FREE EVOLUTION ON ALGEBRAS WITH TWO STATES, II

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Denote by $\mathcal{J}$ the operator of coefficient stripping. We show that for any free convolution semigroup $\{\mu_t : t \geq 0\}$ with finite variance, applying a single stripping produces semicircular evolution with nonzero initial condition, $\mathcal{J}[\mu_t] = \rho \boxplus \sigma_{\beta,\gamma}^{\boxplus t}$, where $\sigma_{\beta,\gamma}$ is the semicircular distribution with mean β and variance γ. For more general freely infinitely divisible distributions τ, expressions of the form $\tilde{\rho} \boxplus \tau^{\boxplus t}$ arise from stripping $\tilde{\mu}_t$, where $\{(\tilde{\mu}_t, \mu_t) : t \geq 0\}$ forms a semigroup under the operation of two-state free convolution. The converse to this statement holds in the algebraic setting. Numerous examples illustrating these constructions are computed. Additional results include the formula for generators of such semigroups.

1. Introduction

A probability measure μ on $\mathbb{R}$ with finite moments can be described by two sequences of Jacobi parameters

$$J(\mu) = \begin{pmatrix} \beta_0, & \beta_1, & \beta_2, & \beta_3, & \dots \\ \gamma_0, & \gamma_1, & \gamma_2, & \gamma_3, & \dots \end{pmatrix}.$$

For example, its Cauchy transform

$$G_\mu(z) = \int_\mathbb{R} \frac{1}{z-x} \, d\mu(x)$$

(which determines the measure) has the continued fraction expansion

$$G_\mu(z) = \cfrac{1}{z - \beta_0 - \cfrac{\gamma_0}{z - \beta_1 - \cfrac{\gamma_1}{z - \beta_2 - \cfrac{\gamma_2}{z - \beta_3 - \cfrac{\gamma_3}{z - \cdots}}}}}$$

Define new measures $\Phi[\mu]$ and $\mathcal{J}[\mu]$ by the right and left shifts on Jacobi parameters

$$J(\Phi[\mu]) = \begin{pmatrix} 0, & \beta_0, & \beta_1, & \beta_2, & \ldots \\ 1, & \gamma_0, & \gamma_1, & \gamma_2, & \ldots \end{pmatrix}$$

and

$$J(\mathcal{J}[\mu]) = \begin{pmatrix} \beta_1, & \beta_2, & \beta_3, & \beta_4, & \ldots \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4, & \ldots \end{pmatrix}.$$

$\mathcal{J}$ is sometimes called coefficient stripping. Actually, both Φ and $\mathcal{J}$ can be defined more generally: Φ for any probability measure, and $\mathcal{J}$ for any probability measure with finite variance. See Definition 2.

Denote by

$$d\sigma_{\beta,\gamma}(x) = \frac{1}{2\pi\gamma}\sqrt{4\gamma - (x-\beta)^2}\, dx$$

the semicircular distribution with mean β and variance γ, $\sigma = \sigma_{0,1}$ the standard semicircular distribution, and $\boxplus$ the operation of free convolution. The semicircular family $\{\sigma_{\beta t, \gamma t} = \sigma_{\beta,\gamma}^{\boxplus t} : t \geq 0\}$ forms a free convolution semigroup. General free convolution semigroups

$$\{\mu_t : t \geq 0\}$$

with mean 0 and variance t are indexed by probability measures ρ. In Proposition 9 of [Anshelevich 2013], we showed that for any such free convolution semigroup,

$$\mathcal{J}[\mu_t] = \rho \boxplus \sigma^{\boxplus t},$$

so that the "once-stripped" free convolution semigroup is always a "free heat evolution" started at ρ. Needless to say, this statement has no analog for semigroups with respect to usual convolution. In the first result of the paper, we extend this formula to the case of general finite variance: for a free convolution semigroup $\{\mu_t\}$ with mean βt and nonzero variance γt,

$$\mathcal{J}[\mu_t] = \rho \boxplus \sigma_{\beta,\gamma}^{\boxplus t}. \tag{1}$$

Since any free convolution semigroup, when stripped, always gives a semicircular evolution, it is natural to ask for which families of measures $\{\tilde{\mu}_t : t \geq 0\}$ is

$$\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \boxplus \tau^{\boxplus t} \tag{2}$$

for other measures τ. The main result of the article is that if this is the case, there exists a free convolution semigroup $\{\mu_t : t \geq 0\}$ such that the family of pairs of measures $\{(\tilde{\mu}_t, \mu_t) : t \geq 0\}$ forms a semigroup under the operation $\boxplus_c$ of *two-state free convolution*. Note that formula (2) can sometimes be assigned a meaning even if τ is not freely infinitely divisible. For example, if $\tilde{\rho} = \nu \boxplus \tau$ for some ν, then for

general probability measures τ, ν, there exists a family of measures forming the first component of the two-state free convolution semigroup such that

$$\mathcal{J}[\tilde{\mu}_t] = \nu \boxplus \tau^{\boxplus(1+t)}$$

(recall that in free probability, $\tau^{\boxplus(1+t)}$ is well-defined for any τ as long as $t \geq 0$). The most general case covered by the main theorem of the article (Theorem 11) is that for some semigroups,

$$\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \boxplus \omega^{\boxplus(t/p)},$$

where $\tau = \omega^{\boxplus(1/p)}$ need not even be a positive measure, but where the *subordination distribution* $\omega \boxplus \tilde{\rho}$ is freely infinitely divisible. It is unclear at this point whether every two-state free convolution semigroup (with finite variance) is of this form. Nevertheless, a large group of examples fit into this framework: free convolution semigroups, Boolean convolution semigroups, two-state free Brownian motions, and two-state free Meixner distributions. Moreover, in the last section of the paper we show that in the algebraic setting, when $(\tilde{\mu}_t, \mu_t)$ are linear functionals on polynomials but do not necessarily come from positive measures, formula (2) always holds for some (not necessarily positive) τ. In that section we also prove a basic formula for the moment-generating function of the multivariate subordination distribution (see below), which really belongs on the long list of properties of that distribution proven in [Nica 2009].

The other aspects of two-state free convolution semigroups are investigated at the end of Section 3. We compute the two-state version of Voiculescu's evolution equation for the Cauchy transform. Then we combine it with the preceding results to find the formula for the generators of two-state free convolution semigroups with finite variance.

Finally, we would like to explain the connection between this article and part I of the same title [Anshelevich 2010]. Belinschi and Nica [2008; 2009] proved that the eponymous family of transformations $\{\mathbb{B}_t : t \geq 0\}$, is related to the free heat evolution via

$$(3) \qquad \qquad \mathbb{B}_t[\Phi[\rho]] = \Phi[\rho \boxplus \sigma^{\boxplus t}].$$

Equation (1) follows from this observation after only a small amount of work. In part I, we constructed a two-variable map $\Phi[\cdot, \cdot]$ and proved that

$$(4) \qquad \qquad \mathbb{B}_t[\Phi[\tau, \tilde{\rho}]] = \Phi[\tau, \tilde{\rho} \boxplus \tau^{\boxplus t}].$$

Moreover, the transformation $\Phi[\cdot, \cdot]$, as defined in [Anshelevich 2010], also comes from two-state free probability theory. Nica [2009] observed that $\Phi[\tau, \tilde{\rho}]$ is closely related to the subordination distribution $\tau \boxplus \tilde{\rho}$, which is a more important object in free probability, and so will be used in computations in this paper.

At this point the evolution formula (4) is only proven for measures with finite moments, while we are interested in a more general class of measures with finite variance. Moreover, the derivation of (1) from (3) does not generalize to a derivation of (2) from (4); the proof of (2) is quite different. Nevertheless, both this article and part I involve two-state free probability theory and generalization of semicircular evolution to more general free convolution semigroups.

2. Background

Notation 1. Denote by $m[\mu]$ and $\text{Var}[\mu]$ the mean and variance of μ,

$$\mathcal{P} = \{\text{probability measures on } \mathbb{R}\},$$
$$\mathcal{P}_2 = \{\mu \in \mathcal{P} : \text{Var}[\mu] < \infty\},$$
$$\mathcal{P}_{0,1} = \{\mu \in \mathcal{P}_2 : m[\mu] = 0, \text{Var}[\mu] = 1\},$$
$$\mathcal{ID}^{\boxplus} = \{\mu \in \mathcal{P} : \mu \text{ is } \boxplus\text{-infinitely divisible}\}.$$

For a probability measure μ on $\mathbb{R}$, its Cauchy transform is

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x} \, d\mu(x),$$

and its F-transform is

$$F_\mu(z) = \frac{1}{G_\mu(z)}$$

(for a function f, f^{-1} will denote its compositional rather than a multiplicative inverse).

