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THE MEASURES OF ASYMMETRY FOR COPRODUCTS OF CONVEX BODIES

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In previous work, we introduced a family of p-measures of asymmetry for convex bodies, which have the well-known Minkowski measure of asymmetry as a particular case. We now reveal more properties of 1-measure and ∞-measure and give some calculating formulas of p-measures, in particular, for the so-called coproducts of convex bodies.

The measures of asymmetry for convex bodies, which in principle can be traced back to an early paper by Minkowski [1897], have been studied for a long time [Asplund et al. 1962; Besicovitch 1948; Chakerian and Stein 1964; Eggleston 1952; Klee 1953; Rogers and Shephard 1958; Stein 1956]. In particular, after B. Grünbaum formulated in his well-known paper [1963] a general definition of measures of (central) asymmetry (or symmetry), many mathematicians have contributed their efforts to this topic: studying the properties/applications of those known measures of asymmetry [Böröczky 2010; Dziechcińska-Halamoda and Szwiec 1985; Ekström 2000; Gluskin and Litvak 2008; Groemer 2000; Groemer and Wallen 2001; Guo 2005; Guo and Kaijser 1999; 2003; 2002; Hug and Schneider 2007; Kaiser 1996; Petitjean 2003; Schneider 2009; Soltan 2005; Mizushima 2000; Toth 2009; 2008], looking for new ones or studying other types of measure of asymmetry [Tuzikov et al. 2000; Tuzikov et al. 1997; Zouaki 2003]. Several such measures, most of which are related to extremal problems, are proposed and investigated.

In [Guo 2012], we found a family of measures of asymmetry $\mathrm{as}_p(\cdot)$ for convex bodies, called the p-measures of asymmetry ($1 \le p \le \infty$) (see definition below), which have the well-known Minkowski measure as a particular case. It turns out that p-measures do share some nice properties with the Minkowski measure and might be useful for further research.

As shown in [Guo 2012], for any convex body C and $1 \le p < \infty$, we have $\mathrm{as}_p(C) \le \mathrm{as}_\infty(C)$ in general, and equality holds if C is a symmetric convex body or a simplex. Equality also holds for some nontrivial (i.e., neither symmetric nor a simplex) convex bodies (see examples in Remark 2.3 below). Since, in some sense,

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$\mathrm{as}_\infty(C)$ is a maximal value and $\mathrm{as}_1(C)$ is a mean of a certain function related to C, defined on the unit sphere, it is interesting to consider the following question: Under what conditions is it true that $\mathrm{as}_1(C) = \mathrm{as}_\infty(C)$ (and therefore that all $\mathrm{as}_p(C)$ coincide)?

In this article, we reveal more properties of 1-measure and ∞-measure and give some calculating formulas of p-measures, in particular, for coproducts of convex bodies (see definition below). We will also formulate some questions related to the question above. From now on, we will simply write asymmetry instead of central asymmetry.

1. Preliminary

Let $\mathbb{R}^n$ denote the usual n-dimensional Euclidean space and $\langle\cdot,\cdot\rangle$ the canonical inner product on $\mathbb{R}^n$. Denote by $\mathcal{K}^n$ the class of all convex bodies (compact convex sets with nonempty interior) in $\mathbb{R}^n$, by $\mathrm{Aff}(\mathbb{R}^n)$ the family of all affine maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, and by $\mathrm{aff}(\mathbb{R}^n)$ the family of all affine functionals on $\mathbb{R}^n$, which forms an $(n+1)$-dimensional linear space under the ordinary addition and scalar multiplication of functions.

We adopt the following notation and terms from [Schneider 1993].

For $C_1, \dots, C_n \in \mathcal{K}^n$, denote by $V(C_1, \dots, C_n)$ the mixed volume of $C_1, \dots, C_n$ and let $V(C[k])$ be an abbreviated notation for

$$V(C, \overset{(k)}{\dots}, C, -C, \overset{(n-k)}{\dots}, -C), \quad 0 \le k \le n.$$

Similarly, denote by $S(C_1, \dots, C_{n-1}, \cdot)$ the mixed area measure (of $C_1, \dots, C_{n-1}$) on $\mathbb{S}^{n-1}$, the $(n-1)$-dimensional unit sphere. It is stated in [Schneider 1993] that $V(C[0]) = V([n]) = V_n(C)$, where $V_n(\cdot)$ denotes the n-dimensional volume, and $S(C, \overset{(n-1)}{\dots}, C, \cdot) = S_{n-1}(C, \cdot)$, the surface area measure of C on $\mathbb{S}^{n-1}$.

For $\alpha \in \mathbb{R}$ and $u \in \mathbb{S}^{n-1}$, set $H_{u,\alpha} = \{x \in \mathbb{R}^n \mid \langle x, u\rangle = \alpha\}$ and notice that $H_{u,\alpha}$ is a hyperplane.

For $C \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$, we define the support function of C based at x by

$$h_x(C, u) := \sup_{x \in C} \langle y - x, u\rangle, \quad \text{for all } u \in \mathbb{S}^{n-1}.$$

Denote $F(C, u) := C \cap H_{u, h_x(C,u)}$, which is independent of x and called the support set (of C) in the direction u.

It is shown in Theorem 5.1.6 of [Schneider 1993] that, for each $x \in \mathbb{R}^n$,

$$(*) \qquad \begin{aligned} V(C[n-1]) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_x(C, -u)\, dS_{n-1}(C, u), \\ V_n(C) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_x(C, u)\, dS_{n-1}(C, u). \end{aligned}$$

Given $C \in \mathcal{K}^n$, for $x \in \operatorname{int}(C)$, we write

$$\mu_p(C,x) := \begin{cases} \left(\int_{\mathbb{S}^{n-1}} \alpha_x(C,u)^p \, dm_x(C,u)\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \sup_{u \in \mathbb{S}^{n-1}} \alpha_x(C,u) & \text{if } p = \infty, \end{cases}$$

where $\alpha_x(C,u) := h_x(C,-u)/h_x(C,u)$ and, for measurable $\omega \subset \mathbb{S}^{n-1}$,

$$m_x(C,\omega) := \frac{\int_\omega h_x(C,u)\, dS_{n-1}(C,u)}{nV_n(C)} = \frac{\int_\omega h_x(C,u)\, dS_{n-1}(C,u)}{\int_{\mathbb{S}^{n-1}} h_x(C,u)\, dS_{n-1}(C,u)},$$

which is a probability measure on $\mathbb{S}^{n-1}$.

Remark 1.1. If C is a polytope with (all) facets $F(C,u_i)$, $i = 1, 2, \ldots, m$, where u_i are outer normal vectors, then the measures $S_{n-1}(C,\cdot)$ and $m_x(C,\cdot)$ are linear combinations of m Dirac measures δ_{u_i}, $i = 1, 2, \ldots, m$, and so the integrals appearing above are just finite sums.

Definition [Guo 2012]. For $C \in \mathcal{K}^n$, we define its p-measure of asymmetry $(1 \le p \le \infty)$ $\operatorname{as}_p(C)$ by

$$\operatorname{as}_p(C) := \inf_{x \in \operatorname{int}(C)} \mu_p(C,x).$$

A point $x \in \operatorname{int}(C)$ satisfying $\mu_p(C,x) = \operatorname{as}_p(C)$ is called a p-critical point of C. The set of all p-critical points is called the p-critical set (of C), denoted by $\mathcal{C}_p(C)$.

