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ON THE DENSITY THEOREM FOR THE SUBDIFFERENTIAL OF CONVEX FUNCTIONS ON HADAMARD SPACES

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We introduce a dual space for any geodesically complete Hadamard space. By using this notion we give a new definition of the subdifferential of convex functions on geodesically complete Hadamard spaces. Some properties of this subdifferential, such as a density theorem, are proved.

1. Introduction

Nondifferentiability appears naturally in different areas of mathematics and arises explicitly in the description of various modern technological systems. Nonsmooth analysis studies the local behavior of nondifferentiable functions and sets lacking smooth boundaries. Generalized gradients or subdifferentials refer to several set-valued replacements for the usual derivative which are used in developing differential calculus for nonsmooth functions.

Nondifferentiable functions are often considered on finite-dimensional or infinite-dimensional Banach spaces. Here, the linear structure plays a central role. Attempts have been made to replace Banach spaces with Riemannian manifolds and develop a subdifferential calculus; see [Hosseini and Pouryayevali 2011; 2013a; 2013b; 2013c]. Shafrir [1992] gave a definition of the coaccretive subdifferential of a convex function defined on a Hilbert ball. His approach involves the structure of (B, ρ) as a Hilbert manifold, where ρ is the hyperbolic metric on B; see also [Kopecká and Reich 2010, p. 188].

Unlike Riemannian manifolds, Hadamard spaces are not equipped with a Riemannian metric. Hence, we need new tools to construct a suitable dual space in order to define subdifferentials of functions on Hadamard spaces. B. Ahmadi Kakavandi and M. Amini [2010] defined a dual space for an Hadamard space using the concept of bound vectors. They defined a pseudometric D on $\mathbb{R} \times \mathcal{X} \times \mathcal{X}$, where $\mathcal{X}$ is an Hadamard space, and considered the pseudometric space $(\mathbb{R} \times \mathcal{X} \times \mathcal{X}, D)$ as a subspace of the pseudometric space $(\mathrm{Lip}(\mathcal{X}, \mathbb{R}), L)$ of all real-valued Lipschitz

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functions. Then, they defined an equivalence relation on $\mathbb{R} \times \mathcal{X} \times \mathcal{X}$, where the equivalence class of (t, a, b) is

$$[t\,\overrightarrow{ab}] := \left\{ s\,\overrightarrow{cd} : t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle \text{ for } x, y \in \mathcal{X} \right\}.$$

After introducing a dual metric space to $\mathcal{X}$,

$$\mathcal{X}^* := \left\{ [t\,\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times \mathcal{X} \times \mathcal{X} \right\},$$

they defined a notion of the subdifferential for a proper function on an Hadamard space.

Here we present a new dual for any Hadamard space and prove a density theorem for the subdifferential of lower semicontinuous convex functions on Hadamard spaces, generalizing the classical one for Hilbert spaces [Clarke et al. 1998]. Our approach differs from the one in [Ahmadi Kakavandi and Amini 2010]: we use the notion of geodesics, defining the dual $\mathcal{X}^*$ as the disjoint union of the sets $\mathcal{X}_x^*$ over $x \in \mathcal{X}$, where $\mathcal{X}_x^*$ contains all unit speed geodesics of $\mathcal{X}$ starting at x. The subdifferential of a function f at a point x is defined as a subset of $\mathcal{X}_x^*$. This property is not visible in Ahmadi Kakavandi and Amini's definition of the subdifferential. This leads us to the claim that the subdifferential of convex functions defined in this paper is an analogue of the concept of the subdifferential of convex functions in Riemannian manifolds and Hilbert balls.

We assume that $\mathcal{X}$ is a geodesically complete Hadamard space with a metric d. Recall that a geodesic in $\mathcal{X}$ is a curve of constant speed which is locally minimizing. We say $\mathcal{X}$ has nonpositive curvature (in the sense of Alexandrov) if every point $p \in \mathcal{X}$ has a neighborhood U with the following properties:

(i) For any two points $x, y \in U$ there is a geodesic $\sigma_x^y : [0, 1] \to U$ from x to y of length $d(x, y)$.

(ii) For any triple of points $x, y, z \in U$, we have

$$d^2(z, m) \le \tfrac{1}{2}(d^2(z, x) + d^2(z, y)) - \tfrac{1}{4} d^2(x, y),$$

where σ_x^y is as in (i) and $m = \sigma_x^y\left(\tfrac{1}{2}\right)$ is the point halfway between x and y.

