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CHRISTINE BREINER AND TOBIAS LAMM

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COMPACTNESS RESULTS FOR SEQUENCES OF APPROXIMATE BIHARMONIC MAPS

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We will prove energy quantization for approximate (intrinsic and extrinsic) biharmonic maps into spheres where the approximate map is in $L \log L$. Moreover, we demonstrate that if the $L \log L$ norm of the approximate maps does not concentrate, the images of the bubbles are connected without necks.

1. Introduction

Critical points to the Dirichlet energy

$$E(u) := \frac{1}{2} \int_\Omega |Du|^2 \, dx$$

are called *harmonic maps*, and the compactness theory for such a sequence in two dimensions is well understood. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and N a smooth, compact Riemannian manifold. For a sequence of harmonic maps $u_k \in W^{1,2}(\Omega, N)$ with uniform energy bounds, Sacks and Uhlenbeck [1981] proved that a subsequence u_k converges weakly to a harmonic u_∞ on Ω and $u_k \to u_\infty$ in $C^\infty(\Omega \setminus \{x_1, \dots, x_\ell\})$ for some finite ℓ depending on the energy bound. For each x_i, they showed that there exist some number of "bubbles", maps $\phi_{ij} : \mathbb{S}^2 \to N$, that result from appropriate conformal scalings of the sequence u_k near x_i. In dimension 2, $E(u)$ is conformally invariant and thus one can ask whether any energy is lost in the limit. Jost [1991] proved that in fact the energy is quantized; there is no unaccounted energy loss:

$$\lim_{k\to\infty} E(u_k) = E(u_\infty) + \sum_{i=1}^{\ell} \sum_{j=1}^{\ell_i} E(\phi_{ij}).$$

Parker [1996] provided the complete description of the C^0 limit or "bubble tree". In particular, he demonstrated that the images of the limiting map u_∞ and the bubbles ϕ_{ij} are connected without necks. Around the same time, various authors

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proved energy quantization and the no-neck property for approximate harmonic maps [Ding and Tian 1995; Wang 1996; Qing and Tian 1997; Lin and Wang 1998; Chen and Tian 1999].

In this paper, we are interested in an analogous compactness problem for a scale-invariant energy in four dimensions. Let (M^4, g) and (N^k, h) be compact Riemannian manifolds without boundary, with N^k isometrically embedded in some $\mathbb{R}^n$. Consider the energy functional

$$E_{\text{ext}}(u) := \int_M |\Delta u|^2 \, dx$$

for $u \in W^{2,2}(M, N)$, where Δ is the Laplace–Beltrami operator. Critical points to this functional are called *extrinsic biharmonic maps*, and the Euler–Lagrange equation satisfied by such maps is of fourth order. Clearly, this functional depends upon the immersion of N into $\mathbb{R}^n$. To avoid such a dependence, one may instead consider critical points to the functional

$$E_{\text{int}}(u) := \int_M \left|(\Delta u)^T\right|^2 dx,$$

where $(\Delta u)^T$ is the projection of Δu onto $T_u N$. Critical points to this functional are called *intrinsic biharmonic maps*. The Euler–Lagrange equations satisfied by extrinsic and intrinsic biharmonic maps have been computed (see, for instance, [Wang 2004b]). We will be interested in approximate critical points.

Definition 1.1. Let $u \in W^{2,2}(B_1, N)$, where $B_1 \subset \mathbb{R}^4$ and N is a C^3 closed submanifold of some $\mathbb{R}^n$. Let $f \in L \log L(B_1, \mathbb{R}^n)$. Then u is an *f-approximate biharmonic map* if

$$\Delta^2 u - \Delta(A(u)(Du, Du)) - 2d^*\langle \Delta u, DP(u)\rangle + \langle \Delta(P(u)), \Delta u\rangle = f.$$

We call u an *f-approximate intrinsic biharmonic map* if

$$\begin{aligned}\Delta^2 u - \Delta(A(u)(Du, Du)) - 2d^*\langle \Delta u, DP(u)\rangle \\ + \langle \Delta(P(u)), \Delta u\rangle - P(u)(A(u)(Du, Du) D_u A(u)(Du, Du)) \\ - 2A(u)(Du, Du) A(u)(Du, DP(u)) = f.\end{aligned}$$

Here A is the second fundamental form of $N \hookrightarrow \mathbb{R}^n$ and $P(u) : \mathbb{R}^n \to T_u N$ is the orthogonal projection from $\mathbb{R}^n$ to the tangent space of N at u.

Recently, Hornung and Moser [2012], Laurain and Rivière [2013], and Wang and Zheng [2012] determined the energy quantization result for sequences of intrinsic biharmonic maps, approximate intrinsic and extrinsic biharmonic maps, and approximate extrinsic biharmonic maps, respectively. (In fact, the result of

[Laurain and Rivière 2013] applies to a broader class of solutions to scaling-invariant variational problems in dimension four.)

As a first result, we demonstrate that when the target manifold is a sphere, the energy quantization result extends to f-approximate biharmonic maps with $f \in L\log L$. For the definition of this Banach space, see the appendix.

Theorem 1.2. *Let $f_k \in L\log L(B_1,\mathbb{R}^{n+1})$ and $u_k \in W^{2,2}(B_1,\mathbb{S}^n)$ a sequence of f_k-approximate biharmonic maps with*

$$\|D^2u_k\|_{L^2(B_1)} + \|Du_k\|_{L^4(B_1)} + \|f_k\|_{L\log L(B_1)} \le \Lambda < \infty. \tag{1-1}$$

If $u_k \rightharpoonup u$ weakly in $W^{2,2}(B_1,\mathbb{S}^n)$, there exists $\{x_1,\dots,x_\ell\}\subset B_1$ such that $u_k \to u$ in $W^{2,2}_{\mathrm{loc}}(B_1\setminus\{x_1,\dots,x_\ell\},\mathbb{S}^n)$.

Moreover, for each $1\le i\le \ell$ there exists an $\ell_i \in \mathbb{N}$ and nontrivial, smooth biharmonic maps $\omega_{ij}\in C^\infty(\mathbb{R}^4,\mathbb{S}^n)$ with finite energy $(1\le j\le \ell_i)$ such that

$$\lim_{k\to\infty}\int_{B_{r_i}(x_i)} |D^2u_k|^2 = \int_{B_{r_i}(x_i)} |D^2u|^2 + \sum_{j=1}^{\ell_i}\int_{\mathbb{R}^4}|D^2\omega_{ij}|^2,$$

$$\lim_{k\to\infty}\int_{B_{r_i}(x_i)} |Du_k|^4 = \int_{B_{r_i}(x_i)} |Du|^4 + \sum_{j=1}^{\ell_i}\int_{\mathbb{R}^4}|D\omega_{ij}|^4.$$

Here $r_i = \frac{1}{2}\min_{1\le j\le \ell, j\ne i}\{|x_i-x_j|, \mathrm{dist}(x_i,\partial B_1)\}$.

As a second result, we demonstrate the no-neck property for approximate biharmonic maps with the approximating functions $L\log L$ norm not concentrating.

Theorem 1.3. *Let $f_k \in L\log L$ such that the $L\log L$ norm does not concentrate. For u_k a sequence of f_k-approximate biharmonic maps satisfying* (1-1), *the images of u and the maps ω_{ij} described above are connected in $\mathbb{S}^n$ without necks.*

In particular, if $f_k \in \phi(L)$, an Orlicz space such that $\lim_{t\to\infty}\phi(t)/(t\log t) = \infty$, the theorem holds. For a definition of an Orlicz space, see the appendix.

Remark 1.4. The theorems also hold for u_k a sequence of f_k-approximate intrinsic biharmonic maps. We will prove the theorems in detail for f_k-approximate biharmonic maps, and point out the necessary changes one must make to prove the intrinsic case.

We consider biharmonic maps into spheres because the symmetry of the target provides structure for the equation that can be exploited to prove higher regularity. For an f-approximate biharmonic map into $\mathbb{S}^n$, the structural equations takes the form (see [Wang 2004a])

$$d^*(D\Delta u\wedge u - \Delta u\wedge Du) = f\wedge u, \tag{1-2}$$

and, for an f-approximate intrinsic biharmonic u,

$$d^*(D\Delta u \wedge u - \Delta u \wedge Du + 2|Du|^2 Du \wedge u) = f \wedge u. \tag{1-3}$$

The structure of the equation for harmonic maps from a compact Riemann surface into $\mathbb{S}^n$ was determined independently by Chen [1989] and Shatah [1988]. They demonstrated that u satisfies the conservation law

$$d^*(Du \wedge u) = 0.$$

Hélein [1990] used the structure of this equation and Wente's inequality [1969] to determine that any weakly harmonic $u \in W^{1,2}$ was in fact C^∞.

Li and Zhu [2011] used this additional structure to determine energy quantization for approximate harmonic maps. In their setting, the equation takes the form $d^*(Du \wedge u) = \tau \wedge u$ for $\tau \in L \log L$. Our proof of energy quantization is similar in spirit to their work and to the recent small-energy compactness result of Sharp and Topping [2013]. Of critical importance are the energy estimates we prove in Section 2. The first estimates, from Proposition 2.1, are used in two ways. First, the L^p estimates of (2-2), (2-3) provide sufficient control to determine a small-energy compactness result away from the bubbles. Second, we use the Lorentz space duality to prove energy quantization and thus require uniform bounds on the appropriate Lorentz energies as in (2-1). In Section 3 we prove the energy quantization result. We point out that since the oscillation bound contains an energy term of the form $\|D\Delta u_k\|_{L^{4/3}}$, we must also prove this energy is quantized. This point justifies the necessity of the estimate (2-4). We prove the energy quantization result, under the presumption of the occurrence of one bubble, in Proposition 3.4.

We next use this stronger energy quantization result for maps into spheres to prove a no-neck property. Zhu [2012] showed the no-neck property for approximate harmonic maps with τ in a space essentially between L^p with $p > 1$ and $L \log L$. For w a cutoff function of the approximate harmonic map u, Zhu considered a Hodge decomposition of the 1-form $\beta := Dw \wedge u$. (This is actually a matrix of 1-forms, but we gloss over that point for now.) He bounded $\|\beta\|_{L^{2,1}}$ by bounding each component of the decomposition, and used this to bound $\|Dw\|_{L^{2,1}}$ by $\|Du\|_{L^2}$ plus a norm of the torsion term, τ. Using ε-compactness and a simple duality argument, he showed the oscillation of u is controlled by $\|Dw\|_{L^{2,1}}$, which in turn implies the desired result.

Like Zhu, we prove the no-neck property by demonstrating that the oscillation of an f-approximate biharmonic map is controlled by norms that tend to zero in the neck region. Using a duality argument, we first determine that the oscillation of u on an annular region is bounded by quantized energy terms plus a third derivative of a cutoff function w. Our main work is in determining an appropriate estimate for $\|D\Delta w\|_{L^{4/3,1}}$. We determine this bound by considering the 1-form

$\beta = D\Delta w \wedge u - \Delta w \wedge Du$, and we bound $D\Delta w$ by bounding β via its Hodge decomposition. In particular, we take advantage of the divergence structure of the equation for biharmonic maps into spheres to show that β not only has good $L^{4/3}$ estimates but in fact has good estimates in $L^{4/3,1}$. This second estimate allows us to prove the necessary oscillation lemma. The proof of the oscillation lemma constitutes the work of Section 4. Coupling the oscillation lemma with energy quantization, we prove Theorem 1.3 in Section 5.

Finally, the arguments we use require a familiarity with Lorentz spaces and the appropriate embedding theorems relevant in dimension four. In the appendix, we describe the various Banach spaces and collect the necessary embeddings and estimates.

Many steps of the proof require the use of cutoff functions, so we set:

Definition 1.5. Let $\phi \in C_0^\infty(B_2)$ with $\phi \equiv 1$ in B_1. For all $r > 0$ set $\phi_r(x) = \phi(x/r)$.

Note added in proof: As we finalized the paper, we noticed a somewhat related preprint [Liu and Yin 2013], in which the authors claim that the no-neck property holds for sequences of biharmonic maps into general targets. Their methods are quite different from ours and we believe our results are of independent interest.

2. Energy estimates

To establish strong convergence away from points of energy concentration, we first prove the necessary energy estimates. The small-energy compactness result relies on the fact that in both (2-2) and (2-3) there is an extra power of the energy on the right-hand side of the inequality. Thus, small energy implies that $\|Du_k\|_{L^4}$ and $\|D^2u_k\|_{L^2}$ must converge to zero on small balls. Measure-theoretic arguments in the next section will then imply strong convergence for these norms to some Du and D^2u respectively.