Remark 1.2. (i) The measure

$$\operatorname{as}_\infty(C) = \inf_{x \in \operatorname{int}(C)} \sup_{u \in \mathbb{S}^{n-1}} \frac{h_x(C,-u)}{h_x(C,u)}$$

is nothing else but the Minkowski measure of asymmetry (of C). The measure $\operatorname{as}_1(C)$ is the (minimal) mean of $h_x(C,-u)/h_x(C,u)$ (against $m_x(C,\cdot)$) among all $x \in \operatorname{int}(C)$, which is in fact independent of x (i.e., $\mu_1(C,x) = \operatorname{as}_1(C)$ for all $x \in \operatorname{int}(C)$; see (*)).

(ii) It is shown in [Guo 2012] that if defining, for any $\varepsilon \ge 0$,

$$\phi(\varepsilon) := \left(\frac{V_n(C - \varepsilon C)}{V_n(C)}\right)^{1/n},$$

then $\operatorname{as}_1(C) = \phi'_+(0)$ (in fact, this is the definition of 1-measure in [Guo 2012]).

(iii) By definition 1 and (*) above, we see that $\operatorname{as}_1(C) = V(C[n-1])/V_n(C)$.

(iv) It is proved in [Guo 2012] that, for $1 < p < \infty$, $\mathcal{C}_p$ is a singleton.

One of the main results in [Guo 2012] is the following theorem.

Theorem 1.3. *For any* $1 \le p, q \le \infty$, *the following statements are true*:

(i) $\mathrm{as}_p(\cdot)$ *is affinely invariant*, *i.e.*, $\mathrm{as}_p(C) = \mathrm{as}_p(\mathrm{T}(C))$ *for any* $C \in \mathcal{K}^n$ *and any invertible* $\mathrm{T} \in \mathrm{Aff}(\mathbb{R}^n)$.

(ii) $\mathrm{as}_p(C) \le \mathrm{as}_q(C)$, *for any* $C \in \mathcal{K}^n$ *and* $1 \le p < q \le \infty$.

(iii) $1 \le \mathrm{as}_p(C) \le n$, $\mathrm{as}_p(C) = 1$ *if and only if* C *is symmetric*, *and* $\mathrm{as}_p(C) = n$ *if and only if* C *is a simplex.*

2. The 1-measure of asymmetry for coproducts of convex bodies

In [Guo 2012] we showed that p-measures do share some nice properties with Minkowski's measure. Here we will present more.

We first recall a conclusion in [Guo and Kaijser 2002]: for any $(n-1)$-dimensional convex set $C \subset \mathbb{R}^n$, $\mathrm{as}_\infty(\widehat{C}_z) = \mathrm{as}_\infty(C) + 1$, where $\widehat{C}_z := \mathrm{conv}(C, z)$ is the convex hull of $C \cup \{z\}$ (called the cone with vertex z and base C) and z is not in the affine hull of C (where $\mathrm{as}_\infty(C)$ is computed in the $(n-1)$-dimensional space). Furthermore, all ∞-critical points x^* of $\widehat{C}_z$ are of the form

$$x^* = \frac{1}{2 + \mathrm{as}_\infty(C)} z + \left(\frac{1 + \mathrm{as}_\infty(C)}{2 + \mathrm{as}_\infty(C)} \right) x',$$

where x' is an ∞-critical point of C.

We show that a similar conclusion holds for 1-measure but not for 2-measure. Further, we extend the result to the so-called coproducts of subsets which are a generalization of cones (see definition below).

Let us start with cones.

Theorem 2.1. *Let* C, z *be as above. Then*

(i) $\mathrm{as}_1(\widehat{C}_z) = \mathrm{as}_1(C) + 1$.

(ii) $\mathrm{as}_2(\widehat{C}_z)^2 = \mathrm{as}_2(C)^2 + 2\sqrt{\mathrm{as}_2(C)^2 + 2\mathrm{as}_1(C) + 1} - 1$. *Consequently we have* $\mathrm{as}_2(\widehat{C}_z) \le \mathrm{as}_2(C) + 1$, *where equality holds if and only if* $\mathrm{as}_2(C) = \mathrm{as}_1(C)$.

To prove Theorem 2.1, the following lemma is needed.

Lemma 2.2. *Let* C, z *be the same as in Theorem 2.1. For* x *in* $\mathrm{ri}(C)$, *the relative interior of* C, *if* $z_\lambda = \lambda z + (1-\lambda)x$ $(0 < \lambda < 1)$, *then*, *for any* $1 \le p < \infty$,

$$\mu_p(\widehat{C}_z, z_\lambda)^p = \lambda \left(\frac{1-\lambda}{\lambda} \right)^p + \frac{1}{(1-\lambda)^{p-1}} \int_{\mathbb{S}^{n-2}} (\lambda + \alpha_x(C, u))^p \, dm_x(C, u).$$

Proof. Since the family of $(n-1)$-dimensional polytopes is dense in $\mathcal{K}^{n-1}$, and $C_i \to C$ implies $\widehat{C}_i \to \widehat{C}$ (with respect to the Hausdorff metric), and $S_{n-1}(C, \cdot)$ is weakly continuous, we may assume, without loss of generality, that

$$C = \mathrm{conv}(v_1, \ldots, v_l)$$

is an $(n-1)$-dimensional polytope, where v_i are (all) vertices of C.

Thus

$$\widehat{C} = \operatorname{conv}(v_1, \dots, v_l, z)$$

is the n-dimensional polytope with vertices $v_1, \dots, v_l$ and z. Furthermore, if all facets of C are F_i $(1 \le i \le m)$, then all facets of $\widehat{C}$ are $\widehat{F_i} = \operatorname{conv}(F_i, z)$ $(1 \le i \le m)$ and C. We denote by $\tilde{u}_0 \in \mathbb{S}^{n-1}$ the outer normal vector of C (as a facet of $\widehat{C}$), by $\tilde{u}_i \in \mathbb{S}^{n-1}$ the outer normal vector of $\widehat{F_i}$, and by $u_i \in \mathbb{S}^{n-2} \equiv H^* \cap \mathbb{S}^{n-1}$, where H^* denotes the $(n-1)$-dimensional subspace parallel to the affine hull H of C, the outer normal vector of F_i (as a facet of C).