We say $\mathcal{X}$ is an Hadamard space if $\mathcal{X}$ is complete and the assertions (i) and (ii) above hold for all points $x, y, z \in \mathcal{X}$. Hadamard spaces are uniquely geodesic, i.e., there exists a unique geodesic between any pair of points.

In this paper, we assume that $\mathcal{X}$ is a geodesically complete Hadamard space, meaning that every geodesic in $\mathcal{X}$ is a subarc of a geodesic which is parametrized on the whole real line. Let $\mathbb{E}^2$ be the Euclidean space equipped with the metric

$$d_{\mathbb{E}^2}((x_1, x_2), (y_1, y_2)) = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{1/2}.$$

A geodesic triangle $\triangle(x, y, z)$ in $\mathcal{X}$ is the union of three points $x, y, z \in \mathcal{X}$ and the geodesic segments joining them. The comparison triangle for $\triangle(x, y, z)$, is a triangle $\triangle(\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{E}^2$ such that $d(x, y) = d_{\mathbb{E}^2}(\bar{x}, \bar{y})$, $d(x, z) = d_{\mathbb{E}^2}(\bar{x}, \bar{z})$ and $d(z, y) = d_{\mathbb{E}^2}(\bar{z}, \bar{y})$. According to this notation: if a is a point on the geodesic segment joining x, y, then $\bar{a}$ is its comparison point provided that $d(x, a) = d_{\mathbb{E}^2}(\bar{x}, \bar{a})$. Also, the comparison angle $\angle_{\bar{x}}(\bar{y}, \bar{z})$ is the interior angle of the comparison triangle $\triangle(\bar{x}, \bar{y}, \bar{z})$ at $\bar{x}$.

The first step in defining a subdifferential for a function defined on an Hadamard space $\mathcal{X}$ is to introduce a dual space $\mathcal{X}^*$ for $\mathcal{X}$. We denote by $\mathcal{X}^*$ the set of all unit speed geodesics of $\mathcal{X}$, i.e., $\mathcal{X}^* = \coprod_{x \in \mathcal{X}} \mathcal{X}_x^*$ where $\mathcal{X}_x^*$ is the set of all unit speed geodesics of $\mathcal{X}$ starting at x. Consider the map $\langle \cdot, \cdot \rangle : \mathcal{X}_x^* \times \mathcal{X}_x^* \to \mathbb{R}$ defined by

$$\langle \gamma_x^y, \gamma_x^z \rangle = \tfrac{1}{2}[d^2(x, z) + d^2(x, y) - d^2(y, z)].$$

It is clear that $(\langle \gamma_x^y, \gamma_x^y \rangle)^{1/2} = d(x, y)$; see [Berg and Nikolaev 2008] for more details. Let $\gamma_x^y \in X_x^*$, $\sigma_z^w \in X_z^*$ and $D := \operatorname{dom}(\sigma_z^w) = \operatorname{dom}(\gamma_x^y)$. Then we say that γ_x^y is parallel to σ_z^w if there exists $C \in \mathbb{R}$ with $d(\sigma_z^w(t), \gamma_x^y(t)) = C$ for all $t \in D$.

2. The subdifferential of a convex function

In this section, we present a new definition of the subdifferential of a convex function on an Hadamard space. Note that the function $f : \mathcal{X} \to \mathbb{R}$ is called convex if, for any geodesic γ, the composition $f \circ \gamma$ is convex (in the usual sense). Let us start with the definition of the directional derivative for functions on geodesically complete Hadamard spaces.

Definition 2.1. Let $f : \mathcal{X} \to \mathbb{R}$ be a real-valued function. The directional derivative $Df(x; \gamma_x^z)$ of f at $x \in \mathcal{X}$ in the direction $\gamma_x^z \in \mathcal{X}_x^*$ for some $z \in \mathcal{X}$ is defined as

$$Df(x; \gamma_x^z) := \lim_{t \downarrow 0} \frac{f(\gamma_x^z(t)) - f(x)}{t}. \tag{2-1}$$

We will use the following remark in the proof of Theorem 2.4.

Remark 2.2. In the case $\mathcal{X} = \mathbb{R}$, the directional derivative of f at x in the direction of γ_x^{x+b} is defined by

$$Df(x; \gamma_x^{x+b}) = \lim_{t \downarrow 0} \frac{f(x+t) - f(x)}{t}$$

for every $b \in (x, \infty)$. This is the same as the usual directional derivative of f at x in the direction 1, denoted by $Df(x; 1)$.

