathrm{SO}(p, q)_0$ with p, q of the same parity. A very detailed study of the minimal representation of $O(p, q)$ is given by T. Kobayashi and B. Ørsted in [2003a; 2003b; 2003c].

1.2. ***Organization of the work.*** We introduce most of the objects that will be fundamental to the study of our problem in Section 2. In particular, the induced, noncompact, and compact pictures are introduced in this section. Since there is a considerable overlap with existing literature, most of the proofs are omitted. In Section 3 we realize the Yamabe operator on $S^1 \times S^n$ as a central element of the universal enveloping algebra of the Lie algebra of $\widetilde{G}$ and study the invariance of the eigenspaces of this operator. In Section 4 we introduce the space of $\widetilde{K}$-finite vectors in the compact picture and give an explicit basis consisting of eigenvectors of the Yamabe operator. In Section 5 we study the structure of the space of $\widetilde{K}$-finite vectors and close with Theorem 1, where the main results are summarized.

2. Preliminary constructions

This section contains a substantial overlap with [Hunziker et al. 2012] due to the similarity of the problems studied. Therefore, most of this section will be dedicated to a quick survey of the results that will be useful for the study of our problem.

2.1. ***Group and subgroups.*** Let $G = \mathrm{SO}(2, n+1)_0$ and let $\mathfrak{g} := \mathfrak{so}(2, n+1)$ denote its Lie algebra. The group G acts naturally on the space $\mathbb{R}^{2,n+1}$. Since G is a group of linear transformations that preserve the signed quadratic form on $\mathbb{R}^{2,n+1}$, the action of G descends to an action on the subspace

$$C^{2,n+1} := \{(a, b) \in \mathbb{R}^{2,n+1} \mid \|a\| = \|b\| \neq 0\}.$$

$C^{2,n+1}$ is a cone in the sense that it is invariant under the action of $\mathbb{R}^\times$. Moreover, since the actions of G and $\mathbb{R}^\times$ commute, G acts on the projectivized cone

$\mathbb{P}(C^{2,n+1}) \cong C^{2,n+1}/\mathbb{R}^\times$. For $[0,1,\pm 1,0,\dots,0] \in \mathbb{P}(C^{2,n+1})$, their stabilizers $Q^\pm := \mathrm{Stab}_G([0,1,\pm 1,0,\dots,0])$ are isomorphic to the minimal parabolic subgroups with Langlands decompositions $Q^\pm = MAN^\pm$. The corresponding parabolic subalgebras $\mathfrak{q}^\pm$ have Langlands decomposition $\mathfrak{q}^\pm = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^\pm$. We describe these subalgebras and subgroups in more detail. The nilpotent subalgebras are

$$\mathfrak{n}^\pm = \{N^\pm_{t,x} \mid (t,x) \in \mathbb{R}^{1,n}\}, \quad \text{where} \quad N^\pm_{t,x} := \left(\begin{array}{ccc|c} 0 & t & \mp t & 0 \\ -t & 0 & 0 & x \\ \mp t & 0 & 0 & \pm x \\ \hline 0 & x^T & \mp x^T & 0_n \end{array}\right),$$

and the maximal abelian subalgebra is

$$\mathfrak{a} = \{H_s \mid s \in \mathbb{R}\}, \quad \text{where} \quad H_s := \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & s & 0 & 0 \\ \hline 0 & 0 & 0 & 0_n \end{array}\right),$$

and if we denote the $+1$ eigenspace of the Cartan involution on $\mathfrak{g}$ by $\mathfrak{k}$, then the centralizer of $\mathfrak{a}$ in $\mathfrak{k} \cong \mathfrak{so}(2) \times \mathfrak{so}(n+1)$ is

$$\mathfrak{m} = \{L_{A,b} \mid A \in \mathfrak{so}(n) \text{ and } b \in \mathbb{R}^n\}, \quad \text{where} \quad L_{A,b} := \left(\begin{array}{ccc|c} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline b^T & 0 & 0 & A \end{array}\right).$$

The corresponding groups in G are

$$N^\pm = \{n^\pm_{t,x} \mid (t,x) \in \mathbb{R}^{1,n}\}, \text{ where } n^\pm_{t,x} := \left(\begin{array}{ccc|c} 1 & t & \mp t & 0 \\ -t & 1+\frac{1}{2}q(t,x) & \mp\frac{1}{2}q(t,x) & x \\ \mp t & \pm\frac{1}{2}q(t,x) & 1-\frac{1}{2}q(t,x) & \pm x \\ \hline 0 & x^T & \mp x^T & I_n \end{array}\right),$$

and

$$A = \{h_s \mid s \in \mathbb{R}\}, \quad \text{where} \quad h_s := \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cosh(s) & \sinh(s) & 0 \\ 0 & \sinh(s) & \cosh(s) & 0 \\ \hline 0 & 0 & 0 & I_n \end{array}\right),$$

with $q(t,x) = -t^2 + \|x\|^2$. If $\mathrm{SO}(1,n)_0$ denotes the identity component of $\mathrm{SO}(1,n)$ and $\mathrm{SO}(1,n)_1$ denotes the remaining component, then

$$M = \{m_{\epsilon,Y} \mid \epsilon = \pm 1\}$$

where

$$m_{\epsilon,Y}=\left(\begin{array}{ccc|c} a & 0 & 0 & b\\ 0 & \epsilon & 0 & 0\\ 0 & 0 & \epsilon & 0\\ \hline c & 0 & 0 & d\end{array}\right) \text{ with } Y=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\begin{cases}\mathrm{SO}(1,n)_0 & \text{if } \epsilon=+1,\\ \mathrm{SO}(1,n)_1 & \text{if } \epsilon=-1.\end{cases}$$

As in [Hunziker et al. 2012], we will look at the representations induced from a character on Q^-. To construct this character, let M_0 denote the connected component of M where $\epsilon=+1$ and let M_1 denote the component where $\epsilon=-1$. For $g_j\in M_j$ define $\mu_M : M\to\mathbb{C}$ by $\mu_M(g_j) := (-1)^j$. Define $\nu_A : A\to\mathbb{C}$ by $\nu_A(h_s)=e^s$. Then, the family of characters from which we will induce our representations is defined by

$$\chi_{m,r}(q^-)=\nu_A(q_A^-)^r\,\mu_M(q_M^-)^m$$

with $q^-=q_M^-q_A^-q_N^-\in Q^-$, $r\in\mathbb{C}$, and $m\in\mathbb{Z}_2$.

