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# ON THE DEGREE OF CERTAIN LOCAL $L$-FUNCTIONS

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**Let $\pi$ be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$, where $F$ is a $p$-adic field. By a result of Bushnell and Kutzko, the group of unramified self-twists of $\pi$ has cardinality $n/e$, where $e$ is the $\mathfrak{o}_F$-period of the principal $\mathfrak{o}_F$-order in $M_n(F)$ attached to $\pi$. This is the degree of the local Rankin–Selberg $L$-function $L(s, \pi \times \pi^\vee)$. In this paper, we compute the degree of the Asai, symmetric square, and exterior square $L$-functions associated to $\pi$. As an application, assuming $p$ is odd, we compute the conductor of the Asai lift of a supercuspidal representation, where we also make use of the conductor formula for pairs of supercuspidal representations due to Bushnell, Henniart, and Kutzko (1998).**

## 1. Introduction

Let $F$ be a $p$-adic field. Let $\mathfrak{o}_F$ denote its ring of integers and let $\mathfrak{p}_F$ be the unique maximal ideal of $\mathfrak{o}_F$. Let $q$ denote the cardinality of the residue field $\mathfrak{o}_F/\mathfrak{p}_F$. Let $W'_F$ denote the Weil–Deligne group of $F$. For a reductive algebraic group $G$ defined over $F$, let ${}^LG$ be its Langlands dual. Given a Langlands parameter $\rho : W'_F \to {}^LG$ and a finite-dimensional representation $r : {}^LG \to \mathrm{GL}(V)$, we have an $L$-function $L(s, \rho, r)$ defined as follows. If $N$ is the nilpotent endomorphism of $V$ associated to $r \circ \rho$, then

$$L(s, \rho, r) = \frac{1}{\det\bigl(1 - (r \circ \rho)(\mathrm{Frob})|_{(\mathrm{Ker}\, N)^I}\, q^{-s}\bigr)}$$

where Frob is the geometric Frobenius and $I$ is the inertia subgroup of the Weil group of $F$. Thus, $L(s, \rho, r) = P(q^{-s})^{-1}$ for some polynomial $P(X)$ with $P(0) = 1$, and by the degree of $L(s, \rho, r)$ we mean the degree of $P(X)$. If $\pi = \pi(\rho)$ denotes the $L$-packet of irreducible admissible representations of $G(F)$ corresponding to $\rho$ under the conjectural Langlands correspondence, then its Langlands $L$-function, denoted by $L(s, \pi, r)$, is expected to coincide with $L(s, \rho, r)$. In many cases, candidates for $L(s, \pi, r)$ can also be obtained either via the Rankin–Selberg method

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of integral representations or by the Langlands–Shahidi method, and in several instances it is known that all these approaches lead to the same $L$-function [Shahidi 1984; 1990; Anandavardhanan and Rajan 2005; Henniart 2010; Matringe 2011; Kewat and Raghunathan 2012].

Let $G = \mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$. If $\pi_i$ is an irreducible admissible representation of $\mathrm{GL}_{n_i}(F)$ $(i = 1, 2)$, and if $r$ is the tensor product representation of ${}^L G = \mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C})$ on $\mathbb{C}^n \otimes \mathbb{C}^n$ given by $r((a, b)) \cdot (x \otimes y) = ax \otimes by$, then the resulting $L$-function is the Rankin–Selberg $L$-function $L(s, \pi_1 \times \pi_2)$ [Jacquet et al. 1983; Shahidi 1984]. If we assume that both $\pi_1$ and $\pi_2$ are supercuspidal representations, then we know that $L(s, \pi_1 \times \pi_2) \equiv 1$ unless $n_1 = n_2$ and $\pi_2^\vee \cong \pi_1 \otimes \chi \circ \det$ for an unramified character $\chi$ of $F^\times$. Here, $\pi^\vee$ denotes the representation contragredient to $\pi$. Moreover, in the latter case, the degree of $L(s, \pi_1 \times \pi_2)$ is equal to the degree of $L(s, \pi_1 \times \pi_1^\vee)$, which in turn equals the cardinality of the group

$$\left\{\eta : F^\times \to \mathbb{C}^\times \mid \pi_1 \otimes \eta \circ \det \cong \pi_1,\ \eta \text{ unramified}\right\}.$$

The result of Bushnell and Kutzko mentioned in the abstract computes the cardinality of the above group of unramified self-twists of $\pi = \pi_1$ [Bushnell and Kutzko 1993, Lemma 6.2.5]. In order to state the result, let $[\mathfrak{A}, m, 0, \beta]$ be the simple stratum defining a maximal simple type occurring in the irreducible supercuspidal representation $\pi$. Here, $\mathfrak{A}$ is a principal $\mathfrak{o}_F$-order in $M_n(F)$, $m \geq 0$ is an integer called the level of $\pi$, and $\beta \in M_n(F)$ is such that $F[\beta]$ is a field with $F[\beta]^\times$ normalizing $\mathfrak{A}$. Let $e = e(\mathfrak{A}|\mathfrak{o}_F)$ be the $\mathfrak{o}_F$-period of $\mathfrak{A}$; this quantity in fact equals the ramification index $e(F[\beta]/F)$ of $F[\beta]/F$. Then $e$ divides $n$, and the cardinality of the group of unramified self-twists of $\pi$ is $n/e$. We mention in passing that the level $m$ of $\pi$ is related to the conductor $f(\pi)$ of $\pi$ by $f(\pi) = n(1 + m/e)$.

The aim of the present work is to analogously compute the degree of some other local $L$-functions in the supercuspidal case. Investigating the supercuspidal case would suffice as the $L$-function of any irreducible admissible representation can usually be built out of $L$-functions associated to supercuspidal representations. The $L$-functions that we study in this paper are the Asai $L$-function, the symmetric square $L$-function, and the exterior square $L$-function.

For the Asai $L$-function, take $G = \mathrm{Res}_{E/F}\,\mathrm{GL}(n)$, the Weil restriction of $\mathrm{GL}(n)$, where $E$ is a quadratic extension of $F$. Thus, $G(F) = \mathrm{GL}_n(E)$. In this case, the dual group is ${}^L G = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$, where the nontrivial element $\sigma$ of the Galois group $\mathrm{Gal}(E/F)$ acts by $\sigma \cdot (a, b) = (b, a)$. The representation $r$ is the Asai representation, also known as the twisted tensor representation, of ${}^L G$ on $\mathbb{C}^n \otimes \mathbb{C}^n$ given by $r((a, b)) \cdot (x \otimes y) = ax \otimes by$ and $r(\sigma) \cdot (x \otimes y) = y \otimes x$. The Asai $L$-function can be studied both by the Rankin–Selberg method (see [Flicker 1993, Appendix; Kable 2004]) and by the Langlands–Shahidi method [Shahidi 1990]. It is also known that all three definitions match [Anandavardhanan and Rajan 2005; Henniart 2010; Matringe 2011].

