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DAVID KALAJ

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QUASICONFORMAL HARMONIC MAPPINGS BETWEEN DINI-SMOOTH JORDAN DOMAINS

DAVID KALAJ

Let D and Ω be Jordan domains with Dini-smooth boundaries. We prove that if $f : D \to \Omega$ is a harmonic homeomorphism and f is quasiconformal, then f is Lipschitz. This extends some recent results, where stronger assumptions on the boundary are imposed. Our result is optimal in that it coincides with the best condition for Lipschitz behavior of conformal mappings in the plane and conformal parametrizations of minimal surfaces.

1. Introduction and statement of the main result

Quasiconformal mappings. By definition, K-quasiconformal mappings (or qc mappings for short) are orientation-preserving homeomorphisms $f : D \to \Omega$ between domains $D, \Omega \subset \mathbb{C}$ that are contained in the Sobolev class $W^{1,2}_{\mathrm{loc}}(D)$ and for which the differential matrix and its determinant are coupled in the distortion inequality

$$|Df(z)|^2 \le K \det Df(z)\,, \quad \text{where } |Df(z)| = \max_{|\xi|=1} |Df(z)\xi|, \tag{1-1}$$

for some $K \ge 1$. Here $\det Df(z)$ is the determinant of the formal derivative $Df(z)$, which will be denoted in the sequel by $J_f(z)$. Note that condition (1-1) can be written in complex notation as

$$(|f_z| + |f_{\bar z}|)^2 \le K(|f_z|^2 - |f_{\bar z}^2|) \quad \text{a.e. on } D, \tag{1-2}$$

or, what is the same,

$$|f_{\bar z}| \le k|f_z| \quad \text{a.e. on } D, \qquad \text{where } k = \frac{K-1}{K+1}, \text{ i.e., } K = \frac{1+k}{1-k}.$$

Harmonic mappings and the Hilbert transform. A mapping f is called *harmonic* in a region D if it has the form $f = u + iv$, where u and v are real-valued harmonic functions in D. If D is simply connected, then there are two analytic functions h and g defined on D such that f has the representation

$$f = h + \bar g.$$

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If f is a harmonic univalent function, then by Lewy's theorem [1936], f has a nonvanishing Jacobian and therefore is a diffeomorphism by the inverse mapping theorem.

Let

$$P(r, x-\varphi) = \frac{1-r^2}{2\pi(1-2r\cos(x-\varphi)+r^2)}$$

denote the Poisson kernel. If $F \in L^1(\mathbb{T})$, where $\mathbb{T}$ is the unit circle, we define the Poisson integral $\mathcal{P}[F]$ of F by

$$\mathcal{P}[F](z) = \int_0^{2\pi} P(r, x-\varphi)F(e^{ix})\,dx, \quad |z|<1,\ z=re^{i\varphi}. \tag{1-3}$$

The function $f(z)=\mathcal{P}[F](z)$ is a harmonic mapping in the unit disk $\mathbb{U}=\{z:|z|<1\}$, which belongs to the Hardy space $h^1(\mathbb{U})$. The mapping f is bounded in $\mathbb{U}$ if and only if $F \in L^\infty(\mathbb{T})$. Standard properties of the Poisson integral show that $\mathcal{P}[F]$ extends by continuity to F on $\overline{\mathbb{U}}$, provided that F is continuous. For these facts and standard properties of harmonic Hardy spaces, we refer to [Axler et al. 1992, Chapter 6; Duren 1970]. With the additional assumption that F is an orientation-preserving homeomorphism of this circle onto a convex Jordan curve γ, $\mathcal{P}[F]$ is an orientation-preserving diffeomorphism of the open unit disk onto the region bounded by γ. This is indeed the celebrated theorem of Choquet–Radó–Kneser [Choquet 1945; Duren 2004]. This theorem is not true for nonconvex domains, but does hold under some additional assumptions. It has been extended in various directions (see for example [Jost 1981; Kalaj 2011b; Duren and Hengartner 1997]).

If $f=u+iv$ is a harmonic function defined in a Dini-smooth Jordan domain D then a harmonic function $\tilde f=\tilde u+i\tilde v$ is called the harmonic conjugate of f if $u+i\tilde u$ and $v+i\tilde v$ are analytic functions. Notice that $\tilde f$ is uniquely determined up to an additive constant. Let $\Phi : D \to \mathbb{U}$ be a conformal mapping, and let $G \in L^1(\partial D)$. Then the Poisson integral of G with respect to the domain D is defined by

$$\mathcal{P}_D[G](z) = \frac{1}{2\pi}\int_{\partial D} \frac{1-|\Phi(z)|^2}{|\Phi(z)-\Phi(\zeta)|^2} G(\zeta)|\Phi'(\zeta)|\,d\zeta.$$

Let χ be the boundary value of f and assume that $\tilde\chi$ is the boundary value of $\tilde f$. Then $\tilde\chi$ is called the Hilbert transform of χ and we also write it as $H(\chi)$. Assume that $\tilde\chi \in L^1(\partial D)$. In particular, the Hilbert transform of a function $\chi \in L^1(\mathbb{T})$ is defined by the formula

$$\tilde\chi(\tau) = H(\chi)(\tau) = -\frac{1}{\pi}\int_{0^+}^{\pi} \frac{\chi(\tau+t)-\chi(\tau-t)}{2\tan(t/2)}\,dt. \tag{1-4}$$