2.2. ***Double cover and induced representations.*** For technical reasons it is necessary to work in a double cover of G that we will denote by $\widetilde{G}$. The maximal compact group of G is $K\cong\mathrm{SO}(2)\times\mathrm{SO}(n+1)$. The double cover $\widetilde{K}$ of K is such that the cover map $\pi:\widetilde{K}\to K$ is given by

$$\pi\begin{pmatrix} R_{\varphi/2} & \\ & u_{n+1}\end{pmatrix}=\begin{pmatrix} R_{\varphi} & \\ & u_{n+1}\end{pmatrix}. \tag{2-1}$$

Up to isomorphism, $\widetilde{K}$ extends uniquely to a group $\widetilde{G}$ that is a double cover of G. Letting $\widetilde{Q}^{\pm}$ denote the parabolic subgroups of $\widetilde{G}$ that cover $Q^{\pm}$, we have that they have Langlands decomposition $\widetilde{Q}^{\pm}=\widetilde{M}\widetilde{A}\widetilde{N}^{\pm}$ and

$$\widetilde{M}\cap\widetilde{K}=\{z_{j,k}\mid k\in O(n),\det k=(-1)^j\}\quad\text{where}\quad z_{j,k}=\left(\begin{array}{cc|c} R^j_{\pi/2} & 0 & 0\\ 0 & (-1)^j & 0\\ \hline 0 & 0 & k\end{array}\right).$$

In particular, $\widetilde{M}$ has four connected components. To define a character on $\widetilde{Q}^-$, we define $\gamma_{\widetilde{M}}:\widetilde{M}\to\mathbb{C}$ by

$$\gamma_{\widetilde{M}}\big|_{\widetilde{M}_j} := i^j$$

and $\nu_{\widetilde{A}}:\widetilde{A}\to\mathbb{C}$ by

$$\nu_{\widetilde{A}}(\widetilde{h}_s)=e^s.$$

In [Hunziker et al. 2012], it is shown that the character defined by

$$\widetilde{\chi}_{m,r}(\widetilde{q}^-) := \nu_{\widetilde{A}}(\widetilde{q}^-_{\widetilde{A}})^r\,\gamma_{\widetilde{M}}(\widetilde{q}^-_{\widetilde{M}})^m$$

with $\widetilde{q}^-=\widetilde{q}^-_{\widetilde{M}}\widetilde{q}^-_{\widetilde{A}}\widetilde{q}^-_{\widetilde{N}}\in\widetilde{Q}^-$, $r\in\mathbb{C}$, and $m\in\mathbb{Z}_4$, satisfies $\widetilde{\chi}_{m,r}=\chi_{m,r}\circ\pi$.

The representation $\widetilde{\chi}_{m,r}$ of $\widetilde{Q}^-$ is used to induce a representation $\mathrm{Ind}_{Q^-}^{\widetilde{G}}(\widetilde{\chi}_{m,r})$ of $\widetilde{G}$. This representation is defined by

$$\text{(2-2)}\quad \mathrm{Ind}_{Q^-}^{\widetilde{G}}(\widetilde{\chi}_{m,r}) = \{\phi \in C^\infty(\widetilde{G}) \mid \varphi(\widetilde{g}\widetilde{q}^-) = \widetilde{\chi}_{m,r}^{-1}(\widetilde{q}^-)\phi(\widetilde{g}) \text{ for } \widetilde{g} \in \widetilde{G}, \widetilde{q}^- \in \widetilde{Q}^-\}.$$

2.3. *Noncompact picture.* For $r \in \mathbb{C}$ and $m \in \mathbb{Z}_4$, define

$$I'_{m,r} := \{f \in C^\infty(\mathbb{R}^{1,n}) \mid f(t,x) = \phi(\widetilde{n}_{t,x}) \text{ for some } \phi \in \mathrm{Ind}_{Q^-}^{\widetilde{G}}(\widetilde{\chi}_{m,r}) \text{ and all } (t,x) \in \mathbb{R}^{1,n}\}.$$

It follows from [Hunziker et al. 2012, Proposition 3.13] that the restriction map is a linear isomorphism and with the appropriate $\widetilde{G}$-action, $I'_{m,r} \cong \mathrm{Ind}_{Q^-}^{\widetilde{G}}(\widetilde{\chi}_{m,r})$ as $\widetilde{G}$-representations.

The action of the group $\widetilde{G}$ and of the corresponding Lie algebra $\mathfrak{g}$ on $I'_{m,r}$ are calculated in [Hunziker et al. 2012, Section 4]. We will record the actions of the Lie algebra for future use.

Proposition 1. *The elements of the Lie algebra $\mathfrak{g}$ act on $I'_{m,r}$ by*

$$\text{(2-3a)}\quad H_s = s(r - t\partial_t - x\partial_x^T),$$

$$\text{(2-3b)}\quad L_{A,b} = -bx^T\partial_t + (xA - tb)\partial_x^T,$$

$$\text{(2-3c)}\quad N_{s,y}^+ = -s\partial_t - y\partial_x^T, \text{ and}$$

$$\text{(2-3d)}\quad N_{s,y}^- = 2(st - yx^T)(r - t\partial_t - x\partial_x^T) - q(t,x)(s\partial_t + y\partial_x^T).$$

Proof. See [Hunziker et al. 2012]. □

A distinguished copy of $\mathfrak{so}(2,1) \cong \mathfrak{sl}_2(\mathbb{R})$ can be embedded in the upper left corner of $\mathfrak{g}$. A standard $\mathfrak{sl}_2$-basis for this Lie algebra is

$$\text{(2-4)}\quad H := H_2 \qquad E := N_{1,0}^+ \qquad F := N_{1,0}^-.$$

The difference of the Casimir element $\Omega_{\mathrm{SL}(2)}$ corresponding to this copy of $\mathfrak{sl}_2(\mathbb{R})$ with the Casimir element $\Omega_{\mathrm{SO}(n)}$ corresponding to $\mathfrak{so}(n)$ (embedded in the lower right corner) will play a special role in this article. The following corollary follows from Proposition 1.

Corollary 1. *The operator $\Omega_{\mathrm{SL}(2)} - \Omega_{\mathrm{SO}(n)}$ acts by*

$$\Omega_{\mathrm{SL}(2)} - \Omega_{\mathrm{SO}(n)} = \|x\|^2\square + (1 - n - 2r)\mathscr{E} + r(r+1)$$

on $I'_{m,r}$, where $\square$ is the wave operator on $\mathbb{R}^{1,n}$ and $\mathscr{E} := \sum_{i=1}^n x_i\partial_{x_i}$ is the Euler operator on $\mathbb{R}^n$.

