Pacific Journal of Mathematics

INEQUALITIES OF ALEXANDROV–FENCHEL TYPE FOR CONVEX HYPERSURFACES IN HYPERBOLIC SPACE AND IN THE SPHERE

YONG WEI AND CHANGWEI XIONG

Volume 277 No. 1 September 2015

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.219

INEQUALITIES OF ALEXANDROV–FENCHEL TYPE FOR CONVEX HYPERSURFACES IN HYPERBOLIC SPACE AND IN THE SPHERE

YONG WEI AND CHANGWEI XIONG

By applying the unit normal flow to well-known inequalities in hyperbolic space $\mathbb{H}^{n+1}$ and in the sphere $\mathbb{S}^{n+1}$, we derive some new inequalities of Alexandrov–Fenchel type for closed convex hypersurfaces in these spaces. We also use the inverse mean curvature flow in the sphere to prove an optimal Sobolev-type inequality for closed convex hypersurfaces in the sphere.

1. Introduction

Let $N^{n+1}(c)$ be the simply connected space form of constant sectional curvature c and $\psi : \Sigma^n \to N^{n+1}(c)$ be a closed hypersurface. Denote the k-th order mean curvature of Σ by p_k (see Section 2A). Inequalities about the integrals $\int_\Sigma p_k \, d\mu$ have attracted much attention for a long time. Among them the most famous one is the classical Minkowski inequality for closed convex surfaces $\Sigma \subset \mathbb{R}^3$, which can be written as

$$\left(\frac{1}{\omega_2}\int_\Sigma p_1 \, d\mu\right)^2 \geq \frac{|\Sigma|}{\omega_2}, \tag{1-1}$$

with equality if and only if Σ is a sphere. Here ω_n is the area of $\mathbb{S}^n(1)$ and $|\Sigma| = \int_\Sigma d\mu$ is the area of Σ with respect to the induced metric from $\mathbb{R}^3$. The general inequality is the Alexandrov–Fenchel inequality [Alexandrov 1937; 1938; Fenchel 1936] which states that for convex hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$,

$$\frac{1}{\omega_n}\int_\Sigma p_k \, d\mu \geq \left(\frac{1}{\omega_n}\int_\Sigma p_l \, d\mu\right)^{(n-k)/(n-l)} \quad \text{for } 0 \leq l < k \leq n, \tag{1-2}$$

with equality if and only if Σ is a sphere. See [Chang and Wang 2011; Guan and Li 2009; McCoy 2005; Schneider 1993] for other references on Alexandrov–Fenchel inequalities for closed hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$.

MSC2010: primary 53C44; secondary 53C42.
Keywords: isoperimetric inequality, convex hypersurface, Alexandrov–Fenchel-type inequality, k-th order mean curvature, Gauss–Bonnet curvature.

It is natural to generalize the Minkowski inequality and Alexandrov–Fenchel inequalities to the hypersurfaces in space forms. See, for example, [Borisenko and Miquel 1999; Gallego and Solanes 2005; Natário 2015]. Recently, the following optimal inequalities of Alexandrov–Fenchel type in $\mathbb{H}^{n+1}$ were obtained (see [Ge et al. 2013; 2014b; Li et al. 2014; Wang and Xia 2014]): for $1 \le k \le n$ and any closed horospherical convex hypersurface $\Sigma \subset \mathbb{H}^{n+1}$,

$$\frac{1}{\omega_n}\int_\Sigma p_k\,d\mu \ge \left(\left(\frac{|\Sigma|}{\omega_n}\right)^{2/k} + \left(\frac{|\Sigma|}{\omega_n}\right)^{2(n-k)/kn}\right)^{k/2}, \tag{1-3}$$

with equality if and only if Σ is a geodesic sphere in $\mathbb{H}^{n+1}$. In particular, when $k=2$, Li, Wei and Xiong [Li et al. 2014] proved that (1-3) holds under the weaker condition that Σ is star-shaped and 2-convex. In the proof of (1-3), the geometric flow was used and was an important tool. However, so far there is no inequality comparing $\int_\Sigma p_k\,d\mu$ and $\int_\Sigma p_l\,d\mu$ in $\mathbb{H}^{n+1}$ like (1-2) in $\mathbb{R}^{n+1}$. And one also wants to know whether there exist other inequalities of Alexandrov–Fenchel type in $\mathbb{H}^{n+1}$ for closed hypersurfaces under a weaker condition than horospherical convex. Besides, in space forms, the integrals $\int_\Sigma p_k\,d\mu$ are essentially the so-called quermassintegrals from convex geometry and integral geometry (see, e.g., [Solanes 2006] for the transformation formula) and many attempts have been devoted to establishing the relationships for quermassintegrals. See [Santaló 1976; Solanes 2003] and the references therein. So in this paper we are interested in obtaining new inequalities between the integrals $\int_\Sigma p_k\,d\mu$.

The Minkowski inequality and the Alexandrov–Fenchel inequalities can be viewed as the generalizations of the classical isoperimetric inequality, which compares the area of the hypersurface Σ and the volume of the domain enclosed by Σ. The Minkowski inequality (1-1) was used by Minkowski himself to prove the isoperimetric inequality for closed convex surfaces (see [Minkowski 1903; Osserman 1978]). Recently, J. Natário [2015] reversed Minkowski's idea and derived a new Minkowski-type inequality for closed convex surfaces in the hyperbolic space $\mathbb{H}^3$ from the isoperimetric inequality by using the unit normal flow. In this paper, first, we deal with the higher dimensional case by adapting Natário's method [2015]. We will derive some new inequalities of Alexandrov–Fenchel type for closed convex hypersurfaces in $\mathbb{H}^{n+1}$ and in $\mathbb{S}^{n+1}$, starting from the isoperimetric inequality.

Let Σ be a closed and convex hypersurface in $\mathbb{H}^{n+1}$. We say a hypersurface Σ is convex if all the principal curvatures of Σ are nonnegative everywhere. Then by the well-known result of do Carmo and Warner [1970], Σ is embedded and bounds a convex body in $\mathbb{H}^{n+1}$. Inspired by [Natário 2015], we flow the initial hypersurface Σ by its unit outer normal ν. The resulting hypersurfaces are $\Sigma_t = \psi_t(\Sigma)$, where $\psi_t(x) = \exp_{\psi(x)}(t\nu(x))$, $x \in \Sigma$. The Σ_t are also called the parallel hypersurfaces of Σ. From Steiner's formula [Allendoerfer 1948], we can compute the area of Σ_t and the volume of the domain Ω_t enclosed by Σ_t. In Natário's paper, the area of Σ_t was obtained by using the first and second variation formulas with the help of the Gauss–Bonnet formula. Steiner's formula can also be obtained by using the precise expressions of the geodesics in space forms (see Section 2C). Since $\mathbb{H}^{n+1}$ has constant negative curvature $c = -1$ and Σ is convex, it follows that Σ_t can be well-defined for all $t \geq 0$. Define a function $r(t)$ such that $|\Sigma_t| = |S_{r(t)}|$. Then the isoperimetric inequality (see [Schmidt 1940; Ros 2005]) implies that $\mathrm{Vol}(\Omega_t) \leq \mathrm{Vol}(B_{r(t)})$, where $S_{r(t)}$ and $B_{r(t)}$ are the geodesic sphere and geodesic ball of radius $r(t)$ in $\mathbb{H}^{n+1}$, respectively. Applying the isoperimetric inequality to Σ_t for sufficiently large t, we obtain the following inequalities of Alexandrov–Fenchel type in $\mathbb{H}^{n+1}$.

Theorem 1.1. *Let Σ^n be a closed and convex hypersurface in $\mathbb{H}^{n+1}$ with $n \geq 3$. Then*

$$\sum_{k=0}^{n} \frac{2k-n}{n\omega_n} \int_\Sigma C_n^k p_k \, d\mu \geq \left(\frac{1}{\omega_n} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \right)^{(n-2)/n}. \tag{1-4}$$

A direct calculation shows that if Σ is a geodesic sphere, then the equality in (1-4) holds. However, we do not obtain the rigidity (i.e.,we don't know whether the equality in (1-4) implies that Σ is a geodesic sphere). In Remark 3.2, we note that when the hypersurface $\Sigma \subset \mathbb{H}^{n+1}$ is sufficiently small, the inequality (1-4) reduces to one of the Alexandrov–Fenchel inequalities in Euclidean space.