For the symmetric square $L$-function (resp. the exterior square $L$-function), take $G = \mathrm{GL}(n)$ and let $r$ be the symmetric square (resp. the exterior square) of the standard representation of ${}^L G = \mathrm{GL}_n(\mathbb{C})$. The Langlands–Shahidi theory of these $L$-functions is satisfactorily understood [Shahidi 1990; 1992] and this definition is known to match with the one via the Langlands formalism [Henniart 2010]. For the Rankin–Selberg theory of these $L$-functions, we refer to [Jacquet and Shalika 1990; Bump and Ginzburg 1992; Kewat and Raghunathan 2012].

These $L$-functions are ubiquitous in number theory and the degree of $L(s, \pi, r)$ often has several meaningful and important interpretations. For instance, these $L$-functions detect functorial lifts from classical groups. In particular, by the work of Shahidi [1992] and Goldberg [1994], for an irreducible supercuspidal representation $\pi$, the degree of $L(s, \pi, r)$ is either the number of unramified twists or half the number of unramified twists of $\pi$ which are functorial lifts from classical groups (see [Shahidi 1992, Theorem 7.7] and [Goldberg 1994, Theorems 5.1 and 5.2]). We refer to Section 2 for some more details in this regard. Since reducibility of parabolic induction is understood in terms of poles of these $L$-functions, the degree of $L(s, \pi, r)$ when $\pi$ is self-dual if $r = \mathrm{Sym}^2$ or $\bigwedge^2$, or when $\pi$ is conjugate self-dual if $r = \mathrm{Asai}$, counts the number of unramified twists or half the number of unramified twists of $\pi$ such that the parabolically induced representation to the relevant classical group is irreducible (see [Shahidi 1992, Theorem 7.6] and [Goldberg 1994, Theorem 6.5]).

These $L$-functions are also related to the theory of distinguished representations. If $\pi$ is a supercuspidal representation of $\mathrm{GL}_n(E)$, then the degree of its Asai $L$-function is the number of unramified characters $\mu$ of $F^\times$ for which $\pi$ is $\mu$-distinguished with respect to $\mathrm{GL}_n(F)$ (see [Anandavardhanan et al. 2004, Corollary 1.5]). Similarly, if $\pi$ is a supercuspidal representation of $\mathrm{GL}_n(F)$, the degree of its exterior square $L$-function is half the number of unramified characters $\mu$ of $F^\times$ such that $\pi \otimes \mu \circ \det$ admits a Shalika functional (see [Jiang et al. 2008, Theorem 5.5]).

Our main theorem computes the degree of $L(s, \pi, r)$, when $\pi$ is a supercuspidal representation, in terms of the simple stratum $[\mathfrak{A}, m, 0, \beta]$ defining a maximal simple type occurring in the irreducible supercuspidal representation $\pi$. Note that $\pi$ is a supercuspidal representation of $\mathrm{GL}_n(E)$, with $E/F$ a quadratic extension, in the Asai case, whereas otherwise it is a supercuspidal representation of $\mathrm{GL}_n(F)$. As before, let $e$ denote the $\mathfrak{o}$-period of $\mathfrak{A}$ where $\mathfrak{o} = \mathfrak{o}_E$ in the Asai case and $\mathfrak{o} = \mathfrak{o}_F$ otherwise.

Let $\omega = \omega_{E/F}$ be the quadratic character of $F^\times$ associated to the extension $E/F$ and let $\kappa$ be an extension of $\omega$ to $E^\times$. For the purposes of this paper, let us say that a supercuspidal representation, and more generally a discrete series representation, $\pi$ of $\mathrm{GL}_n(E)$ is distinguished (resp. $\omega$-distinguished) if its Asai $L$-function $L(s,\pi,r)$ (resp. $L(s,\pi\otimes\kappa,r)$) has a pole at $s=0$. Strictly speaking, this is not how distinction is usually defined, but the property above does characterize distinction for the pair $(\mathrm{GL}_n(E),\mathrm{GL}_n(F))$ (see [Anandavardhanan et al. 2004, Corollary 1.5]). It follows that a supercuspidal representation, and more generally a discrete series representation, cannot be both distinguished and $\omega$-distinguished because of the identity

$$L(s,\pi\times\pi^\sigma)=L(s,\pi,r)L(s,\pi\otimes\kappa,r).$$

Here, $\sigma$ is the nontrivial element of the Galois group $\mathrm{Gal}(E/F)$.

Recall also that a supercuspidal representation $\pi$, and more generally a discrete series representation, of $\mathrm{GL}_n(F)$ which is self-dual is said to be orthogonal (resp. symplectic) if its symmetric square $L$-function $L(s,\pi,\mathrm{Sym}^2)$ (resp. its exterior square $L$-function $L(s,\pi,\bigwedge^2)$) has a pole at $s=0$. Thus, a supercuspidal representation, and more generally a discrete series representation, cannot be both orthogonal and symplectic, since we have the factorization

$$L(s,\pi\times\pi)=L(s,\pi,\mathrm{Sym}^2)L(s,\pi,\textstyle\bigwedge^2).$$

Thanks to the above factorizations, if $\pi$ is a supercuspidal representation of $\mathrm{GL}_n(E)$, we can conclude that

$$\deg L(s,\pi,r)+\deg L(s,\pi\otimes\kappa,r)=\begin{cases}2n/e & \text{if } E/F \text{ is unramified,}\\ n/e & \text{if } E/F \text{ is ramified.}\end{cases}$$

Similarly, if $\pi$ is a supercuspidal representation of $\mathrm{GL}_n(F)$, then

$$\deg L(s,\pi,\mathrm{Sym}^2)+\deg L(s,\pi,\textstyle\bigwedge^2)=n/e$$

by the result of Bushnell and Kutzko mentioned earlier. Our main results assert that if both the degrees on the left-hand side of the above identities are nonzero, then they are equal.

