REGULARITY AND ANALYTICITY OF SOLUTIONS IN A DIRECTION FOR ELLIPTIC EQUATIONS

Yongyang Jin, Dongsheng Li and Xu-Jia Wang
In this paper, we study the regularity and analyticity of solutions to linear elliptic equations with measurable or continuous coefficients. We prove that if the coefficients and inhomogeneous term are Hölder-continuous in a direction, then the second-order derivative in this direction of the solution is Hölder-continuous, with a different Hölder exponent. We also prove that if the coefficients and the inhomogeneous term are analytic in a direction, then the solution is analytic in that direction.

1. Introduction

We study the regularity and analyticity of solutions in a given direction to the elliptic equation

\[(1-1) \sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u = f(x) \quad \text{in} \quad \Omega,\]

assuming that the coefficients \(a_{ij}, b_i, c\) and the inhomogeneous term \(f\) are smooth or analytic along the direction, where \(\Omega\) is a bounded domain in the Euclidean space \(\mathbb{R}^n\). We assume that the equation is uniformly elliptic, namely, that there exist positive constants \(\Lambda > \lambda > 0\) such that

\[(1-2) \quad \lambda |\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \Lambda |\xi|^2 \quad \text{for all} \quad x \in \Omega.\]

We also assume that \(b_i, c \in L^\infty(\Omega)\), and \(f \in L^n(\Omega)\).

The regularity of solutions is a fundamental issue in the study of partial differential equations. Most regularity theories, such as the Schauder estimate and the \(W^{2,p}\) estimate, are isotropic; namely, the solution is uniformly regular in all directions. An interesting question is whether the solution to (1-1) is smooth in a direction if the coefficients \(a_{ij}, b_i, c\) and the inhomogeneous term \(f\) are smooth in this direction only. This question can be asked for more general nonlinear elliptic and parabolic...
equations. One may also consider the regularity when the coefficients $a_{ij}, b_i, c$ and the inhomogeneous term $f$ are smooth in a submanifold of high codimensions.

This is a significant problem in partial differential equations as it is not only stronger than the Schauder estimate but also has applications in areas such as fluid mechanics, partial differential systems, manifolds with nonsmooth metric tensors, and other physical problems such as the propagation of singularities [Taylor 2000; Kukavica and Ziane 2007; Cao and Titi 2008; 2011]. For many PDE systems if one can first prove the regularity of solutions in a direction, one may be able to obtain the full regularity. At a first glance, one may feel that an affirmative answer would be too good to be true, even for an expert in the area. However in this paper we show that this is indeed true at least in dimension two, and also in higher dimensions if the coefficients are continuous. At the moment we are not aware of a counterexample without the continuity. This question is also open for most nonlinear equations and deserves further investigations.

The analyticity of solutions is also an important topic in the regularity theory of partial differential equations. For the linear elliptic equation (1-1), it is well known that if the coefficients $a_{ij}, b_i, c$ and the inhomogeneous term $f$ are analytic, then the solution is also analytic. A similar question is whether the solution is analytic in a direction if $a_{ij}, b_i, c$ and $f$ are analytic only in the given direction.

Let us first state our results on the analyticity of solutions in a given direction:

**Theorem 1.1.** Let $u \in W^{2,n}(\Omega)$ be a strong solution to (1-1). Assume that the coefficients $a_{ij}, b_i, c$ and the inhomogeneous term $f$ are independent of the variable $x_n$. Then the solution $u$ is analytic in $x_n$.

The proof of Theorem 1.1 is based on the Krylov–Safonov Hölder-continuity of linear elliptic equations. Using the $W^{2,p}$ estimate, we also have:

**Theorem 1.2.** Let $u \in W^{2,n}(\Omega)$ be a strong solution to (1-1). Assume that the coefficients $a_{ij}$ are continuous, and $a_{ij}, b_i, c$ and $f$ are analytic in the variable $x_n$. Then the solution $u$ is analytic in $x_n$.

In Theorem 1.1, we do not assume the continuity of the coefficients $a_{ij}, b_i, c$ but in Theorem 1.2 we do. An interesting question is whether one can remove the continuity of the $a_{ij}$ in Theorem 1.2. An affirmative answer can be given in dimension two:

**Theorem 1.3.** Let $u \in W^{2,2}(\Omega)$ be a strong solution to (1-1). Assume that $n = 2$ and $a_{ij}, b_i, c$ and $f$ are analytic in the variable $x_2$. Then the solution $u$ is analytic in $x_2$.

Our results are stronger than the classical results on the analyticity of solutions to linear elliptic equations. In the classical theory the coefficients $a_{ij}, b_i, c$ and the inhomogeneous term $f$ are assumed to be analytic in all directions.

When the coefficients are Hölder-continuous in a given direction, we have the following directional $C^{2,\alpha}$ regularity:
Theorem 1.4. Let \( u \in W^{2,n}(\Omega) \) be a strong solution to (1-1). Suppose that \( a_{ij}, b_i, c \) are \( C^\alpha \) in the \( \xi \)-direction for some \( 0 < \alpha < 1 \) and \( a_{ij} \in C^0(\Omega) \) and satisfy (1-2). Suppose \( f \in L^p(\Omega) \) for some \( p > n/\alpha \). Then for any \( 0 < \beta < \alpha - n/p \) and any \( y, z \in \Omega_\delta \), we have the estimate

\[
|\partial_\xi \partial_x u(y) - \partial_\xi \partial_x u(z)| \leq C d^\beta \left[ \sup_{\Omega} |u| + \|f\|_{L^p(\Omega)} + \int_0^d \frac{\omega_{f,\xi}(r)}{r^{1+\beta}} \right] + C \int_0^d \frac{\omega_{f,\xi}(r)}{r} + C \|a_{ij}\|_{C^\alpha(\Omega)} (\|f\|_{L^p(\Omega)} + \sup_{\Omega} |u|) d^{\alpha-n/p},
\]

where \( \Omega_\delta = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \delta \} \) and \( d = |y - z| \). The constant \( C \) depends on \( n, \alpha, \beta, \delta, p, \lambda, \Lambda \) and the modulus of continuity of \( a_{ij} \).

In Theorem 1.4, \( \xi \) is a given unit vector, and the notation \( \omega_{f,\xi} \) is defined at the beginning of Section 4. The continuity assumption of the \( a_{ij} \) is for the use of the \( W^{2,p} \) estimate, hence it suffices to assume that the \( a_{ij} \) are in the VMO space [Chiarenza et al. 1993], or the \( a_{ij} \) are continuous in \( n-1 \) variables [Kim and Krylov 2007]. In particular, in dimension two, by the \( W^{2,p} \) estimate in the latter reference, the continuity of the \( a_{ij} \) is not needed. Hence we have:

Corollary 1.5. Let \( u \in W^{2,2}(\Omega) \) be a strong solution to (1-1). Assume that \( n = 2 \) and \( a_{ij}, b_i, c \) and \( f \) are Hölder-continuous in direction \( \xi \). Then \( \partial_\xi \partial_x u \) is Hölder-continuous.

Note that the Hölder-continuity of \( \partial_\xi \partial_x u \) in Theorem 1.4 and Corollary 1.5 is uniform in all directions. But the Hölder exponent of the second derivative is smaller than that of the coefficients and we need to assume \( f \in L^p \) for a large \( p \).