Besides the isoperimetric inequality, there are many other known inequalities in hyperbolic space. If we use the warped product model for the hyperbolic space $\mathbb{H}^{n+1}$, i.e., $\mathbb{H}^{n+1} = \mathbb{R}^+ \times \mathbb{S}^n$ with the metric $g = dr^2 + \sinh^2 r \, g_{\mathbb{S}^n}$, then there are two important functions on the hypersurface Σ in $\mathbb{H}^{n+1}$. One is the weight function $f = \cosh r$, and the other one is the support function $u = \langle Df, \nu \rangle$. Recently, the following inequality of Alexandrov–Fenchel type with weight f was proved by de Lima and Girão [2015]: for any mean convex and star-shaped closed hypersurface Σ in $\mathbb{H}^{n+1}$,

$$\frac{1}{\omega_n} \int_\Sigma f p_1 \, d\mu \geq \left(\frac{|\Sigma|}{\omega_n} \right)^{(n+1)/n} + \left(\frac{|\Sigma|}{\omega_n} \right)^{(n-1)/n}, \tag{1-5}$$

with equality if and only if Σ is a geodesic sphere centered at the origin in $\mathbb{H}^{n+1}$. For more weighted inequalities of Alexandrov–Fenchel type in different ambient spaces, readers can refer to the recent papers [Brendle et al. 2014; Ge et al. 2014a; 2015]. We remark that in [Ge et al. 2014a], the weighted Alexandrov–Fenchel-type inequalities were used to prove the Penrose-type inequality for the new Gauss–Bonnet–Chern mass in asymptotically hyperbolic graphs. Thus it is an interesting question to establish new inequalities with weight.

Applying the same method as in Theorem 1.1 to inequality (1-5), we can obtain a new inequality as follows:

Theorem 1.2. *Let Σ^n be a closed and convex hypersurface in $\mathbb{H}^{n+1}$. Then*

$$\frac{1}{\omega_n}\int_\Sigma (f+u)\sum_{k=0}^{n} C_n^k p_k \, d\mu \geq \left(\frac{1}{\omega_n}\int_\Sigma \sum_{k=0}^{n} C_n^k p_k \, d\mu\right)^{(n+1)/n}. \tag{1-6}$$

We remark that if Σ is a geodesic sphere centered at the origin, then the equality in (1-6) holds. But as before we do not obtain the rigidity.

Next we will use the same method to derive inequalities for closed convex hypersurfaces in $\mathbb{S}^{n+1}$. In this case, we can prove the rigidity result.

Theorem 1.3. *Let Σ^n be a closed and convex hypersurface in $\mathbb{S}^{n+1}$ with $n \geq 2$. Then*

$$\omega_n \leq \sum_{s=\frac{1-(-1)^n}{2},+2}^{n} \sqrt{(E(s))^2+(F(s))^2}, \tag{1-7}$$

where "+2" means that the step-length of the summation for s is 2 and

$$E(s) = \sum_{\substack{p+q=(n\pm s)/2\\ p,q\geq 0}} \sum_{\substack{q\leq k\leq n-p\\ 2|k}} C_n^k \frac{1}{2^n} C_{n-k}^p C_k^q (-1)^{\frac{k}{2}+k-q} \int_\Sigma p_k \, d\mu,$$

$$F(s) = \sum_{\substack{p+q=(n\pm s)/2\\ p,q\geq 0}} \sum_{\substack{q\leq k\leq n-p\\ 2\nmid k}} C_n^k \frac{1}{2^n} C_{n-k}^p C_k^q (-1)^{[\frac{k}{2}]+k-q} \times (-1)^{\chi_{\{2(p+q)-n\leq 0\}}} \int_\Sigma p_k \, d\mu,$$

Moreover, the equality holds in (1-7) *if and only if Σ^n is a geodesic sphere.*

When $n=2$, it is easy to check that

$$E(0) = 2\pi, \qquad F(0) = 0,$$
$$E(2) = |\Sigma| - 2\pi, \qquad F(2) = \int_\Sigma p_1 \, d\mu,$$

using the Gauss–Bonnet theorem $|\Sigma| + \int_\Sigma p_2 \, d\mu = 4\pi$ (see Section 2B). So (1-7) implies the Minkowski-type inequality in the sphere

$$\left(\int_\Sigma p_1 \, d\mu\right)^2 \geq |\Sigma|(4\pi - |\Sigma|), \tag{1-8}$$

which is just Theorem 0.2 in [Natário 2015]. See also [Blaschke 1938; Knothe 1952; Santaló 1963]. Makowski and Scheuer [2013] proved (1-8) by using the inverse curvature flow in sphere. To get a better feeling of the inequality (1-7), we also give the precise expressions of (1-7) in the case of $n=3$ and $n=4$; see Remark 3.3.

Finally, in the last part of this paper, we use the inverse mean curvature flow in the sphere [Makowski and Scheuer 2013; Gerhardt 2015] to prove the following optimal inequalities for strictly convex hypersurfaces in sphere $\mathbb{S}^{n+1}$.

Theorem 1.4. *Let Σ^n be a closed and strictly convex hypersurface in $\mathbb{S}^{n+1}$. Then we have the optimal inequality*

$$\int_\Sigma L_k \, d\mu \geq C_n^{2k}(2k)!\,\omega_n^{2k/n}|\Sigma|^{(n-2k)/n} \quad \text{for } k \leq n/2. \tag{1-9}$$

Equality holds in (1-9) *if and only if Σ is a geodesic sphere. Here L_k is the Gauss–Bonnet curvature of the induced metric on Σ* (*see Section 2B for details*).

The proof of Theorem 1.4 uses a similar idea as in [Brendle et al. 2014; de Lima and Girão 2015; Guan and Li 2009; Ge et al. 2013; 2014b; Li et al. 2014]. We define a curvature quantity $Q(t)$ which is monotone nonincreasing under the inverse mean curvature flow in the sphere. Then we obtain the inequality (1-9) by comparing the initial value $Q(0)$ with the limit $\lim_{t\to T^*} Q(t)$. We remark that since Σ is a closed and strictly convex hypersurface in $\mathbb{S}^{n+1}$, a well-known result due to do Carmo and Warner [1970] implies that Σ is embedded and is contained in an open hemisphere.

When $k = 1$, the inequality (1-9) reduces to

$$\int_\Sigma p_2 \, d\mu + |\Sigma| \geq \omega_n^{2/n}|\Sigma|^{(n-2)/n}, \tag{1-10}$$

which was already proved by Makowski and Scheuer [2013]. One can compare (1-10) with the case $k=2$ of the Alexandrov–Fenchel-type inequality (1-3) in $\mathbb{H}^{n+1}$; that is,

$$\int_\Sigma p_2 \, d\mu - |\Sigma| \geq \omega_n^{2/n}|\Sigma|^{(n-2)/n}, \tag{1-11}$$

which was proved by Li, Wei and Xiong [Li et al. 2014] for star-shaped and 2-convex hypersurfaces in $\mathbb{H}^{n+1}$. For $k > 1$, inequalities of the same type as (1-9) were proved by Ge, Wang and Wu [Ge et al. 2013; 2014b] for horospherical convex hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$.