To state the result more precisely, we introduce the following notion. Let $[\pi]$ denote the inertial equivalence class of $\pi$; thus $[\pi]$ consists of all the unramified twists of $\pi$. We say that $[\pi]$ is $\mu$-distinguished (resp. orthogonal, symplectic) if there is an unramified twist of $\pi$ which is $\mu$-distinguished (resp. orthogonal, symplectic). Now we state the main results of this paper.

**Theorem 1.1.** *Let $\pi$ be a supercuspidal representation of $\mathrm{GL}_n(E)$, with $E/F$ a quadratic extension. Let $e$ be the $\mathfrak{o}_E$-period of the principal $\mathfrak{o}_E$-order in $M_n(E)$ attached to $\pi$. Let $L(s,\pi,r)$ be the Asai $L$-function of $\pi$.*

(1) *Suppose $E/F$ is unramified. Then the degree of $L(s,\pi,r)$ is*

$$d(\mathrm{Asai}) = \begin{cases} 0 & \textit{if } [\pi] \textit{ is not distinguished,} \\ n/e & \textit{if } [\pi] \textit{ is distinguished.} \end{cases}$$

(2) *Suppose $E/F$ is ramified. Then the degree of $L(s,\pi,r)$ is*

$$d(\mathrm{Asai}) = \begin{cases} 0 & \textit{if } [\pi] \textit{ is not distinguished,} \\ n/2e & \textit{if } [\pi] \textit{ is both distinguished and } \omega\textit{-distinguished,} \\ n/e & \textit{if } [\pi] \textit{ is distinguished but not } \omega\textit{-distinguished.} \end{cases}$$

**Theorem 1.2.** *Let $\pi$ be a supercuspidal representation of $\mathrm{GL}_n(F)$. Let $e$ be the $\mathfrak{o}_F$-period of the principal $\mathfrak{o}_F$-order in $M_n(F)$ attached to $\pi$. Then the degree of its symmetric square $L$-function $L(s,\pi,\mathrm{Sym}^2)$ is*

$$d(\mathrm{Sym}^2) = \begin{cases} 0 & \textit{if } [\pi] \textit{ is not orthogonal,} \\ n/2e & \textit{if } [\pi] \textit{ is both orthogonal and symplectic,} \\ n/e & \textit{if } [\pi] \textit{ is orthogonal but not symplectic.} \end{cases}$$

**Theorem 1.3.** *Let $\pi$ be a supercuspidal representation of $\mathrm{GL}_n(F)$. Let $e$ be the $\mathfrak{o}_F$-period of the principal $\mathfrak{o}_F$-order in $M_n(F)$ attached to $\pi$. Then the degree of its exterior square $L$-function $L(s,\pi,\bigwedge^2)$ is*

$$d(\textstyle\bigwedge^2) = \begin{cases} 0 & \textit{if } [\pi] \textit{ is not symplectic,} \\ n/2e & \textit{if } [\pi] \textit{ is both symplectic and orthogonal,} \\ n/e & \textit{if } [\pi] \textit{ is symplectic but not orthogonal.} \end{cases}$$

**Remark.** As mentioned earlier, a consequence of Theorems 1.2 and 1.3 is that

$$\deg L(s,\pi,\mathrm{Sym}^2) = \deg L(s,\pi,\textstyle\bigwedge^2)$$

if both these $L$-functions are not identically 1. In this context, we also refer to the remark following Theorem 2.1 in Section 2, which places the above observation in the framework of the work of Shahidi [1992].

Finally, in Section 6, we prove the following theorem. We stress that the assumption of odd residue characteristic is essential in its proof.

**Theorem 1.4.** *Let $E/F$ be a quadratic extension of $p$-adic fields. If it is ramified, assume also that $p \neq 2$. Let $\kappa$ be a character of $E^\times$ which restricts to the quadratic character $\omega_{E/F}$ of $F^\times$ associated to $E/F$. Let $\pi$ be an irreducible supercuspidal representation of $\mathrm{GL}_n(E)$ and let $r(\pi)$ be its Asai lift to $\mathrm{GL}_{n^2}(F)$. Then*

$$f(r(\pi)) + \deg L(s,\pi,r) = f(r(\pi)\otimes\omega_{E/F}) + \deg L(s,\pi\otimes\kappa,r).$$

**Remark.** The conductor formula of Bushnell, Henniart, and Kutzko [1998, Theorem 6.5] gives an explicit formula for $f(\pi\times\pi^\sigma)$ (see Section 5). Thus, together with Theorem 1.1 and this explicit conductor formula for pairs of supercuspidal representations of general linear groups, Theorem 1.4 in fact produces an explicit conductor formula for the Asai lift. Since the statement of such an explicit formula involves introducing further notations, we leave the precise formula to Section 6 (see Theorem 6.1).

## 2. Results of Shahidi and Goldberg

We recall the results of [Shahidi 1992; Goldberg 1994] to place our Theorems 1.1, 1.2, and 1.3 in context. For the unexplained definitions in the following, we refer to [Shahidi 1992, Definitions 7.4 and 7.5].

**Theorem 2.1** [Shahidi 1992, Theorem 7.7]. *Let $\pi$ be an irreducible supercuspidal representation of* $\mathrm{GL}_n(F)$.

(1) *The $L$-function $L(s,\pi,\bigwedge^2)$ is identically* 1 *unless some unramified twist of $\pi$ is self-dual. Assume $\pi$ is self-dual. Let $S$ be the* (*possibly empty*) *set of all the unramified characters $\eta$, no two of which have equal squares, for which* $\pi\otimes\eta\circ\det$ *comes from* $\mathrm{SO}_{n+1}(F)$. *Then*
$$L(s,\pi,\textstyle\bigwedge^2)=\displaystyle\prod_{\eta\in S}\bigl(1-\eta^2(\varpi)q^{-s}\bigr)^{-1}.$$

(2) *The $L$-function $L(s,\pi,\mathrm{Sym}^2)$ is identically* 1 *unless some unramified twist of $\pi$ is self-dual. Assume $\pi$ is self-dual. If $\pi$ comes from* $\mathrm{Sp}_{n-1}(F)$, *then*
$$L(s,\pi,\mathrm{Sym}^2)=(1-q^{-rs})^{-1},$$
*where $r$ is the number of unramified self-twists of $\pi$. Otherwise, let $S'$ be the* (*possibly empty*) *set of all the unramified characters $\eta$, no two of which have equal squares, for which* $\pi\otimes\eta\circ\det$ *comes from* $\mathrm{SO}^*_n(F)$. *Then*
$$L(s,\pi,\mathrm{Sym}^2)=\prod_{\eta\in S'}\bigl(1-\eta^2(\varpi)q^{-s}\bigr)^{-1}.$$

**Remark.** A consequence of Theorem 1.2 and Theorem 1.3 is that $S$ and $S'$ have the same cardinality if both these sets are nonempty.