Theorem 1.4 improves [Tian and Wang 2010, Theorem 3.2], where the coefficients \( a_{ij} \) were assumed to be Lipschitz in \( \xi \), and the directional \( C^{2,\alpha} \) regularity was obtained by differentiating (1-1). We point out that Corollary 1.5 was also obtained in [Dong 2012, Section 6]. By the \( W^{2,p} \) estimate [Kim and Krylov 2007], related result holds in higher dimension too. That is, if \( u \) is a strong solution to (1-1) and if \( a_{ij}, b_i, c \) and \( f \) are Hölder-continuous in \( x' = (x_1, \ldots, x_{n-1}) \), then \( \partial_{x'} \partial_x u \) is Hölder-continuous. The \( C^{2,\alpha} \) regularity of solutions in a given direction was also investigated in [Dong and Kim 2011]. See also [Tian and Wang 2010] for discussions.

To prove Theorems 1.1–1.3, we introduce appropriate function spaces and establish related interpolation inequalities. We will prove Theorem 1.1 in Section 2, Theorems 1.2 and 1.3 in Section 3, and Theorem 1.4 in Section 4. In Section 5, we give a brief discussion on equations of divergence form.
2. Proof of Theorem 1.1

For simplicity we assume \( b_i = c = 0 \); namely, we consider the equation

\[
L[u] := \sum_{i,j=1}^{n} a_{i,j}(x')u_{ij} = f(x) \quad \text{in } \Omega,
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( x' = (x_1, \ldots, x_{n-1}) \), and \( u_{ij} = u_{x_i x_j} \). The proof is similar if \( b_i \neq 0 \) and \( c \neq 0 \), provided they satisfy the conditions specified in the introduction. We assume that the coefficients \( a_{ij} \) are measurable and satisfy the uniformly elliptic condition (1-2), \( f \in L^n(\Omega) \), and the \( a_{ij} \) and \( f \) are analytic in the \( x_n \) variable.

Set \( u' = u_{x_n}, u'' = u_{x_n x_n} \),

\[
u^{(k)} = \frac{\partial^k u}{\partial x_n^k}, \quad k = 1, 2, \ldots,
\]

\[
\langle u \rangle_{\alpha, \Omega} = \sup_{x, y \in \Omega} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \right\}
\]

and

\[
|u|_{k+\alpha, \Omega} = \sup_{x \in \Omega} |u| + \langle u^{(k)} \rangle_{\alpha, \Omega}, \quad k = 0, 1, 2, \ldots,
\]

\[
\|u\|_{k+\alpha, \Omega} = \sup_{x \in \Omega} |u| + \sup_{x, y \in \Omega} \frac{|D^k u(x) - D^k u(y)|}{|x - y|^{\alpha}},
\]

where \( 0 < \alpha \leq 1 \) and \( (y - x)/e_n \) means the vector \( y - x \) is parallel to the vector \( e_n = (0, \ldots, 0, 1) \). We also set

\[
\langle u^{(k)} \rangle_{\alpha, \Omega}^\beta = \sup_{Q_r(x) \subset \Omega} r^{k+\alpha+\beta} \langle u^{(k)} \rangle_{\alpha, Q_r(x)}, \quad \beta \in \mathbb{R},
\]

\[
|u|_{k+\alpha, \Omega}^\beta = \sup_{Q_r(x) \subset \Omega} \left[ r^\beta \|u\| + r^{k+\alpha+\beta} \langle u^{(k)} \rangle_{\alpha, Q_r(x)} \right],
\]

and

\[
\|u\|_{k+\alpha, \Omega}^\beta = \sup_{Q_r(x) \subset \Omega} \left[ r^\beta \|u\|_{L^\infty(Q_r(x))} + r^{k+\alpha+\beta} \sup_{y, z \in Q_r(x)} \frac{|D^k u(y) - D^k u(z)|}{|y - z|^{\alpha}} \right],
\]

where \( Q_r(x) \) denotes the open cube with center \( x \) and side-length \( 2r \). We can extend the above definition to \( \alpha = 0 \) by letting

\[
|u|_{k, \Omega}^{(k)} = \sup_{Q_r(x) \subset \Omega} \left[ r^\beta \|u\|_{L^\infty(Q_r(x))} + r^{k+\beta} \langle u^{(k-1)} \rangle_{1, Q_r(x)} \right] \quad \text{if } k > 0,
\]

\[
|u|_{0, \Omega}^{(k)} = \sup_{Q_r(x) \subset \Omega} r^\beta \|u\|_{L^\infty(Q_r(x))} \quad \text{if } k = 0.
\]
We point out the equivalence of the norm $|u|_{k+\alpha,\Omega}^{(\beta)}$ given in (2-3) and the norm
\[
[u]_{k+\alpha,\Omega}^{(\beta)} := \sup_{Q_{(1+\sigma)r}(x) \subset \Omega} \left[ r^\beta \| u \|_{L^\infty(Q_r(x))} + r^{k+\alpha+\beta}(u^{(k)})_{\alpha, Q_r(x)} \right],
\]
where $\sigma > 0$ is a constant. Namely,
\[
C^{-1} |u|_{k+\alpha,\Omega}^{(\beta)} \leq [u]_{k+\alpha,\Omega}^{(\beta)} \leq C |u|_{k+\alpha,\Omega}^{(\beta)},
\]
for some constant $C$ depending only on $n$, $k$, $\alpha$, $\beta$ and $\sigma$. To prove the above inequalities, it suffices to divide the cube $Q_{3r/2}$ into $2^n$ disjoint smaller cubes if $\sigma \in [\frac{1}{2}, 2]$, and divide into more, smaller cubes for other $\sigma$. Note that if $\beta = -k$, the constant $C$ is independent of $k$.

We also point out three differences between our definition of the norms $|u|_{k+\alpha,\Omega}^{(\beta)}$ and the usual one [Gilbarg and Trudinger 1998]. That is, (i) the derivative in the former one is taken only on the $x_n$-direction; (ii) in the Hölder seminorm (2-2) we assume that $(y-x)/e_n$; and (iii) the supremum in (2-3) is taken among all cubes $Q_r(x)$ satisfying the condition $Q_{2r}(x) \subset \Omega$. The reason of choosing the cubes with the property $Q_{2r}(x) \subset \Omega$ is that the norm is homogeneous under rescaling.

First we prove an interpolation inequality for the norm $\|u\|_{k+\alpha,\Omega}^{(\beta)}$.