2.4. ***Compact picture.*** In [Hunziker et al. 2012] it is shown that the space $I''_{m,r}$ of functions $F \in C^\infty(\mathbb{R} \times \mathbb{R} \times S^{n-1})$ that satisfy

$$\begin{aligned} F(\varphi, \theta+\pi, -\hat{x}) &= F(\varphi, \theta, \hat{x}) \\ F(\varphi+\pi, \theta+\pi, \hat{x}) &= i^{-m} F(\varphi, \theta, \hat{x}) \\ F(\varphi, 0, \hat{x}) &= F(\varphi, 0, \hat{x}') \end{aligned}$$

for all $\varphi, \theta \in R$ and $\hat{x}, \hat{x}' \in S^{n-1}$, is a $\widetilde{K}$-representation. Moreover, this representation is isomorphic to $\mathrm{Ind}_{Q^-}^{\widetilde{G}}(\widetilde{\chi}_{m,r})$, hence to $I'_{m,r}$. If $f \in I'_{m,r}$ and $F \in I''_{m,r}$ correspond under the canonical isomorphism between $I'_{m,r}$ and $I''_{m,r}$, then they are related by

$$F(\varphi, \theta, \hat{x}) = i^{mj} \left| \frac{\cos\varphi + \cos\theta}{2} \right|^r f\left(\frac{\sin\varphi}{\cos\varphi+\cos\theta}, \frac{\hat{x}\sin\theta}{\cos\varphi+\cos\theta} \right), \tag{2-5}$$

where j is given by

$$j = \begin{cases} 0 & \text{if } \cos\varphi - \cos\theta > 0 \quad \text{and} \quad \frac{\varphi}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \pmod{2\pi}, \\ 1 & \text{if } \cos\varphi - \cos\theta < 0 \quad \text{and} \quad \frac{\varphi}{2} \in (0, \pi) \pmod{2\pi}, \\ 2 & \text{if } \cos\varphi - \cos\theta > 0 \quad \text{and} \quad \frac{\varphi}{2} \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \pmod{2\pi}, \\ 3 & \text{if } \cos\varphi - \cos\theta < 0 \quad \text{and} \quad \frac{\varphi}{2} \in (\pi, 2\pi) \pmod{2\pi}, \end{cases}$$

and

$$f(t, x) = \lambda(t, x)^r F\left(\operatorname{sgn} t \cos^{-1} \frac{1+q(t,x)}{\lambda(t,x)}, \cos^{-1} \frac{1-q(t,x)}{\lambda(t,x)}, \frac{x}{\|x\|} \right) \tag{2-6}$$

where $\lambda(t, x) = (4t^2 + (1+q(t,x))^2)^{1/2}$ and $q(t, x) = -t^2 + \|x\|^2$.

Define a function $\widetilde{F} \in C^\infty(\mathbb{R} \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}))$ by

$$\widetilde{F}(\varphi, \theta, \hat{x}) := F\left(\varphi, \theta, \frac{x}{\|x\|}\right),$$

associated to $F \in C^\infty(\mathbb{R} \times \mathbb{R} \times S^{n-1})$, and also the derivative

$$\partial_{\hat{x}_i} F(\varphi, \theta, \hat{x}) = \partial_{x_i} F(\varphi, \theta, x)\big|_{x=\hat{x}}.$$

With this convention, the actions of the Lie algebra are given by the formulas in the following proposition.

Proposition 2. *The Lie algebra action of $\mathfrak{g}$ on $I''_{m,r}$ is given by*:

(2-7a) $$H_s = s(r\cos\theta\cos\varphi - \cos\theta\sin\varphi\partial_\varphi - \sin\theta\cos\varphi\partial_\theta),$$

(2-7b) $$\begin{aligned} L_{A,b} &= -bx^T(r\sin\varphi\sin\theta + \cos\varphi\sin\theta\partial_\varphi + \sin\varphi\cos\theta\partial_\theta) \\ &\quad + (\hat{x}A - \frac{\sin\varphi}{\sin\theta}b)\partial_{\hat{x}}^T, \end{aligned}$$

(2-7c) $$\begin{aligned} N^+_{s,y} &= -r(y\hat{x}^T\sin\theta\cos\varphi + s\cos\theta\sin\varphi) \\ &\quad + (y\hat{x}^T\sin\theta\sin\varphi - s(\cos\theta\cos\varphi + 1))\partial_\varphi \\ &\quad - (y\hat{x}^T(\cos\theta\cos\varphi + 1) - s\sin\theta\sin\varphi)\partial_\theta - \frac{\cos\varphi+\cos\theta}{\sin\theta}y\partial_{\hat{x}}^T, \end{aligned}$$

(2-7d) $$\begin{aligned} N^-_{s,y} &= -r(y\hat{x}^T\sin\theta\cos\varphi - s\cos\theta\sin\varphi) \\ &\quad + (y\hat{x}^T\sin\theta\sin\varphi + s(\cos\theta\cos\varphi - 1))\partial_\varphi \\ &\quad - (y\hat{x}^T(\cos\theta\cos\varphi - 1) + s\sin\theta\sin\varphi)\partial_\theta - \frac{\cos\varphi-\cos\theta}{\sin\theta}y\partial_{\hat{x}}^T \end{aligned}$$

Proof. See [Hunziker et al. 2012]. □

Now that we have introduced the spaces, mappings, and groups that we will use in the rest of the article, we can start studying the problem that concerns us.

3. Yamabe operator

Recall that the maximal compact group $\widetilde{K}$ is isomorphic to $\mathrm{SO}(2) \times \mathrm{SO}(n+1)$ with $\mathrm{SO}(2)$ embedded in the upper left corner and $\mathrm{SO}(n+1)$ embedded in the lower right corner. It will prove profitable to investigate the action of the Casimir operators associated to these groups.

Proposition 3. *When $r = (1-n)/2$, the element*

$$\Omega := \Omega_{\mathrm{SO}(2)} - \Omega_{\mathrm{SO}(n+1)} - r^2$$

of $\mathfrak{g}$ acts on $I''_{m,r}$ as the Yamabe operator $\widetilde{\Delta}_{S^1\times S^n}$ on the manifold $S^1 \times S^n$ as a subset of $\mathbb{R}^{2,n+1}$. In particular, when $r = (1-n)/2$, $\ker(\Omega - \mu) = \ker(\widetilde{\Delta}_{S^1\times S^n} - \mu)$ as subsets of $I''_{m,r}$, for any constant $\mu \in \mathbb{R}$.