2.1. *Convolutions.* For $\mu \in \mathcal{P}$, define its Voiculescu transform ϕ_μ by

$$(\phi_\mu \circ F_\mu)(z) + F_\mu(z) = z.$$

See [Bercovici and Voiculescu 1993; Voiculescu et al. 1992]. The free convolution of two measures $\mu \boxplus \nu$ is determined by the equality

$$\phi_{\mu \boxplus \nu} = \phi_\mu + \phi_\nu$$

on a domain. A free convolution semigroup is a weakly continuous family $\{\mu_t : t \geq 0\} \subset \mathcal{P}$ satisfying

$$\mu_t \boxplus \mu_s = \mu_{t+s}.$$

In this case, we denote $\mu_t = \mu^{\boxplus t}$. A measure μ is $\boxplus$-infinitely divisible if $\mu = \mu_1$ for some free convolution semigroup. A fundamental result in [Nica and Speicher 1996], extended to measures with unbounded support in [Belinschi and Bercovici 2004], is that for any $\mu \in \mathcal{P}$, $\mu^{\boxplus t}$ is defined for $t \geq 1$.

We will refer to the set

$$\{(\beta, \gamma, \rho) : \beta \in \mathbb{R}, \gamma > 0, \rho \in \mathcal{P}\} \cup \{(\beta, 0, \cdot) : \beta \in \mathbb{R}\}$$

as the set of *canonical triples*. By a result of Maassen [1992], $\boxplus$-convolution semigroups with finite variance

$$\{\mu_t : t \geq 0, \mathrm{Var}[\mu_1] < \infty\}$$

are in bijection with canonical triples, with the bijection given by

(5) $$\phi_{\mu_t}(z) = \beta t + \gamma t G_\rho(z).$$

Here $\beta = m[\mu_1]$ and $\gamma = \mathrm{Var}[\mu_1]$. The $\boxplus$-convolution semigroups with zero variance are of the form $\mu_t = \delta_{\beta t}$, and so correspond to $(\beta, 0, \cdot)$ with $\gamma = 0$ and ρ undefined.

Similarly, for $\tilde{\mu}, \mu \in \mathcal{P}$, define the two-state Voiculescu transform $\phi_{\tilde{\mu},\mu}$ by

(6) $$(\phi_{\tilde{\mu},\mu} \circ F_\mu)(z) + F_{\tilde{\mu}}(z) = z.$$

See [Krystek 2007; Wang 2011]. The two-state free convolution of two pairs of measures

$$(\rho, \mu \boxplus \nu) = (\tilde{\mu}, \mu) \boxplus_c (\tilde{\nu}, \nu)$$

is determined by the equality

$$\phi_{\rho, \mu \boxplus \nu} = \phi_{\tilde{\mu},\mu} + \phi_{\tilde{\nu},\nu}$$

on a domain. A two-state free convolution semigroup is a componentwise weakly continuous family $\{(\tilde{\mu}_t, \mu_t) : t \geq 0\}$ satisfying

$$(\tilde{\mu}_t, \mu_t) \boxplus_c (\tilde{\mu}_s, \mu_s) = (\tilde{\mu}_{t+s}, \mu_{t+s}).$$

In this case, we denote $(\tilde{\mu}_t, \mu_t) = (\tilde{\mu}, \mu)^{\boxplus_c t}$. The pair $(\tilde{\mu}, \mu)$ is $\boxplus_c$-infinitely divisible if $(\tilde{\mu}, \mu) = (\tilde{\mu}_1, \mu_1)$ for some two-state free convolution semigroup.

For a fixed free convolution semigroup $\{\mu_t : t \geq 0\}$, the $\boxplus_c$-convolution semigroups $\{(\tilde{\mu}_t, \mu_t)\}$ such that $\tilde{\mu}_1$ has finite variance are in bijection with (relative) canonical triples $(\tilde{\beta}, \tilde{\gamma}, \tilde{\rho})$, with the bijection given by

(7) $$\phi_{\tilde{\mu}_t, \mu_t}(z) = \tilde{\beta} t + \tilde{\gamma} t G_{\tilde{\rho}}(z).$$

Here $\tilde{\beta} = m[\tilde{\mu}_1]$ and $\tilde{\gamma} = \mathrm{Var}[\tilde{\mu}_1]$. This does not appear to be stated explicitly, but follows from the description of general two-state freely infinitely divisible distributions in Theorem 4.1 of [Wang 2011]. Again, the case $\mathrm{Var}[\tilde{\mu}_1] = 0$ can be included by setting $\tilde{\gamma} = 0$ and leaving $\tilde{\rho}$ undefined.

The Boolean convolution $\mu \uplus \nu$ is defined by

$$(\mu, \delta_0) \boxplus_c (\nu, \delta_0) = (\mu \uplus \nu, \delta_0).$$

More explicitly, $\phi_{\mu,\delta_0}(z) = z - F_\mu(z)$, so

$$z - F_{\mu \uplus \nu}(z) = (z - F_\mu(z)) + (z - F_\nu(z)).$$

Any distribution is $\uplus$-infinitely divisible, so $\mu^{\uplus t}$ is always defined for any $t \geq 0$.

Finally, a few arguments in the article simplify with the use of the monotone convolution $\mu \rhd \nu$, defined by

$$F_{\mu \rhd \nu} = F_\mu \rhd F_\nu.$$

Definition 2. For measures with finite moments, the transformations Φ and $\mathcal{J}$ were defined in the introduction. Here are the more general definitions. Φ is the bijection

$$\Phi : \mathcal{P} \to \mathcal{P}_{0,1}$$

defined by

$$F_{\Phi[\nu]}(z) = z - G_\nu(z).$$

See [Belinschi and Nica 2008]. For $\mu \in \mathcal{P}_2$ with $m[\mu] = \beta$ and $\mathrm{Var}[\mu] = \gamma > 0$, define $\mathcal{J}[\mu]$ by

$$F_\mu(z) = z - \beta - \gamma G_{\mathcal{J}[\mu]}(z).$$

Then

$$\mathcal{J} : \mathcal{P}_2 \to \mathcal{P},$$

and $\mathcal{J} \circ \Phi$ is the identity map, while $\Phi \circ \mathcal{J}$ is the identity on $\mathcal{P}_{0,1}$.

Definition 3. Recall that all probability measures are infinitely divisible in the Boolean sense. The Boolean-to-free bijection of Bercovici and Pata [1999, Section 6]

$$\mathbb{B} : \mathcal{P} \to \mathcal{ID}^{\boxplus}$$

is defined by

$$\phi_{\mathbb{B}[\mu]}(z) = z - F_\mu(z).$$

More generally, define the Belinschi–Nica transformations [2008] $\{\mathbb{B}_t : t \geq 0\}$ on $\mathcal{P}$ by

$$\mathbb{B}_t[\mu] = \left(\mu^{\boxplus(1+t)}\right)^{\uplus(1/1+t)}.$$

These transformations form a semigroup under composition, and $\mathbb{B}_1 = \mathbb{B}$.

Remark 4. Note that

$$\phi_{\mathbb{B}[\Phi[\rho]]}(z) = z - F_{\Phi[\rho]}(z) = G_\rho(z).$$

So for a free convolution semigroup $\{\mu_t : t \geq 0\}$, equation (5) is equivalent to

$$(8) \qquad\qquad \mu_t = \delta_{\beta t} \boxplus \mathbb{B}[\Phi[\rho]]^{\boxplus \gamma t}.$$

Definition 5. For $\mu, \nu \in \mathcal{P}$, the subordination distribution [Lenczewski 2007; Nica 2009] $\mu \boxplus \nu$ is the unique probability measure such that

$$G_{\mu \boxplus \nu}(z) = G_\nu(F_{\mu \boxplus \nu}(z)).$$

Here $F_{\mu \boxplus \nu}$ is the corresponding subordination function of $\mu \boxplus \nu$ with respect to ν. If $\mu \boxplus \nu \in \mathcal{ID}^{\boxplus}$, we may define (cf. [Anshelevich 2010])

$$\Phi[\mu, \nu] = \mathbb{B}^{-1}[\mu \boxplus \nu].$$

Lemma 6. *On a common domain,*

$$\phi_{\mu \boxplus \nu}(z) = (\phi_\mu \circ F_\nu)(z).$$

Also, whenever $\Phi[\mu, \nu]$ *is defined,*

$$z - F_{\Phi[\mu, \nu]}(z) = (\phi_\mu \circ F_\nu)(z)$$

and

$$\phi_\mu = \phi_{\Phi[\mu, \nu], \nu}.$$

Proof. We compute

$$\phi_{\mu \boxplus \nu}(z) = F_{\mu \boxplus \nu}^{-1}(z) - z = (F_{\mu \boxplus \nu}^{-1} \circ F_\nu)(z) - z$$
$$= \big(\phi_{\mu \boxplus \nu}(F_\nu(z)) + F_\nu(z)\big) - \big(\phi_\nu(F_\nu(z)) + F_\nu(z)\big) = (\phi_\mu \circ F_\nu)(z).$$

The second property follows by combining this with the definition of $\mathbb{B}$. Finally,

$$(\phi_\mu \circ F_\nu)(z) + F_{\Phi[\mu, \nu]}(z) = z,$$

which implies the third property after comparison with (6). □

The following result is the analog of Corollary 4.13 in [Nica 2009] for single-variable, unbounded distributions.

Lemma 7. *If* $\mu \in \mathcal{ID}^{\boxplus}$, *or if* $\nu = \mu \boxplus \nu'$, *then* $\mu \boxplus \nu \in \mathcal{ID}^{\boxplus}$.