**Lemma 2.1.** Suppose that $j + \beta < k + \alpha$, where $j, k = 0, 1, 2, \ldots$ and $0 \leq \alpha, \beta \leq 1$. Assume that $u \in C^{k,\alpha}(\Omega)$. Then there exists a positive constant $C$ depending on $j$, $k$, $\alpha$, $\beta$, such that
\[
\|u\|_{j+\beta,\Omega}^{(\gamma)} \leq C [\|u\|_{k+\alpha,\Omega}^{(\gamma)}]^{(j+\beta)/(k+\alpha)} [\|u\|_{k+\alpha,\Omega}^{(\gamma)}]^{1-(j+\beta)/(k+\alpha)}.
\]

**Proof.** It is well known [Hörmander 1976] that there is a positive constant $C = C(j, k, \alpha, \beta)$ such that
\[
\|u\|_{j+\beta, Q_1(0)} \leq C [\|u\|_{k+\alpha, Q_1(0)}]^{(j+\beta)/(k+\alpha)} [\|u\|_{L^\infty(Q_1(0))}^{(j+\beta)/(k+\alpha)}]^{1-(j+\beta)/(k+\alpha)}.
\]
For any $Q_r(x) \subset \Omega$, by rescaling, we obtain
\[
\|u\|_{L^\infty(Q_r(x))} + r^{j+\beta} \langle D^j u \rangle_{\beta, Q_r(x)} \\
\quad \leq C [\|u\|_{L^\infty(Q_r(x))}]^{1-(j+\beta)/(k+\alpha)} \\
\quad \times (\|u\|_{L^\infty(Q_r(x))} + r^{k+\alpha} \langle D^k u \rangle_{\alpha, Q_r(x)})^{(j+\beta)/(k+\alpha)}.
\]
That is,
\[
r^\gamma \|u\|_{L^\infty(Q_r(x))} + r^{j+\beta+\gamma} \langle D^j u \rangle_{\beta, Q_r(x)} \\
\quad \leq C (r^\gamma \|u\|_{L^\infty(Q_r(x))})^{1-(j+\beta)/(k+\alpha)} \\
\quad \times (r^\gamma \|u\|_{L^\infty(Q_r(x))} + r^{k+\alpha+\gamma} \langle D^k u \rangle_{\alpha, Q_r(x)})^{(j+\beta)/(k+\alpha)}.
\]
Taking the supremum of all cubes $Q_r(x)$ with $Q_{2r}(x) \subset \Omega$, we obtain (2-4). \qed

Next we extend the inequality (2-4) to the norm $|u|_{k+\alpha,\Omega}^{(\beta)}$:
Lemma 2.2. Suppose that \( j + \beta < k + \alpha \), where \( j, k = 0, 1, 2, \ldots \) and \( 0 \leq \alpha, \beta \leq 1 \). Assume that \( u \in L^\infty(\Omega) \) and \( u^{(k)} \in C^\alpha(\Omega) \). Then there exists a positive constant \( C \) depending on \( j, k, \alpha, \beta \), such that

\[
|u|_{j+\beta, \Omega}^{(y)} \leq C [ |u|_{k+\alpha, \Omega}^{(y)} ]^{(j+\beta)/(k+\alpha)} [ |u|_{0, \Omega}^{(y)} ]^{1-(j+\beta)/(k+\alpha)}.
\]

Proof. By the rescaling argument in the proof of Lemma 2.1, it suffices to prove

\[
|u|_{j+\beta, Q_1(0)} \leq C (|u|_{k+\alpha, Q_1(0)})^{(j+\beta)/(k+\alpha)} (|u|_{L^\infty(Q_1(0))})^{1-(j+\beta)/(k+\alpha)}.
\]

By the definition (2-3), it suffices to prove

\[
\langle \gamma \rangle_{j+\beta, Q_1(0)} \leq C (|u|_{k+\alpha, Q_1(0)})^{(j+\beta)/(k+\alpha)} (|u|_{L^\infty(Q_1(0))})^{1-(j+\beta)/(k+\alpha)}.
\]

Again, by the definition of (2-3), there exists \( x'_0 \) such that

\[
\langle \gamma \rangle_{j+\beta, Q_1(0)} \leq 2 \sup \left\{ \frac{|u(x'_0, x_n) - u(y'_0, y_n)|}{|x_n - y_n|^{\beta}} : -1 < x_n, y_n < 1 \right\}
\]

where \( I = (-1, 1) \subset \mathbb{R}^1 \) is the unit interval. By (2-5) in the one-dimensional case, the right-hand side is bounded by

\[
\langle \gamma \rangle_{j+\beta, Q_1(0)} \leq (|u(x'_0, \cdot)|_{k+\alpha, I})^{(j+\beta)/(k+\alpha)} (|u(y'_0, \cdot)|_{L^\infty(I)})^{1-(j+\beta)/(k+\alpha)}
\]

\[
\leq (|u|_{k+\alpha, Q_1(0)})^{(j+\beta)/(k+\alpha)} (|u|_{L^\infty(Q_1(0))})^{1-(j+\beta)/(k+\alpha)}. \quad \square
\]

Theorem 2.3. Let \( u \in W^{2,n}(\Omega) \) be a strong solution of (2-1), where the coefficients \( a_{ij} \) are measurable and independent of \( x_n \) and satisfy the uniformly elliptic condition (1-2). Assume that \( f \) is analytic in \( x_n \). Then there exists a constant \( C = C(n, \lambda, \Lambda) \) such that, for any \( Q_R(x_0) \subset \Omega \), the following inequality holds:

\[
|u^{(k)}(x_0)| \leq \left( \frac{Ck}{R} \right) |u|_{L^\infty(Q_R(x_0))} + 1.
\]

Proof. As the coefficients \( a_{ij} \) are independent of \( x_n \) and \( u \) is a strong solution, one sees that

\[
u'_\delta := \frac{1}{\delta} (u(x + \delta e_n) - u(x))
\]

is a strong solution to \( L[u] = f'_\delta \), where \( L \) is the elliptic operator in (2-1). Hence the Krylov–Safonov Hölder estimate holds for \( u'_\delta \), uniformly in \( \delta \). Similarly,

\[
u''_\delta := \frac{1}{\delta^2} (u(x + \delta e_n) + u(x - \delta e_n) - 2u(x))
\]

is a strong solution to \( L[u] = f''_\delta \), and is uniformly Hölder-continuous as \( \delta \to 0 \). Sending \( \delta \to 0 \), we see that \( u'' \) is Hölder-continuous. By induction, we see that for
any $k > 0$, $u^{(k)}$ is Hölder-continuous, and

$$\langle u^{(k)} \rangle_{\alpha, Q_{1/4}(x)} \leq C \left( \| u^{(k)} \|_{L^\infty(Q_{1/2}(x))} + \| f^{(k)} \|_{L^\infty(Q_{1/2}(x))} \right)$$

for all $k = 1, 2, \ldots$, and the constant $C$ is independent of $k$.

Set $Q_0 = Q_R(x_0)$. Let $Q_{2r}(\hat{x}) \subset Q_R(x_0)$ be any given cube. Then there exist $x_1, x_2 \in Q_r(\hat{x})$ with $(x_2 - x_1)/e_n$ such that

$$r^{1+\alpha} \langle u' \rangle_{\alpha, Q_r(\hat{x})} \leq 2r^{1+\alpha} |u'(x_2) - u'(x_1)| / |x_2 - x_1|^\alpha.$$  

If $|x_2 - x_1| \geq \frac{1}{4} r$, then, by Lemma 2.2 with $j = 1$, $\beta = 0$, $k = 1$,

$$r^{1+\alpha} \langle u' \rangle_{\alpha, Q_r(\hat{x})} \leq 2 \cdot 4^\alpha r |u'(x_1) - u'(x_2)| \leq 4^{1+\alpha} r \| u' \|_{L^\infty(Q_r(\hat{x}))} \leq C \left( r^{1+\alpha} \langle u' \rangle_{\alpha, Q_r(\hat{x})} + \| u \|_{L^\infty(Q_r(\hat{x}))}^{1/(1+\alpha)} \right)^{1/(1+\alpha)} \| u \|_{L^\infty(Q_r(\hat{x}))}^{\alpha/(1+\alpha)} \leq C \left[ (r^{1+\alpha} \langle u' \rangle_{\alpha, Q_r(\hat{x})})^{1/(1+\alpha)} \| u \|_{L^\infty(Q_r(\hat{x}))}^{\alpha/(1+\alpha)} + \| u \|_{L^\infty(Q_r(\hat{x}))} \right].$$