Proposition 2.1. *Let $u \in W^{2,2}(B_2, \mathbb{S}^n)$ be an f-approximate (intrinsic) biharmonic map, where $f \in L\log L(B_2, \mathbb{R}^{n+1})$. Then there exists $C > 0$ such that*

$$\begin{aligned}\text{(2-1)}\quad \|D^3u\|_{L^{4/3,1}(B_1)} + \|D^2u\|_{L^{2,1}(B_1)} + \|Du\|_{L^{4,1}(B_1)} \\ \le C(\|D^2u\|^2_{L^2(B_2)} + \|Du\|^2_{L^2(B_2)} + \|Du\|_{L^2(B_2)} + \|f\|_{L\log L(B_2)}).\end{aligned}$$

Moreover, there exists $\tilde{\varepsilon} > 0$ such that, if

$$\|D^2u\|_{L^2(B_2)} + \|Du\|_{L^4(B_2)} < \tilde{\varepsilon},$$

then, for every $0 < r < \frac{1}{2}$,

$$\begin{aligned}\text{(2-2)}\quad \|D^2u\|^2_{L^2(B_r)} &\le Cr^2\|D^2u\|^2_{L^2(B_2)} \\ &\quad + C(\|D^2u\|^4_{L^2(B_2)} + \|Du\|^4_{L^4(B_2)} + \|f\|^2_{L^1(B_2)}\|f\|_{L\log L(B_2)}),\end{aligned}$$

$$\|Du\|^4_{L^4(B_r)} \le Cr^4\|Du\|^4_{L^2(B_2)} + C(\|D^2u\|^8_{L^2(B_2)} + \|Du\|^8_{L^4(B_2)} + \|f\|^3_{L^1(B_2)}\|f\|_{L\log L(B_2)}), \tag{2-3}$$

$$\|D\Delta u\|^{4/3}_{L^{4/3}(B_r)} \le Cr^{4/3}\|D^2u\|^{4/3}_{L^2(B_2)} + C(\|D^2u\|^{8/3}_{L^2(B_2)} + \|Du\|^{8/3}_{L^4(B_2)} + \|f\|^{1/3}_{L^1(B_2)}\|f\|_{L\log L(B_2)}). \tag{2-4}$$

Remark 2.2. In point of fact, we do not need the full strength of (2-4) in application. We use instead the estimate

$$\|D\Delta u\|^{4/3}_{L^{4/3}(B_r)} \le C(\|D^2u\|^{4/3}_{L^2(B_{8r})} + \|Du\|^{4/3}_{L^4(B_{8r})} + \|f\|^{1/3}_{L^1(B_{8r})}\|f\|_{L\log L(B_{8r})}),$$

which can be immediately proven via the method outlined below.

Proof. First, find $v \in W^{1,2}_0(B_2, \mathrm{so}(n+1)) \cap W^{2,2}(B_2, \mathrm{so}(n+1))$ such that

$$\Delta v = \Delta u \wedge u.$$

Thus, for each $i, j \in \{1, \dots, n+1\}$, $\Delta v^{ij} = u^j \Delta u^i - u^i \Delta u^j$. It follows from (1-2) that

$$\Delta^2 v = \Delta(\Delta u \wedge u) = 2d^*(\Delta u \wedge Du) + f \wedge u.$$

Next we let $\phi \in W^{2,2}_0(B_2, \mathrm{so}(n+1) \otimes \Omega^1\mathbb{R}^4)$ be the solution of

$$\Delta^2\phi = d^*(2\Delta u \wedge Du).$$

Here $\mathrm{so}(n+1) \otimes \Omega^1\mathbb{R}^4$ denotes the space of 1-forms tensored with $(n+1)\times(n+1)$-antisymmetric matrices. Using Calderón–Zygmund theory coupled with interpolation, and using the estimates from Section A.2, we determine that

$$\|D^3\phi\|_{L^{4/3,1}(B_2)} + \|D^2\phi\|_{L^{2,1}(B_2)} + \|D\phi\|_{L^{4,1}(B_2)} \le c(\|D^2u\|^2_{L^2(B_2)} + \|Du\|^2_{L^2(B_2)}). \tag{2-5}$$

Moreover, letting $\psi \in W^{2,2}_0(B_2, \mathrm{so}(n+1))$ be the solution of

$$\Delta^2\psi = f \wedge u,$$

we conclude that

$$\|D\psi\|_{L^{4,1}(B_2)} + \|D^2\psi\|_{L^{2,1}(B_2)} + \|D^3\psi\|_{L^{4/3,1}(B_2)} \le c\|f\|_{L\log L(B_2)}. \tag{2-6}$$

Defining

$$B := v - \phi - \psi$$

and using the above equation for v, we conclude that each B^{ij} is a biharmonic function on B_2. Now every biharmonic function satisfies the mean value property

$$B(x) = c_1 \fint_{B_r(x)} B(y)\, dy - c_2 \fint_{B_{2r}(x)} B(y)\, dy,$$

for every $B_{2r}(x) \subset B_2$ (see, e.g., [Huilgol 1971]). Hence we estimate

$$\begin{aligned}\|D^2 B\|_{L^{2,1}(B_{3/2})} &+ \|D^3 B\|_{L^{4/3,1}(B_{3/2})}\\ &\le c\|DB\|_{L^2(B_2)}\\ &\le c(\|Dv\|_{L^2(B_2)} + \|f\|_{L\log L(B_2)} + \|D^2u\|^2_{L^2(B_2)} + \|Du\|^2_{L^2(B_2)}).\end{aligned}$$

Since $v = 0$ on ∂B_2, we can use the divergence theorem and Cauchy–Schwarz to show that

$$\begin{aligned}\int_{B_2} |Dv^{ij}|^2 = -\int_{B_2} v^{ij}\,\Delta v^{ij} &= -\int_{B_2} Dv^{ij}\cdot (Du\wedge u)^{ij}\\ &\le \frac{1}{2}\int_{B_2} |Dv^{ij}|^2 + C\int_{B_2} |Du|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\|D^2 B\|_{L^{2,1}(B_{3/2})} &+ \|D^3 B\|_{L^{4/3,1}(B_{3/2})}\\ &\le c(\|Du\|_{L^2(B_2)} + \|f\|_{L\log L(B_2)} + \|D^2u\|^2_{L^2(B_2)} + \|Du\|^2_{L^2(B_2)}).\end{aligned}$$

Now we observe that, since $\Delta v = \Delta u \wedge u$,

$$\Delta u = (\Delta u \wedge u).u + \langle \Delta u, u\rangle u = \Delta v.u - |Du|^2 u,$$

where here $\Omega.u$ represents matrix multiplication. Therefore,

$$\Delta^2 u = \Delta(\Delta v.u - |Du|^2 u) = d^*(D\Delta v.u + \Delta v.Du - D(|Du|^2 u)).$$

To get the second- and third-derivative estimates in (2-1), we first observe that

$$\begin{aligned}\|D^2 v\|_{L^{2,1}(B_{3/2})} &+ \|D^3 v\|_{L^{4/3,1}(B_{3/2})}\\ &\le c(\|Du\|_{L^2(B_2)} + \|f\|_{L\log L(B_2)} + \|D^2u\|^2_{L^2(B_2)} + \|Du\|^2_{L^2(B_2)}).\end{aligned}$$

Using the previous estimates and Section A.2, we observe that the 1-form in the parentheses is in $L^{4/3,1}$. Lemma A.3 in [Lamm and Rivière 2008] implies that

$$\begin{aligned}\|D^2 u\|_{L^{2,1}(B_1)} &+ \|D^3 u\|_{L^{4/3,1}(B_1)}\\ &\le c(\|D^3 v\|_{L^{4/3,1}(B_{3/2})} + \|D^2 v\|^2_{L^2(B_2)} + \|D^2u\|^2_{L^2(B_2)} + \|Du\|^2_{L^2(B_2)})\\ &\le c(\|Du\|_{L^2(B_2)} + \|f\|_{L\log L(B_2)} + \|D^2u\|^2_{L^2(B_2)} + \|Du\|^2_{L^2(B_2)}).\end{aligned}$$

Finally, Sobolev embedding for Lorentz spaces implies that

$$\begin{aligned}\|Du\|_{L^{4,1}(B_1)} &\le c(\|D^2u\|_{L^{2,1}(B_2)} + \|Du\|_{L^{2,1}(B_2)})\\ &\le c(\|D^2u\|_{L^{2,1}(B_2)} + \|Du\|_{L^{2}(B_2)}).\end{aligned}$$

Combining this with the previous estimates finishes the proof of (2-1).

To prove the small-energy estimates, we observe that u satisfies (see, for instance, [Lamm and Rivière 2008, Equations 1.4 and 1.14])

$$\Delta^2 u = \Delta(V\cdot Du) + d^*(wDu) + W\cdot Du + f, \tag{2-7}$$

where $V^{ij} = u^i Du^j - u^j Du^i$, $w^{ij} = -d^*(V^{ij}) - 2|Du|^2\delta_{ij}$, and

$$W^{ij} = -D(d^*(V^{ij})) + 2(\Delta u^i Du^j - \Delta u^j Du^i).$$

Let $\mathcal{M}_m$ denote the space of $m\times m$ matrices and $\mathcal{M}_m \otimes \Omega^k\mathbb{R}^4$ the space of k-forms tensored with $m\times m$ matrices. Then $V \in W^{1,2}(B_2, \mathcal{M}_{n+1}\otimes\Omega^1\mathbb{R}^4)$, $w\in L^2(B_2,\mathcal{M}_{n+1})$, and $W\in W^{-1,2}(B_2, \mathcal{M}_{n+1}\otimes\Omega^1\mathbb{R}^4)$.

Without loss of generality we extend f by zero outside of B_2. The small-energy hypothesis implies (see, for instance, [Lamm and Rivière 2008]) that there exist $A\in L^\infty\cap W^{2,2}(B_1,\mathrm{GL}_{n+1})$ and $\widetilde{B}\in W^{1,4/3}(B_1,\mathcal{M}_{n+1}\otimes\Omega^2\mathbb{R}^4)$ such that

$$D\Delta A + \Delta A V - DAw + AW = D\widetilde{B}$$

and

$$\begin{aligned}&\Delta(A\Delta u)\\ &\quad = d^*(2DA\Delta u - \Delta A Du + AwD - DA(V\cdot Du) + AD(V\cdot Du) + \widetilde{B}\cdot Du)) + Af\\ &\quad := d^*(K) + Af.\end{aligned}$$

Moreover,

$$\begin{aligned}&\|DA\|_{W^{1,2}(B_1)} + \|\operatorname{dist}(A,\mathrm{SO}(n+1))\|_{L^\infty(B_1)} + \|\widetilde{B}\|_{W^{1,4/3}(B_1)}\\ &\qquad\qquad \le c(\|D^2u\|_{L^2(B_2)} + \|Du\|_{L^4(B_2)}).\end{aligned}$$

First, we determine $E, F\in W_0^{1,2}(B_1)$ such that

$$\Delta E = d^*(K),\quad \Delta F = Af.$$

Interpolating on standard L^p theory, we get the estimates

$$\begin{aligned}\|E\|_{L^{2,1}(B_1)} + \|DE\|_{L^{4/3,1}(B_1)} &\le c\|K\|_{L^{4/3,1}(B_2)}\\ &\le c(\|D^2u\|^2_{L^2(B_2)} + \|Du\|^2_{L^4(B_2)}).\end{aligned}$$

Note that the estimate on K comes from considering the form of (2-7) and the estimates on V, w, W and consequently those on A, $\widetilde{B}$.

To determine estimates on F, we first observe that the estimates of Section A.2 imply that for G the fundamental solution to $\Delta^2 G = \delta_0$,

$$\|F\|_{L^{2,\infty}(B_1)} \le c\|D^2 G * (Af)\|_{L^{2,\infty}(B_1)} \le c\|f\|_{L^1(B_2)},$$
$$\|DF\|_{L^{4/3,\infty}(B_1)} \le c\|D^3 G\|_{L^{4/3,\infty}(B_2)}\|f\|_{L^1(B_2)}.$$

Also, since $\Delta F = Af \in \mathcal{H}^1(\mathbb{R}^4)$, standard theory implies that $D^2 F \in L^1(\mathbb{R}^4)$ and thus, by the embedding of $W^{1,1}$ into $L^{4/3,1}$ and Sobolev embeddings in $\mathbb{R}^4$,

$$\|F\|_{L^{2,1}(B_1)} + \|DF\|_{L^{4/3,1}(B_1)} \le c\|f\|_{L\log L(B_2)}.$$

Using a duality argument, we conclude that

$$\begin{aligned}\|F\|^2_{L^2(B_1)} &\le c\|F\|_{L^{2,\infty}(B_1)}\|F\|_{L^{2,1}(B_1)}\\ &\le c\|f\|_{L^1(B_2)}\|f\|_{L\log L(B_2)},\\ \|DF\|^{4/3}_{L^{4/3}(B_1)} &\le c\|(DF)^{1/3}\|_{L^{4,\infty}(B_1)}\|DF\|_{L^{4/3,1}(B_1)}\\ &\le c\|DF\|^{1/3}_{L^{4/3,\infty}(B_2)}\|f\|_{L\log L(B_2)}\\ &\le c\|f\|^{1/3}_{L^1(B_2)}\|f\|_{L\log L(B_2)}.\end{aligned}$$

Now, set $H = A\Delta u - E - F$. Then $\Delta H = 0$ in B_1, and, using standard estimates on harmonic functions, we determine that for all $0 < r < \frac{1}{2}$

$$\|H\|_{L^2(B_r)} + \|DH\|_{L^{4/3}(B_r)} \le cr\|H\|_{W^{1,\infty}(B_{1/2})} \le cr\|H\|_{L^2(B_1)}.$$

The previous estimates imply that

$$\|H\|^2_{L^2(B_1)} \le c(\|D^2 u\|^2_{L^2(B_2)} + \|Du\|^4_{L^4(B_2)} + \|f\|_{L^1(B_2)}\|f\|_{L\log L(B_2)}).$$

Since

$$\Delta u = A^{-1}(E + F + H),$$

the estimates for $D^2 u$ now follow from a standard cutoff argument and the previous estimates.