Theorem 2.3. *Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function on $\mathcal{X}$ and consider $\gamma_x^z \in \mathcal{X}_x^*$.*

(i) *The function* $Q : \mathrm{dom}(\gamma_x^z) \cap (0, \infty) \to \mathbb{R}$ *defined by*

$$Q(t) = \frac{f(\gamma_x^z(t)) - f(x)}{t}$$

is increasing.

(ii) $Df(x; \gamma_x^z)$ *exists and is equal to* $\inf_t Q(t)$.

(iii) $Df(x; \gamma_x^x) = 0$.

Proof. (i) Since f is convex, the function $g(t) = f(\gamma_x^z(t))$, defined on $\mathrm{dom}(\gamma_x^z)$, is convex. If $0 < t_1 < t_2$, we have

$$\frac{g(t_1) - g(0)}{t_1} \leq \frac{g(t_2) - g(0)}{t_2}.$$

This implies that

$$\frac{f(\gamma_x^z(t_1)) - f(x)}{t_1} \leq \frac{f(\gamma_x^z(t_2)) - f(x)}{t_2},$$

which means that Q is increasing.

(ii) Assertion (i) implies that for any decreasing sequence of positive numbers $\{t_n\}$ which converges to zero, the sequence $\{Q(t_n)\}$ is increasing. Hence, $\{Q(t_n)\}$ has a limit, namely $Df(x; \gamma_x^z) = \inf_t Q(t)$.

(iii) For every $x \in \mathcal{X}$ and t, we have $\gamma_x^x(t) = x$. Hence

$$Df(x; \gamma_x^x) = \lim_{t \downarrow 0} \frac{f(\gamma_x^x(t)) - f(x)}{t} = 0. \qquad \square$$

Theorem 2.4 (mean value theorem). *Suppose that* $x, y \in \mathcal{X}$, *and that* $f : \mathcal{X} \longrightarrow \mathbb{R}$ *is convex. Then there exists* $t_0 \in (0, d(x, y))$ *such that*

$$\frac{f(y) - f(x)}{d(x, y)} \leq Df(\gamma_x^y(t_0); \sigma^y_{\gamma_x^y(t_0)}).$$

Proof. Let γ_x^y be the unit speed geodesic joining x to y. Then, $f \circ \gamma_x^y$ is a real-valued convex function on $[0, d(x, y)]$. By the mean value theorem for convex functions from $\mathbb{R}$ to $\mathbb{R}$, there exist $t_0 \in (0, d(x, y))$ and $z \in \partial f \circ \gamma_x^y(t_0)$ such that

$$\frac{f \circ \gamma_x^y(d(x, y)) - f \circ \gamma_x^y(0)}{d(x, y)} = z,$$

where $\partial f \circ \gamma_x^y(t_0)$ denotes the subdifferential of the real-valued function $f \circ \gamma_x^y$ at t_0. We set $w = \gamma_x^y(t_0)$. For the unit speed geodesic σ_w^y,

$$Df(w; \sigma_w^y) = \lim_{t \downarrow 0} \frac{f \circ \sigma_w^y(t) - f \circ \sigma_w^y(0)}{t} = Df \circ \sigma_w^y(0; 1).$$

Since the geodesic connecting w and y is unique, we have $\sigma_w^y(t) = \gamma_x^y(t_0 + t)$ for every $t \in [0, d(w, y)]$. Hence, $Df \circ \sigma_w^y(0; 1) = Df \circ \gamma_x^y(t_0; 1)$ and $z \le Df \circ \gamma_x^y(t_0; 1)$. Therefore,

$$\frac{f(y) - f(x)}{d(x, y)} \le Df(\gamma_x^y(t_0); \sigma^y_{\gamma_x^y(t_0)}). \qquad \square$$

Definition 2.5. Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function. A geodesic $\gamma_x^z \in \mathcal{X}_x^*$ is called the subgradient of f at x if

$$f(y) \ge f(x) + \langle \gamma_x^z, \sigma_x^y \rangle, \quad \forall y \in \mathcal{X}, \quad \forall \sigma_x^y \in \mathcal{X}_x^*.$$

The set-valued map $\partial f : \mathcal{X} \to \mathcal{X}^*$ is called the subdifferential of f and we call $\partial f(x)$ the subdifferential of f at x: it is the set of all subgradients of f at x.

