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CYCLICITY IN DIRICHLET-TYPE SPACES AND EXTREMAL POLYNOMIALS II: FUNCTIONS ON THE BIDISK

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We study Dirichlet-type spaces $\mathfrak{D}_\alpha$ of analytic functions in the unit bidisk and their cyclic elements. These are the functions f for which there exists a sequence $(p_n)_{n=1}^\infty$ of polynomials in two variables such that $\|p_n f - 1\|_\alpha \to 0$ as $n \to \infty$. We obtain a number of conditions that imply cyclicity, and obtain sharp estimates on the best possible rate of decay of the norms $\|p_n f - 1\|_\alpha$, in terms of the degree of p_n, for certain classes of functions using results concerning Hilbert spaces of functions of one complex variable and comparisons between norms in one and two variables.

We give examples of polynomials with no zeros on the bidisk that are not cyclic in $\mathfrak{D}_\alpha$ for $\alpha > 1/2$ (including the Dirichlet space); this is in contrast with the one-variable case where all nonvanishing polynomials are cyclic in Dirichlet-type spaces that are not algebras ($\alpha \leq 1$). Further, we point out the necessity of a capacity zero condition on zero sets (in an appropriate sense) for cyclicity in the setting of the bidisk, and conclude by stating some open problems.

1. Introduction

Dirichlet-type spaces on the bidisk. We consider a scale of Hilbert spaces of holomorphic functions on the bidisk

$$\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\},$$

indexed by a parameter $\alpha \in (-\infty, \infty)$. A holomorphic function $f : \mathbb{D}^2 \to \mathbb{C}$ belongs

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to the *Dirichlet-type space* $\mathfrak{D}_\alpha$ if its power series expansion

$$f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} z_1^k z_2^l$$

satisfies

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (k+1)^\alpha (l+1)^\alpha |a_{k,l}|^2 < \infty. \tag{1-1}$$

Recall that a function of two complex variables is said to be *holomorphic* if it is holomorphic in each variable separately. A review of the definitions and basic properties such as power series expansions can be found in [Hörmander 1990, Chapter 2]. Since zero sets on the boundary of functions $f \in \mathfrak{D}_\alpha$ will play a role later on, we point out that the topological boundary of the bidisk is much larger than the so-called *distinguished boundary*

$$\mathbb{T}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\},$$

which is still large enough to support standard integral representations and the maximum principle on the bidisk.

The spaces $\mathfrak{D}_\alpha$ are a natural generalization to two variables of the classical Dirichlet-type spaces D_α, $-\infty < \alpha < \infty$, consisting of functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

that are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy

$$\|f\|_{D_\alpha}^2 = \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty;$$

see, for instance, [Taylor 1966; Brown and Shields 1984], and the references therein. As a remark on notation, we will continue to use $\|\cdot\|_\alpha$ for the norm of two variable functions in $\mathfrak{D}_\alpha$ while $\|\cdot\|_{D_\alpha}$ will denote the norm of one variable functions in D_α. We point out that the particular choice $\alpha = 0$ in D_α and $\mathfrak{D}_\alpha$ leads to the classical Hardy spaces H^2 on the disk and bidisk, respectively, while

$$D_{-1} = A^2(\mathbb{D}) \quad \text{and} \quad \mathfrak{D}_{-1} = A^2(\mathbb{D}^2)$$

are the canonical Bergman spaces of the disk and bidisk, and D_1 and $\mathfrak{D}_1$ are the Dirichlet spaces of the disk and bidisk, respectively.

The spaces $\mathfrak{D}_\alpha$ were studied in detail by Jupiter and Redett [2006]. Spaces of this type appear in the earlier work of Kaptanoğlu [1994], which focuses on Möbius invariance and boundary behavior in Dirichlet-type spaces, and Hedenmalm [1988], which concentrates on closed ideals in function algebras. We note here (compare [Kaptanoğlu 1994, p. 343; Hedenmalm 1988, Section 4]) that an equivalent norm for $\mathfrak{D}_\alpha$ is given by

$$\begin{aligned}\|f\|_\alpha^2 = |f(0,0)|^2 &\\ &+ \int_{\mathbb{D}} |\partial_{z_1} f(z_1, 0)|^2 (1-|z_1|^2)^{1-\alpha}\, dA(z_1)\\ &+ \int_{\mathbb{D}} |\partial_{z_2} f(0, z_2)|^2 (1-|z_2|^2)^{1-\alpha}\, dA(z_2)\\ &+ \int_{\mathbb{D}^2} |\partial_{z_2}\partial_{z_1} f(z_1, z_2)|^2 (1-|z_1|^2)^{1-\alpha}(1-|z_2|^2)^{1-\alpha}\, dA(z_1)\, dA(z_2),\end{aligned}$$

where $dA(z) = \pi^{-1}\, dx\, dy$ denotes area measure. The proof involves computations with power series, and is omitted.

Extending the earlier one-variable work of G. D. Taylor [1966] and Stegenga [1980], Jupiter and Redett identified multipliers on $\mathfrak{D}_\alpha$ and studied restriction properties of these spaces. It was also shown in [Jupiter and Redett 2006] that evaluation at a point in $\mathbb{D}^2$ is a bounded linear functional, and hence $\mathfrak{D}_\alpha$ is a *reproducing kernel Hilbert space* for all α. When $\alpha > 1$, the spaces $\mathfrak{D}_\alpha$ are actually *algebras* (viz. the proof of [op. cit., Theorem 3.10]) that are contained (as sets) in $H^\infty(\mathbb{D}^2)$, the algebra of bounded holomorphic functions.

It is clear from the definition of the norm in (1-1) that any polynomial $p = p(z_1, z_2)$ belongs to $\mathfrak{D}_\alpha$. Moreover, any $f \in D_\alpha$ lifts to $\mathfrak{D}_\alpha$ when regarded as constant in one of the variables. In fact, if $g \in D_\alpha$ and $h \in D_\alpha$, then the function

$$f(z_1, z_2) = g(z_1)h(z_2), \quad (z_1, z_2) \in \mathbb{D}^2,$$

is analytic in the bidisk and belongs to $\mathfrak{D}_\alpha$ [op. cit., Proposition 4.7], and so $\mathfrak{D}_\alpha$ certainly contains nontrivial holomorphic functions.

Shift operators and cyclic functions. In this paper, we are interested in a natural pair $\{S_1, S_2\}$ of bounded linear operators acting on the spaces $\mathfrak{D}_\alpha$. The *shift operators* S_1 and S_2 are defined by setting, for $f \in \mathfrak{D}_\alpha$,

$$S_1 f(z_1, z_2) = z_1 f(z_1, z_2) \quad \text{and} \quad S_2 f(z_1, z_2) = z_2 f(z_1, z_2).$$

It is then clear that S_1 and S_2 are linear, and it follows from (1-1) that, for every α, $\{S_1, S_2\}$ forms a pair of bounded operators mapping $\mathfrak{D}_\alpha$ into itself.

It is a standard problem of operator theory to describe the invariant subspaces of an operator. In the present context, we are interested in closed subspaces $\mathcal{M} \subset \mathfrak{D}_\alpha$ such that

$$S_1\mathcal{M} \subset \mathcal{M} \quad \text{and} \quad S_2\mathcal{M} \subset \mathcal{M}.$$

As a first step towards understanding the invariant subspaces of the pair $\{S_1, S_2\}$, we seek conditions under which a function $f \in \mathfrak{D}_\alpha$ is *cyclic*, that is,

$$[f] = \overline{\operatorname{span}\{z_1^k z_2^l f : k = 0, 1, \dots; l = 0, 1, \dots\}} = \mathfrak{D}_\alpha.$$

It is easy to see that there exists at least one cyclic function in each $\mathfrak{D}_\alpha$, namely the function $f(z_1, z_2) = 1$. This follows from the fact that polynomials in two variables are dense in $\mathfrak{D}_\alpha$. On the other hand, since norm convergence implies uniform convergence on compact subsets, every $g \in [f]$ inherits any zeros f may have inside $\mathbb{D}^2$, and so a necessary condition for cyclicity is that $f(z_1, z_2) \neq 0$, $(z_1, z_2) \in \mathbb{D}^2$. Note that since $g \in [f]$ implies $[g] \subset [f]$, an equivalent condition for f to be cyclic in $\mathfrak{D}_\alpha$ is that there exists a sequence of polynomials $(p_n)_{n=1}^\infty$ of two variables with

$$\|p_n f - 1\|_\alpha \to 0, \quad n \to \infty.$$

Since point evaluation is a bounded linear functional, this latter condition is equivalent to the existence of a sequence of polynomials (p_n) such that

$$p_n(z_1, z_2) f(z_1, z_2) - 1 \to 0, \quad (z_1, z_2) \in \mathbb{D}^2,$$

and

$$\|p_n f - 1\|_\alpha \leq C.$$

When $\alpha > 1$ the spaces D_α and $\mathfrak{D}_\alpha$ are algebras, and cyclic functions have to be nonvanishing on $\overline{\mathbb{D}}$ and $\overline{\mathbb{D}^2}$, respectively.