Next we state Theorems 5.1 and 5.2 of [Goldberg 1994]. Here, $E/F$ is a quadratic extension of $p$-adic fields and $\sigma$ denotes the nontrivial element of $\mathrm{Gal}(E/F)$. For an irreducible admissible representation of $\mathrm{GL}_n(E)$, let $L(s,\pi,r)$ denote its Asai $L$-function. In the following, $q=q_F$ is the residue cardinality of $F$. For the unexplained definitions in the following two theorems, we refer to Definitions 1.11 and 1.12 of [Goldberg 1994].

**Theorem 2.2.** *Let $n$ be odd. Suppose that $\pi$ is an irreducible supercuspidal representation of* $\mathrm{GL}_n(E)$ *such that $\pi^\vee\cong\pi^\sigma$. Let $S$ be the set of all unramified* *characters $\eta$ of $E^\times$, no two of which have equal squares, such that $\pi \otimes \eta \circ \det$ is a stable lift from $U(n, E/F)$.*

(1) *Suppose $E/F$ is ramified. Then*
$$L(s, \pi, r) = \prod_{\eta \in S} \bigl(1 - \eta(\varpi_F) q^{-s}\bigr)^{-1}.$$

(2) *Suppose $E/F$ is unramified. Then*
$$L(s, \pi, r) = \prod_{\eta \in S} \bigl(1 - \eta^2(\varpi_F) q^{-s}\bigr)^{-1}.$$

**Theorem 2.3.** *Let $n$ be even. Suppose that $\pi$ is an irreducible supercuspidal representation of $\mathrm{GL}_n(E)$ such that $\pi^\vee \cong \pi^\sigma$. Let $S$ be the set of all unramified characters $\eta$ of $E^\times$, no two of which have equal value at $\varpi_F$, such that $\pi \otimes \eta \circ \det$ is an unstable lift from $U(n, E/F)$. Then*
$$L(s, \pi, r) = \prod_{\eta \in S} \bigl(1 - \eta(\varpi_F) q^{-s}\bigr)^{-1}.$$

**Remark.** Theorem 1.1 computes explicitly the cardinality of $S$ in Theorems 2.2 and 2.3.

## 3. The Asai lift

We collect together various results on the Asai representation in this section.

Let $H$ be a subgroup of index two in a group $G$. Let $\rho$ be a finite dimensional representation of $H$ of dimension $n$. Its Asai lift, which we do not define here, is a representation of $G$ of dimension $n^2$. Let $r(\rho)$ denote the Asai lift of $\rho$ to $G$. The following proposition summarizes the key properties of the Asai lift (see [Prasad 1999; Murty and Prasad 2000]).

**Proposition 3.1.** *The Asai lift satisfies*:

(1) $r(\rho_1 \otimes \rho_2) \cong r(\rho_1) \otimes r(\rho_2)$.

(2) $r(\rho)^\vee \cong r(\rho^\vee)$.

(3) *$r(\chi)$ for a character $\chi$ is $\chi \circ \mathrm{tr}$, where $\mathrm{tr}$ is the transfer map from $G$ to the abelianization of $H$.*

(4) *$r(\rho^\sigma) \cong r(\rho)$, where $\sigma$ is the nontrivial element of $G/H$.*

(5) $r(\rho)|_H \cong \rho \otimes \rho^\sigma$.

(6) *For a representation $\tau$ of $G$, we have $r(\tau|_H) \cong \mathrm{Sym}^2 \tau \oplus \omega_{G/H} \bigwedge^2 \tau$, where $\omega_{G/H}$ is the nontrivial character of $G/H$.*

(7) *Let $\mathrm{Ind}_H^G \rho$ denote the representation of $G$ induced from $\rho$. Then*:

(a) $\mathrm{Sym}^2(\mathrm{Ind}_H^G \rho) \cong \mathrm{Ind}_H^G \mathrm{Sym}^2 \rho \oplus r(\rho)$.

(b) $\bigwedge^2(\mathrm{Ind}_H^G \rho) \cong \mathrm{Ind}_H^G \bigwedge^2 \rho \oplus r(\rho) \otimes \omega_{G/H}$.

**Remark.** We have assumed $[G : H] = 2$ since that is the case of interest to us. The Asai lift can be more generally defined when $H$ is of any finite index in $G$.

## 4. Proofs of Theorems 1.1–1.3

We now prove Theorems 1.1, 1.2, and 1.3. We first prove (1) of Theorem 1.1, use this to prove Theorems 1.2 and 1.3, and finally prove (2) of Theorem 1.1. We will appeal to a result mentioned in Section 1, which we formally state now for ease of reference.

**Theorem 4.1** [Bushnell and Kutzko 1993, Lemma 6.2.5]. *Let $\pi$ be an irreducible supercuspidal representation of $\mathrm{GL}_n(E)$. Let $[\mathfrak{A}, m, 0, \beta]$ be the simple stratum defining a maximal simple type occurring in $\pi$, where $\mathfrak{A}$ is a principal $\mathfrak{o}_E$-order in $M_n(E)$, $m \geq 0$ is the level of $\pi$, and $\beta \in M_n(E)$ is such that $E[\beta]$ is a field with $E[\beta]^\times$ normalizing $\mathfrak{A}$. Let $e = e(\mathfrak{A}|\mathfrak{o}_E)$ be the $\mathfrak{o}_E$-period of $\mathfrak{A}$ (which is the same as the ramification index $e(E[\beta]/E)$ of $E[\beta]/E$). Then $e$ divides $n$, and the cardinality of the group of unramified self-twists of $\pi$ is $n/e$.*