If $|x_2 - x_1| < \frac{1}{4} r$, then, by (2-10) and Lemma 2.2,

$$r^{1+\alpha} \langle u' \rangle_{\alpha, Q_r(\hat{x})} \leq 2 \cdot r^{1+\alpha} \langle u' \rangle_{\alpha, Q_{r/4}(x_1)} \leq C \left[ r \| u' \|_{L^\infty(Q_{r/2}(x_1))} + r \| f' \|_{L^\infty(Q_{r/2}(x_1))} \right] \leq C \left[ (r^{1+\alpha} \langle u' \rangle_{\alpha, Q_{r/2}(x_1)})^{1/(1+\alpha)} \| u \|_{L^\infty(Q_{r/2}(x_1))}^{\alpha/(1+\alpha)} + \| u \|_{L^\infty(Q_{r/2}(x_1))} + r \| f' \|_{L^\infty(Q_{r/2}(x_1))} \right].$$

Taking the supremum among all the cubes $Q_r(\hat{x})$ with $Q_{2r}(\hat{x}) \subset Q_R(x_0)$, we obtain from the above estimates (2-11) and (2-12) that

$$\langle u' \rangle_{\alpha, Q_0}^{(0)} \leq C \left[ (\langle u' \rangle_{\alpha, Q_0})^{(0)} \right]^{1/(1+\alpha)} \| u \|_{L^\infty(Q_0)}^{\alpha/(1+\alpha)} + \| u \|_{L^\infty(Q_0)} + R \| f' \|_{L^\infty(Q_0)} \right],$$

which implies

$$|u'|_{1+\alpha, Q_0}^{(0)} \leq C \left[ \| u \|_{L^\infty(Q_0)} + R \| f' \|_{L^\infty(Q_0)} \right].$$

By Lemma 2.2 it follows that

$$\| u' \|_{L^\infty(Q_{R/2}(x_0))} \leq C \left[ \| u \|_{L^\infty(Q_0)} + R \| f' \|_{L^\infty(Q_0)} \right].$$

Hence we obtain

$$|u'(x_0)| \leq C \left[ \| u \|_{L^\infty(Q_0)} + R \| f' \|_{L^\infty(Q_0)} \right] \leq C \left[ \| u \|_{L^\infty(Q_0)} + 1 \right],$$

where we used the analyticity of $f$ in $x_n$. 

Next we estimate higher derivatives of $u$ at $x_0$. Suppose by induction that

$$u^{(k)}(x_0) \leq \left( \frac{C}{R} \right)^k (\|u\|_{L^\infty(Q_0)} + 1).$$

By (2-13), (2-14), and observing that for any $x \in Q_{R/(k+1)}(x_0)$, $Q_{kR/(k+1)}(x) \subset Q_R(x_0)$, we have

$$u^{(k+1)}(x_0) = |(u^{(k)})'(x_0)|$$

$$\leq \frac{C}{R} \left( \|u^{(k)}\|_{L^\infty(Q_{R/(k+1)}(x_0))} + \frac{R}{k+1} \|f^{(k+1)}\|_{L^\infty(Q_{R/(k+1)}(x_0))} \right)$$

$$\leq \frac{C(k+1)}{R} \left\{ \left( \frac{C}{R} \right)^k (\|u\|_{L^\infty(Q_0)} + 1) + \frac{R}{k+1} \|f^{(k+1)}\|_{L^\infty(Q_0)} \right\}$$

$$\leq \left( \frac{C}{R} \right)^{k+1} (k+1)^{k+1} (\|u\|_{L^\infty(Q_0)} + 1).$$

In the last inequality we used the analyticity of $f$ in $x_n$. 

**Theorem 2.4.** Let $u \in W^{2,n}(\Omega)$ be a strong solution to (2-1). Assume that the coefficients $a_{ij}$ are measurable and independent of $x_n$ and satisfy (1-2). Assume that $f$ is analytic in $x_n$. Then the solution $u$ is analytic in $x_n$.

**Proof.** For any given point $x_0 = (x_0', x_{0,n})$ in $\Omega$, let $r_0 = \frac{1}{4} \text{dist}(x_0, \partial\Omega)$. Consider the Taylor expansion of $u$ in $Q_{r_0}(x_0)$

$$u(x_0', x_n) = \sum_{k=0}^n \frac{u^{(k)}(x_0)}{k!} (x_n - x_{0,n})^k + \frac{u^{(n+1)}(x_0', \xi)}{(n+1)!} (x_n - x_{0,n})^{n+1},$$

where $\xi = t x_{0,n} + (1-t)x_n$ for some $t \in (0, 1)$. By Theorem 2.3, we know that

$$|u^{(k)}(x_0)| \leq \left( \frac{C}{r_0} \right)^k M,$$

$$|u^{(k+1)}(x_0', \xi)| \leq \left( \frac{C(k+1)}{r_0} \right)^{k+1} M,$$

where $M := \|u\|_{L^\infty(Q_{2r_0}(x_0))} + 1$. By Stirling’s formula we have

$$(k+1)^{k+1} < e^{k+1} (k+1)!. $$

Hence when $|x - x_0| \leq r_0/2Ce$ we have

$$\frac{|u^{(k)}(x_0)|}{k!} |x_n - x_{0,n}|^k \leq \frac{M}{2^k} \to 0 \quad \text{as} \quad k \to \infty.$$

Hence $u$ is analytic in the $x_n$ direction. 

□
3. Proof of Theorem 1.2

In this section we prove the analyticity of solutions in $x_n$ to the equation

$$L[u] := \sum_{i,j=1}^n a_{ij}(x) u_{ij} = f(x) \quad \text{in } \Omega,$$

where the coefficients $a_{ij}$ also depend on $x_n$. We assume that the $a_{ij}$ are in $C^0(\Omega)$ and satisfy (1-2) and $f \in L^p(\Omega)$ ($p \geq n$). We also assume that $a_{ij}$ and $f$ are analytic in $x_n$ and satisfy

$$\|\partial_{x_n}^k a_{ij} \| + |\partial_{x_n}^k f| \leq B^k k!$$

for all $k \geq 1$, where $B > 0$ is a constant.

As before, we set $u' = u_{x_n}$, $u'' = u_{x_n x_n}$ and $u^{(k)} = \frac{\partial^k u}{\partial x_n^k}$ for all integer $k \geq 1$. In this section we also set

$$[u]_{W^{2,p}(\Omega)} = \sum_{|\alpha| = 2} \|D^\alpha u\|_{L^p(\Omega)},$$

$$\begin{align*}
\left[u^{(\ell)}\right]_{W^{2,p}(\Omega)}^{(\beta)} &= \sup_{Q_r(x) \subset \Omega} d_{Q_r(x)}^{\ell+2-n/p+\beta} [u^{(\ell)}]_{W^{2,p}(Q_r(x))}, \\
\|u^{(\ell)}\|_{L^p(\Omega)}^{(\beta)} &= \sup_{Q_r(x) \subset \Omega} d_{Q_r(x)}^{\ell-n/p+\beta} \|u^{(\ell)}\|_{L^p(Q_r(x))},
\end{align*}$$

for $\ell = 0, 1, 2, \ldots$ and $\beta \in \mathbb{R}$, where $d_{Q_r(x)} = \text{dist}(Q_r(x), \partial \Omega)$.

