Pacific Journal of Mathematics

Volume 277 No. 1 September 2015

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2015 is US $420/year for the electronic version, and $570/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.1

REAL POSITIVITY AND APPROXIMATE IDENTITIES IN BANACH ALGEBRAS

DAVID P. BLECHER AND NARUTAKA OZAWA

Blecher and Read recently introduced and studied a new notion of positivity in operator algebras, with an eye to extending certain C^*-algebraic results and theories to more general algebras. In the present paper, we generalize some part of this, and some other facts, to larger classes of Banach algebras.

1. Introduction

An *operator algebra* is a closed subalgebra of $B(H)$ for a complex Hilbert space H. Blecher and Read [2011; 2013a; 2014] and Read [2011] recently introduced and studied a new notion of positivity in operator algebras (see also [Blecher and Neal 2012a; 2012b; Bearden et al. 2014; Blecher et al. 2008]), with an eye to extending certain C^*-algebraic results and theories to more general algebras. Over the last several years, we have mentioned in lectures on this work that most of the results of those papers make sense for bigger classes of Banach algebras, and

Bleher was supported by a grant from the NSF. Ozawa was supported by JSPS KAKENHI grant number 26400114. Some of this material was presented at the 7th Conference on Function Spaces, May 2014, and at the AMS National Meeting in January 2015.
MSC2010: primary 46H10, 46H99, 46J99, 47L10, 47L30; secondary 47L75.
Keywords: Banach algebra, approximate identity, unitization, real-positive, states, quasistate, ideals, hereditary subalgebra, ordered linear spaces, M-ideal, accretive operator, sectorial operator, operator roots, noncommutative Tietze theorem.

that many of the tools and techniques exist there. In the present paper we initiate this direction. Thus we generalize a number of the main results from the series of papers mentioned above, and some other facts, to a larger class of Banach algebras. In the process we give simplifications of several facts in these earlier papers. We will also point out some of the main results from the series of papers mentioned above which do not seem to generalize, or are less tidy if they do. (We will not spend much time discussing aspects from that series concerning noncommutative peak interpolation, or generalizations of noncommutative topology such as the noncommutative Urysohn lemma; these seem unlikely to generalize much farther.)

Before we proceed we make an editorial/historical note: the preprint [Blecher and Read 2013b], which contains many of the basic ideas and facts we use here, has been split into several papers, which have each taken on a life of their own (e.g., [Blecher and Read 2014] which focuses on operator algebras, and the present paper in the setting of Banach algebras).

In this paper we are interested in Banach algebras A (over the complex field) with a bounded approximate identity (bai). In fact, often there will be a contractive approximate identity (cai), and, in this case, we call A an *approximately unital* Banach algebra. A Banach algebra with an identity of norm 1 will be called *unital*. Most of our results are stated for approximately unital algebras. Frequently this is simply because algebras in this class have an especially nice "multiplier unitization" A^1, defined below, and a large portion of our constructs are defined in terms of A^1. Also, approximately unital algebras constitute a strong platform for the simultaneous generalization of as much as possible from the series of papers referenced above. However, as one might expect, for algebras without any kind of approximate identity it is easy to derive variants of a large portion of our results (namely, almost all of Sections 3, 4, and 7), by viewing the algebra as a subalgebra of a unital Banach algebra (any unitization, for example). We will discuss this point in more detail in Section 9 and in a forthcoming conference proceedings survey article [Blecher 2015].

Indeed many of our results are stated for special classes of Banach algebras, for example, for Banach algebras with a sequential cai or which are Hahn–Banach smooth in a sense defined later. Several of the results are sharper for *M-approximately unital Banach algebras*, which means that A is an M-ideal in its multiplier unitization A^1 (see Section 2). This is equivalent to saying that A is approximately unital and for all $x \in A^{**}$, we have $\|1-x\|_{(A^1)^{**}} = \max\{\|e-x\|_{A^{**}}, 1\}$. Here e is the identity for A^{**}, if it has one (otherwise it is a "mixed identity" of norm 1 — see below for the definition of this). However, as will be seen from the proofs, some of the results involving the M-approximately unital hypothesis will work under weaker assumptions, for example, *strong proximinality* of A in A^1 at 1 (that is, given $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ with $\|1-y\| < 1+\delta$ then there is a $z \in A$ with $\|1-z\| = 1$ and $\|y-z\| < \epsilon$).