Here $\int_{0^+}^{\pi}\Phi(t)\,dt := \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\pi}\Phi(t)\,dt$. This integral is improper and converges for a.e. $\tau \in [0, 2\pi]$. This and other facts concerning the operator H used in this paper can be found in [Zygmund 1959, Chapter VII]. Assume that χ, $\tilde{\chi}$ are in $L^1(\mathbb{T})$. Then

$$\mathcal{P}[\tilde{\chi}] = (\mathcal{P}[\chi])^{\sim}, \tag{1-5}$$

where $(k)^{\sim}$ is the harmonic conjugate of k (see for instance [Pavlović 2004, Theorem 6.1.3]).

If $f = h + \bar{g} : \mathbb{U} \to \Omega$ is a harmonic mapping then the radial and tangential derivatives at $z = re^{it}$ are defined by

$$\partial_r f(z) = \frac{1}{r}(h' + \bar{g}') \quad \text{and} \quad \partial_t f(z) = i(h' - \bar{g}').$$

So $r\partial_r f$ is the harmonic conjugate of $\partial_t f$. We generalize this definition for a mapping $f = h + \bar{g}$ defined in a Jordan domain D. In order to do so, let $\Phi = Re^{i\Theta}$ be a conformal mapping of the domain D onto the unit disk. Then the radial derivative and tangent derivative of f in a point $w \in D$ are defined by

$$\partial_R f(w) = \frac{1}{|\Phi(w)|} Df(w)\left(\frac{\Phi(w)}{\Phi'(w)}\right) \quad \text{and} \quad \partial_\Theta f(w) = Df(w)\left(i\frac{\Phi(w)}{\Phi'(w)}\right).$$

Here $\Phi(w)/\Phi'(w)$ and $i(\Phi(w)/\Phi'(w))$ are treated as two vectors from $\mathbb{R}^2 \cong \mathbb{C}$. Then it is easy to show that

$$R\partial_R f(w) = \frac{h'(w)}{\Phi'(w)} + \frac{\overline{g'(w)}}{\overline{\Phi'(w)}} \quad \text{and} \quad \partial_\Theta f(w) = i\left(\frac{h'(w)}{\Phi'(w)} - \frac{\overline{g'(w)}}{\overline{\Phi'(w)}}\right).$$

This implies that $R\partial_R f(w)$ and $\partial_\Theta f(w)$ are harmonic functions in D and $R\partial_R f(w)$ is the harmonic conjugate of $\partial_\Theta f(w)$. Notice also that these derivatives are uniquely determined up to a conformal mapping Φ. Assume further that D and Ω have Dini-smooth boundaries. If $F : \partial D \to \partial\Omega$ is the boundary function of f, and if $\partial_\Theta f(w)$ is a bounded harmonic function, then

$$\lim_{w \to w_0} \partial_\Theta f(w) = F'(w_0),$$

where the limit is nontangential. Here

$$F'(w_0) := \frac{\partial(F \circ \Phi^{-1})(e^{it})}{\partial t},$$

where $\Phi(w_0) = e^{it}$. If $F' \in L^1(\partial D)$, then the harmonic function $R\partial_R f(w)$ has nontangential limits in almost every point of ∂D and its boundary value is the Hilbert transform of F', namely

$$H(F')(w_0) = \lim_{w \to w_0} R\partial_R f(w).$$

From now on the boundary value of f will be denoted by F. We will focus on orientation-preserving harmonic quasiconformal mappings between smooth domains and investigate their Lipschitz character up to the boundary. For future reference, we will say that a qc mapping $f : \mathbb{U} \to \Omega$ of the unit disk onto the Jordan domain Ω with rectifiable boundary is *normalized* if $f(1) = w_0$, $f(e^{2\pi i/3}) = w_1$ and $f(e^{4\pi i/3}) = w_2$, where w_0w_1, w_1w_2 and w_2w_0 are arcs of $\gamma = \partial\Omega$ having the same length $|\gamma|/3$.

Background. Let Ω be a Jordan domain with rectifiable boundary, and let γ be an arc-length parametrization of $\partial\Omega$. We say that $\partial\Omega$ is C^1 if $\gamma \in C^1$. Then $\arg\gamma'$ is continuous and we let ω be its modulus of continuity. If ω satisfies

$$\int_0^\delta \frac{\omega(t)}{t}\,dt < \infty, \qquad \delta > 0, \tag{1-6}$$

we say that $\partial\Omega$ is Dini-smooth. Denote by $C^{1,\varpi}$ the class of all Dini-smooth Jordan curves. The derivative of a conformal mapping f of the unit disk onto Ω is continuous and nonvanishing in $\overline{D}$ [Pommerenke 1975, Theorem 10.2] (see also [Warschawski 1961]). This implies that f is bi-Lipschitz continuous. For later reference we refer to this result as Kellogg's theorem, see [Kellogg 1912; Goluzin 1969, p. 374]. Kellogg was the first to consider this type of result for $C^{1,\alpha}$ domains, where $0 < \alpha < 1$. Warschawski [1970] proved the same result for a conformal parametrization of a minimal surface.