We estimate $\|D\Delta u\|_{L^{4/3}(B_r)}$ by using the previous estimates and noting that

$$\begin{aligned}&\|D(A^{-1}(E + F + H))\|_{L^{4/3}(B_r)}\\ &\qquad\le C(\|E + F + H\|_{L^2(B_r)}\|DA\|_{L^4(B_r)} + \|D(E + F + H)\|_{L^{4/3}(B_r)}).\end{aligned}$$

To estimate Du, we first consider $\alpha \in W^{2,2}(B_1)$, $\beta \in W_0^{1,2} \cap W^{2,2}(B_1, \Omega^1\mathbb{R}^4)$ such that

$$ADu = d\alpha + d^*\beta.$$

Then

$$\Delta^2\alpha = \Delta d^*(ADu) = \Delta(A\Delta u + DA.Du) = d^*(\widetilde{K}) + Af \quad \text{on } B_1$$

and

$$\Delta\beta = DA \wedge Du \quad \text{on } B_1.$$

Here $\widetilde{K}$ is the appropriate modification of K to include the additional term. We first observe that

$$\|D\beta\|_{L^4(B_r)} \le c(\|D^2\beta\|_{L^2(B_1)} + \|D\beta\|_{L^2(B_1)}).$$

Standard L^p theory implies that

$$\|D^2\beta\|_{L^2(B_2)} \le c\|DA\|_{W^{1,2}(B_1)}\|Du\|_{W^{1,2}(B_1)}.$$

Moreover, using a weighted Cauchy–Schwarz inequality and the Poincaré inequality, we note that

$$\begin{aligned}\int_{B_1} |D\beta^{ij}|^2 &= -\int_{B_1} \beta^{ij}(DA \wedge Du)^{ij} \\ &\le c\|DA\|^2_{L^4(B_1)}\|Du\|^2_{L^4(B_1)} + \tfrac{1}{2}\|D\beta\|^2_{L^2(B_1)}.\end{aligned}$$

Combining this with previous estimates implies that

$$\|D\beta\|_{L^4(B_r)} \le c(\|D^2u\|^2_{L^2(B_2)} + \|Du\|^2_{L^4(B_2)}).$$

For the α term, we follow the ideas used to prove (2-1). Indeed, first determine $\phi, \psi \in W^{2,2}_0(B_2)$ such that $\Delta^2\phi = d^*(K)$ and $\Delta^2\psi = Af$. Then by (2-5), (2-6), and appropriate duality arguments, we conclude that, for any $0 < r < 1$,

$$\begin{aligned}\|D\phi\|_{L^4(B_r)} &\le c(\|D^2u\|^2_{L^2(B_2)} + \|Du\|^2_{L^4(B_2)}), \\ \|D\psi\|^4_{L^4(B_r)} &\le c\|f\|^3_{L^1(B_2)}\|f\|_{L\log L(B_2)}.\end{aligned}$$

Setting $B = \alpha - \psi - \phi$, we have $\Delta^2 B = 0$ on B_1, and we use the mean value property to show that for any $0 < r < \frac{1}{2}$

$$\|DB\|_{L^4(B_r)} \le cr\|DB\|_{L^\infty(B_{3/4})} \le cr\|DB\|_{L^4(B_{7/8})}.$$

Noting that

$$\begin{aligned}\|DB\|^4_{L^4(B_{7/8})} \le c\big(\|D\alpha\|^4_{L^4(B_{7/8})} + \|Du\|^4_{L^4(B_1)} + \|D^2u\|^8_{L^2(B_2)} \\ + \|Du\|^8_{L^4(B_2)} + \|f\|^3_{L^1(B_2)}\|f\|_{L\log L(B_2)}\big),\end{aligned}$$

we combine the previous estimates to get the result for Du. □

Remark 2.3. When u is intrinsic, the strategy is the same, except for two things. In the first part of the argument, the equation for u will have the additional term $-d^*(|Du|^2 Du \wedge u)$ on the right side. But this term doesn't change the estimates. In the second part of the argument, W^{ij} will include the term $|Du|^2(u^i Du^j - u^j Du^i)$. This gives the same value for $d^*(W^{ij})$, and all estimates going forward are the same.

We will prove the energy quantization results by appealing to Lorentz duality. In Proposition 2.1, we determined uniform estimates for Lorentz norms of the form $L^{p,1}$. The next lemma provides the necessary small-energy estimates for the $L^{p,\infty}$ norms on the annular region, presuming small energy on all dyadic annuli:

Lemma 2.4. *Let $u \in W^{2,2}(B_1, \mathbb{S}^n)$ be an f-approximate biharmonic map with $f \in L\log L(B_1, \mathbb{R}^{n+1})$. Given $\varepsilon > 0$, suppose that for all ρ such that $B_{2\rho} \setminus B_\rho \subset B_{2\delta} \setminus B_{t/2}$ we have*

$$\int_{B_{2\rho}\setminus B_\rho} |Du|^4 + |D^2u|^2 + |D\Delta u|^{4/3} < \varepsilon. \tag{2-8}$$

Then,

$$\|Du\|_{L^{4,\infty}(B_\delta\setminus B_t)} + \|D^2u\|_{L^{2,\infty}(B_\delta\setminus B_t)} + \|D\Delta u\|_{L^{4/3,\infty}(B_\delta\setminus B_t)} \le C(\varepsilon^{\frac{1}{8}} + (\log(1/\delta))^{-1}).$$

Proof. Let $\widetilde{\phi}_k := \phi_{2^{k+2}t}(1-\phi_{2^{k-2}t})$ be the annular cutoff supported on $A_k := B_{2^{k+3}t} \setminus B_{2^{k-2}t}$ which is identically 1 on $B_{2^{k+2}t} \setminus B_{2^{k-1}t}$. Let G be the distribution such that $\Delta^2 G = \delta_0$ in $\mathbb{R}^4$. Then $|DG(x)| = C|x|^{-1}$. Note that operator bounds on $D^k G$ can be found in the appendix. Let $\bar{u}_k := \fint_{A_k} u$. Set $\tilde{u}_k(x) := \widetilde{\phi}_k(u - \bar{u}_k)(x)$. Therefore on $B_{2^{k+1}t} \setminus B_{2^k t}$

$$\Delta^2 \tilde{u}_k = (\Delta^2\widetilde{\phi}_k)(u-\bar{u}_k) + 4D\Delta\widetilde{\phi}_k \cdot D(u-\bar{u}_k) + 2\Delta\widetilde{\phi}_k \Delta u + 4D\widetilde{\phi}_k \cdot D\Delta u + \widetilde{\phi}_k \Delta^2 u.$$

Using the facts that $\Delta^2 u = \Delta(\Delta u \wedge u.u - |Du|^2 u)$ and that $\Delta^2 u \wedge u = f \wedge u$, we note that

$$\begin{aligned}\widetilde{\phi}_k \Delta^2 u = {} & d^*\big(\widetilde{\phi}_k(2\Delta u \wedge Du.u + 2\Delta u \wedge u.Du - D(u|Du|^2))\big) \\ & - D\widetilde{\phi}_k \cdot (2\Delta u \wedge Du.u + 2\Delta u \wedge u.Du - D(u|Du|^2)) \\ & + \widetilde{\phi}_k (f \wedge u.u - 2\Delta u \wedge Du.Du - \Delta u \wedge u.\Delta u).\end{aligned}$$

And thus,

$$\begin{aligned}\Delta^2 \tilde{u}_k = {} & (\Delta^2\widetilde{\phi}_k)(u-\bar{u}_k) + 4D\Delta\widetilde{\phi}_k \cdot D(u-\bar{u}_k) + 2\Delta\widetilde{\phi}_k \Delta u + 4D\widetilde{\phi}_k \cdot D\Delta u \\ & - D\widetilde{\phi}_k \cdot (2\Delta u \wedge Du.u + 2\Delta u \wedge u.Du - D(u|Du|^2)) \\ & + d^*\big(\widetilde{\phi}_k(2\Delta u \wedge Du.u + 2\Delta u \wedge u.Du - D(u|Du|^2))\big) \\ & + \widetilde{\phi}_k (f \wedge u.u - 2\Delta u \wedge Du.Du - \Delta u \wedge u.\Delta u).\end{aligned}$$

For ease of notation, we let I_k denote the first four terms above, and $\mathrm{II}_k, \mathrm{III}_k, \mathrm{IV}_k$ denote the last three terms, respectively. Then on each $B_{2^{k+1}t} \setminus B_{2^k t}$

$$\begin{aligned}|Du(x)| &= |D(\widetilde{\phi}_k(u-\bar{u}_k))(x)| = |\Delta^2 G * D(\widetilde{\phi}_k(u-\bar{u}_k))(x)| \\ &= |DG * \Delta^2(\widetilde{\phi}_k(u-\bar{u}_k))(x)| = |DG * (\mathrm{I}_k + \mathrm{II}_k + \mathrm{III}_k + \mathrm{IV}_k)(x)|.\end{aligned}$$

We consider each of these estimates separately. First, note that

$$\begin{aligned}&|DG * \mathrm{I}_k(x)| \\ &\le C \Big| \int_{(B_{2^{k+3}t}\setminus B_{2^{k+2}t})\cup(B_{2^{k-1}t}\setminus B_{2^{k-2}t})} \frac{1}{|x-y|} \\ &\qquad \times \big((2^k t)^{-4}(u-\bar{u}_k) + (2^k t)^{-3} D(u-\bar{u}_k) + (2^k t)^{-2}\Delta u + (2^k t)^{-1} D\Delta u\big)\, dy \Big| \\ &\le C \Big| \int_{A_k} (2^k t)^{-1}\big((2^k t)^{-4}(u-\bar{u}_k) + (2^k t)^{-3} Du \\ &\qquad\qquad + (2^k t)^{-2}\Delta u + (2^k t)^{-1} D\Delta u\big)\, dy \Big| \\ &\le C \int_{A_k} (2^k t)^{-4}|Du| + (2^k t)^{-3}|D^2 u| + (2^k t)^{-2}|D\Delta u| \\ &\le C(2^k t)^{-1}(\|Du\|_{L^4} + \|D^2 u\|_{L^2} + \|D\Delta u\|_{L^{4/3}}) \\ &\le C(\varepsilon^{1/4} + \varepsilon^{1/2} + \varepsilon^{3/4})|x|^{-1}.\end{aligned}$$

Using the same ideas as previously, we bound

$$\begin{aligned}&|DG * \mathrm{II}_k(x)| \\ &\qquad \le C(2^k t)^{-2} \int_{A_k} |2\Delta u \wedge Du.u + 2\Delta u \wedge u.Du - D(u|Du|^2)| \\ &\qquad \le C(2^k t)^{-1} \|2\Delta u \wedge Du.u + 2\Delta u \wedge u.Du - D(u|Du|^2)\|_{L^{4/3}(A_k)} \\ &\qquad \le C(2^k t)^{-1}(\|D^2 u\|_{L^2}\|Du\|_{L^4} + \|Du\|_{L^4}^3) \\ &\qquad \le C(\varepsilon^{1/8} + \varepsilon^{3/4})|x|^{-1}.\end{aligned}$$

Using the estimates from the appendix, we note that

$$\begin{aligned}&\|DG * \mathrm{III}_k\|_{L^{4,\infty}(A_k)} \\ &\qquad \le C\|D^2 G * \widetilde{\phi}_k(2\Delta u \wedge Du.u + 2\Delta u \wedge u.Du - D(u|Du|^2)\|_{L^{4,\infty}(A_k)} \\ &\qquad \le C\|\widetilde{\phi}_k(2\Delta u \wedge Du.u + 2\Delta u \wedge u.Du - D(u|Du|^2))\|_{L^{4/3}(A_k)}\end{aligned}$$

and

$$\|DG * \mathrm{IV}_k\|_{L^{4,\infty}(A_k)} \le C\|\widetilde{\phi}_k(f \wedge u.u - 2\Delta u \wedge Du.Du - \Delta u \wedge u.\Delta u)\|_{L^1(A_k)}.$$

Thus

$$
\begin{aligned}
&|\{x : |DG * (\mathrm{III}_k + \mathrm{IV}_k)(x)| > \lambda\}| \\
&\qquad \le \lambda^{-4} \| DG * (\mathrm{III}_k + \mathrm{IV}_k)\|^4_{L^{4,\infty}(\mathbb{R}^4)} \\
&\qquad \le C\lambda^{-4}\big(\|\widetilde{\phi}_k(f \wedge u.u - 2\Delta u \wedge Du.Du - \Delta u \wedge u.\Delta u)\|^4_{L^1(A_k)} \\
&\qquad\qquad + \|\widetilde{\phi}_k(2\Delta u \wedge Du.u + 2\Delta u \wedge u.Du - D(u|Du|^2))\|^4_{L^{4/3}(A_k)}\big) \\
&\qquad \le C\lambda^{-4}\bigg(\bigg(\int \widetilde{\phi}_k |D^2u|^2\bigg)^2 \int \widetilde{\phi}_k |Du|^4 + \bigg(\int \widetilde{\phi}_k |Du|^4\bigg)^3\bigg) \\
&\qquad\qquad + C\lambda^{-4}\|\widetilde{\phi}_k(f \wedge u.u - 2\Delta u \wedge Du.Du - \Delta u \wedge u.\Delta u)\|^4_{L^1(A_k)}.
\end{aligned}
$$

Thus, if $\delta = 2^M t$, then (letting $S_k := B_{2^{k+1}t} \setminus B_{2^k t}$ for ease of notation)

$$
\begin{aligned}
&|\{x \in B_\delta \setminus B_t : |Du(x)| > 3\lambda\}| \\
&\qquad \le \sum_{k=0}^{M-1} |\{x \in S_k : |Du(x)| > 3\lambda\}| \\
&\qquad \le \sum_{k=0}^{M-1} |\{x \in S_k : |DG * \mathrm{I}_k| > \lambda\}| + \sum_{k=0}^{M-1} |\{x \in S_k : |DG * \mathrm{II}_k| > \lambda\}| \\
&\qquad\quad + \sum_{k=0}^{M-1} |\{x \in S_k : |DG * (\mathrm{III}_k + \mathrm{IV}_k)| > \lambda\}| \\
&\qquad \le \sum_{k=0}^{M-1} |\{|DG * (\mathrm{III}_k + \mathrm{IV}_k)| > \lambda\}| + \left|\left\{x \in B_1 : C\frac{\varepsilon^{\frac{1}{8}}}{|x|} > \lambda\right\}\right| \\
&\qquad \le C\lambda^{-4}\Bigg(\varepsilon^{\frac{1}{2}} + \sum_{k=0}^{M-1} \|\widetilde{\phi}_k(f \wedge u.u)\|^4_{L^1(A_k)} + \sum_{k=0}^{M-1}\bigg(\bigg(\int \widetilde{\phi}_k |D^2u|^2\bigg)^4 \\
&\qquad\qquad + \bigg(\int \widetilde{\phi}_k |Du|^4\bigg)^4 + \bigg(\int \widetilde{\phi}_k |Du|^4\bigg)^3 + \bigg(\int \widetilde{\phi}_k |Du|^4\bigg)^2\bigg)\Bigg) \\
&\qquad \le C\lambda^{-4}(\varepsilon^{\frac{1}{2}} + (\log(1/\delta))^{-4}\|f\|^4_{L\log L(B_{2\delta})} + \varepsilon^2).
\end{aligned}
$$

For the estimate on $\|f \wedge u.u\|_{L^1}$ we use Lemma A.2, and for the rest of the L^1 estimate we just use Cauchy–Schwarz. This proves the estimate for Du. The estimates for D^2u and $D\Delta u$ work in much the same way. In the case of D^2u, the terms like III_k and IV_k require the fact that $D^3G : L^{4/3} \to L^{2,\infty}$ and $D^2G : L^1 \to L^{2,\infty}$ are bounded operators, where the operation is convolution. For the term $D\Delta u$ we observe that $D^3G : L^1 \to L^{4/3,\infty}$ and $D^4G : L^{4/3} \to L^{4/3,\infty}$ are also bounded operators. □

3. Energy quantization — proof of Theorem 1.2

We now determine a weak convergence result which will give small-energy compactness and help us complete the proof of the energy quantization. We follow the ideas of [Li and Zhu 2011; Sharp and Topping 2013], which in turn follow the arguments of [Evans 1990], with appropriate minor modifications. Throughout this lemma and its proof, we consider a measurable function f as both a function and a Radon measure.