In one variable, Beurling characterized the cyclic vectors of $H^2(\mathbb{D})$: a function f is cyclic if and only if it is outer. In the bidisk, one can show that if $f \in H^2(\mathbb{D}^2)$ or indeed if f belongs to the Nevanlinna class, then f has (nonzero) radial limits f^* at almost every $(\zeta_1, \zeta_2) \in \mathbb{T}^2$. Thus, we can declare $f \in H^2(\mathbb{D}^2)$ to be *outer* if

$$\log|f(z_1, z_2)| = \int_{\mathbb{T}^2} \log|f^*(e^{i\theta}, e^{i\eta})| \, P((z_1, z_2); (e^{i\theta}, e^{i\eta})) \, d\theta d\eta;$$

here, P is the product Poisson kernel

$$P((z_1, z_2); (e^{i\theta}, e^{i\eta})) = P_{|z_1|}(\arg z_1 - \theta) \, P_{|z_2|}(\arg z_2 - \eta),$$

where $(z_1, z_2) \in \mathbb{D}^2$ and $\theta, \eta \in [0, 2\pi)$. As usual,

$$P_r(\theta) = \frac{1 - r^2}{(r^2 - 2r\cos(\theta) + 1)^2}$$

denotes the Poisson kernel of the unit disk.

The cyclicity of $f \in H^2(\mathbb{D}^2)$ does imply that f is an outer function. But this condition is no longer sufficient: there are outer functions that are not cyclic [Rudin 1969, Theorem 4.4.6]; this is another example of how the higher-dimensional theory is somewhat different. (See, however, [Mandrekar 1988; Douglas and Yang 2000; Redett and Tung 2010] for some positive results.)

Polynomials in two variables with no zeros in $\mathbb{D}^2$ are outer functions, and are therefore candidates for being cyclic in $\mathfrak{D}_\alpha$ for $\alpha \geq 0$. Indeed, Gelca [1995] proved that polynomials f with $\mathcal{Z}(f) \cap \mathbb{D}^2 = \varnothing$ are cyclic in $H^2(\mathbb{D}^2)$, the Hardy space of the bidisk, and hence in $\mathfrak{D}_\alpha$ for all $\alpha \leq 0$.

Overview of results. In [Bénéteau et al. 2015], the problem of cyclicity in Dirichlet-type spaces in the unit disk was studied. More specifically, the authors identified some subclasses of cyclic functions and derived sharp estimates on the rate of decay of the norms $\|p_n f - 1\|_\alpha$ for such $f \in D_\alpha$. It seems natural to investigate to what extent these results can be extended to functions $f \in \mathfrak{D}_\alpha$.

To make the notion of best possible norm decay precise, we let $\mathfrak{P}_n$, $n = 1, 2, \dots,$ be the subspaces of $\mathfrak{D}_\alpha$ consisting of polynomials of two variables of the form

$$p_n = \sum_{k=0}^{n} \sum_{l=0}^{n} c_{k,l} z_1^k z_2^l.$$

Note that we regard a monomial $z_1^k z_2^l$ in two variables as having degree $k + l$, meaning that members of $\mathfrak{P}_n$ are polynomials of degree at most $2n$. Similarly, we denote by $\mathcal{P}_n$ the space of polynomials of one complex variable having degree at most n. We now make the following definition.

Definition 1.1. Let $f \in \mathfrak{D}_\alpha$. We say that a polynomial $p_n \in \mathfrak{P}_n$ is an *optimal approximant* of order n to $1/f$ if p_n minimizes $\|pf - 1\|_\alpha$ among all polynomials $p \in \mathfrak{P}_n$. We call $\|p_n f - 1\|_\alpha$ the *optimal norm* of order n associated with f.

Stated differently, p_n is an optimal approximant to $1/f$ if we have

$$\|p_n f - 1\|_\alpha = \operatorname{dist}_{\mathfrak{D}_\alpha}(1, f \cdot \mathfrak{P}_n);$$

here, $\operatorname{dist}_X(x, A) = \inf\{\|x - a\|_X : a \in A\}$ is the usual distance function between a point and a subset $A \subset X$ of a normed space X.

Sharp estimates on the unit disk analog of $\operatorname{dist}_{\mathfrak{D}_\alpha}(1, f \cdot \mathfrak{P}_n)$ were obtained for certain classes of functions in [Bénéteau et al. 2015]. To state these estimates, we define $\varphi_1(s) = \log^+(s)$ for $s \in [0, \infty)$ and, when $\alpha < 1$,

$$\varphi_\alpha(s) = s^{1-\alpha}, \quad s \in [0, \infty).$$

Theorem 1.2 [Bénéteau et al. 2015, Theorem 3.6]. *Let $\alpha \leq 1$. If f is a function admitting an analytic continuation to the closed unit disk and whose zeros lie in $\mathbb{C} \setminus \mathbb{D}$, then there exists a constant $C = C(\alpha, f)$ such that*

$$\operatorname{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_m) \leq C \varphi_\alpha^{-1}(m + 1)$$

holds for all sufficiently large m. This estimate is sharp in the sense that if such a function f has at least one zero on $\mathbb{T}$, there exists a constant $\tilde{C} = \tilde{C}(\alpha, f)$ such that

$$\tilde{C}\varphi_\alpha^{-1}(m+1) \le \operatorname{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_m).$$

In this paper, we obtain analogous theorems for certain subclasses of functions in $\mathfrak{D}_\alpha$. We begin Section 2 with some general remarks concerning cyclicity in $\mathfrak{D}_\alpha$. For instance, if f is cyclic, then each slice function f_{z_j} obtained when fixing the variable z_j, $j = 1$ or 2, has to be cyclic in D_α. Then the problem of cyclicity and rates associated with optimal approximants is addressed for separable functions, i.e., for functions f of the form $f(z_1, z_2) = g(z_1)h(z_2)$. We prove that such a function is cyclic if and only if the factors g and h are cyclic in the one-variable space D_α, and then obtain, in Theorem 2.6, sharp estimates on $\operatorname{dist}_{\mathfrak{D}_\alpha}(1, f \cdot \mathfrak{P}_n)$ under the assumption that g and h admit analytic continuation to the closed disk and have no zeros in $\mathbb{D}$.

In Section 3, we turn our attention to functions of the form $f(z_1, z_2) = f(z_1^M \cdot z_2^N)$, for integers $M, N \ge 1$, and again obtain cyclicity results and sharp estimates in Theorem 3.1. Our proofs are based on the fact that certain restriction operators furnish isomorphisms between our subclasses of functions in $\mathfrak{D}_\alpha$ and the one-variable spaces $D_{2\alpha}$, and on comparisons between the associated norms.

In [Bénéteau et al. 2015], a key role was played by certain Riesz-type means of the power series expansion of $1/f$, which turned out to produce optimal, or near optimal, approximants to $1/f$. The one-variable construction extends to the bidisk setting as follows. Suppose $1/f$ has formal power series expansion

$$\frac{1}{f(z_1, z_2)} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{k,l} z_1^k z_2^l.$$

We then set

$$p_n(z_1, z_2) = \sum_{k=0}^{n}\sum_{l=0}^{n}\left(1 - \frac{\varphi_\alpha(\max\{k, l\})}{\varphi_\alpha(n+1)}\right) b_{k,l} z_1^k z_2^l. \tag{1-2}$$

Note that when $\alpha = 0$, the polynomials p_n are simply the n-th Cesàro means of the Taylor series of $1/f$:

$$\begin{aligned} C_n(1/f)(z_1, z_2) &= \sum_{k=0}^{n}\sum_{l=0}^{n}\left(1 - \frac{\max\{k, l\}}{n+1}\right) b_{k,l} z_1^k z_2^l \\ &= \frac{1}{n+1}\sum_{m=0}^{n} t_m(1/f)(z_1, z_2), \end{aligned}$$

where t_m denotes the m-th order Taylor polynomial. In Section 4, we take a closer look at some concrete polynomials in two variables, and show that in some cases the polynomials (1-2) are indeed close to optimal.