If f is merely quasiconformal and maps the unit disk onto itself, then Mori's theorem implies that $|f(z) - f(w)| \le M_1(K)|z - w|^{1/K}$. The constant $1/K$ is the best possible. If f is a conformal mapping of the unit disk onto a Jordan domain with a C^1 boundary, then the function f is not necessarily Lipschitz (see for example [Lesley and Warschawski 1978, p. 277]). This is why we need to add some assumption, other than quasiconformality, as well as some smoothness of the image curve that is better than C^1 in order to obtain that the resulting mapping is Lipschitz or bi-Lipschitz.

Since every conformal mapping in the plane is harmonic and quasiconformal, it is an interesting question to ask to what extent the smoothness of the boundary of a Jordan domain Ω implies that a quasiconformal harmonic mapping of the unit disk onto Ω is Lipschitz. The first study of harmonic quasiconformal mappings of the unit disk onto itself was done by O. Martio [1968]. This paper has been generalized in [Kalaj 2004] for qc mappings from the unit disk onto a convex Jordan domain. Pavlović [2002] proved in a very interesting way that every qc harmonic mapping of the unit disk onto itself is Lipschitz. Kalaj [2008] proved that every qc harmonic mapping between two Jordan domains with $C^{1,\alpha}$ boundary is Lipschitz. This result has its counterpart for non-Euclidean metrics [Kalaj and Mateljević 2006]. For a generalization of the last result to the several-dimensional case we refer to [Kalaj 2013]. The problem of bi-Lipschitz continuity of a quasiconformal mapping of the unit disk onto a Jordan domain with C^2 boundary has been solved in [Kalaj 2011a]. The object of this paper is to extend some of these results.

New results. The following theorem is such an extension in which the Hölder continuity is replaced by the more general Dini condition.

Theorem 1.1. *Let $f = \mathcal{P}[F](z)$ be a harmonic normalized K-quasiconformal mapping between the unit disk and the Jordan domain Ω with $\gamma = \partial\Omega \in C^{1,\varpi}$. Then there exists a constant $C' = C'(\gamma, K)$ such that*

$$\left|\frac{\partial F(e^{i\varphi})}{\partial\varphi}\right| \le C' \quad \text{for almost every } \varphi \in [0, 2\pi], \tag{1-7}$$

and

$$|f(z_1) - f(z_2)| \le KC'|z_1 - z_2| \quad \text{for } z_1, z_2 \in \mathbb{U}. \tag{1-8}$$

By using Theorem 1.1, we obtain the following improvement of [Kalaj 2008, Theorem 3.1].

Theorem 1.2. *Let D and Ω be Jordan domains such that ∂D and $\partial\Omega$ are contained in $C^{1,\varpi}$ and let $f : D \mapsto \Omega$ be a harmonic homeomorphism. The following statements hold true.*

(a) *If f is qc, then f is Lipschitz.*

(b) *If Ω is convex and f is qc, then f is bi-Lipschitz.*

(c) *If Ω is convex, then f is qc if and only* $\log|F'|$ *and $H(F')$ are in $L^\infty(\partial D)$.*

Proof of Theorem 1.2. (a) Choose a conformal mapping $\Phi : \mathbb{U} \to D$ so that the qc mapping $f_1 = f \circ \Phi$ is normalized. Then f_1 is a qc harmonic mapping of the unit disk onto Ω that satisfies the conditions of Theorem 1.1. This implies in particular that f_1 is Lipschitz. In view of Kellogg's theorem, the mapping Φ is bi-Lipschitz. Thus $f = f_1 \circ \Phi^{-1}$ is Lipschitz.

(b) If Ω is a convex domain, and if $D = \mathbb{U}$, then by [Kalaj 2003], we have that

$$|Df(z)| \ge \tfrac{1}{4}\,\mathrm{dist}(f(0), \partial\Omega)$$

for $z \in \mathbb{U}$. If D is not the unit disk, then we make use of the conformal mapping $\Phi : \mathbb{U} \to D$ as in the proof of (a). Then we obtain

$$|Df(z)| = |Df_1(z)|/|\Phi'(z)| \ge c.$$

Now by using the quasiconformality of f, we have that

$$|Df(z)|^2 \le KJ_f(z).$$

Therefore

$$J_{f^{-1}(f(z))} = \frac{1}{J_f(z)} \le \frac{K}{c^2}.$$

Since f^{-1} is K-quasiconformal, we have further that

$$|Df^{-1}(w)|^2 \le K J_{f^{-1}}(w) \le \frac{K^2}{c^2}.$$

This implies that f^{-1} is Lipschitz. This finishes the proof of (b).

(c) If f is harmonic and quasiconformal, then by (b) it is bi-Lipschitz, and so its boundary function F is bi-Lipschitz. Furthermore, $R\partial_R f$ is a bounded harmonic function and this is equivalent with the fact that $\log|F'| \in L^\infty(\partial D)$. Since $H(F')$ is its boundary function, it is bounded, i.e., it belongs to $L^\infty(\partial D)$.

