Pacific Journal of Mathematics

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JEAN-BAPTISTE CASTÉRAS

Volume 276 No. 2 August 2015

PACIFIC JOURNAL OF MATHEMATICS
Vol. 276, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.276.321

A MEAN FIELD TYPE FLOW II: EXISTENCE AND CONVERGENCE

JEAN-BAPTISTE CASTÉRAS

This paper is the continuation of (Castéras 2015), in which we investigated a gradient flow related to the mean field type equation. First, we show that this flow exists for all time. Next, using the compactness result of Castéras (2015), we prove, under a suitable hypothesis on its energy, the convergence of the flow to a solution of the mean field type equation. We also get a divergence result if the energy of the initial data is largely negative.

Introduction

Let (M, g) be a compact Riemannian surface without boundary. We will study an evolution problem associated to a mean field type equation

$$-\Delta v + Q = \rho \frac{e^v}{\int_M e^v \, dV}, \tag{0-1}$$

where ρ is a real number, $Q \in C^\infty(M)$ is a given function such that $\int_M Q \, dV = \rho$ and Δ is the Laplacian with respect to the metric g. Equation (0-1) is equivalent to the mean field equation

$$-\Delta u + \rho\left(\frac{-f e^u}{\int_M f e^u \, dV} + \frac{1}{|M|}\right) = 0, \tag{0-2}$$

where $|M|$ stands for the volume of M with respect to the metric g and $f \in C^\infty(M)$ is a positive function. Indeed, if v is a solution of (0-1), by setting $v = u + \log f$, we recover that u is a solution of (0-2) with $Q = \rho/|M| + \Delta \log f$.

The mean field equation appears in conformal geometry but also in statistical mechanics from Onsager's vortex model for turbulent Euler flows. More precisely, in this setting the solution u of the mean field equation is the stream function in the infinite vortex limit (see [Caglioti et al. 1992]). It also arises in the abelian Chern–Simons–Higgs model (see for example [Caffarelli and Yang 1995; Han 2003; Tarantello 1996; Yang 2001]).

The author was supported by the CNPq (Brazil) project 501559/2012-4.
MSC2010: 35B33, 35J20, 53C44, 58E20.
Keywords: mean field equation, blow-up analysis, geometric flow.

Equation (0-2) has a variational structure and its solutions can be found as the critical points of the functional

$$(0\text{-}3)\quad I_\rho(u)=\frac{1}{2}\int_M|\nabla u|^2\,dV+\frac{\rho}{|M|}\int_M u\,dV-\rho\log\left(\int_M fe^u\,dV\right),\ u\in H^1(M).$$

When $\rho<8\pi$, from the Moser–Trudinger inequality one can easily prove that the functional I_ρ is bounded from below and coercive; thus one can find a solution of (0-2) by minimizing I_ρ. The existence of solutions becomes more delicate if $\rho\geq 8\pi$. When $\rho=8\pi$, I_ρ admits a lower bound but is no longer coercive, while for $\rho>8\pi$, I_ρ is unbounded from below and above. The existence of solutions of (0-1) has been widely studied in recent decades. Many partial existence results have been obtained for $\rho\neq 8k\pi,\ k\in\mathbb{N}^*$, and according to the Euler characteristic of M (see for example [Brezis and Merle 1991; Chen and Lin 2003; Ding et al. 1999; Li 1999; Li and Shafrir 1994; Malchiodi 2008; Struwe and Tarantello 1998]). Finally, when $\rho\neq 8k\pi$, $k\in\mathbb{N}^*$, Djadli [2008] has generalized the previous results, establishing the existence of solutions for all surfaces M by studying the topology of sublevels $\{I_\rho\leq -C\}$ to achieve a min-max scheme (already introduced in [Djadli and Malchiodi 2008]).

In this paper, we consider the evolution problem associated to (0-1), namely the equation

$$(0\text{-}4)\qquad \begin{cases}\dfrac{\partial}{\partial t}e^v=\Delta v-Q+\rho\dfrac{e^v}{\int_M e^v\,dV},\\ v(x,0)=v_0(x),\end{cases}$$

with initial data $v_0\in C^{2+\alpha}(M)$, $\alpha\in(0,1)$ and a function $Q\in C^\infty(M)$ such that $\int_M Q dV=\rho$. It is a gradient flow with respect to the following functional, which will be called energy:

$$(0\text{-}5)\quad J_\rho(v)=\frac{1}{2}\int_M|\nabla v|^2\,dV+\int_M Qv(t)\,dV-\rho\ln\left(\int_M e^v\,dV\right),\quad v\in H^1(M).$$

This functional is unbounded from below (except in the case $\rho<8\pi$) and above. The interest of this flow is that it satisfies some important geometrical properties useful for its convergence (see in particular estimate (2-2) of Section 2). When Q is a constant equal to the scalar curvature of M with respect to the metric g, the flow (0-4) (normalized) has been studied by Struwe [2002] (we note that in this case $\rho=\int_M Q\,dV\leq 8\pi$). For other curvature flows, we refer to [Baird et al. 2004; Brendle 2003; 2005; 2006; Castéras 2013; Hamilton 1988; Lamm et al. 2009; Malchiodi and Struwe 2006; Schwetlick and Struwe 2003; Struwe 2005] and the references therein.

We begin by studying the global existence of the flow (0-4). We prove:

Theorem 0.1. *For all initial data $v_0 \in C^{2+\alpha}(M)$, $\alpha \in (0,1)$, all $\rho \in \mathbb{R}$ and all functions $Q \in C^\infty(M)$ such that $\int_M Q\,dV = \rho$, there exists a unique global solution $v \in C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(M \times [0,+\infty))$ of (0-4).*

Next, we investigate the convergence of the flow. Let $v(t) : M \to \mathbb{R}$ denote the function defined by $v(t)(x) = v(x,t)$. We show that if the energy $J_\rho(v(t))$ of the global solution is bounded from below uniformly in time (when $\rho > 8\pi$), then as $t \to +\infty$, $v(t)$ converges to a function v_∞ which is a solution of (0-1). More precisely, we have:

Theorem 0.2. *Let $v(t)$ be the solution of (0-4).*

(i) *If $\rho < 8\pi$, then $v(t)$ converges in $H^2(M)$ to a solution $v_\infty \in C^\infty(M)$ of*

$$-\Delta v_\infty + Q = \rho \frac{e^{v_\infty}}{\int_M e^{v_\infty}\,dV}.$$

(ii) *If $\rho > 8\pi$, $\rho \neq 8k\pi$, $k \in \mathbb{N}^*$, and if there exists a constant $C > 0$ not depending on t such that for all $t \geq 0$,*

$$J_\rho(v(t)) \geq -C, \tag{0-6}$$

then $v(t)$ converges in $H^2(M)$ to a solution $v_\infty \in C^\infty(M)$ of

$$-\Delta v_\infty + Q = \rho \frac{e^{v_\infty}}{\int_M e^{v_\infty}\,dV}.$$

Moreover, we prove that there exist initial data $v_0 \in C^\infty(M)$ such that the energy of the global solution $v(t)$ of the flow, with $v(0)(x) = v_0(x)$ for all $x \in M$, stays uniformly bounded from below, and hence, thanks to Theorem 0.2, such that the flow converges.

Theorem 0.3. *Let $\rho \neq 8k\pi$, $k \in \mathbb{N}^*$. There exist initial data $v_0 \in C^\infty(M)$ such that the global solution $v(t)$ of (0-4) with $v(x,0) = v_0(x)$, for all $x \in M$, satisfies (0-6), i.e., such that the global solution $v(t)$ of (0-4) converges in $H^2(M)$ to a solution $v_\infty \in C^\infty(M)$ of (0-1):*

$$-\Delta v_\infty + Q = \rho \frac{e^{v_\infty}}{\int_M e^{v_\infty}\,dV}.$$

Finally, we show that if the energy of the initial data v_0 of (0-4) is largely negative then the flow diverges when $t \to +\infty$.