Lemma 3.1. *Suppose $\{V_k\} \subset W^{1,4/3}(B_1)$ is a bounded sequence in $B_1 \subset \mathbb{R}^4$. Then there exist at most countable $\{x_i\} \subset B_1$ and $\{a_i > 0\}$ with $\sum_i a_i < \infty$ and $V \in W^{1,4/3}(B_1)$ such that, after passing to a subsequence,*

$$V_k^2 \rightharpoonup V^2 + \sum_i a_i \delta_{x_i}$$

weakly as measures.

Proof. As $W^{1,4/3}$ embeds continuously into L^2 in four dimensions, after taking a subsequence, by Rellich compactness there exists some $V \in L^2$ such that $V_k \to V$ strongly in L^p for $1 \le p < 2$ and $V_k \rightharpoonup V$ weakly in L^2. Moreover, since $\{DV_k\}$ is uniformly bounded in $L^{4/3}$, it follows that $DV_k \rightharpoonup f \in L^{4/3}$ and f is necessarily DV.

Set $g_k := V_k - V$. Then $g_k \in L^2$ and $Dg_k \in L^{4/3}$ with uniform bounds. Thus, in the weak-$*$ topology, both $|Dg_k|^{4/3}$ and g_k^2 converge to nonnegative Radon measures with finite total mass. (We denote this space by $M(B)$). Then $g_k^2 \rightharpoonup \nu \in M(B)$ and $|Dg_k|^{4/3} \rightharpoonup \mu \in M(B)$ where ν, μ are both nonnegative. Now consider $\phi \in C_0^1(B_1)$, and observe that the Sobolev embedding of $W^{1,4/3}$ into L^2 implies that

$$\left(\int (\phi g_k)^2 \, dx\right)^{\frac{1}{2}} \le C\left(\int |D(\phi g_k)|^{\frac{4}{3}} \, dx\right)^{\frac{3}{4}}.$$

Taking $k \to \infty$ and noting that $g_k \to 0$ in $L^{4/3}$, we use the weak convergence to observe that

$$\int \phi^2 \, d\nu \le C\left(\int |\phi|^{\frac{4}{3}} \, d\mu\right)^{\frac{3}{2}}.$$

Let ϕ approximate $\chi_{B_r(x)}$ for $B_r(x) \subset B_1$. Then

$$\nu(B_r(x)) \le C\big(\mu(B_r(x))\big)^{\frac{3}{2}}.$$

By standard results on the differentiation of measures (see [Evans and Gariepy 1992, Section 1.6]), for any Borel set E

$$\nu(E) = \int_E D_\mu \nu \, d\mu,$$

where

$$D_\mu \nu(x) = \lim_{r\to 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^4.$$

Now, as μ is a finite, nonnegative Radon measure, there exist at most countably many $x_i \in B_1$ such that $\mu(\{x_i\}) > 0$. Moreover, for all $x \in B$ such that $\mu(\{x\}) = 0$, we note that

$$D_\mu \nu(x) = \lim_{r\to 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} \le C \lim_{r\to 0} \mu(B_r(x))^{\frac{1}{2}} = 0.$$

For every x_j such that $\mu(\{x_j\}) > 0$, set $a_j = D_\mu \nu(x_j)\mu(\{x_j\})$. Then

$$\nu(E) = \int_E D_\mu \nu \, d\mu = \sum_{\{j : x_j \in E\}} a_j \quad \text{or} \quad \nu = \sum_j a_j \delta_{x_j}.$$

Since $g_k^2 \rightharpoonup \nu$ as measures, for $\phi \in C_0^0(B_1)$,

$$\sum_j a_j \phi(x_j) = \lim_{k\to\infty} \int_{B_1} g_k^2 \phi \, dx = \lim_{k\to\infty} \int_{B_1} (V_k - V)^2 \, dx.$$

Since $(V_k - V)^2 = V_k^2 - V^2 + 2V(V - V_k)$ and $V - V_k = g_k \rightharpoonup 0$ in L^2, we have the result. □

Corollary 3.2. *For $\{V_k\}$ as in Lemma 3.1, if*

$$\lim_{r\to 0} \limsup_{k\to\infty} \|V_k\|_{L^2(B_r(x))} = 0 \tag{3-1}$$

for all $x \in B$, then

$$V_k \to V \text{ strongly in } L^2_{\mathrm{loc}}(B).$$

Proof. Notice the condition (3-1) implies that $|V_k|^2 \rightharpoonup |V|^2$ weakly as bounded Radon measures. Then, by [Evans and Gariepy 1992, Section 1.9], for any $B_r(x) \subset B_1$, we have $\|V_k\|_{L^2(B_r(x))} \to \|V\|_{L^2(B_r(x))}$ strongly for all $B_r(x) \subset B_1$. Then, again using the fact that $(V_k - V)^2 = V_k^2 - V^2 + 2V(V - V_k)$ and

$$\int_{B_r(x)} V_k^2 - V^2 \, dx + \int_{B_r(x)} 2V(V - V_k) \, dx \to 0 \quad \text{as } k \to \infty,$$

we conclude that $V_k \to V$ strongly in $L^2_{\mathrm{loc}}(B_1)$. □

We now use the energy estimates of Proposition 2.1 to prove a small-energy compactness result:

Lemma 3.3. *Let u_k be a sequence of f_k-approximate biharmonic maps in B_2 with $f_k \in L \log L(B_2)$ satisfying* (1-1). *There exists $\varepsilon_0 > 0$ such that if*

$$\|Du_k\|_{L^4(B_2)} + \|D^2 u_k\|_{L^2(B_2)} < \varepsilon_0,$$

then there exists $u \in W^{2,2}_{\mathrm{loc}}(B_2)$ *such that*

$$Du_k \to Du \text{ strongly in } L^4_{\mathrm{loc}}(B_1) \quad \text{and} \quad D^2u_k \to D^2u \text{ strongly in } L^2_{\mathrm{loc}}(B_1).$$

Proof. We will first prove convergence of Du_k to Du and D^2u_k to D^2u in L^2_{loc} and then use Gagliardo–Nirenberg interpolation to get the L^4 convergence.

Begin by choosing $0 < \varepsilon_0 < \tilde{\varepsilon}$ from Proposition 2.1. First note that the uniform bounds on u_k in $W^{2,2}(B_2)$ imply that there exists a $u \in W^{2,2}_{\mathrm{loc}}(B_2)$ such that $u_k \rightharpoonup u$ in $W^{2,2}_{\mathrm{loc}}(B_2)$. We now show strong convergence for the derivatives indicated.

Pick any $x_0 \in B_1$ and $2R \in (0, \frac{3}{4}]$. Then $B_{2R}(x_0) \subset B_2$. Let $\hat{u}_k(x) := u_k(x_0 + 2Rx)$ and $\hat{f}_k(x) := (2R)^4 f_k(x_0 + 2Rx)$. Then $\hat{u}_k$ is an $\hat{f}_k$-approximate biharmonic map on B_1. From (2-2), (2-3), we note that, for any $r \in (0, \frac{1}{2}]$,

$$\begin{aligned}
&\|D\hat{u}_k\|_{L^4(B_r)} + \|D^2\hat{u}_k\|_{L^2(B_r)} \\
&\quad \le Cr(\|D\hat{u}_k\|_{L^4(B_2)} + \|D^2\hat{u}_k\|_{L^2(B_2)}) \\
&\qquad + C\big(\|D\hat{u}_k\|^2_{L^4(B_2)} + \|D^2\hat{u}_k\|^2_{L^2(B_2)} + (\|\hat{f}_k\|_{L^1(B_2)}\|\hat{f}_k\|_{L\log L(B_2)})^{\frac{1}{2}} \\
&\qquad\qquad + (\|\hat{f}_k\|^3_{L^1(B_2)}\|\hat{f}_k\|_{L\log L(B_2)})^{\frac{1}{4}}\big).
\end{aligned}$$

Using the scaling relations listed in Section A.3 and Lemma A.3 we observe that

$$\begin{aligned}
&\|Du_k\|_{L^4(B_{r2R}(x_0))} + \|D^2u_k\|_{L^2(B_{r2R}(x_0))} \\
&\quad \le Cr(\|Du_k\|_{L^4(B_{2R}(x_0))} + \|D^2u_k\|_{L^2(B_{2R}(x_0))}) \\
&\qquad + C\big(\|Du_k\|^2_{L^4(B_{2R}(x_0))} + \|D^2u_k\|^2_{L^2(B_{2R}(x_0))} \\
&\qquad\qquad + (\|f_k\|_{L^1(B_{2R}(x_0))}\|f_k\|_{L\log L(B_{2R}(x_0))})^{\frac{1}{2}} \\
&\qquad\qquad\qquad + (\|f_k\|^3_{L^1(B_{2R}(x_0))}\|f_k\|_{L\log L(B_{2R}(x_0))})^{\frac{1}{4}}\big).
\end{aligned}$$

Lemma A.2 and (1-1) together imply that

$$\|f_k\|_{L^1(B_{2R}(x_0))} \le C\Big(\log\frac{1}{2R}\Big)^{-1}\|f_k\|_{L\log L(B_{2R}(x_0))} \le C\Lambda\Big(\log\frac{1}{2R}\Big)^{-1}.$$

Note that the right-hand side goes to zero as $R \to 0$. Therefore, the small-energy hypothesis implies that

$$\begin{aligned}
&\lim_{R\to 0}\lim_{r\to 0}\lim_{k\to\infty}(\|Du_k\|_{L^4(B_{r2R}(x_0))} + \|D^2u_k\|_{L^2(B_{r2R}(x_0))}) \\
&\qquad \le C\varepsilon_0 \lim_{R\to 0}\lim_{r\to 0}\lim_{k\to\infty}(\|Du_k\|_{L^4(B_{2R}(x_0))} + \|D^2u_k\|_{L^2(B_{2R}(x_0))}).
\end{aligned}$$

Decreasing ε_0, if necessary, so that $\varepsilon_0 < 1/C$, implies that

$$\lim_{r\to 0}\lim_{k\to\infty}(\|Du_k\|_{L^4(B_r(x_0))} + \|D^2u_k\|_{L^2(B_r(x_0))}) = 0$$

for all $x_0 \in B_1$. Let $V_k = D^2 u_k$ and $V = D^2 u$. Since $V_k \rightharpoonup V$ weakly in L^2 as measures and V_k satisfies the hypotheses of Lemma 3.1 and Corollary 3.2 on B_1, $V_k \to V$ strongly in $L^2_{\mathrm{loc}}(B_1)$.

Since $Du_k \rightharpoonup Du$ weakly as measures in $L^2(B_2)$ and

$$\lim_{r\to 0}\lim_{k\to\infty} \|Du_k\|_{L^2(B_r(x_0))} \le \lim_{r\to 0}\lim_{k\to\infty} r\|Du_k\|_{L^4(B_r(x_0))} = 0$$

for all $x_0 \in B_1$, Corollary 3.2 again implies that $Du_k \to Du$ strongly in $L^2_{\mathrm{loc}}(B_1)$.

Now, for any $B_r(x) \subset B_1$, we consider the functions

$$w_k := (u_k - u) - \fint_{B_r(x)} (u_k - u).$$

Then, $Dw_k = D(u_k - u)$ and $D^2 w_k = D^2(u_k - u)$. We apply the Gagliardo–Nirenberg interpolation inequality for w_k and then the Poincaré inequality for the L^2 estimates on w_k to conclude that

$$\|Dw_k\|_{L^4(B_r(x))} \le C\|D^2 w_k\|_{L^2(B_r(x))}\|Dw_k\|_{L^2(B_r(x))} + C\|Dw_k\|_{L^2(B_r(x))}.$$

Then, using the strong convergence of $D^2 u_k \to Du$ in L^2_{loc} and $Du_k \to Du$ in L^2_{loc}, we conclude $Du_k \to Du$ in $L^4_{\mathrm{loc}}(B_1)$. □

Finally, we prove the energy quantization result under the presumption of one bubble at the origin.