We now prove the opposite implication. Since

$$\partial_\Theta f = \mathcal{P}_D[F'] \quad \text{and} \quad R\partial_R f = \mathcal{P}_D[H(F')],$$

it follows that $\partial_\Theta f$ and $R\partial_R f$ are bounded harmonic functions. This means that $|Df|$ is bounded by a constant M. In order to show that f is quasiconformal, it is enough to show that the Jacobian of f is bigger than a positive constant in D. Let $f_1 = f \circ \Phi^{-1}$, and let $\delta = \operatorname{dist}(f_1(0), \partial\Omega)$ and $\kappa = \min|\partial_t f_1(e^{it})|$. Then by [Kalaj 2004, Corollary 2.9], we have

$$J_f(\Phi(w))|\Phi'(w)|^2 = J_{f_1}(w) \ge \frac{\kappa\delta}{2}.$$

So

$$J_f(z) \ge c > 0, \quad z \in D.$$

We conclude that

$$\frac{|Df(z)|^2}{J_f(z)} \le \frac{M^2}{c}. \qquad \square$$

2. Preliminary results

Definition 2.1. Let $\xi : [a, b] \to \mathbb{C}$ be a continuous function. The modulus of continuity of ξ is

$$\omega(t) = \omega_\xi(t) = \sup_{|x-y|\le t} |\xi(x) - \xi(y)|.$$

The function ξ is called Dini-continuous if

$$\int_0^{b-a} \frac{\omega_\xi(t)}{t}\, dt < \infty. \tag{2-1}$$

Let γ be a C^1 Jordan curve γ with the length $l = |\gamma|$ and assume that $g : [0, l] \to \gamma$ is its arc-length parametrization . We say that γ is Dini-smooth if g' is Dini-continuous on $[0, l]$. If $\omega(t)$ is the modulus of continuity of g' for $0 \le t \le l$, then we extend ω by $\omega(t) = \omega(l)$ for $t \ge l$.

A function $F : \mathbb{T} \to \gamma$ is called Dini-smooth if the function $\Phi(t) = F(e^{it})$ is Dini-smooth, i.e.,

$$|\Phi'(t) - \Phi'(s)| \le \omega(|t - s|),$$

where ω is Dini-continuous. Observe that every smooth $C^{1,\alpha}$ Jordan curve is Dini-smooth.

Let

$$\mathcal{K}(s, t) = \operatorname{Re}\,[\overline{(g(t) - g(s))} \cdot i g'(s)] \tag{2-2}$$

be a function defined on $[0, l] \times [0, l]$. By $\mathcal{K}(s \pm l, t \pm l) = \mathcal{K}(s, t)$ we extend it to $\mathbb{R} \times \mathbb{R}$. Suppose now that $\Psi : \mathbb{R} \mapsto \gamma$ is an arbitrary 2π-periodic Lipschitz function such that $\Psi|_{[0,2\pi)} : [0, 2\pi) \mapsto \gamma$ is an orientation-preserving bijective function. Then there exists an increasing continuous function $\psi : [0, 2\pi] \mapsto [0, l]$ such that

$$\Psi(\tau) = g(\psi(\tau)). \tag{2-3}$$

We have for a.e. $e^{i\tau} \in \mathbb{T}$ that

$$\Psi'(\tau) = g'(\psi(\tau)) \cdot \psi'(\tau),$$

and therefore

$$|\Psi'(\tau)| = |g'(\psi(\tau))| \cdot |\psi'(\tau)| = \psi'(\tau).$$

Along with the function $\mathcal{K}$ we will also consider the function $\mathcal{K}_F$ defined by

$$\mathcal{K}_F(t, \tau) = \operatorname{Re}\,[\overline{(\Psi(t) - \Psi(\tau))} \cdot i\Psi'(\tau)].$$

Here $F(e^{it}) = \Psi(t)$. It is easy to see that

$$\mathcal{K}_F(t, \tau) = \psi'(\tau)\mathcal{K}(\psi(t), \psi(\tau)). \tag{2-4}$$

Lemma 2.2. *Let γ be a Dini-smooth Jordan curve and let $g : [0, l] \mapsto \gamma$ be a natural parametrization of a Jordan curve with g' having modulus of continuity ω. Assume further that $\Psi : [0, 2\pi] \mapsto \gamma$ is an arbitrary parametrization of γ and let $F(e^{it}) = \Psi(t)$. Then*

$$|\mathcal{K}(s, t)| \le \int_0^{\min\{|s-t|, l-|s-t|\}} \omega(\tau)\, d\tau \tag{2-5}$$

and

$$|\mathcal{K}_F(\varphi, x)| \le |\psi'(\varphi)| \int_0^{d_\gamma(\Psi(\varphi), \Psi(x))} \omega(\tau)\, d\tau. \tag{2-6}$$

Here $d_\gamma(\Psi(\varphi), \Psi(x)) := \min\{|s(\varphi) - s(x)|, (l - |s(\varphi) - s(x)|)\}$ is the (shortest) *distance between* $\Psi(\varphi)$ *and* $\Psi(x)$ *along* γ, *and it satisfies*

$$|\Psi(\varphi)-\Psi(x)|\le d_\gamma(\Psi(\varphi),\Psi(x))\le B_\gamma|\Psi(\varphi)-\Psi(x)|.$$