2. Preliminaries

2A. ***k-th order mean curvature.*** Let Σ be a closed hypersurface in $N^{n+1}(c)$ with unit outward normal ν. The second fundamental form h of Σ is defined by

$$h(X, Y) = \langle \overline{\nabla}_X \nu, Y\rangle$$

for any two tangent vector fields X, Y on Σ. For an orthonormal basis $\{e_1, \dots, e_n\}$ of Σ, the components of the second fundamental form are given by $h_{ij} = h(e_i, e_j)$ and $h_i^j = g^{jk}h_{ki}$, where g is the induced metric on Σ. The principal curvatures

$\kappa = (\kappa_1, \dots, \kappa_n)$ are the eigenvalues of h with respect to g. The k-th order mean curvature of Σ for $1 \le k \le n$ is defined as

$$p_k = \frac{1}{C_n^k}\sigma_k(\kappa) = \frac{1}{C_n^k}\sum_{i_1<i_2<\cdots<i_k} \kappa_{i_1}\cdots\kappa_{i_k}, \tag{2-1}$$

or equivalently as

$$p_k = \frac{1}{C_n^k}\sigma_k(h_i^j) = \frac{1}{C_n^k k!}\delta_{j_1\cdots j_k}^{i_1\cdots i_k} h_{i_1}^{j_1}\cdots h_{i_k}^{j_k}, \tag{2-2}$$

where $\delta_{j_1\cdots j_k}^{i_1\cdots i_k}$ is the generalized Kronecker delta given by

$$\delta_{j_1\cdots j_k}^{i_1\cdots i_k} = \det\begin{pmatrix} \delta_{j_1}^{i_1} & \delta_{j_1}^{i_2} & \cdots & \delta_{j_1}^{i_k} \\ \delta_{j_2}^{i_1} & \delta_{j_2}^{i_2} & \cdots & \delta_{j_2}^{i_k} \\ \vdots & \vdots & \vdots & \vdots \\ \delta_{j_k}^{i_1} & \delta_{j_k}^{i_2} & \cdots & \delta_{j_k}^{i_k} \end{pmatrix}.$$

We have the following Newton–MacLaurin inequalities (see, e.g., [Guan 2006]).

Lemma 2.1. *For $\kappa \in \overline{\Gamma}_k^+$, $1 \le k \le n$, where $\overline{\Gamma}_k^+$ is the closure of the Garding cone*

$$\Gamma_k^+ = \{\kappa \in \mathbb{R}^n \mid p_j(\kappa) > 0, \forall j \le k\},$$

we have the following Newton–MacLaurin inequalities

$$\begin{aligned} p_1 p_{k-1} &\ge p_k, \\ p_1 \ge p_2^{1/2} &\ge \cdots \ge p_k^{1/k}. \end{aligned}$$

Moreover, equalities above hold for some $\kappa \in \Gamma_k^+$ if and only if $\kappa = c(1, \dots, 1)$, where c is a constant.

2B. ***Gauss–Bonnet curvature L_k.*** Given an n-dimensional Riemannian manifold (M, g), the Gauss–Bonnet curvature L_k, where $k \le n/2$, is defined by (see, e.g., [Ge et al. 2014b; 2014c])

$$L_k = \frac{1}{2^k}\delta_{j_1 j_2\cdots j_{2k-1} j_{2k}}^{i_1 i_2\cdots i_{2k-1} i_{2k}} R_{i_1 i_2}{}^{j_1 j_2}\cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}. \tag{2-3}$$

For a closed hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$, recall the Gauss equation

$$R_{ij}{}^{kl} = h_i^k h_j^l - h_i^l h_j^k.$$

Then the Gauss–Bonnet curvature of the induced metric on $\Sigma^n \subset \mathbb{R}^{n+1}$ is

$$\begin{aligned} L_k &= \delta_{j_1 j_2\cdots j_{2k-1} j_{2k}}^{i_1 i_2\cdots i_{2k-1} i_{2k}} h_{i_1}^{j_1} h_{i_2}^{j_2}\cdots h_{i_{2k-1}}^{j_{2k-1}} h_{i_{2k}}^{j_{2k}} \\ &= (2k)!\,C_n^{2k} p_{2k}. \end{aligned} \tag{2-4}$$

For a closed hypersurface $\Sigma^n \subset \mathbb{S}^{n+1}$, the Gauss equations are

$$R_{ij}{}^{kl} = (h_i^k h_j^l - h_i^l h_j^k) + (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k). \tag{2-5}$$

Then by a straightforward calculation, we have

$$\begin{aligned} L_k &= \delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} (h_{i_1}^{j_1} h_{i_2}^{j_2} + \delta_{i_1}^{j_1} \delta_{i_2}^{j_2}) \cdots (h_{i_{2k-1}}^{j_{2k-1}} h_{i_{2k}}^{j_{2k}} + \delta_{i_{2k-1}}^{j_{2k-1}} \delta_{i_{2k}}^{j_{2k}}) \\ &= \sum_{i=0}^{k} C_k^i (n-2k+1)(n-2k+2)\cdots(n-2k+2i)(2k-2i)!\,\sigma_{2k-2i} \\ &= \sum_{i=0}^{k} C_k^i (n-2k+1)(n-2k+2)\cdots(n-2k+2i)(2k-2i)!\,C_n^{2k-2i}\, p_{2k-2i} \\ &= \sum_{i=0}^{k} C_k^i \frac{n!}{(n-2k)!} p_{2k-2i} \\ &= C_n^{2k}(2k)! \sum_{i=0}^{k} C_k^i p_{2k-2i}. \end{aligned}$$

Similarly, for a closed hypersurface $\Sigma^n \subset \mathbb{H}^{n+1}$, its Gauss–Bonnet curvature is

$$L_k = C_n^{2k}(2k)! \sum_{i=0}^{k} C_k^i (-1)^i p_{2k-2i}. \tag{2-6}$$

Finally, note that throughout our paper, we assume that the hypersurface $\Sigma \subset N^{n+1}(c)$ is closed and convex. It follows that Σ is homeomorphic to the n-sphere (see [do Carmo and Warner 1970]). Then if the dimension of Σ is even, the Gauss–Bonnet–Chern theorem [Chern 1944] implies that

$$\int_\Sigma L_{n/2}\, d\mu = n!\, \omega_n. \tag{2-7}$$

Equation (2-7) will be used in the following sections. Also (2-7) shows that when $2k = n$, the inequality (1-9) is an equality.

2C. ***Unit normal flow and Steiner's formula.*** Let $\psi : \Sigma \to N^{n+1}(c)$ be a closed and convex hypersurface in the simply connected space form $N^{n+1}(c)$ of constant sectional curvature c. Denote by Ω the domain enclosed by Σ. The area of Σ is denoted by $|\Sigma|$ and the volume of Ω is denoted by $|V|$. As we mentioned in Section 1, we flow the initial hypersurface Σ by its unit outer normal ν. The resulting hypersurfaces are $\Sigma_t = \psi_t(\Sigma)$, where $\psi_t(x) = \exp_{\psi(x)}(t\nu(x)), x \in \Sigma$. The Σ_t are also called the parallel hypersurfaces of Σ. Denote by Ω_t the domain bounded by Σ_t. The convexity assumption of Σ and the curvature of $N^{n+1}(c)$ guarantee that the Σ_t are well-defined in the following range:

$$\begin{aligned} &t \in \left[0, \tfrac{\pi}{2}\right) && \text{for } c = 1, \\ &t \geq 0 && \text{for } c = 0 \text{ or } -1. \end{aligned}$$

Further, denote the area of Σ_t and the volume of Ω_t by $|\Sigma_t|$ and $|V_t|$, respectively. Then Steiner's formula [Allendoerfer 1948] implies that

$$|\Sigma_t| = \begin{cases} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \, t^k & \text{if } c = 0, \\ \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \cosh^{n-k} t \sinh^k t & \text{if } c = -1, \\ \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \cos^{n-k} t \sin^k t & \text{if } c = 1. \end{cases} \tag{2-8}$$

and

$$|V_t| = \begin{cases} |V| + \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \frac{1}{k+1} t^{k+1} & \text{if } c = 0, \\ |V| + \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \int_0^t \cosh^{n-k} s \sinh^k s \, ds & \text{if } c = -1, \\ |V| + \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \int_0^t \cos^{n-k} s \sin^k s \, ds & \text{if } c = 1. \end{cases} \tag{2-9}$$

We give a simple proof of (2-8) and (2-9) here. First, when $c = 0$, the parallel hypersurface can be expressed as $\psi_t = \psi + t\nu$ (see [Montiel and Ros 1991]). So $(\psi_t)_*(e_i) = (1 + t\kappa_i)e_i$. Therefore the area element of Σ_t is

$$d\mu_t = (1 + t\kappa_1) \cdots (1 + t\kappa_n) \, d\mu,$$

which implies that the areas $|\Sigma_t|$ of the parallel hypersurfaces Σ_t are equal to

$$|\Sigma_t| = \int_\Sigma (1 + t\kappa_1) \cdots (1 + t\kappa_n) \, d\mu = \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \, t^k.$$