*Proof of Theorem 1.1(1).* Let $E/F$ be quadratic unramified. Let $\pi$ be a supercuspidal representation of $\mathrm{GL}_n(E)$. Let $\rho_\pi : W_E \to \mathrm{GL}_n(\mathbb{C})$ be its Langlands parameter. We assume that its Asai lift $r(\rho_\pi) : W_F \to \mathrm{GL}_{n^2}(\mathbb{C})$ contains the trivial character of $W_F$, which in particular implies that $\rho_\pi^\sigma \cong \rho_\pi^\vee$. Since $\omega = \omega_{E/F}$ is unramified, clearly the number of unramified characters in $r(\rho_\pi)$ and $r(\rho_\pi) \otimes \omega$ is the same. Since

$$\deg L(s, \pi, r) + \deg L(s, \pi \otimes \kappa, r) = \deg L(s, \pi \times \pi^\vee) = 2n/e$$

by Theorem 4.1, item (1) of Theorem 1.1 is immediate. □

*Proof of Theorems 1.2 and 1.3.* Let $\pi$ be a supercuspidal representation of $\mathrm{GL}_n(F)$. Let $\rho_\pi : W_F \to \mathrm{GL}_n(\mathbb{C})$ be its Langlands parameter. We assume that $r(\rho_\pi)$ contains the trivial character of $W_F$, which in particular implies that $\rho_\pi \cong \rho_\pi^\vee$. Here, $r$ is either the symmetric square representation or the exterior square representation of $\mathrm{GL}_n(\mathbb{C})$. Thus, the dimension of $r(\rho_\pi)$ is either $n(n+1)/2$ or $n(n-1)/2$. We have the identity

$$L(s, \pi \times \pi) = L(s, \pi, \mathrm{Sym}^2) L(s, \pi, \textstyle\bigwedge^2),$$

and we know that the left-hand side $L$-function has degree $n/e$ by Theorem 4.1.

If $n/e = 1$, then the trivial character of $W_F$ is the only unramified character appearing in $\rho_\pi \otimes \rho_\pi^\vee$ and hence in $r(\rho_\pi)$. Therefore, in this case there is nothing to prove. Otherwise, there is a nontrivial unramified character $\chi : W_F \to \mathbb{C}^\times$ such that $\rho_\pi \otimes \chi \cong \rho_\pi$. Thus,

$$\rho_\pi = \mathrm{Ind}_{W_{F'}}^{W_F} \tau$$

for some irreducible representation $\tau$ of $W_{F'}$, where $F'/F$ is the unramified extension of degree $n/e$. Let $\sigma$ denote a generator of $\mathrm{Gal}(F'/F)$.

We know that

$$\rho_\pi \otimes \rho_\pi = \mathrm{Ind}_{W_{F'}}^{W_F}(\tau \otimes \tau) \oplus \mathrm{Ind}_{W_{F'}}^{W_F}(\tau \otimes \tau^\sigma) \oplus \cdots \oplus \mathrm{Ind}_{W_{F'}}^{W_F}\left(\tau \otimes \tau^{\sigma^{n/e-1}}\right).$$

If $n/e$ is an odd integer, then observe that each summand other than the first one on the right-hand side of the above identity appears twice. This is indeed the case since

$$\mathrm{Ind}_{W_{F'}}^{W_F}\left(\tau \otimes \tau^{\sigma^a}\right) = \mathrm{Ind}_{W_{F'}}^{W_F}\left(\tau \otimes \tau^{\sigma^{n/e-a}}\right)$$

for every $1 \le a \le n/e$. Since the trivial character of $W_F$ appears exactly once on the left-hand side, it follows that

$$1 \in \mathrm{Ind}_{W_{F'}}^{W_F}(\tau \otimes \tau)$$

when $n/e$ is odd. Therefore, precisely one of $\mathrm{Sym}^2 \tau$ or $\bigwedge^2 \tau$ contains the trivial character of $W_{F'}$, and hence precisely one of $\mathrm{Ind}_{W_{F'}}^{W_F}(\mathrm{Sym}^2 \tau)$ or $\mathrm{Ind}_{W_{F'}}^{W_F}(\bigwedge^2 \tau)$ contains all the unramified self-twists of $\rho_\pi$. Thus, Theorems 1.2 and 1.3 follow in the case when $n/e$ is an odd integer.

If $n/e$ is an even integer, we proceed by induction on $\dim \rho_\pi$. We start by writing $\rho_\pi = \mathrm{Ind}_{W_E}^{W_F} \tau$ for an irreducible representation $\tau$ of $W_E$, where $E$ is the quadratic unramified extension of $F$. This can always be done because an unramified extension of even degree necessarily has the quadratic unramified subextension. By (7) of Proposition 3.1, we have

$$r(\rho_\pi) \cong \begin{cases} \mathrm{Ind}_{W_E}^{W_F} \mathrm{Sym}^2 \tau \oplus \mathrm{Asai}(\tau) & \text{if } r = \mathrm{Sym}^2, \\ \mathrm{Ind}_{W_E}^{W_F} \bigwedge^2 \tau \oplus \mathrm{Asai}(\tau) \otimes \omega_{E/F} & \text{if } r = \bigwedge^2. \end{cases}$$

Now either $\tau \cong \tau^\vee$ or $\tau^\sigma \cong \tau^\vee$ but not both, since $\rho_\pi$ is an irreducible representation of $W_F$. Here, $\sigma$ is the element of order two in $\mathrm{Gal}(E/F)$. We claim that $\mathrm{Asai}(\tau)$ (resp. $\mathrm{Asai}(\tau) \otimes \omega_{E/F}$) contains an unramified character of $W_F$ only if $\mathrm{Sym}^2 \tau$ (resp. $\bigwedge^2 \tau$) does not contain an unramified character of $W_F$. Indeed, if $\mathrm{Asai}(\tau)$ contains an unramified character of $W_F$, the total number of unramified characters in

$$\mathrm{Asai}(\tau) \oplus \mathrm{Asai}(\tau) \otimes \omega_{E/F}$$

is $n/2e + n/2e = n/e$, by applying part (1) of Theorem 1.1 to the representation $\tau$ which has dimension $n/2$, and by observing that $\omega_{E/F}$ is unramified. Note also that $e = e(\rho_\pi) = e(\tau)$, since the extension $E/F$ is unramified. Since this number equals the number of unramified characters contained in

$$\rho_\pi \otimes \rho_\pi = \mathrm{Sym}^2 \rho_\pi \oplus \bigwedge^2 \rho_\pi,$$

the claim follows.