Proof. If $\mu \in \mathcal{ID}^{\boxplus}$, then for any $t \geq 0$,

$$\phi_{\mu \boxplus t \boxplus \nu}(z) = \phi_{\mu \boxplus t}(F_\nu(z)) = t\phi_\mu(F_\nu(z)) = \phi_{(\mu \boxplus \nu)^{\boxplus t}}(z),$$

and so $(\mu \boxplus \nu)^{\boxplus t} = \mu^{\boxplus t} \boxplus \nu$ is well-defined.

If $\nu = \mu \boxplus \nu'$, then

$$\phi_{\mu \boxplus \nu}(z) = \phi_{\mu \boxplus (\mu \boxplus \nu')}(z) = \phi_\mu(F_{\mu \boxplus \nu'}(z))$$
$$= \phi_{\mu \boxplus \nu'}(F_{\mu \boxplus \nu'}(z)) - \phi_{\nu'}(F_{\mu \boxplus \nu'}(z)) = z - F_{\mu \boxplus \nu'}(z) - \phi_{\nu'}(F_{\mu \boxplus \nu'}(z))$$
$$= z - F_{\nu'}^{-1}(F_{\mu \boxplus \nu'}(z)) = z - F_{\mu \boxplus \nu'}(z) = \phi_{\mathbb{B}[\mu \boxplus \nu']}(z),$$

and so $\mu \boxplus \nu = \mathbb{B}[\mu \boxplus \nu'] \in \mathcal{ID}^{\boxplus}$. □

Lemma 8. *For a canonical triple (β, γ, ρ) and $t \geq 0$,*

$$\mathbb{B}_t[\delta_\beta \uplus \Phi[\rho]^{\uplus \gamma}] = \delta_\beta \uplus \Phi[\rho \boxplus \delta_{\beta t} \boxplus \sigma^{\boxplus \gamma t}]^{\uplus \gamma}.$$

Proof. For $\gamma = 0$, the identity reduces to $\mathbb{B}_t[\delta_\beta] = \delta_\beta$. The argument for $\gamma > 0$ is a slight modification of Remark 4.4 (proof of Theorem 1.6) from [Belinschi and Nica 2008]. Following that paper, denote by θ the subordination function of $\rho \boxplus \delta_{\beta t} \boxplus \sigma^{\boxplus \gamma t}$ with respect to ρ, and by ω the subordination function of $(\delta_\beta \uplus \Phi[\rho]^{\uplus \gamma})^{\boxplus(t+1)}$ with respect to $(\delta_\beta \uplus \Phi[\rho]^{\uplus \gamma})$. On the one hand,

$$G_{\rho \boxplus \delta_{\beta t} \boxplus \sigma^{\boxplus \gamma t}}(z) = G_\rho(\theta(z))$$

and

$$z - F_{\delta_\beta \uplus \Phi[\rho]^{\uplus \gamma}}(z) = \beta + \gamma G_\rho(z).$$

Therefore

(9) $$\theta(z) - F_{\delta_\beta \uplus \Phi[\rho]^{\uplus \gamma}}(\theta(z)) = \beta + \gamma G_{\rho \boxplus \delta_{\beta t} \boxplus \sigma^{\boxplus \gamma t}}(z).$$

On the other hand, denoting by $\tilde{\theta}$ the subordination function of $\rho \boxplus \sigma^{\boxplus \gamma t}$ with respect to ρ, by equation (4.8) in [Belinschi and Nica 2008],

$$\tilde{\theta}(z) = z - \gamma t G_{\rho \boxplus \sigma^{\boxplus \gamma t}}(z).$$

But

$$G_{\rho \boxplus \delta_{\beta t} \boxplus \sigma^{\boxplus \gamma t}}(z) = G_{\rho \boxplus \sigma^{\boxplus \gamma t}}(z - \beta t) = G_\rho(\tilde{\theta}(z - \beta t)) = G_\rho(\theta(z)).$$

Thus

$$\theta(z) = \tilde{\theta}(z - \beta t) = z - \beta t - \gamma t G_{\rho \boxplus \sigma^{\boxplus \gamma t}}(z - \beta t) = z - \beta t - \gamma t G_{\rho \boxplus \delta_{\beta t} \boxplus \sigma^{\boxplus \gamma t}}(z).$$

Combining this with (9), we see that

$$t\theta(z) - t F_{\delta_\beta \uplus \Phi[\rho]^{\uplus \gamma}}(\theta(z)) = z - \theta(z)$$

and

$$\theta(z) = \frac{1}{t+1} z + \left(1 - \frac{1}{t+1}\right) F_{\delta_\beta \uplus \Phi[\rho]^{\uplus \gamma}}(\theta(z)).$$

Then (see [Belinschi and Nica 2008]) it follows that $\theta = \omega$, and so the argument concludes as in that paper:

$$\begin{aligned}
z - F_{\mathbb{B}_t[\delta_\beta \uplus \Phi[\rho]^{\uplus \gamma}]}(z) &= z - \left(\left(1 - \frac{1}{t}\right)z + \frac{1}{t}\omega(z)\right) \\
&= \frac{1}{t}(z - \omega(z)) = \frac{1}{t}(z - \theta(z)) \\
&= \beta + \gamma G_{\rho \boxplus \delta_{\beta t} \boxplus \sigma^{\gamma t}}(z) \\
&= z - F_{\delta_\beta \uplus \Phi[\rho \boxplus \delta_{\beta t} \boxplus \sigma^{\boxplus \gamma t}]^{\uplus \gamma}}(z). \qquad \square
\end{aligned}$$

3. Single-variable, complex-analytic results

Proposition 9. *For any a canonical triple* (β, γ, ρ), *the corresponding free convolution semigroup is*

$$\mu_t = \delta_{\beta t} \uplus \Phi[\rho \boxplus \sigma_{\beta,\gamma}^{\boxplus t}]^{\uplus \gamma t}.$$

In particular, for any free convolution semigroup with nonzero, finite variance,

$$\mathcal{J}[\mu_t] = \rho \boxplus \sigma_{\beta,\gamma}^{\boxplus t}.$$

Proof. A free convolution semigroup with finite variance $\{\mu_t\}$ can be rewritten as

$$
\begin{aligned}
\mu_t &= \delta_{\beta t} \boxplus \mathbb{B}[\Phi[\rho]]^{\boxplus \gamma t} \qquad \text{(by the Maassen representation (8))}\\
&= \mathbb{B}_{t-1}[\delta_\beta \boxplus \mathbb{B}[\Phi[\rho]]^\gamma]^{\uplus t} \qquad \text{(by definition of } \mathbb{B}_{t-1})\\
&= \mathbb{B}_t[\delta_\beta \uplus \Phi[\rho]^{\uplus \gamma}]^{\uplus t} \qquad \text{(by definition of } \mathbb{B} = \mathbb{B}_1)\\
&= \delta_{\beta t} \uplus \Phi[\rho \boxplus \delta_{\beta t} \boxplus \sigma^{\boxplus \gamma t}]^{\uplus \gamma t} \qquad \text{(by Lemma 8)}\\
&= \delta_{\beta t} \uplus \Phi[\rho \boxplus \sigma_{\beta,\gamma}^{\boxplus t}]^{\uplus \gamma t} \qquad \text{(by definition of } \sigma_{\beta,\gamma}).
\end{aligned}
$$

For $\gamma = 0$, we have $\mu_t = \delta_{\beta t} = \sigma_{\beta t,0}$, so the equation still holds. $\qquad\square$

Lemma 10. *A family* $\{(\tilde{\mu}_t, \mu_t) : t \geq 0\}$ *is the two-state free convolution semigroup with the relative canonical triple* $(\tilde{\beta}, \tilde{\gamma}, \tilde{\rho})$ *if and only if* $\{\mu_t : t \geq 0\}$ *forms a free convolution semigroup and*

$$\tilde{\mu}_t = \delta_{\tilde{\beta} t} \uplus \Phi[\tilde{\rho} \rhd \mu_t]^{\uplus \tilde{\gamma} t}.$$

In particular, whenever $\tilde{\gamma} > 0$, *such a family satisfies*

$$\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \rhd \mu_t.$$

Proof. Using the properties of Boolean and monotone convolutions and the definition of Φ,

$$z - F_{\delta_{\tilde{\beta} t} \uplus \Phi[\tilde{\rho} \rhd \mu_t]^{\uplus \tilde{\gamma} t}}(z) = \tilde{\beta} t + \tilde{\gamma} t G_{\tilde{\rho} \rhd \mu_t}(z) = \tilde{\beta} t + \tilde{\gamma} t G_{\tilde{\rho}}(F_{\mu_t}(z)).$$

On the other hand, by formulas (6) and (7), for the two-state free convolution semigroup with the relative canonical triple $(\tilde{\beta}, \tilde{\gamma}, \tilde{\rho})$,

$$z - F_{\tilde{\mu}_t}(z) = (\phi_{\tilde{\mu}_t, \mu_t} \circ F_{\mu_t})(z) = \tilde{\beta} t + \tilde{\gamma} t G_{\tilde{\rho}}(F_{\mu_t}(z)).$$

Comparing these, we obtain the result. $\qquad\square$

Theorem 11. *Fix* $\tilde{\beta} \in \mathbb{R}$, $\tilde{\gamma} > 0$, *and* $p > 0$. *Let* $\omega, \tilde{\rho} \in \mathcal{P}$ *be measures such that*

$$\omega \boxplus \tilde{\rho} \in \mathcal{ID}^{\boxplus}.$$

(a) *For any $t \geq 0$,*

$$\phi_{\tilde{\rho}} + (t/p)\phi_{\omega}$$

is a Voiculescu transform of a probability measure, and so

$$\tilde{\rho} \boxplus \omega^{\boxplus(t/p)}$$

is well-defined.