We now outline the structure of this paper, describing each section briefly. Because our paper is rather diverse, to help the readers focus we will also mention at least one highlight from each section. In Section 2 we discuss unitization and states, and also introduce some classes of Banach algebras. A key result in this section ensures the existence of a "real positive" cai in Banach algebras with a countable cai satisfying a reasonable extra condition. We also characterize this extra condition, and the related property that the quasistate space be weak* closed and convex. In the latter setting, by the bipolar theorem, there exists a "Kaplansky density theorem". (Conversely, such a density result often immediately gives a real positive approximate identity by weak* approximating an identity in the bidual by real positive elements in A, and using, e.g., Lemma 2.1 below.) Section 3 starts by generalizing many of the basic ideas from the papers of Blecher and Read cited above involving cais, roots, and positivity. With these in place, we give several applications of the kind found in those papers; for example, we characterize when xA is closed in terms of the "generalized invertibility" of the real positive element x, and show that these are the right ideals qA for a real positive idempotent q in A. We also list several examples illustrating some of the things from the cited series of papers that will break down without further restrictions on the class of Banach algebras considered. The main advance in Section 4 is the introduction of the concept of *hereditary subalgebras* (HSAs), an important tool in C^*-algebra theory, to Banach algebras, and establishing the basics of their theory. In particular, we study the relationship between HSAs and one-sided ideals with one-sided approximate identities. Some aspects of this relationship are problematic for general Banach algebras, but it works much better in separable algebras, as we shall see. We characterize the HSAs, and the associated class of one-sided ideals, as increasing unions of "principal" ones; and indeed in the separable case they are exactly the "principal" ones. Indeed it is obvious that in a Banach algebra A, every closed right ideal with a real positive left bai is of the form $\overline{EA}$ for a set E of real positive elements of A. Section 4 contains an Aarnes–Kadison-type theorem for Banach algebras, and related results that use the Cohen's factorization proof technique. Some similar results and ideas have been found by Sinclair (in [Sinclair 1978], for example), but these are somewhat different, and were not directly connected to "positivity". It is interesting though that Sinclair was inspired by papers of Esterle based on the Cohen's factorization proof technique, and one of these does have some connection to our notion of positivity [Esterle 1978].

In Section 5 we consider the better behaved class of M-approximately unital Banach algebras. The main result here is the generalization of Read's theorem [Read 2011] to this class. That is, such algebras have cais (e_t) satisfying $\|1-2e_t\| \leq 1$. This may be the class to which the most results from our previous operator algebra papers will generalize, as we shall see at points throughout our paper. In Section 6 we show that basic aspects and notions from the classical theory of ordered linear spaces correspond to interesting facts about our positivity for our various classes of approximately unital Banach algebras (for example, for M-approximately unital algebras, or certain algebras with a sequential cai). Indeed the highlight of this section is the revealing of interesting connections between Banach algebras and this classical ordered linear theory (see also [Blecher and Read 2014] for more, and clearer, such connections if the algebras are in addition operator algebras). In the process we generalize several basic facts about C^*-algebras. For example, we give the aforementioned variant of Kaplansky's density theorem, and variants of several well-known order-theoretic properties of the unit ball of a C^*-algebra and its dual.

In Sections 7 and 8 we find variants for approximately unital Banach algebras of several other results about two-sided ideals from [Blecher and Read 2011; 2013a; 2014]. In Section 7 we assume that A is commutative, and in this case we are able to establish the converse of the last result mentioned in our description of Section 4 above. Thus closed ideals having a real positive bai, in a commutative Banach algebra A, are precisely the spaces $\overline{EA}$ for sets E of real positive elements of A. In Section 8 we only consider ideals that are M-ideals in A (this does generalize the operator algebra case at least for two-sided ideals, since the closed two-sided ideals with cais in an operator algebra are exactly the M-ideals [Effros and Ruan 1990]). The lattice theoretic properties of such ideals behave considerably more like the C^*-algebra case and are related to faces in the quasistate space. Section 8 may be considered to be a continuation of the study of M-ideals in Banach algebras initiated in [Smith and Ward 1978; 1979; Smith 1979] and, e.g., [Harmand et al. 1993, Chapter V]. At the end of this section, we give a "noncommutative peak interpolation" result reminiscent of Tietze's extension theorem, which is based on a remarkable result of Chui, Smith, Smith, and Ward [Chui et al. 1977]. This solves an open problem from [Blecher and Read 2013b], or earlier, concerning real positive elements in a quotient. Finally, in Section 9 we discuss which results from earlier sections generalize to algebras without a cai; more details on this are given in [Blecher 2015]. The latter is a survey article which also contains a few additional details on some of the material in the present paper, as well as some small improvements found after this paper was in press.