Use the fact that $\langle z - x, \tilde{u}_i \rangle = h_x(\widehat{C}, \tilde{u}_i)$ to observe that

$$\begin{aligned} h_{z_\lambda}(\widehat{C}, \tilde{u}_i) &= \sup_{y \in \widehat{C}} \langle y - z_\lambda, \tilde{u}_i \rangle = \sup_{y \in \widehat{C}} \langle (y + x - z_\lambda) - x, \tilde{u}_i \rangle \\ &= \sup_{y' \in \widehat{C} + x - z_\lambda} \langle y' - x, \tilde{u}_i \rangle = h_x(\widehat{C} + x - z_\lambda, \tilde{u}_i) \\ &= h_x(\widehat{C} + \lambda(x - z), \tilde{u}_i) = h_x(\widehat{C}, \tilde{u}_i) - \lambda \langle z - x, \tilde{u}_i \rangle \\ &= (1 - \lambda) h_x(\widehat{C}, \tilde{u}_i). \end{aligned}$$

This, together with the fact that $h_x(\widehat{C}, \tilde{u}_i) + h_x(\widehat{C}, -\tilde{u}_i)$ is just the width of $\widehat{C}$ along $\tilde{u}_i$ and does not depend on the choice of x, in turn leads to

$$h_{z_\lambda}(\widehat{C}, -\tilde{u}_i) = \lambda h_x(\widehat{C}, \tilde{u}_i) + h_x(\widehat{C}, -\tilde{u}_i).$$

Finally we get

$$\begin{aligned} \alpha_{z_\lambda}(\widehat{C}, \tilde{u}_i) &= \frac{h_{z_\lambda}(\widehat{C}, -\tilde{u}_i)}{h_{z_\lambda}(\widehat{C}, \tilde{u}_i)} = \frac{\lambda}{1-\lambda} + \frac{1}{1-\lambda} \frac{h_x(\widehat{C}, -\tilde{u}_i)}{h_x(\widehat{C}, \tilde{u}_i)} \\ &= \frac{\lambda}{1-\lambda} + \frac{1}{1-\lambda} \frac{h_x(C, -u_i)}{h_x(C, u_i)} = \frac{\lambda}{1-\lambda} + \frac{1}{1-\lambda} \alpha_x(C, u_i). \end{aligned} \tag{2-1}$$

Furthermore,

$$\begin{aligned} h_{z_\lambda}(\widehat{C}, \tilde{u}_i) V_{n-1}(\widehat{F_i}) &= (1-\lambda) h_x(\widehat{C}, \tilde{u}_i) V_{n-1}(\widehat{F_i}) = (1-\lambda) n V_n(\widehat{C_i}) \\ &= (1-\lambda)(h_{z_\lambda}(\widehat{C}, u_0) + h_{z_\lambda}(\widehat{C}, -u_0)) V_{n-1}(C_i) \\ &= (1-\lambda)(h_{z_\lambda}(\widehat{C}, u_0) + h_{z_\lambda}(\widehat{C}, -u_0)) \frac{h_x(C, u_i)}{n-1} V_{n-2}(F_i), \end{aligned} \tag{2-2}$$

where $\widehat{C_i}$ and C_i denote, respectively, the n-dimensional body $\operatorname{conv}(z, x, F_i)$ and the $(n-1)$-dimensional body $\operatorname{conv}(x, F_i)$.

Now, by (2-1), (2-2) and the fact that

$$n V_n(\widehat{C}) = (h_{z_\lambda}(\widehat{C}, u_0) + h_{z_\lambda}(\widehat{C}, -u_0)) V_{n-1}(C),$$

it follows that

$$
\begin{aligned}
&\mu_p(\widehat{C}, z_\lambda)^p \\
&\quad = \int_{\mathbb{S}^{n-1}} \alpha_{z_\lambda}(\widehat{C}, u)^p \, dm_{z_\lambda}(\widehat{C}, u) \\
&\quad = \frac{\alpha_{z_\lambda}(\widehat{C}, u_0)^p h_{z_\lambda}(\widehat{C}, u_0) V_{n-1}(C) + \sum_{i=1}^m \alpha_{z_\lambda}(\widehat{C}, \tilde{u}_i)^p h_{z_\lambda}(\widehat{C}, \tilde{u}_i) V_{n-1}(\widehat{F}_i)}{n V_n(\widehat{C})} \\
&\quad = \frac{\alpha_{z_\lambda}(\widehat{C}, u_0)^p h_{z_\lambda}(\widehat{C}, u_0)}{h_{z_\lambda}(\widehat{C}, u_0) + h_{z_\lambda}(\widehat{C}, -u_0)} + \frac{\frac{1-\lambda}{n-1} \sum_{i=1}^m \alpha_{z_\lambda}(\widehat{C}, \tilde{u}_i)^p h_x(C, u_i) V_{n-2}(F_i)}{V_{n-1}(C)} \\
&\quad = \lambda \Big(\frac{1-\lambda}{\lambda}\Big)^p + \frac{1}{(1-\lambda)^{p-1}} \int_{\mathbb{S}^{n-2}} (\lambda + \alpha_x(C, u))^p \, dm_x(C, u),
\end{aligned}
$$

where we used the equalities $h_{z_\lambda}(\widehat{C}, u_0)/(h_{z_\lambda}(\widehat{C}, u_0) + h_{z_\lambda}(\widehat{C}, -u_0)) = \lambda$ and $\alpha_{z_\lambda}(\widehat{C}, u_0) = (1-\lambda)/\lambda$. $\square$

Proof of Theorem 2.1. (i) In Lemma 2.2, taking $p = 1$, we have, for any $x \in \mathrm{ri}(C)$, $0 < \lambda < 1$,

$$
\begin{aligned}
\mathrm{as}_1(\widehat{C}_z) &= 1 - \lambda + \int_{\mathbb{S}^{n-2}} (\lambda + \alpha_x(C, u)) \, dm_x(C, u) \\
&= 1 + \int_{\mathbb{S}^{n-2}} \alpha_x(C, u) \, dm_x(C, u) = 1 + \mathrm{as}_1(C).
\end{aligned}
$$

(ii) In Lemma 2.2, taking $p = 2$ and noticing $\mu_1(C, x) = \mathrm{as}_1(C)$, we have

$$
\begin{aligned}
\mu_2(\widehat{C}, z_\lambda)^2 &= \frac{(1-\lambda)^2}{\lambda} + \frac{1}{1-\lambda} \int_{\mathbb{S}^{n-2}} (\lambda + \alpha_x(C, u))^2 \, dm_x(C, u) \\
&= \frac{(1-\lambda)^2}{\lambda} + \frac{\lambda^2}{1-\lambda} + \frac{2\lambda}{1-\lambda} \mathrm{as}_1(C) + \frac{1}{1-\lambda} \mu_2(C, x)^2 =: A(\lambda).
\end{aligned}
$$

Letting

$$
A'(\lambda) = \frac{(\mu_2(C, x)^2 + 2\mathrm{as}_1(C))\lambda^2 + 2\lambda - 1}{\lambda^2 (1-\lambda)^2} = 0,
$$

we get $\lambda_0 = \big(\sqrt{\mu_2(C, x)^2 + 2\mathrm{as}_1(C) + 1} + 1\big)^{-1}$. Thus, with an elementary computation, it follows that

$$
\begin{aligned}
\min_{0<\lambda<1} \mu_2(\widehat{C}, z_\lambda)^2 &= A(\lambda_0) = \mu_2(\widehat{C}, z_{\lambda_0})^2 \\
&= \mu_2(C, x)^2 + 2\sqrt{\mu_2(C, x)^2 + 2\mathrm{as}_1(C) + 1} - 1.
\end{aligned}
$$

Therefore

$$
\mathrm{as}_2(\widehat{C}_z)^2 = \min_{x \in \mathrm{ri}(C)} \min_{0<\lambda<1} \mu_2(\widehat{C}, z_\lambda)^2 = \mathrm{as}_2(C)^2 + 2\sqrt{\mathrm{as}_2(C)^2 + 2\mathrm{as}_1(C) + 1} - 1.
$$