Proof. The explicit form of the Yamabe operator on $S^1 \times S^n$ embedded in the Minkowski space $\mathbb{R}^{2,n+1}$ is calculated in [Kobayashi and Ørsted 2003a] and is equal to

$$\widetilde{\Delta}_{S^1\times S^n} = \Delta_{S^1} - \Delta_{S^n} - \frac{(n-1)^2}{4}.$$

The Casimir elements in the universal enveloping algebras of $\mathrm{SO}(2)$ and $\mathrm{SO}(n+1)$ act on $I''_{m,r}$ by

$$\Omega_{\mathrm{SO}(2)} = -\tfrac{1}{4}(N^+_{1,0} + N^-_{1,0})^2 = -\partial_\varphi^2$$

and

$$\Omega_{\mathrm{SO}(n+1)} = -\tfrac{1}{4}\sum_{i=1}^{n}(N^+_{0,e_i} - N^-_{0,e_i})^2 - \sum_{1\le i<j\le n}(L_{E_{i,j}-E_{j,i},0})^2$$
$$= -\sum_{i=1}^{n}(\hat{x}_i\partial_\theta + \cot\theta\,\partial_{\hat{x}_i})^2 - \sum_{1\le i<j\le n}(\hat{x}_i\partial_{\hat{x}_j} - \hat{x}_j\partial_{\hat{x}_i})^2 = -\Delta_{S^n}$$

by Equations (2-7) and the recursion formula

$$\Delta_{S^n} = \partial_\theta^2 + (n-1)\cot\theta\,\partial_\theta - \csc^2\theta\,\Delta_{S^{n-1}}.$$

Then, Ω acts on $I''_{m,r}$ by the Yamabe operator $\widetilde{\Delta}_{S^1\times S^n}$. Therefore, the solution space of the equation

$$\widetilde{\Delta}_{S^1\times S^n}F(\varphi,\theta,\hat{x}) = \mu F(\varphi,\theta,\hat{x})$$

is equal to $\ker(\Omega-\mu)$ in $I''_{m,r}$. □

For the rest of the article, unless otherwise specified, let

$$r = \frac{1-n}{2}.$$

We will now determine the maximal subgroup of $\widetilde{G}$ that leaves $\ker(\Omega-\mu)\subset I''_{m,r}$ invariant. To do that, we will use the fact that, on $I''_{m,r}$,

$$\Omega_{\mathrm{SL}(2)} - \Omega_{\mathrm{SO}(n)} - r(r+1) = \sin^2\theta\,(\Omega_{\mathrm{SO}(2)} - \Omega_{\mathrm{SO}(n+1)} - r^2)$$

(see [Hunziker et al. 2012]). From this, it follows that

$$\ker(\Omega_{\mathrm{SL}(2)} - \Omega_{\mathrm{SO}(n)} - r(r+1) - \mu\sin^2\theta) = \ker(\Omega-\mu)$$

when viewed as subspaces of $I''_{m,r}$. Using the canonical isomorphism between $I''_{m,r}$ and $I'_{m,r}$, more specifically (2-6), we obtain that

$$\ker(\Omega-\mu) = \ker\left(\Omega_{\mathrm{SL}(2)} - \Omega_{\mathrm{SO}(n)} - r(r+1) - \frac{4\mu\|x\|^2}{\lambda(t,x)^2}\right) = \ker(\lambda(t,x)^2\Box - 4\mu)$$

as subspaces of $I'_{m,r}$, where $\Box$ is the wave operator on $\mathbb{R}^{1,n}$. We will use this fact to show the invariance of $\ker(\Omega-\mu)$.

Proposition 4. *If $\mu\neq 0$, then the stabilizer of $\ker(\Omega-\mu)\subset I''_{m,r}$ in $\widetilde{G}$ is the maximal compact subgroup $\widetilde{K}$.*

Proof. Since $\widetilde{G}$ is connected, it suffices to find the maximal subalgebra of $\mathfrak{g}$ that leaves $\ker(\Omega-\mu)\subset I''_{m,r}$ invariant. Since $I'_{m,r}$ and $I''_{m,r}$ are isomorphic, it suffices to determine the maximal subalgebra that leaves $\ker(\lambda(t,x)^2\Box - 4\mu)\subset I'_{m,r}$ invariant.

A necessary and sufficient condition for $\ker(\lambda(t,x)^2\Box - 4\mu)$ to be invariant under the action of an element $X \in \mathfrak{g}$ is that

$$[X, \lambda(t,x)^2\Box - 4\mu] = \beta(t,x)(\lambda(t,x)^2\Box - 4\mu) \tag{3-1}$$

for some function $\beta(t,x)$. Straightforward calculations using the formulas in Proposition 1 give the following equations:

$$\begin{aligned}
[H_1, \lambda(t,x)^2\Box - 4\mu] &= 2(1 - q(t,x)^2)\Box,\\
[L_{E_{ij}-E_{ji},0}, \lambda(t,x)^2\Box - 4\mu] &= 0,\\
[L_{0,e_i}, \lambda(t,x)^2\Box - 4\mu] &= -8x_i t\Box,\\
[N^+_{1,0}, \lambda(t,x)^2\Box - 4\mu] &= 4t(q(t,x) - 1)\Box,\\
[N^+_{0,e_i}, \lambda(t,x)^2\Box - 4\mu] &= -4x_i(q(t,x) + 1)\Box,\\
[N^-_{1,0}, \lambda(t,x)^2\Box - 4\mu] &= 4t(q(t,x) - 1)\Box,\\
[N^-_{0,e_i}, \lambda(t,x)^2\Box - 4\mu] &= -4x_i(q(t,x) + 1)\Box.
\end{aligned}$$

From these equations, it can be seen that the condition (3-1) is only satisfied when $\beta(t,x) = 0$. Therefore, the maximal invariance subalgebra of $\ker(\Omega - \mu)$ is spanned by the set $\{L_{E_{ij}-E_{ji}} \mid 1 \le i < j \le n\} \cup \{N^+_{1,0} + N^-_{0,1}\} \cup \{N^+_{0,e_i} - N^-_{0,e_i} \mid 1 \le i \le n\}$ which is isomorphic to the maximal compact subalgebra $\mathfrak{k}$. This yields the desired result. $\Box$

It is worth remarking that in the noncompact picture,

$$\ker(\Omega - \mu) = \ker\Big(\Box - \frac{4\mu}{\lambda(t,x)^2}\Big)$$

Therefore, $\ker(\Omega - \mu)$ corresponds to the space of solutions of

$$-u_{tt} + \Delta_n u = \frac{4\mu}{\lambda(t,x)^2}u$$

in $I'_{m,r}$. In particular, $\ker\Omega$ corresponds to the space of solutions of the wave equation in $I'_{m,r}$.