We now list some of our notation and general facts: We write $\mathrm{Ball}(X)$ for the set $\{x \in X : \|x\| \leq 1\}$. If E, F are sets then EF denotes the span of products xy for $x \in E, y \in F$. If $x \in A$ for a Banach algebra A, then $\mathrm{ba}(x)$ denotes the closed subalgebra generated by x. For two spaces X, Y which are in duality, for a subset E of X, we use the polar $E^\circ = \{y \in Y : \langle x, y\rangle \geq -1 \text{ for all } x \in E\}$.

For us, Banach algebras satisfy $\|xy\| \leq \|x\|\|y\|$. We recall that a nonunital Banach algebra A is Arens regular if and only if its unitization is Arens regular (any unitization will do here). In the rest of this paragraph, we consider an Arens regular approximately unital Banach algebra A. For such an algebra, we will always write e for the unique identity of A^{**}. Indeed if A is an Arens regular Banach algebra with cai (e_t), and $e_{t_\mu} \to \eta$ weak* in A^{**}, then $e_{t_\mu} a \to \eta a$ weak* for all $a \in A$. So $\eta a = a$, and similarly $a\eta = a$. Therefore η is the unique identity e of A^{**}, and $e_t \to e$ weak*. We will show at the end of this section that the multiplier unitization A^1 is isometrically isomorphic to the subalgebra $A + \mathbb{C}e$ of A^{**}.

If A is a Banach algebra which is not Arens regular, then the multiplication we usually use on A^{**} is the "second Arens product" ($\diamond$ in the notation of [Dales 2000]). This is weak* continuous in the second variable. If A is a nonunital, not necessarily Arens regular, Banach algebra with a bai, then A^{**} has a so-called "mixed identity" [Dales 2000; Palmer 1994; Doran and Wichmann 1979], which we will again write as e. This is a right identity for the first Arens product, and a left identity for the second Arens product. A mixed identity need not be unique; indeed, mixed identities are just the weak* limit points of bais for A.

We will also use the theory of M-ideals. These were invented by Alfsen and Effros, and [Harmand et al. 1993] is the basic text for their theory. We recall, a subspace E of a Banach space X is an M-ideal in X if $E^{\perp\perp}$ is complemented in X^{**} via a contractive projection P so that $X^{**} = E^{\perp\perp} \oplus^\infty \operatorname{Ker} P$. In this case, there is a unique contractive projection onto $E^{\perp\perp}$. M-ideals have many beautiful properties, some of which will be mentioned below.

We will need the following result several times:

Lemma 1.1. *Let X be a Banach space, and suppose that (x_t) is a bounded net in X with $x_t \to \eta$ weak* in X^{**}. Then*

$$\|\eta\| = \lim_t \inf\big\{\|y\| : y \in \operatorname{conv}\{x_j : j \geq t\}\big\}.$$

Proof. It is easy to see that $\|\eta\| \leq \lim_t \inf\big\{\|y\| : y \in \operatorname{conv}\{x_j : j \geq t\}\big\}$, for example, by using the weak* semicontinuity of the norm, and noting that for every t and any choice $y_t \in \operatorname{conv}\{x_j : j \geq t\}$, we have $y_t \to \eta$ weak*. By way of contradiction, suppose that

$$\|\eta\| < C < \lim_t \inf\big\{\|y\| : y \in \operatorname{conv}\{x_j : j \geq t\}\big\}.$$

Then there exists t_0 such that the norm closure of $\operatorname{conv}\{x_j : j \geq t\}$ is disjoint from $C\,\operatorname{Ball}(X)$ for all $t \geq t_0$. By the Hahn–Banach theorem, there exists $\varphi \in X^*$ with

$$C\,\|\varphi\| < K < \operatorname{Re}\varphi(x_j), \qquad j \geq t,$$

so that $C\,\|\varphi\| < K \leq \operatorname{Re}\varphi(\eta)$. This contradicts $\|\eta\| < C$. □

Any nonunital operator algebra has a unique operator algebra unitization (see [Blecher and Le Merdy 2004, Section 2.1]), but of course this is not true for Banach algebras. We will choose to use the unitization that typically has the smallest norm among all unitizations, and which we now describe. If A is an approximately unital Banach algebra, then the left regular representation embeds A isometrically in $B(A)$. We will always write A^1 for the *multiplier unitization* of A; that is, we identify A^1 isometrically with $A + \mathbb{C}I$ in $B(A)$. For $a \in A, \lambda \in \mathbb{C}$, we have

$$\|a + \lambda 1\| = \sup\big\{\|ac + \lambda c\| : c \in \mathrm{Ball}(A)\big\} = \sup_t \|ae_t + \lambda e_t\| = \lim_t \|ae_t + \lambda e_t\|,$$