Therefore, if $\mathrm{Asai}(\tau)$ contains an unramified character, the proof is complete by appealing to part (1) of Theorem 1.1. Otherwise, since $\dim \tau = \frac{1}{2}\dim \rho_\pi$, the proof is complete by appealing to the induction hypothesis. Note that the base case of the induction is easily verified since there are at most two unramified characters to consider when $\dim \rho_\pi = 2$, i.e., when $\dim \tau = 1$. □

*Proof of Theorem 1.1(2).* Now let $E/F$ be a ramified quadratic extension, and let $\pi$ be a supercuspidal representation of $\mathrm{GL}_n(E)$. Let $\rho_\pi : W_E \to \mathrm{GL}_n(\mathbb{C})$ be its Langlands parameter. We may assume that $r(\rho_\pi) \ni 1$, where $r$ denotes the Asai lift from $W_E$ to $W_F$. Note that this implies that $r(\rho_\pi)$ does not contain $\omega_{E/F}$, the nontrivial character of $W_F/W_E$. In what follows, we use this assumption many times to reduce the number of cases that we need to analyze.

Consider the $2n$-dimensional representation $\mathrm{Ind}_{W_E}^{W_F} \rho_\pi$ of $W_F$. We have

$$\mathrm{Sym}^2(\mathrm{Ind}_{W_E}^{W_F} \rho_\pi) \cong \mathrm{Ind}_{W_E}^{W_F} \mathrm{Sym}^2 \rho_\pi \oplus r(\rho_\pi), \tag{1}$$

$$\textstyle\bigwedge^2(\mathrm{Ind}_{W_E}^{W_F} \rho_\pi) \cong \mathrm{Ind}_{W_E}^{W_F} \bigwedge^2 \rho_\pi \oplus r(\rho_\pi) \otimes \omega_{E/F}. \tag{2}$$

We divide the proof into two cases.

First, we assume that $\pi \ncong \pi^\sigma$ so that $\mathrm{Ind}_{W_E}^{W_F} \rho_\pi$ is irreducible. Let $\mathrm{Ind}_E^F \pi$ denote the corresponding supercuspidal representation of $\mathrm{GL}_{2n}(F)$. Note that by our assumption that $r(\rho_\pi) \ni 1$, $\mathrm{Ind}_E^F \pi$ is orthogonal and not symplectic by (1). Therefore, it follows from Theorems 1.2 and 1.3 that

$$x = \deg L(s, \mathrm{Ind}_E^F \pi, \mathrm{Sym}^2) - \deg L(s, \mathrm{Ind}_E^F \pi, \textstyle\bigwedge^2)$$

is given by

$$x = \begin{cases} \deg L(s, \mathrm{Ind}_E^F \pi, \mathrm{Sym}^2) & \text{if } [\mathrm{Ind}_E^F \pi] \text{ is orthogonal but not symplectic}, \\ 0 & \text{if } [\mathrm{Ind}_E^F \pi] \text{ is orthogonal and symplectic}. \end{cases} \tag{3}$$

Since the extension $E/F$ is ramified, the period associated to $\mathrm{Ind}_E^F \pi$ may be $e$ or $2e$, and thus the degree of $L(s, \mathrm{Ind}_E^F \pi, \mathrm{Sym}^2)$ is either $2n/e$ or $n/e$.

On the other hand, the difference

$$y = \deg L(s, \pi, \mathrm{Sym}^2) - \deg L(s, \pi, \textstyle\bigwedge^2) \tag{4}$$

could be, a priori, $n/e$ or $0$ or $-n/e$.

Now we do a case-by-case analysis to list all the possible candidates for the pair $(x, y)$. To this end, note that:

(i) In (1) and (2), possible values for the degree of the first summand on the right-hand side are $0$, $n/e$, and $n/2e$ (by Theorems 1.2 and 1.3).

(ii) In (1) and (2), the second summand on the right-hand side cannot have degree more than $n/e$ (since $\deg L(s, \pi, r) + \deg L(s, \pi \otimes \kappa, r) = n/e$).

(iii) In addition, in (1), the second summand on the right hand side has nonzero degree (by the assumption that $r(\rho_\pi) \ni 1$).

We have already observed, using (3), that when $x \neq 0$, it is either $2n/e$ or $n/e$, and the degree of $L(s, \mathrm{Ind}_E^F \pi, \bigwedge^2)$ is 0. In particular, when $x \neq 0$, all the terms in (2) have degree 0. When $x = 2n/e$, both the summands in (1) have degree $n/e$ by (i) and (ii), and thus $y = n/e$. When $x = n/e$, we claim that the degree of the first summand in (1) is 0 (and that of the second summand is $n/e$), and thus $y = 0$. Indeed, if the first summand had nonzero degree it would have to be either $n/e$ or $n/2e$ by (i). But it cannot be $n/e$ by (iii), and it cannot be $n/2e$ since this would imply that the second summand in (2), which we know to be 0, would have degree $n/2e$ as well.

When $x = 0$, the degrees of the left-hand sides in both (1) and (2) are equal by (3), and are either $n/e$ or $n/2e$. When this degree is $n/e$, the degree of the first summand in (1) is either 0 or $n/2e$ by (iii). Note that the degree of the first summand in (2) would then be either $n/e$ or $n/2e$ respectively, and thus $y = -n/e$ or 0 respectively. In the preceding argument, we have made use of the identity

$$\deg L(s, \pi, r) + \deg L(s, \pi \otimes \kappa, r) = n/e. \tag{5}$$

When the degrees of the left-hand sides in both (1) and (2) are $n/2e$, the degrees of the first summands are both 0. Thus $y = 0$, once again by arguing with (i), (iii), and (5).

Observe that since $E/F$ is ramified, the number of unramified characters in $\mathrm{Ind}_{W_E}^{W_F} \mathrm{Sym}^2 \rho_\pi$ (resp. in $\mathrm{Ind}_{W_E}^{W_F} \bigwedge^2 \rho_\pi$) is the same as the number of unramified characters in $\mathrm{Sym}^2 \rho_\pi$ (resp. in $\bigwedge^2 \rho_\pi$). It follows that

$$\deg L(s, \pi, r) - \deg L(s, \pi \otimes \kappa, r) = x - y$$

is either $n/e$ or 0. This proves (2) of Theorem 1.1 in this case.