Proposition 3.4. *Let $f_k \in L\log L(B_1, \mathbb{R}^{n+1})$, and let $u_k \in W^{2,2}(B_1, \mathbb{S}^n)$ be a sequence of f_k-approximate biharmonic maps with bounded energy such that*

$$\begin{aligned} u_k &\to u && \text{in } W^{2,2}_{\mathrm{loc}}(B_1 \setminus \{0\}, \mathbb{S}^n),\\ \tilde{u}_k(x) := u_k(\lambda_k x) &\to \omega(x) && \text{in } W^{2,2}_{\mathrm{loc}}(\mathbb{R}^4, \mathbb{S}^n). \end{aligned}$$

Presume further that ω is the only "bubble" at the origin. Let

$$A_k(\delta, R) := \{x : \lambda_k R \le |x| \le \delta\}.$$

Then

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{k\to\infty} \big(\|D^2 u_k\|_{L^2(A_k(\delta,R))} + \|Du_k\|_{L^4(A_k(\delta,R))} + \|D\Delta u_k\|_{L^{4/3}(A_k(\delta,R))}\big) = 0.$$

The proposition also holds if u_k is a sequence of f_k-approximate intrinsic biharmonic maps.

Proof. We first prove that for any $\varepsilon > 0$ there exists K sufficiently large and δ small so that, for all $k \geq K$ and $\rho_k > 0$ such that $B_{4\rho_k} \setminus B_{\rho_k/2} \subset A_k(\delta, R)$,

$$\|D^2 u_k\|_{L^2(B_{2\rho_k}\setminus B_{\rho_k})} + \|Du_k\|_{L^4(B_{2\rho_k}\setminus B_{\rho_k})} + \|D\Delta u_k\|_{L^{4/3}(B_{2\rho_k}\setminus B_{\rho_k})} < \varepsilon. \tag{3-2}$$

Since $\{0\}$ is the only point of energy concentration, the strong convergence of $D^2u_k \to D^2u$ in L^2 and $Du_k \to Du$ in L^4 implies that for any $\varepsilon > 0$ and any $m \in \mathbb{Z}^+$ and δ sufficiently small, there exists $K := K(m)$ sufficiently large such that, for all $k \geq K(m)$,

$$\|D^2 u_k\|_{L^2(B_{2\delta}\setminus B_{\delta 2^{-m-1}})} + \|Du_k\|_{L^4(B_{2\delta}\setminus B_{\delta 2^{-m-1}})} \leq \frac{\varepsilon}{C\Gamma^{m+3}}. \tag{3-3}$$

Here C is an appropriately large constant determined by the bounds of Proposition 2.1 and Γ is the number of balls of radius $r/32$ needed to cover $B_r \setminus B_{r/2}$. By (2-4), for any $x \in B_{2\delta} \setminus B_{\delta 2^{-m-1}}$ and $0 < r < \delta 2^{-m-1}$,

$$\|D\Delta u_k\|_{L^{4/3}(B_{r/32}(x))} \leq C\big(\|D^2u_k\|_{L^2(B_{r/2}(x))} + \|Du_k\|_{L^4(B_{r/2}(x))} + \|f_k\|^{1/4}_{L^1(B_{r/2}(x))}\|f_k\|^{3/4}_{L\log L(B_{r/2}(x))}\big) \tag{3-4}$$

Since Lemma A.2 and (1-1) imply that

$$\|f_k\|_{L^1(B_{r/2}(x))} \leq C\Big(\log\frac{1}{r}\Big)^{-1}\|f_k\|_{L\log L(B_{r/2}(x))}, \tag{3-5}$$

for sufficiently small δ, (3-3), (3-4), and (3-5) together imply that for $k \geq K(m)$

$$\|D\Delta u_k\|_{L^{4/3}(B_{2\delta}\setminus B_{\delta 2^{-m-1}})} + \|Du_k\|_{L^4(B_{2\delta}\setminus B_{\delta 2^{-m-1}})} + \|D^2u_k\|_{L^2(B_{2\delta}\setminus B_{\delta 2^{-m-1}})} \leq \tfrac{1}{2}\varepsilon. \tag{3-6}$$

A similar argument (perhaps requiring a larger K) implies that

$$\|D\Delta u_k\|_{L^{4/3}(B_{2^m\lambda_k R}\setminus B_{\lambda_k R})} + \|Du_k\|_{L^4(B_{2^m\lambda_k R}\setminus B_{\lambda_k R})} + \|D^2u_k\|_{L^2(B_{2^m\lambda_k R}\setminus B_{\lambda_k R})} \leq \tfrac{1}{2}\varepsilon. \tag{3-7}$$

Now suppose there exists a sequence t_k with $\lambda_k R < t_k < \delta$ such that

$$\|D^2u_k\|_{L^2(B_{2t_k}\setminus B_{t_k})} + \|Du_k\|_{L^4(B_{2t_k}\setminus B_{t_k})} + \|D\Delta u_k\|_{L^{4/3}(B_{2t_k}\setminus B_{t_k})} \geq \varepsilon.$$

By (3-6) and (3-7), $t_k \to 0$ and $B_{\delta/t_k} \setminus B_{\lambda_k R/t_k} \to \mathbb{R}^4\setminus\{0\}$. Define $v_k(x) = u_k(t_k x)$ and $\tilde{f}_k(x) := t_k^4 f_k(t_k x)$. Then v_k is an $\tilde{f}_k$-approximate biharmonic map, defined on

$B_{t_k^{-1}}$. We first observe that $v_k \to v_\infty$ weakly in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^4, \mathbb{S}^n)$. Notice for any $R > 0$

$$\begin{aligned}
\int_{B_R} |\tilde{f}_k(x)|\, dx &= \int_{B_{Rt_k}} |f_k(s)|\, ds \\
&\le \int_0^{|B_{Rt_k}|} (f_k)^*(t)\, dt \\
&\le c\left(\log\left(2 + \frac{1}{Rt_k}\right)\right)^{-1} \int_0^\infty (f_k)^*(t) \log\left(2 + \frac{1}{t}\right) dt \\
&= c\left(\log\left(2 + \frac{1}{Rt_k}\right)\right)^{-1} \|f_k\|_{L \log L(B_1)}.
\end{aligned}$$

By (1-1), $\tilde{f}_k \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^4)$. Moreover, for all k,

$$\|D^2 v_k\|_{L^2(B_2 \setminus B_1)} + \|Dv_k\|_{L^4(B_2 \setminus B_1)} + \|D\Delta v_k\|_{L^{4/3}(B_2 \setminus B_1)} \ge \varepsilon.$$

If $v_k \to v_\infty$ strongly in $W^{2,2}(B_{16} \setminus B_{1/16}, \mathbb{S}^n)$, then v_∞ is a nonconstant biharmonic map into $\mathbb{S}^n$. Note that by Proposition 2.1 we get

$$\|D^2 v_\infty\|_{L^2(B_2 \setminus B_1)} + \|Dv_\infty\|_{L^4(B_2 \setminus B_1)} > 0.$$

This contradicts the fact that there is only one bubble at $\{0\}$. If the convergence is not strong, then Lemma 3.3 implies that the energy must concentrate. That is, there exists a subsequence v_k such that $\|D^2 v_k\|_{L^2(B_r(x))} + \|Dv_k\|_{L^4(B_r(x))} \ge \varepsilon_0^2$ for all $r > 0$. This also contradicts the existence of only one bubble. Thus, (3-2) holds.

Using the duality of Lorentz spaces and the estimates of Section A.2, we get the bounds

$$\begin{aligned}
\|D^2 u_k\|_{L^2}^2 &\le C \|D^2 u_k\|_{L^{2,\infty}} \|D^2 u_k\|_{L^{2,1}}, \\
\|Du_k\|_{L^4}^4 &\le C \|\, |Du_k|^3 \|_{L^{4/3,\infty}} \|Du_k\|_{L^{4,1}} \\
&\le C \|Du_k\|_{L^{4,\infty}}^3 \|Du_k\|_{L^{4,1}}, \\
\|D\Delta u_k\|_{L^{4/3}}^{4/3} &\le C \|(D\Delta u_k)^{1/3}\|_{L^{4,\infty}} \|D\Delta u_k\|_{L^{4/3,1}} \\
&\le C \|D\Delta u_k\|_{L^{4/3,\infty}}^{1/3} \|D\Delta u_k\|_{L^{4/3,1}}.
\end{aligned} \tag{3-8}$$

Using (1-1) and (2-1), we observe that

$$\|D^2 u_k\|_{L^{2,1}} + \|Du_k\|_{L^{4,1}} + \|D\Delta u_k\|_{L^{4/3,1}} \le C\Lambda.$$

Since (3-2) allows us to apply Lemma 2.4, appealing to (3-8) implies the result. □

The full proof of Theorem 1.2 now follows immediately from the uniform energy bounds of (1-1), the small-energy compactness results of this section, and standard induction arguments on the bubbles.

4. Oscillation bounds

The proof of the following oscillation lemma will constitute the work of this section:

Lemma 4.1. *Let $u \in W^{2,2}(B_1, \mathbb{S}^n)$ be an f-approximate biharmonic map for $f \in L\log L(B_1, \mathbb{R}^{n+1})$ with*

$$\|D^2u\|_{L^2(B_1)} + \|Du\|_{L^4(B_1)} + \|f\|_{L\log L(B_1)} \leq \Lambda < \infty.$$

Then for $0 < 2t < \delta/2 < 1/16$,

$$\begin{aligned}\sup_{x,y\in B_{\delta/2}\setminus B_{2t}} &|u(x)-u(y)| \\ &\leq C\big(\|D^2u\|_{L^2(B_{2\delta}\setminus B_t)} + \|Du\|_{L^4(B_{2\delta}\setminus B_t)} + \|f\|_{L\log L(B_{2\delta})} \\ &\qquad + \|D\Delta u\|_{L^{4/3}(B_{2\delta}\setminus B_t)} + \|D\Delta u\|_{L^{4/3,1}(B_{2t}\setminus B_t)} + |B_{4\delta}|\big).\end{aligned}$$

The lemma also holds if u is an f-approximate intrinsic biharmonic map.

Consider the map $u_1 : B_1 \to \mathbb{R}^{n+1}$ such that $u_1(\boldsymbol{x}) = \boldsymbol{b} + A\boldsymbol{x}$, where $\boldsymbol{b} \in \mathbb{R}^{n+1}$ and A is an $(n+1)\times 4$ matrix with

$$A := \fint_{B_{2t}\setminus B_t} Du \quad \text{and} \quad \boldsymbol{b} := \fint_{B_{2t}\setminus B_t} (u(\boldsymbol{x}) - A\boldsymbol{x})\, d\mathrm{Vol}(\boldsymbol{x}).$$

Then by construction

$$\fint_{B_{2t}\setminus B_t} u - u_1 = 0, \quad \fint_{B_{2t}\setminus B_t} Du - Du_1 = 0, \quad D^k u_1 \equiv 0 \text{ for all } k \geq 2.$$

Set $w = (1-\phi_t)(u-u_1)$. Let $w_1 : B_1 \to \mathbb{R}^{n+1}$ such that $w_1(\boldsymbol{x}) = \boldsymbol{m} + N\boldsymbol{x}$, where

$$N := \fint_{B_\delta\setminus B_{\delta/2}} Dw \quad \text{and} \quad \boldsymbol{m} := \fint_{B_\delta\setminus B_{\delta/2}} (w(\boldsymbol{x}) - N\boldsymbol{x})\, d\mathrm{Vol}(\boldsymbol{x}).$$

Let $\widetilde{w} = (w - w_1)\phi_{\delta/2}$, so $\widetilde{w} = w - w_1$ on $B_{\delta/2}$ and the support of $\widetilde{w}$ is contained in B_δ.

By definition,

$$\begin{aligned}\sup_{x,y\in B_{\delta/2}\setminus B_{2t}} |u(x)-u(y)| &= \sup_{x,y\in B_{\delta/2}\setminus B_{2t}} |w(x)-w(y)+u_1(x)-u_1(y)| \\ &= \sup_{x,y\in B_{\delta/2}\setminus B_{2t}} |(\widetilde{w}+u_1+w_1)(x)-(\widetilde{w}+u_1+w_1)(y)| \\ &\leq 2 \sup_{x\in B_{\delta/2}\setminus B_{2t}} |\widetilde{w}(x)-\widetilde{w}(0)+(A+N)x|.\end{aligned}$$

We first observe that, outside of B_{2t}, $w = u - u_1$ so the definition of N implies that

$$A + N = A + \fint_{B_\delta\setminus B_{\delta/2}} Du - \fint_{B_\delta\setminus B_{\delta/2}} A = \fint_{B_\delta\setminus B_{\delta/2}} Du.$$

Thus, for $x \in B_{\delta/2}$, Hölder's inequality implies that

$$|(A+N)x| \le C\delta^{-3}\int_{B_\delta\setminus B_{\delta/2}} |Du| \le C\|Du\|_{L^4(B_\delta\setminus B_{\delta/2})}.$$

As before, let G be the distribution in $\mathbb{R}^4$ such that $\Delta^2 G = \delta_0$. Then $G(x) = C\log|x|$, and recall that $DG \in L^{4,\infty}(\mathbb{R}^4)$. It is enough to show that:

Claim 4.2. $$\left|\widetilde{w}(x) - \fint_{\mathbb{R}^4} \widetilde{w}\right| \le C\|D\Delta\widetilde{w}\|_{L^{4/3,1}(\mathbb{R}^4)}.$$

Since all of the above quantities are translation-invariant, we may assume $x = 0$. Then

$$\begin{aligned}\left|\widetilde{w}(0) - \fint \widetilde{w}\right| &= \left|\int_{\mathbb{R}^4} \Delta^2 G(y)\left(\widetilde{w}(y) - \fint \widetilde{w}\right) dV(y)\right| \\ &= \left|\int_{\mathbb{R}^4} DG(y) D\Delta\widetilde{w}(y)\, dV(y)\right| \\ &\le C\|DG\|_{L^{4,\infty}(\mathbb{R}^4)}\|D\Delta\widetilde{w}\|_{L^{4/3,1}(\mathbb{R}^4)}.\end{aligned}$$