Since $\mathrm{as}_1(C) \le \mathrm{as}_2(C)$ by Theorem 1.3, we get

$$\mathrm{as}_2(\widehat{C}_z)^2 \le \mathrm{as}_2(C)^2 + 2\sqrt{\mathrm{as}_2(C)^2 + 2\mathrm{as}_2(C) + 1} - 1 = (\mathrm{as}_2(C) + 1)^2$$

which implies that $\mathrm{as}_2(\widehat{C}_z) \le \mathrm{as}_2(C) + 1$ and that equality holds if and only if $\mathrm{as}_2(C) = \mathrm{as}_1(C)$. □

Remark 2.3. Theorem 2.1 indicates that there are nontrivial $C \in \mathcal{K}^n$ $(n \ge 4)$ such that $\mathrm{as}_1(C) = \mathrm{as}_\infty(C)$: in $\mathbb{R}^n$, taking a symmetric $D \in \mathcal{K}^{n-2}$ and forming $\widehat{D}_y \in \mathcal{K}^{n-1}$, for each $C := \mathrm{conv}(z, \widehat{D}_y) \in \mathcal{K}^n$, we have, by Theorems 1.3 and 2.1 and Theorem 2 in [Guo and Kaijser 2002], $\mathrm{as}_1(C) = \mathrm{as}_2(C) = \mathrm{as}_\infty(C) = 3$, while clearly C is neither a symmetric convex body nor a simplex.

Now we introduce the so-called coproduct of subsets in different spaces and then generalize Theorem 2.1.

Definition. Given $C \subset \mathbb{R}^m$ and $D \subset \mathbb{R}^n$ $(m, n \ge 0)$, we define the coproduct body $C \amalg D \subset \mathbb{R}^{m+n+1}$ as

$$C \amalg D := \bigcup_{0 \le \lambda \le 1} (1-\lambda)C \times \lambda D \times \{\lambda\} = \{((1-\lambda)x, \lambda y, \lambda) \mid x \in C, y \in D, 0 \le \lambda \le 1\}.$$

Remark 2.4. (i) If both C and D are convex, then $C \amalg D = \mathrm{conv}(C \cup \tilde{D})$, where $\tilde{D} = \{0\} \times D \times \{1\} = \{(0, y, 1) \mid y \in D\}$ (in particular, $C \amalg D$ is convex). For example, $[-1, 1] \amalg [-1, 1] = \mathrm{conv}\{(1, 0, 0), (-1, 0, 0), (0, 1, 1), (0, -1, 1)\}$, a 3-dimensional simplex. In general, $[a, b] \amalg [c, d]$ is a 3-dimensional simplex.

(ii) If $C = \{v\}$ is a singleton and D is convex, then $C \amalg D$ reduces to the cone with vertex v and base D.

The next proposition, which may be checked easily, shows that, in a sense, the coproduct operation is the dual of product operation. For $C \in \mathcal{K}^n$, denote

$$C^A = \{f \in \mathrm{aff}(\mathbb{R}^n) \mid f(C) \subset [-1, 1]\}.$$

Proposition 2.5. *For any $C \in \mathcal{K}^m$ and $D \in \mathcal{K}^n$ under the correspondence*

$$(C \amalg D)^A \ni f \longleftrightarrow (f_{|C}, f_{|D}) \in C^A \times D^A,$$

where $(f_{|C}, f_{|D})\big(((1-\lambda)x, \lambda y, \lambda)\big) := (1-\lambda) f_{|C}(x) + \lambda f_{|D}(y)$, *we have*

$$(C \amalg D)^A = C^A \times D^A.$$

Now we can generalize (i) in Theorem 2.1 to the coproduct bodies.

Theorem 2.6. *For any $C \in \mathcal{K}^m$ and $D \in \mathcal{K}^n$ $(m, n \ge 0)$,*

$$\mathrm{as}_1(C \amalg D) = \mathrm{as}_1(C) + \mathrm{as}_1(D) + 1,$$

where we take the convention that $\mathrm{as}_1(C)$ *(or* $\mathrm{as}_1(D)) = 0$ *if* C *(or* $D) \in \mathcal{K}^0$.

In order to prove Theorem 2.6, more lemmas are needed. For any $\lambda \in \mathbb{R}$, $\varepsilon \geq 0$ and $A \subset \mathbb{R}^{m+n+1}$, denote

$$\mathbb{H}_\lambda := \mathbb{R}^m \times \mathbb{R}^n \times \{\lambda\}, \quad A_\lambda := A \cap \mathbb{H}_\lambda, \quad A_{\lambda,\varepsilon} := [A - \varepsilon A]_\lambda.$$

Lemma 2.7. *For any $C \in \mathcal{K}^m$ and $D \in \mathcal{K}^n$ $(m, n \geq 1)$,*

$$V_{m+n+1}(C \amalg D) = B(m+1, n+1) V_m(C) V_n(D),$$

where $B(\cdot, \cdot)$ is the Beta function.

Proof. Since $[C \amalg D]_\lambda = (1-\lambda)C \times \lambda D \times \{\lambda\}$ for $0 \leq \lambda \leq 1$, we have

$$V_{m+n}([C \amalg D]_\lambda) = V_{m+n}((1-\lambda)C \times \lambda D) = (1-\lambda)^m \lambda^n V_m(C) V_n(D).$$

Hence

$$V_{m+n+1}(C \amalg D) = \int_0^1 (1-\lambda)^m \lambda^n V_m(C) V_n(D)\, d\lambda = B(m+1, n+1) V_m(C) V_n(D).$$

□

Lemma 2.8. *If $o \in C$ and $o \in D$, then for any $0 \leq \varepsilon < 1$ and $0 \leq \lambda \leq 1-\varepsilon$,*

$$V_{m+n}([C \amalg D]_{\lambda,\varepsilon}) = V_{m+n}\big(((1-\lambda)C - \varepsilon C)) \times ((\lambda+\varepsilon)D - \varepsilon D)\big) - \varepsilon^2 P^*(\lambda, \varepsilon),$$

where $P^*(\lambda, \varepsilon)$ is a polynomial of λ and ε.

Proof. Since

$$\begin{aligned} C \amalg D - \varepsilon(C \amalg D) &= \bigcup_{0 \leq \mu, \nu \leq 1} ((1-\mu)C \times \mu D \times \{\mu\} - \varepsilon(1-\nu)C \times \varepsilon\nu D \times \{\varepsilon\nu\}) \\ &= \bigcup_{0 \leq \mu, \nu \leq 1} ((1-\mu)C - \varepsilon(1-\nu)C) \times (\mu D - \varepsilon\nu D) \times \{\mu - \varepsilon\nu\}, \end{aligned}$$

we have

$$[C \amalg D]_{\lambda,\varepsilon} = \bigcup_{\mu - \varepsilon\nu = \lambda} ((1-\mu)C - \varepsilon(1-\nu)C) \times (\mu D - \varepsilon\nu D) \times \{\lambda\}.$$

Thus,

$$[C \amalg D]_{\lambda,\varepsilon} \supset ((1-(\lambda+\varepsilon))C) \times ((\lambda+\varepsilon)D - \varepsilon D) \times \{\lambda\} =: E_1 \tag{2-3}$$

(the set when $\nu = 1$ and so $\mu = \lambda + \varepsilon$) and

$$[C \amalg D]_{\lambda,\varepsilon} \supset ((1-\lambda)C - \varepsilon C) \times (\lambda D) \times \{\lambda\} =: E_2 \tag{2-4}$$