Proof. Note that the estimate (2-5) has been proved in [Kalaj 2011b, Lemma 2.3]. Now (2-6) follows from (2-5) and (2-4). □

A closed rectifiable Jordan curve γ satisfies a B-chord-arc condition for some constant $B>1$ if for all $z_1,z_2\in\gamma$ we have

$$d_\gamma(z_1,z_2)\le B|z_1-z_2|. \tag{2-7}$$

Here $d_\gamma(z_1,z_2)$ is the length of the shorter arc of γ with endpoints z_1 and z_2. It is clear that if $\gamma\in C^1$, then γ satisfies a chord-arc condition for some $B_\gamma>1$. The following lemma is proved in [Kalaj 2012].

Lemma 2.3. *Assume that* γ *satisfies a chord-arc condition for some* $B>1$. *Then for every normalized K-qc mapping* f *between the unit disk* $\mathbb{U}$ *and the Jordan domain* $\Omega=\operatorname{int}\gamma$ *we have*

$$|f(z_1)-f(z_2)|\le\Lambda_\gamma(K)|z_1-z_2|^\alpha,\quad z_1,z_2\in\mathbb{T},$$

where

$$\alpha=\frac{2}{K(1+2B)^2},\quad \Lambda_\gamma(K)=4\cdot 2^\alpha(1+2B)\sqrt{\frac{2\pi K|\Omega|}{\log 2}}.$$

Next we recall some estimates for the Jacobian of a harmonic univalent function.

Lemma 2.4 [Kalaj 2011b, Lemma 3.1]. *Suppose* $f=\mathcal{P}[F]$ *is a harmonic mapping such that* F *is a Lipschitz homeomorphism from the unit circle onto a Dini-smooth Jordan curve* γ. *Let* g *be an arc-length parametrization of* γ, *let* $\psi(t)=g^{-1}(F(e^{it}))$, *and define* $\Psi(t)=F(e^{it})=g(\psi(t))$. *Then for almost every* $\tau\in[0,2\pi]$, *the limit*

$$J_f(e^{i\tau}):=\lim_{r\to 1}J_f(re^{i\tau})$$

exists and we have

$$J_f(e^{i\tau})=\psi'(\tau)\int_0^{2\pi}\frac{\operatorname{Re}\big[\overline{(g(\psi(t))-g(\psi(\tau)))}\cdot ig'(\psi(\tau))\big]}{2\sin^2((t-\tau)/2)}\frac{dt}{2\pi}. \tag{2-8}$$

From Lemma 2.2 and Lemma 2.4 we obtain

Lemma 2.5. *Under the conditions and notation of Lemma 2.4 we have*

$$J_f(e^{i\varphi})\le\frac{\pi}{4}|\Psi'(\varphi)|\int_{-\pi}^{\pi}\frac{1}{x^2}\int_0^{d_\gamma(F(e^{i(\varphi+x)}),F(e^{i\varphi}))}\omega(\tau)\,d\tau\,dx \tag{2-9}$$

for a.e. $e^{i\varphi}\in\mathbb{T}$. *Here* ω *is the modulus of continuity of* g'.

Lemma 2.6. *Let $f = \mathcal{P}[F](z)$ be a harmonic mapping between the unit disk $\mathbb{U}$ and the Jordan domain Ω, with $F \in C^{1,\varpi}(\mathbb{T})$. Then the partial derivatives of f have a continuous extension to the boundary of the unit disk.*

Proof. In the proof of this lemma we denote $\partial_t \Psi(e^{it})$ by $\Psi'(t)$. If F is Lipschitz-continuous, then $\Phi = \Psi' \in L^\infty(\mathbb{T})$, and by the famous Marcel Riesz theorem (see for example [Garnett 1981, Theorem 2.3]) there is a constant A_p such that

$$\|H(\Psi')\|_{L^p(\mathbb{T})} \le A_p \|\Psi'\|_{L^p(\mathbb{T})}$$

for $1 < p < \infty$. It follows that $\tilde{\Phi} = H(\Psi') \in L^1$. Since rf_r is the harmonic conjugate of f_τ, we have $rw_r = \mathcal{P}[H(\Psi')]$ according to (1-5). By again using Fatou's theorem, we have

$$\lim_{r\to 1^-} f_r(re^{i\tau}) = H(\Psi')(\tau) \quad \text{a.e.} \tag{2-10}$$

By (1-4), and by following the proof of Privaloff's theorem [Zygmund 1959], we obtain that if $|\Psi'(x) - \Psi'(y)| \le \omega(|x-y|)$ for the Dini-continuous function, then

$$|H(\Psi')(x+h) - H(\Psi')(x)| \le A \int_0^{2h} \frac{\omega(t)}{t}\,dt + Bh \int_h^{2\pi} \frac{\omega(t)}{t^2}\,dt + C\omega(h),$$

for some absolute constants A, B and C. The detailed proof of the last fact can be found in [Garnett 1981, Theorem III 1.3.]. This implies that $rw_r(re^{it})$ and $f_t(re^{it})$ have continuous extensions to the boundary and this is what we needed to prove. □

We now prove the following lemma needed in the sequel.