It is worth pointing out that $\partial f(x) \subset \mathcal{X}_x^*$ for every $x \in \mathcal{X}$. A roughly analogous concept of subdifferential is introduced and investigated on the Hilbert ball in [Reich and Shafrir 1990].

Theorem 2.6. *Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function. Then $\gamma_x^x \in \partial f(x)$ if and only if x is a minimum point of f.*

Proof. We know that $\langle \gamma_x^x, \sigma_x^y \rangle = 0$ for every $x, y \in \mathcal{X}$ and $\sigma_x^y \in \mathcal{X}_x^*$. Hence, if $\gamma_x^x \in \partial f(x)$, then

$$f(y) \ge f(x) + \langle \gamma_x^x, \sigma_x^y \rangle = f(x), \quad \forall y \in \mathcal{X},\ \forall \sigma_x^y \in \mathcal{X}_x^*,$$

which means that x is a minimum point of f.

Now assume that x is a minimum point of f, so $f(y) \ge f(x)$ for every $y \in \mathcal{X}$. Then

$$f(y) \ge f(x) + \langle \gamma_x^x, \sigma_x^y \rangle = f(x), \quad \forall y \in \mathcal{X},\ \forall \sigma_x^y \in \mathcal{X}_x^*,$$

and the proof is complete. $\square$

Theorem 2.7. *Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function. If $Df(x; \sigma_x^y) \ge \langle \gamma_x^z, \sigma_x^y \rangle$ for all $y \in \mathcal{X}$ and $\sigma_x^y \in \mathcal{X}_x^*$, then $\gamma_x^z \in \partial f(x)$.*

Proof. The relations $Df(x; \sigma_x^y) \ge \langle \gamma_x^z, \sigma_x^y \rangle$ and

$$f(y) - f(x) \ge \frac{f(\sigma_x^y(s)) - f(x)}{s} \ge Df(x; \sigma_x^y)$$

imply $f(y) - f(x) \ge \langle \gamma_x^z, \sigma_x^y \rangle$, and hence $\gamma_x^z \in \partial f(x)$. $\square$

Corollary 2.8. *Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function. Then x is a minimum point of f if and only if $Df(x; \gamma_x^z) \ge 0$ for each $\gamma_x^z \in \mathcal{X}_x^*$.*

Proof. If x is a minimum point, then $f(\gamma_x^z(t)) \ge f(x)$ for each $z \in \mathcal{X}$ and $t \in \mathrm{dom}\gamma_x^z$. Hence, $Df(x; \gamma_x^z) \ge 0$. The converse is obvious by Theorem 2.7. $\square$

Lemma 2.9. *For each triple of points $x, y, z \in \mathcal{X}$, there exists $w \in \mathcal{X}$ such that $d(x, y) = d(z, w)$ and γ_x^y is parallel to σ_z^w.*

Proof. Since $\mathcal{X}$ is geodesically complete, there is a unit speed geodesic ray γ_x connecting x and y. By Proposition 9.2.28 in [Burago et al. 2001], there exists a unique unit speed geodesic ray σ_z starting at z, parallel to γ_x. Define $w \in \mathcal{X}$ by $w = \sigma_z(d(x, y))$. Then: $d(x, y) = d(w, z)$ and γ_x^y is parallel to σ_z^w. Suppose that σ_z^v is another geodesic segment parallel to γ_x^y. Since it is also parallel to σ_z^w and $d(\sigma_z^w(0), \sigma_z^v(0)) = 0$, we have $d(\sigma_z^w(t), \sigma_z^v(t)) = 0$ for each $t \in [0, d(x, y)]$. □

We use the notation $\gamma_x^y \parallel \gamma_z^w$ when γ_x^y is parallel to γ_z^w for $x, y, z, w \in \mathcal{X}$. We also denote by xy the line segment between $x, y \in \mathbb{E}^2$.

Definition 2.10. (i) The function $P_{xy} : \mathcal{X}_x^* \longrightarrow \mathcal{X}_y^*$ defined by $P_{xy}(\gamma_x^w) = \gamma_y^v$ is called the parallel translation of γ_x^w along γ_x^y. Here, v is selected such that $d(x, w) = d(y, v)$ and γ_x^w is parallel to γ_y^v.