(the set when $\nu = 0$ and so $\mu = \lambda$). We also have

$$[C \amalg D]_{\lambda,\varepsilon} \subset ((1-\lambda)C - \varepsilon C) \times ((\lambda+\varepsilon)D - \varepsilon D) \times \{\lambda\} =: E_3, \tag{2-5}$$

since, for $0 \le \mu,\ \nu \le 1$ with $\mu - \varepsilon\nu = \lambda$ (notice that $\lambda \le \mu \le \lambda + \varepsilon$ and that $o \in C$, $o \in D$),

$$\begin{aligned}(1-\mu)C - \varepsilon(1-\nu)C &\subset (1-\lambda)C - \varepsilon C,\\ \mu D - \varepsilon\nu D \subset \mu D - \varepsilon D &\subset (\lambda+\varepsilon)D - \varepsilon D.\end{aligned}\tag{2-6}$$

Now, setting

$$P(\lambda,\varepsilon) := V_{m+n}(E_3) - V_{m+n}([C \amalg D]_{\lambda,\varepsilon}),$$

which is a polynomial of λ and ε, we have by (2-3), (2-4), (2-5) and the fact that $(1-\lambda-\varepsilon)C \subset (1-\lambda)C - \varepsilon C$, $\lambda D \subset (\lambda+\varepsilon)D - \varepsilon D$,

$$\begin{aligned}0 &\le P(\lambda,\varepsilon)\\ &\le V_{m+n}(E_3) - V_{m+n}(E_1 \cup E_2)\\ &= V_{m+n}(E_3) - V_{m+n}(E_1) - V_{m+n}(E_2) + V_{m+n}(E_1 \cap E_2).\end{aligned}$$

By the polynomial expansion of the Minkowski sum (see Theorem 5.1.6 in [Schneider 1993]),

$$\begin{aligned}V_{m+n}(E_3) &= V_m((1-\lambda)C - \varepsilon C)V_n((\lambda+\varepsilon)D - \varepsilon D)\\ &= ((1-\lambda)^m V_m(C) + m\varepsilon(1-\lambda)^{m-1}V(C[m-1]) + \varepsilon^2 P_1'(\lambda,\varepsilon))\\ &\qquad \times((\lambda+\varepsilon)^n V_n(D) + n\varepsilon(\lambda+\varepsilon)^{n-1}V(D[n-1]) + \varepsilon^2 P_1''(\lambda,\varepsilon))\\ &= (1-\lambda)^m(\lambda+\varepsilon)^n V_m(C)V_n(D)\\ &\qquad + n\varepsilon(1-\lambda)^m(\lambda+\varepsilon)^{n-1}V_m(C)V(D[n-1])\\ &\qquad\quad + m\varepsilon(1-\lambda)^{m-1}(\lambda+\varepsilon)^n V(C[m-1])V_n(D) + \varepsilon^2 P_1(\lambda,\varepsilon),\end{aligned}\tag{2-7}$$

$$\begin{aligned}V_{m+n}(E_1) &= V_m((1-\lambda-\varepsilon)C)V_n((\lambda+\varepsilon)D - \varepsilon D)\\ &= (1-\lambda-\varepsilon)^m V_m(C)\\ &\qquad \times((\lambda+\varepsilon)^n V_n(D) + n\varepsilon(\lambda+\varepsilon)^{n-1}V(D[n-1]) + \varepsilon^2 P_2'(\lambda,\varepsilon))\\ &= (1-\lambda-\varepsilon)^m(\lambda+\varepsilon)^n V_m(C)V_n(D)\\ &\qquad + n\varepsilon(1-\lambda-\varepsilon)^m(\lambda+\varepsilon)^{n-1}V_m(C)V(D[n-1]) + \varepsilon^2 P_2(\lambda,\varepsilon),\end{aligned}$$

$$\begin{aligned}V_{m+n}(E_2) &= V_m((1-\lambda)C - \varepsilon C)V_n(\lambda D)\\ &= ((1-\lambda)^m V_m(C) + m\varepsilon(1-\lambda)^{m-1}V(C[m-1]) + \varepsilon^2 P_3'(\lambda,\varepsilon))\lambda^n V_n(D)\\ &= (1-\lambda)^m\lambda^n V_m(C)V_n(D)\\ &\qquad + m\varepsilon(1-\lambda)^{m-1}\lambda^n V(C[m-1])V_n(D) + \varepsilon^2 P_3(\lambda,\varepsilon),\end{aligned}$$

$$V_{m+n}(E_1 \cap E_2) = V_m((1-\lambda-\varepsilon)C)V_n(\lambda D) = (1-\lambda-\varepsilon)^m\lambda^n V_m(C)V_n(D),$$

where P_i', P_i'', P_i are polynomials of λ and ε, and

$$\begin{aligned}(1-\lambda)^m(\lambda+\varepsilon)^n &- (1-\lambda-\varepsilon)^m(\lambda+\varepsilon)^n - (1-\lambda)^m\lambda^n + (1-\lambda-\varepsilon)^m\lambda^n \\ &= ((1-\lambda)^m - (1-\lambda-\varepsilon)^m)(\lambda+\varepsilon)^n - ((1-\lambda)^m - (1-\lambda-\varepsilon)^m)\lambda^n \\ &= ((1-\lambda)^m - (1-\lambda-\varepsilon)^m)((\lambda+\varepsilon)^n - \lambda^n) \\ &= \varepsilon^2 Q_1(\lambda, \varepsilon),\end{aligned}$$

$$\begin{aligned}n\varepsilon(1-\lambda)^m(\lambda+\varepsilon)^{n-1} &- n\varepsilon(1-\lambda-\varepsilon)^m(\lambda+\varepsilon)^{n-1} \\ &= n\varepsilon((1-\lambda)^m - (1-\lambda-\varepsilon)^m)(\lambda+\varepsilon)^{n-1} \\ &= \varepsilon^2 Q_2(\lambda, \varepsilon),\end{aligned}$$

$$m\varepsilon(1-\lambda)^{m-1}((\lambda+\varepsilon)^n - \lambda^n) = \varepsilon^2 Q_3(\lambda, \varepsilon),$$

where Q_i are polynomials of λ and ε. Thus

$$V_{m+n}(E_3) - V_{m+n}(E_1) - V_{m+n}(E_2) + V_{m+n}(E_1 \cap E_2) = \varepsilon^2 Q(\lambda, \varepsilon)$$

for some polynomial $Q(\lambda, \varepsilon)$, and in turn $P(\lambda, \varepsilon) = \varepsilon^2 P^*(\lambda, \varepsilon)$ for some polynomial $P^*(\lambda, \varepsilon)$. □

Lemma 2.9. *For any $C \in \mathcal{K}^m$ and $D \in \mathcal{K}^n$ with $o \in C$, $o \in D$ $(m, n \geq 1)$,*

$$\begin{aligned}\frac{d}{d\varepsilon}\left(\int_0^{1-\varepsilon} V_{m+n}([C \amalg D]_{\lambda,\varepsilon})\, d\lambda\right)_{|\varepsilon=0} &\\ = (m+n+1)B(m+1, n+1)& \\ \times (V_m(C)V_n(D) + V(C[m-1])&V_n(D) + V_m(C)V(D[n-1])).\end{aligned}$$