Note that $\Sigma_t = \psi_t(\Sigma)$ are parallel hypersurfaces of Σ given by

$$\psi_t(x) = \exp_{\psi(x)}(t\nu(x))$$

for $x \in \Sigma$. By integrating and using the co-area formula, we obtain

$$|V_t| = |V| + \int_0^t |\Sigma_s| \, ds = |V| + \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \frac{1}{k+1} t^{k+1}.$$

Similarly, when $c = -1$, $\psi_t = \cosh t \, \psi + \sinh t \, \nu$ (see [Montiel and Ros 1991]) and so $(\psi_t)_*(e_i) = (\cosh t + \sinh t \, \kappa_i)e_i$. Therefore the area element of Σ_t is

$$d\mu_t = (\cosh t + \sinh t \, \kappa_1) \cdots (\cosh t + \sinh t \, \kappa_n) \, d\mu,$$

which implies

$$\begin{aligned}|\Sigma_t| &= \int_\Sigma (\cosh t + \sinh t\,\kappa_1)\cdots(\cosh t + \sinh t\,\kappa_n)\,d\mu\\ &= \sum_{k=0}^{n} \int_\Sigma C_n^k\,p_k\,d\mu \cosh^{n-k} t \sinh^k t.\end{aligned}$$

Then by integrating, we obtain

$$|V_t| = |V| + \int_0^t |\Sigma_s|\,ds = |V| + \sum_{k=0}^{n} \int_\Sigma C_n^k\,p_k\,d\mu \int_0^t \cosh^{n-k} s \sinh^k s\,ds.$$

Finally, the case $c = 1$ can also be proved by noting that $\psi_t = \cos t\,\psi + \sin t\,\nu$, where $t \in \left[0, \frac{\pi}{2}\right)$.

3. The results by the method of unit normal flow

3A. ***The Euclidean case.*** To demonstrate the method which will be used to prove Theorems 1.1–1.3, in this subsection we first consider the simple case that Σ is a closed and convex hypersurface in $\mathbb{R}^{n+1}$. Let Σ_t be the parallel hypersurfaces of Σ and Ω_t be the domain enclosed by Σ_t. Then Σ_t is well-defined for all $t \geq 0$. For all $t \geq 0$, the isoperimetric inequality (see [Osserman 1978]) in Euclidean space $\mathbb{R}^{n+1}$ implies

$$\left(\frac{|\Sigma_t|}{\omega_n}\right)^{n+1} \geq \left((n+1)\frac{|V_t|}{\omega_n}\right)^n. \tag{3-1}$$

Substitute Steiner's formulas (2-8), (2-9) into (3-1). If n is odd, then comparing the coefficient of $t^{n(n+1)}$ in (3-1) yields

$$\int_\Sigma p_n\,d\mu \geq \omega_n, \tag{3-2}$$

which is a special Alexandrov–Fenchel inequality.

If n is even, (2-4) and the Gauss–Bonnet–Chern theorem (2-7) imply that $\int_\Sigma p_n\,d\mu = \omega_n$. Thus expanding the two sides of (3-1) and comparing the coefficients of $t^{n(n+1)}$, $t^{n(n+1)-1}$ and $t^{n(n+1)-2}$, we can get

$$\left(\frac{1}{\omega_n}\int_\Sigma p_{n-1}\,d\mu\right)^2 \geq \frac{1}{\omega_n}\int_\Sigma p_{n-2}\,d\mu, \tag{3-3}$$

which is also an Alexandrov–Fenchel inequality. In particular, when $n = 2$, (3-3) reduces to the classical Minkowski inequality (1-1).

3B. ***The hyperbolic case, I.*** In this subsection, we prove Theorem 1.1. Assume that Σ is a closed and convex hypersurface in $\mathbb{H}^{n+1}$. Then the parallel hypersurfaces Σ_t are well-defined for all $t \geq 0$. Recall that the area of a geodesic sphere S_r and the volume of a geodesic ball B_r with radius r in the hyperbolic space $\mathbb{H}^{n+1}$ are

$$\begin{aligned} S(r) &:= |S_r| = \omega_n \sinh^n r, \\ V(r) &:= \mathrm{Vol}(B_r) = \omega_n \int_0^r \sinh^n s \, ds. \end{aligned}$$

Now define a function $r(t)$ such that $|\Sigma_t| = S(r(t))$. That is,

$$\sum_{k=0}^{n} \cosh^{n-k} t \sinh^k t \int_\Sigma C_n^k p_k \, d\mu = \omega_n \sinh^n r(t). \tag{3-4}$$

Then the isoperimetric inequality (see [Schmidt 1940; Ros 2005]) implies

$$|V_t| \leq V(r(t)) \quad \text{for } t \geq 0. \tag{3-5}$$

From this inequality, we can get some information for Σ.

First we get a rough estimate for $r(t)$. When $t \to +\infty$, $\cosh^{n-k} t \sinh^k t = \sinh^n t \, (1 + o(1))$. Thus from $|\Sigma_t| = S(r(t))$, we get

$$\sinh^n t \, (1 + o(1)) \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu = \omega_n \sinh^n r(t),$$

which implies

$$r(t) = t + \frac{1}{n} \ln\biggl(\frac{1}{\omega_n} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \biggr) + o(1). \tag{3-6}$$

However, this estimate for $r(t)$ is not enough. For our purposes, we should make better use of $|\Sigma_t| = S(r(t))$ as follows. The case of $n = 2$ was considered by Natário [2015], so we assume that $n \geq 3$ in the following calculation. Since we will examine (3-5) for sufficiently large t, we only care about the terms involving e^{nt} and $e^{(n-2)t}$. The other terms are $o(e^{(n-2)t})$. It is straightforward to check that

$$\cosh^{n-k} t \sinh^k t = \frac{1}{2^n} e^{nt} + \frac{1}{2^n}(n-2k) e^{(n-2)t} + \cdots.$$

Consequently (3-4) implies

$$\frac{1}{2^n} \sum_{k=0}^{n} (e^{nt} + (n-2k) e^{(n-2)t} + \cdots) \int_\Sigma C_n^k p_k \, d\mu = \omega_n \biggl(\frac{1}{2^n} e^{nr} - \frac{1}{2^n} n e^{(n-2)r} + \cdots \biggr). \tag{3-7}$$

On the other hand, from Steiner's formula (2-9), we have

$$
\begin{aligned}
|V_t| &= |V| + \sum_{k=0}^{n} \int_0^t \cosh^{n-k} s \sinh^k s \, ds \int_\Sigma C_n^k p_k \, d\mu \\
&= |V| + \frac{1}{2^n} \sum_{k=0}^{n} \int_0^t (e^{ns} + (n-2k)e^{(n-2)s} + \cdots) \, ds \int_\Sigma C_n^k p_k \, d\mu \\
&= |V| + \frac{1}{2^n} \sum_{k=0}^{n} \left(\frac{1}{n} e^{nt} + \frac{n-2k}{n-2} e^{(n-2)t} + \cdots \right) \int_\Sigma C_n^k p_k \, d\mu \\
&= \frac{1}{2^n} \frac{1}{n} e^{nt} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu + \frac{1}{2^n} e^{(n-2)t} \sum_{k=0}^{n} \frac{n-2k}{n-2} \int_\Sigma C_n^k p_k \, d\mu + \cdots ,
\end{aligned}
$$

and

$$
\begin{aligned}
V(r(t)) &= \omega_n \int_0^r \sinh^n s \, ds \\
&= \omega_n \int_0^r \left(\frac{1}{2^n} e^{ns} - \frac{1}{2^n} n e^{(n-2)s} + \cdots \right) ds \\
&= \frac{\omega_n}{2^n} \frac{1}{n} e^{nr} - \frac{\omega_n}{2^n} \frac{n}{n-2} e^{(n-2)r} + \cdots .
\end{aligned}
$$