Lemma 2.7. *Let A be a positive integrable function in $[0, B]$ and assume that $q, Q > 0$. Then there exists a continuous increasing function χ of $(0, +\infty)$ into itself, depending on A, B, q and Q, such that the following hold*: $\lim_{x\to\infty} \chi(x) = \infty$, *the function $g(x) = x\chi(x)$ is convex, and*

$$\int_0^B A(x)\chi(Qx^{-q})\,dx \le 4 \int_0^B A(x)dx.$$

Proof. First define inductively a sequence $x_0 = B$, $x_k > 0$ for $k > 0$, such that $x_{k+1} < x_k/2$, and

$$\int_0^{x_k} A(x)\,dx \le M2^{-k} \quad \text{where } M = \int_0^B A(x)dx.$$

This is possible because A is integrable.

Then define a continuous function ξ in $[0, B]$ by $\xi(x_k) = k$, and by extending it linearly on each interval $[x_{k+1}, x_k]$, that is

$$\xi(x) = k + \frac{x_k - x}{x_k - x_{k+1}}, \quad x \in [x_{k+1}, x_k].$$

It is easy to see that this function is convex, decreasing and tends to $+\infty$ as $x \to \infty$. Moreover

$$\int_0^B A(x)\xi(x)\,dx \le M \sum_{k=0}^{\infty}(k+1)2^{-k} = 4M.$$

Now set $\chi(x) = \xi((Q/x)^\tau)$ for $\tau = 1/q$. It remains to verify that $x\chi(x)$ is convex. This we do by differentiation:

$$(x\chi(x))' = \xi(Q^\tau x^{-\tau}) - Q^\tau \tau x^{-\tau}\xi'(Q^\tau x^{-\tau}).$$

Since both summands are increasing, $x\chi(x)$ is convex. □

3. The proof of Theorem 1.1

By assumption of the theorem, the derivative of an arc-length parametrization g' has a Dini-continuous modulus of continuity ω. We consider two cases.

(i) $F(e^{it}) = \Psi(t) \in C^{1,\varpi}(\mathbb{T})$. Then by Lemma 2.6 the mapping $f(z) = \mathcal{P}[F](z)$ is C^1 up to the boundary. First we notice that for $L = \sup|\Psi'(t)|$, it is clear that $L < \infty$. We will prove more. We will show that L is bounded by a constant not depending a priori on F. According to Lemma 2.6 and to (1-1), we have

$$\begin{aligned} |Df(e^{i\varphi})|^2 &= (|f_z(e^{i\varphi})| + |f_{\bar z}(e^{i\varphi})|)^2 \\ &= \lim_{z\to e^{i\varphi}} (|f_z(z)| + |f_{\bar z}(z)|)^2 \\ &\le K \lim_{z\to e^{i\varphi}} (|f_z(z)|^2 - |f_{\bar z}(z)|^2) \\ &= K(|f_z(e^{i\varphi})|^2 - |f_{\bar z}(e^{i\varphi})|^2) = K J_f(e^{i\varphi}). \end{aligned} \tag{3-1}$$

Furthermore, we have

$$|Df(re^{i\varphi})| = \sup_{|\xi|=1} |Df(re^{i\varphi})\xi| \ge |Df(re^{i\varphi})(ie^{i\varphi})| = |\partial_\varphi f(re^{i\varphi})|. \tag{3-2}$$

This implies that

$$|Df(e^{i\varphi})|^2 \ge |\partial_\varphi f(e^{i\varphi})|^2 = |\Psi'(\varphi)|^2. \tag{3-3}$$

From (2-9), (3-3) and (3-1), we obtain:

$$|\Psi'(\varphi)|^2 \le K C_1 |\Psi'(\varphi)| \int_{-\pi}^{\pi} \frac{1}{x^2} \int_0^{\rho(x,\varphi)} \omega(\tau)\,d\tau\,dx,$$

where

$$\rho(x,\varphi) = d_\gamma\big(F(e^{i(\varphi+x)}), F(e^{i\varphi})\big),$$

which is the same as

$$|\Psi'(\varphi)| \le KC_1 \int_{-\pi}^{\pi} \frac{\rho(\varphi, x)}{x^2} \int_0^1 \omega(\tau\rho(\varphi, x))\, d\tau\, dx.$$

Thus

$$|\Psi'(\varphi)| \le KC_1 \int_{-\pi}^{\pi} \frac{\rho(\varphi, x)}{x^2} \omega(\rho(\varphi, x))\, dx.$$

Let

$$L := \max_{x\in[0,2\pi]} |\Psi'(x)| = \max_{x\in[0,2\pi]} \psi'(x) = \psi'(\varphi). \tag{3-4}$$

Then

$$L \le KC_1 \int_{-\pi}^{\pi} \frac{\rho(\varphi, x)}{x^2} \omega(\rho(\varphi, x))\, dx.$$

Furthermore, we have

$$M := \frac{L}{2\pi KC_1} \le \int_{-\pi}^{\pi} M(x, \varphi) \frac{dx}{2\pi},$$

where

$$M(x, \varphi) = \frac{\rho(\varphi, x)}{x^2} \omega(\rho(\varphi, x)).$$

The idea is to make use of Lemma 2.7 with a convex function depending only on K to be found below.