(b) *Define*

$$\mu = (\omega \boxplus \tilde{\rho})^{\boxplus(1/p)},$$

$\mu_t = \mu^{\boxplus t}$, *and*

$$\tilde{\mu}_t = \delta_{\tilde{\beta}t} \uplus \Phi[\tilde{\rho} \boxplus \omega^{\boxplus(t/p)}]^{\uplus\tilde{\gamma}t}.$$

Then $\tilde{\mu}_1$ has finite, nonzero variance, the family $\{(\tilde{\mu}_t, \mu_t) : t \geq 0\}$ forms a two-state free convolution semigroup, and

$$\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \boxplus \omega^{\boxplus(t/p)}.$$

Proof. For part (a), using Lemma 6,

$$
\begin{aligned}
\phi_{\tilde{\rho}}(z) + (t/p)\phi_{\omega}(z) &= \phi_{\tilde{\rho}}(z) + (t/p)\phi_{\omega \boxplus \tilde{\rho}}(F_{\tilde{\rho}}^{-1}(z)) \\
&= F_{\tilde{\rho}}^{-1}(z) - z + \phi_{(\omega \boxplus \tilde{\rho})^{\boxplus(t/p)}}(F_{\tilde{\rho}}^{-1}(z)) \\
&= F_{(\omega \boxplus \tilde{\rho})^{\boxplus(t/p)}}^{-1}(F_{\tilde{\rho}}^{-1}(z)) - z \\
&= F_{\tilde{\rho} \triangleright (\omega \boxplus \tilde{\rho})^{\boxplus(t/p)}}^{-1}(z) - z \\
&= \phi_{\tilde{\rho} \triangleright (\omega \boxplus \tilde{\rho})^{\boxplus(t/p)}}(z).
\end{aligned}
$$

(10)

Since the monotone convolution is known to preserve positivity, this implies part (a). Next, it is clear that in part (b), $\{\mu_t : t \geq 0\}$ forms a free convolution semigroup. From (10), it follows that

$$\tilde{\rho} \boxplus \omega^{\boxplus(t/p)} = \tilde{\rho} \triangleright \mu_t.$$

Part (b) now follows from Lemma 10.   □

See Proposition 33 for a partial converse to the theorem.

The following corollary is an immediate consequence of Lemma 7.

Corollary 12. *The assumptions of Theorem 11 are satisfied in the following cases:*

(a) $\tilde{\rho} \in \mathcal{P}$ *is arbitrary and $\omega = \tau \in \mathcal{ID}^{\boxplus}$. In this case one can, without loss of generality, take $p = 1$.*

(b) $\omega \in \mathcal{P}$ *is arbitrary, and $\tilde{\rho} = \nu \boxplus \omega$ for some $\nu \in \mathcal{P}$.*

In particular, for any $\tilde{\rho} \in \mathcal{P}$ and $\tau \in \mathcal{ID}^{\boxplus}$, there exists a two-state free convolution semigroup $\{(\tilde{\mu}_t, \mu_t) : t \geq 0\}$ such that

$$\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \boxplus \tau^{\boxplus t}.$$

I am grateful to Serban Belinschi for a discussion leading to the following example.

Example 13. Recall that the analytic R-transform is defined by $R_\mu(z) = \phi_\mu(1/z)$. Let

$$\tau_\varepsilon = \tfrac{1}{2}(\delta_{-\varepsilon} + \delta_\varepsilon).$$

Then

$$R_{\tau_\varepsilon}(z) = \frac{2\varepsilon^2 z}{\sqrt{1 + 4\varepsilon^2 z^2} + 1}$$

is analytic for $|z| < (2\varepsilon)^{-1}$ and grows as $|R_{\tau_\varepsilon}(z)| \approx \varepsilon^2 |z|$. It follows from Theorem 2 of [Bercovici and Voiculescu 1995] that for sufficiently small ε,

$$z + t R_{\tau_\varepsilon}(z)$$

is an R-transform of a positive measure for all $t \in [0, 1]$. On the other hand, $\tau_\varepsilon^{\boxplus t}$ is well-defined and positive for all $t \geq 1$. It follows that $\sigma \boxplus \tau_\varepsilon^{\boxplus t}$ is well-defined and positive for all $t \geq 0$. However, $\tau_\varepsilon \notin \mathcal{ID}^{\boxplus}$, and $\sigma \neq \nu \boxplus \tau_\varepsilon^{\boxplus p}$ for any $p > 0$, so this family is not covered by the preceding corollary. Nevertheless, $F_\sigma(\mathbb{C}^+) = \mathbb{C}^+ \setminus \{z : |z| \leq 2\}$, and

$$\phi_{\tau_\varepsilon}(z) = \frac{\sqrt{z^2 + 4\varepsilon^2} - z}{2}$$

is analytic on this image for $\varepsilon < 1$. It follows that

$$\phi_{\tau_\varepsilon \boxplus \sigma} = \phi_{\tau_\varepsilon} \circ F_\sigma$$

analytically extends to $\mathbb{C}^+$, and so $\tau_\varepsilon \boxplus \sigma \in \mathcal{ID}^{\boxplus}$. So this family is still covered by the preceding theorem.

Question. Can the hypothesis of Theorem 11 be weakened to the assumption in the following proposition? In other words, does this assumption imply that the (equivalent) statements in the following proposition necessarily hold?

Proposition 14. *Let $\tilde{\rho} \in \mathcal{P}$, $\tau \in \mathcal{P}$, and suppose that $\tilde{\rho} \boxplus \tau^{\boxplus t}$ is defined for all $t \geq 0$. The following are equivalent:*

(a) $\tau \boxplus \tilde{\rho} \in \mathcal{ID}^{\boxplus}$.

(b) $F_{\tilde{\rho} \boxplus \tau^{\boxplus t}}$ *is subordinate to $F_{\tilde{\rho}}$ for all $t \geq 0$, in the sense that there exist analytic transformations $\theta_t : \mathbb{C}^+ \to \mathbb{C}^+$ such that $F_{\tilde{\rho} \boxplus \tau^{\boxplus t}}(z) = F_{\tilde{\rho}}(\theta_t(z))$.*

(c) $\{\Phi[\tilde{\rho} \boxplus \tau^{\boxplus t}] : t \geq 0\}$ *is the first component of a two-state free convolution*
   semigroup.

Proof. Calculations in the proof of Theorem 11 show that if θ_t exists, then
$\theta_t = F_{(\tau \boxminus \tilde{\rho})^{\boxplus t}}$. This shows that (a) and (b) are equivalent. The same calculations
also imply that if $\{(\tilde{\mu}_t = \Phi[\tilde{\rho} \boxplus \tau^{\boxplus t}], \mu_t) : t \geq 0\}$ is a two-state free convolution
semigroup, then $\mu_t = (\tau \boxminus \tilde{\rho})^{\boxplus t}$. Thus (a) and (c) are equivalent. $\square$

Lemma 15. *Subordination distributions have the following properties*:

$$(\mu \boxplus \nu) \boxminus \rho = (\mu \boxminus \rho) \boxplus (\nu \boxminus \rho),$$
$$\sigma \boxminus \mu = \mathbb{B}[\Phi[\mu]],$$
$$\mu \boxminus \delta_0 = \mu,$$
$$\mu \boxminus \mu = \mathbb{B}[\mu],$$
$$\delta_a \boxminus \mu = \delta_a.$$

There is a corresponding list of properties for $\Phi[\cdot, \cdot]$.

Proof. All of these properties follow immediately from

$$\phi_{\mu \boxminus \nu}(z) = (\phi_\mu \circ F_\nu)(z). \qquad \square$$

   In a number of the following examples, free convolution semigroups $\{\mu_t : t \geq 0\}$
will have finite variance, and so will be associated with canonical triples (β, γ, ρ);
in all cases, the relative canonical triple is $(\tilde{\beta}, \tilde{\gamma}, \tilde{\rho})$.

Example 16. Let $\tilde{\rho} = \rho \in \mathcal{P}$. Then the first component of the corresponding
two-state free convolution semigroup satisfies

$$\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \boxplus \sigma_{\beta, \gamma}^{\boxplus t},$$

so that in Corollary 12, $\tau = \sigma_{\beta, \gamma} \in \mathcal{ID}^{\boxplus}$ is a semicircular distribution. Indeed,

$$\sigma_{\beta, \gamma} \boxminus \tilde{\rho} = (\delta_\beta \boxplus \sigma^{\boxplus \gamma}) \boxminus \tilde{\rho} = \delta_\beta \boxplus \mathbb{B}[\Phi[\tilde{\rho}]]^{\boxplus \gamma} = \mu.$$

In the particular case when $\tilde{\beta} = \beta$ and $\tilde{\gamma} = \gamma$, it follows that $\tilde{\mu}_t = \mu_t$ form a free
convolution semigroup, and we are in the Belinschi–Nica setting of Proposition 9.