By the $W^{2,p}$ estimate, we have:

Lemma 3.1. Let $u \in W^{2,n}(\Omega)$ be a strong solution to (3-1). Assume that the $a_{ij}$ are in $C^0(\Omega)$ and satisfy (1-2), $f \in L^p(\Omega)$ ($p \geq 1$), and $Q_R(x_0) \subset \Omega$. There exists a constant $C$ such that, if $0 < r < r + \delta < R$, then

$$\|u\|_{W^{2,p}(Q_r(x_0))} \leq C \left\{ \frac{1}{\delta^2} \|u\|_{L^p(Q_{r+\delta}(x_0))} + \|f\|_{L^p(Q_{r+\delta}(x_0))} \right\},$$

where $C$ depends only on $n$, $p$, $\lambda$, $\Lambda$ and the moduli of the continuity of the coefficients $a_{ij}$.

Proof. When $r \leq \delta$, by the $W^{2,p}$ estimate for elliptic equations [Gilbarg and Trudinger 1998] and a rescaling argument, we have

$$\|D^2 u\|_{L^p(Q_r(x_0))} \leq C \left( \frac{1}{\delta^2} \|u\|_{L^p(Q_{r+\delta}(x_0))} + (r + \delta)^2 \|f\|_{L^p(Q_{r+\delta}(x_0))} \right),$$

$$\leq C \left( \frac{1}{\delta^2} \|u\|_{L^p(Q_{r+\delta}(x_0))} + \|f\|_{L^p(Q_{r+\delta}(x_0))} \right).$$
When $\delta < r$, we choose $m \geq 2$ such that $r/m \leq \delta < r/(m-1)$, and equally divide the cube $Q_r(x_0)$ into smaller cubes with side-length $r/m$. Then

$$\|D^2 u\|_{L^p(Q_r(x_0))} = \sum_i \|D^2 u\|_{L^p(Q_r/m(x_i))}.$$  

By (3-5),

$$(3-6) \quad \|D^2 u\|_{L^p(Q_{r/m}(x_i))} \leq C\left(\frac{1}{\delta^{2p}}\|u\|_{L^p(Q_{2r/m}(x_i))} + \|f\|_{L^p(Q_{2r/m}(x_i))}\right).$$

Note that for each $Q_{r/m}(x_i)$ there are at most $3^n$ cubes of the form $Q_{2r/m}(x_j)$ intersecting with it. Hence, summing up, we obtain

$$(3-7) \quad \|D^2 u\|_{L^p(Q_r(x_0))} \leq C\left(\frac{1}{\delta^{2p}}\|u\|_{L^p(Q_{r+4}(x_0))} + \|f\|_{L^p(Q_{r+4}(x_0))}\right).$$

We obtain (3-4). \hfill \Box

We remark that in Lemma 3.1 the assumption $u \in W^{2,n}(\Omega)$ implies that $f \in L^n(\Omega)$. But the inequality (3-4) holds for all $p \geq 1$.

**Theorem 3.2.** Let $u \in W^{2,n}(\Omega)$ be a solution to (3-1). Assume that the $a_{ij}$ are in $C^0(\Omega)$ and satisfy (1-2). Assume also that the $a_{ij}$ and $f$ are analytic in $x_n$ and satisfy (3-2). Then $u$ is analytic in $x_n$.

**Proof.** By (3-1), we have

$$(3-8) \quad \sum a_{ij}(x + \delta e_n)[u_E']_{ij} = -\sum [a_{ij}']_{ij} u_{ij} + f',$$

where $u_E' = (1/\delta)[u(x + \delta e_n) - u(x)]$, $[a_{ij}]' = (1/\delta)[a_{ij}(x + \delta e_n) - a_{ij}(x)]$, and $e_n = (0, \ldots, 0, 1)$ is the unit vector on the $x_n$-axis. Since the $a_{ij}$ are continuous, by the $W^{2,p}$ estimate, we see that $u_E' \in W^{2,p}(\Omega')$ ($p = n$) for any $\Omega' \subset \Omega$. Sending $\delta \to 0$, we obtain that $u' \in W^{2,p}_{\text{loc}}(\Omega)$ and is a solution to $L[u'] = f' - a_{ij}' u_{ij}$. Similarly $u^{(k)} \in W^{2,p}_{\text{loc}}(\Omega)$ and is a solution to

$$(3-9) \quad L[u^{(k)}] = f^{(k)} - \sum_{\ell=1}^k \binom{k}{\ell} a_{ij}^{(\ell)} u^{(k-\ell)} := f^{(k)} - \phi \quad \text{in} \ \Omega,$$

where $\binom{k}{\ell} = k!/(\ell!(k-\ell)!)$.

We will prove Theorem 3.2 by induction. There is no loss of generality in assuming that $\Omega = Q_0$ is the cube of side-length two centered at the origin. By the definition of $[u]_{W^{2,p}(Q_0)}^{(n/p)}$, there exists a cube $Q_{r_0}(x_0) \subset Q_0$ such that

$$[u]_{W^{2,p}(Q_0)}^{(n/p)} \leq 2d_0^2 [u]_{W^{2,p}(Q_{r_0}(x_0))},$$
where \(d_0 = \text{dist}(Q_{r_0}(x_0), \partial Q_0)\). We may assume that the center of \(Q_{r_0}\) is the origin; otherwise we may replace \(Q_{r_0}(x_0)\) by the larger cube \(Q_{1-d_0}(0)\). Therefore the last inequality becomes

\[
[u]^{(n/p)}_{W^{2,p}(Q_0)} \leq 2(1 - r_0)^2[u]_{W^{2,p}(Q_{r_0})},
\]

where \(Q_{r_0}\) is centered at the origin. Thanks to Lemma 3.1, there is a constant \(C\) independent of \(r_0\) such that

\[
[u]_{W^{2,p}(Q_{r_0})} \leq C\{4(1 - r_0)^{-2}\|u\|_{L^p(Q_{r_0})} + \|f\|_{L^p(Q_{r_0})}\}
\leq C\{4(1 - r_0)^{-2}\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)}\},
\]

where \(Q'_{r_0} = Q_{r_0+(1-r_0)/2} \subset Q_0\). Hence we obtain

\[
[u]^{(n/p)}_{W^{2,p}(Q_0)} \leq C(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)}).
\]

Next we consider the \(W^{2,p}\) estimate for \(u'\). Similarly to (3-10), there exists a cube \(Q_{r_1}\), centered at the origin, such that

\[
[u']^{(n/p)}_{W^{2,p}(Q_0)} \leq 2(1 - r_1)^3[u']_{W^{2,p}(Q_{r_1})}.
\]

By (3-9) and Lemma 3.1,

\[
[u']_{W^{2,p}(Q_{r_1})} \leq C\left\{\frac{9}{(1-r_1)^2}\|u'\|_{L^p(Q_{r_1})} + \|f'\|_{L^p(Q_{r_1})} + \sum_{i,j=1}^n a'_{ij}u_{ij} L^p(Q_{r_1})\right\},
\]

where \(Q'_{r_1} = Q_{r_1+(1-r_1)/3}\) is a cube centered at the origin. By the interpolation inequality, the right-hand side of the above formula is

\[
\leq C\{ (1-r_1)^{-3}\|u\|_{L^p(Q_{r_1})} + (1-r_1)^{-1}\|D^2u\|_{L^p(Q_{r_1})} + \|f'\|_{L^p(Q_{r_1})} + B\|D^2u\|_{L^p(Q_{r_1})}\}
\leq CB(1-r_1)^{-3}\{\|u\|_{L^p(Q_0)} + \|f'\|_{L^p(Q_0)} + [u]^{(n/p)}_{W^{2,p}(Q_0)}\}.
\]

Therefore we obtain

\[
[u']^{(n/p)}_{W^{2,p}(Q_0)} \leq CB(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1),
\]

where the number 1 arises in \(\|f'\|_{L^p(Q_0)}\).