Assume that $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function to be determined in the sequel such that the function $\Phi(t) = t\chi(t)$ is convex. By using Jensen's inequality to the previous integral with respect to the convex function Φ, we obtain

$$\Phi(M) \le \int_{-\pi}^{\pi} \Phi(M(x, \varphi))\, \frac{dx}{2\pi},$$

or equivalently,

$$M\chi(M) \le \int_{-\pi}^{\pi} M(x, \varphi)\chi(M(x, \varphi))\, \frac{dx}{2\pi}. \tag{3-5}$$

From (2-7) and (3-4) we deduce that

$$\rho(\varphi, x) \le B_\gamma L|x|. \tag{3-6}$$

On the other hand, since f is a normalized qc mapping, we have by Lemma 2.3 that

$$\rho(\varphi, x) \le B_\gamma \Lambda_\gamma(K)|x|^\alpha. \tag{3-7}$$

Notice that this time we used the boundary normalization. This implies that

$$M(x,\varphi)=\frac{\rho(\varphi,x)}{x^2}\omega(\rho(\varphi,x))\le\frac{B_\gamma L}{x}\omega(B_\gamma\Lambda_\gamma(K)|x|^\alpha), \tag{3-8}$$

and

$$M(x,\varphi)=\frac{\rho(\varphi,x)}{x^2}\omega(\rho(\varphi,x))\le\frac{B_\gamma\Lambda_\gamma(K)}{x^{2-\alpha}}\omega(B_\gamma\Lambda_\gamma(K)|x|^\alpha). \tag{3-9}$$

So, in view of Definition 2.1 we have

$$M(x,\varphi)\le\frac{B_\gamma\Lambda_\gamma(K)}{x^{2-\alpha}}\omega(|\gamma|). \tag{3-10}$$

From (3-5) and (3-8), we obtain

$$\begin{aligned}\chi\left(\frac{L}{2\pi KC_1}\right)&\le\int_{-\pi}^{\pi}\frac{KC_1B_\gamma}{x}\omega(B_\gamma\Lambda_\gamma(K)|x|^\alpha)\chi\left(\frac{B_\gamma\Lambda_\gamma(K)\omega(|\gamma|)}{|x|^{2-\alpha}}\right)dx\\&=2\int_0^{\pi}\frac{KC_1B_\gamma}{x}\omega(B_\gamma\Lambda_\gamma(K)|x|^\alpha)\chi\left(\frac{B_\gamma\Lambda_\gamma(K)\omega(|\gamma|)}{|x|^{2-\alpha}}\right)dx\\&=\frac{2KC_1B_\gamma}{B_\gamma\Lambda_\gamma(K)\alpha}\int_0^B\frac{\omega(y)}{y}\chi(Qy^{1-2/\alpha})\,dy,\end{aligned} \tag{3-11}$$

where

$$B=B_\gamma\Lambda_\gamma(K)\pi^\alpha\quad\text{and}\quad Q=\omega(|\gamma|)(B_\gamma\Lambda_\gamma(K))^{2-2/\alpha}.$$

In view of the last term of (3-11), now is the time to determine the function χ. Lemma 2.7 with $q=2/\alpha-1$ and $A(y)=\omega(y)/y$, provides us with a function χ such that Φ is convex and such that the estimate

$$\int_0^B\frac{\omega(y)}{y}\chi(Qy^{1-2/\alpha})dy\le 4\int_0^B\frac{\omega(y)}{y}dy$$

holds. From (3-11), we have

$$\chi\left(\frac{L}{2\pi KC_1}\right)\le\frac{8KC_1B_\gamma}{B_\gamma\Lambda_\gamma(K)\alpha}\int_0^B\frac{\omega(y)}{y}dy=:\Upsilon(K,\Omega).$$

Since χ is increasing, we infer finally that

$$L\le 2\pi KC_1\cdot\chi^{-1}(\Upsilon(K,\Omega))=\frac{\pi^2}{2}K\cdot\chi^{-1}(\Upsilon(K,\Omega)). \tag{3-12}$$

By the maximum principle, for $z=re^{i\varphi}$, we further have

$$|\partial_\varphi f(z)|\le L.$$

Since f is K-quasiconformal, we have

$$|Dw(z)|\le K|\partial_\varphi f(z)|.$$

This and the mean value inequality imply that

$$|f(z)-f(z')| \le KL|z-z'|, \quad |z|<1, |z'|<1. \tag{3-13}$$

(ii) $F \notin C^{1,\varpi}(\mathbb{T})$. In order to deal with nonsmooth F, we make use of an approximation argument. We begin by this definition.

Definition 3.1. Let G be a domain in $\mathbb{C}$ and let $a \in \partial G$. We will say that $G_a \subset G$ is a neighborhood of a if there exists a disk $D(a,r) := \{z : |z-a| < r\}$ such that $(D(a,r)\cap G) \subset G_a$.