Example 17. Let $\tilde{\rho} = \delta_0$ and $\rho \in \mathcal{P}$. Then the first component of the corresponding
two-state free convolution semigroup satisfies

$$\mathcal{J}[\tilde{\mu}_t] = \mu_t,$$

so that in Corollary 12, $\tau = \mu \in \mathcal{ID}^{\boxplus}$ is arbitrary. Indeed,

$$\mu \boxminus \delta_0 = \mu.$$

These are the (distributions of) two-state free Brownian motions (in [Anshelevich 2011], they were called algebraic two-state free Brownian motions).

Example 18. Let $\tilde{\rho} \in \mathcal{P}$ and $\gamma = 0$, so that $\mu_t = \delta_{\beta t}$. Then the first component of the corresponding two-state free convolution semigroup satisfies

$$\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \boxplus \delta_{\beta t},$$

so that in Corollary 12, $\tau = \delta_\beta \in \mathcal{ID}^{\boxplus}$. Indeed,

$$\delta_\beta \boxplus \tilde{\rho} = \delta_\beta = \mu.$$

For general β and measures with finite moments,

$$\tilde{\mu}_t = \delta_{\tilde{\beta} t} \uplus \Phi[\tilde{\rho} \boxplus \delta_{\beta t}]^{\uplus \tilde{\gamma} t}$$

are precisely the families constructed in [Anshelevich and Młotkowski 2012, Proposition 7]. For $\beta = 0$, this is a Boolean convolution semigroup, and an arbitrary Boolean convolution semigroup (with finite variance) arises in this way.

On the other hand, if $\tilde{\mu}_t = \delta_{\tilde{\beta} t}$, for any free convolution semigroup $\{\mu_t : t \geq 0\}$, the measures $(\delta_{\tilde{\beta} t}, \mu_t)$ form a two-state free convolution semigroup.

Example 19. Let $\tilde{\rho}, \rho \in \mathcal{P}$ such that $\mathcal{J}[\tilde{\rho}] = \rho$. That is, for some $\tilde{b}$ and $\tilde{c} > 0$,

$$\tilde{\rho} = \delta_{\tilde{b}} \uplus \Phi[\rho]^{\uplus \tilde{c}}.$$

Let

$$p = \tilde{c}/\gamma, \quad u = \tilde{b} - \beta \tilde{c}/\gamma.$$

Then the first component of the corresponding two-state free convolution semigroup satisfies

$$\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \boxplus \omega^{\boxplus pt},$$

where in Corollary 12, $\omega = \delta_{-u} \boxplus \tilde{\rho} \in \mathcal{P}$ but, in general, is not freely infinitely divisible. Indeed,

$$((\delta_{-u} \boxplus \tilde{\rho}) \boxplus \tilde{\rho})^{\boxplus(1/p)} = \delta_{-(u/p)} \boxplus \mathbb{B}[\tilde{\rho}]^{\boxplus(1/p)} = \mathbb{B}[\delta_{-(u/p)} \uplus \tilde{\rho}^{\uplus(1/p)}]$$
$$= \mathbb{B}[\delta_{-(u/p)} \uplus (\delta_{\tilde{b}} \uplus \Phi[\rho]^{\uplus \tilde{c}})^{\uplus(1/p)}] = \mathbb{B}[\delta_\beta \uplus \Phi[\rho]^{\uplus \gamma}] = \mu.$$

If $\tau = \omega^{\boxplus(\gamma/\tilde{c})} \in \mathcal{P}$, then

$$\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \boxplus \tau^{\boxplus t}.$$

Remark 20. A free Meixner distribution $\mu_{b,c,\beta,\gamma}$ with parameters $b, \beta \in \mathbb{R}$, $c + \gamma, \gamma \geq 0$ is the probability measure with Jacobi parameters

$$J(\mu_{b,c,\beta,\gamma}) = \begin{pmatrix} \beta, & b+\beta, & b+\beta, & b+\beta, & \dots \\ \gamma, & c+\gamma, & c+\gamma, & c+\gamma, & \dots \end{pmatrix}.$$

For other values of c, γ, these Jacobi parameters determine a unital, linear, but not positive definite functional. Normalized free Meixner distributions $\mu_{b,c} = \mu_{b,c,0,1}$ have mean 0 and variance 1, and are positive for $c \geq -1$.

Free Meixner distributions form a two-parameter semigroup with respect to $\boxplus$:

$$\mu_{b,c,\beta,\gamma} \boxplus \mu_{b,c,\beta',\gamma'} = \mu_{b,c,\beta+\beta',\gamma+\gamma'}.$$

See Definition 2 of [Anshelevich and Młotkowski 2012]. In particular,

$$\mu_{b,c,\beta,\gamma}^{\boxplus t} = \mu_{b,c,\beta t,\gamma t}.$$

Also,

$$\mathbb{B}_t[\mu_{b,c,\beta,\gamma}] = \mu_{b+\beta t,c+\gamma t,\beta,\gamma}.$$

Lemma 21. *The subordination distribution of two Meixner distributions with special parameters*

$$\mu_{b,c,\beta',\gamma'} \boxplus \mu_{b,c,\beta,\gamma} = \mu_{b+\beta,c+\gamma,\beta',\gamma'}$$

is again a free Meixner distribution.

Proof. Using Lemma 15 and the properties from the preceding remark, we compute

$$
\begin{aligned}
\mu_{b,c,\beta',\gamma'} \boxplus \mu_{b,c,\beta,\gamma} &= \left(\delta_{\beta'-\beta\gamma'/\gamma} \boxplus \mu_{b,c,\beta,\gamma}^{\boxplus(\gamma'/\gamma)} \right) \boxplus \mu_{b,c,\beta,\gamma} \\
&= \delta_{\beta'-\beta\gamma'/\gamma} \boxplus \mu_{b,c,\beta,\gamma} \boxplus \mu_{b,c,\beta,\gamma}^{\boxplus(\gamma'/\gamma)} \\
&= \delta_{\beta'-\beta\gamma'/\gamma} \boxplus \mathbb{B}[\mu_{b,c,\beta,\gamma}]^{\boxplus(\gamma'/\gamma)} \\
&= \delta_{\beta'-\beta\gamma'/\gamma} \boxplus \mu_{b+\beta,c+\gamma,\beta,\gamma}^{\boxplus(\gamma'/\gamma)} \\
&= \delta_{\beta'-\beta\gamma'/\gamma} \boxplus \mu_{b+\beta,c+\gamma,\beta\gamma'/\gamma,\gamma'} \\
&= \mu_{b+\beta,c+\gamma,\beta',\gamma'}. \qquad \square
\end{aligned}
$$

Remark 22. Since $\nu \rhd (\mu \boxplus \nu) = \mu \boxplus \nu$, the preceding lemma implies a monotone convolution identity

$$\mu_{b,c,\beta,\gamma} \rhd \mu_{b+\beta,c+\gamma,\beta',\gamma'} = \mu_{b,c,\beta+\beta',\gamma+\gamma'}.$$

This result can also be proved directly using the F-transforms, but the computation is rather surprising. Since $\mu_{0,0,\beta,\gamma}$ are semicircular, $\mu_{b,0,\beta,\gamma}$ are free Poisson, $\mu_{b,-\gamma,\beta,\gamma}$ are Bernoulli, and $\mu_{0,-\gamma,0,2\gamma}$ are arcsine distributions, we get various identities between them involving the monotone convolution. For example,

$$\mu_{b,c} \rhd \mu_{b,c+1} = \mu_{b,c}^{\boxplus 2},$$

which for $b = 0$, $c = -1$ gives Bernoulli $\rhd$ semicircle = arcsine. See [Młotkowski 2010] for related results.

Example 23. For a particular case of Example 19, let $\tilde{c} > 0$ and $c \geq 0$. The two-state free Meixner semigroups from [Anshelevich and Młotkowski 2012] satisfy

$$J(\tilde{\mu}_t) = \begin{pmatrix} \tilde{\beta}t, & \tilde{b} + \beta t, & b + \beta t, & b + \beta t, & \ldots \\ \tilde{\gamma}t, & \tilde{c} + \gamma t, & c + \gamma t, & c + \gamma t, & \ldots \end{pmatrix}$$

and

$$J(\mu_t) = \begin{pmatrix} \beta t, & b + \beta t, & b + \beta t, & \ldots \\ \gamma t, & c + \gamma t, & c + \gamma t, & \ldots \end{pmatrix}.$$

Thus

$$J(\mathcal{J}[\tilde{\mu}_t]) = \begin{pmatrix} \tilde{b} + \beta t, & b + \beta t, & b + \beta t, & \ldots \\ \tilde{c} + \gamma t, & c + \gamma t, & c + \gamma t, & \ldots \end{pmatrix},$$

so $\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \boxplus \omega^{\boxplus(\gamma/\tilde{c})t}$, where

$$J(\tilde{\rho}) = \begin{pmatrix} \tilde{b}, & b, & b, & b, & \ldots \\ \tilde{c}, & c, & c, & c, & \ldots \end{pmatrix}$$

and

$$J(\omega) = \begin{pmatrix} \beta\tilde{c}/\gamma, & \beta\tilde{c}/\gamma + b - \tilde{b}, & \beta\tilde{c}/\gamma + b - \tilde{b}, & \beta\tilde{c}/\gamma + b - \tilde{b}, & \ldots \\ \tilde{c}, & c, & c, & c, & \ldots \end{pmatrix}.$$

In particular, $\tilde{\rho} = \delta_{\tilde{b} - \beta\tilde{c}/\gamma} \boxplus \omega$. Note that both $\tilde{\rho}$ and ω are free Meixner distributions. Also,

$$J(\rho) = \begin{pmatrix} b, & b, & b, & b, & \ldots \\ c, & c, & c, & c, & \ldots \end{pmatrix},$$

so $\rho = \sigma_{b,c} = \delta_b \boxplus \sigma^{\boxplus c}$ and $\mathcal{J}[\tilde{\rho}] = \rho$. Finally, for $\tau = \omega^{\boxplus(\gamma/\tilde{c})}$,

$$J(\tau) = \begin{pmatrix} \beta, & \beta + b - \tilde{b}, & \beta + b - \tilde{b}, & \beta + b - \tilde{b}, & \ldots \\ \gamma, & \gamma + c - \tilde{c}, & \gamma + c - \tilde{c}, & \gamma + c - \tilde{c}, & \ldots \end{pmatrix}.$$

So $\tau \in \mathcal{ID}^{\boxplus}$ for $c \geq \tilde{c}$, $\tau \in \mathcal{P}$ for $\gamma + c \geq \tilde{c}$, and for $\gamma + c < \tilde{c}$, τ is not a positive functional.