Proof. By Lemma 2.8 and (2-7), we have

$$\begin{aligned}&V_{m+n}([C \amalg D]_{\lambda,\varepsilon}) \\ &\quad = V_{m+n}(E_3) - \varepsilon^2 P^*(\lambda, \varepsilon) \\ &\quad = (1-\lambda)^m(\lambda+\varepsilon)^n V_m(C)V_n(D) + m\varepsilon(1-\lambda)^{m-1}(\lambda+\varepsilon)^n V(C[m-1])V_n(D) \\ &\qquad + n\varepsilon(\lambda+\varepsilon)^{n-1}(1-\lambda)^m V_m(C)V(D[n-1]) + \varepsilon^2 P_1(\lambda, \varepsilon) - \varepsilon^2 P^*(\lambda, \varepsilon).\end{aligned}$$

Thus, since

$$\frac{d}{d\varepsilon}\left(\int_0^{1-\varepsilon} (1-\lambda)^m(\lambda+\varepsilon)^n d\lambda\right)_{|\varepsilon=0} = nB(m+1, n),$$

$$\frac{d}{d\varepsilon}\left(\varepsilon\int_0^{1-\varepsilon} (1-\lambda)^{m-1}(\lambda+\varepsilon)^n d\lambda\right)_{|\varepsilon=0} = B(m, n+1),$$

$$\frac{d}{d\varepsilon}\left(\varepsilon\int_0^{1-\varepsilon}(1-\lambda)^m(\lambda+\varepsilon)^{n-1}d\lambda\right)_{|\varepsilon=0}=B(m+1,n),$$
$$\frac{d}{d\varepsilon}\left(\varepsilon^2\int_0^{1-\varepsilon}(P_1(\lambda,\varepsilon)-P^*(\lambda,\varepsilon))d\lambda\right)_{|\varepsilon=0}=0,$$

and because $mB(m,n+1)=nB(m+1,n)=(m+n+1)B(m+1,n+1)$, we get

$$\begin{aligned}\frac{d}{d\varepsilon}&\left(\int_0^{1-\varepsilon}V_{m+n}([C\amalg D]_{\lambda,\varepsilon})\,d\lambda\right)_{|\varepsilon=0}\\&=nB(m+1,n)V_m(C)V_n(D)+mB(m,n+1)V(C[m-1])V_n(D)\\&\qquad\qquad+nB(m+1,n)V_m(C)V(D[n-1])\\&=(m+n+1)B(m+1,n+1)\\&\qquad\times(V_m(C)V_n(D)+V(C[m-1])V_n(D)+V_m(C)V(D[n-1])).\qquad\square\end{aligned}$$

The following simple fact will be needed in the proof of Theorem 2.6.

Fact 2.10. *Suppose* $0\le u(t)\le v(t)$ *and* $u(0)=v(0)=0$. *If*

$$\frac{dv(t)}{dt^+}_{|t=0}=0,$$

then

$$0\le\lim_{t\to0^+}\frac{u(t)-u(0)}{t-0}\le\lim_{t\to0^+}\frac{v(t)-v(0)}{t-0}=0,$$

i.e.,

$$\frac{du(t)}{dt^+}_{|t=0}=0\quad\left(or\ \frac{du(t)}{dt}_{|t=0}=0\ \textit{if it exists}\right),$$

where d/dt^+ *denotes the right derivative.*

Proof of Theorem 2.6. If $m=n=0$, then $\mathrm{as}_1(C)=\mathrm{as}_1(D)=0$ and $C\amalg D$ is just the segment with ends o and $(0,0,1)$. Hence $\mathrm{as}_1(C\amalg D)=1=\mathrm{as}_1(C)+\mathrm{as}_1(D)+1$.

If $m=0$, $n\ge1$ (or $m\ge1$, $n=0$), it reduces to (i) in Theorem 2.1 (see (ii) in Remark 2.4).

Now we assume $m,n\ge1$ and $o\in C$, $o\in D$ (since $\mathrm{as}_1(\cdot)$ is affine invariant). Notice that $C\amalg D-\varepsilon(C\amalg D)$ is located in between $\mathbb{H}_{-\varepsilon}$ and $\mathbb{H}_1$ since $C\amalg D$ is located in between $\mathbb{H}_0$ and $\mathbb{H}_1$.

By the polynomial expansion of the Minkowski sum, we have that

$$\begin{aligned}(2\text{-}8)\quad&(m+n+1)V((C\amalg D)[m+n])\\&=\frac{d}{d\varepsilon}V_{m+n+1}(C\amalg D-\varepsilon C\amalg D)_{|\varepsilon=0}\\&=\frac{d}{d\varepsilon}\left[\left(\int_{-\varepsilon}^0+\int_0^{1-\varepsilon}+\int_{1-\varepsilon}^1\right)V_{m+n}([C\amalg D]_{\lambda,\varepsilon})\,d\lambda\right]_{|\varepsilon=0}.\end{aligned}$$

In order to compute

$$\frac{d}{d\varepsilon}\left(\int_{-\varepsilon}^{0} V_{m+n}([C \amalg D]_{\lambda,\varepsilon})\,d\lambda\right)_{|\varepsilon=0},$$

we observe that if $-\varepsilon \le \lambda \le 0$ and $\mu - \varepsilon\nu = \lambda$, then

$$((1-\mu)C - \varepsilon(1-\nu)C) \times (\mu D - \varepsilon\nu D) \subset (C - \varepsilon C) \times ((\lambda+\varepsilon)(D-D) + \lambda D),$$

since $(1-\mu)C - \varepsilon(1-\nu)C \subset C - \varepsilon C$ and

$$\mu D - \varepsilon\nu D = \mu D - (\mu-\lambda)D = \mu(D-D) - \lambda D \subset (\lambda+\varepsilon)(D-D) + \lambda D$$

(notice that $-\lambda \ge 0$, $o \in D-D$ and that $\mu - \varepsilon\nu = \lambda$ implies $\mu \le \lambda + \varepsilon$). So

(2-9)
$$0 \le \int_{-\varepsilon}^{0} V_{m+n}([C \amalg D]_{\lambda,\varepsilon})\,d\lambda \le \int_{-\varepsilon}^{0} V_m(C-\varepsilon C)V_n((\lambda+\varepsilon)(D-D)+\lambda D)\,d\lambda.$$

Denote $f(\lambda,\varepsilon) := V_m(C-\varepsilon C)V_n((\lambda+\varepsilon)(D-D)+\lambda D)$, which is a polynomial of λ and ε by the polynomial expansion of the Minkowski sum, and $f(0,0)=0$. Thus

$$\frac{d}{d\varepsilon}\left(\int_{-\varepsilon}^{0} f(\lambda,\varepsilon)\,d\lambda\right)_{|\varepsilon=0} = f(0,0) = 0,$$

which, together with (2-9) and Fact 2.10, leads to

$$\frac{d}{d\varepsilon}\left(\int_{-\varepsilon}^{0} V_{m+n}([C \amalg D]_{\lambda,\varepsilon})\,d\lambda\right)_{|\varepsilon=0} = 0. \tag{2-10}$$

Similarly, we have

$$\frac{d}{d\varepsilon}\left(\int_{1-\varepsilon}^{1} V_{m+n}([C \amalg D]_{\lambda,\varepsilon})\,d\lambda\right)_{|\varepsilon=0} = 0. \tag{2-11}$$