Let $t = e^{ix} \in \mathbb{T}$. Then $F(t) = \Psi(x) \in \partial\Omega$. Let g be an arc-length parametrization of $\partial\Omega$ with $g(\psi(x)) = F(e^{ix})$, where $\psi : [0, 2\pi] \to [0, |\gamma|]$ is as in the first part of the proof. Put $s = \psi(x)$. Since the modulus of continuity of g' is a Dini-continuous function ω, there exists a neighborhood Ω_t of $\Psi(t)$ such that the derivative of its arc-length parametrization g_t' has modulus of continuity $C_t \cdot \omega$. Moreover, there exist positive numbers r_t and R_t such that

$$\Omega_t^\tau := \Omega_t + ig'(s)\cdot\tau \subset \Omega, \quad \tau \in (0, R_t), \tag{3-14}$$

$$\partial\Omega_t^\tau \subset \Omega, \quad \tau \in (0, R_t), \tag{3-15}$$

$$g[s-r_t, s+r_t] \subset \partial\Omega_t. \tag{3-16}$$

An example of a family Ω_t^τ such that $\partial\Omega_t^\tau \in C^{1,\alpha}$ for $0<\alpha<1$ with property (3-14) has been given in [Kalaj 2008]. The same construction yields the family $\partial\Omega_t^\tau$ with the above mentioned properties.

Take $U_\tau = f^{-1}(\Omega_t^\tau)$. Let η_t^τ be a conformal mapping of the unit disk onto U_τ with normalized boundary condition: $\eta_t^\tau(e^{i2k\pi/3}) = f^{-1}(\zeta_k)$ for $k = 0, 1, 2$, where $\zeta_0, \zeta_1, \zeta_2$ are three points of $\partial\Omega_t^\tau$ of equal distance. Then the mapping

$$f_t^\tau(z) := f(\eta_t^\tau(z)) - ig'(s)\cdot\tau$$

is a harmonic K-quasiconformal mapping of the unit disk onto Ω_t satisfying the boundary normalization. Moreover,

$$f_t^\tau = \mathcal{P}[F_t^\tau] \in C^1(\overline{\mathbb{U}})$$

for some function $F_t^\tau \in C^1(\mathbb{T})$.

Since $[0, l]$ is compact, there exists a finite family of Jordan arcs

$$\gamma_j = g(s_j - r_{s_j}/2, s_j + r_{s_j}/2), \quad j = 1, \dots, n,$$

covering γ. Assume that $F(t_j) = s_j$. Let

$$F_{j,\tau} := F_{t_j}^\tau, \quad a_{j,\tau} := \eta_{t_j}^\tau \quad \text{and} \quad f_{j,\tau} := f_{t_j}^\tau.$$

Using the case $F \in C^{1,\varpi}$, it follows that there exists a constant $C'_j = C'(K, \gamma_j)$ such that

$$|\partial_\varphi F'_{j,\tau}(e^{i\varphi})| \le C'_j$$

and

$$|f_{j,\tau}(z_1) - f_{j,\tau}(z_2)| \le KC'_j |z_1 - z_2|. \tag{3-17}$$

Since $a_{j,\tau}(z)$ converges uniformly on compact subsets of $\mathbb{U}$ to the function $a_{j,0}(z)$ when $\tau \to 0$, and since $f_{j,\tau} = f \circ a_{j,\tau}$, inequality (3-17) implies

$$|f_j(z_1) - f_j(z_2)| \le KC'_j |z_1 - z_2| \quad \text{for } z_1, z_2 \in \overline{\mathbb{U}}, \tag{3-18}$$

where $f_j = f \circ a_{j,0} = \mathcal{P}[F_j]$. For $z_1 = e^{it}$ and $z_2 = e^{i\varphi}$ for $t \to \varphi$, we obtain that $|\partial_\varphi F_j(e^{i\varphi})| \le KC'_j$ a.e. Since the mapping $b_j = a_{0,j}^{-1}$ can be extended conformally across the arc $S_j = f^{-1}(\lambda_j)$, where $\lambda_j = g(s_j - t_{s_j}, s_j + t_{s_j})$, there exists a constant L_j such that $|b_j(z)| \le L_j$ on $S'_j = \mathbb{T} \cap f^{-1}(\gamma_j)$ for $j = 1, \dots, n$. Hence $|\partial_\varphi F(e^{i\varphi})| \le KC'_j \cdot L_j$ on S'_j. Let $C' = \max\{KC'_j \cdot L_j : j = 1, \dots, n\}$. Inequalities (1-7) and (1-8) easily follow from $\mathbb{T} = \bigcup_{j=1}^n S'_j$.

Notice that we can now repeat the first part of the proof for a Lipschitz $f = \mathcal{P}[F]$ in order to obtain a more concrete Lipschitz constant, i.e., the constant L satisfying (3-12). The proof is complete.

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DAVID KALAJ
FACULTY OF NATURAL SCIENCES AND MATHEMATICS
UNIVERSITY OF MONTENEGRO
DZORDZA VASINGTONA B.B.
81000 PODGORICA
MONTENEGRO

davidk@ac.me

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