4. $\widetilde{K}$-finite vectors

In this section we will determine the space $\widetilde{K}$-finite vectors of the representation $\ker(\Omega - \mu) \subset I''_{m,r}$. In order to do this, we will first determine the $\widetilde{K}$-finite vectors in $I''_{m,r}$ explicitly by using the following realization of $I''_{m,r}$:

$$I''_{m,r} \cong \{\phi \in C^\infty(S^1 \times S^n) \mid \phi(c \cdot w) = i^{-m}\phi(c) \text{ for every } c \in S^1 \times S^n\},$$

where

$$w = \begin{pmatrix} R_{\pi/2} & \\ & -I_{n+1} \end{pmatrix}.$$

Let the space of $\widetilde{K}$-finite vectors in $C^\infty(S^1 \times S^n)$ be denoted by $C^\infty(S^1 \times S^n)_{\widetilde{K}}$. Then it is well known that

$$C^\infty(S^1 \times S^n)_{\tilde{K}} \cong \bigoplus_{\substack{p,k\in\mathbb{Z}\\ k\geq 0}} \mathbb{C}e^{ip\varphi/2} \otimes \mathcal{H}_k(S^n).$$

Where $\mathcal{H}_k(S^n)$ denotes the space of homogeneous harmonic polynomials of degree k on $\mathbb{R}^{n+1}$ restricted to S^n. Then,

$$\mathcal{H}_k(S^n) = \{h \in C^\infty(S^n) \mid \Omega_{\mathrm{SO}(n+1)} h = k(k+n-1)h\}.$$

Since $\phi \in I''_{m,r}$ must satisfy $\phi(c \cdot w) = i^{-m}\phi(c)$, the space of $\widetilde{K}$ finite vectors in $I''_{m,r}$ is given by

$$(I''_{m,r})_{\widetilde{K}} \cong \bigoplus_{\substack{(p,k)\in\mathbb{Z}\times\mathbb{Z}^{\geq 0}\\ p+2k\equiv -m \pmod 4}} \mathbb{C}e^{ip\varphi/2} \otimes \mathcal{H}_k(S^n). \tag{4-1}$$

Proposition 5. *With* $r = (1-n)/2$, *let* $\ker(\Omega - \mu)_{\widetilde{K}}$ *denote the space of* $\widetilde{K}$*-finite vectors in* $\ker(\Omega - \mu) \subset I''_{m,r}$. *Then,*

$$\ker(\Omega - \mu)_{\widetilde{K}} \cong \bigoplus_{\substack{(p,k)\in\mathbb{Z}\times\mathbb{Z}^{\geq 0}\\ p+2k\equiv -m \pmod 4\\ \mu=(p/2)^2-(k-r)^2}} \mathbb{C}e^{ip\frac{\varphi}{2}} \otimes \mathcal{H}_k(S^n).$$

Proof. From the decomposition (4-1) and the well-known fact that $\Omega_{\mathrm{SO}(n)}$ acts on $\mathcal{H}_k(S^n)$ by $k(k-1+n)$, it is easy to see that the operator $\Omega - \mu$ acts on $\mathbb{C}e^{ip\varphi/2} \otimes \mathcal{H}_k(S^n)$ by

$$\left(\frac{p}{2}\right)^2 - k(k-2r) - r^2 = \left(\frac{p}{2}\right)^2 - (k-r)^2.$$

The proposition follows from this and (4-1). $\square$

The following lemma is proved in [Hunziker et al. 2012] and will be used to write a basis for $(I''_{m,r})_{\widetilde{K}}$ explicitly.

Lemma 1. *Let* $\mathrm{SO}(n) \subset \mathrm{SO}(n+1)$ *be the stabilizer of* $(\pm 1, 0, \ldots, 0) \in S^n$. *Then, as an* $\mathrm{SO}(n)$*-module,*

$$\mathcal{H}_k(S^n) \cong \bigoplus_{l=0}^{k} \mathcal{H}_l(S^{n-1}),$$

where the isomorphism is given by

$$(h_0(\hat{x}), \dots, h_k(\hat{x})) \mapsto \sum_{l=0}^{k} \widetilde{C}_{k-l}^{(l-r)}(\cos\theta)\sin^l\theta\, h_l(\hat{x}),$$

where $r = (1-n)/2$ and $\widetilde{C}_d^{(\lambda)}$ is the degree d normalized Gegenbauer polynomial with parameter λ.

Let $\{h_{l,j}\}$ be a basis for the homogeneous harmonic polynomials on $\mathbb{R}^n$ of degree l such that when restricted to S^{n-1} they form an orthonormal basis for $L^2(S^{n-1})$. Then, Lemma 1 implies that the functions of the form

$$F_{p,d,l,j}(\varphi,\theta,\hat{x}) := e^{ip/2\varphi} C_d^{(l-r)}(\cos\theta)\sin^l\theta\, h_{l,j}(\hat{x}) \tag{4-2}$$

with $p, d, l, j \in \mathbb{Z}$, and $d, l, j \geq 0$ such that $p + 2(d+l) \equiv -m \pmod 4$, form a basis of $(I''_{m,r})_{\widetilde{K}}$, where $r = (1-n)/2$. For later use, we note that if we define

$$k := d + l,$$

then the function $C_d^{(l-r)}(\cos\theta)\sin^l\theta\, h_{l,j}(\hat{x}) \in \mathscr{H}_k(S^n)$.

By Proposition 5 we know that $F_{p,d,l,j}$ is an eigenvector for Ω with eigenvalue

$$\mu_{p,d,l} := \left(\frac{p}{2}\right)^2 - \left(\frac{2d+2l-n+1}{2}\right)^2.$$

Definition. We will say that $\mu \in \mathbb{R}$ is an *admissible* eigenvalue of Ω if and only if $\mu = \mu_{p,d,l}$ for some $p, d, l \in \mathbb{Z}$ such that $d \geq 0$, $l \geq 0$, and $p + 2(d+l) \equiv -m \pmod 4$. Let S be the set of admissible eigenvalues of Ω.

Then, it is clear that

$$(I''_{m,r})_{\widetilde{K}} \cong \bigoplus_{\mu \in S} \ker(\Omega - \mu)_{\widetilde{K}}.$$

Since $(I''_{m,r})_{\widetilde{K}}$ has the structure of a $(\mathfrak{g}, \widetilde{K})$-module, the direct sum of all eigenspaces of Ω with admissible eigenvalues does too, and as such it can be completed to a representation of $\widetilde{G}$.

One last note to close this section is to contrast the functions $F_{p,l,j}$ constructed in [Hunziker et al. 2012] and the functions $F_{p,l,j,d}$ that we just defined. Even though they are very similar, the main difference is that here we are allowing the parameter $d \geq 0$ to vary, in contrast with $F_{p,l,j}$ where the parameter $d = |p|/2 + r - l$ was fixed for each choice of l. In a sense, the introduction of the parameter d is counting for the fact that different eigenvalues $\mu_{p,d,l} \neq 0$ of the Yamabe operator are being admitted. In [loc. cit.], the only eigenvalue that was considered admissible was $\mu = 0$, in this sense we are generalizing their result.