Theorem 0.4. *Let $\rho \in (8k\pi, 8(k+1)\pi)$, $k \geq 1$. There exists a constant $C > 0$ depending on M, Q and ρ such that for all $v_0 \in C^{2+\alpha}(M)$ satisfying $J_\rho(v_0) \leq -C$, the global solution $v(t)$ of (0-4) satisfies*

$$J_\rho(v(t)) \underset{n \to +\infty}{\longrightarrow} -\infty.$$

To prove these convergence results, we use the compactness result of [Castéras 2015]. There we studied the compactness property of solutions $(v_n)_n \subseteq H^2(M)$ of the perturbed elliptic mean field type equation

$$-\Delta v_n = Q_0 + h_n e^{v_n} + \rho e^{v_n}, \tag{0-7}$$

where $\rho > 0$, $Q_0 \in C^0(M)$ and $(h_n)_n \subseteq C^0(M)$. The term h_n corresponds to the parabolic term of (0-4). We also assume that there exists a constant $C > 0$ not depending on n such that

$$\begin{aligned} &\text{(i)}\ \lim_{n\to+\infty} \int_M h_n^2 e^{v_n}\, dV = 0, \\ &\text{(ii)}\ h_n(x) e^{v_n(x)} + \rho e^{v_n(x)} \geq -C, \quad \forall x \in M, \quad \forall n \geq 0. \end{aligned} \tag{0-8}$$

We will see that these conditions are satisfied by the solution of the flow (0-4). We have established in [Castéras 2015] the following compactness result:

Theorem 0.5. *Let $(v_n)_n \subseteq H^2(M)$ be a sequence of solutions of (0-7) such that $\int_M e^{v_n}\, dV = 1$ for all $n \geq 0$, and satisfying (0-8). If $\rho \neq 8k\pi$, $k \in \mathbb{N}^*$, then there exists a constant C not depending on n such that*

$$\|v_n\|_{H^2(M)} \leq C.$$

The paper is organized as follows. In Section 1, we prove the global existence of a solution of (0-4). We also show the continuity of the flow with respect to its initial data. In Section 2, we study the convergence of the flow (0-4). We begin by proving Theorem 0.2. We first show that the global solution $v(t)$ of (0-4) is uniformly (with respect to t) bounded in $H^1(M)$ when $v(t)$ satisfies condition (0-6). The proof involves the compactness result obtained in Theorem 0.5. We point out that, when $\rho < 8\pi$, condition (0-6) is always satisfied. Then, we show that the parabolic term of (0-4), $\partial v(t)/\partial t$, tends to 0 as $t \to +\infty$ in the $L^2(M)$ norm with respect to the metric $g_1(t) = e^{v(t)} g$. This implies that $v(t)$ is uniformly bounded in $H^2(M)$. Next we prove Theorem 0.3, i.e., there exists initial data in $C^\infty(M)$ such that condition (0-6) is satisfied. Our proof is based on the study of the topology of the level set

$$\{v \in X \ :\ J_\rho(v) \leq -L\},$$

where X is the space of $C^\infty(M)$ functions endowed with the $C^{2+\alpha}(M)$ norm, $\alpha \in (0, 1)$. The end of Section 2 is devoted to the proof of Theorem 0.4.

1. Global existence of the flow

We begin by noticing that since the flow is parabolic, standard methods (see for example [Friedman 1964]) provide short time existence. Thus, there exists $T_1 > 0$ such that $v \in C^{2+\alpha,1+\alpha/2}(M \times [0, T_1])$ is a solution of (0-4). We give two basic properties of the flow: the conservation of the volume of M endowed with the metric $g_1(t) = e^{v(t)}g$, and the decreasing along the flow of the functional $J_\rho(v(t))$.

Proposition 1.1. (i) *For all* $t \in [0, T_1]$, *we have*

$$\int_M e^{v(t)}\,dV = \int_M e^{v_0}\,dV. \tag{1-1}$$

(ii) *If* $0 \le t_0 \le t_1 \le T_1$, *we have*

$$J_\rho(v(t_1)) \le J_\rho(v(t_0)). \tag{1-2}$$

Proof. To see that (1-1) holds, it is sufficient to integrate (0-4) on M. Differentiating $J_\rho(v(t))$ with respect to t and integrating by parts, one finds, for all $t \in [0, T_1]$,

$$\frac{\partial}{\partial t} J_\rho(v(t)) = -\int_M \Bigl(\frac{\partial v(t)}{\partial t}\Bigr)^2 e^{v(t)}\,dV \le 0. \tag{1-3}$$

This implies (1-2). □

Proof of Theorem 0.1. To prove the global existence of the flow, we set

$$T = \sup\bigl\{\overline{T} > 0 : C^{2+\alpha,1+\alpha/2}(M \times [0, \overline{T}]) \text{ contains a solution } v \text{ of (0-4)}\bigr\},$$

and suppose that $T < +\infty$. From the definition of T, we must have a solution $v \in C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(M \times [0, T))$. We show that there exists a constant $\widetilde{C}_T > 0$, depending on T, M, Q, ρ, α and $\|v_0\|_{C^{2+\alpha}(M)}$, such that

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(M\times[0,T))} \le \widetilde{C}_T. \tag{1-4}$$

This estimate allows us to extend v beyond T, contradicting the definition of T.

In the following, C denotes constants depending on M, Q, ρ, α and $\|v_0\|_{C^{2+\alpha}(M)}$, while C_T represents constants depending on M, Q, ρ, α, $\|v_0\|_{C^{2+\alpha}(M)}$ and T. They are allowed to vary from line to line.

Proposition 1.2. *For all* $\rho \in \mathbb{R}$, *there exists a constant* $\widetilde{C}_{T,1}$ *depending on* M, Q, ρ, $\|v_0\|_{C^{2+\alpha}(M)}$ *and* T, *such that*

$$\|v(t)\|_{H^1(M)} \le \widetilde{C}_{T,1}, \quad \forall t \in [0, T). \tag{1-5}$$

Moreover, if $\rho < 8\pi$, *then there exists a constant* $\widetilde{C}_1$ *depending on* M, Q, ρ *and* $\|v_0\|_{C^{2+\alpha}(M)}$ *but not on* T *such that*

$$\|v(t)\|_{H^1(M)} \le \widetilde{C}_1, \quad \forall t \in [0, T). \tag{1-6}$$

Proof. We decompose the proof into three steps.

Step 1. Let $\rho \geq 8\pi$. There exists a constant C'_T, depending on M, Q, ρ, $\|v_0\|_{C^{2+\alpha}(M)}$ and T, such that

$$v(x,t) \leq C'_T, \quad \forall x \in M, \quad \forall t \in [0,T). \tag{1-7}$$

Proof of Step 1. Define $v_{\max}(t) = \max_{x \in M} v(x,t) = v(x_t,t)$ where $x_t \in M$. Consider the upper derivative of $v_{\max}(t)$, i.e.,

$$\frac{\partial}{\partial t} v_{\max}(t) = \limsup_{h \to 0^+} \frac{v(x_{t+h}, t+h) - v(x_t,t)}{h}. \tag{1-8}$$

We can assume that $v_{\max}(t)$ is differentiable. By the maximum principle, and since v satisfies (0-4), we find

$$\frac{\partial}{\partial t} e^{v_{\max}(t)} \leq \frac{\rho}{\int_M e^{v_0}\, dV} \left(\|Q\|_{L^\infty(M)} \frac{\int_M e^{v_0}\, dV}{\rho} + e^{v_{\max}(t)} \right),$$

where we use the fact that $\int_M e^{v(t)}\, dV = \int_M e^{v_0}\, dV$ for all $t \in [0,T)$. Integrating this last inequality between 0 and t, we get

$$e^{v_{\max}(t)} + \|Q\|_{L^\infty(M)} \frac{\int_M e^{v_0}\, dV}{\rho} \leq \left(e^{v_{\max}(0)} + \|Q\|_{L^\infty(M)} \frac{\int_M e^{v_0}\, dV}{\rho} \right) e^{\frac{\rho t}{\int_M e^{v_0}\, dV}},$$

and (1-7) follows.

Step 2. Let $\rho \geq 8\pi$. There exists a subset A of M, with volume satisfying $|A| > C_T$ for some constant $C_T > 0$, and a constant δ depending on M, Q, ρ, $\|v_0\|_{C^{2+\alpha}(M)}$ and T, such that

$$|v(x,t)| \leq \delta, \quad \forall x \in A, \quad \forall t \in [0,T). \tag{1-9}$$

Proof of Step 2. Fix $t \in [0,T)$ and set

$$M_\varepsilon = \{x \in M : e^{v(x,t)} < \varepsilon\},$$

where $\varepsilon > 0$ is a real number which will be determined later. Setting $\int_M e^{v_0}\, dV = a$, by the conservation of the volume and (1-7), we have

$$a = \int_M e^{v(t)}\, dV = \int_{M_\varepsilon} e^{v(t)}\, dV + \int_{M \setminus M_\varepsilon} e^{v(t)}\, dV \leq \varepsilon |M_\varepsilon| + e^{C'_T} |M \setminus M_\varepsilon|.$$

Taking $\varepsilon = \dfrac{a}{2|M|}$, we find

$$|M \setminus M_\varepsilon| \geq \frac{a}{2} e^{-C'_T} > 0. \tag{1-10}$$

Setting $A = M \setminus M_\varepsilon$, by definition of M_ε we have $v(x,t) \geq \ln \varepsilon = \ln(a/2|M|)$ for all $x \in A$ and $t \in [0,T)$. On the other hand, by Step 1, $v(x,t) \leq C'_T$ for all $x \in M$ and $t \in [0,T)$. Therefore we find that there exists a constant δ such that

$$|v(x,t)| \leq \delta, \quad \forall x \in A, \quad \forall t \in [0,T).$$

Step 3. Let $\rho \geq 8\pi$. For all $t \in [0, T)$, we have

$$\int_M v^2(t)\,dV \leq C_1 \int_M |\nabla v(t)|^2\,dV + C_2, \tag{1-11}$$

where C_1, C_2 are constants depending on T, Q, $|v_0\|_{C^{2+\alpha}(M)}$, M and A (where A is the set defined in Step 2).