Now taking (3-7) into account yields

$$
\begin{aligned}
V(r(t)) &= \frac{\omega_n}{2^n} e^{(n-2)r} + \frac{1}{2^n} \frac{1}{n} \sum_{k=0}^{n} (e^{nt} + (n-2k)e^{(n-2)t} + \cdots) \int_\Sigma C_n^k p_k \, d\mu \\
&\qquad - \frac{\omega_n}{2^n} \frac{n}{n-2} e^{(n-2)r} \\
&= \frac{\omega_n}{2^n} \left(\frac{-2}{n-2} \right) e^{(n-2)r} + \frac{1}{2^n} \frac{1}{n} e^{nt} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \\
&\qquad + \frac{1}{2^n} \frac{1}{n} e^{(n-2)t} \sum_{k=0}^{n} (n-2k) \int_\Sigma C_n^k p_k \, d\mu + \cdots .
\end{aligned}
$$

Noting (3-6), we have

$$
\begin{aligned}
V(r(t)) &= \frac{\omega_n}{2^n} \left(\frac{-2}{n-2} \right) e^{(n-2)t} \left(\frac{1}{\omega_n} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \right)^{(n-2)/n} \\
&\qquad + \frac{1}{2^n} \frac{1}{n} e^{nt} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu + \frac{1}{2^n} \frac{1}{n} e^{(n-2)t} \sum_{k=0}^{n} (n-2k) \int_\Sigma C_n^k p_k \, d\mu + \cdots .
\end{aligned}
$$

Now $|V_t| \le V(r(t)), t \to +\infty$ gives us

$$\frac{1}{2^n}\sum_{k=0}^{n}\left(\frac{n-2k}{n-2}-\frac{n-2k}{n}\right)\int_\Sigma C_n^k p_k \, d\mu \le \frac{\omega_n}{2^n}\frac{-2}{n-2}\left(\frac{1}{\omega_n}\sum_{k=0}^{n}\int_\Sigma C_n^k p_k \, d\mu\right)^{(n-2)/n},$$

or equivalently

$$\sum_{k=0}^{n}\frac{2k-n}{n}\int_\Sigma C_n^k p_k \, d\mu \ge \omega_n\left(\frac{1}{\omega_n}\sum_{k=0}^{n}\int_\Sigma C_n^k p_k \, d\mu\right)^{(n-2)/n} \quad \text{for } n \ge 3. \tag{3-8}$$

Hence we complete the proof of Theorem 1.1.

Remark 3.1. It is easy to check that for a geodesic sphere in $\mathbb{H}^{n+1}$, the equality holds in (3-8). However this method can not yield the rigidity result; i.e., we cannot conclude that Σ is a geodesic sphere if the equality holds in (3-8).

Remark 3.2. We also remark that for a small hypersurface $\Sigma \subset \mathbb{H}^{n+1}$ (i.e., with small diameter), the inequality (3-8) can reduce to the Euclidean inequalities (3-2) and (3-3). For example, we first assume $n = 4$. For 4-dimensional hypersurface $\Sigma \subset \mathbb{H}^5$, the Gauss–Bonnet–Chern formula (2-7) implies

$$\int_\Sigma (p_4 - 2p_2 + 1)\, d\mu = \frac{1}{4!}\int_\Sigma L_2 \, d\mu = \omega_4. \tag{3-9}$$

Substituting (3-9) into the inequality (3-8) gives that

$$\left(1 + \frac{2}{\omega_4}\int_\Sigma (p_3 + p_2 - p_1 - 1)\, d\mu\right)^2 \ge 1 + \frac{4}{\omega_4}\int_\Sigma (p_3 + 2p_2 + p_1)\, d\mu.$$

Expanding the left-hand side of the above inequality, and comparing both sides by orders (note that Σ is a small hypersurface), we obtain that

$$\left(\frac{1}{\omega_4}\int_\Sigma p_3 \, d\mu\right)^2 \ge \frac{1}{\omega_4}\int_\Sigma p_2 \, d\mu. \tag{3-10}$$

This is just the inequality (3-3) for hypersurfaces in Euclidean space $\mathbb{R}^5$. For the general even-dimensional case, by using the Gauss–Bonnet–Chern formula,

$$\int_\Sigma \sum_{k=0}^{n/2} C_{n/2}^k (-1)^k p_{n-2k}\, d\mu = \frac{1}{n!}\int_\Sigma L_{n/2}\, d\mu = \omega_n.$$

We can also reduce the inequality (3-8) to the Euclidean version (3-3) for small hypersurfaces $\Sigma \subset \mathbb{H}^{n+1}$. For the odd-dimensional case, the argument is similar.

3C. ***The hyperbolic case, II.*** In this subsection, we will prove Theorem 1.2. Since the method is similar to that of the last subsection, we just sketch it.

Here we need the following model of the hyperbolic space. Let $\mathbb{R}^{n+2}_1$ be the Minkowski space with the Lorentzian metric

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_{n+1} y_{n+1} - x_{n+2} y_{n+2}.$$

Then the $(n+1)$-dimensional hyperbolic space can be defined by

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+2}_1 \mid \langle x, x\rangle = -1, x_{n+2} \geq 1\}$$

with the induced metric from $\mathbb{R}^{n+2}_1$.

Fix a point $a = (0, \dots, 0, -1)$. Then it is easy to check that the weight function and the support function can be written down as

$$f = \langle \psi, a\rangle,$$
$$u = \langle \nu, a\rangle.$$

Next define a family of parallel hypersurfaces $\Sigma_t = \psi_t(\Sigma)$, where $\psi_t(x) = \exp_{\psi(x)}(t\nu(x))$, $x \in \Sigma$, and $\nu(x)$ is the outward unit normal of Σ. In fact, $\psi_t = \cosh t\, \psi + \sinh t\, \nu$. And since the initial hypersurface is convex, Σ_t is well-defined for all $t \geq 0$. Then $(\psi_t)_*(e_i) = (\cosh t + \kappa_i \sinh t)e_i$ and

$$\kappa_i(t) = \frac{\tanh t + \kappa_i}{1 + \kappa_i \tanh t}.$$

For convenience, we define a function $Q(t)$ by

$$Q_n(t) = (1 + t\kappa_1)\cdots(1 + t\kappa_n) = 1 + C^1_n p_1 t + \cdots + C^n_n p_n t^n.$$

Then the mean curvature of Σ_t is

$$p_1(t) = \frac{n \cosh t \sinh t\, Q_n(\tanh t) + Q'_n(\tanh t)}{n \cosh^2 t\, Q_n(\tanh t)}.$$

Note that $p_1(t) \to 1$ as $t \to +\infty$. So for sufficiently large t, Σ_t is mean convex. And $\langle \nu_t, a\rangle = \langle \sinh t\, \psi + \cosh t\, \nu, a\rangle \geq 0$ for sufficiently large t, which implies Σ_t is star-shaped for these t. Thus, we can apply (1-5) to Σ_t:

$$\frac{1}{\omega_n}\int_\Sigma \langle \cosh t\, \psi + \sinh t\, \nu, a\rangle\, p_1(t) \cosh^n t\, Q_n(\tanh t)\, d\mu$$
$$\geq \left(\frac{1}{\omega_n}\int_\Sigma \cosh^n t\, Q_n(\tanh t)\, d\mu\right)^{\frac{n+1}{n}} + \left(\frac{1}{\omega_n}\int_\Sigma \cosh^n t\, Q_n(\tanh t)\, d\mu\right)^{\frac{n-1}{n}}.$$

Let $t \to +\infty$. Taking into account that $\tanh t \to 1$, $p_1(t) \to 1$ and $\sinh t = \cosh t\, (1 + o(1))$, we obtain (1-6). So we have finished the proof.