By induction, let us assume for \(\ell = 0, 1, 2, \ldots, k\) that

\[
[u]^{(\ell)}_{W^{2,p}(Q_0)} \leq A^\ell \ell!(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1).
\]

Then, similarly to (3-10), there exists a cube \(Q_{r_{k+1}} \subset Q_0\), centered at the origin, such that

\[
[u]^{(k+1)}_{W^{2,p}(Q_0)} \leq 2(1 - r_{k+1})^{k+3}[u]^{(k+1)}_{W^{2,p}(Q_{r_{k+1}})}.
\]
where \( Q_{r_{k+1}} \) is a cube with center at the origin. By Lemma 3.1, with \( \delta = \frac{1 - r_{k+1}}{k + 3} \),
\[
(1 - r_{k+1})^{k+3}[u^{(k+1)}]_{W^2,p(Q_{r_{k+1}})}
\leq C(1 - r_{k+1})^{k+3} \left\{ \frac{(k + 3)^2}{(1 - r_{k+1})^2} \left[ \|u^{(k+1)}\|_{L^p(Q'_{r_{k+1}})} + \|f^{(k+1)}\|_{L^\infty(Q'_{r_{k+1}})} \right] + \sum_{i,j=1}^{n} \sum_{m=0}^{k} \left( \frac{m}{k+1} \right) \|a_{ij}^{(k+1-m)}u_{ij}^{(m)}\|_{L^p(Q'_{r_{k+1}})} \right\},
\]
where \( Q'_{r_{k+1}} := Q_{r_{k+1} + (1 - r_{k+1})/(k + 3)} \). Note that \( \text{dist}(Q'_{r_{k+1}}, \partial Q_0) = \frac{k + 2}{k + 3}(1 - r_{k+1}) \).

We have
\[
(k + 3)^2(1 - r_{k+1})^{k+1}\|u^{(k+1)}\|_{L^p(Q'_{r_{k+1}})}
\leq (k + 3)^2\left( \frac{k + 2}{k + 3}(1 - r_{k+1}) \right)^{k+1}\|u^{(k+1)}\|_{W^2,p(Q'_{r_{k+1}})}
\leq 4(k + 3)^2[u^{(k-1)}]_{W^2,p(Q_0)}^{(n/p)}.
\]

Similarly,
\[
(1 - r_{k+1})^{k+3}\|a_{ij}^{(k+1-m)}u_{ij}^{(m)}\|_{L^p(Q'_{r_{k+1}})}
\leq \|a_{ij}^{(k+1-m)}\|_{L^\infty(Q_0)}(1 - r_{k+1})^{m+2}\|u^{(m)}\|_{W^2,p(Q'_{r_{k+1}})}
\leq 4\|a_{ij}^{(k+1-m)}\|_{L^\infty(Q_0)}[u^{(m)}]_{W^2,p(Q_0)}^{(n/p)}.
\]

Hence for fixed \( i, j \), by the induction assumptions,
\[
(1 - r_{k+1})^{k+3}\sum_{m=0}^{k}\left( \frac{m}{k+1} \right)\|a_{ij}^{(k+1-m)}u_{ij}^{(m)}\|_{L^p(Q'_{r_{k+1}})}
\leq 4\sum_{m=0}^{k}\left( \frac{m}{k+1} \right)\|a_{ij}^{(k+1-m)}\|_{L^\infty(Q_0)}[u^{(m)}]_{W^2,p(Q_0)}^{(n/p)}
\leq 4(k + 1)!A^m B^{k+1-m}(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1)
\leq 4(k + 1)!A^k B(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1).
\]

Hence by (3-13) we obtain
\[
[u^{(k+1)}]_{W^2,p(Q_0)}^{(n/p)} \leq C\left\{ (k + 3)^2[u^{(k-1)}]_{W^2,p(Q_0)}^{(n/p)} + \|f^{(k+1)}\|_{L^\infty(Q_0)}
+ (k + 1)!A^k B(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1) \right\}.
\]

By (3-2) and the induction assumption (3-12), we then obtain
\[
[u^{(k+1)}]_{W^2,p(Q_0)}^{(n/p)} \leq C(k + 1)!(A^{k-1} + A^k B + B^{k+1})(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1).
\]
Choosing \( A \gg B \), we obtain (3-12) for \( k + 1 \).
From (3-12), we obtain that
\[ [u^{(k+1)}]_{W^{2,p}(Q_{1/2}(0))} \leq 2^{k+1} A^{k+1}(k+1)! (\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1). \]
By the Sobolev embedding and since \( p > n \), we have
\[ |u^{(k+1)}(0)| \leq C 2^{k+1} A^{k+1}(k+1)!. \]
Hence \( u \) is analytic in \( x_n \) at the origin. \( \square \)

As we remarked in Section 1, the continuity assumption on the \( a_{ij} \) can be relaxed. The continuity is used for the \( W^{2,p} \) estimate; it suffices to assume that the \( a_{ij} \) are continuous in any \( n-1 \) variables [Kim and Krylov 2007]. In particular, in the dimension-two case, we can remove the continuity of \( a_{ij} \) in Theorem 1.2, as the analyticity of \( a_{ij} \) automatically implies that they are continuous in one variable. Therefore, for the equation

(3-14) \[ \sum_{i,j=1}^{2} a_{ij}(x) u_{ij} = f(x) \quad \text{in} \; \Omega, \]
where the coefficients \( a_{ij} \) satisfy the uniformly elliptic condition (1-2), we have:

**Theorem 3.3.** Let \( u \in W^{2,2}(\Omega) \) be a strong solution to (3-14). Assume that the \( a_{ij} \) satisfy (1-2) and assume that \( a_{ij} \) and \( f \) are analytic in \( x_2 \). Then under the above conditions, \( u \) is analytic in \( x_2 \).