(ii) To define the sum of γ_x^a and γ_x^b, we pick a point c such that by $P_{xa}(\gamma_x^b) = \gamma_x^c$ and put $\gamma_x^a + \gamma_x^b := \gamma_x^c$.

(iii) We define

$$-\gamma_x^y := P_{yx}(\gamma_y^x),$$
$$\gamma_x^a - \gamma_x^b := \gamma_x^a + (-\gamma_x^b).$$

Theorem 2.11. *Suppose that $\gamma_x^y = P_{ax}(\gamma_a^b)$ and $\gamma_x^z = P_{ax}(\gamma_a^c)$. Then:*

(i) $d(b, c) = d(y, z)$,

(ii) $\angle_a(b, c) = \angle_x(y, z)$,

(iii) $\langle \gamma_a^b, \gamma_a^c \rangle = \langle \gamma_x^y, \gamma_x^z \rangle$,

(iv) $\langle -\gamma_x^y, \gamma_x^z \rangle = \langle \gamma_x^y, -\gamma_x^z \rangle$.

Proof. Let $\triangle(\bar{a}, \bar{b}, \bar{c})$ and $\triangle(\bar{x}, \bar{y}, \bar{z})$ be the comparison triangles for $\triangle(a, b, c)$ and $\triangle(x, y, z)$ respectively. By definition, $d(\gamma_a^b(t), \gamma_x^y(t)) = d_{\mathbb{E}^2}(\overline{\gamma_a^b(t)}, \overline{\gamma_x^y(t)}) = C$ where C is constant for each t. We can assume that $\bar{a}\bar{b} \parallel \bar{x}\bar{y}$ and $\bar{a}\bar{c} \parallel \bar{x}\bar{z}$.

This means that $\angle_{\bar{a}}(\bar{b}, \bar{c})$ and $\angle_{\bar{x}}(\bar{y}, \bar{z})$ are two angles with parallel sides. They are therefore congruent or supplementary. But since $d_{\mathbb{E}^2}(\overline{\gamma_a^b(t)}, \overline{\gamma_x^y(t)})$ is constant for each t, the two angles are congruent.

By a similar argument, we get $\angle_{\bar{a}}(\overline{\gamma_a^b(t)}, \overline{\gamma_a^c(t)}) = \angle_{\bar{x}}(\overline{\gamma_x^y(t)}, \overline{\gamma_x^z(t)})$ for each t. Thus, by definition, $\angle_a(b, c) = \angle_x(y, z)$. Moreover, $\triangle(\bar{a}, \bar{b}, \bar{c})$ is congruent to $\triangle(\bar{x}, \bar{y}, \bar{z})$. Then: $d_{\mathbb{E}^2}(\bar{b}, \bar{c}) = d_{\mathbb{E}^2}(\bar{y}, \bar{z})$ and hence $d(b, c) = d(y, z)$. Now by (i) and the definition of $\langle \cdot, \cdot \rangle$, (iii) is obvious.

To prove (iv), suppose that $-\gamma_x^z = \gamma_x^{z'}$ and $-\gamma_x^y = \gamma_x^{y'}$. Let $\triangle_1 = \triangle(\overline{x_1}, \overline{y'}, \bar{z})$ and $\triangle_2 = \triangle(\overline{x_2}, \bar{y}, \overline{z'})$ be the comparison triangles for $\triangle(x, y', z)$ and $\triangle(x, y, z')$ respectively. Since $\gamma_x^{y'} \parallel \gamma_y^x$ and $\gamma_x^{z'} \parallel \gamma_z^x$, we can consider $\triangle_1$ and $\triangle_2$ such that

$\overline{x_1}\,\overline{y'}$ is parallel to $\overline{y}\,\overline{x_2}$ and $\overline{x_2}\,\overline{z'}$ is parallel to $\overline{z}\,\overline{x_1}$. Then: $\angle_{\overline{x_1}}(\bar{z}, \overline{y'}) = \angle_{\overline{x_2}}(\overline{y}, \overline{z'})$. Therefore, $\triangle_1$ and $\triangle_2$ are congruent. Hence, $d_{\mathbb{E}^2}(\overline{z'}, \overline{y}) = d_{\mathbb{E}^2}(\overline{y'}, \bar{z})$. It means that $d(z', y) = d(y', z)$. Now we have

$$\begin{aligned}\langle -\gamma_x^y, \gamma_x^z\rangle &= \langle \gamma_x^{y'}, \gamma_x^z\rangle = \tfrac{1}{2}[d^2(x,z) + d^2(y',x) - d^2(y',z)]\\ &= \tfrac{1}{2}[d^2(x,z') + d^2(x,y) - d^2(z',y)] = \langle \gamma_x^y, \gamma_x^{z'}\rangle = \langle \gamma_x^y, -\gamma_x^z\rangle. \qquad \square\end{aligned}$$