Next, suppose that $\pi \cong \pi^\sigma \cong \pi^\vee$. Since $\pi \cong \pi^\sigma$, it follows that

$$\rho_\pi \cong \tau|_{W_E}$$

for an irreducible representation $\tau$ of $W_F$. In this case,

$$\mathrm{Ind}_{W_E}^{W_F} \rho_\pi \cong \tau \oplus \tau \otimes \omega_{E/F}.$$

Thus, we get

$$\mathrm{Sym}^2 \tau \oplus \mathrm{Sym}^2 \tau \oplus \tau \otimes \tau \otimes \omega_{E/F} \cong \mathrm{Sym}^2(\mathrm{Ind}_{W_E}^{W_F} \rho_\pi) \cong \mathrm{Ind}_{W_E}^{W_F} \mathrm{Sym}^2 \rho_\pi \oplus r(\rho_\pi) \tag{6}$$

and

$$\textstyle\bigwedge^2 \tau \oplus \bigwedge^2 \tau \oplus \tau \otimes \tau \otimes \omega_{E/F} \cong \bigwedge^2(\mathrm{Ind}_{W_E}^{W_F} \rho_\pi) \cong \mathrm{Ind}_{W_E}^{W_F} \bigwedge^2 \rho_\pi \oplus r(\rho_\pi) \otimes \omega_{E/F}. \tag{7}$$

By our assumption that $r(\rho_\pi) \ni 1$, we conclude that the irreducible representation $\tau$ is not symplectic. This is because if $\bigwedge^2 \tau \ni 1$, then the left-hand side of (7) contains the trivial character at least twice whereas the right-hand side can contain the trivial character at most once since $r(\rho_\pi) \otimes \omega_{E/F} \not\ni 1$.

As before, we now do a case-by-case analysis to list all possible pairs $(a, b)$ where

$$a = \deg L(s, \tau, \mathrm{Sym}^2) - \deg L(s, \tau, \textstyle\bigwedge^2), \tag{8}$$

$$b = \deg L(s, \rho_\pi, \mathrm{Sym}^2) - \deg L(s, \rho_\pi, \textstyle\bigwedge^2), \tag{9}$$

and we verify that

$$\deg L(s, \rho_\pi, r) - \deg L(s, \rho_\pi \otimes \kappa, r) = 2a - b$$

is either $n/e$ or 0.

Since we have observed that the irreducible representation $\tau$ is not symplectic, $a \geq 0$ and it is either $n/e$ or 0 by Theorems 1.2 and 1.3. Now the possible values for $b$ could be, a priori, $n/e$ or 0 or $-n/e$.

When $a = n/e$, considering the sum of (6) and (7), we can conclude that all the terms in (7) are of degree 0. Also, note that both the terms on the right-hand side of (6) will have degree $n/e$, and in particular $b = n/e$. When $a = 0$, the left-hand sides of both (6) and (7) are each of total degree $n/e$. Since $r(\rho_\pi) \ni 1$, the degree of $L(s, \rho_\pi, \mathrm{Sym}^2)$ is either 0 or $n/2e$. It follows that the value of $b$ is either $-n/e$ or 0 respectively. Thus, in all cases $2a - b$ is $n/e$ or 0, and the result follows. $\square$

## 5. The conductor formula of Bushnell, Henniart, and Kutzko

We state the explicit conductor formula for pairs of supercuspidal representation due to Bushnell, Henniart, and Kutzko. This section closely follows [Bushnell et al. 1998, § 6].

Let $\pi$ be a supercuspidal representation of $\mathrm{GL}_n(F)$. Following [Bushnell and Kutzko 1993], let $[\mathfrak{A}, m, 0, \beta]$ be a simple stratum of a maximal simple type occurring in $\pi$. Here, $\mathfrak{A}$ is a principal $\mathfrak{o}_F$-order in $M_n(F)$, $m$ is the level of $\pi$, and $\beta \in M_n(F)$ is such that $E = F[\beta]$ is a field with $E^\times$ normalizing $\mathfrak{A}$. If $e$ denotes the $\mathfrak{o}_F$-period of $\mathfrak{A}$, then the number of unramified self-twists of $\pi$ is $n/e$ by Theorem 4.1. As mentioned in the introduction, the conductor $f(\pi)$ of $\pi$ is given by

$$f(\pi) = n\Big(1 + \frac{m}{e}\Big).$$

Let $\pi_i$ be two supercuspidal representations of $\mathrm{GL}_{n_i}(F)$ for $i = 1, 2$. There are three distinct possibilities: (i) $\pi_1$ and $\pi_2$ are unramified twists of each other, (ii) $\pi_1$ and $\pi_2$ are *completely distinct*, and (iii) $\pi_1$ and $\pi_2$ admit a *common approximation*. We do not get into defining these notions and refer to [Bushnell et al. 1998, § 6] instead. Suffice to say that when $\pi_1$ and $\pi_2$ admit a common approximation, there is a best common approximation and this is an object of the form $([\Lambda, m, 0, \gamma], l, \vartheta)$, where the stratum $[\Lambda, m, 0, \gamma]$ is determined by $\pi_1$ and $\pi_2$, $0 \le l < m$ is an integer, and $\vartheta$ is a character of a compact group attached to the data coming from $\pi_1$ and $\pi_2$.

Another ingredient in the conductor formula is an integer $\mathfrak{c}(\beta)$ associated to $\beta$. This comes from the "generalized discriminant", say $C(\beta)$, associated to the exact sequence

$$0 \longrightarrow E \longrightarrow \mathrm{End}_F(E) \xrightarrow{a_\beta} \mathrm{End}_F(E) \xrightarrow{s_\beta} E \longrightarrow 0,$$

where $s_\beta$ is a tame corestriction relative to $E/F$ [Bushnell and Kutzko 1993, § 1.3] and $a_\beta$ is the adjoint map $x \mapsto \beta x - x\beta$. The constant $\mathfrak{c}(\beta)$ is defined such that

$$C(\beta) = q^{\mathfrak{c}(\beta)}.$$

Now we state the conductor formula of [Bushnell et al. 1998].

**Theorem 5.1** (Bushnell, Henniart, and Kutzko). *For $i = 1, 2$, let $\pi_i$ be an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(F)$. Define quantities $m_i$, $e_i$, $\beta_i$ as above. Let $e = \mathrm{lcm}(e_1, e_2)$ and $m/e = \max\{m_1/e_1, m_2/e_2\}$.*

(1) *Suppose that $n_1 = n_2 = n$ and $\pi_1$ and $\pi_2$ are unramified twists of each other. Let $\beta = \beta_1$ and $d = [F[\beta] : F]$. Then*

$$f(\pi_1^\vee \times \pi_2) = n^2\Big(1 + \frac{\mathfrak{c}(\beta)}{d^2}\Big) - \deg L(s, \pi_1^\vee \times \pi_2).$$