Using the definition of $\widetilde{w}$,

$$\begin{aligned}&\|D\Delta\widetilde{w}\|_{L^{4/3,1}(\mathbb{R}^4)} \\ &\quad\le C\|(\delta^{-3}|w-w_1| + \delta^{-2}|D(w-w_1)| + \delta^{-1}|D^2 w|)\|_{L^{4/3,1}(B_\delta\setminus B_{\delta/2})} \\ &\qquad + C\|D\Delta w\|_{L^{4/3,1}(B_\delta)}.\end{aligned}$$

Interpolation techniques and Poincaré's inequality imply that

$$\begin{aligned}\|\delta^{-3}(w-w_1)\|_{L^{4/3,1}(B_\delta\setminus B_{\delta/2})} &\le C\|\delta^{-2}D(w-w_1)\|_{L^{4/3,1}(B_\delta\setminus B_{\delta/2})} \\ &\le C\|\delta^{-1}D^2 w\|_{L^{4/3,1}(B_\delta\setminus B_{\delta/2})}.\end{aligned}$$

Moreover, the embedding theorems for Lorentz spaces imply that

$$\|\delta^{-1}D^2 w\|_{L^{4/3,1}(B_\delta\setminus B_{\delta/2})} \le C\|D^2 w\|_{L^2(B_\delta\setminus B_{\delta/2})}.$$

Therefore,

$$\|D\Delta\widetilde{w}\|_{L^{4/3,1}(\mathbb{R}^4)} \le C\|D^2 w\|_{L^2(B_\delta\setminus B_{\delta/2})} + C\|D\Delta w\|_{L^{4/3,1}(B_\delta)}. \tag{4-1}$$

Since $D^2 w = D^2 u$ on $B_\delta \setminus B_{2t}$, we conclude that

$$\begin{aligned}&\operatorname{osc}_{B_{\delta/2}\setminus B_{2t}} u \\ &\quad\le C(\|D\Delta w\|_{L^{4/3,1}(B_\delta)} + \|D^2 u\|_{L^2(B_\delta\setminus B_{\delta/2})} + \|Du\|_{L^4(B_\delta\setminus B_{\delta/2})}).\end{aligned} \tag{4-2}$$

The remainder of the proof will be devoted to bounding the $D\Delta w$ term.

We define $\beta = D\Delta w \wedge u - \Delta w \wedge Du$. Then

$$\beta^{ij} := u^j D\Delta w^i - u^i D\Delta w^j - \Delta w^i Du^j + \Delta w^j Du^i \in \Omega^1\mathbb{R}^4$$

for $i, j = 1, \dots, n+1$. By definition $\beta = D\Delta u \wedge u - \Delta u \wedge Du$ in $B_\delta \setminus B_{2t}$ and thus $d^*\beta = f \wedge u$ in $B_\delta \setminus B_{2t}$. We will require an $L^{4/3}$ bound for β, and to that end note that

$$
\begin{aligned}
\text{(4-3)} \qquad \|\beta\|_{L^{4/3}(B_{2\delta})} &\le C(\|D\Delta w\|_{L^{4/3}(B_{2\delta})} + \|\Delta w \wedge Du\|_{L^{4/3}(B_{2\delta})}) \\
&\le C(\|D\Delta w\|_{L^{4/3}(B_{2\delta})} + \|\Delta w\|_{L^{2}(B_{2\delta})}\|Du\|_{L^{4}(B_{2\delta})}) \\
&\le C(\|D\Delta u\|_{L^{4/3}(B_{2\delta}\setminus B_t)} + \|D^2 u\|_{L^{2}(B_{2\delta}\setminus B_t)}).
\end{aligned}
$$

For the last inequality, $\|Du\|_{L^4(B_{2\delta})}$ is bounded and is absorbed into the constant. In addition, we use the definition of w and repeated applications of Poincaré and Hölder to determine

$$
\begin{aligned}
\|D\Delta w\|_{L^{4/3}(B_{2\delta})} &\le C(\|D^2 u\|_{L^{2}(B_{2t}\setminus B_t)} + \|(1-\phi_t)D\Delta u\|_{L^{4/3}(B_{2\delta})}), \\
\|\Delta w\|_{L^{2}(B_{2\delta})} &\le C\|D^2 u\|_{L^{2}(B_{2\delta}\setminus B_t)}.
\end{aligned}
$$

Set

$$\gamma := d^*(D\Delta(w-u) \wedge u - \Delta(w-u) \wedge Du).$$

Then

$$
\begin{gathered}
d^*\beta = f \wedge u + \gamma, \qquad\qquad d\beta = -2D\Delta w \wedge Du, \\
\Delta\beta = (dd^* + d^*d)\beta = d(f \wedge u + \gamma) + d^*(-2D\Delta w \wedge Du).
\end{gathered}
$$

We consider a decomposition $\beta^{ij} = H^{ij} + d\Psi^{ij} + d^*\Phi^{ij}$ for each component β^{ij}, where H^{ij} is a harmonic 1-form and Φ, Ψ satisfy appropriate partial differential equations. Our objective is to bound $\|D\Delta w\|_{L^{4/3,1}}$ by $\|\beta\|_{L^{4/3,1}}$, and to that end we determine such bounds for $d\Psi$, $d^*\Phi$, and H.

Remark 4.3. For the intrinsic case, we modify a few definitions. Let $\beta_I := \beta + 2|Du|^2 Dw_I \wedge u$, where $w_I = (1-\phi_t)(u-\boldsymbol{d})$ and $\boldsymbol{d} := \fint_{B_{2t}\setminus B_t} u$. Using the definition of w_I, we get the bound $\|\beta_I\|_{L^{4/3}(B_{2\delta})} \le \|\beta\|_{L^{4/3}(B_{2\delta})} + C\|Du\|_{L^4(B_{2\delta}\setminus B_t)}$ by using Hölder's inequality and Poincaré's inequality. We then define $\gamma_I := \gamma + d^*(2|Du|^2 D(w_I - u) \wedge u)$, and thus

$$d^*\beta_I = f \wedge u + \gamma_I \quad \text{and} \quad d\beta_I = d\beta + D(|Du|^2)Dw_I \wedge u - |Du|^2 Dw_I \wedge Du.$$

We now continue with the proof for the extrinsic case:

Proposition 4.4. *Let Ψ^{ij} be a function on $B_{2\delta}$ satisfying*

$$
\begin{cases}
\Delta\Psi^{ij} = f^i u^j - f^j u^i + \gamma^{ij} & \text{in } B_{2\delta}, \\
\Psi^{ij} = 0 & \text{on } \partial B_{2\delta}.
\end{cases}
$$

Then

$$\|d\Psi^{ij}\|_{L^{4/3,1}(B_{2\delta})} \leq C\big(\|D^2u\|_{L^2(B_{2t}\setminus B_t)} + \|Du\|_{L^4(B_{2t}\setminus B_t)} + \|D\Delta u\|_{L^{4/3}(B_{2t}\setminus B_t)} + \|f\|_{L\log L(B_{2\delta})} + |B_{4\delta}|\big).$$

Proof. We decompose $\Psi^{ij} = \Psi_1^{ij} + \Psi_2^{ij}$ so that

$$\begin{cases} \Delta\Psi_1^{ij} = \gamma^{ij} & \text{in } B_{2\delta}, \\ \Psi_1^{ij} = 0 & \text{on } \partial B_{2\delta}. \end{cases}$$

Following classical arguments,

$$\|D^2\Psi_1^{ij}\|_{L^1(B_{2\delta})} \leq C\|\gamma^{ij}\|_{\mathcal{H}^1(B_{2\delta})}.$$

Thus the embedding theorems imply that $\|D\Psi_1^{ij}\|_{L^{4/3,1}(B_{2\delta})} \leq C\|\gamma^{ij}\|_{\mathcal{H}^1(B_{2\delta})}$. Now we consider the $\mathcal{H}^1$ norm of γ^{ij}. By definition,

$$\begin{aligned}\gamma^{ij} &= d^*(D\Delta(w^i-u^i)u^j - D\Delta(w^j-u^j)u^i - [\Delta(w^i-u^i)Du^j - \Delta(w^j-u^j)Du^i]) \\ &= \Delta^2(w^i-u^i)u^j - \Delta^2(w^j-u^j)u^i - (\Delta(w^i-u^i)\Delta u^j - \Delta(w^j-u^j)\Delta u^i).\end{aligned}$$

Recall that $w := (1-\phi_t)(u-u_1)$. So

$$\begin{aligned}\Delta(w^j-u^j) &= -\Delta\phi_t(u^j-u_1^j) - 2D\phi_t\cdot D(u^j-u_1^j) - \phi_t\Delta u^j, \\ \Delta^2(w^j-u^j) &= -\Delta^2\phi_t(u^j-u_1^j) - \Delta\phi_t\Delta u^j - 2D\Delta\phi_t D(u^j-u_1^j) \\ &\quad - 2\Delta(D\phi_t\cdot D(u^j-u_1^j)) - \Delta\phi_t\Delta u^j - 2D\phi_t D\Delta u^j - \phi_t\Delta^2u^j.\end{aligned}$$

Combining all of the terms, we estimate

$$\begin{aligned}|\gamma^{ij}| &\leq C|D^4\phi_t|\,|u-u_1| + C|D^3\phi_t|\,|D(u-u_1)| + C|D^2\phi_t|\,|D^2u| \\ &\quad + C|D\phi_t|(|D\Delta u| + |D(u-u_1)|\,|\Delta u|) + |\phi_t||u^i\Delta^2u^j - u^j\Delta^2u^i|.\end{aligned}$$

The definition of γ^{ij} implies that $\gamma^{ij} = 0$ on $\mathbb{R}^4\setminus B_{2t}$ and

$$\int_{\mathbb{R}^4}\gamma^{ij} = \int_{\partial B_{2t}}(D\Delta(w-u)\wedge u - \Delta(w-u)\wedge Du)^{ij}\cdot\boldsymbol{n} = 0.$$

The estimate from Lemma A.1 implies that

$$\|\gamma^{ij}\|_{\mathcal{H}^1(B_{2\delta})} \leq c\big(t\|\gamma^{ij} - \phi_t(u^j\Delta^2u^i - u^i\Delta^2u^j)\|_{L^{4/3}(B_{2t})} + \|\phi_t(u^j\Delta^2u^i - u^i\Delta^2u^j)\|_{L\log L(B_{2\delta})} + |B_{4\delta}|\big).$$

Repeating techniques used previously, we bound the first three terms of $|\gamma^{ij}|$:

$$t^{-4}\|u-u_1\|_{L^{4/3}(B_{2t}\setminus B_t)} \le Ct^{-3}\|D(u-u_1)\|_{L^{4/3}(B_{2t}\setminus B_t)} \le Ct^{-2}\|D^2u\|_{L^{4/3}(B_{2t}\setminus B_t)} \le Ct^{-1}\|D^2u\|_{L^2(B_{2t}\setminus B_t)}.$$

We will preserve the term

$$t^{-1}\|D\Delta u\|_{L^{4/3}(B_{2t}\setminus B_t)},$$

as our energy quantization result implies that this term will vanish when taking limits. Hölder's inequality and the fact that $\|D(u-u_1)\|_{L^4(B_{2t}\setminus B_t)} \le C\|Du\|_{L^4(B_{2t}\setminus B_t)}$ imply that

$$\|D(u-u_1)\Delta u\|_{L^{4/3}(B_{2t}\setminus B_t)} \le C\|D(u-u_1)\|_{L^4(B_{2t}\setminus B_t)}\|D^2u\|_{L^2(B_{2t}\setminus B_t)} \le C\|D^2u\|_{L^2(B_{2t}\setminus B_t)}.$$

For the last term, since u is an f-approximate biharmonic map into $\mathbb{S}^n$,

$$\|\phi_t(\Delta^2 u\wedge u)\|_{L\log L(B_{2\delta})} \le \|f\wedge u\|_{L\log L(B_{2\delta})} \le \|f\|_{L\log L(B_{2\delta})}.$$

All of the above estimates imply that

$$\|\gamma^{ij}\|_{\mathcal{H}^1(B_{2\delta})} \le C\big(\|D^2u\|_{L^2(B_{2t}\setminus B_t)} + \|Du\|_{L^4(B_{2t}\setminus B_t)} + \|D\Delta u\|_{L^{4/3}(B_{2t}\setminus B_t)} + \|f\|_{L\log L(B_{2\delta})} + |B_{4\delta}|\big).$$

Finally, consider

$$\begin{cases} \Delta\Psi_2^{ij} = f^iu^j - u^if^j & \text{in } B_{2\delta}, \\ \Psi_2^{ij} = 0 & \text{on } \partial B_{2\delta}. \end{cases}$$

Then classical results give $\|\Psi_2^{ij}\|_{W^{2,1}(B_{2\delta})} \le C\|f\|_{\mathcal{H}^1(B_{2\delta})} \le C\|f\|_{L\log L(B_{2\delta})}$. Thus

$$\|d\Psi_2^{ij}\|_{W^{1,1}(B_{2\delta})} \le C\|f\|_{L\log L(B_{2\delta})},$$

and the embedding theorems in $\mathbb{R}^4$ imply that

$$\|d\Psi_2^{ij}\|_{L^{4/3,1}(B_{2\delta})} \le C\|f\|_{L\log L(B_{2\delta})}. \qquad \square$$

Remark 4.5. For the intrinsic case, we define

$$\begin{aligned}\gamma_I &= \gamma + d^*(2|Du|^2D(w_I-u)\wedge u)\\ &= \gamma - 2\phi_t d^*(|Du|^2Du\wedge u) + 2|Du|^2(\Delta\phi_t(\boldsymbol{d}-u)\wedge u - D\phi_t\cdot Du\wedge(\boldsymbol{d}+u))\\ &\qquad + 2D|Du|^2\cdot D\phi_t(\boldsymbol{d}-u)\wedge u.\end{aligned}$$

We bound $\|\gamma_I\|_{\mathcal{H}^1}$ by making some observations: First, $-2\phi_t d^*(|Du|^2Du\wedge u)$ is added to the term $-\phi_t\Delta^2u\wedge u$ that appears in the expansion of γ. We then make the substitution $-\phi_t f\wedge u$ as in the extrinsic case. Second, using Poincaré's inequality, Hölder's inequality, and the global energy bound for u, the $L^{4/3}$ norm of what remains is bounded by $Ct^{-1}(\|Du\|_{L^4(B_{2t}\setminus B_t)} + \|D^2u\|_{L^2(B_{2t}\setminus B_t)})$. Finally, observe that, by construction, γ_I is supported on B_{2t} and $\int_{\mathbb{R}^4}\gamma_I = 0$, so the estimate used for $\|\gamma\|_{\mathcal{H}^1}$ still applies.