3D. ***The spherical case.*** We now prove Theorem 1.3. Assume that Σ is a closed and convex hypersurface in $\mathbb{S}^{n+1}$. Then the parallel hypersurface Σ_t is well-defined for $t \in \left[0, \frac{\pi}{2}\right)$. Recall that the area of a geodesic sphere S_r and the volume of a geodesic ball B_r with radius r in the sphere $\mathbb{S}^{n+1}$ are

$$\begin{aligned} S(r) &= \omega_n \sin^n r, \\ V(r) &= \omega_n \int_0^r \sin^n s \, ds. \end{aligned}$$

Now since $|V_t|$ is increasing in t, when t satisfies

$$|V_t| = V\left(\frac{\pi}{2}\right) = \omega_n \int_0^{\pi/2} \sin^n r \, dr,$$

the isoperimetric inequality (see [Ros 2005]) implies $|\Sigma_t| \geq S(\frac{\pi}{2}) = \omega_n$ for this t. Therefore, a weaker requirement is

$$\max_{t \in [0,\pi/2)} |\Sigma_t| \geq \omega_n. \tag{3-11}$$

Then the key point is to estimate $\max_{t\in[0,\pi/2)}|\Sigma_t|$. Direct computation shows that

$$\begin{aligned} \cos^{n-k} t \sin^k t &= \left(\frac{e^{it}+e^{-it}}{2}\right)^{n-k}\left(\frac{e^{it}-e^{-it}}{2i}\right)^k \\ &= \frac{1}{2^n}\sum_{p=0}^{n-k}\sum_{q=0}^{k} C_{n-k}^p C_k^q \cos\left((2(p+q)-n)t - \frac{k\pi}{2}\right)(-1)^{k-q}. \end{aligned}$$

Then Steiner's formula (2-8) implies

$$\begin{aligned} |\Sigma_t| &= \sum_{k=0}^{n} C_n^k \cos^{n-k} t \sin^k t \int_\Sigma p_k \, d\mu \\ &= \sum_{k=0}^{n} C_n^k \frac{1}{2^n}\sum_{p=0}^{n-k}\sum_{q=0}^{k} C_{n-k}^p C_k^q \cos\left((2(p+q)-n)t - \frac{k\pi}{2}\right)(-1)^{k-q}\int_\Sigma p_k \, d\mu \\ &= \sum_{\substack{0\leq k\leq n \\ 2|k}} C_n^k \frac{1}{2^n}\sum_{p=0}^{n-k}\sum_{q=0}^{k} C_{n-k}^p C_k^q (-1)^{\frac{k}{2}} \cos((2(p+q)-n)t)(-1)^{k-q}\int_\Sigma p_k \, d\mu \\ &\quad + \sum_{\substack{0\leq k\leq n \\ 2\nmid k}} C_n^k \frac{1}{2^n}\sum_{p=0}^{n-k}\sum_{q=0}^{k} C_{n-k}^p C_k^q (-1)^{[\frac{k}{2}]} \sin((2(p+q)-n)t)(-1)^{k-q}\int_\Sigma p_k \, d\mu. \end{aligned}$$

Next let $2(p+q)-n=\pm s$ and sum up in terms of s first. We get

$$
\begin{aligned}
(3\text{-}12)\quad |\Sigma_t| = & \sum_{s=\frac{1-(-1)^n}{2},+2}^{n} \sum_{\substack{p+q=(n\pm s)/2\\ p,q\ge 0}} \sum_{\substack{q\le k\le n-p\\ 2|k}} C_n^k \frac{1}{2^n} C_{n-k}^p C_k^q \\
& \qquad\qquad \times (-1)^{\frac{k}{2}+k-q} \cos(st) \int_\Sigma p_k \, d\mu \\
& + \sum_{s=\frac{1-(-1)^n}{2},+2}^{n} \sum_{\substack{p+q=(n\pm s)/2\\ p,q\ge 0}} \sum_{\substack{q\le k\le n-p\\ 2\nmid k}} C_n^k \frac{1}{2^n} C_{n-k}^p C_k^q (-1)^{[\frac{k}{2}]+k-q} \\
& \qquad\qquad \times (-1)^{\chi_{\{2(p+q)-n\le 0\}}} \sin(st) \int_\Sigma p_k \, d\mu \\
\le & \sum_{s=\frac{1-(-1)^n}{2},+2}^{n} \sqrt{(E(s))^2+(F(s))^2},
\end{aligned}
$$

in the notation of Theorem 1.3.

Next we show that for the geodesic sphere with radius $r\in\left[0,\frac{\pi}{2}\right)$, the equality holds. For this special hypersurface, $\int_\Sigma p_k \, d\mu=\omega_n \sin^n r \cot^k r=\omega_n \sin^{n-k} r \cos^k r$. Thus

$$
\begin{aligned}
|\Sigma_t| &= \omega_n \sum_{k=0}^{n} C_n^k (\cos t \sin r)^{n-k} (\sin t \cos r)^k = \omega_n \sin^n(r+t) \\
&= \omega_n \frac{1}{2^n} \sum_{q=0}^{n} C_n^q \cos\Bigl((2q-n)(r+t)-\frac{n\pi}{2}\Bigr)(-1)^{n-q}.
\end{aligned}
$$

For simplicity, we assume n is even. Then

$$
\begin{aligned}
|\Sigma_t| &= \omega_n \frac{1}{2^n} \sum_{q=0}^{n} C_n^q \cos((2q-n)(r+t))(-1)^{n/2+n-q} \\
&= \omega_n \sum_{s=0,2,\dots,n} \sum_{2q-n=\pm s} \frac{1}{2^n} C_n^q \cos(s(r+t))(-1)^{3n/2-q} \\
&= \omega_n \sum_{s=0,2,\dots,n} \sum_{2q-n=s} 2\frac{1}{2^n} C_n^q \cos(s(r+t))(-1)^{3n/2-q},
\end{aligned}
$$

where we note that the coefficients of $\cos(s(r+t))$ for the two choices of q are the same.

Now expand $\cos(s(r+t))=\cos sr \cos st-\sin sr \sin st$. We find that all the inequalities in (3-12) become equalities for $t=\frac{\pi}{2}-r$ and $|\Sigma_t|=\omega_n \sin^n \frac{\pi}{2}=\omega_n$. Thus for the geodesic sphere, the equality in (3-12) holds.

On the other hand, assume the equality holds. Then when some t satisfies $|V_t|=V(\frac{\pi}{2})=\omega_n \int_0^{\pi/2} \sin^n r \, dr$, we must have $|\Sigma_t|=S(\frac{\pi}{2})=\omega_n$ for this t. So the isoperimetric inequality implies that $\Sigma_t = \mathbb{S}^n(1)$. Then the initial hypersurface must be a geodesic sphere.

Thus Theorem 1.3 is proved.

Remark 3.3. In Section 1, we discussed the special case $n = 2$ of (1-7), which is just the Minkowski-type inequality for convex surfaces in $\mathbb{S}^3$. Here, to get a better feeling of the inequality (1-7), we give the precise expressions for $n = 3$ and $n = 4$. For $n = 3$, we have

$$\omega_3 \le \sqrt{\left(\frac{1}{4}\left(|\Sigma| - 3\int_\Sigma p_2\, d\mu\right)\right)^2 + \left(\frac{1}{4}\left(3\int_\Sigma p_1\, d\mu - \int_\Sigma p_3\, d\mu\right)\right)^2} + \sqrt{\left(\frac{3}{4}\left(|\Sigma| + \int_\Sigma p_2\, d\mu\right)\right)^2 + \left(\frac{3}{4}\left(\int_\Sigma p_1\, d\mu + \int_\Sigma p_3\, d\mu\right)\right)^2}.$$

And for $n = 4$, we have

$$\begin{aligned}\omega_4 \le &\sqrt{\left(\frac{1}{8}\left(|\Sigma| - 6\int_\Sigma p_2\, d\mu + \int_\Sigma p_4\, d\mu\right)\right)^2 + \left(\frac{1}{2}\left(\int_\Sigma p_1\, d\mu - \int_\Sigma p_3\, d\mu\right)\right)^2} \\ &+ \sqrt{\left(\frac{1}{2}\left(|\Sigma| - \int_\Sigma p_4\, d\mu\right)\right)^2 + \left(\int_\Sigma p_1\, d\mu + \int_\Sigma p_3\, d\mu\right)^2} \\ &+ \frac{3}{8}\left(|\Sigma| + 2\int_\Sigma p_2\, d\mu + \int_\Sigma p_4\, d\mu\right).\end{aligned} \tag{3-13}$$

For a 4-dimensional hypersurface Σ in $\mathbb{S}^5$, we have the Gauss–Bonnet–Chern formula

$$\int_\Sigma (p_4 + 2p_2 + 1)\, d\mu = \frac{1}{4!}\int_\Sigma L_2\, d\mu = \omega_4. \tag{3-14}$$

Therefore, the inequality (3-13) can be further simplified by using the formula (3-14).