5. Structure theorems

In this section, we will study the structure of $\oplus_{\mu\in S}\ker(\Omega-\mu)_{\widetilde{K}}$ as a representation of $\mathfrak{g}$. To do this, we will introduce a new basis $\{\kappa, e^+, e^-\}$ for the complexification of the distinguished Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ introduced in Section 2.3. This basis is defined by

$$\kappa := i(E-F), \quad e^+ := \tfrac{1}{2}(H-i(E+F)), \quad e^- := \tfrac{1}{2}(H+i(E+F)). \tag{5-1}$$

Using (2-4) and (2-7) it can be shown that this $\mathfrak{sl}(2,\mathbb{C})$-triple acts on $I''_{m,r}$ by

$$\kappa = -2i\,\partial_\varphi \qquad\qquad e^\pm = e^{\pm i\varphi}(r\cos\theta \pm i\cos\theta\,\partial_\varphi - \sin\theta\,\partial_\theta). \tag{5-2}$$

Lemma 2. *The differential operators e^+ and e^- on $I''_{m,r}$ can be written as linear combinations of the form $e^\pm = A^\pm + B^\pm$ such that $A^\pm F_{p,d,l,j}$ and $B^\pm F_{p,d,l,j}$ are all eigenvectors of Ω.*

Proof. Notice that $\Omega_{\mathrm{SO}(2)}$ and $e^\pm$ commute. Since $F_{p,d,l,j}$ is an eigenvector for $\Omega_{\mathrm{SO}(2)}$, so are $e^\pm F_{p,d,l,j}$. Moreover, if $A^\pm$ and $B^\pm$ are a linear combinations of $\{r\cos\theta, \cos\theta\,\partial_\varphi, \sin\theta\,\partial_\theta\}$, then $A^\pm F_{p,d,l,j}$ and $B^\pm F_{p,d,l,j}$ are all eigenvectors of $\Omega_{\mathrm{SO}(2)}$. So, by the definition of Ω, it suffices to determine $A^\pm$ and $B^\pm$ so that $A^\pm F_{p,d,l,j}$ and $B^\pm F_{p,d,l,j}$ are eigenvectors of $\Omega_{\mathrm{SO}(n+1)}$.

Now, $A^\pm F_{p,d,l,j}$ and $B^\pm F_{p,d,l,j}$ are eigenvectors of $\Omega_{\mathrm{SO}(n+1)}$ if and only if $F_{p,d,l,j}$ is an eigenvector of $[\Omega_{\mathrm{SO}(n+1)}, A^\pm]$ and of $[\Omega_{\mathrm{SO}(n+1)}, B^\pm]$. Writing these conditions out explicitly gives the following form for the operators $A^\pm$ and $B^\pm$:

$$A^\pm = e^{\pm i\varphi}\Big(\frac{r}{2}\cos\theta - \frac{4r^2+2(d+l)(2(d+l)\mp p-4r)}{4p(d+l-r)} i\cos\theta\,\partial_\varphi - \frac{2(d+l)\mp p-2r}{4((d+l)-r)}\sin\theta\,\partial_\theta\Big) \tag{5-3}$$

and

$$B^\pm = e^{\pm i\varphi}\Big(\frac{r}{2}\cos\theta + \frac{4r(r\mp p)+2(d+l)(2(d+l)\pm p-4r)}{4p(d+l-r)} i\cos\theta\,\partial_\varphi - \frac{2(d+l)\pm p-2r}{4(d+l-r)}\sin\theta\,\partial_\theta\Big). \tag{5-4}$$

These operators satisfy the required conditions. □

By making the change of variables $s=\cos\theta$ and considering the fact that $\partial_\varphi F_{p,d,l,j} = ip/2$, when restricted to the basis elements

$$F_{p,d,l,j}(\varphi, s, \hat{x}) = e^{ip\varphi/2}(1-s^2)^{l/2} C_d^{(l-r)}(s)\, h_{l,j}(\hat{x}),$$

the operators in (5-3) and (5-4) act by

$$A^{\pm} = \frac{\mp p+2(d+l-r)}{4(d+l-r)} e^{\pm i\varphi}((d+l)s+(1-s^2)\partial_s) \tag{5-5}$$

and

$$B^{\pm} = \frac{\pm p-2(d+l-r)}{4(d+l-r)} e^{\pm i\varphi}((2r-d-l)s+(1-s^2)\partial_s), \tag{5-6}$$

respectively. We are now in a position to calculate the actions of $e^{\pm}$ on the functions $F_{p,d,l,j}(\varphi, s, \hat{x})$. We start with the following proposition.

Proposition 6. *Let $F_{p,d,l,j}$ be defined as in (4-2) and let $A^{\pm}$ and $B^{\pm}$ be defined by (5-3) and (5-4) respectively. Then,*

$$A^{\pm}F_{p,d,l,j} = \frac{\mp p+2(d+l-r)}{4(d+l-r)}(d+2(l-r)-1)F_{p\pm 2,d-1,l,j} \tag{5-7}$$

and

$$B^{\pm}F_{p,d,l,j} = \frac{\mp p-2(d+l-r)}{4(d+l-r)}(d+1)F_{p\pm 2,d+1,l,j}. \tag{5-8}$$

Proof. Firstly, we will need to state two well-known identities for the Gegenbauer polynomials:

$$(1-s^2)\frac{d}{ds}C_d^{(\lambda)}(s) = -dsC_d^{(\lambda)}(s)+(d+2\lambda-1)C_{d-1}^{(\lambda)}(s) \tag{5-9}$$

and

$$(1-s^2)\frac{d}{ds}C_d^{(\lambda)}(s) = (d+2\lambda)sC_d^{(\lambda)}(s)-(d+1)C_{d+1}^{(\lambda)}(s) \tag{5-10}$$

(see [Abramowitz and Stegun 1964, Formulas 22.8.2 and 22.7.3]). Now, (5-7) follows from (5-5) and (5-9), and analogously (5-8) is obtained by combining (5-6) and (5-10). □

Corollary 2. *Let $F_{p,d,l,j}$ be defined as in (4-2). Then,*

$$\begin{aligned} e^{+}F_{p,d,l,j} = \frac{-p+2(d+l-r)}{4(d+l-r)}&(d+2(l-r)-1)F_{p+2,d-1,l,j} \\ &+\frac{-p-2(d+l-r)}{4(d+l-r)}(d+1)F_{p+2,d+1,l,j} \end{aligned} \tag{5-11}$$

and

$$\begin{aligned} e^{-}F_{p,d,l,j} = \frac{p+2(d+l-r)}{4(d+l-r)}&(d+2(l-r)-1)F_{p-2,d-1,l,j} \\ &+\frac{p-2(d+l-r)}{4(d+l-r)}(d+1)F_{p-2,d+1,l,j}, \end{aligned} \tag{5-12}$$

with the convention that $F_{p,-1,l,j} \equiv 0$.