Proposition 4.6. *Let $\Phi^{ij} \in \Omega^2\mathbb{R}^4$ be the solution to the system*

$$\begin{cases} \Delta\Phi^{ij} = -2(D\Delta w^i\, Du^j - D\Delta w^j\, Du^i) & \text{in } B_{2\delta}, \\ \Phi^{ij} = 0 & \text{on } \partial B_{2\delta}. \end{cases}$$

Then

$$\|d^*\Phi^{ij}\|_{L^{4/3,1}(B_{2\delta})} \le C(\|D^2u\|_{L^2(B_{2t}\setminus B_t)} + \|D\Delta u\|_{L^{4/3}(B_{2\delta}\setminus B_t)}). \tag{4-4}$$

Proof. Using the same techniques and estimates as in the previous proposition, we note that

$$\begin{aligned}\|d\Phi^{ij}\|_{L^{4/3,1}(B_{2\delta})} &\le C\|D\Delta w \wedge Du\|_{\mathcal{H}^1(B_{2\delta})} \\ &\le C\|D\Delta w\|_{L^{4/3}(B_{2\delta})}\|Du\|_{L^4(B_{2\delta})} \\ &\le C(\|D^2u\|_{L^2(B_{2t}\setminus B_t)} + \|D\Delta u\|_{L^{4/3}(B_{2\delta}\setminus B_t)}). \qquad \square\end{aligned}$$

Remark 4.7. In the intrinsic setting the steps of the proof are the same, though the equation for $\Delta\Phi_I^{ij}$ includes the terms $D(|Du|^2)Dw_I \wedge u - |Du|^2 Dw_I \wedge Du$. Since $\|Dw_I\|_{L^4(B_{2\delta})} \le C\|Du\|_{L^4(B_{2\delta}\setminus B_t)}$, one can quickly show the intrinsic bound has the form

$$\|d^*\Phi_I\|_{L^{4/3,1}(B_{2\delta})} \le \|d^*\Phi\|_{L^{4/3,1}(B_{2\delta})} + C\|Du\|_{L^4(B_{2\delta}\setminus B_t)}.$$

Now consider the harmonic 1-form

$$H^{ij} = \beta^{ij} - d^*\Phi^{ij} - d\Psi^{ij}.$$

Propositions 4.4 and 4.6, along with (4-3), imply that

$$\begin{aligned}\|H\|_{L^{4/3}(B_{2\delta})} &\le \|\beta\|_{L^{4/3}(B_{2\delta})} + \|d^*\Phi\|_{L^{4/3}(B_{2\delta})} + \|d\Psi\|_{L^{4/3}(B_{2\delta})} \\ &\le C\big(\|D^2u\|_{L^2(B_{2\delta}\setminus B_t)} + \|Du\|_{L^4(B_{2\delta}\setminus B_t)} \\ &\qquad + \|D\Delta u\|_{L^{4/3}(B_{2\delta}\setminus B_t)} + \|f\|_{L\log L(B_{2\delta})} + |B_{4\delta}|\big).\end{aligned}$$

The mean value property and Hölder's inequality together imply that

$$\begin{aligned}\|H^{ij}\|_{C^0(B_\delta)} \le \frac{C}{\delta^3}\big(&\|D^2u\|_{L^2(B_{2\delta}\setminus B_t)} + \|Du\|_{L^4(B_{2\delta}\setminus B_t)} \\ &+ \|D\Delta u\|_{L^{4/3}(B_{2\delta}\setminus B_t)} + \|f\|_{L\log L(B_{2\delta})} + |B_{4\delta}|\big).\end{aligned}$$

Moreover, a straightforward calculation implies that

$$\|H^{ij}\|_{L^{4/3,1}(B_\delta)} \le C\delta^3\|H^{ij}\|_{C^0(B_\delta)}.$$

Thus,

$$\|\beta\|_{L^{4/3,1}(B_\delta)} \le C\big(\|D^2u\|_{L^2(B_{2\delta}\setminus B_t)} + \|Du\|_{L^4(B_{2\delta}\setminus B_t)} + \|D\Delta u\|_{L^{4/3}(B_{2\delta}\setminus B_t)} + \|f\|_{L\log L(B_{2\delta})} + |B_{4\delta}|\big).$$

Using the appropriate harmonic 1-form H_I, we produce an identical estimate for β_I.

We now use the definitions of w and β to determine a bound on $\|D\Delta w\|_{L^{4/3,1}(B_\delta)}$. First we consider the function on B_{2t}

$$\begin{aligned}\|D\Delta w\|_{L^{4/3,1}(B_{2t})} &\le \|C(t^{-3}|u-u_1| + t^{-2}|D(u-u_1)| + t^{-1}|D^2u|)\|_{L^{4/3,1}(B_{2t}\setminus B_t)} \\ &\qquad + \|(1-\phi_t)D\Delta u\|_{L^{4/3,1}(B_{2t})} \\ &\le C\|D^2u\|_{L^2(B_{2t}\setminus B_t)} + C\|D\Delta u\|_{L^{4/3,1}(B_{2t}\setminus B_t)}.\end{aligned}$$

On $B_\delta \setminus B_{2t}$, $w = u - u_1$ so $D\Delta w \equiv D\Delta u$. We first decompose $D\Delta u$ into tangential and normal parts with tangency relative to the target manifold $\mathbb{S}^n$. Then

$$D\Delta u = D\Delta u^T + D\Delta u^N = D\Delta u \wedge u.u + \langle D\Delta u, u\rangle u.$$

Here we define $\langle Dv, u\rangle := \sum_{i,k}(\partial v^k/\partial x_i)u^k\, dx_i$. On $B_\delta \setminus B_{2t}$, $D\Delta u \wedge u = \beta + \Delta u \wedge Du$, and thus

$$|(D\Delta u)^T| \le |\beta| + |\Delta u|\,|Du|.$$

Since

$$\begin{aligned}\langle D\Delta u, u\rangle = D\langle \Delta u, u\rangle - \langle \Delta u, Du\rangle &= D(d^*\langle Du, u\rangle - |Du|^2) - \langle \Delta u, Du\rangle \\ &= -D|Du|^2 - \langle \Delta u, Du\rangle,\end{aligned}$$

we estimate

$$\begin{aligned}\|D\Delta w\|_{L^{4/3,1}(B_\delta\setminus B_{2t})} &\le C\|\beta\|_{L^{4/3,1}(B_\delta)} + C\|D^2u\|_{L^2(B_\delta\setminus B_{2t})}\|Du\|_{L^4(B_\delta\setminus B_{2t})} \\ &\le C\big(\|D^2u\|_{L^2(B_{2\delta}\setminus B_t)} + \|Du\|_{L^4(B_{2\delta}\setminus B_t)} + \|D\Delta u\|_{L^{4/3}(B_{2\delta}\setminus B_t)} \\ &\qquad + \|f\|_{L\log L(B_{2\delta})} + |B_{4\delta}|\big).\end{aligned}$$

Thus,

$$\begin{aligned}\|D\Delta w\|_{L^{4/3,1}(B_\delta)} \le C(&\|D^2u\|_{L^2(B_{2\delta}\setminus B_t)} + \|Du\|_{L^4(B_{2\delta}\setminus B_t)} + \|f\|_{L\log L(B_{2\delta})} \\ &+ \|D\Delta u\|_{L^{4/3,1}(B_{2t}\setminus B_t)} + \|D\Delta u\|_{L^{4/3}(B_{2\delta}\setminus B_t)} + |B_{4\delta}|).\end{aligned}$$

Inserting this inequality into (4-2) proves the oscillation lemma.

Remark 4.8. To complete the proof in the intrinsic case, observe that, on $B_\delta \setminus B_{2t}$, $D\Delta w \wedge u = D\Delta u \wedge u = \beta + \Delta u \wedge Du + 2|Du|^2 Du \wedge u$. This changes the L^∞ estimate for $|(D\Delta u)^T|$ on $B_\delta \setminus B_{2t}$, but using embedding theorems for Lorentz spaces we note that the $L^{4/3,1}$ estimate is unchanged.

5. No-neck property — proof of Theorem 1.3

The proof of the no-neck property now follows easily from combining the energy quantization and the oscillation bounds.

Proof. As we may use induction to deal with the case of multiple bubbles, we prove the theorem for one bubble. Let λ_k be such that $\tilde{u}_k(x) := u_k(\lambda_k x) \to \omega(x) \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^4, \mathbb{S}^n)$. Since each of the $u_k \in W^{2,2}(B_1, \mathbb{S}^n)$ are f_k-approximate biharmonic maps with $f_k \in L\log L(B_1, \mathbb{R}^{n+1})$ and have uniform energy bounds, Lemma 4.1 implies that

$$\begin{aligned}\sup_{x,y\in B_{\delta/2}\setminus B_{2\lambda_k R}} &|u_k(x)-u_k(y)| \\ &\le C\big(\|D^2u_k\|_{L^2(B_{2\delta}\setminus B_{\lambda_k R/2})} + \|Du_k\|_{L^4(B_{2\delta}\setminus B_{\lambda_k R/2})} \\ &\qquad + \|f_k\|_{L\log L(B_{2\delta})} + \|D\Delta u_k\|_{L^{4/3,1}(B_{\lambda_k R}\setminus B_{\lambda_k R/2})} \\ &\qquad\qquad + \|D\Delta u_k\|_{L^{4/3}(B_{2\delta}\setminus B_{\lambda_k R/2})} + |B_{4\delta}|\big).\end{aligned}$$

Theorem 1.2 implies that

$$\begin{aligned}\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{k\to\infty}\big(&\|D^2u_k\|_{L^2(B_{2\delta}\setminus B_{\lambda_k R/2})} + \|Du_k\|_{L^4(B_{2\delta}\setminus B_{\lambda_k R/2})} \\ &+ \|D\Delta u_k\|_{L^{4/3}(B_{2\delta}\setminus B_{\lambda_k R/2})}\big) = 0.\end{aligned}$$

Further, (2-1) and Hölder's inequality imply that

$$\begin{aligned}\|D\Delta u_k\|_{L^{4/3,1}(B_{\lambda_k R}\setminus B_{\lambda_k R/2})} &\le C(\|Du_k\|_{L^4(B_{2\lambda_k R}\setminus B_{\lambda_k R/4})} \\ &\quad + \|D^2u_k\|_{L^2((B_{2\lambda_k R}\setminus B_{\lambda_k R/4})} + \|f_k\|_{L\log L(B_{2\lambda_k R})}).\end{aligned}$$

Since we presume the $L\log L$ norm of f_k does not concentrate,

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{k\to\infty}\|f_k\|_{L\log L(B_{2\delta})} = 0.$$

Therefore,

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{k\to\infty}\|D\Delta u_k\|_{L^{4/3,1}(B_{\lambda_k R}\setminus B_{\lambda_k R/2})} = 0.$$

Taking all of the estimates together implies that

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{k\to\infty}\sup_{x,y\in B_{\delta/2}\setminus B_{2\lambda_k R}}|u_k(x)-u_k(y)| = 0.$$

Thus, no neck occurs in the blowup. □

Remark 5.1. For $f_k \in \phi(L)$, we use the estimate

$$
\begin{aligned}
&\|f_k\|_{L\log L(B_{2\delta})}\\
&\quad= \int_{B_{2\delta}\cap\{|f_k|\le\delta^{-1}\}} |f_k|\log(2+|f_k|)\,dx + \int_{|f_k|>\delta^{-1}} |f_k|\log(2+|f_k|)\,dx\\
&\quad\le C\delta^3\log(2+\delta^{-1}) + \sup_{t>\delta^{-1}} \frac{t\log(2+t)}{\phi(t)} \int_{|f_k|>\delta^{-1}} \phi(|f_k|)\,dx\\
&\quad\le C\delta^3\log(2+\delta^{-1}) + \sup_{t>\delta^{-1}} \frac{t\log(2+t)}{\phi(t)}\Lambda.
\end{aligned}
$$

Since we presumed $\lim_{t\to\infty}\phi(t)/(t\log t)=\infty$, we determine

$$\lim_{\delta\to 0}\sup_k \|f_k\|_{L\log L(B_{2\delta})} = 0.$$

Appendix: Necessary background

A.1. ***Hardy spaces, Lorentz spaces,*** $L\log L$***, and Orlicz spaces.*** Let

$$T := \{\Phi \in C^\infty(\mathbb{R}^4) : \mathrm{spt}(\Phi) \subset B_1,\ \|\nabla\Phi\|_{L^\infty(\mathbb{R}^4)} \le 1\}.$$

For any $\Phi \in T$, let $\Phi_t(x) := t^{-4}\Phi(x/t)$. For each $f \in L^1(\mathbb{R}^4)$, let

$$f_*(x) = \sup_{\Phi\in T}\sup_{t>0} |(\Phi_t * f)(x)|.$$

Then f is in the *Hardy space* $\mathcal{H}^1(\mathbb{R}^4)$ if $f_* \in L^1(\mathbb{R}^4)$ and

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^4)} = \|f_*\|_{L^1(\mathbb{R}^4)}.$$

Thus, one has the continuous embedding $\mathcal{H}^1 \hookrightarrow L^1$.