Proposition 24. *Let $(\tilde{\mu}_t, \mu_t)$ be a general two-state free convolution semigroup. Then we have two evolution equations,*

$$(11) \qquad \partial_t F_{\tilde{\mu}_t} = \phi_\mu(F_{\mu_t}) - \phi_{\tilde{\mu},\mu}(F_{\mu_t}) - \phi_\mu(F_{\mu_t})\partial_z F_{\tilde{\mu}_t}$$

and

$$\partial_t F_{\mu_t} = -\phi_\mu(F_{\mu_t})\partial_z F_{\mu_t}.$$

Proof. The second equation is standard; see equation (3.18) in [Voiculescu et al. 1992]. Using (6),

$$\partial_t F_{\tilde{\mu}_t} + \phi_{\tilde{\mu},\mu}(F_{\mu_t}) + t\phi'_{\tilde{\mu},\mu}(F_{\mu_t})\partial_t F_{\mu_t}$$
$$= \partial_t F_{\tilde{\mu}_t} + \phi_{\tilde{\mu},\mu}(F_{\mu_t}) - t\phi'_{\tilde{\mu},\mu}(F_{\mu_t})\phi_\mu(F_{\mu_t})\partial_z F_{\mu_t} = 0$$

and

$$\partial_z F_{\tilde{\mu}_t} + t\phi'_{\tilde{\mu},\mu}(F_{\mu_t})\partial_z F_{\mu_t} = 1.$$

Plugging in, we get

$$\partial_t F_{\tilde{\mu}_t} = -\phi_{\tilde{\mu},\mu}(F_{\mu_t}) + \frac{1 - \partial_z F_{\tilde{\mu}_t}}{\phi'_{\tilde{\mu},\mu}(F_{\mu_t})\partial_z F_{\mu_t}}\phi'_{\tilde{\mu},\mu}(F_{\mu_t})\phi_\mu(F_{\mu_t})\partial_z F_{\mu_t}$$

$$= \phi_\mu(F_{\mu_t}) - \phi_{\tilde{\mu},\mu}(F_{\mu_t}) - \phi_\mu(F_{\mu_t})\partial_z F_{\tilde{\mu}_t}. \qquad \square$$

Definition 25. The functional L_t is the *generator* of the family $\{\mu_t : t \geq 0\}$ of functionals at time t, with domain $\mathcal{D}$, if for any $f \in \mathcal{D}$,

$$\langle L_t, f \rangle = \frac{d}{dt}\langle \mu_t, f \rangle.$$

Proposition 26. *Let $(\tilde{\mu}_t, \mu_t)$ be a general two-state free convolution semigroup with finite variance, with canonical triples $\{(\tilde{\beta}, \tilde{\gamma}, \tilde{\rho}), (\beta, \gamma, \rho)\}$. Let $\mathcal{J}[\tilde{\mu}_t] = \tilde{\nu}_t$ and $\mathcal{J}[\mu_t] = \nu_t$. Note that*

$$\nu_t = \rho \boxplus \sigma_{\beta,\gamma}^{\boxplus t},$$

and for measures covered in Theorem 11, *$\tilde{\nu}_t = \tilde{\rho} \boxplus \tau^{\boxplus t}$.*

Then the generators of the families $\{\tilde{\mu}_t\}$ and $\{\mu_t\}$ with domain

$$\mathcal{D} = \mathrm{Span}\left(\left\{\frac{1}{z-x} : z \in \mathbb{C} \setminus \mathbb{R}\right\}\right)$$

are, respectively,

$$\tilde{L}_t = \tilde{\gamma}(\tilde{\mu}_t \otimes \tilde{\mu}_t \otimes \tilde{\nu}_t)\partial^2 - \gamma(\tilde{\mu}_t \otimes \tilde{\mu}_t \otimes \nu_t)\partial^2$$
$$+ (\tilde{\beta} - \beta)(\tilde{\mu}_t \otimes \tilde{\mu}_t)\partial + \gamma(\tilde{\mu}_t \otimes \nu_t)(\partial_x \otimes 1)\partial + \beta\tilde{\mu}_t\partial_x$$

and

$$L_t = \gamma(\mu_t \otimes \nu_t)(\partial_x \otimes 1)\partial + \beta\mu_t\partial_x.$$

Here $\partial : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ is the difference quotient operation

$$(\partial f)(x, y) = \frac{f(x) - f(y)}{x - y}.$$

Proof. Note first that

$$(\phi_{\tilde{\mu},\mu} \circ F_{\mu_t})(z) = \frac{1}{t}(\phi_{\tilde{\mu}_t,\mu_t} \circ F_{\mu_t})(z) = \frac{1}{t}(z - F_{\tilde{\mu}_t}(z)) = \tilde{\beta} + \tilde{\gamma}G_{\tilde{\nu}_t}(z),$$

and similarly $\phi_\mu \circ F_{\mu_t} = \beta + \gamma G_{\nu_t}$. Therefore in this case, (11) gives

$$\partial_t F_{\tilde{\mu}_t} = \beta + \gamma G_{\nu_t} - \tilde{\beta} + \tilde{\gamma}G_{\tilde{\nu}_t} - (\beta + \gamma G_{\nu_t})\partial_z F_{\tilde{\mu}_t}.$$

Equivalently,

$$\partial_t G_{\tilde{\mu}_t} = -(\beta + \gamma G_{\nu_t} - \tilde{\beta} + \tilde{\gamma} G_{\tilde{\nu}_t}) G_{\tilde{\mu}_t}^2 - (\beta + \gamma G_{\nu_t}) \partial_z G_{\tilde{\mu}_t}.$$

In other words,

$$\partial_t \left\langle \tilde{\mu}_t, \frac{1}{z-x} \right\rangle = -\gamma \left\langle \tilde{\mu}_t \otimes \tilde{\mu}_t \otimes \nu_t, \partial^2 \frac{1}{z-x} \right\rangle - \beta \left\langle \tilde{\mu}_t \otimes \tilde{\mu}_t, \partial \frac{1}{z-x} \right\rangle$$

$$+ \tilde{\gamma} \left\langle \tilde{\mu}_t \otimes \tilde{\mu}_t \otimes \tilde{\nu}_t, \partial^2 \frac{1}{z-x} \right\rangle + \tilde{\beta} \left\langle \tilde{\mu}_t \otimes \tilde{\mu}_t, \partial \frac{1}{z-x} \right\rangle$$

$$+ \gamma \left\langle \tilde{\mu}_t \otimes \nu_t, (\partial_x \otimes 1) \partial \frac{1}{z-x} \right\rangle + \beta \left\langle \tilde{\mu}_t, \partial_x \frac{1}{z-x} \right\rangle.$$

The formula for the generator $\tilde{L}_t$ of $\{\tilde{\mu}_t\}$ on the span of such functions follows. The formula for L_t follows by setting $\tilde{\mu}_t = \mu_t$. □

Remark 27. Setting $t = 0$ in the preceding proposition, $\mu_0 = \tilde{\mu}_0 = \delta_0$, $\tilde{\nu}_0 = \tilde{\rho}$, and $\nu_0 = \rho$. Thus

$$L_0 f = \gamma \langle \delta_0 \otimes \rho, (\partial_x \otimes 1) \partial f \rangle + \beta \langle \delta_0, \partial_x f \rangle$$

$$= \gamma \int_{\mathbb{R}} \frac{f(y) - f(0) - y f'(0)}{y^2} d\rho(y) + \beta f'(0),$$

and

$$\tilde{L}_0 f = \tilde{\gamma} \langle \delta_0 \otimes \delta_0 \otimes \tilde{\rho}, \partial^2 f \rangle - \gamma \langle \delta_0 \otimes \delta_0 \otimes \rho, \partial^2 f \rangle + (\tilde{\beta} - \beta) \langle \delta_0 \otimes \delta_0, \partial f \rangle$$

$$+ \gamma \langle \delta_0 \otimes \rho, (\partial_x \otimes 1) \partial f \rangle + \beta \langle \delta_0, \partial_x f \rangle$$

$$= \int_{\mathbb{R}} \frac{f(y) - f(0) - y f'(0)}{y^2} d(\tilde{\gamma} \tilde{\rho} - \gamma \rho)(y) + (\tilde{\beta} - \beta) f'(0)$$

$$+ \int_{\mathbb{R}} \frac{f(y) - f(0) - y f'(0)}{y^2} d(\gamma \rho)(y) + \beta f'(0)$$

$$= \tilde{\gamma} \int_{\mathbb{R}} \frac{f(y) - f(0) - y f'(0)}{y^2} d\tilde{\rho}(y) + \tilde{\beta} f'(0)$$

has exactly the same form as in Proposition 3 of [Anshelevich 2013]; see also Remark 11 of that paper.