4. Proof of Theorem 1.4

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Let \( \xi \) be a unit vector in \( \mathbb{R}^n \) and \( \phi \) a function defined in \( \Omega \). Set
\[ \omega_{\phi,\xi}(r) = \sup\{|\phi(x) - \phi(x + t\xi)| \mid x, x + t\xi \in \Omega, \; |t| \leq r \}. \]
We say \( \phi \) is Hölder-continuous in the \( \xi \) direction with Hölder exponent \( \alpha \) if \( \omega_{\phi,\xi} \in C^\alpha \), and write \( \phi \in C^\alpha_{\xi}(\Omega) \), with the norm
\[ \|\phi\|_{C^\alpha_{\xi}(\Omega)} = \sup_{x \in \Omega} |\phi(x)| + \sup_{t > 0} \frac{\omega_{\phi,\xi}(t)}{t^\alpha}. \]
To prove Theorem 1.4, we assume for simplicity that \( b_i = c = 0 \) and consider the equation

(4-1) \[ L[u] := \sum_{i,j=1}^{n} a_{ij}(x) u_{ij} = f(x) \quad \text{in} \; \Omega, \]
where the coefficients \( a_{ij} \) satisfies the uniformly elliptic condition (1-2). The proof below is based on a perturbation argument and follows closely that of [Wang 2006].
Proof of Theorem 1.4. Without loss of generality we assume $\xi = e_1 = (1, 0, \ldots, 0)$ and $\Omega = B_1(0)$, the unit ball. We set

$$B_k = B_{2^{-k}}(0), \quad \hat{a}_{ij}(x) = a_{ij}(0, x_2, \ldots, x_n), \quad \hat{f}(x) = f(0, x_2, \ldots, x_n).$$

For $k = 0, 1, 2, \ldots$, let $u_k$ be the solution of

$$\sum_{i,j=1}^{n} \hat{a}_{ij}(x)(u_k)_{x_i x_j} = \hat{f}(x) \quad \text{in} \ B_k,$$

$$u_k = u \quad \text{on} \ \partial B_k.$$

Then

$$\sum_{i,j=1}^{n} \hat{a}_{ij}(x)(u_k - u)_{x_i x_j} = \sum_{i,j=1}^{n} (a_{ij}(x) - \hat{a}_{ij}(x))u_{x_i x_j} + \hat{f}(x) - f(x) \quad \text{in} \ B_k,$$

$$u_k - u = 0 \quad \text{on} \ \partial B_k.$$

Hence, by the Alexandrov maximum principle, for $k \geq 1$,

$$\sup_{B_k} |u - u_k| \leq C 2^{-k} \left[ \int_{B_k} |(a_{ij}(x) - \hat{a}_{ij}(x))u_{x_i x_j}|^n \, dx \right]^{1/n} + C 2^{-2k} \omega_{f,\xi}(2^{-k})$$

$$\leq C 2^{-k} \|a_{ij}\|_{C^a(B_k)} \left[ \int_{B_k} |x|^n |u_{x_i x_j}|^n \, dx \right]^{1/n} + C 2^{-2k} \omega_{f,\xi}(2^{-k})$$

$$\leq C 2^{-k} \|a_{ij}\|_{C^a(B_k)} \left[ \left( \int_{B_k} |x|^{n \alpha / (p-n)} \, dx \right)^{(p-n)/p} \left( \int_{B_k} |u_{x_i x_j}|^p \, dx \right)^{n/p} \right]^{1/n} + C 2^{-2k} \omega_{f,\xi}(2^{-k})$$

$$\leq C 2^{-k} \|a_{ij}\|_{C^a(B_k)} (2^{-k})^{\alpha + 1 - n/p} \|u\|_{W^{2,p}(B_k)} + C 2^{-2k} \omega_{f,\xi}(2^{-k})$$

$$\leq C (A \cdot (2^{-k})^{2 + \alpha - n/p} + 2^{-2k} \omega_{f,\xi}(2^{-k})),$$

where

$$A = \|u\|_{W^{2,p}(B_k)} \|a_{ij}\|_{C^a(\Omega)}.$$

Since the $a_{ij}$ are continuous and satisfy the uniformly elliptic condition, by the $W^{2,p}$ estimate,

$$A \leq C (\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}) \|a_{ij}\|_{C^a(\Omega)}.$$

Hence

$$\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} \leq C \{ A \cdot (2^{-k})^{2 + \alpha - n/p} + 2^{-2k} \omega_{f,\xi}(2^{-k}) \}$$

$$= C 2^{-2k} \{ A \cdot (2^{-k})^{\alpha - n/p} + \omega_{f,\xi}(2^{-k}) \}.$$
Since \( w_k := u_{k+1} - u_k \) satisfies

\[
\hat{a}_{ij}(x) w_{x_i x_j} = 0
\]

in \( B_{k+1} \), where the coefficients \( \hat{a}_{ij}(x) \) are independent of \( x_1 \), by differentiating the equation and by the \( W^{2,p} \) estimate, we have

\[
\| \partial_x w_k \|_{W^{2,p}(B_{k+2})} \leq C 2^{3k} \| w_k \|_{L^\infty(B_{k+1})} \quad \text{for all } p > 1.
\]

Hence by the Sobolev embedding theorem,

\[
\| \partial_x w_k \|_{C^{1,\beta}(B_{k+2})} \leq C 2^{2k+2\beta} \| w_k \|_{L^\infty(B_{k+1})} \quad \text{for all } \beta \in (0, 1).
\]

Therefore by rescaling,

\[
\| \partial_x \partial_x w \|_{L^\infty(B_{k+2})} \leq C [ A \cdot (2^{-k})^{\alpha-n/p} + \omega_{f,\xi}(2^{-k}) ],
\]

(4-6) \[
\| \partial_x \partial_x w \|_{C^\beta(B_{k+2})} \leq C 2^{k\beta} [ A \cdot (2^{-k})^{\alpha-n/p} + \omega_{f,\xi}(2^{-k}) ].
\]

As the coefficients \( a_{ij} \) are continuous, the solution can be approximated by smooth solutions. Hence, to prove Theorem 1.4, we may assume that \( u \) is smooth, so that

\[
D^2 u_k(0) \to D^2 u(0).
\]

For \( y \) near 0, let \( m \geq 1 \) be such that

\[
2^{-m-4} \leq |y| < 2^{-m-3}.
\]

Then

(4-7) \[
|\partial_x \partial_x u(y) - \partial_x \partial_x u(0)| \leq |\partial_x \partial_x u_m(y) - \partial_x \partial_x u_m(0)| + |\partial_x \partial_x u_m(0) - \partial_x \partial_x u(0)| + |\partial_x \partial_x u(y) - \partial_x \partial_x u_m(y)|.
\]

We have

(4-8) \[
|\partial_x \partial_x u_m(0) - \partial_x \partial_x u(0)| \leq \sum_{k=m}^{\infty} |\partial_x \partial_x u_k(0) - \partial_x \partial_x u_{k+1}(0)|
\]

\[
\leq C \sum_{k=m}^{\infty} [ A \cdot (2^{-k})^{\alpha-n/p} + \omega_{f,\xi}(2^{-k}) ]
\]

\[
\leq C \left\{ A \cdot (2^{-m})^{\alpha-n/p} + \int_0^{\|y\|} \frac{\omega_{f,\xi}(r)}{r} \right\}
\]

\[
\leq C \left\{ A \cdot \|y\|^{\alpha-n/p} + \int_0^{\|y\|} \frac{\omega_{f,\xi}(r)}{r} \right\}.
\]

Similarly,

\[
|\partial_x \partial_x u(y) - \partial_x \partial_x u_m(y)| \leq C \left\{ A \cdot \|y\|^{\alpha-n/p} + \int_0^{\|y\|} \frac{\omega_{f,\xi}(r)}{r} \right\}.
\]
By (4-6) we have

\[ |\partial_\xi \partial_x u_k(y) - \partial_\xi \partial_x u_k(0)| \leq \|\partial_\xi \partial_x w_k\|_{C^\beta(B_{k+2})}|y|^\beta \]

\[ \leq C|y|^\beta 2^{k\beta}[A \cdot (2^{-k})^{\alpha-n/p} + \omega_{f,\xi}(2^{-k})]. \]