Remark 3.4. As in the hyperbolic case, when the hypersurface $\Sigma \subset \mathbb{S}^5$ is small, the inequality (3-13) reduces to the Euclidean version (3-3). This can be seen using a similar argument to that in Remark 3.2.

4. The results by the method of inverse mean curvature flow

In this section we give the proof of Theorem 1.4 using a different method from the one in the previous section.

4A. ***Evolution equations.*** Considering Σ as the initial hypersurface, we flow Σ in $\mathbb{S}^{n+1}$ under the flow equation $X : \Sigma \times [0, T^*) \to \mathbb{S}^{n+1}$,

$$\partial_t X = F\nu,$$

where F is a curvature function and ν is the unit normal to the flow hypersurfaces Σ_t. First we recall the following evolution equations.

Lemma 4.1 [Makowski and Scheuer 2013]. *Under the curvature flow $\partial_t X = F\nu$ in $\mathbb{S}^{n+1}$, we have*

$$\frac{d}{dt}|\Sigma_t| = n\int_{\Sigma_t} Fp_1\,d\mu_t, \tag{4-1}$$

$$\frac{d}{dt}\int_{\Sigma_t} p_m\,d\mu_t = (n-m)\int_{\Sigma_t} Fp_{m+1}\,d\mu_t - m\int_{\Sigma_t} Fp_{m-1}\,d\mu_t. \tag{4-2}$$

To simplify the notation, in the following we define

$$\widetilde{L}_k = \frac{1}{C_n^{2k}(2k)!}L_k = \sum_{i=0}^{k} C_k^i p_{2k-2i}. \tag{4-3}$$

Using Lemma 4.1, we obtain the following.

Lemma 4.2. *Under the curvature flow $\partial_t X = F\nu$ in $\mathbb{S}^{n+1}$, we have*

$$\frac{d}{dt}\int_{\Sigma_t} \widetilde{L}_k\,d\mu_t = (n-2k)\sum_{i=0}^{k} C_k^i \int_{\Sigma_t} Fp_{2k-2i+1}\,d\mu_t.$$

Proof. The proof is by a direct calculation:

$$\begin{aligned}
\frac{d}{dt}\int_{\Sigma_t} \widetilde{L}_k\,d\mu_t &= \sum_{i=0}^{k} C_k^i \frac{d}{dt}\int_{\Sigma_t} p_{2k-2i}\,d\mu_t \\
&= \sum_{i=0}^{k} C_k^i \int_{\Sigma_t} \big((n-2k+2i)Fp_{2k-2i+1} - 2(k-i)Fp_{2k-2i-1}\big)\,d\mu_t \\
&= \sum_{i=0}^{k} C_k^i \int_{\Sigma_t} (n-2k+2i)Fp_{2k-2i+1}\,d\mu_t \\
&\qquad\qquad - \sum_{i=1}^{k} C_k^{i-1} \int_{\Sigma_t} 2(k-i+1)Fp_{2k-2i+1}\,d\mu_t \\
&= (n-2k)\sum_{i=0}^{k} C_k^i \int_{\Sigma_t} Fp_{2k-2i+1}\,d\mu_t. \qquad \square
\end{aligned}$$

4B. ***Proof of Theorem 1.4.*** Recently, Makowski and Scheuer [2013] and Gerhardt [2015] studied the curvature flows in the sphere. If the initial hypersurface $\Sigma \subset \mathbb{S}^{n+1}$ is closed and strictly convex, then under the inverse mean curvature flow

$$\partial_t X = \frac{1}{p_1}\nu,$$

there exists a finite time $T^* < \infty$ such that the flow hypersurface Σ_t converges to an equator in $\mathbb{S}^{n+1}$ and the mean curvature of Σ_t converges to zero almost everywhere in the sense of (see Theorem 1.4 in [Makowski and Scheuer 2013])

$$\lim_{t\to T^*}\int_{\Sigma_t} p_1^{\alpha}\, d\mu_t = 0 \quad \text{for all } 1 \le \alpha < \infty. \tag{4-4}$$

For each $t \in [0, T^*)$, define the quantity $Q(t)$ by

$$Q(t) = |\Sigma_t|^{-(n-2k)/n}\int_{\Sigma_t} \widetilde{L}_k\, d\mu_t. \tag{4-5}$$

On the one hand, by Lemmas 4.2 and 2.1 (note that strictly convex implies all principal curvatures of Σ_t are positive, and certainly belong to Γ_k^+), we have

$$\begin{aligned}\frac{d}{dt}\int_{\Sigma_t}\widetilde{L}_k d\mu_t &= (n-2k)\sum_{i=0}^{k} C_k^i \int_{\Sigma_t}\frac{p_{2k-2i+1}}{p_1}\, d\mu_t\\ &\le (n-2k)\sum_{i=0}^{k} C_k^i\int_{\Sigma_t} p_{2k-2i}\, d\mu_t\\ &= (n-2k)\int_{\Sigma_t}\widetilde{L}_k\, d\mu_t.\end{aligned}$$

Equality holds if and only if Σ_t is totally umbilical. On the other hand, the area of the flow hypersurface evolves as

$$\frac{d}{dt}|\Sigma_t| = n|\Sigma_t|.$$

Therefore we obtain that the quantity $Q(t)$ is monotone nonincreasing in t; i.e.,

$$\frac{d}{dt}Q(t) \le 0. \tag{4-6}$$

Since under the inverse mean curvature flow, the flow hypersurfaces converge to an equator in $\mathbb{S}^{n+1}$ and the mean curvature of Σ_t converges to zero almost everywhere in the sense of (4-4), we have

$$\lim_{t\to T^*} Q(t) = \omega_n^{2k/n}. \tag{4-7}$$

Combining (4-6) and (4-7), we have

$$Q(0) = |\Sigma|^{-(n-2k)/n}\int_{\Sigma}\widetilde{L}_k\, d\mu \ge \lim_{t\to T^*} Q(t) = \omega_n^{2k/n}.$$

Hence noting (4-3), we obtain that

$$\int_{\Sigma} L_k\, d\mu \ge C_n^{2k}(2k)!\,\omega_n^{2k/n}|\Sigma|^{(n-2k)/n}. \tag{4-8}$$

Equality holds in (4-8) if and only if $Q(t)$ is constant in t. Then Σ_t is totally umbilical for each $t \in [0, T^*)$, and, in particular, Σ is totally umbilical and hence a geodesic sphere. The inequality (4-8) says that the induced metric of convex hypersurfaces in $\mathbb{S}^{n+1}$ satisfies the optimal Sobolev inequalities. See [Ge et al. 2014b] for further information about the Sobolev inequalities of the same type.

Acknowledgements

The authors would like to thank Professor Haizhong Li for constant encouragement and help. They also thank the referee for some helpful comments which made this paper more readable. The research of the authors was supported by NSFC No. 11271214.

References

[Alexandrov 1937] A. D. Alexandrov, "Zur Theorie der gemischten Volumina von konvexen Körpern, II: Neue Ungleichungen zwischen den gemischten Volumina und ihre Anwendungen", *Mat. Sb.* (*N.S.*) **2(44)**:6 (1937), 1205–1238. In Russian. Zbl 0018.27601

[Alexandrov 1938] A. D. Alexandrov, "Zur Theorie der gemischten Volumina von konvexen Körpern, III: Die Erweiterung zweeier Lehrsatze Minkowskis über die konvexen Polyeder auf beliebige konvexe Flachen", *Mat. Sb.* (*N.S.*) **3(45)**:1 (1938), 27–46. In Russian. Zbl 0018.42402

[Allendoerfer 1948] C. B. Allendoerfer, "Steiner's formulae on a general S^{n+1}", *Bull. Amer. Math. Soc.* **54** (1948), 128–135. MR 9,607j Zbl 0033.39503

[Blaschke 1938] W. Blaschke, *Über eine geometrische Frage von Euklid bis heute*, Hamburger Mathematiches Einzelschriften **23**, B. G. Teubner, Berlin, 1938. Zbl 0018.23404

[Borisenko and Miquel 1999] A. A. Borisenko and V. Miquel, "Total curvatures of convex hypersurfaces in hyperbolic space", *Illinois J. Math.* **43**:1 (1999), 61–78. MR 2000h:53100 Zbl 0981.53046

[Brendle et al. 2014] S. Brendle, P. K. Hung, and M. T. Wang, "Minkowski-type inequality for hypersurfaces in the Anti-de Sitter–Schwarzschild manifold", *Comm. Pure Appl. Math.* (online publication December 2014).