Proof. The corollary follows at once from Proposition 6. □

Now we can read much information from the coefficients in (5-11) and (5-12). Firstly, by definition $d, l \geq 0$ and $r < 0$, so $d + l - r > 0$. Therefore, these equations are well defined for every $F_{p,d,l,j}$ as in (4-2). Secondly, we can look for highest/lowest weight vectors for the action of $\mathfrak{sl}(2, \mathbb{C})$. In order for $F_{p,d,l,j}$ to be a highest or lowest weight vector, we would need either e^+ or e^- to annihilate $F_{p,d,l,j}$ respectively. By inspecting the coefficients we conclude that this can only occur whenever $p = 2(d+l-r)$ or $p = -2(d+l-r)$. However, this can occur only if $\mu = 0$. In this case, the actions of $e^\pm$ would have only one term depending on the sign of p. This corresponds with the action calculated in [Hunziker et al. 2012]. Moreover, the highest and lowest weight vectors are precisely the ones calculated therein.

In more detail, if $p = -2(d+l-r)$, then

$$e^+ F_{-2(d+l-r),d,l,j} = \frac{-p+2(d+l-r)}{4(d+l-r)}(d+2(l-r)-1)F_{-2(d+l-r-1),d-1,l,j}.$$

In particular,

$$e^+ F_{-2(l-r),0,l,j} = 0.$$

Similarly,

$$e^- F_{2(l-r),0,l,j} = 0$$

for $l, j \in \mathbb{Z}^{\geq 0}$. So, for each combination of parameters $l, j \in \mathbb{Z}^{\geq 0}$ there exists a highest weight $\mathfrak{sl}(2, \mathbb{C})$-module in $(I''_{m,r})_{\widetilde{K}}$ with highest weight vector $F_{-2(l-r),0,l,j}$. There also exists a lowest weight module with lowest weight vector $F_{2(l-r),0,l,j}$.

In Figure 1 we show the projection to the plane $l = 0$ of the representation $(I''_{m,r})_{\widetilde{K}}$ pictorially. However, the pictorial representation of all of $(I''_{m,r})_{\widetilde{K}}$ would require a third axis for the l parameter. In this picture, the span of highest weight vector $F_{2r,0,0,0}$ of $(\mathcal{H}^-)_{\widetilde{K}}$ corresponds to the point $(0, 2r)$. For a fixed $l \in \mathbb{Z}^{\geq 0}$, the span of the highest weight vectors $F_{-2(l-r),0,l,j}$ corresponds to the dot in the position $(0, -2(l-r))$ in that particular plane. To illustrate, Figure 2 shows the projection onto the plane $l = 1$. The highest weight vectors there are $F_{2r-2,0,1,j}$ and represented by the point $(0, 2r-2)$.

In Figure 1 we also describe the actions of $\mathfrak{sl}(2, \mathbb{C})$. In this figure, a dot at the (d, p) coordinate represents the span of $\{F_{p,d,l,j} \mid l, j \in \mathbb{Z}^{\geq 0}\}$. As shown in Corollary 2, the action of e^+ sends a multiple of $F_{p,d,l,j}$ into the span of $F_{p+2,d\pm1,l,j}$, thus the northeast and northwest arrows. Similarly, the action of e^- sends a multiple of $F_{p,d,l,j}$ into the span of $F_{p-2,d\pm1,l,j}$, hence the arrows pointing in the southeast and southwest directions. Lastly, the semisimple element κ acts by the scalar p on $F_{p,d,l,j}$, thus leaving each point fixed.

To finish this analysis we introduce two spaces that correspond to the positive and negative energy representations for the zero eigenspace. These representations

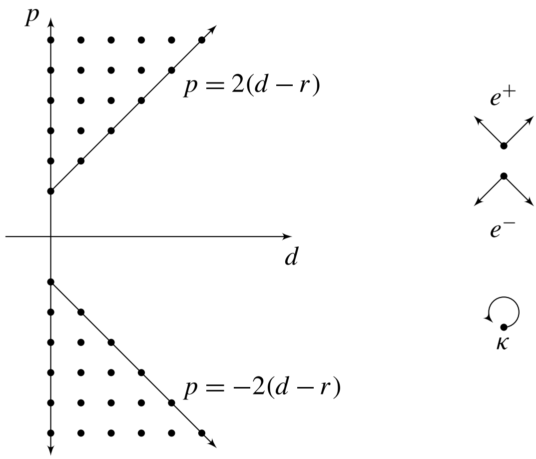

Figure 1. Summary of the action of $\mathfrak{sl}(2, \mathbb{C})$ on $(I''_{m,r})_{\widetilde{K}}$ and $l = 0$.

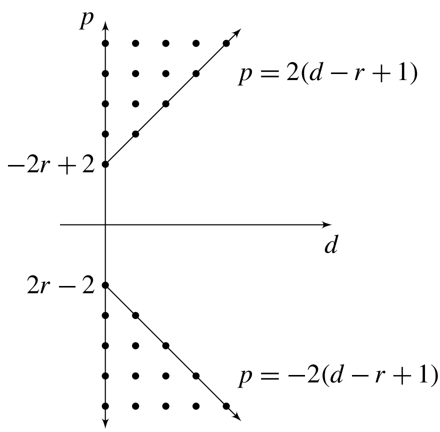

Figure 2. Pictorial representation of the plane $l = 1$ of $(I''_{m,r})_{\widetilde{K}}$.

are related to positive and negative energy solutions of the wave equation. For more information in this direction, see [Hunziker et al. 2012]. Let

$$(\mathcal{H}^-)_{\widetilde{K}} := \bigoplus_{\substack{p\in\mathbb{Z},k\in\mathbb{Z}^{\geq 0}\\ p=-2(k-r)\\ p+2k\equiv -m \pmod 4}} \mathbb{C}e^{ip\varphi/2} \otimes \mathcal{H}_k(S^n)$$

and

$$(\mathcal{H}^+)_{\widetilde{K}} := \bigoplus_{\substack{p\in\mathbb{Z},k\in\mathbb{Z}^{\geq 0}\\ p=2(k-r)\\ p+2k\equiv -m \pmod 4}} \mathbb{C}e^{ip\varphi/2} \otimes \mathcal{H}_k(S^n).$$

In the pictorial representation, $(\mathcal{H}^-)_{\widetilde{K}}$ would live in the $p=-2(d+l-r)$ plane and $(\mathcal{H}^+)_{\widetilde{K}}$ would live in the plane $p=2(d+l-r)$. Each projection onto a fixed l would look essentially the same as Figure 1, with the intercepts at $\pm 2(l-r)=\pm(2l-n+1)$.