Lemma 2.12. *Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function. Then $\partial f : \mathcal{X} \to \mathcal{X}^*$ is monotone; that is*

$$\langle \gamma_x^y, \sigma_x^z - P_{yx}(\eta_y^n)\rangle \le 0, \quad \forall x, y \in \mathcal{X}, \quad \forall \eta_y^n \in \partial f(y), \quad \forall \sigma_x^z \in \partial f(x).$$

Proof. Suppose that $\eta_y^n \in \partial f(y)$ and $\sigma_x^z \in \partial f(x)$. Thus $f(y) - f(x) \ge \langle \gamma_x^y, \sigma_x^z\rangle$ and $f(x) - f(y) \ge \langle \gamma_y^x, \eta_y^n\rangle$. Note that

$$\langle \gamma_y^x, \eta_y^n\rangle = \langle P_{yx}(\eta_y^n), P_{yx}(\gamma_y^x)\rangle = \langle -P_{yx}(\eta_y^n), \gamma_x^y\rangle.$$

Therefore

$$\langle \gamma_x^y, \sigma_x^z - P_{yx}(\eta_y^n)\rangle \le 0, \quad \forall x, y \in \mathcal{X}, \quad \forall \eta_y^n \in \partial f(y), \quad \forall \sigma_x^z \in \partial f(x). \qquad \square$$

Let S be a nonempty closed convex subset of $\mathcal{X}$ and $\pi_S : \mathcal{X} \longrightarrow S$ be the nearest point map onto S.

Now we need some lemmas to prove the density theorem for the subdifferential of a convex lower semicontinuous function on $\mathcal{X}$.

Lemma 2.13. *Let $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be a proper, convex and lower semicontinuous function. Suppose that $(e, r_e) \in (\mathrm{epi}(f))^c$ and $X_0 = (x_0, f(x_0)) = \pi_{\mathrm{epi}(f)}(e, r_e)$ with $f(x_0) - r_e = 1$. Then: $\partial f(x_0) \ne \varnothing$.*

Proof. Set $E = (e, r_e)$. By Proposition 2.4 in [Bridson and Haefliger 1999], for each $A = (a, r_a) \in \mathrm{epi}(f)$ not equal to X_0, we have $\angle_{X_0}(E, A) \ge \frac{\pi}{2}$. Consequently: $\rho^2(A, X_0) + \rho^2(X_0, E) \le \rho^2(A, E)$, where ρ is the metric of the space $\mathcal{X} \times \mathbb{R}$ defined by

$$\rho^2((x_1, r_1), (x_2, r_2)) = d^2(x_1, x_2) + (r_2 - r_1)^2.$$

Thus

$$d^2(a, x_0) + d^2(x_0, e) + (f(x_0) - r_a)^2 + (f(x_0) - r_e)^2 \le d^2(a, e) + (r_e - r_a)^2.$$

Therefore, we can easily find

$$\tfrac{1}{2}[d^2(a, x_0) + d^2(x_0, e) - d^2(a, e)] \le (r_a - f(x_0))(f(x_0) - r_e). \tag{2-2}$$

Since $f(x_0) - r_e = 1$, we get

$$\langle \gamma_{x_0}^e, \gamma_{x_0}^a\rangle \le r_a - f(x_0)$$

for all $a \in \operatorname{dom} f$. Put $r_a = f(a)$. Clearly, the above inequality holds for each $a \notin \operatorname{dom} f$. Hence, $\gamma_{x_0}^e \in \partial f(x_0)$. □

It is worth pointing out that since r_a in (2-2) (in the proof of Lemma 2.13) can be selected large enough, we get $f(x_0) \geq r_e$.