Recall that in the case of the unit disk, any polynomial that is zero-free in $\mathbb{D}$ is cyclic in D_α for all $\alpha \leq 1$. However, the analogous statement for the bidisk need not hold. In fact, we give examples of polynomials whose zero sets lie in $\mathbb{T}^2$ that are noncyclic for $\alpha > \frac{1}{2}$, and also polynomials with zeros on the boundary of the bidisk that are cyclic for all $\alpha \leq 1$; in fact, such polynomials can have zero sets that intersect $\mathbb{T}^2$, and extend into $\partial\mathbb{D}^2 \setminus \mathbb{T}^2$.

The existence of noncyclic polynomials in Hilbert spaces of analytic functions in higher dimensions has also been observed by Richter and Sundberg in the setting of the Drury–Arveson space in the unit ball of $\mathbb{C}^d$ when $d \geq 4$; see [Richter and Sundberg 2012] for this and other results on cyclic vectors in that context.

Many of our results and arguments carry over to the d-dimensional polydisk $\mathbb{D}^d$, but as notation becomes much more cumbersome, we restrict our attention to functions on the bidisk.

2. Classes of cyclic vectors in $\mathfrak{D}_\alpha$

In this section, we present some examples of cyclic functions in the bidisk. As a preliminary example, we have already observed that $f(z_1, z_2) = 1$ is cyclic in $\mathfrak{D}_\alpha$ for all α, and that cyclic functions cannot vanish inside the bidisk. Moreover, it is not difficult to see that if both f and $1/f$ extend to a larger bidisk, then f is nonvanishing on the closure $\overline{\mathbb{D}^2}$, and f is cyclic; indeed, if (p_n) is a sequence of polynomials such that $\|p_n - 1/f\|_\alpha$ tends to 0, the estimate

$$\|p_n f - 1\|_\alpha \leq \|f\|_{M(\mathfrak{D}_\alpha)} \|p_n - 1/f\|_\alpha,$$

where $\|\cdot\|_{M(\mathfrak{D}_\alpha)}$ denotes the multiplier norm, implies that $1 \in [f]$ and so f is cyclic.

However, there do exist cyclic functions in $\mathfrak{D}_\alpha$ that vanish on the boundary of the bidisk, as in the one variable case. In this section, we focus on three different ways of building functions in the bidisk from one variable functions in the unit disk, and explore the relationship between the cyclicity in two variables versus that in one variable. First, let us make some preliminary remarks.

Slices of a function. For a function $f = f(z_1, z_2)$ in the bidisk, we can fix the variable z_2, say, and consider the *slice*

$$f_{z_2}(z_1) = f(z_1, z_2), \quad z_1 \in \mathbb{D},$$

as a function in the unit disk. The slice f_{z_1} is defined in an analogous manner.

Proposition 2.1. *If f is cyclic in $\mathfrak{D}_\alpha$, then the slices f_{z_2} and f_{z_1} are cyclic in D_α.*

Proof. As a consequence of the Cauchy–Schwarz inequality applied to the coefficients of f_{z_2} we obtain

$$\|f_{z_2}\|_{D_\alpha} \le \|k_{z_2}\|_{D_\alpha} \cdot \|f\|_\alpha,$$

where k_{z_2} denotes the reproducing kernel at z_2 for D_α. Therefore, for any polynomial $p = p(z_1, z_2)$, we get

$$\|p_{z_2} f_{z_2} - 1\|_{D_\alpha} \le \|k_{z_2}\|_{D_\alpha} \cdot \|pf - 1\|_\alpha.$$

If f is cyclic in $\mathfrak{D}_\alpha$, then this last norm tends to 0 as the degree of p approaches ∞, and therefore for fixed z_2, $\|p_{z_2} f_{z_2} - 1\|_{D_\alpha}$ approaches 0 as well. Consequently, the slice f_{z_2} is cyclic in D_α. An analogous argument applies to the slices in z_1, and thus the result follows. □

Note that the converse of the above statement does not hold: consider, for example, $f(z_1, z_2) = 1 - z_1 z_2$. Then each slice f_{z_2} and f_{z_1} is nonvanishing in the closed unit disk (for a fixed z_2 and a fixed z_1, respectively), and thus each is cyclic in every D_α, but it turns out that f is only cyclic in $\mathfrak{D}_\alpha$ for $\alpha \le \frac{1}{2}$; see Remark 3.2.

Let us now consider three different natural ways to construct a one variable function from a two variable function and examine issues of cyclicity.

Diagonal restrictions. The *restriction to the diagonal* of a holomorphic function on the bidisk produces a function on the disk, and it turns out that these functions often inherit properties that allow us to transfer information between one and two variable spaces; see, e.g., [Horowitz and Oberlin 1975; Rudin 1969]. For instance, Massaneda and Thomas [2013] were able to use restriction arguments to show that it is not possible to characterize cyclic functions in $H^2(\mathbb{D}^2)$ in terms of decay at the boundary.

We define the restriction operator R_{diag} on $f \in \mathfrak{D}_\alpha$ by

$$R_{\text{diag}} : f \mapsto (\oslash f)(z) = f(z, z), \quad z \in \mathbb{D}.$$

To rigorously define which spaces this restriction operator acts on, we define the map

$$\beta(\alpha) = \begin{cases} \alpha - 1 & \text{for } \alpha \ge 0, \\ 2\alpha - 1 & \text{for } \alpha < 0. \end{cases}$$

In order to shorten notation, we use the abbreviation $\beta = \beta(\alpha)$. In the context of the Dirichlet-type spaces, the following restriction estimate holds.

Proposition 2.2. *For all* $f \in \mathfrak{D}_\alpha$,

$$\|\oslash f\|_{D_\beta} \le \|f\|_\alpha.$$

This result is probably known to the experts, and can be proved by appealing to the theory of reproducing kernels. For the convenience of the reader, we give an elementary proof.

Proof of Proposition 2.2. Let $f(z_1, z_2) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{k,l} z_1^k z_2^l$, which converges absolutely for every $|z_1| < 1$ and $|z_2| < 1$. Then

$$\oslash f(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{k,l} z^{k+l}$$

converges absolutely for every $|z| < 1$, hence can be rewritten as $\oslash f(z) = \sum_{n=0}^{\infty} b_n z^n$, where $b_n = \sum_{k+l=n} a_{k,l} = \sum_{k=0}^{n} a_{k,n-k}$. Thus,

$$\|\oslash f\|_{D_\beta}^2 = \sum_{n=0}^{\infty} |b_n|^2 (n+1)^\beta = \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} a_{k,n-k} \right|^2 (n+1)^\beta$$

and

$$\|f\|_\alpha^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} |a_{k,n-k}|^2 (k+1)^\alpha (n-k+1)^\alpha.$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|\sum_{k=0}^{n} a_{k,n-k}\right|^2 &\le \left(\sum_{k=0}^{n} |a_{k,n-k}|^2 (k+1)^\alpha (n-k+1)^\alpha\right)\left(\sum_{k=0}^{n} (k+1)^{-\alpha}(n-k+1)^{-\alpha}\right) \\ &\le \left(\sum_{k=0}^{n} |a_{k,n-k}|^2 (k+1)^\alpha (n-k+1)^\alpha\right)(n+1)^{-\beta}.\end{aligned}$$

In summary, our observations yield, as required,

$$\begin{aligned}\|\oslash f\|_{D_\beta}^2 &= \sum_{n=0}^{\infty} \left|\sum_{k=0}^{n} a_{k,n-k}\right|^2 (n+1)^\beta \\ &\le \sum_{n=0}^{\infty}\sum_{k=0}^{n} |a_{k,n-k}|^2 (k+1)^\alpha (n-k+1)^\alpha = \|f\|_\alpha^2. \qquad \square\end{aligned}$$