Remark 28. Boolean evolution corresponds to $\beta = \gamma = 0$, $\mu_t = \delta_0$. Then

$$\partial_t F_{\tilde{\mu}_t}(z) = -\phi_{\tilde{\mu}, \delta_0}(z).$$

In fact, since $\phi_{\tilde{\mu}, \delta_0}(z) = z - F_{\tilde{\mu}}(z)$, this is easy to see directly. It follows that in this case,

$$\tilde{L}_t = \tilde{\gamma}(\tilde{\mu}_t \otimes \tilde{\mu}_t \otimes \tilde{\nu}_t) \partial^2 + \tilde{\beta}(\tilde{\mu}_t \otimes \tilde{\mu}_t) \partial.$$

For $t = 0$, we again get the formula from the preceding remark.

Similarly, distributions of analytic two-state free Brownian motions correspond to $\tilde{\gamma} = \gamma = 1$, $\tilde{\beta} = 0$, $\tilde{\mu} = \Phi[\mu]$ and $\mu = \delta_\beta \boxplus \sigma$, so that $\tilde{\nu}_t = \nu_t = \mu_t$. Then the generator formula reduces to

$$\tilde{L}_t = -\beta(\tilde{\mu}_t \otimes \tilde{\mu}_t)\partial + (\tilde{\mu}_t \otimes \mu_t)(\partial_x \otimes 1)\partial + \beta\tilde{\mu}_t\partial_x$$
$$= \tilde{\mu}_t\big(-\beta(1 \otimes \tilde{\mu}_t)\partial + \partial_x(1 \otimes \mu_t)\partial + \beta\partial_x\big),$$

consistent with the result of Proposition 24 in [Anshelevich 2013].

4. Background, II

4.1. *Multivariate polynomials.* The number $d \in \mathbb{N}$ will be fixed throughout the remainder of the article. Denote by

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$$

a d-tuple of variables, and define $\boldsymbol{z}$, etc., similarly. Let

$$\mathbb{C}\langle\boldsymbol{x}\rangle = \mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle$$

be the algebra of polynomials in d noncommuting variables. For $k \geq 1$ and

$$\vec{u} = (u(1), u(2), \ldots, u(k)) \in \{1, \ldots, d\}^k$$

a multi-index, let

$$x_{\vec{u}} = x_{u(1)}x_{u(2)} \cdots x_{u(k)}.$$

Let

$$\mathcal{D}_{\text{alg}}(d) = \{\mu : \mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle \to \mathbb{C} \text{ unital, linear functionals}\}.$$

For $\beta \in \mathbb{R}^d$, the element $\delta_\beta \in \mathcal{D}_{\text{alg}}(d)$ is

$$\delta_\beta[x_{\vec{u}}] = \beta_{\vec{u}}.$$

4.2. *Free, Boolean, and two-state free convolutions.* Let $\mu \in \mathcal{D}_{\text{alg}}(d)$. Denote its moment-generating function by

$$M^\mu(\boldsymbol{z}) = \sum_{\vec{u}} \mu[x_{\vec{u}}]z_{\vec{u}}.$$

The (combinatorial) R-transform R^μ of μ is determined by

$$(12) \qquad R^\mu\big(z_1(1 + M^\mu(\boldsymbol{z})), \ldots, z_d(1 + M^\mu(\boldsymbol{z}))\big) = M^\mu(\boldsymbol{z}).$$

See Lecture 16 of [Nica and Speicher 2006]. The free convolution of two functionals $\mu \boxplus \nu$ is determined by the equality

$$R^{\mu\boxplus\nu} = R^\mu + R^\nu.$$

In the algebraic setting, any functional is $\boxplus$-infinitely divisible.

Similarly, the η-transform η^μ is

$$\eta^\mu(z) = (1 + M^\mu(z))^{-1} M^\mu(z)$$

(for a multivariate power series F, F^{-1} will denote its multiplicative inverse). The Boolean convolution of two functionals $\mu \uplus \nu$ is determined by the equality

$$\eta^{\mu \uplus \nu} = \eta^\mu + \eta^\nu.$$

Finally, for $\tilde{\mu}, \mu \in \mathcal{D}_{\text{alg}}(d)$, the two-state R-transform $R^{\tilde{\mu},\mu}$ is determined by

$$\eta^{\tilde{\mu}}(z) = R^{\tilde{\mu},\mu}\big(z_1(1 + M^\mu(z)), \ldots, z_d(1 + M^\mu(z))\big)(1 + M^\mu(z))^{-1},$$

and the two-state free convolution of two pairs of functionals

$$(\rho, \mu \boxplus \nu) = (\tilde{\mu}, \mu) \boxplus_c (\tilde{\nu}, \nu)$$

is determined by the equality

$$R^{\rho, \mu \boxplus \nu} = R^{\tilde{\mu},\mu} + R^{\tilde{\nu},\nu}.$$

See Section 2.5 of [Anshelevich 2010].

If $d = 1$ and μ is a compactly supported probability measure on $\mathbb{R}$, it can be identified with an element of $\mathcal{D}_{\text{alg}}(1)$. In this case, the complex function transforms from Section 2 have power series expansions related to the power series from this section by

$$1 + M^\mu(z) = \frac{1}{z} G_\mu\left(\frac{1}{z}\right), \quad R^\mu(z) = z R_\mu(z) = z \phi_\mu\left(\frac{1}{z}\right),$$

$$\eta^\mu(z) = \frac{1}{z} - F_\mu\left(\frac{1}{z}\right), \quad R^{\tilde{\mu},\mu}(z) = z \phi_{\tilde{\mu},\mu}\left(\frac{1}{z}\right).$$

4.3. Transformations. For $\nu \in \mathcal{D}_{\text{alg}}(d)$, the functional $\Phi[\nu]$ is determined by

$$\eta^{\Phi[\nu]}(z) = \sum_{i=1}^{d} z_i(1 + M^\nu(z))z_i.$$

See [Belinschi and Nica 2009; Anshelevich 2009].

In the algebraic setting, $\mathbb{B}$ is a bijection from $\mathcal{D}_{\text{alg}}(d)$ to itself determined by

$$R^{\mathbb{B}[\mu]} = \eta^\mu.$$

Finally, for $\mu, \nu \in \mathcal{D}_{\text{alg}}(d)$, the multivariate subordination distribution $\mu \boxdot \nu \in \mathcal{D}_{\text{alg}}(d)$ is defined via

$$(13) \quad R^{\mu \boxdot \nu}(z) = R^\mu\big(z_1(1 + M^\nu(z)), \ldots, z_d(1 + M^\nu(z))\big)(1 + M^\nu(z))^{-1}.$$

See Definition 1.1 in [Nica 2009].

5. Multivariate, algebraic results

The following proposition is the analog of the single-variable relation $G_{\mu \boxplus \nu}(z) = G_\nu(F_{\mu \boxdot \nu}(z))$.

Proposition 29. *The subordination distribution $\mu \boxdot \nu$ satisfies*

$$
\begin{aligned}
1 + M^{\mu \boxdot \nu}(z) \\
= (1 + M^{\mu \boxdot \nu}(z))\big(1 + M^\nu\big(z_1(1 + M^{\mu \boxdot \nu}(z)), \ldots, z_d(1 + M^{\mu \boxdot \nu}(z))\big)\big).
\end{aligned}
$$

Consequently, for a fixed ν, the map $\mu \mapsto \mu \boxdot \nu$ is a bijection on $\mathcal{D}_{\mathrm{alg}}(d)$.

Proof. Note first that the equation

$$
1 + M^{\mu \boxdot \nu}(z) = (1 + M^\lambda(z))\big(1 + M^\nu\big(z_1(1 + M^\lambda(z)), \ldots, z_d(1 + M^\lambda(z))\big)\big)
$$

has a unique solution λ. Indeed,

$$
(\mu \boxdot \nu)[x_{\vec{u}}] = \lambda[x_{\vec{u}}] + \nu[x_{\vec{u}}] + P_{\vec{u}}(\lambda[x_{\vec{v}}], \nu[x_{\vec{v}}] : |\vec{v}| < |\vec{u}|)
$$

for some polynomial $P_{\vec{u}}$.