Proof of Step 3. By Poincaré's inequality,

$$\int_M v^2(t)\,dV \leq \frac{1}{\lambda_1}\int_M |\nabla v(t)|^2\,dV + \frac{1}{|M|}\left(\int_M v(t)\,dV\right)^2, \tag{1-12}$$

where λ_1 is the first eigenvalue of the Laplacian. Now, using Young's inequality and (1-9), we find

$$\begin{aligned} \frac{1}{|M|}\left(\int_M v(t)\,dV\right)^2 &= \frac{1}{|M|}\left(\int_A v(t)\,dV\right)^2 \\ &\qquad + \frac{1}{|M|}\left(\int_{M\setminus A} v(t)\,dV\right)^2 + \frac{2}{|M|}\left(\int_A v(t)\,dV\right)\left(\int_{M\setminus A} v(t)\,dV\right) \\ &\leq \frac{\delta^2|A|^2}{|M|} + \frac{1}{|M|}\left(\int_{M\setminus A} v(t)\,dV\right)^2 + \frac{2\delta^2|A|^2}{\varepsilon|M|} + \frac{2\varepsilon}{|M|}\left(\int_{M\setminus A} v(t)\,dV\right)^2, \end{aligned} \tag{1-13}$$

where ε is a positive constant which will be determined later. By the Cauchy–Schwarz inequality,

$$\left(\int_{M\setminus A} v(t)\,dV\right)^2 \leq |M\setminus A| \int_{M\setminus A} v^2(t)\,dV. \tag{1-14}$$

Thus, (1-12), (1-13) and (1-14) yield

$$\begin{aligned} \int_M v^2(t)\,dV \leq \frac{1}{\lambda_1}\int_M |\nabla v(t)|^2\,dV + \left(1 - \frac{|A|}{|M|} + \frac{2\varepsilon}{|M|}|M\setminus A|\right)\int_M v^2(t)\,dV + \widetilde{C}, \end{aligned} \tag{1-15}$$

where

$$\widetilde{C} = \frac{\delta^2|A|^2}{|M|} + \frac{2\delta^2|A|^2}{\varepsilon|M|}.$$

Choosing ε such that the factor in parentheses in (1-15) equals $\alpha < 1$, we deduce

$$(1-\alpha)\int_M v^2(t)\,dV \leq \frac{1}{\lambda_1}\int_M |\nabla v(t)|^2\,dV + \widetilde{C},$$

establishing (1-11).

Proof of Proposition 1.2. We consider separately the cases $\rho < 8\pi$ and $\rho \geq 8\pi$.

In the first case, we prove that the constant $\widetilde{C}_1$ of estimate (1-6) is independent of T. Using Poincaré's and Young's inequalities, we have

$$C\int_M |v(t)-\bar{v}(t)|\,dV \leq \varepsilon\int_M |\nabla v(t)|^2\,dV + C,$$

where $\varepsilon > 0$ is a small constant to be chosen later. This implies that

$$\begin{aligned}(1\text{-}16)\quad J_\rho(v(t)) &= \\ &\frac{1}{2}\int_M |\nabla v(t)|^2\,dV + \int_M Q(v(t)-\bar{v}(t))\,dV - \rho\log\Big(\int_M e^{v(t)-\bar{v}(t)}\,dV\Big) \\ &\geq \Big(\frac{1}{2}-\varepsilon\Big)\int_M |\nabla v(t)|^2\,dV - C - \rho\log\Big(\int_M e^{v(t)-\bar{v}(t)}\,dV\Big).\end{aligned}$$

By Jensen's inequality, we have

$$\log\Big(\int_M e^{v(t)-\bar{v}(t)}\,dV\Big) \geq C, \quad \forall t \in [0,T), \tag{1-17}$$

where $\bar{v}(t) = (\int_M v(t)\,dV)/|M|$. Hence, using (1-16) and (1-17), and setting $\rho_1 = \max\{\rho, 0\}$, we deduce that

$$J_\rho(v(t)) \geq \Big(\frac{1}{2}-\varepsilon\Big)\int_M |\nabla v(t)|^2\,dV - C - \rho_1\log\Big(\int_M e^{v(t)-\bar{v}(t)}\,dV\Big).$$

By the Moser–Trudinger inequality (see [Moser 1970/71; Trudinger 1967]), one has

$$\log\int_M e^{(v(t)-\bar{v}(t))}\,dV \leq \frac{1}{16\pi}\int_M |\nabla v(t)|^2\,dV + C. \tag{1-18}$$

Therefore

$$J_\rho(v(t)) \geq \Big(\frac{1}{2}-\frac{\rho_1}{16\pi}-\varepsilon\Big)\int_M |\nabla v(t)|^2\,dV - C.$$

Thus, by taking $\varepsilon = (8\pi-\rho_1)/32\pi$ and using the fact that $J_\rho(v(t)) \leq J_\rho(v_0)$ for all $t \in [0,T)$, we find that

$$\int_M |\nabla v(t)|^2\,dV \leq C, \quad \forall \rho < 8\pi. \tag{1-19}$$

Now, using (1-19) and Poincaré's inequality, we obtain

$$\|v(t)-\bar{v}(t)\|_{H^1(M)} \leq C, \quad \forall \rho < 8\pi. \tag{1-20}$$

Since $\int_M e^{v(t)}\,dV = \int_M e^{v_0}\,dV$ for all $t \in [0,T)$, using Jensen's inequality (1-17), the Moser–Trudinger inequality (1-18) and (1-19), we deduce that

$$|\bar{v}(t)| \leq C.$$

Finally, from (1-20) and the previous inequality, we find that for all $\rho < 8\pi$, there exists a constant $\widetilde{C}_1$ independent of T such that

$$\|v(t)\|_{H^1(M)} \le \widetilde{C}_1, \quad \forall\, 0 \le t < T. \tag{1-21}$$

We now consider the second case, $\rho \ge 8\pi$. Since $\int_M e^{v(t)}\,dV = \int_M e^{v_0}$ for all $t \in [0, T)$, by Young's inequality we have

$$\begin{aligned} J_\rho(v_0) \ge J_\rho(v) &\ge \frac{1}{2}\int_M |\nabla v|^2\,dV + \int_M Qv\,dV - C \\ &\ge \frac{1}{2}\int_M |\nabla v|^2\,dV - \varepsilon\int_M v^2(t)\,dV - C, \end{aligned} \tag{1-22}$$

where ε is a positive constant which will be chosen later. Thanks to estimate (1-11) of Step 3, inequality (1-22) leads to

$$\frac{1}{2}\int_M |\nabla v(t)|^2\,dV \le C + C_1\varepsilon\int_M |\nabla v(t)|^2\,dV.$$

Choosing ε such that $1/2 - \varepsilon C_1 > 0$, we find that for all $t \in [0, T)$, there exists a constant $C_T > 0$ such that

$$\int_M |\nabla v(t)|^2\,dV \le C_T. \tag{1-23}$$

Combining (1-11) and (1-23), we obtain $\int_M v^2(t)\,dV \le C_T$. Finally, for all $\rho \in \mathbb{R}$, there exists a constant $\widetilde{C}_{T,1} > 0$ such that

$$\|v(t)\|_{H^1(M)} \le \widetilde{C}_{T,1}, \quad \forall t \in [0, T). \qquad \square$$

Proposition 1.3. *There is a constant* $\widetilde{C}_{T,2} > 0$, *depending on* M, Q, ρ, $\|v_0\|_{C^{2+\alpha}(M)}$ *and* T, *such that*

$$\|v(t)\|_{H^2(M)} \le \widetilde{C}_{T,2}, \quad \forall\, 0 \le t < T.$$

Proof. Since $\|v\|_{H^1(M)} \le \widetilde{C}_{T,1}$, we just need to bound $\int_M (\Delta v(t))^2\,dV$ for all $t \in [0, T)$. To this purpose, set

$$w(t) = \frac{\partial v(t)}{\partial t} e^{v(t)/2}.$$

By differentiating with respect to t and integrating by parts on M, we have

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\int_M (\Delta v(t))^2\,dV &= \int_M \Big(w(t)e^{v(t)/2} + Q - \frac{\rho e^{v(t)}}{\int_M e^{v_0}\,dV}\Big)\Delta\big(w(t)e^{-v(t)/2}\big)\,dV \\ &= -\int_M |\nabla w(t)|^2\,dV + \frac{1}{4}\int_M w^2(t)|\nabla v(t)|^2\,dV + \int_M \Delta Q\big(w(t)e^{-v(t)/2}\big)\,dV \\ &\quad + \frac{\rho}{\int_M e^{v_0}\,dV}\Big(\int_M \nabla v(t)\nabla w(t)e^{v(t)/2}\,dV - \frac{1}{2}\int_M w(t)e^{v(t)/2}|\nabla v(t)|^2\,dV\Big). \end{aligned}$$