So far we have studied the structure of $(I''_{m,r})_{\widetilde{K}}$ as an $\mathfrak{sl}(2,\mathbb{C})$-module. To analyze the structure of $(I''_{m,r})_{\widetilde{K}}$ as a $(\mathfrak{g}_{\mathbb{C}}, \widetilde{K})$-module, we start by fixing a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{k}_{\mathbb{C}}=\mathfrak{so}(2)\times\mathfrak{so}(n+1)$ given by

$$\mathfrak{h}_{\mathbb{C}} := \{H_{h_1,\dots,h_\ell} \mid h_1,\dots,h_\ell\in\mathbb{C}\},$$

where

$$H_{h_1,\dots,h_\ell} = \left(\begin{array}{cc|cccccc} 0 & ih_0 & & & & & & \\ -ih_0 & 0 & & & & & & \\ \hline & & \ddots & & & & & \\ & & & 0 & ih_2 & & & \\ & & & -ih_2 & 0 & & & \\ & & & & & & 0 & ih_1 \\ & & & & & & -ih_1 & 0 \end{array}\right)$$

where $\ell=[(n+1)/2]$. Let $\epsilon_j : \mathfrak{h}_{\mathbb{C}}\to\mathbb{C}$ be the functional such that $\epsilon_j(H)=h_j$. As it turns out, the roots $\epsilon_0+\epsilon_1$ and $\epsilon_0-\epsilon_1$ are the noncompact simple root and the highest root respectively. The corresponding root vectors $X_{\epsilon_0\pm\epsilon_1}$ are given by

$$X_{\epsilon_0\pm\epsilon_1} := L_{0,e_{n-1}\mp ie_n} + \tfrac{1}{2}(N^+_{0,-ie_{n-1}\mp e_n} + N_{0,-ie_{n-1}\mp e_n}) \in \mathfrak{p}^+$$

and the complex conjugates $\overline{X}_{\epsilon_0\pm\epsilon_1}$ are the root vectors for the respective negative roots. The proof of [Hunziker et al. 2012, Proposition 9.6] implies that for $k\ge 0$

$$(X_{\epsilon_0\pm\epsilon_1})^k\cdot F_{-2r,0,0,0}\in\operatorname{span}\{F_{-2r+2k,0,k,j}\mid j\in\mathbb{Z}^{\ge 0}\}$$

and

$$(\overline{X}_{\epsilon_0\pm\epsilon_1})^k\cdot F_{2r,0,0,0}\in\operatorname{span}\{F_{2r-2k,0,k,j}\mid j\in\mathbb{Z}^{\ge 0}\}.$$

Moreover, they show that the functions $(X_{\epsilon_0+\epsilon_1})^k F_{-2r,0,0,0}$ are $\mathfrak{k}_{\mathbb{C}}$-lowest weight vectors and $(\overline{X}_{\epsilon_0-\epsilon_1})^k F_{2r,0,0,0}$ are $\mathfrak{k}_{\mathbb{C}}$-lowest weight vectors. Since these vectors are also annihilated by e^+ and e^- respectively, the vectors $F_{-2r,0,0,0}$ and $F_{2r,0,0,0}$ are $\mathfrak{g}_{\mathbb{C}}$-lowest and highest weight vectors respectively. Intuitively, in the pictorial representation of $(I''_{m,r})_{\widetilde{K}}$ these root vectors $X_{\epsilon_0\pm\epsilon_1}$ and $\overline{X}_{\epsilon_0\pm\epsilon_1}$ allow us to move from one l plane to the preceding $l-1$ and superseding $l+1$ planes.

Putting all this information together, we obtain our main result.

Theorem 1. *Let $r=(1-n)/2$. Let $(I''_{m,r})_{\widetilde{K}}$ denote the space of $\widetilde{K}$-finite vectors of $I''_{m,r}$ and let*

$$(I''_{m,r})^{\pm}_{\widetilde{K}} := \bigoplus_{\substack{\pm p\in\mathbb{Z}^{>0}\\ k\in\mathbb{Z}^{\ge 0}\\ p+2k\equiv -m \ (\mathrm{mod}\ 4)}} \mathbb{C}e^{ip\varphi/2}\otimes\mathcal{H}_k(S^n).$$

Then, as $(\mathfrak{g}_{\mathbb{C}}, \widetilde{K})$*-modules:*

(1) *The submodules* $(\mathcal{H}^{\pm})_{\widetilde{K}}$ *are irreducible lowest/highest weight modules with weight vectors* $F_{\pm 2r,0,0,0}$ *respectively.*

(2) *The quotient modules* $(I''_{m,r})^{+}_{\widetilde{K}}/(\mathcal{H}^{+})_{\widetilde{K}}$ *and* $(I''_{m,r})^{-}_{\widetilde{K}}/(\mathcal{H}^{-})_{\widetilde{K}}$ *are irreducible lowest/highest weight modules with the weight vectors being the cosets corresponding to* $F_{-2r+2,0,0,0}$ *and* $F_{2r-2,0,0,0}$.

(3) *The following is a composition series of* $(I''_{m,r})_{\widetilde{K}}$*:*

$$\{0\} \subset (\mathcal{H}^{\pm})_{\widetilde{K}} \subset (\mathcal{H}^{-})_{\widetilde{K}} \oplus (\mathcal{H}^{+})_{\widetilde{K}} \subset (I''_{m,r})_{\widetilde{K}}.$$

Proof. The only statement left to be shown is (2), as this implies the composition series in (3). This statement follows from the fact that for $k > 0$ the nonzero vectors

$$(X_{\epsilon_0 \pm \epsilon_1})^k \cdot F_{-2r+2,0,0,0} \in \operatorname{span}\{F_{-2(r-k-1),0,k,j} \mid j \in \mathbb{Z}^{\geq 0}\}$$

and

$$(\overline{X}_{\epsilon_0 \pm \epsilon_1})^k \cdot F_{2r-2,0,0,0} \in \operatorname{span}\{F_{2(r-k-1),0,k,j} \mid j \in \mathbb{Z}^{\geq 0}\},$$

which is proved using the explicit actions of $X_{\epsilon_0 \pm \epsilon_1}$ and $\overline{X}_{\epsilon_0 \pm \epsilon_1}$ and induction. □

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