Let $w_i = z_i(1 + M^\lambda(z))$. Then

$$
\begin{aligned}
M^{\mu \boxdot \nu}(z) &= R^{\mu \boxdot \nu}\big(z_1(1 + M^{\mu \boxdot \nu}(z)), \ldots, z_d(1 + M^{\mu \boxdot \nu}(z))\big) \\
&= R^{\mu \boxdot \nu}\big(w_1(1 + M^\nu(w)), \ldots, w_d(1 + M^\nu(w))\big) \\
&= R^\mu\big(w_1(1 + M^\nu(w)), \ldots, w_d(1 + M^\nu(w))\big) + M^\nu(w).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
M^{\mu \boxdot \nu}(z) &= (1 + M^\lambda(z))(1 + M^\nu(w)) - 1 \\
&= M^\lambda(z)(1 + M^\nu(w)) + M^\nu(w) \\
&= R^\lambda(w)(1 + M^\nu(w)) + M^\nu(w).
\end{aligned}
$$

Combining these two equations,

$$
R^\mu\big(w_1(1 + M^\nu(w)), \ldots, w_d(1 + M^\nu(w))\big) = R^\lambda(w)(1 + M^\nu(w)).
$$

Comparing with (13), we see that $\lambda = \mu \boxdot \nu$.

The equation in the proposition shows that given ν and λ, $\mu \boxdot \nu$, and consequently μ, is uniquely determined. Conversely, the uniqueness statement above shows that given ν and μ, $\mu \boxdot \nu$ is uniquely determined. $\qquad\square$

In the multivariate, algebraic setting, all the results in Lemma 15 were proved in [Nica 2009]; see Remark 1.2, Theorem 1.8, equation (5.7), and Proposition 5.3. We will use them without proof.

Proposition 30. *Let $\tilde{\beta} \in \mathbb{R}^d$, $\tilde{\gamma} > 0$, $\tilde{\rho} \in \mathcal{D}_{\mathrm{alg}}(d)$, and $\{\mu_t : t \geq 0\} \subset \mathcal{D}_{\mathrm{alg}}(d)$ be a free convolution semigroup. Define a two-state free convolution semigroup $\{(\tilde{\mu}_t, \mu_t) : t \geq 0\}$ by*

$$R^{\tilde{\mu}_t, \mu_t}(z) = t\tilde{\beta} \cdot z + t\tilde{\gamma} \sum_{i=1}^{d} z_i(1 + M^{\tilde{\rho}}(z))z_i.$$

Define $\tau \in \mathcal{D}_{\mathrm{alg}}(d)$ via

$$\mu = \mu_1 = \tau \boxplus \tilde{\rho}.$$

Then

$$\tilde{\mu}_t = \delta_{t\tilde{\beta}} \uplus \Phi[\tilde{\rho} \boxplus \tau^{\boxplus t}]^{\uplus \tilde{\gamma} t}.$$

Proof. By the preceding proposition, for $\nu = \tau^{\boxplus t}$,

$$1 + M^{\nu \boxplus \tilde{\rho}}(z) = (1 + M^{\nu \boxplus \tilde{\rho}}(z))(1 + M^{\tilde{\rho}}(z_1(1 + M^{\nu \boxplus \tilde{\rho}}(z)), \ldots, z_d(1 + M^{\nu \boxplus \tilde{\rho}}(z)))).$$

Since

$$\mu_t = \tau^{\boxplus t} \boxplus \tilde{\rho} = \nu \boxplus \tilde{\rho},$$

we have

$$1 + M^{\tilde{\rho} \boxplus \tau^{\boxplus t}}(z) = (1 + M^{\mu_t}(z))(1 + M^{\tilde{\rho}}(z_1(1 + M^{\mu_t}(z)), \ldots, z_d(1 + M^{\mu_t}(z)))).$$

On the other hand,

$$\eta^{\tilde{\mu}_t}(z) = R^{\tilde{\mu}_t, \mu_t}(z_1(1 + M^{\mu_t}(z)), \ldots, z_d(1 + M^{\mu_t}(z)))(1 + M^{\mu_t}(z))^{-1}$$

$$= t\tilde{\beta} \cdot z + t\tilde{\gamma} \sum_{i=1}^{d} z_i(1 + M^{\mu_t}(z))$$

$$\left(1 + M^{\tilde{\rho}}(z_1(1 + M^{\mu_t}(z)), \ldots, z_d(1 + M^{\mu_t}(z)))\right)z_i.$$

Combining these two equations, it follows that

$$\eta^{\tilde{\mu}_t}(z) = t\tilde{\beta} \cdot z + t\tilde{\gamma} \sum_{i=1}^{d} z_i(1 + M^{\tilde{\rho} \boxplus \tau^{\boxplus t}}(z))z_i$$

and

$$\tilde{\mu}_t = \delta_{t\tilde{\beta}} \uplus \Phi[\tilde{\rho} \boxplus \tau^{\boxplus t}]^{\uplus \tilde{\gamma} t}. \qquad \square$$

I am grateful to Hari Bercovici for a discussion leading to the following observations.

Corollary 31. *Let $\{(\tilde{\mu}_t, \mu_t) : t \geq 0\}$ be a two-state free convolution semigroup of compactly supported probability measures such that $\tilde{\mu}_1$ has nonzero variance. Then*

$$\mu = \tau \boxplus \tilde{\rho}$$

and

$$\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \boxplus \tau^{\boxplus t}$$

for some $\tilde{\rho}$ a compactly supported probability measure, and τ a unital, not necessarily positive linear functional with nonnegative variance such that $|\tau[x^n]| \leq C^n$ for some C.

Proof. The result follows by applying the preceding proposition in the case $d = 1$, when every compactly supported two-state free convolution semigroup is of the form specified in that proposition. Since for each t,

$$\mathrm{Var}[\tilde{\rho}] + t\,\mathrm{Var}[\tau] = \mathrm{Var}[\mathcal{J}[\tilde{\mu}_t]] \geq 0,$$

it follows that $\mathrm{Var}[\tau] \geq 0$. The positivity of $\tilde{\rho}$ follows from the positivity of $(\tilde{\mu}_t, \mu_t)$, and the compact support of $\tilde{\rho}$ and the growth conditions on τ follow from the compact support of $(\tilde{\mu}_t, \mu_t)$. $\qquad\square$

Lemma 32. *Let τ be a unital linear functional with positive variance such that $|\tau[x^n]| \leq C^n$ for some C. Then $\tau^{\boxplus t}$ is positive definite, and so can be identified with a compactly supported measure, for sufficiently large t.*

Proof. Without loss of generality, we may assume that τ has mean 0 and variance 1. By assumption, the moments of τ, and so also its free cumulants, grow no faster than exponentially. Therefore the R-transform of τ can be identified with an analytic function whose power series expansion at zero starts with z. It follows that for sufficiently large t, the R-transform of $D_{1/\sqrt{t}}\tau^{\boxplus t}$ satisfies the conditions of Theorem 2 of [Bercovici and Voiculescu 1995]. Applying that theorem, we conclude that $D_{1/\sqrt{t}}\tau^{\boxplus t}$, and so also $\tau^{\boxplus t}$, can be identified with a positive measure. $\qquad\square$

Proposition 33. *Let $\{(\tilde{\mu}_t, \mu_t) : t \geq 0\}$ be a two-state free convolution semigroup of compactly supported measures such that $\tilde{\mu}_1$ has nonzero variance. Then there exist $\tilde{\beta} \in \mathbb{R}$, $\tilde{\gamma} > 0$, $p > 0$ and $\omega, \tilde{\rho} \in \mathcal{P}$ such that $\omega \boxplus \tilde{\rho} \in \mathcal{ID}^\boxplus$, $\tilde{\rho} \boxplus \omega^{\boxplus(t/p)}$ is well-defined (in the sense of part (a) of Theorem 11) for all $t \geq 0$, and*

$$\mu_t = (\omega \boxplus \tilde{\rho})^{\boxplus(t/p)}$$

and

$$\tilde{\mu}_t = \delta_{\tilde{\beta}t} \uplus \Phi[\tilde{\rho} \boxplus \omega^{\boxplus(t/p)}]^{\uplus\tilde{\gamma}t},$$

so that in particular,

$$\mathcal{J}[\tilde{\mu}_t] = \tilde{\rho} \boxplus \omega^{\boxplus(t/p)}.$$

Proof. First suppose that $\mu_t = \delta_{\beta t}$ has zero variance. Then all the relations hold if we set $p = 1$, $\omega = \delta_\beta$, and $\tilde{\rho}$ to be the measure in the relative canonical triple of $\{(\tilde{\mu}_t, \mu_t) : t \geq 0\}$.

In the remainder of the argument, we assume that μ_1 has nonzero variance. Let $\tilde{\rho}$ be the measure in the relative canonical triple of $\{(\tilde{\mu}_t, \mu_t) : t \geq 0\}$. By Corollary 31, there exists a linear, unital, not necessarily positive functional τ such that

$$(14) \qquad\qquad \tau \boxplus \tilde{\rho} = \mu$$

is $\boxplus$-infinitely divisible, and

$$\tilde{\rho} \boxplus \tau^{\boxplus p} = \mathcal{J}[\tilde{\mu}_t]$$

can be identified with a positive measure. Moreover, it follows from (14) that $\mathrm{Var}[\tau] = \mathrm{Var}[\mu] > 0$. So by Lemma 32, for sufficiently large p, $\omega = \tau^{\boxplus p}$ can itself be identified with a positive measure. The result follows. $\qquad\square$

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