This result implies that a function $g \in D_\beta$ that arises as the restriction to the diagonal of a cyclic function in $\mathfrak{D}_\alpha$ is itself cyclic. Viewed differently, a function of two variables cannot be cyclic in $\mathfrak{D}_\alpha$ unless its restriction $\oslash f$ is cyclic in D_β, though it can happen that $\oslash f$ is cyclic, and $f \in \mathfrak{D}_\alpha$ is not: the functions considered in the examples in Section 4 are not cyclic in $\mathfrak{D}_2$, but their restrictions $\oslash f$ are cyclic in the Dirichlet space D (see also [Massaneda and Thomas 2013] for a discussion in the context of $H^2(\mathbb{D}^2)$). Moreover, together with the second assertion in Theorem 1.2, Proposition 2.2 immediately implies a lower bound for the decay rate of $\|p_n f - 1\|_\alpha^2$ for certain "nice" functions f:

Corollary 2.3. *Let $\alpha \leq 2$. Suppose $f \in \mathfrak{D}_\alpha$ is such that the diagonal restriction $\oslash f$ satisfies the hypotheses of Theorem 1.2. Then,*

$$\|p_n f - 1\|_\alpha^2 \geq C\varphi_\beta^{-1}(n+1), \quad \textit{for all } p_n \in \mathfrak{P}_n.$$

Here, we have used that $\varphi_\beta^{-1}(2n+1)$ is comparable to $\varphi_\beta^{-1}(n+1)$. We will see later (see Proposition 2.4 and Theorem 3.1) that this decay rate is not optimal in general. Note that the diagonal restrictions of the functions $f(z_1, z_2) = 1 - z_1 z_2$, $f(z_1, z_2) = (1 - z_1)(1 - z_2)$, and $f(z_1, z_2) = 1 - z_1$ all satisfy the hypotheses.

The above remarks show how, given a cyclic function of two variables, one can easily obtain examples of cyclic functions of one variable (although we might need to change the index α of the space in which cyclicity is being considered!) In the next two subsections we examine how to obtain some classes of cyclic functions of two variables from cyclic functions of one variable, and we obtain *sharp* rates of decay in some cases.

Separable functions. Let us now consider functions of two variables that can be written as products of two functions of one variable:

$$f(z_1, z_2) = g(z_1)h(z_2). \tag{2-1}$$

We shall refer to such functions as *separable*. Note that for such products, it follows from (1-1) that $\|f\|_\alpha = \|g\|_{D_\alpha}\|h\|_{D_\alpha}$.

Proposition 2.4. *Let $\alpha \in \mathbb{R}$ and f be defined as in* (2-1), *where $g, h \in D_\alpha$. Then, f is cyclic in $\mathfrak{D}_\alpha$ if and only if g and h are cyclic in D_α.*

Proof. First notice that by Proposition 2.1, if f is cyclic in $\mathfrak{D}_\alpha$, then g and h are constant multiples (with respect to the fixed variable) of the slices of f, and thus are cyclic in D_α.

For the converse, suppose both g and h are cyclic in D_α. Let (p_n) and (q_n) be sequences of polynomials such that $\|p_n g - 1\|_{D_\alpha} \to 0$ and $\|q_n h - 1\|_{D_\alpha} \to 0$, respectively. Since the expression $p_n gh - h = (p_n(z_1)g(z_1) - 1)h(z_2)$ is separable, we obtain

$$\|p_n f - h\|_\alpha = \|p_n g - 1\|_{D_\alpha}\|h\|_{D_\alpha}.$$

Hence, we get that $h \in [f]$, where $[\,\cdot\,]$ denotes the cyclicity class in $\mathfrak{D}_\alpha$, and so $[h] \subset [f]$. Since $\|q_n h - 1\|_\alpha = \|q_n h - 1\|_{D_\alpha}$, the function h is cyclic in $\mathfrak{D}_\alpha$ and D_α simultaneously, and the assertion follows. □

It seems natural to ask whether the growth of the extremal polynomials for separable functions is the same as for functions in the unit disk. As we will see in Theorem 2.6, this is indeed the case. Let us first prove a lemma that will help to establish the sharp growth restrictions.

Lemma 2.5. *Suppose $f = g \cdot h \in \mathfrak{D}_\alpha$ for $g, h \in \mathfrak{D}_\alpha$, and suppose that g admits a nonvanishing analytic continuation to the closed bidisk. Then, there exists a constant C, independent of n, such that*

$$\operatorname{dist}_{\mathfrak{D}_\alpha}(1, f \cdot \mathfrak{P}_n) \geq C \operatorname{dist}_{\mathfrak{D}_\alpha}(1, h \cdot \mathfrak{P}_{2n}).$$

Proof. Notice first that since the power series for g converges in a larger polydisk than the unit bidisk, there exists $R > 1$ such that if g_n are the Taylor polynomials of degree n approximating g, the multiplier norm $\|g - g_n\|_{M(\mathfrak{D}_\alpha)}$ decays exponentially like $R^{-(n+1)}$. Moreover, since in addition g has no zeros in the closed disk, the multiplier norm $\|1/g\|_{M(\mathfrak{D}_\alpha)}$ is bounded.

Now let $p_n(z_1, z_2)$ be the optimal approximant to $1/f$ of degree n. Then by the above remarks, we have

$$\|p_n h - 1/g\|_\alpha \leq \|1/g\|_{M(\mathfrak{D}_\alpha)} \|p_n f - 1\|_\alpha,$$

which goes to 0 as $n \to \infty$, and therefore, in particular, the norms $\|p_n h\|_\alpha$ are bounded by some constant C_1. Moreover,

$$\begin{aligned}\|p_n f - 1\|_\alpha &= \|p_n h(g - g_n) + g_n p_n h - 1\|_\alpha \\ &\geq \|g_n p_n h - 1\|_\alpha - \|p_n h\|_\alpha \|g - g_n\|_{M(\mathfrak{D}_\alpha)}.\end{aligned}$$

Since $\|p_n h\|_\alpha$ is bounded and $\|g - g_n\|_{M(\mathfrak{D}_\alpha)}$ decays exponentially, we obtain that there exists a constant C such that

$$\|p_n f - 1\|_\alpha \geq C \operatorname{dist}_{\mathfrak{D}_\alpha}(1, h \cdot \mathfrak{P}_{2n}).$$

□

Using Lemma 2.5, we obtain sharp estimates on the decay of norms.

Theorem 2.6. *Let $\alpha \leq 1$ and $g, h \in D_\alpha$. Suppose that g and h admit analytic continuations to $\overline{\mathbb{D}}$ and have no zeros in $\mathbb{D}$. Define $f(z_1, z_2) = g(z_1)h(z_2)$. Then there exists a constant $C = C(g, h, \alpha)$ such that*

$$\operatorname{dist}^2_{\mathfrak{D}_\alpha}(1, f \cdot \mathfrak{P}_n) \leq C \varphi_\alpha^{-1}(n+1),$$

for all sufficiently large n. Moreover, this estimate is sharp in the sense that if h has at least one zero on $\mathbb{T}$ and g has no zeros in the closed disk $\mathbb{D}$ (or vice versa), then there exists a constant $\tilde{C} = \tilde{C}(g, h, \alpha)$ such that

$$\tilde{C} \varphi_\alpha^{-1}(n+1) \leq \operatorname{dist}^2_{\mathfrak{D}_\alpha}(1, f \cdot \mathfrak{P}_n).$$

Proof. By Theorem 1.2, for any polynomials $p_n(z_1)$ and $q_n(z_2)$ of degree less than or equal to n, there exist constants C_1 and C_2 such that

$$\begin{aligned}\|p_n(z_1)g(z_1) - 1\|_{D_\alpha} &\leq C_1 \varphi_\alpha^{-1/2}(n+1), \\ \|q_n(z_2)h(z_2) - 1\|_{D_\alpha} &\leq C_2 \varphi_\alpha^{-1/2}(n+1).\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\|p_n(z_1)q_n(z_2)g(z_1)h(z_2)-1\|_\alpha \\
&\qquad \le \|q_n(z_2)h(z_2)(p_n(z_1)g(z_1)-1)\|_\alpha + \|q_n(z_2)h(z_2)-1\|_\alpha \\
&\qquad \le \|q_n h\|_\alpha \, \|p_n g-1\|_\alpha + \|q_n h-1\|_\alpha \\
&\qquad = \|q_n h\|_{D_\alpha} \, \|p_n g-1\|_{D_\alpha} + \|q_n h-1\|_{D_\alpha} \\
&\qquad \le (\|q_n h-1\|_{D_\alpha}+1) \, \|p_n g-1\|_{D_\alpha} + \|q_n h-1\|_{D_\alpha} \\
&\qquad \le C_2 C_1 \varphi_\alpha^{-1}(n+1) + (C_1+C_2)\varphi_\alpha^{-1/2}(n+1) \\
&\qquad \le C\varphi_\alpha^{-1/2}(n+1)
\end{aligned}
$$

for some constant C. Therefore,

$$\operatorname{dist}^2_{\mathfrak{D}_\alpha}(1, f\cdot\mathfrak{P}_n) \le C\varphi_\alpha^{-1}(n+1),$$

for all sufficiently large n, as desired.