Now, (2-8), (2-10), (2-11) and Lemma 2.9 show that

$$\begin{aligned}(m+n+1)&V((C \amalg D)[m+n])\\ &= \frac{d}{d\varepsilon}\left(\int_{0}^{1-\varepsilon} V_{m+n}([C \amalg D]_{\lambda,\varepsilon})\,d\lambda\right)_{|\varepsilon=0}\\ &= (m+n+1)B(m+1,n+1)\\ &\qquad \times (V_m(C)V_n(D) + V(C[m-1])V_n(D) + V_m(C)V(D[n-1])),\end{aligned}$$

which, together with (iii) in Remark 1.2 and Lemma 2.7, leads to

$$\begin{aligned}\mathrm{as}_1(C \amalg D) &= \frac{V((C \amalg D)[m+n])}{V_{m+n+1}(C \amalg D)} \\ &= \frac{V_m(C)V_n(D) + V(C[m-1])V_n(D) + V_m(C)V(D[n-1])}{V_m(C)V_n(D)} \\ &= 1 + \mathrm{as}_1(C) + \mathrm{as}_1(D). \qquad \square\end{aligned}$$

3. The Minkowski measure of coproducts of convex bodies

In this section, we will show that Theorem 2.6 also holds for the well-known Minkowski measure as_∞.

First, given $C \in \mathcal{K}^n$, for any fixed $x \in \mathrm{int}(C)$, define

$$\gamma(C, x) := \sup\{f(x) \mid f \in C^a\},$$

where $C^a := \{f \in \mathrm{aff}(R^n) \mid f(C) = [-1, 1]\}$. It is easy to check (see [Guo 2005]) that $\mu_\infty(C, x) = (1 + \gamma(C, x))/(1 - \gamma(C, x))$. Defining a measure of asymmetry $\mathrm{As}(C)$ of C by

$$\mathrm{As}(C) = \inf_{x \in \mathrm{int}(C)} \gamma(C, x),$$

we have $0 \le \mathrm{As}(C) \le (n-1)/(n+1)$ for $C \in \mathcal{K}^n$ and

$$\mathrm{As}(C) = \frac{\mathrm{as}_\infty(C) - 1}{\mathrm{as}_\infty(C) + 1} \quad \text{or} \quad \mathrm{as}_\infty(C) = \frac{1 + \mathrm{As}(C)}{1 - \mathrm{As}(C)}.$$

Then x is an ∞-critical point if and only if it is an As-critical point, and it is reasonable to study the Minkowski measure $\mathrm{as}_\infty(C)$ in terms of $\gamma(C, x)$ and $\mathrm{As}(C)$.

Definition. For $C \in \mathcal{K}^m$ and $D \in \mathcal{K}^n$, we define the affine direct sum of C^a and D^a, $C^a \boxplus D^a \subset \mathrm{aff}(\mathbb{R}^m) \times \mathrm{aff}(\mathbb{R}^n)$, by

$$C^a \boxplus D^a := (\{1_C\} \times D^a) \cup (C^a \times \{1_D\}),$$

where 1_C and 1_D denote the constant function 1 respectively on $\mathbb{R}^m$ and $\mathbb{R}^n$.

Under the same correspondence as in Proposition 2.5, $C^a \boxplus D^a$ can be identified with a subset of $(C \amalg D)^a$, and it is easy to check that

$$C^a \boxplus D^a \subsetneq (C \amalg D)^a \subsetneq (C \amalg D)^A.$$

Lemma 3.1. *Given $C \in \mathcal{K}^m$ and $D \in \mathcal{K}^n$, for any fixed $z = ((1-\lambda)x, \lambda y, \lambda)$ in $\mathrm{int}(C \amalg D)$ (i.e., $x \in \mathrm{ri}(C)$, $y \in \mathrm{ri}(D)$ and $0 < \lambda < 1$),*

$$\gamma_z := \gamma(C \amalg D, z) = \sup_{(f,g) \in (C \amalg D)^a} (f, g)(z) = \sup_{(f,g) \in C^a \boxplus D^a} (f, g)(z),$$

where $f \in \mathrm{aff}(\mathbb{R}^m)$, $g \in \mathrm{aff}(\mathbb{R}^n)$ and $(f, g)(z) := (1-\lambda)f(x) + \lambda g(y)$.

Proof. By a standard compactness argument, there is $(f_0, g_0) \in (C \amalg D)^a$ such that

$$\gamma_z = (f_0, g_0)(z) = (1-\lambda) f_0(x) + \lambda g_0(y). \tag{3-1}$$

Now we will show that $f_0 = 1_C$ and $g_0 \in D^a$ or $g_0 = 1_C$ and $f_0 \in C^a$.

To see this, we observe first that $f_0(C) \subset [-1, 1]$, $g_0(D) \subset [-1, 1]$ and

$$(f_0, g_0)(C \amalg D) = \operatorname{conv}(f_0(C) \cup g_0(D)), \tag{3-2}$$

which can be easily checked by the definition of $C \amalg D$ (in fact, this holds for any $(f, g) \in (C \amalg D)^a$).

Then we claim that $1 \in f_0(C)$ and $1 \in g_0(D)$. Suppose it is not true that, say, $1 \notin f_0(C)$. Then (3-2) implies that $1 \in g_0(D)$ since $(f_0, g_0)(C \amalg D) = [-1, 1]$, and either -1 is in $f_0(C)$ or $g_0(D)$. However, we will see that in either case there is a contradiction.

If $-1 \in g_0(D)$, then $g_0 \in D^a$. Thus $(1_C, g_0) \in (C \amalg D)^a$ and we have the inequality $(1_C, g_0)(z) > (f_0, g_0)(z)$, which contradicts (3-1).

If $-1 \in f_0(C)$, then we can find $f_1 \in C^a$ such that $\{f_1 = -1\} = \{f_0 = -1\}$, which implies that $f_1(x) > f_0(x)$ (since $1 \notin f_0(C)$). Thus $(f_1, g_0) \in (C \amalg D)^a$ and $(f_1, g_0)(z) > (f_0, g_0)(z)$ which contradicts (3-1) too. Hence we have confirmed our claim.

Now, with a similar argument, we can show that -1 is in $f_0(C)$ or $g_0(D)$.

If $-1 \in g_0(D)$, then $g_0 \in D^a$, and we must have $f_0 = 1_C$ since $(1_C, g_0) \in (C \amalg D)^a$ and $(1_C, g_0)(z) > (f, g_0)(z)$ for all $f \neq 1_C$. Thus $(f_0, g_0) = (1_C, g_0) \in C^a \boxplus D^a$.

Similarly, if $-1 \in f_0(C)$, then $g_0 = 1_D$ and so $(f_0, g_0) = (f_0, 1_D) \in C^a \boxplus D^a$. □

Now we can prove the following generalization of Theorem 2 in [Guo and Kaijser 2002].

Theorem 3.2. *For any $C \in \mathcal{K}^m$ and $D \in \mathcal{K}^n$ $(m, n \geq 0)$,*

$$\operatorname{as}_\infty(C \amalg D) = \operatorname{as}_\infty(C) + \operatorname{as}_\infty(D) + 1,$$

where we take the convention that $\operatorname{as}_\infty(C) = 0$ *for $C \in \mathcal{K}^0$. Moreover, all ∞-critical points z^* of $C \amalg D$ have the form*

$$z^* = \frac{1-\gamma_y}{2-\gamma_x-\gamma_y} x^* + \frac{1-\gamma_x}{2-\gamma_x-\gamma_y} y^*,$$

where $x^* = (x, 0, 0)$ with x being an ∞-critical point of C, and $y^* = (0, y, 1)$ with y being an ∞-critical point of D, and $\gamma_x := \gamma(C, x)$, $\gamma_y := \gamma(D, y)$.