Since $Q \in C^\infty(M)$ and $w(t) = \dfrac{\partial v(t)}{\partial t} e^{v(t)/2}$, we find

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_M (\Delta v(t))^2\,dV &\\ \le -\int_M |\nabla w(t)|^2\,dV + \frac{1}{4}\int_M w^2(t)|\nabla v(t)|^2\,dV + C\Big\|\frac{\partial v(t)}{\partial t}\Big\|_{L^1(M)} &\\ + C\Big(\int_M e^{v(t)/2}\big(|\nabla w(t)||\nabla v(t)| + |\nabla v(t)|^2|w(t)|\big)\,dV\Big).&\end{aligned} \tag{1-24}$$

We now estimate the positive terms on the right of (1-24). From the Gagliardo–Nirenberg inequality (see for example [Brouttelande 2003]), for all $f \in H^1(M)$,

$$\|f\|^2_{L^4(M)} \le C\|f\|_{L^2(M)}\|f\|_{H^1(M)}.$$

Using the Cauchy–Schwarz inequality and (1-5), we have

$$\begin{aligned}\int_M w^2(t)|\nabla v(t)|^2\,dV &\le \|w(t)\|^2_{L^4(M)}\|\nabla v(t)\|^2_{L^4(M)}\\ &\le C_T\|w(t)\|_{L^2(M)}\|w(t)\|_{H^1(M)}\|v(t)\|_{H^2(M)}.\end{aligned} \tag{1-25}$$

Using (1-5) and the Moser–Trudinger inequality (1-18), we deduce that there exists a constant C_T such that, for all $t \in [0, T)$ and $p \in \mathbb{R}$,

$$\int_M e^{pv(t)}\,dV \le C_T. \tag{1-26}$$

By the same reasoning used to prove (1-25), from (1-5) and (1-26) we have

$$\begin{aligned}\int_M |\nabla v(t)|^2|w(t)|e^{v(t)/2}\,dV &\\ \le \Big(\int_M |\nabla v(t)|^4\,dV\Big)^{1/2}\Big(\int_M w^4(t)\,dV\Big)^{1/4}\Big(\int_M e^{2v(t)}\,dV\Big)^{1/4} &\\ \le C_T\|v(t)\|_{H^2(M)}\|w(t)\|^{1/2}_{H^1(M)}\|w(t)\|^{1/2}_{L^2(M)},&\end{aligned} \tag{1-27}$$

$$\begin{aligned}\int_M |\nabla w(t)||\nabla v(t)|e^{v(t)/2}\,dV &\\ \le \Big(\int_M |\nabla w(t)|^2\,dV\Big)^{1/2}\Big(\int_M |\nabla v(t)|^4\,dV\Big)^{1/4}\Big(\int_M e^{2v(t)}\,dV\Big)^{1/4} &\\ \le C_T\|w(t)\|_{H^1(M)}\|v(t)\|^{1/2}_{H^2(M)},&\end{aligned} \tag{1-28}$$

$$\begin{aligned}\int_M \Big|\frac{\partial v(t)}{\partial t}\Big|\,dV &\le \Big(\int_M \Big(\frac{\partial v(t)}{\partial t}\Big)^2 e^{v(t)}\,dV\Big)^{1/2}\Big(\int_M e^{-v(t)}\,dV\Big)^{1/2}\\ &\le C_T\|w(t)\|_{L^2(M)}.\end{aligned} \tag{1-29}$$

Finally, putting (1-25), (1-27), (1-28) and (1-29) in (1-24), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_M(\Delta v(t))^2\,dV &\le -\int_M|\nabla w(t)|^2\,dV \\ &\quad + C_T\|w(t)\|_{H^1(M)}\|w(t)\|_{L^2(M)}\|v(t)\|_{H^2(M)} + C_T\|w(t)\|_{L^2(M)} \\ &\quad + C_T\big(\|w(t)\|_{H^1(M)}\|v(t)\|^{1/2}_{H^2(M)} + \|w(t)\|^{1/2}_{H^1(M)}\|w(t)\|^{1/2}_{L^2(M)}\|v(t)\|_{H^2(M)}\big).\end{aligned}$$

Using Young's inequality, we get

$$\frac{\partial}{\partial t}\left(\int_M(\Delta v(t))^2\,dV+1\right) \le C_T\left(\int_M(\Delta v(t))^2\,dV+1\right)\big(\|w(t)\|^2_{L^2(M)}+1\big). \tag{1-30}$$

On the other hand, by (1-3), we have for all $t\in[0,T)$

$$\begin{aligned}\int_0^t\|w(s)\|^2_{L^2(M)}\,ds &= \int_0^t\int_M\Big(\frac{\partial v(s)}{\partial s}\Big)^2 e^{v(s)}\,dV\,ds \\ &= -\int_0^t\frac{\partial}{\partial s}J_\rho(v(s))\,ds = J_\rho(v_0)-J_\rho(v(t)) \le C_T,\end{aligned} \tag{1-31}$$

where we use the fact that $\|v(t)\|_{H^1(M)} \le \widetilde{C}_{T,1}$ from Proposition 1.2. Integrating (1-30) with respect to t and using (1-31), we have

$$\int_M(\Delta v(t))^2\,dV \le C_T, \quad \forall t\in[0,T).$$

Since $\|v(t)\|_{H^1(M)} \le \widetilde{C}_{T,1}$, we deduce that there exists a constant $\widetilde{C}_{T,2}$ such that

$$\|v(t)\|_{H^2(M)} \le \widetilde{C}_{T,2}, \quad \forall t\in[0,T). \qquad \square$$

Proof of Theorem 0.1. We recall that to prove the global existence of the flow it is sufficient to prove (1-4), i.e., there exists a constant $\widetilde{C}_T$ depending on T and $\alpha\in(0,1)$ such that

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(M\times[0,T))} \le \widetilde{C}_T.$$

First, we claim that for all $\alpha\in(0,1)$, there exists a constant C_T such that

$$|v(x_1,t_1)-v(x_2,t_2)| \le C_T(|t_1-t_2|^{\alpha/2}+|x_1-x_2|^\alpha), \tag{1-32}$$

for all $x_1,x_2\in M$ and $t_1,t_2\in[0,T)$. Here $|x_1-x_2|$ stands for the geodesic distance from x_1 to x_2 with respect to the metric g. From Proposition 1.3, for all $t\in[0,T)$ we have $\|v(t)\|_{H^2(M)} \le \widetilde{C}_{T,2}$. Thus, by Sobolev's embedding theorem (see [Hebey 1997]), we find for $\alpha\in(0,1)$, $v(t)\in C^\alpha(M)$ and for all $x,y\in M$,

$$|v(x,t)-v(y,t)| \le C_T|x-y|^\alpha. \tag{1-33}$$

If $t_2 - t_1 \geq 1$, using (1-33) it is easy to see that (1-32) holds. Thus, from now on we assume that $0 < t_2 - t_1 < 1$. Since $v(t)$ is a solution of (0-4) and $\|e^{v(t)}\|_{C^\alpha(M)} \leq C_T$, for all $t \in [0, T)$ one has

$$\left|\frac{\partial v(t)}{\partial t}\right|^2 \leq C_T |\Delta v(t)|^2 + C_T.$$

Integrating on M, we obtain for all $t \in [0, T)$

$$\int_M \left|\frac{\partial v(t)}{\partial t}\right|^2 dV \leq C_T \|v(t)\|^2_{H^2(M)} + C_T \leq C_T. \tag{1-34}$$

Now, we write

$$\begin{aligned} |v(x,t_1) - v(x,t_2)| &= \frac{1}{|B_{\sqrt{t_2-t_1}}(x)|} \int_{B_{\sqrt{t_2-t_1}}(x)} |v(x,t_1) - v(x,t_2)|\, dV(y)\,, \\ &\leq P_1 + P_2 + P_3 \end{aligned} \tag{1-35}$$

where $B_{\sqrt{t_2-t_1}}(x)$ stands for the geodesic ball of center x and radius $\sqrt{t_2 - t_1}$ and

$$\begin{aligned} P_1 &= \frac{C}{t_2-t_1} \int_{B_{\sqrt{t_2-t_1}}(x)} |v(x,t_1) - v(y,t_1)|\, dV(y), \\ P_2 &= \frac{C}{t_2-t_1} \int_{B_{\sqrt{t_2-t_1}}(x)} |v(y,t_1) - v(y,t_2)|\, dV(y), \\ P_3 &= \frac{C}{t_2-t_1} \int_{B_{\sqrt{t_2-t_1}}(x)} |v(y,t_2) - v(x,t_2)|\, dV(y), \end{aligned}$$