Moreover, the inequality is sharp. To see this, suppose h has at least one zero on $\mathbb{T}$ and g has no zeros in the closed unit disk. Then, by Lemma 2.5, there exists a constant C_1 such that

$$\operatorname{dist}_{\mathfrak{D}_\alpha}(1, f\cdot\mathfrak{P}_n) \ge C_1 \operatorname{dist}_{\mathfrak{D}_\alpha}(1, h\cdot\mathfrak{P}_{2n}). \tag{2-2}$$

Note that $h = h(z_2)$, and so, by orthogonality of monomials in $\mathfrak{D}_\alpha$, the quantity $\operatorname{dist}_{\mathfrak{D}_\alpha}(1, h\cdot\mathfrak{P}_{2n})$ is bounded from below by $\operatorname{dist}_{\mathfrak{D}_\alpha}(1, h\cdot\mathcal{P}_{2n}) = \operatorname{dist}_{D_\alpha}(1, h\cdot\mathcal{P}_{2n})$. Now, by Theorem 1.2 applied to h, and again since $\varphi_\alpha(2n+1)$ is comparable to $\varphi_\alpha(n+1)$, there exists a constant C_2 such that

$$\operatorname{dist}^2_{D_\alpha}(1, h\cdot\mathcal{P}_n) \ge C_2\varphi_\alpha^{-1}(n+1). \tag{2-3}$$

Thus, the inequalities in (2-2) and (2-3) imply the desired result. □

3. Norm comparisons and sharp decay of norms for the subspaces $\mathcal{J}_{\alpha,M,N}$

Let us now consider a third way of relating two variable cyclic functions to one variable cyclic functions. In particular, we shall show that the polynomials in (1-2) furnish optimal approximants for a certain subclass of functions.

The subspaces $\mathcal{J}_{\alpha,M,N}$. In order to formulate our results, we need some notation. For $-\infty < \alpha < \infty$ and integers $M, N \ge 1$, we consider the closed subspaces

$$\mathcal{J}_{\alpha,M,N} = \left\{ f \in \mathfrak{D}_\alpha : f = \sum_{k=0}^{\infty} a_k z_1^{Mk} z_2^{Nk} \right\}.$$

For instance, $\mathcal{J}_\alpha = \mathcal{J}_{\alpha,1,1}$ consists of the functions f whose Taylor coefficients $(a_{k,l})$ vanish off the diagonal $k = l$, meaning that $f(z_1, z_2) = f(z_1 \cdot z_2)$.

We shall write D_{α,z_1} for the set of functions in D_α in the variable z_1, viewed as a subspace of $\mathfrak{D}_\alpha$.

Theorem 3.1. *Let $\alpha \le \frac{1}{2}$ and suppose that $f \in \mathcal{J}_{\alpha,M,N}$ has the property that $R(f)(z) = f(z^{1/M}, 1)$ is a function that admits an analytic continuation to the closed unit disk, whose zeros lie in $\mathbb{C} \setminus \mathbb{D}$. Then, f is cyclic in $\mathfrak{D}_\alpha$, and there exists a constant $C = C(\alpha, f, M, N)$ such that*

$$\operatorname{dist}^2_{\mathfrak{D}_\alpha}(1, f \cdot \mathfrak{P}_n) \le C\varphi^{-1}_{2\alpha}(n+1).$$

This result is sharp in the sense that, if $R(f)$ has at least one zero on $\mathbb{T}$, then there exists a constant $c = c(\alpha, f, M, N) > 0$ such that, for large n,

$$c\varphi^{-1}_{2\alpha}(n+1) \le \operatorname{dist}^2_{\mathfrak{D}_\alpha}(1, f \cdot \mathfrak{P}_n).$$

The same conclusions remain valid for $f \in D_{\alpha,z_1}$, with the rate $\varphi^{-1}_{2\alpha}$ replaced by φ^{-1}_{α}.

We should point out that the hypotheses of Theorem 3.1 imply that f is nonvanishing in $\mathbb{D}^2$. Suppose $f \in \mathcal{J}_{\alpha,M,N}$ and $f(z_1, z_2) = 0$, for some $(z_1, z_2) \in \mathbb{D}^2$. Then, the function $R(f)$ will have a zero at $z = z_1^M z_2^N \in \mathbb{D}$.

Remark 3.2. It is straightforward to check that functions like $f(z_1, z_2) = 1 - z_1$, $f(z_1, z_2) = (1 - z_1 z_2)^N$, $N \in \mathbb{N}$, and $f(z_1, z_2) = z_1^2 z_2^2 - 2\cos\theta z_1 z_2 + 1$, $\theta \in \mathbb{R}$, satisfy the assumptions of Theorem 3.1.

The arguments used in the proof of Theorem 3.1 imply a function $f \in \mathcal{J}_{\alpha,M,N}$ can *fail* to be cyclic in $\mathfrak{D}_\alpha$ when $\alpha > \frac{1}{2}$. For instance, the function $f(z_1, z_2) = 1 - z_1 z_2$ is cyclic if and only if $\alpha \le \frac{1}{2}$ (see Example 2 below), and the Riesz polynomials (1-2) are optimal approximants to $1/f$ when $\alpha \le \frac{1}{2}$.

Liftings, restrictions, and norm comparisons. The proof of Theorem 3.1 ultimately relies on Theorem 1.2, and comparison between the norm of $\mathfrak{D}_\alpha$ and that of $D_{2\alpha}$.

Suppose that for some real α, the function $F = \sum_{k=0}^{\infty} a_k z^k$ belongs to D_α, a Dirichlet-type space on the unit disk. We define $E : D_\alpha \to \mathfrak{D}_\alpha$ by

$$E(F)(z_1, z_2) = F(z_1).$$

In addition, if $f \in D_{\alpha,z_1}$, the mapping $C : \mathfrak{D}_\alpha \to D_\alpha$ given by $C(f)(z) = f(z, 1)$ is well-defined, and we have $E \circ C|_{D_{\alpha,z_1}} = \mathrm{id}_{D_{\alpha,z_1}}$. Moreover, it is immediate that

$$\|E(F)\|_\alpha = \|F\|_{D_\alpha}, \quad F \in D_\alpha$$

and

$$\|f\|_\alpha = \|C(f)\|_{D_\alpha}, \quad f \in D_{\alpha,z_1}.$$

Another embedding is the following one. For $\alpha \in \mathbb{R}$ fixed, define the mappings

$$L_{M,N} : D_{2\alpha} \to \mathfrak{D}_\alpha \quad \text{via} \quad L_{M,N}(F)(z_1, z_2) = F(z_1^M \cdot z_2^N),$$

and

$$R_{M,N} : \mathcal{J}_{\alpha,M,N} \to D_{2\alpha} \quad \text{via} \quad R_{M,N}(f)(z) = f(z^{1/M}, 1).$$

We initially view $f(z^{1/M}, 1)$ as a formal expression, but the assumption that

$$\sum_k (k+1)^{2\alpha} |a_k|^2 < \infty$$

implies that $f(z_1^{1/M}, 1)$ is actually a well-defined holomorphic function on $\mathbb{D}$; this will become apparent below. By definition, we again have $L \circ R|_{\mathcal{J}_{\alpha,M,N}} = \mathrm{id}_{\mathcal{J}_{\alpha,M,N}}$.

Lemma 3.3. *For $F \in D_{2\alpha}$ and $f \in \mathcal{J}_{\alpha,M,N}$, there are constants $c_1 = c_1(\alpha, M, N)$ and $c_2 = c_2(\alpha, M, N)$ such that*

$$\|L_{M,N}(F)\|_\alpha \le c_1 \|F\|_{D_{2\alpha}} \quad \textit{and} \quad c_2 \|R(f)\|_{D_{2\alpha}} \le \|f\|_\alpha.$$

In particular, if $f \in \mathcal{J}_{\alpha,M,N}$, then

$$c_2 \|R(f)\|_{D_{2\alpha}} \le \|f\|_\alpha \le c_1 \|R(f)\|_{D_{2\alpha}}. \tag{3-1}$$

Proof. We provide the proof of the second inequality; the proof of the first is analogous.