Proof. If $m = n = 0$, the same argument as in the proof of Theorem 2.6 can be applied.

If $m = 0$, $n \geq 1$ or $m \geq 1$, $n = 0$, it reduces to Theorem 2 in [Guo and Kaijser 2002].

Now assume $m \geq 1$, $n \geq 1$. We first prove a general result: for any $\bar{x} := (x, 0, 0)$ with $x \in \mathrm{ri}(C)$ and $\bar{y} := (0, y, 1)$ with $y \in \mathrm{ri}(D)$,

$$\min_{z\in(\bar{x},\bar{y})} \gamma(C \amalg D, z) = \gamma(C \amalg D, z_0) = \frac{1-\gamma_x\gamma_y}{2-\gamma_x-\gamma_y}, \tag{3-3}$$

where $(\bar{x}, \bar{y})$ is the open interval with $\bar{x}, \bar{y}$ as ends and

$$z_0 = \frac{1-\gamma_y}{2-\gamma_x-\gamma_y}\bar{x} + \frac{1-\gamma_x}{2-\gamma_x-\gamma_y}\bar{y}.$$

In fact, for any $(1, g), (f, 1) \in C^a \boxplus D^a$,

$$\begin{aligned}(1, g)(z_0) &= \frac{1-\gamma_y}{2-\gamma_x-\gamma_y} + \frac{1-\gamma_x}{2-\gamma_x-\gamma_y}g(y)\\ &\leq \frac{1-\gamma_y}{2-\gamma_x-\gamma_y} + \frac{1-\gamma_x}{2-\gamma_x-\gamma_y}\gamma_y = \frac{1-\gamma_x\gamma_y}{2-\gamma_x-\gamma_y},\end{aligned}$$

$$\begin{aligned}(f, 1)(z_0) &= \frac{1-\gamma_y}{2-\gamma_x-\gamma_y}f(x) + \frac{1-\gamma_x}{2-\gamma_x-\gamma_y}\\ &\leq \frac{1-\gamma_y}{2-\gamma_x-\gamma_y}\gamma_x + \frac{1-\gamma_x}{2-\gamma_x-\gamma_y} = \frac{1-\gamma_x\gamma_y}{2-\gamma_x-\gamma_y},\end{aligned}$$

with equality in the first formula if $g \in D^a$ such that $g(y) = \gamma_y$ and equality in the second formula if $f \in C^a$ such that $f(x) = \gamma_x$. So by Lemma 3.1, we have $\gamma_{z_0} = (1-\gamma_x\gamma_y)/(2-\gamma_x-\gamma_y)$.

Now, for $z = \lambda\bar{x} + (1-\lambda)\bar{y} \in [\bar{x}, \bar{y}]$, if $\lambda > (1-\gamma_y)/(2-\gamma_x-\gamma_y)$, we choose $g_0 \in D^a$ such that $g_0(y) = \gamma_y$. Then

$$\begin{aligned}(1, g_0)(z) &= \lambda + (1-\lambda)\gamma_y = (1-\gamma_y)\lambda + \gamma_y\\ &> (1-\gamma_y)\frac{1-\gamma_y}{2-\gamma_x-\gamma_y} + \gamma_y = \frac{1-\gamma_x\gamma_y}{2-\gamma_x-\gamma_y} = \gamma_{z_0},\end{aligned}$$

which implies that $\gamma_z > \gamma_{z_0}$.

If $\lambda < (1-\gamma_y)/(2-\gamma_x-\gamma_y)$, then, noticing that $\gamma_x - 1 < 0$, we have

$$\begin{aligned}(f_0, 1)(z) &= \lambda\gamma_x + (1-\lambda) = (\gamma_x-1)\lambda + 1\\ &> (\gamma_x-1)\frac{1-\gamma_y}{2-\gamma_x-\gamma_y} + 1 = \frac{1-\gamma_x\gamma_y}{2-\gamma_x-\gamma_y} = \gamma_{z_0},\end{aligned}$$

which also implies that $\gamma_z > \gamma_{z_0}$. Hence (3-3) is confirmed.

Finally, since it is easy to check that $\mu_\infty(C, x) = (1+\gamma(C, x))/(1-\gamma(C, x))$, we can use the fact that the function $(1+t)/(1-t)$ is increasing on [0, 1), to get

$$\begin{aligned}\mu_\infty(C \amalg D, z_0) &= \min_{z\in(\bar{x},\bar{y})} \mu_\infty(C \amalg D, z) = \frac{1+\gamma_{z_0}}{1-\gamma_{z_0}} \\ &= \Bigl(1+\frac{1-\gamma_x\gamma_y}{2-\gamma_x-\gamma_y}\Bigr)\Bigl(1-\frac{1-\gamma_x\gamma_y}{2-\gamma_x-\gamma_y}\Bigr)^{-1} \\ &= \frac{3-\gamma_x-\gamma_y-\gamma_x\gamma_y}{1-\gamma_x-\gamma_y+\gamma_x\gamma_y} = \frac{1+\gamma_x}{1-\gamma_x}+\frac{1+\gamma_y}{1-\gamma_y}+1 \\ &= \mu_\infty(C, x)+\mu_\infty(D, y)+1.\end{aligned}$$

It follows that

$$\mathrm{as}_\infty(C \amalg D) = \min_{x\in \mathrm{ri}(C),\, y\in\mathrm{ri}(D)} (\mu_\infty(C, x)+\mu_\infty(D, y)+1) = \mathrm{as}_\infty(C)+\mathrm{as}_\infty(D)+1$$

and all ∞-critical points z^* of $C \amalg D$ have the form

$$z^* = \frac{1-\gamma_y}{2-\gamma_x-\gamma_y}x^* + \frac{1-\gamma_x}{2-\gamma_x-\gamma_y}y^*,$$

where $x^* = (x, 0, 0)$ with x being an ∞-critical point of C and $y^* = (0, y, 1)$ with y being an ∞-critical point of D. □

Remark 3.3. Let $\mathcal{A} := \{C \in \mathcal{K}^k \mid \mathrm{as}_1(C) = \mathrm{as}_\infty(C),\ k = 0, 1, 2, \dots\}$ be the class of convex bodies whose p-measures coincide for all p, in all dimensions. Then $\mathcal{A}$ is closed under invertible affine transformations and coproducts of convex bodies, as follows from Theorems 2.6 and 3.2. Observe also that a simplex in k dimensions can be considered as the $(k+1)$-fold coproduct of its vertices (trivially symmetric convex bodies in 0 dimensions). Thus, we have naturally the following questions:

Question 1. Is the class of symmetric convex bodies a generating set for $\mathcal{A}$ under invertible affine transformations and coproducts?

Question 2. Does $\mathrm{as}_1(C) = \mathrm{as}_\infty(C)$ hold if $\mathrm{as}_1(C) = \mathrm{as}_2(C)$ (or, generally, if $\mathrm{as}_{p_1}(C) = \mathrm{as}_{p_2}(C)$ for distinct p_1, p_2)?

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