[do Carmo and Warner 1970] M. P. do Carmo and F. W. Warner, "Rigidity and convexity of hypersurfaces in spheres", *J. Differential Geometry* **4** (1970), 133–144. MR 42 #1014 Zbl 0201.23702

[Chang and Wang 2011] S.-Y. A. Chang and Y. Wang, "On Aleksandrov–Fenchel inequalities for k-convex domains", *Milan J. Math.* **79**:1 (2011), 13–38. MR 2012h:52005 Zbl 1229.52002

[Chern 1944] S.-s. Chern, "A simple intrinsic proof of the Gauss–Bonnet formula for closed Riemannian manifolds", *Ann. of Math.* (2) **45**:4 (1944), 747–752. MR 6,106a Zbl 0060.38103

[Fenchel 1936] W. Fenchel, "Inégalités quadratiques entre les volumes mixtes des corps convexes", *C. R. Acad. Sci., Paris* **203** (1936), 647–650. Zbl 0015.12006

[Gallego and Solanes 2005] E. Gallego and G. Solanes, "Integral geometry and geometric inequalities in hyperbolic space", *Differential Geom. Appl.* **22**:3 (2005), 315–325. MR 2006g:28011 Zbl 1077.52010

[Ge et al. 2013] Y. Ge, G. Wang, and J. Wu, "Hyperbolic Alexandrov–Fenchel quermassintegral inequalities, I", preprint, 2013. arXiv 1303.1714

[Ge et al. 2014a] Y. Ge, G. Wang, and J. Wu, "The GBC mass for asymptotically hyperbolic manifolds", *C. R. Math. Acad. Sci. Paris* **352**:2 (2014), 147–151. MR 3151884 Zbl 1285.53026

[Ge et al. 2014b] Y. Ge, G. Wang, and J. Wu, "Hyperbolic Alexandrov–Fenchel quermassintegral inequalities II", *J. Differential Geom.* **98**:2 (2014), 237–260. MR 3263518 Zbl 1301.53077

[Ge et al. 2014c] Y. Ge, G. Wang, and J. Wu, "A new mass for asymptotically flat manifolds", *Adv. Math.* **266** (2014), 84–119. MR 3262355 Zbl 1301.53032

[Ge et al. 2015] Y. Ge, G. Wang, J. Wu, and C. Xia, "A Penrose inequality for graphs over Kottler space", *Calc. Var. Partial Differential Equations* **52**:3-4 (2015), 755–782. MR 3311913 Zbl 1308.83093

[Gerhardt 2015] C. Gerhardt, "Curvature flows in the sphere", *J. Differential Geom.* **100**:2 (2015), 301–347. MR 3343834 Zbl 06451271

[Guan 2006] P. Guan, "Topics in geometric fully nonlinear equations", Lecture notes, 2006, Available at http://www.math.mcgill.ca/guan/notes.html.

[Guan and Li 2009] P. Guan and J. Li, "The quermassintegral inequalities for k-convex starshaped domains", *Adv. Math.* **221**:5 (2009), 1725–1732. MR 2010i:52021 Zbl 1170.53058

[Knothe 1952] H. Knothe, "Zur Theorie der konvexen Körper im Raum konstanter positiver Krümmung", *Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat.* (2) **2** (1952), 336–348. MR 15,819c Zbl 0049.12301

[Li et al. 2014] H. Li, Y. Wei, and C. Xiong, "A geometric inequality on hypersurface in hyperbolic space", *Adv. Math.* **253** (2014), 152–162. MR 3148549 Zbl 06285046

[de Lima and Girão 2015] L. L. de Lima and F. Girão, "An Alexandrov–Fenchel-type inequality in hyperbolic space with an application to a Penrose inequality", *Ann. Henri Poincaré* (online publication May 2015).

[Makowski and Scheuer 2013] M. Makowski and J. Scheuer, "Rigidity results, inverse curvature flows and Alexandrov–Fenchel type inequalities in the sphere", 2013. To appear in *Asian J. Math.* arXiv 1307.5764

[McCoy 2005] J. A. McCoy, "Mixed volume preserving curvature flows", *Calc. Var. Partial Differential Equations* **24**:2 (2005), 131–154. MR 2006g:53098 Zbl 1079.53099

[Minkowski 1903] H. Minkowski, "Volumen und Oberfläche", *Math. Ann.* **57**:4 (1903), 447–495. MR 1511220 JFM 34.0649.01

[Montiel and Ros 1991] S. Montiel and A. Ros, "Compact hypersurfaces: The Alexandrov theorem for higher order mean curvatures", pp. 279–296 in *Differential geometry*, edited by B. Lawson and K. Tenenblat, Pitman Monogr. Surveys Pure Appl. Math. **52**, Longman Sci. Tech., Harlow, 1991. MR 93h:53062 Zbl 0723.53032

[Natário 2015] J. Natário, "A Minkowski-type inequality for convex surfaces in the hyperbolic 3-space", *Differential Geom. Appl.* **41** (2015), 102–109. MR 3353742 Zbl 06448541

[Osserman 1978] R. Osserman, "The isoperimetric inequality", *Bull. Amer. Math. Soc.* **84**:6 (1978), 1182–1238. MR 58 #18161 Zbl 0411.52006

[Ros 2005] A. Ros, "The isoperimetric problem", pp. 175–209 in *Global theory of minimal surfaces* (Berkeley, CA, 2001), edited by D. Hoffman, Clay Math. Proc. **2**, Amer. Math. Soc., Providence, RI, 2005. MR 2006e:53023 Zbl 1125.49034

[Santaló 1963] L. A. Santaló, "A relation between the mean curvatures of parallel convex bodies in spaces of constant curvature", *Rev. Un. Mat. Argentina* **21** (1963), 131–137. In Spanish. MR 29 #6423 Zbl 0131.37801

[Santaló 1976] L. A. Santaló, *Integral geometry and geometric probability*, Encyclopedia of Mathematics and its Applications **1**, Addison-Wesley, Reading, MA, 1976. MR 55 #6340 Zbl 0342.53049

[Schmidt 1940] E. Schmidt, "Die isoperimetrischen Ungleichungen auf der gewöhnlichen Kugel und für Rotationskörper im n-dimensionalen sphärischen Raum", *Math. Z.* **46** (1940), 743–794. MR 2,262e Zbl 0023.38101

[Schneider 1993] R. Schneider, *Convex bodies: the Brunn–Minkowski theory*, Encyclopedia of Mathematics and its Applications **44**, Cambridge Univ. Press, 1993. MR 94d:52007 Zbl 0798.52001

[Solanes 2003] G. Solanes, *Integrals de curvatura i geometria integral a l'espai hiperbolic*, Ph.D. thesis, Univ. Aut. Barcelona, 2003, Available at http://mat.uab.cat/~egallego/gil/tesi.pdf.

[Solanes 2006] G. Solanes, "Integral geometry and the Gauss–Bonnet theorem in constant curvature spaces", *Trans. Amer. Math. Soc.* **358**:3 (2006), 1105–1115. MR 2006h:53075 Zbl 1082.53075

[Wang and Xia 2014] G. Wang and C. Xia, "Isoperimetric type problems and Alexandrov–Fenchel type inequalities in the hyperbolic space", *Adv. Math.* **259** (2014), 532–556. MR 3197666 Zbl 1292.52008

Received October 6, 2013. Revised May 8, 2014.

YONG WEI
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
100084, BEIJING
CHINA

and

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE LONDON
LONDON WC1E 6BT
UNITED KINGDOM

wei-y09@mails.tsinghua.edu.cn

yong.wei@ucl.ac.uk

CHANGWEI XIONG
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
100084, BEIJING
CHINA

xiongcw10@mails.tsinghua.edu.cn

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2015 is US $420/year for the electronic version, and $570/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 277 No. 1 September 2015

0030-8730(201509)277:1;1-V