Remark 2.14. The notation $(1-t)a \oplus tb$ is used for some results on Hilbert balls in [Shafrir 1992], on hyperbolic spaces in [Goebel and Reich 1984; Reich and Shafrir 1990] and on Hadamard spaces in [Dhompongsa and Panyanak 2008] to denote the unique point a_t such that $d(a, a_t) = td(a, b)$ and $d(a_t, b) = (1-t)d(a, b)$. Now, if (x_0, y_0) and (x_1, y_1) are two points in $\mathcal{X} \times \mathcal{Y}$ and (x, y) is a point on the unique geodesic joining them, (x, y) is the unique point satisfying the equations

$$\rho((x_0, y_0), (x, y)) = t\rho((x_0, y_0), (x_1, y_1)),$$
$$\rho((x_1, y_1), (x, y)) = (1-t)\rho((x_0, y_0), (x_1, y_1))$$

for some $t \in [0, 1]$. The point

$$(\gamma_{x_0}^{x_1}(td(x_0, x_1)), \gamma_{y_0}^{y_1}(td(y_0, y_1))) = ((1-t)x_0 \oplus tx_1, (1-t)y_0 \oplus ty_1)$$

has the same property. Hence

$$(1-t)(x_0, y_0) \oplus t(x_1, y_1) = ((1-t)x_0 \oplus tx_1, (1-t)y_0 \oplus ty_1)$$

for all $t \in [0, 1]$.

If $x, y \in \mathcal{X}$, we denote by $[\![x, y]\!]$ the set $\{\gamma_x^y(t) : t \in \operatorname{dom}\gamma_x^y\}$.

Lemma 2.15. *Let $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be a proper, convex and lower semicontinuous function. Suppose that $(y_0, r_0) \in (\operatorname{epi}(f))^c$ and $X_0 = (x_0, f(x_0)) = \pi_{\operatorname{epi}(f)}((y_0, r_0))$ and $x_0 \in \operatorname{int}(\operatorname{dom} f)$, where $\operatorname{dom} f = \{x \in \mathcal{X} \mid f(x) < \infty\}$. Then: $r_0 \neq f(x_0)$.*

Proof. Assume by contradiction that $r_0 = f(x_0)$. Put $Y_0 = (y_0, r_0)$. Let r be a positive number so that $B(x_0, r) \subseteq \operatorname{dom} f$. There is a $\lambda_0 \in [0, 1]$ such that $\gamma_{x_0}^{y_0}(\lambda d(x_0, y_0)) \in B(x_0, r)$ for the unit speed geodesic $\gamma_{x_0}^{y_0}$ and for each $\lambda \in [0, \lambda_0]$. First suppose that there exists $x_1 \in B(x_0, r) \cap [\![x_0, y_0]\!]$ such that $f(x_0) < f(x_1)$. Hence, $x_1 = \gamma_{x_0}^{y_0}(\lambda_1 d(x_0, y_0))$ for some $\lambda_1 \in (0, \lambda_0)$. Put $X_1 = (x_1, f(x_1)) \in \operatorname{epi} f$. Then,

$$\rho^2(X_1, X_0) + \rho^2(X_0, Y_0) \leq \rho^2(X_1, Y_0).$$

Putting $\alpha = (f(x_1) - f(x_0))^2$, we have

$$\rho^2(X_0, X_1) = d^2(x_0, x_1) + \alpha = \lambda_1^2 d^2(x_0, y_0) + \alpha, \tag{2-3}$$

$$\rho^2(Y_0, X_1) = d^2(y_0, x_1) + \alpha = (1-\lambda_1)^2 d^2(x_0, y_0) + \alpha, \tag{2-4}$$

$$\rho^2(X_0, Y_0) = d^2(x_0, y_0). \tag{2-5}$$

Hence, by (2-3), (2-4) and (2-5), we have

$$\lambda_1^2 d^2(x_0, y_0) + \alpha + d^2(x_0, y_0) \le (1-\lambda_1)^2 d^2(x_0, y_0) + \alpha.$$