Using (1-33), we obtain

$$P_1 \;\leq\; \frac{C_T}{t_2-t_1} \int_{B_{\sqrt{t_2-t_1}}(x)} |x-y|^\alpha\, dV(y) \;\leq\; C_T (t_2-t_1)^{\alpha/2}. \tag{1-36}$$

In the same way, we have

$$P_3 \leq C_T (t_2 - t_1)^{\alpha/2}. \tag{1-37}$$

From Hölder's inequality and (1-34) it follows that

$$\begin{aligned} P_2 &\leq C \sup_{t_1 \leq \tau \leq t_2} \int_{B_{\sqrt{t_2-t_1}}(x)} \left|\frac{\partial v}{\partial s}\right|(y,\tau)\, dV(y) \\ &\leq C\sqrt{t_2-t_1} \sup_{t_1 \leq \tau \leq t_2} \left(\int_{B_{\sqrt{t_2-t_1}}(x)} \left|\frac{\partial v}{\partial s}\right|^2 (y,\tau)\, dV(y) \right)^{1/2} \\ &\leq C_T \sqrt{t_2 - t_1}. \end{aligned} \tag{1-38}$$

Putting (1-36), (1-37) and (1-38) in (1-35), and noticing that $\sqrt{t_2 - t_1} \leq (t_2 - t_1)^{\alpha/2}$

for all $0 < t_2 - t_1 < 1$, we find

$$|v(x, t_1) - v(x, t_2)| \le C_T (t_2 - t_1)^{\alpha/2}. \tag{1-39}$$

Therefore, from (1-33) and (1-39), we see that (1-32) holds. In view of (1-32), we may apply the standard regularity theory for parabolic equations (see for example [Friedman 1964]) to derive the existence of a constant $\widetilde{C}_T$ depending on T and $\alpha \in (0, 1)$ such that

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(M\times[0,T))} \le \widetilde{C}_T.$$

This establishes the existence part of Theorem 0.1. The uniqueness follows from Proposition 1.5. □

Remark 1.4. Following the proof of Theorem 0.1, we see that, for all $T > 0$ fixed, if $\|u_0\|_{C^{2+\alpha}(M)} \le K$ for some constant $K > 0$, then there exists a constant $C_T > 0$ depending on K and T such that

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(M\times[0,T])} \le C_T.$$

Continuity of the flow with respect to its initial data. We now state the continuity of the flow with respect to its initial data, which will be useful for the proof of Theorem 0.3 (see Section 2). The proof is standard and we omit it.

Proposition 1.5. *Let $u, v \in C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(M \times [0, +\infty))$, $\alpha \in (0, 1)$ be solutions of*

$$\begin{cases} \dfrac{\partial}{\partial t} e^v = \Delta v - Q + \rho \dfrac{e^v}{\int_M e^v \, dV}, \\ v(x, 0) = v_0(x), \end{cases}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t} e^u = \Delta u - Q + \rho \dfrac{e^u}{\int_M e^u \, dV}, \\ u(x, 0) = u_0(x), \end{cases}$$

where $u_0, v_0 \in C^{2+\alpha}(M)$. Then for all $T > 0$, there exists a constant $C_T > 0$, depending on $\|u_0\|_{C^{2+\alpha}(M)}$, $\|v_0\|_{C^{2+\alpha}(M)}$ and T, such that

$$\|u - v\|_{C^{2+\alpha,1+\alpha/2}(M\times[0,T])} \le C_T \|u_0 - v_0\|_{C^{2+\alpha}(M)}. \tag{1-40}$$

Remark 1.6. One can also prove that, for all $T > 0$ fixed, if $\|u_0\|_{C^{2+\alpha}(M)} \le K_1$ and $\|v_0\|_{C^{2+\alpha}(M)} \le K_2$ for some constants $K_1, K_2 > 0$, then there exists a constant $C_T > 0$ depending on K_1, K_2 and T such that

$$\|u - v\|_{C^{2+\alpha,1+\alpha/2}(M\times[0,T])} \le C_T \|u_0 - v_0\|_{C^{2+\alpha}(M)}.$$

2. Convergence of the flow

This section is devoted to the proof of Theorems 0.2, 0.3 and 0.4.

Proof of Theorem 0.2. Let $v : M \times [0, +\infty) \to \mathbb{R}$ be the global solution of (0-4). Throughout this subsection, we assume without loss of generality that $\int_M e^{v(t)}\, dV = 1$ for all $t \geq 0$. C will denote constants not depending on t.

In order to prove Theorem 0.2, we need to bound $\|v(t)\|_{H^2(M)}$, $t \geq 0$ uniformly in time. For this, we first bound $\|v(t)\|_{H^1(M)}$, $t \geq 0$ uniformly in time. To bound $\|v(t)\|_{H^1(M)}$, we use the compactness result of Theorem 0.5. More precisely, using Theorem 0.5, we first prove that there exists a sequence $(t_n)_n$ with $\lim_{n\to+\infty} t_n = +\infty$ such that

$$\|v(t_n)\|_{H^2(M)} \leq C, \quad \forall n \geq 0.$$

Therefore we aim to prove that there exists a sequence $(t_n)_n$, $\lim_{n\to+\infty} t_n = +\infty$, such that, setting $v_n = v(t_n)$ and $h_n = -(\partial v/\partial t)(t_n)$, the sequence $(v_n)_n \subseteq H^2(M)$ satisfies conditions (0-8) of Theorem 0.5. First, we show that there exists a sequence $(t_n)_n$, $\lim_{n\to+\infty} t_n = +\infty$, such that (0-8)(i) is satisfied for $v_n = v(t_n)$. Recall that for all $T > 0$,

$$\int_0^T \int_M \Big(\frac{\partial v(t)}{\partial t}\Big)^2 e^{v(t)}\, dV\, dt = J_\rho(v(0)) - J_\rho(v(T)).$$

Using hypothesis (2-5), we deduce that there exists a sequence $(t_n)_n$ such that $n \leq t_n \leq n+1$, for all $n \in \mathbb{N}$, and

$$\lim_{n\to+\infty} \int_M \Big|\frac{\partial v(t_n)}{\partial t}\Big|^2 e^{v(t_n)}\, dV = 0. \tag{2-1}$$

The next proposition shows that condition (0-8)(ii) of Theorem 0.5 is satisfied.

Proposition 2.1. *We have*

$$-\frac{\partial e^{v(x,t)}}{\partial t} + \rho e^{v(x,t)} \geq -C, \quad \forall t \geq 0, \quad \forall x \in M. \tag{2-2}$$

Proof. Set

$$R(x,t) = e^{-v(x,t)}(-\Delta v(x,t) + Q(x)).$$

We can rewrite equation (0-4), satisfied by v, in the form

$$\frac{\partial v(x,t)}{\partial t} = -(R(x,t) - \rho).$$

Hence

$$\frac{\partial R(x,t)}{\partial t} = R(x,t)(R(x,t) - \rho) + e^{-v(x,t)} \Delta R(x,t).$$

Define $R_{\min}(t) = \min_{x\in M} R(x,t)$. Using the maximum principle (as in (1-8), we may assume that $R_{\min}(t)$ is differentiable), we find

$$\frac{\partial R_{\min}(t)}{\partial t} \geq -\rho R_{\min}(t).$$

Integrate between 0 and t to obtain

$$R_{\min}(t) \geq e^{-\rho t} R_{\min}(0).$$

This implies that

$$-\frac{\partial e^{v(x,t)}}{\partial t} + \rho e^{v(x,t)} \geq -|R_{\min}(0)| e^{-\rho t + v(x,t)}. \tag{2-3}$$

Set $v_{\max}(t) = \max_{x \in M} v(x,t)$. By the maximum principle, we have

$$\frac{\partial}{\partial t} e^{v_{\max}(t)} \leq \rho \Big(\frac{1}{\rho} \|Q\|_{L^\infty(M)} + e^{v_{\max}(t)} \Big).$$

Integrating again between 0 and t, we get

$$e^{v_{\max}(t) - \rho t} \leq e^{v_{\max}(0)} + \frac{1}{\rho} \|Q\|_{L^\infty(M)} - \frac{1}{\rho} \|Q\|_{L^\infty(M)} e^{-\rho t} \leq C. \tag{2-4}$$

Combining (2-3) and (2-4), we finally conclude

$$-\frac{\partial e^{v(x,t)}}{\partial t} + \rho e^{v(x,t)} \geq -C |R_{\min}(0)| \geq -C. \qquad \square$$

We are now in position to bound $\|v(t)\|_{H^1(M)}$, $t \geq 0$, uniformly in time.