We first observe that for any $\alpha \in \mathbb{R}$ and $M \ge 1$, there exist constants $c_1(\alpha, M)$ and $c_2(\alpha, M)$ such that

$$c_1(\alpha, M)(k+1)^\alpha \le (Mk+1)^\alpha \le c_2(\alpha, M)(k+1)^\alpha,$$

for any $k \in \mathbb{N}$. Thus, writing $R(f)(z) = \sum_{k=0}^\infty a_k z^k$, we have

$$\begin{aligned}
\|R(f)\|_{D_{2\alpha}}^2 &= \sum_{k=0}^\infty (k+1)^{2\alpha} |a_k|^2 = \sum_{k=0}^\infty (k+1)^\alpha (k+1)^\alpha |a_k|^2 \\
&\le [c_1(\alpha, M) c_1(\alpha, N)]^{-1} \sum_{k=0}^\infty (Mk+1)^\alpha (Nk+1)^\alpha |a_k|^2 \\
&= [c_1(\alpha, M) c_1(\alpha, N)]^{-1} \|f\|_\alpha^2,
\end{aligned}$$

which proves the assertion. The two-sided bound (3-1) follows from the one-sided bounds and the fact that $f = L(R(f))$. □

In particular, we see from the proof of Lemma 3.3 that in the case $M = N = 1$, the equalities

$$\|L(F)\|_\alpha = \|F\|_{D_{2\alpha}} \quad \text{and} \quad \|R(f)\|_{D_{2\alpha}} = \|f\|_\alpha$$

hold; hence, R is an isometric isomorphism between $\mathcal{J}_\alpha$ and $D_{2\alpha}$.

Sharpness of norm decay. We shall use Lemma 3.3, along with the following lemma, to prove Theorem 3.1.

Lemma 3.4. *Suppose that $f \in \mathcal{J}_{\alpha,M,N}$ for some $\alpha \in \mathbb{R}$ and some integers $M, N \geq 1$. Let $r_n = \sum_{k=0}^{n} \sum_{l=0}^{n} c_{k,l} z_1^k z_2^l$ be an arbitrary polynomial, and let s_n be its projection onto $\mathcal{J}_{\alpha,M,N}$,*

$$s_n = \sum_{\{k:\, Mk, Nk \leq n\}} c_{Mk,Nk} z_1^{Mk} z_2^{Nk}.$$

Then,

$$\|r_n f - 1\|_\alpha \geq \|s_n f - 1\|_\alpha.$$

Proof. We begin by noting again that monomials of the form $\{z_1^k z_2^l\}$ form an orthogonal basis for $\mathfrak{D}_\alpha$. Next, setting $\tilde{s}_n = r_n - s_n$, we have $s_n f \in \mathcal{J}_{\alpha,M,N}$, and $\tilde{s}_n f \notin \mathcal{J}_{\alpha,M,N}$. Then, by the previous observation, $s_n f - 1 \perp \tilde{s}_n f$.

This means that

$$\begin{aligned} \|r_n f - 1\|_\alpha^2 &= \|s_n f - 1 + \tilde{s}_n f\|_\alpha^2 \\ &= \|s_n f - 1\|_\alpha^2 + \|\tilde{s}_n f\|_\alpha^2 \\ &\geq \|s_n f - 1\|_\alpha^2. \end{aligned}$$

□

An analogous result holds for functions in the subspace D_{α,z_1}.

Proof of Theorem 3.1. We present the details for functions $f \in \mathcal{J}_\alpha$; the same type of arguments work for $\mathcal{J}_{\alpha,M,N}$, with the appropriate inequalities from Lemma 3.3 in place of equalities, and also for $f \in D_{\alpha,z_1}$.

We begin by establishing the lower bound. Let $r_n = \sum_k \sum_l c_{k,l} z_1^k z_2^l$ be any polynomial, and extract the diagonal part s_n from r_n as in the preceding lemma. Note that by construction, $s_n f - 1 \in \mathcal{J}_\alpha$ for each α. By Lemma 3.4 and the norm inequality (3-1), we obtain

$$\|r_n f - 1\|_\alpha \geq \|s_n f - 1\|_\alpha = \|R(s_n f - 1)\|_{D_{2\alpha}} = \|R(s_n) R(f) - 1\|_{D_{2\alpha}}.$$

It is assumed that $R(f)$ satisfies the hypotheses of Theorem 1.2; the theorem then asserts that $\operatorname{dist}^2_{D_{2\alpha}}(1, R(f) \cdot \mathcal{P}_n) \geq \tilde{C} \varphi_{2\alpha}^{-1}(n+1)$. In particular, this yields a lower bound for $\|R(s_n)R(f) - 1\|_{D_{2\alpha}}$, and the lower bound on $\operatorname{dist}_{\mathfrak{D}_\alpha}(1, f \cdot \mathfrak{P}_n)$ follows.

To obtain the upper bound, it is enough to exhibit a concrete sequence (p_n) of polynomials having $\|p_n f - 1\|_\alpha^2 \leq C(\alpha, f) \varphi_{2\alpha}^{-1}(n+1)$. However, since $R(f)$ satisfies the hypotheses of Theorem 1.2, there exists a sequence (q_n) of polynomials in one variable that achieves

$$\|q_n R(f) - 1\|^2_{D_{2\alpha}} \leq C(\alpha, f) \varphi_{2\alpha}^{-1}(n+1)$$

for large enough n. But then we can define $p_n = L(q_n) \in \mathcal{J}_\alpha$, and the desired estimate follows since

$$\|L(q_n)f - 1\|_\alpha^2 = \|R(L(q_n))R(f) - 1\|_{D_{2\alpha}}^2 = \|q_n R(f) - 1\|_{D_{2\alpha}}^2$$

by Lemma 3.3. □

Note that if $R(f)$ is a polynomial with only simple zeros on the unit circle $\mathbb{T}$, then it is shown in [Bénéteau et al. 2015, Section 3] that the one-variable Riesz polynomials achieve the norm decay obtained above. In the situation $M = N = 1$ then, we have $L(q_n)(z_1, z_2) = p_n(z_1, z_2)$, where p_n are the Riesz-type polynomials defined in (1-2).

4. Polynomials with zeros on $\partial\mathbb{D}^2$ and measures of finite energy

Let us now examine the relationship between cyclicity and boundary zero sets of functions in $\mathfrak{D}_\alpha$. Surprisingly, some functions with large zero sets in some sense *are* cyclic while others with smaller zero sets are not.

Examples. Let us examine a few simple examples.

Example 1. Set $f(z_1, z_2) = 1 - z_1$. Then f has zero set

$$\mathcal{Z}(f) = \{1\} \times \overline{\mathbb{D}},$$

a (real) 2-dimensional subset of the topological boundary of $\mathbb{D}^2$ which meets the distinguished boundary along the 1-dimensional curve $\{1\} \times \mathbb{T}$. Note that f is an example of a function of the product type $g(z_1)h(z_2)$ with $g(z_1) = 1 - z_1$ and $h(z_2) = 1$, and therefore by Proposition 2.4, f is cyclic in $\mathfrak{D}_\alpha$ if and only if $\alpha \le 1$.

Example 2. Consider the function $f(z_1, z_2) = 1 - z_1 z_2$. The part of the zero set of f that lies on the boundary of the bidisk,

$$\mathcal{Z}(f) = \{(e^{i\theta}, e^{-i\theta}) : \theta \in [0, 2\pi)\},$$

can be seen as a 1-dimensional real curve contained in the distinguished boundary $\mathbb{T}^2$. One verifies that all the points in $\mathcal{Z}(f)$ are simple zeros. Since

$$\frac{1}{f(z_1, z_2)} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \delta_{k,l} z_1^k z_2^l = \sum_{k=0}^{\infty} z_1^k z_2^k,$$

we have $\|1/f\|_{-1}^2 = \sum_{k=0}^{\infty}(1+k)^{-2} < \infty$ but $\|1/f\|_0^2 = \sum_{k=0}^{\infty} 1 = +\infty$, so f is invertible in the Bergman space, and indeed in $\mathfrak{D}_\alpha$ whenever $\alpha < -\frac{1}{2}$, but not in the Hardy space of the bidisk.

Nevertheless, by Theorem 3.1, f is cyclic in $\mathfrak{D}_\alpha$ if and only if $\alpha \le \frac{1}{2}$. Note in particular that this function is *not* cyclic in the classical Dirichlet space of the bidisk!