For a measurable function $f:\Omega\to\mathbb{R}$, let f^* denote the nonincreasing rearrangement of $|f|$ on $[0,|\Omega|)$ such that

$$|\{x\in\Omega : |f(x)|\ge s\}| = |\{t\in(0,|\Omega|) : f^*(t)\ge s\}|.$$

Let

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,ds.$$

For $p\in(1,\infty)$, let

$$\|f\|_{L^{p,q}} = \begin{cases} \int_0^\infty t^{1/p-1} f^{**}(t)\,dt & \text{if } q=1,\\ \sup_{t>0} t^{1/p} f^{**}(t) & \text{if } q=\infty. \end{cases}$$

We will also occasionally exploit the fact that one may understand $\|f\|_{L^{p,\infty}}$ by understanding instead its seminorm

$$\|f\|^*_{L^{p,\infty}} := \sup_{\lambda>0} \lambda|\{x : |f(x)| > \lambda\}|^{1/p}.$$

We define the Banach spaces

$$L^{p,q} := \{f : \|f\|_{L^{p,q}} < \infty\}.$$

The spaces $L^{p,1}$ and $L^{p,\infty}$ are examples of *Lorentz spaces*, and can be thought of as interpolation spaces between the standard L^p spaces. For example, one observes that the following embeddings are all continuous

$$L^r(B_1) \hookrightarrow L^{p,1}(B_1) \hookrightarrow L^{p,p}(B_1) = L^p(B_1) \hookrightarrow L^{p,\infty}(B_1) \hookrightarrow L^q(B_1)$$

for all $q < p < r$ [Hélein 1990].

We define

$$L \log L := \left\{ f : \int |f(x)| \log(2 + |f(x)|)\, dx < \infty \right\}.$$

Since this is nonlinear, we will use the following seminorm which is equivalent to the norm for $L \log L$

$$\|f\|_{L \log L} := \int f^*(t) \log\Big(2 + \frac{1}{t}\Big)\, dt.$$

We also note that $L^p(B_1) \hookrightarrow L \log L(B_1) \hookrightarrow L^1(B_1)$ are continuous embeddings for all $p > 1$. Finally, we say f is in $\mathcal{H}^1(B_1)$ if

$$\Big(f - \fint_{B_1} f(x)\, dx \Big) \chi_{B_1} \in \mathcal{H}^1(\mathbb{R}^4).$$

We record here the often-used estimate

$$\|f\|_{\mathcal{H}^1(B_1)} \le C \|f\|_{L \log L(B_1)}. \tag{A-1}$$

Finally, for any increasing function $\phi : [0, \infty) \to [0, \infty)$ we define the Orlicz space

$$\phi(L) := \left\{ f : \|f\|_{\phi(L)} := \int \phi(|f|)\, dx < \infty \right\}.$$

Examples include the L^p spaces for $\phi(t) = t^p$ and $L \log L$ when $\phi(t) = t \log(2+t)$.

A.2. ***Embeddings and estimates for Lorentz spaces.*** We will frequently use the following facts about Lorentz spaces:

(1) $L^{p,q}\cdot L^{p',q'}$ continuously embeds into $L^{r,s}$ for $1/p+1/p'\le 1$ where

$$\frac{1}{r}=\frac{1}{p}+\frac{1}{p'}\quad\text{and}\quad\frac{1}{s}=\frac{1}{q}+\frac{1}{q'},$$

with

$$\|fg\|_{L^{r,s}}\le C\|f\|_{L^{p,q}}\|g\|_{L^{p',q'}}.$$

(2) For $f\in L^2$ and $g\in W^{1,2}$,

$$\|fg\|_{L^{4/3,1}}\le C\|f\|_{L^2}\|g\|_{W^{1,2}}.$$

(3) $W^{1,1}(\mathbb{R}^4)\hookrightarrow L^{4/3,1}(\mathbb{R}^4)$ and $W^{1,2}(\mathbb{R}^4)\hookrightarrow L^{4,2}(\mathbb{R}^4)$ are continuous embeddings.

(4) $L^{2,1}$ and $L^{2,\infty}$ are dual spaces, as are $L^{4,\infty}$, $L^{4/3,1}$ and $L^{4,1}$, $L^{4/3,\infty}$.

(5) For all $0<p,r<\infty$ and $0<q\le\infty$ (see [Grafakos 2008], Section 1.4.2),

$$\|f^r\|_{L^{p,q}}=\|f\|^r_{L^{pr,qr}}.$$

(6) Let $f\in L^{p,q}(\mathbb{R}^4)$ and $g\in L^{p',q'}(\mathbb{R}^4)$ with $1/p+1/p'>1$. Then $h=f*g\in L^{r,s}(\mathbb{R}^4)$ where $1/r=1/p+1/p'-1$ and s is a number such that $1/q+1/q'\ge 1/s$. Moreover,

$$\|h\|_{L^{r,s}(\mathbb{R}^4)}\le c\|f\|_{L^{p,q}(\mathbb{R}^4)}\|g\|_{L^{p',q'}(\mathbb{R}^4)}.$$

For a proof, see [Ziemer 1989].

Let G be the distribution such that $\Delta^2G=\delta_0$. Then, $D^2G\in L^{2,\infty}(\mathbb{R}^4)$ and $D^3G\in L^{4/3,\infty}(\mathbb{R}^4)$. Moreover, $DG\in L^{4,\infty}(\mathbb{R}^4)$.

Using (6), and considering D^2G, D^3G as operators by convolution, we have:

(7) $D^2G : L^{4/3,1}(\mathbb{R}^4)\to L^{4,1}(\mathbb{R}^4)$ and $D^3G : L^{4/3,1}(\mathbb{R}^4)\to L^{2,1}(\mathbb{R}^4)$ are bounded operators.

A.3. ***Scaling and estimates for $L\log L$ and $\mathcal{H}^1$.*** We first prove an essential but technical lemma that is probably well known, though we have not found a reference in the literature. (We prove the lemma for our particular setting, though a more general result is true.)

Lemma A.1. *Let $f=f_1+f_2$, where $f_1\in L^{4/3}(B_R)$ and $f_2\in L\log L(B_R)$, be a compactly supported function with $\operatorname{spt}(f)\subset B_R$ and $\int_{\mathbb{R}^4}f(x)\,dx=0$. Then $f\in\mathcal{H}^1(B_R)$ and there exists $C>0$ such that*

$$\|f\|_{\mathcal{H}^1(B_R)}\le C(R\|f_1\|_{L^{4/3}(B_R)}+\|f_2\|_{L\log L(B_R)}+|B_{2R}|). \tag{A-2}$$

Proof. First note that

$$\|f_*\|_{L^1} = \int_{B_{2R}} f_*(x)\,dx + \int_{\mathbb{R}^4\setminus B_{2R}} f_*(x)\,dx. \tag{A-3}$$

Since $f_1 \in L^{4/3}(\mathbb{R}^4)$ and $f_2 \in L\log L(\mathbb{R}^4)$, we see that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^4)$ and therefore $f_*(x) \le cMf(x)$ for every $x \in \mathbb{R}^4$. Here $Mf : \mathbb{R}^4 \to \mathbb{R}$ is the maximal function defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\,dy.$$

Using the above, Hölder's inequality and the estimates $\|Mf_1\|_{L^{4/3}} \le c\|f_1\|_{L^{4/3}}$ and $\|Mf_2\|_{L^1(B_{2R})} \le c\|f_2\|_{L\log L(B_{2R})} + c|B_{2R}|$,

$$\begin{aligned}\int_{B_{2R}} f_*(x)\,dx &\le cR\|(f_1)_*\|_{L^{4/3}} + \|(f_2)_*\|_{L^1}\\ &\le cR\|Mf_1\|_{L^{4/3}} + c\|Mf_2\|_{L^1}\\ &\le cR\|f_1\|_{L^{4/3}} + c\|f_2\|_{L\log L} + c|B_{2R}|.\end{aligned} \tag{A-4}$$

Now we calculate for $\phi \in T$ and $x \in \mathbb{R}^4$:

$$\begin{aligned}|\phi_t \star f(x)| &= \left|\int_{B_R} \phi_t(x-y)f(y)\,dy\right|\\ &= \left|\int_{B_R} (\phi_t(x-y)-\phi_t(x))f(y)\,dy\right|\\ &\le \|\nabla\phi_t\|_{L^\infty} \int_{B_R} |y||f(y)|\,dy,\end{aligned}$$

where we used the mean value theorem and the cancellation property $\int_{\mathbb{R}^4} f(y)\,dy = 0$. Since $\|\nabla\phi_t\|_{L^\infty} \le 1/t^5$, for $t > 0$, we estimate

$$\begin{aligned}|\phi_t \star f(x)| &\le \frac{R}{t^5}\int_{B_R} |f(y)|\,dy\\ &\le \frac{cR^2}{t^5}\|f_1\|_{L^{4/3}} + \frac{cR}{t^5}\|f_2\|_{L\log L}.\end{aligned} \tag{A-5}$$

Assuming now that $|x| \ge 2R$, we can apply a technical result to get

$$f_*(x) = \sup_{\phi\in T}\sup_{t>|x|/2} |\phi_t \star f(x)| \le \frac{cR^2}{|x|^5}\|f_1\|_{L^{4/3}} + \frac{cR}{|x|^5}\|f_2\|_{L\log L}. \tag{A-6}$$

Inserting (A-4) and (A-6) into (A-3), we conclude that

$$\begin{aligned}\|f_*\|_{L^1} &\le cR\|f_1\|_{L^{4/3}} + c\|f_2\|_{L\log L} + c|B_{2R}| \\ &\quad + (cR^2\|f_1\|_{L^{4/3}} + cR\|f_2\|_{L\log L})\int_{\mathbb{R}^4\setminus B_{2R}} \frac{1}{|x|^5}\,dx \\ &\le cR\|f_1\|_{L^{4/3}} + c\|f_2\|_{L\log L} + c|B_{2R}|.\end{aligned} \tag{A-7}$$

This concludes the proof. □

We also note two important inequalities (with proofs following those of [Sharp and Topping 2013]):

Lemma A.2. *Let $f \in L\log L(B_r(x_0))$ for $r \in (0, 1/2]$. There exists $C > 0$ such that*

$$\|f\|_{L^1(B_r(x_0))} \le C(\log(1/r))^{-1}\|f\|_{L\log L(B_r(x_0))}. \tag{A-8}$$

Proof. Start by observing that

$$\begin{aligned}0 &\le r^4\int_0^{|B_1|} f^*(r^4t)\log\left(2+\frac{1}{t}\right)dt \\ &= \int_0^{|B_r(x_0)|} f^*(s)\log\left(2+\frac{r^4}{s}\right)ds \\ &= \int_0^{|B_r(x_0)|} f^*(s)\log(r^4)\,ds + \int_0^{|B_r(x_0)|} f^*(s)\log\left(\frac{2}{r^4}+\frac{1}{s}\right)ds \\ &\le -4\log(1/r)\|f\|_{L^1(B_r(x_0))} + C\|f\|_{L\log L(B_r(x_0))}.\end{aligned}$$

The last inequality follows from the fact that there exists a fixed C such that

$$\frac{2}{r^4}+\frac{1}{s} \le \frac{2\omega_4+1}{s} \le \left(2+\frac{1}{s}\right)^C$$

for all $s \le \omega_4 r^4$. □

Let u be an f-approximate biharmonic map on B_1 with $f \in L\log L(B_1)$. For $x_0 \in B_1$ and $R > 0$ such that $B_R(x_0) \subset B_1$, define $\hat{u}(x) := u(x_0 + Rx)$ and $\hat{f}(x) := R^4 f(x_0 + Rx)$. Then $\hat{u}$ is an $\hat{f}$-approximate biharmonic map. Moreover, we note that for any $r \in (0, 1)$, $p \ge 1$, and $k = 1, 2, 3$:

(1) $\|D^k\hat{u}\|_{L^{4/k}(B_r)} = \|D^k u\|_{L^{4/k}(B_{rR}(x_0))}$.

(2) $\|\hat{f}\|_{L^p(B_r)} = R^{4(1-1/p)}\|f\|_{L^p(B_{rR}(x_0))}$.

Lemma A.3. *Let $f \in L\log L(B_r(x_0))$, where $r \in (0, 1/2]$ and define $\hat{f}(x) := r^4 f(x_0 + rx)$. Then there exists $C > 0$ such that*

$$\|\hat{f}\|_{L\log L(B_1)} \le C\|f\|_{L\log L(B_r(x_0))}.$$

Proof. First note that, using the definition of $\hat{f}$, one can immediately show that $\hat{f}^*(t) = r^4 f^*(r^4 t)$. Thus,

$$\begin{aligned}\int_0^{|B_1|} \hat{f}^*(t)\log\left(2+\frac{1}{t}\right)dt &= \int_0^{|B_1|} r^4 f^*(r^4 t)\log\left(2+\frac{1}{t}\right)dt \\ &= \int_0^{|B_r(x_0)|} f^*(s)\log\left(2+\frac{r^4}{s}\right)ds \\ &\le \int_0^{|B_r(x_0)|} f^*(s)\log\left(2+\frac{1}{s}\right)ds. \qquad \square\end{aligned}$$

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CHRISTINE BREINER
DEPARTMENT OF MATHEMATICS
FORDHAM UNIVERSITY
BRONX, NY 10458
UNITED STATES
cbreiner@fordham.edu

TOBIAS LAMM
INSTITUTE FOR ANALYSIS
KARLSRUHE INSTITUTE OF TECHNOLOGY
KAISERSTRASSE 89-93
D-76133 KARLSRUHE
GERMANY
tobias.lamm@kit.edu

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