Thus $\lambda_1^2 + 1 \le (1-\lambda_1)^2$, and we get the contradiction $\lambda_1 \le 0$. Next, consider the case that $f(x) \le f(x_0)$ for each $x \in B(x_0, r) \cap [\![x_0, y_0]\!]$. Let

$$Y_n = (1-\tfrac{1}{n})X_0 \oplus \tfrac{1}{n}Y_0 = (y_n, r_n).$$

By Proposition 2.4 in [Bridson and Haefliger 1999], X_0 is the nearest point of $\operatorname{epi}(f)$ to each Y_n, and $\{Y_n\}$ is a sequence converging to X_0. If $y_0 \in B(x_0, r)$, then $f(y_0) \le f(x_0)$. Thus $(y_0, r_0) = (y_0, f(x_0)) \in \operatorname{epi}(f)$, which is a contradiction. Therefore, $y_0 \in (B(x_0, r))^c$. By Remark 2.14 we have $r_n = f(x_0)$ for every n, so a similar argument for each Y_n shows that $y_n \in (B(x_0, r))^c$. This means that $\{y_n\}$ is a sequence in $(B(x_0, r))^c$ converging to x_0. Thus we get the contradiction $x_0 \notin B(x_0, r)$. □

Lemma 2.16. *Let* $E' \in (\operatorname{epi}(f))^c$ *and* $X_0 = (x_0, f(x_0)) = \pi_{\operatorname{epi}(f)}(E')$. *Then there exists* $E = (e, r_e) \in (\operatorname{epi}(f))^c$ *such that* $f(x_0) - r_e = 1$ *and* $X_0 = \pi_{\operatorname{epi}(f)}(E)$.

Proof. Let γ be the geodesic joining X_0 to E'. Put $E' = (e', r_{e'})$. First suppose that $f(x_0) - r_{e'} \ge 1$. Since γ is continuous by the intermediate value theorem, the assertion is obvious.

Next, suppose that $f(x_0) - r_{e'} < 1$. Put

$$l = \rho(X_0, E') \quad \text{and} \quad s = \frac{l}{f(x_0) - r_{e'}}.$$

Let $\overline{\gamma}$ be the extension of γ to $[0, \infty)$ that is the unit speed geodesic ray emanating from X_0. Put $E = \overline{\gamma}(s)$. We claim that E is the desired point. If $E = (e, r_e)$, then one has $E' = (1-\frac{l}{s})X_0 \oplus \frac{l}{s}E$. By Remark 2.14, $e' = (1-\frac{l}{s})x_0 \oplus \frac{l}{s}e$ and $r_{e'} = (1-\frac{l}{s})f(x_0) + \frac{l}{s}r_e$. Hence, $f(x_0) - r_{e'} = \frac{l}{s}(f(x_0) - r_e)$. Therefore,

$$f(x_0) - r_e = \frac{s}{l}(f(x_0) - r_{e'}) = s \times \frac{f(x_0) - r_{e'}}{l} = 1.$$

Now we prove that $\pi_{\operatorname{epi}(f)}(E) = X_0$. Suppose for a contradiction that $\pi_{\operatorname{epi}(f)}(E) = X'$ and $X_0 \ne X'$. Then $\angle_{X_0}(X', E') \ge \frac{\pi}{2}$ and $\angle_{X'}(X_0, E) \ge \frac{\pi}{2}$. Then the sum of the angles of $\triangle(X', X_0, E)$ is more than π, which is a contradiction. □

The next theorem is a generalization of the density theorem on geodesically complete Hadamard spaces. For a density theorem on Hilbert spaces see [Clarke et al. 1998].

Theorem 2.17. *Suppose that* f *is a proper, convex and lower semicontinuous function. Then* $\operatorname{dom}(\partial f(x))$ *is dense in* $\operatorname{int}(\operatorname{dom} f)$.

Proof. Given $x_0 \in \mathrm{int}(\mathrm{dom} f)$, the point $X_0 = (x_0, f(x_0))$ is a boundary point of $\mathrm{epi}(f)$. So, there exists a sequence $Y_n = (y_n, r_n)$ in the complement of $\mathrm{epi}(f)$ that converges to X_0. Since $\mathrm{epi}(f)$ is convex and closed in $\mathcal{X} \times \mathbb{R}$ there exists a unique point $X_n = (x_n, f(x_n)) \in \mathrm{epi}(f)$ such that $\pi_{\mathrm{epi}(f)}(Y_n) = X_n$ for each Y_n. Moreover,

$$\rho(X_n, X_0) \leq \rho(X_n, Y_n) + \rho(Y_n, X_0) \leq 2\rho(Y_n, X_0),$$

which implies that X_n converges to X_0. Therefore, the sequence $\{x_n\}$ converges to x_0 and for every neighborhood U of x_0, there exists $x_n \in U$. By Lemma 2.16, one can assume that $f(x_n) - r_n = 1$, so by Lemma 2.13, $\partial f(x_n) \neq \varnothing$. $\square$

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