Proposition 2.2. *Let $\rho \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}^*$ and $v(t) : M \to \mathbb{R}$ be the solution of (0-4). Suppose that*

$$J_\rho(v(t)) \geq -C, \quad \forall t \geq 0. \tag{2-5}$$

Then there exists a constant $\widetilde{C}$, depending on M, Q, ρ, α and $\|v_0\|_{C^{2+\alpha}(M)}$ but not on T, such that

$$\|v(t)\|_{H^1(M)} \leq \widetilde{C}, \quad \forall t \geq 0. \tag{2-6}$$

Proof. Thanks to (2-1) and (2-2), from Theorem 0.5 there exists a constant $C > 0$ such that

$$\|v(t_n)\|_{H^2(M)} \leq C,$$

where $(t_n)_n$ is the sequence defined in (2-1). By Sobolev's embedding theorem, it follows that $\|v(t_n)\|_{C^\alpha(M)} \leq C$ for all $\alpha \in (0,1)$. Since $\lim_{n \to +\infty} t_n = +\infty$, for all sufficiently large $t \geq 0$ there exists $n \in \mathbb{N}$ such that $t_n \leq t \leq t_{n+1}$. Moreover, since $|t_{n+1} - t_n| \leq 2$, we have $|t - t_n| \leq 2$. We claim that for all $p > 1$,

$$\int_M e^{p v(t)} \, dV \leq C, \quad \forall t \geq 0. \tag{2-7}$$

Since $v(t)$ satisfies (0-4), integrating by parts and using Young's inequality, we see that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_M e^{pv(t)}\,dV\\
&\quad = -p(p-1)\int_M |\nabla v(t)|^2 e^{(p-1)v(t)}\,dV - p\int_M Qe^{(p-1)v(t)}\,dV + p\frac{\rho}{a}\int_M e^{pv(t)}\,dV\\
&\quad \leq C\int_M e^{(p-1)v(t)}\,dV + p\frac{\rho}{a}\int_M e^{pv(t)}\,dV\\
&\quad \leq C + C\int_M e^{pv(t)}\,dV.
\end{aligned}
$$

Setting $y(t) = \int_M e^{pv(t)}\,dV$ and integrating the previous inequality between t_n and t, it follows that

$$y(t) \leq e^{C(t-t_n)}y(t_n) + C\big(e^{C(t-t_n)} - 1\big).$$

Since $\|v(t_n)\|_{C^\alpha(M)} \leq C$, $\alpha \in (0,1)$, and $|t - t_n| \leq 2$, we have that (2-7) is satisfied.

Fix $t \geq 0$ and set

$$M_\varepsilon = \{x \in M : e^{v(x,t)} < \varepsilon\},$$

where $\varepsilon > 0$ is a real number which will be determined shortly. We have

$$
\begin{aligned}
1 = \int_M e^{v(t)}\,dV &= \int_{M_\varepsilon} e^{v(t)}\,dV + \int_{M\setminus M_\varepsilon} e^{v(t)}\,dV\\
&\leq \varepsilon|M_\varepsilon| + |M\setminus M_\varepsilon|^{1-1/p}\bigg(\int_M e^{pv(t)}\,dV\bigg)^{1/p}.
\end{aligned}
\tag{2-8}
$$

Thus, taking $\varepsilon = \dfrac{1}{2|M|}$, (2-7) implies

$$\tfrac{1}{2} \leq C|M\setminus M_\varepsilon|^{1-1/p}.$$

Since $p > 1$, we get

$$|M\setminus M_\varepsilon| \geq \Big(\frac{1}{2C}\Big)^{p/(p-1)} > 0. \tag{2-9}$$

Set $A = M\setminus M_\varepsilon$, so that

$$\int_A v(t)\,dV \geq \ln\Big(\frac{1}{2|M|}\Big)|A|. \tag{2-10}$$

On the other hand, we have

$$\int_A v(t)\,dV \leq \int_A e^{v(t)}\,dV \leq 1.$$

From this inequality and (2-10), we deduce that there exists a constant C such that

$$\left|\int_A v(t)\,dV\right| \le C. \tag{2-11}$$

Arguing the same way as in Proposition 1.2, (2-9) and (2-11) imply that there exists a constant $\widetilde{C}$ not depending on t such that, for all $t \ge 0$,

$$\|v(t)\|_{H^1(M)} \le \widetilde{C}. \qquad \square$$

Proof of Theorem 0.2. First, we prove that

$$\int_M (\Delta v(t))^2\,dV \le C, \quad \forall t \ge 0 \tag{2-12}$$

following the arguments of Brendle [2003]. Set

$$V(t) = \frac{\partial v(t)}{\partial t}$$

and

$$y(t) = \int_M V^2(t) e^{v(t)}\,dV.$$

We claim that $\lim_{t\to+\infty} y(t) = 0$. By (2-6), we have for all $T \ge 0$

$$\int_0^T \int_M \left(\frac{\partial v(t)}{\partial t}\right)^2 e^{v(t)}\,dV dt \le J_\rho(v(0)) - J_\rho(v(T)) \le C, \tag{2-13}$$

where C is a constant not depending on T. Let ε be some positive real number. From (2-13), we deduce that there exists $t_0 \ge 0$ such that $y(t_0) \le \varepsilon$.

We want to prove that

$$y(t) \le 3\varepsilon, \quad \forall t \ge t_0.$$

Otherwise, define

$$t_1 = \inf\{t \ge t_0 : y(t) \ge 3\varepsilon\} < +\infty.$$

This implies that

$$y(t) \le 3\varepsilon, \quad \forall t_0 \le t \le t_1. \tag{2-14}$$

Since $\dfrac{\partial v(t)}{\partial t} = e^{-v(t)}(\Delta v(t) - Q) + \rho$, using (2-14) we arrive at

$$\int_M e^{-v(t)}(\Delta v(t) - Q)^2\,dV = y(t) + \rho^2 \le C_1, \quad \forall t_0 \le t \le t_1, \tag{2-15}$$

where C_1 denotes a constant depending on ε, and thus on t_1. From (2-7), we have for all $t \ge 0$

$$\int_M e^{3v(t)}\,dV \le C, \tag{2-16}$$

with C independent of t_1. Using Hölder's inequality, (2-15) and (2-16), we obtain for all $t_0 \le t \le t_1$

$$\int_M |\Delta v(t) - Q|^{3/2}\, dV \le \left(\int_M e^{-v(t)}(\Delta v(t) - Q)^2\, dV\right)^{3/4} \left(\int_M e^{3v(t)}\, dV\right)^{1/4} \le C_1.$$

Thus, $\int_M |\Delta v(t)|^{3/2}\, dV \le C_1$ for all $t_0 \le t \le t_1$. From Sobolev's embedding theorem, we get

$$|v(t)| \le C_1, \quad \forall t_0 \le t \le t_1. \tag{2-17}$$

On the other hand, we see that $V(t) = \partial v(t)/\partial t$ satisfies

$$\frac{\partial V(t)}{\partial t} = -V(t)e^{-v(t)}\Delta v(t) + e^{-v(t)}\Delta V(t) + QV(t)e^{-v(t)}. \tag{2-18}$$

Now, using (2-18), we have for all $t_0 \le t \le t_1$

$$\begin{aligned}\frac{\partial y(t)}{\partial t} &= \frac{\partial}{\partial t}\left(\int_M V^2(t)e^{v(t)}\, dV\right)\\ &= 2\int_M V(t)e^{v(t)}\big(e^{-v(t)}\Delta V(t) - V(t)e^{-v(t)}\Delta v(t) + QV(t)e^{-v(t)}\big)\, dV\\ &\qquad + \int_M V^3(t)e^{v(t)}\, dV\end{aligned}$$

Integrating by parts, we obtain

$$\frac{\partial y(t)}{\partial t} = -2\int_M |\nabla V(t)|^2\, dV - \int_M V^3(t)e^{v(t)}\, dV + 2\rho\int_M V^2(t)e^{v(t)}\, dV. \tag{2-19}$$