Write

\[ u_m = u_1 + \sum_{k=1}^{m-1} w_k. \]

We have, for \( \beta < \alpha - n/p \),

\[ |\partial_\xi \partial_x u_m(y) - \partial_\xi \partial_x u_m(0)| \]

\[ \leq |\partial_\xi \partial_x u_1(y) - \partial_\xi \partial_x u_1(0)| + \sum_{k=1}^{m-1} |\partial_\xi \partial_x w_k(y) - \partial_\xi \partial_x w_k(0)| \]

\[ \leq C|y|^\beta \left( \|u_1\|_{L^\infty(\Omega)} + \sum_{k=1}^{m-1} 2^{k\beta} (A \cdot (2^{-k})^{\alpha-n/p} + \omega_{f,\xi}(2^{-k})) \right) \]

\[ \leq C|y|^\beta \left( \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \int_{|y|}^1 \frac{\omega_{f,\xi}(r)}{r^{1+\beta}} \right). \]

This completes the proof of Theorem 1.4. \( \square \)

5. Equation of divergence form

We consider the following linear elliptic equation of divergence form:

\[ Lu = \text{div}(A(x)\nabla u(x)) = \text{div} f(x) \quad \text{in} \ \Omega, \]

where the coefficient matrix \( A(x) = (a_{ij}(x))_{n \times n} \) satisfies the uniformly elliptic condition (1-2) and \( f(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \in [L^p(\Omega)]^n \) for \( p > 1 \). We assume also that \( a_{ij} \) and \( f \) are analytic in \( x_n \), and that there exists a constant \( B > 0 \) such that

\[ |\partial_{x_n}^k a_{ij}| + |\partial_{x_n}^k f| \leq B^k k! \]

for all \( k \geq 1 \).

**Definition 5.1.** Let \( 1 < p < \infty \). We say that \( u \) is a solution to (5-1) if \( u \in W^{1,p}_{\text{loc}}(\Omega) \) and satisfies

\[ \int_{\Omega} a_{ij}(x) u_{x_j} \phi_{x_i} \ dx = \int_{\Omega} f(x) \phi_{x_i} \ dx \]

for all \( \phi \in C_0^\infty(\Omega) \).
As before, we set $u' = u_{x_n}$, $u'' = u_{x_n x_n}$ and $u^{(k)} = \frac{\partial^k u}{\partial x^k_n}$ for all integers $k \geq 1$. We also define

$$[u]_{W^{1,p}(\Omega)} = \|Du\|_{L^p(\Omega)},$$

(5.3) 

$$\|u^{(k)}\|_{W^{1,p}(\Omega)} = \sup_{Q_r(x) \subset \Omega} d_{Q_r(x)}^{k+1-n/p+\beta} \|u^{(k)}\|_{W^{1,p}(Q_r(x))},$$

$$\|u^{(k)}\|_{L^p(\Omega)} = \sup_{Q_r(x) \subset \Omega} d_{Q_r(x)}^{k-n/p+\beta} \|u^{(k)}\|_{L^p(Q_r(x))},$$

where $d_{Q_r(x)} = \text{dist}(Q_r(x), \partial \Omega)$, $k$ is a nonnegative integer, and $p > 1$ is a constant.

By the $W^{1,p}$ estimate for the divergence form (5.1) in [Di Fazio 1996], we have:

**Lemma 5.2.** Let $u$ be a solution to (5.1). Assume that the $a_{ij}$ satisfy (1-2), $f \in [L^p(\Omega)]^n$ ($p > 1$) and $Q_R(x_0) \subset \Omega$. There exists a constant $C$ such that, if $0 < r < r + \delta < R$, then

(5.4) \n
$$[u]_{W^{1,p}(Q_r(x_0))} \leq C \left\{ \frac{1}{\delta} \|u\|_{L^p(Q_{r+\delta}(x_0))} + \sum_{i=1}^n \|f_i\|_{L^p(Q_{r+\delta}(x_0))} \right\},$$

where the constant $C$ depends only on $n$, $p$, $\lambda$, $\Lambda$.

By **Lemma 5.2** we then have:

**Theorem 5.3.** Let $u$ be a solution to (5.1). Assume that the $a_{ij}$ satisfy (1-2) and $f \in [L^p(\Omega)]^n$ ($p > n$). Assume that the $a_{ij}$ and $f$ are analytic in the variable $x_n$. Then $u$ is analytic in $x_n$.

The proofs of **Lemma 5.2** and **Theorem 5.3** are similar to those in Section 3 and are omitted here. Note that the assumption $p > n$ in **Theorem 5.3** is for the use of Sobolev embedding; namely, by the estimate $\|u^{(k)}\|_{W^{1,p}(Q_r(0))} \leq C$ one infers that $|u^{(k)}(0)| \leq C_1$.

**References**


Received July 14, 2014. Revised October 27, 2014.

YONGYANG JIN

DEPARTMENT OF APPLIED MATHEMATICS

ZHEJIANG UNIVERSITY OF TECHNOLOGY

HANGZHOU, 310023

CHINA

yongyang@zjut.edu.cn

DONGSHENG LI

SCHOOL OF MATHEMATICS AND STATISTICS

XI’AN JIAOTONG UNIVERSITY

XI’AN, 710049

CHINA

lidsh@xjtu.edu.cn

XU-JIA WANG

CENTRE FOR MATHEMATICS AND ITS APPLICATIONS

AUSTRALIAN NATIONAL UNIVERSITY

CANBERRA ACT 0200

AUSTRALIA

xu-jia.wang@anu.edu.au
Free evolution on algebras with two states, II
MICHAEL ANSHELEVICH

Systems of parameters and holonomicity of A-hypergeometric systems
CHRISTINE BERKESCH ZAMAERE, STEPHEN GRIFFETH and EIZRA MILLER

Complex interpolation and twisted twisted Hilbert spaces
FÉLIX CABELLO SÁNCHEZ, JESÚS M. F. CASTILLO and NIGEL J. KALTON

The ramification group filtrations of certain function field extensions
JEFFREY A. CASTAÑEDA and QINGQUAN WU

A mean field type flow, II: Existence and convergence
JEAN-BAPTISTE CASTÉRAS

Isometric embedding of negatively curved complete surfaces in Lorentz–Minkowski space
BING-LONG CHEN and LE YIN

The complex Monge–Ampère equation on some compact Hermitian manifolds
JIANCHUN CHU

Topological and physical link theory are distinct
ALEXANDER COWARD and JOEL HASS

The measures of asymmetry for coproducts of convex bodies
QI GUO, JINFENG GUO and XUNLI SU

Regularity and analyticity of solutions in a direction for elliptic equations
YONGYANG JIN, DONGSHENG LI and XU-JIA WANG

On the density theorem for the subdifferential of convex functions on Hadamard spaces
MINA MOVAHEDI, DARYOUSH BEHMARDI and SEYEDEH SOMAYEH HOSSEINI

$L^p$ regularity of weighted Szegő projections on the unit disc
SAMANGI MUNASINGHE and YUNUS E. ZEYTUNCU

Topology of complete Finsler manifolds admitting convex functions
SORIN V. SABAU and KATSUHIRO SHIOHAMA

Variations of the telescope conjecture and Bousfield lattices for localized categories of spectra
F. LUKE WOLCOTT