Explicit computations with the Riesz polynomials in (1-2) recover the upper bound in Theorem 3.1. Namely, we have

$$p_n(z_1,z_2)f(z_1,z_2)-1=-\frac{1}{\varphi_\alpha(n+1)}\sum_{k=1}^{n+1}[\varphi_\alpha(k)-\varphi_\alpha(k-1)](z_1z_2)^k,$$

and then, since $|\varphi_\alpha(k)-\varphi_\alpha(k-1)|^2\le C(\alpha)(k-1)^{-2\alpha}$, we obtain

$$\|p_nf-1\|_\alpha^2\le\frac{C_1(\alpha)}{(n+1)^{1-2\alpha}}.$$

Thus $\|p_nf-1\|_\alpha^2\to 0$ as $n\to\infty$ and f is cyclic, provided that $\alpha\le\frac{1}{2}$.

In fact, considering functions of the form $f=1-z_1^Mz_2^N$ for integer $M,N\ge 1$ instead, and performing the analogous computations, we obtain

$$\|p_nf-1\|_\alpha^2\le\frac{C_1(\alpha,M,N)}{(n+1)^{1-2\alpha}} \tag{4-1}$$

with a constant $C_1(\alpha,M,N)$ which does not depend on n.

Example 3. We examine $f(z_1,z_2)=1-z_1-z_2+z_1z_2=(1-z_1)(1-z_2)$. The zero set of f is

$$\mathcal{Z}(f)=(\{1\}\times\overline{\mathbb{D}})\cup(\overline{\mathbb{D}}\times\{1\}),$$

a 2-dimensional set that extends into the topological boundary of the bidisk. Its intersection with $\mathbb{T}^2$ consists of the curves

$$\mathcal{Z}(f)=(\{1\}\times\mathbb{T})\cup(\mathbb{T}\times\{1\}).$$

All zeros of f are simple, except the point (1, 1), which has order 2. Since

$$\frac{1}{f(z_1,z_2)}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}z_1^kz_2^l,$$

it follows that $1/f\notin A^2(\mathbb{D}^2)$. Note that again, f is separable with $g(z_1)=1-z_1$ and $h(z_2)=1-z_2$, and therefore f is cyclic in $\mathfrak{D}_\alpha$ if and only if $\alpha\le 1$.

In this case, computing with the Riesz polynomials leads to misleading estimates. Defining polynomials p_n, as before, via (1-2), we compute

$$p_nf=-\frac{1}{(n+1)^{1-\alpha}}\sum_{k=1}^{n+1}[k^{1-\alpha}-(k-1)^{1-\alpha}](z_1^k+z_2^k)$$
$$+\frac{1}{(n+1)^{1-\alpha}}\sum_{k=1}^{n+1}[k^{1-\alpha}-(k-1)^{1-\alpha}]z_1^kz_2^k.$$

We use the estimates from the previous example, and exploit the one-variable estimates from [Bénéteau et al. 2015], to obtain

$$\begin{aligned}\|p_n f - 1\|^2_{\mathfrak{D}_\alpha} &= \frac{2}{(n+1)^{2-2\alpha}} \sum_{k=1}^{n+1} (k+1)^{\alpha} [k^{1-\alpha} - (k-1)^{1-\alpha}]^2 \\ &\quad + \frac{1}{(n+1)^{2-2\alpha}} \sum_{k=1}^{n+1} (k+1)^{2\alpha} [k^{1-\alpha} - (k-1)^{1-\alpha}]^2 \\ &\le \frac{c_1(\alpha)}{(n+1)^{1-\alpha}} + \frac{c_2(\alpha)}{(n+1)^{1-2\alpha}}.\end{aligned}$$

The first term in the right-hand side dominates when $\alpha < 0$, whereas the second is larger when $\alpha > 0$. In particular, the estimate does show that f is cyclic in $\mathfrak{D}_\alpha$ provided $\alpha \le \frac{1}{2}$. However, as we have seen, the rate is not optimal, and f remains cyclic when $\alpha > \frac{1}{2}$.

Note the interesting contrast between Example 2 and Example 3: the function in Example 2 is not cyclic in the (classical) Dirichlet space of the bidisk, and yet in some sense has a much smaller zero set than the function in Example 3, which is cyclic! On the other hand, as a kind of dual phenomenon, $f = 1 - z_1 z_2$ exhibits a faster rate of decay of norms $\|p_n f - 1\|_\alpha$ for $\alpha < 0$ than does $f = (1 - z_1)(1 - z_2)$.

Example 4. The polynomial $f(z_1, z_2) = 1 - (z_1 + z_2)/2$ has no zeros in $\mathbb{D}^2$, and vanishes at a single boundary point: $\mathcal{Z}(f) = \{(1, 1)\} \subset \mathbb{T}^2$.

In [Hedenmalm 1988, Section 4], it is proved that if $f \in \mathfrak{D}_2$ has $\mathcal{Z}(f) = \{(1, 1)\}$, and both $f(\cdot, 1)$ and $f(1, \cdot)$ are outer functions, then the closure of the principal ideal generated by f coincides with the closed ideal

$$\mathcal{I}\big(\{(1, 1)\}\big) = \{f \in \mathfrak{D}_2 : f(1, 1) = 0\}.$$

(Hedenmalm's norm is defined using the weights $(1 + k^2)(1 + l^2)$ but is equivalent to the norm in $\mathfrak{D}_2$.) Since the norm of $\mathfrak{D}_1$ is weaker than that of $\mathfrak{D}_2$, it follows that such functions are cyclic in $\mathfrak{D}_\alpha$ for $\alpha \le 1$ as the $\mathfrak{D}_1$-closure of the invariant subspace $\mathcal{I}\big(\{(1, 1)\}\big) \subset \mathfrak{D}_2$ coincides with $[f]$, and contains the cyclic function $1 - z_1$.

In particular, the polynomial $f(z_1, z_2) = 1 - (z_1 + z_2)/2$ is cyclic in $\mathfrak{D}_\alpha$, for all $\alpha \le 1$. (An independent proof of this fact has been given by T. J. Ransford [personal communication, 2014].) Computing with polynomials of the form

$$p_n(z_1, z_2) = \sum_{k=0}^{n} \left(1 - \frac{\varphi_\alpha(k)}{\varphi_\alpha(n+1)}\right) \frac{(z_1 + z_2)^k}{2^k}$$

and using the fact that $(z_1+z_2)^{k_1} \perp (z_1+z_2)^{k_2}$ when $k_1 \neq k_2$, one finds that

$$\|p_n f - 1\|_\alpha^2 = \sum_{k=1}^{n+1} 4^{-k} \left(\frac{k^{1-\alpha} - (k-1)^{1-\alpha}}{(n+1)^{1-\alpha}} \right)^2 \sum_{j=0}^{k} \binom{k}{j}^2 (j+1)^\alpha (k-j+1)^\alpha.$$

Using the bound

$$(j+1)^\alpha (k-j+1)^\alpha \le C(k+1)^{2\alpha}, \quad 0 \le j \le k,$$

together with the identity

$$\sum_{j=0}^{k} \binom{k}{j}^2 = \binom{2k}{k}$$

and standard estimates on binomial coefficients, we obtain the estimate

$$\operatorname{dist}^2_{\mathfrak{D}_\alpha}(1, (2-z_1-z_2)\cdot \mathfrak{P}_n) \le C\varphi_{2\alpha-1/2}(n+1).$$

Unfortunately, we have not been able to obtain a sharp estimate, but the above bound shows that the optimal rate is different from the two rates we have seen previously.

Measures of finite energy. It would be interesting to understand the relationship between cyclicity and boundary zero sets — in particular, given a function f, to find a measure whose support lies on the zero set of the boundary values of f that relates to the cyclicity properties of f.

We now give a necessary condition for a function to be cyclic. This condition involves the notion of capacity, and represents a straightforward generalization of results of Brown and Shields in the one-variable case.

Definition 4.1. Let $E \subset \mathbb{T}^2$ be a Borel set. We say that a probability measure μ supported in E has *finite logarithmic energy* if

$$I[\mu] = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \log \frac{e}{|e^{i\theta_1} - e^{i\vartheta_1}|} \log \frac{e}{|e^{i\theta_2} - e^{i\vartheta_2}|} \, d\mu(\theta_1, \theta_2) \, d\mu(\vartheta_1, \vartheta_2) < \infty.$$

If E supports no such measure, we say that E has *logarithmic capacity* 0.