The Gagliardo–Nirenberg inequality now gives

$$\|V(t)\|_{L^3_{g_1}(M)} \le C\|V(t)\|^{2/3}_{L^2_{g_1}(M)}\|V(t)\|^{1/3}_{H^1_{g_1}(M)},$$

where the norms are taken with respect to the metric $g_1(t) = e^{v(t)}g$. From (2-17), notice that the first eigenvalue of the Laplacian $\tilde{\lambda}_1(t)$ with respect to the metric $g_1(t)$ satisfies, for all $t_0 \le t \le t_1$,

$$\tilde{\lambda}_1(t) \ge C_1. \tag{2-20}$$

Combining $\int_M Ve^v\, dV = 0$, Poincaré's inequality and (2-20), we have

$$\int_M e^v|V|^3\, dV \le C_1\left(\int_M V^2e^v\, dV\right)\left(\int_M |\nabla V|^2\, dV\right)^{1/2}. \tag{2-21}$$

Thus we obtain, from (2-19), (2-21) and Young's inequality,

$$\frac{\partial}{\partial t}\left(\int_M V^2e^v\, dV\right) \le C_1\left(\int_M V^2e^v\, dV\right)^2 + C\left(\int_M V^2e^v\, dV\right),$$

i.e.,

$$\frac{\partial}{\partial t} y(t) \le C_1 y^2(t) + C y(t).$$

Since $y(t_0) \le \varepsilon$ and $y(t_1) = 3\varepsilon$, we find

$$2\varepsilon \le y(t_1) - y(t_0) \le (C_1 + C) \int_{t_0}^{t_1} y(t)\, dt.$$

Choosing t_0 large enough, we have $(C_1 + C) \int_{t_0}^{+\infty} y(t)\, dt \le \varepsilon$, and thus we obtain a contradiction. We conclude that

$$y(t) \underset{t\to+\infty}{\longrightarrow} 0,$$

and thereby find $t_1 = +\infty$. This implies that all previous estimates hold for all $t \ge 0$. Thus, for all $t \ge 0$ we have $|v(t)| \le C$ and

$$\int_M e^{-v(t)} (\Delta v(t) - Q)^2\, dV \le C.$$

It follows that, for all $t \ge 0$, $\int_M (\Delta v(t))^2\, dV \le C$.

Thus, using (2-6), for all $t \ge 0$ we have $\|v(t)\|_{H^2(M)} \le C$. Therefore, there exist a function $v_\infty \in H^2(M)$ and a sequence $(t_n)_n$ with $\lim_{n\to+\infty} t_n = +\infty$ such that

$$v(t_n) \underset{n\to+\infty}{\longrightarrow} v_\infty \text{ weakly in } H^2(M)$$

and

$$v(t_n) \underset{n\to+\infty}{\longrightarrow} v_\infty \text{ in } C^\alpha(M), \alpha \in (0, 1).$$

It is easy to check that v_∞ is a solution to

$$-\Delta v_\infty + Q = \rho \frac{e^{v_\infty}}{\int_M e^{v_\infty}\, dV},$$

and, by bootstrap regularity arguments, we have $v_\infty \in C^\infty(M)$. To obtain that $\|v(t_n) - v_\infty\|_{H^2(M)} \underset{n\to+\infty}{\longrightarrow} 0$, notice that

$$\begin{aligned}
&\int_M (\Delta v(t_n) - \Delta v_\infty)^2\, dV \\
&\qquad = \int_M \left(\frac{\rho}{a}\left(e^{v_\infty} - e^{v(t_n)}\right) + \frac{\partial e^{v(t_n)}}{\partial t} \right)^2 dV \\
&\qquad \le C \int_M \left(e^{v_\infty} - e^{v(t_n)}\right)^2 dV + C \int_M \left| \frac{\partial v}{\partial t}(t_n) \right|^2 e^{v(t_n)}\, dV \underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

Since the flow is a gradient flow for the functional J_ρ, which is real analytic, from a general result of Simon [1983] we finally obtain that

$$\|v(t) - v_\infty\|_{H^2(M)} \underset{n\to+\infty}{\longrightarrow} 0. \qquad \square$$

Proof of Theorem 0.3. We prove the existence of an initial data $v_0 \in C^\infty(M)$ for the flow (0-4) such that the functional $J_\rho(v(t))$, $t \geq 0$, is uniformly bounded from below. From standard parabolic theory, it is easy to see that for $v_0 \in C^\infty(M)$, the solution v of (0-4) belongs to $C^\infty(M \times [0, +\infty))$.

Let X be the space of functions $C^\infty(M)$ endowed with the norm $\|\cdot\|_{C^{2+\alpha}(M)}$, and define

$$\Phi : X \times [0, +\infty) \longrightarrow C^\infty(M \times [0, +\infty))$$

by letting $\Phi(v, t)$ be a solution of

$$\begin{cases} \dfrac{\partial \Phi(v,t)}{\partial t} = e^{-\Phi(v,t)} \Delta \Phi(v,t) - e^{-\Phi(v,t)} Q + \dfrac{\rho}{\int_M e^{\Phi(v,t)}\, dV}, \\ \Phi(v, 0) = v. \end{cases}$$

Suppose that for all $v \in X$, we have

$$J_\rho(\Phi(v,t)) \underset{t \to +\infty}{\longrightarrow} -\infty. \tag{2-22}$$

Let $L > 0$. Following the same arguments as in [Malchiodi 2008], one can show that there exists $L_1 > 0$ such that $\{v \in X : J_\rho(v) \leq -L_1\}$ is not contractible. However, we prove that if (2-22) is satisfied then $\{v \in X : J_\rho(v) \leq -L\}$ is contractible. We proceed in two steps.

Step 1. Let $L > 0$ be fixed and

$$T_v = \inf\{t \geq 0 : J_\rho(\Phi(v,t)) \leq -L\},$$

then the function $T : C^{2+\alpha}(M) \to \mathbb{R}$, $v \mapsto T_v$ is continuous.

Proof of Step 1. From (2-22), we have

$$\{t \geq 0 : J_\rho(\Phi(v,t)) \leq -L\} \neq \varnothing,$$

and from the uniqueness of solutions of (0-4) having the same initial data, one can prove that $J_\rho(\Phi(v,t))$ is strictly decreasing on $[0, +\infty)$. Let $\bar{v} \in C^\infty(M)$ and $(v_n)_n \in C^\infty(M)$ be a sequence such that $\lim_{n\to+\infty} v_n = \bar{v}$ in $C^{2+\alpha}(M)$. We claim that $\lim_{n\to+\infty} T_{v,n} = T_{\bar{v}}$. To prove this, we consider two cases depending on the value of $J_\rho(\bar{v})$.

First case. Suppose that $J_\rho(\bar{v}) < -L$. Since the function $t \to J_\rho(\Phi(\bar{v}, t))$ is decreasing, we have $J_\rho(\Phi(\bar{v}, t)) < -L$ for all $t \geq 0$. We deduce that $T_{\bar{v}} = 0$. Since $\lim_{n\to+\infty} v_n = \bar{v}$ in $C^{2+\alpha}(M)$, it is easy to see that

$$J_\rho(v_n) \underset{n \to +\infty}{\longrightarrow} J_\rho(\bar{v}).$$

Thus, there exists $n_0 \in \mathbb{N}$ such that $J_\rho(v_n) \le -L$ for all $n \ge n_0$. So, we obtain that $T_{v_n} = 0 = T_{\bar{v}}$ for all $n \ge n_0$. This implies that

$$T_{v_n} \underset{n\to+\infty}{\longrightarrow} T_{\bar{v}}.$$

Second case. Suppose that $J_\rho(\bar{v}) \ge -L$. In this case, $T_{\bar{v}}$ verifies $J_\rho(\Phi(\bar{v}, T_{\bar{v}})) = -L$. Setting $T_n := T_{v_n}$ and supposing that T_n does not converge to $T_{\bar{v}}$, then, up to extracting a subsequence, there exists $\varepsilon_0 > 0$ such that $|T_n - T_{\bar{v}}| \ge \varepsilon_0$. So we have $T_n \ge \varepsilon_0 + T_{\bar{v}}$ or $T_n \le -\varepsilon_0 + T_{\bar{v}}$. Suppose, without loss of generality, that

$$T_n \ge \varepsilon_0 + T_{\bar{v}}. \tag{2-23}$$

Set $T = T_{\bar{v}} + \varepsilon_0 + 1$. Since $\lim_{n\to+\infty} v_n = \bar{v}$ in $C^{2+\alpha}(M)$ by Proposition 1.5, it is easy to see that

$$J_\rho(\Phi(v_n, t)) \underset{n\to+\infty}{\longrightarrow} J_\rho(\Phi(\bar{v}, t)), \tag{2-24}$$

for all t fixed in $[0, T]$. Since $t \to J_\rho(\Phi(\bar{v}, t))$ is strictly decreasing, we have

$$\alpha_1 = J_\rho(\Phi(\bar{v}, T_{\bar{v}})) - J_\rho(\Phi(\bar{v}, T_{\bar{v}} + \varepsilon_0)) > 0.$$

From (2-24), since $T_{\bar{v}} + \varepsilon_0 \in [0, T]$, we get

$$J_\rho(\Phi(v_n, T_{\bar{v}} + \varepsilon_0)) \underset{n\to+\infty}{\longrightarrow} J_\rho(\Phi(\bar{v}, T_{\bar{v}} + \varepsilon_0)) = -L - \alpha_1,$$

and from (2-23),

$$J_\rho(\Phi(v_n, T_n)) \le J_\rho(\Phi(v_n, T_{\bar{v}} + \varepsilon_0)).$$

This implies that, if n tends to $+\infty$, $-L \le -L - \alpha_1$. Thus we obtain a contradiction.