(2) *Suppose that $\pi_1$ and $\pi_2$ are completely distinct. Then*

$$f(\pi_1^\vee \times \pi_2) = n_1 n_2\Big(1 + \frac{m}{e}\Big).$$

(3) *Suppose that $\pi_2$ is not equivalent to an unramified twist of $\pi_1$, but that $\pi_1$ and $\pi_2$ are not completely distinct. Let $([\Lambda, m, 0, \gamma], l, \vartheta)$ be a best common approximation to the $\pi_i$, and assume that the stratum $[\Lambda, m, l, \gamma]$ is simple. Put $d = [F[\gamma] : F]$. Then*

$$f(\pi_1^\vee \times \pi_2) = n_1 n_2\Big(1 + \frac{\mathfrak{c}(\gamma)}{d^2} + \frac{l}{de}\Big).$$

**Remark.** Observe that in (2) and (3), $\deg L(s, \pi_1^\vee \times \pi_2) = 0$.

## 6. Conductor of the Asai lift

Let $E/F$ be a quadratic extension of $p$-adic fields. Let $\pi$ be a supercuspidal representation of $\mathrm{GL}_n(E)$. Let $\rho_\pi : W_E \to \mathrm{GL}_n(\mathbb{C})$ be its Langlands parameter. Let $r(\rho_\pi) : W_F \to \mathrm{GL}_{n^2}(\mathbb{C})$ be the Asai lift of $\rho_\pi$. In this section, we compute the

Artin conductor of $r(\rho_\pi)$. Throughout this section, we assume that $p$ is odd. For a representation $\tau$ of the Weil–Deligne group, let $f(\tau)$ denote its Artin conductor.

Our formula for the Asai lift is a consequence of the conductor formula for pairs of supercuspidal representations due to Bushnell, Henniart, and Kutzko [1998, Theorem 6.5]. Since

$$r(\rho_\pi)|_{W_E} \cong \rho_\pi \otimes \rho_\pi^\sigma,$$

it follows that

$$f(\rho_\pi \otimes \rho_\pi^\sigma) = \begin{cases} f(r(\rho_\pi)) & \text{if } E/F \text{ is unramified,} \\ f(r(\rho_\pi)) + f(r(\rho_\pi)\otimes \omega_{E/F}) - n^2 & \text{if } E/F \text{ is ramified.} \end{cases}$$

In the second case of the above, we have made use of the fact that $E/F$ is tamely ramified, which is true since $p$ is odd by our assumption. Since the formula of Bushnell, Henniart, and Kutzko [1998] computes the left hand side, in order to derive a formula for the Asai lift, it suffices to compute $f(r(\rho_\pi)) - f(r(\rho_\pi)\otimes \omega_{E/F})$.

Let

$$r(\rho_\pi) \cong \bigoplus_i \rho_i$$

be the direct sum decomposition of $r(\rho_\pi)$ into irreducible representations. Now

$$r(\rho_\pi)\otimes \omega_{E/F} \cong \bigoplus_i \rho_i \otimes \omega_{E/F},$$

and since the Artin conductor is additive, it follows that

$$f(r(\rho_\pi)) - f(r(\rho_\pi)\otimes \omega_{E/F}) = \bigoplus_i [f(\rho_i) - f(\rho_i \otimes \omega_{E/F})].$$

We know that

$$f(\rho\otimes\chi) \le \max\{f(\rho), \dim\rho\cdot f(\chi)\},$$

with equality in the above identity if $f(\rho) \neq \dim\rho\cdot f(\chi)$. Thus,

$$f(\rho_i\otimes\omega_{E/F}) = f(\rho_i)$$

unless $\rho_i$ is a one-dimensional character with Artin conductor one, in which case $f(\rho_i\otimes\omega_{E/F})$ can be 0 or 1.

Observe that the contribution to

$$f(r(\rho_\pi)) - f(r(\rho_\pi)\otimes\omega_{E/F})$$

from tamely ramified characters $\rho_i$ in $r(\rho_\pi)$ such that $\rho_i\otimes\omega_{E/F}$ is unramified is

$$\deg L(s, \pi\otimes\kappa, r),$$

whereas the contribution from unramified characters $\rho_i$ in $r(\rho_\pi)$ such that $\rho_i\otimes\omega_{E/F}$ is tamely ramified is

$$-\deg L(s, \pi, r).$$

Therefore, it follows that

$$f(r(\rho_\pi)) - f(r(\rho_\pi) \otimes \omega_{E/F}) = \deg L(s, \pi \otimes \kappa, r) - \deg L(s, \pi, r).$$

Now making use of Theorem 5.1, we get the following conductor formula for the Asai lift.

**Theorem 6.1.** *Let $E/F$ be a quadratic extension of $p$-adic fields, where $p$ is odd, with ramification index $e(E/F)$. Let $\sigma$ denote the nontrivial element of* $\mathrm{Gal}(E/F)$. *Let $\pi$ be a supercuspidal representation of* $\mathrm{GL}_n(E)$. *Let $e$ be the $\mathfrak{o}_E$-period of the principal $\mathfrak{o}_E$-order in $M_n(E)$ attached to $\pi$. Let $r(\pi)$ be its Asai lift to* $\mathrm{GL}_{n^2}(F)$ *and let $L(s, \pi, r)$ be the Asai $L$-function attached to $\pi$.*

(1) *Suppose $\pi^\vee$ and $\pi^\sigma$ are unramified twists of each other. Then*

$$f(r(\pi)) = n^2\Big(1 + \frac{\mathfrak{c}(\beta)}{e(E/F)d^2}\Big) - \deg L(s, \pi, r).$$

(2) *Suppose $\pi^\vee$ and $\pi^\sigma$ are completely distinct. Then*

$$f(r(\pi)) = n^2\Big(1 + \frac{m}{e(E/F)e}\Big).$$

(3) *Suppose that $\pi^\vee$ is not equivalent to an unramified twist of $\pi^\sigma$ and that they are not completely distinct. Let $([\Lambda, m, 0, \gamma], l, \vartheta)$ be a best common approximation to $\pi^\vee$ and $\pi^\sigma$, and assume that the stratum $[\Lambda, m, l, \gamma]$ is simple. Set $d = [F[\gamma] : F]$. Then*

$$f(r(\pi)) = n^2\Big(1 + \frac{\mathfrak{c}(\gamma)}{e(E/F)d^2} + \frac{l}{e(E/F)de}\Big).$$

**Remark.** Together with Theorem 1.1, Theorem 6.1 gives an explicit conductor formula for the Asai lift. As in the case of Theorem 5.1, $\deg L(s, \pi, r) = 0$ in cases (2) and (3).

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