The integral defining the energy $I[\mu]$ can be seen as a convolution with kernel

$$h(s,t) = \log \frac{e}{|1-e^{is}|} \log \frac{e}{|1-e^{it}|}.$$

Replacing the logarithmic product kernel in the definitions above with

$$h_\alpha(s,t) = \frac{1}{|1-e^{is}|^{1-\alpha}} \frac{1}{|1-e^{it}|^{1-\alpha}},$$

one obtains the notions of *Riesz energy*, denoted by $I_\alpha[\mu]$, and *Riesz capacity* of order $0 < \alpha < 1$.

The α-energy of μ can be expressed in terms of its Fourier coefficients

$$\hat{\mu}(k,l)=\int_{\mathbb{T}^2} e^{-i(k\theta_1+l\theta_2)}\,d\mu(\theta_1,\theta_2),\qquad k,l\in\mathbb{Z}.$$

Namely, we have (compare [El-Fallah et al. 2014, Chapter 2], for instance)

$$I_\alpha[\mu]=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\hat{h}_\alpha(k,l)|\hat{\mu}(k,l)|^2.$$

Computing the Fourier coefficients $\hat{h}(k,l)$ (see [Brown and Shields 1984, p. 294] for details), we find that

$$I[\mu]=1+\sum_{k=1}^{\infty}\frac{|\hat{\mu}(k,0)|^2}{k}+\sum_{l=1}^{\infty}\frac{|\hat{\mu}(0,l)|^2}{l}+\frac{1}{2}\sum_{k\in\mathbb{Z}\setminus\{0\}}\sum_{l=1}^{\infty}\frac{|\hat{\mu}(k,l)|^2}{|k|l}. \tag{4-2}$$

Similarly, one can show (again see [El-Fallah et al. 2014, Chapter 2]) that the Fourier coefficients of h_α satisfy

$$c_1(|k|+1)^{-\alpha}(|l|+1)^{-\alpha}\le|\hat{h}_\alpha(k,l)|\le c_2(|k|+1)^{-\alpha}(|l|+1)^{-\alpha}$$

for some constants $0<c_1<c_2<\infty$.

The notion of energy now allows us to identify some noncyclic $f\in\mathfrak{D}_\alpha$ by looking at their boundary zero sets. To make this notion precise, we note that one can show that functions $f\in\mathfrak{D}_\alpha$ have radial limits

$$f^*(e^{i\theta_1},e^{i\theta_2})=\lim_{r\to 1^-}f(re^{i\theta_1},re^{i\theta_2})$$

quasi-everywhere with respect to the appropriate capacity. That is, the limit exists for all points outside a set of capacity 0, and hence it makes sense to speak of the capacity of the set $\mathcal{Z}(f^*)$. (In fact, Kaptanoğlu considers more general approach regions in [Kaptanoğlu 1994], but we do not need this here.)

Proposition 4.2. *If $f\in\mathfrak{D}$ and $\mathcal{Z}(f^*)$ has positive logarithmic capacity, then f is not cyclic.*

Proof. The proof is completely analogous to that of [Brown and Shields 1984, Theorem 5]; we refer the reader to the paper of Brown and Shields for details and present the arguments in condensed form here.

The key idea is to identify the Bergman space $A^2(\mathbb{D}^2)$ with the dual of $\mathfrak{D}$ via the pairing

$$\langle f,g\rangle=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a_{k,l}b_{k,l},$$

where $f=\sum_{k,l}a_{k,l}z_1^kz_2^l\in\mathfrak{D}$ and $g=\sum_{k,l}b_{k,l}z_1^kz_2^l\in A^2(\mathbb{D}^2)$. We then consider the

Cauchy integral

$$C[\mu]=\int_{\mathbb{T}^2}(1-e^{i\theta_1}z_1)^{-1}(1-e^{i\theta_2}z_2)^{-1}\,d\mu(\theta_1,\theta_2)$$

of μ, a measure of finite logarithmic energy with $\mathrm{supp}(\mu)\subset\mathcal{Z}(f^*)$. A comparison with (4-2) then reveals that

$$\|C[\mu]\|^2_{A^2(\mathbb{D}^2)}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{|\hat{\mu}(k,l)|^2}{(k+1)(l+1)}<\infty$$

so that $C[\mu]$ induces a nontrivial element of $\mathfrak{D}^*$. On the other hand, since the measure μ is supported on $\mathcal{Z}(f^*)$ by assumption, the functional induced by $C[\mu]$ annihilates $[f]$, and so f is not cyclic. □

For $0<\alpha<1$, the same result holds once we replace logarithmic capacity with Riesz capacity and make the identification $(\mathfrak{D}_\alpha)^*=\mathfrak{D}_{-\alpha}$ in the proof.

The argument used in the proof of Proposition 4.2 can be used to give another proof of the noncyclicity of the function $f(z_1,z_2)=1-z_1z_2$ in $\mathfrak{D}$. Namely, consider the probability measure $\mu_{\mathcal{Z}}$ on $\mathbb{T}^2$ induced by the (normalized) *integration current* associated with the variety $\mathcal{Z}(1-z_1z_2)\cap\mathbb{T}^2$ (see [Lelong and Gruman 1986, Chapter 2] for the relevant definitions). A quick computation reveals that $\hat{\mu}_{\mathcal{Z}}(k,l)=\delta_{kl}$, so that $C[\mu_{\mathcal{Z}}](z_1,z_2)=1/(1-z_1z_2)$, a function in the Bergman space of the bidisk which satisfies

$$\left\langle z_1^kz_2^lf,\,C[\mu_{\mathcal{Z}}]\right\rangle=0,\quad \text{for all } k,l\geq 0.$$

In fact, $\mathcal{Z}(1-z_1z_2)\cap\mathbb{T}^2$ has positive Riesz capacity precisely when $\alpha>\frac{1}{2}$.

5. Concluding remarks and open problems

It appears to be a difficult task to characterize the cyclic elements of $\mathfrak{D}_\alpha$ for $\alpha\leq 1$, and many basic questions remain. For instance, it is natural to ask whether the *Brown–Shields conjecture* is true for functions on the bidisk.

Problem 5.1. Is the condition that $f\in\mathfrak{D}$ is outer and $\mathcal{Z}(f^*)$ has logarithmic capacity 0 sufficient for f to be cyclic?

This question remains open for the Dirichlet space of the unit disk, and is widely considered to be a challenging problem. A first step towards understanding cyclic functions in $\mathfrak{D}_\alpha$ might be to solve the following natural problem.

Problem 5.2. Characterize the *cyclic polynomials* $f\in\mathfrak{D}_\alpha$ for each $\alpha\in(0,1]$.

An obvious necessary condition for f to be cyclic is that $\mathcal{Z}(f)\cap\mathbb{D}^2=\varnothing$, and if f is a polynomial that does not vanish in $\overline{\mathbb{D}^2}$, then f is cyclic because both f and $1/f$ extend analytically to a larger polydisk. But the problem appears to be open for polynomials with $\mathcal{Z}(f)\cap\partial\mathbb{D}^2\neq\varnothing$: we would at least like to identify the polynomials whose zero sets have positive capacity. We have proved that polynomials that are products of polynomials in one variable are cyclic, and so the zero sets associated with such functions must all have zero capacity.

As we have seen in our examples, it can happen that a polynomial with a larger zero set, in the topological sense and in the sense of measure, is cyclic in $\mathfrak{D}_\alpha$ for some α, while a polynomial with a smaller zero set is not. We have also noted that a polynomial that *fails* to be cyclic in $\mathfrak{D}_\alpha$ when $\alpha>\frac{1}{2}$ can be "more" cyclic in $\mathfrak{D}_\alpha$, for $\alpha<0$, than polynomials that are cyclic in all $\mathfrak{D}_\alpha$. We mean this in the sense that

$$\operatorname{dist}^2_{\mathfrak{D}_\alpha}(1,\,(1-z_1z_2)\cdot\mathfrak{P}_n)\asymp C\varphi_{2\alpha}^{-1}(n+1)$$

while

$$\operatorname{dist}^2_{\mathfrak{D}_\alpha}(1,\,(1-z_1)(1-z_2)\cdot\mathfrak{P}_n)\asymp C\varphi_{\alpha}^{-1}(n+1).$$

It would be interesting to develop a rigorous understanding of this phenomenon.

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