Step 2. If (2-22) holds, then the set $\{v \in X : J_\rho(v) \le -L\}$ is contractible.

Proof of Step 2. We construct a deformation retract from $\{v \in X\}$ into $\{v \in X : J_\rho(v) \le -L\}$. Since $\{v \in X\}$ is contractible, $\{v \in X : J_\rho(v) \le -L\}$ must also be contractible. We denote by h the one-to-one function defined by

$$h(t) : [0, 1) \to [0, +\infty), \quad t \mapsto \frac{t}{1-t},$$

and by $\eta(v, t) : X \times [0, 1] \to X$ the function defined by

$$\eta(v, t) = \begin{cases} \Phi(v, h(t)) & \text{if } h(t) \le T_v, \\ \Phi(v, T_v) & \text{if } h(t) \ge T_v. \end{cases}$$

First we prove that $\eta = \Phi \circ \Phi_1 : X \times [0, 1) \to X$ is continuous, in which $\Phi_1 : X \times [0, 1) \to X \times [0, +\infty)$ is the function defined by

$$\Phi_1(v,t)=\begin{cases}(v,h(t)) & \text{if } h(t)\le T_v,\\ (v,T_v) & \text{if } h(t)\ge T_v.\end{cases}$$

From Step 1, $\Phi_1 : X \times [0,1) \to X \times [0,+\infty)$ is a continuous function. Therefore, to prove that η is a continuous map from $X \times [0,1) \to X$, it is sufficient to prove that, for $T>0$ fixed, $\Phi : X \times [0,T] \to X$ is continuous.

Let $(v_n,t_n) \in C^\infty(M)\times[0,T]$ be such that $\lim_{n\to+\infty} v_n = v$ in $C^{2+\alpha}(M)$, where $v \in C^\infty(M)$ and $\lim_{n\to+\infty} t_n = t \in [0,T]$. Then we have

$$\begin{aligned}&\|\Phi(v_n,t_n)-\Phi(v,t)\|_{C^{2+\alpha}(M)}\\ &\qquad\le \|\Phi(v_n,t_n)-\Phi(v_n,t)\|_{C^{2+\alpha}(M)}+\|\Phi(v_n,t)-\Phi(v,t)\|_{C^{2+\alpha}(M)}.\end{aligned}\tag{2-25}$$

Since $\Phi(v_n,\cdot\,) \in C^\infty(M\times[0,T])$, Theorem 0.1 implies that for all $t\in[0,T]$,

$$\left\|\frac{\partial\Phi(v_n,t)}{\partial t}\right\|_{C^{2+\alpha}(M)} \le C_T,$$

where C_T denotes a constant not depending on n. We deduce that

$$\begin{aligned}\|\Phi(v_n,t_n)-\Phi(v_n,t)\|_{C^{2+\alpha}(M)} &= \left\|\int_{t_n}^{t}\frac{\partial\Phi(v_n,s)}{\partial s}\,ds\right\|_{C^{2+\alpha}(M)}\\ &\le |t_n-t|\max_{s\in[t_n,t]}\left\|\frac{\partial\Phi(v_n,s)}{\partial s}\right\|_{C^{2+\alpha}(M)} \underset{n\to+\infty}{\longrightarrow} 0.\end{aligned}\tag{2-26}$$

On the other hand, using Proposition 1.5, we have for all $t\in[0,T]$

$$\|\Phi(v_n,t)-\Phi(v,t)\|_{C^{2+\alpha}(M)} \le C_T\|v_n-v\|_{C^{2+\alpha}(M)} \underset{n\to+\infty}{\longrightarrow} 0.\tag{2-27}$$

Combining (2-25) , (2-26) and (2-27), we find that

$$\|\Phi(v_n,t_n)-\Phi(v,t)\|_{C^{2+\alpha}(M)} \underset{n\to+\infty}{\longrightarrow} 0.$$

Thus η is continuous from $X\times[0,1)\to X$. It remains to prove that it is continuous on $X\times[0,1]$. Let $(v_n,t_n)\in C^\infty(M)\times[0,1]$ be such that $\lim_{n\to+\infty} v_n=\bar v$ in $C^{2+\alpha}(M)$, where $\bar v\in C^\infty(M)$, and $\lim_{n\to+\infty} t_n=1$. From Step 1, we have

$$T_{v_n}=T_n \underset{n\to+\infty}{\longrightarrow} T_{\bar v}.$$

Since T_n is finite and $\lim_{n\to+\infty} t_n=1$, it follows that $\lim_{n\to+\infty} h(t_n)=+\infty$. So, for sufficiently large n, $h(t_n)\ge T_n$ and thus $\eta(v_n,t_n)=\Phi(v_n,T_n)$. We have, in the same way as (2-26) and (2-27), that

$$\begin{aligned}&\|\eta(v_n,t_n)-\eta(\bar v,1)\|_{C^{2+\alpha}(M)}\\ &\quad=\|\Phi(v_n,T_n)-\Phi(\bar v,T_{\bar v})\|_{C^{2+\alpha}(M)}\\ &\quad\le\|\Phi(v_n,T_n)-\Phi(\bar v,T_n)\|_{C^{2+\alpha}(M)}+\|\Phi(\bar v,T_n)-\Phi(\bar v,T_{\bar v})\|_{C^{2+\alpha}(M)} \underset{n\to+\infty}{\longrightarrow} 0.\end{aligned}$$

Therefore η is continuous from $X \times [0, 1] \to X$.

Now it is easy to check that η is a deformation retract from X into the set $\{v \in X : J_\rho(v) \le -L\}$. Hence this set is contractible. $\square$

Nonconvergence of the flow: proof of Theorem 0.4. To prove Theorem 0.4, it is sufficient to prove that there exists a real number $C > 0$ depending on M, Q and ρ such that, for all $v_0 \in C^{2+\alpha}(M)$ satisfying $J_\rho(v_0) \le -C$, the solution $v(t)$ of the flow (0-4), with $v(x, 0) = v_0(x)$ for all $x \in M$, satisfies

$$J_\rho(v(t)) \underset{t\to+\infty}{\longrightarrow} -\infty.$$

We recall (see [Li 1999]) that there exists a constant $C_0 \ge 0$ depending on M, Q and ρ such that

$$\|w\|_{C^{2+\alpha}(M)} \le C_0 \tag{2-28}$$

for any solution $w \in C^{2+\alpha}(M)$, $\alpha \in (0, 1)$, of

$$-\Delta w + Q = \frac{\rho e^w}{\int_M e^w \, dV}.$$

Since $J_\rho(v(t))$ is decreasing, if $\lim_{t\to+\infty} J_\rho(v(t)) \ne -\infty$ then there exists $L \in \mathbb{R}$ such that

$$J_\rho(v(t)) \ge L, \quad \forall t \in [0, +\infty).$$

From Theorem 0.2, there is a function $v_\infty \in C^\infty(M)$ such that

$$\|v(t) - v_\infty\|_{H^2(M)} \underset{t\to+\infty}{\longrightarrow} 0$$

which is a solution of

$$-\Delta v_\infty + Q = \frac{\rho e^{v_\infty}}{\int_M e^{v_\infty} \, dV}. \tag{2-29}$$

It follows that

$$\|v_\infty\|_{C^{2+\alpha}(M)} \le C_0,$$

where C_0 is the constant defined in (2-28). This implies that there exists a constant $\overline{C}$ depending on M, Q, ρ and C_0 such that

$$J(w_\infty) \ge -\overline{C}.$$

Since $J_\rho(v(t_1)) \le J_\rho(v(t_2))$, for all $t_1 \ge t_2$, we have

$$J_\rho(v_0) \ge J_\rho(v_\infty) \ge -\overline{C}.$$

However, $J_\rho(v_0) \le -C$ by hypothesis. Therefore, by choosing $C > \overline{C}$, we get a contradiction. $\square$

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Received June 14, 2014. Revised December 2, 2014.

JEAN-BAPTISTE CASTÉRAS
UFRGS, INSTITUTO DE MATEMÁTICA
AV. BENTO GONCALVES 9500
91540-000 PORTO ALEGRE-RS
BRAZIL
jean-baptiste.casteras@univ-brest.fr

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PACIFIC JOURNAL OF MATHEMATICS

Volume 276 No. 2 August 2015