(see [loc. cit., A.4.3], for example). If A is actually nonunital then the map $\chi_0(a + \lambda 1) = \lambda$ on A^1 is contractive, as is any character on a Banach algebra. We call this the *trivial character*. Below, 1 will almost always denote the identity of A^1, if A is not already unital. Note that the multiplier unitization also makes sense for the so-called *self-induced* Banach algebras, namely those for which the left regular representation embeds A isometrically in $B(A)$.

If A is a nonunital, approximately unital Banach algebra then the multiplier unitization A^1 may also be identified with a subalgebra of A^{**}. Indeed if e is a mixed identity of norm 1 for A^{**} then $A + \mathbb{C}e$ is then a unitization of A (by basic facts about the Arens product). To see that this is isometric to A^1 above, note that for any $c \in \mathrm{Ball}(A), a \in A, \lambda \in \mathbb{C}$, we have

$$\|ac + \lambda c\| \le \|a + \lambda e\|_{A^{**}} = \|e(a + \lambda 1)\|_{(A^1)^{**}} \le \|a + \lambda 1\|_{A^1}.$$

Thus by the displayed equation in the last paragraph, $\|a + \lambda e\|_{A^{**}} = \|a + \lambda 1\|_{A^1}$ as desired.

2. Unitization and states

If A is an approximately unital Banach algebra, then we may view A in its multiplier unitization A^1, and write

$$\mathfrak{F}_A = \{a \in A : \|1 - a\| \le 1\} = \{a \in A : \|e - a\| \le 1\},$$

where e is as in the last paragraph (or set $e = 1$ if A is unital). So

$$\tfrac{1}{2}\mathfrak{F}_A = \{a \in A : \|1 - 2a\| \le 1\}.$$

If $x \in \frac{1}{2}\mathfrak{F}_A$ then $x, 1 - x \in \mathrm{Ball}(A^1)$. Also, $\mathfrak{F}_A = \mathfrak{F}_{A^1} \cap A$, and $\mathfrak{F}_A$ is closed under the quasiproduct $a + b - ab$. (It is interesting that cones containing $\mathfrak{F}_A$ were used to obtain nice results about "order" in unital Banach algebras and their duals in Section 1 of the historically important paper [Kelley and Vaught 1953], based on a 1951 ICM talk. Slightly earlier, $\mathfrak{F}_A$ also appeared in a memoir by Kadison.)

If $\eta \in A^{**}$ then an expression such as $\lambda 1 + \eta$ will usually need to be interpreted as an element of $(A^1)^{**}$, with 1 interpreted as the identity for A^1 and $(A^1)^{**}$. Thus $\|1 - \eta\|$ denotes $\|1 - \eta\|_{(A^1)^{**}}$. We define

$$\mathfrak{F}_{A^{**}} = \{\eta \in A^{**} : \|1 - \eta\| \le 1\} = A^{**} \cap \mathfrak{F}_{(A^1)^{**}}.$$

We write $\mathfrak{r}_A$ for the set of $a \in A$ whose numerical range in A^1 is contained in the right half-plane. That is,

$$\mathfrak{r}_A = \{a \in A : \operatorname{Re} \varphi(a) \geq 0 \text{ for all } \varphi \in S(A^1)\},$$

where $S(A^1)$ denotes the states on A^1. Note that $\mathfrak{r}_A$ is a closed cone in A, but it is not proper (hence it is what is sometimes called a *wedge*). We write $a \preceq b$ if $b - a \in \mathfrak{r}_A$. It is easy to see that $\mathbb{R}^+\mathfrak{F}_A \subset \mathfrak{r}_A$. Conversely, if A is a unital Banach algebra and $a \in \mathfrak{r}_A$ then $a + \epsilon 1 \in \mathbb{R}^+\mathfrak{F}_A$ for every $\epsilon > 0$. Indeed $a + \epsilon 1 \in C\mathfrak{F}_A$, where $C = \|a\|^2/\epsilon + \epsilon$, as can be easily seen from the well-known fact that the numerical range of a is contained in the right half-plane if and only if $\|1 - ta\| \leq 1 + t^2\|a\|^2$ for all $t > 0$ (see, e.g., [Magajna 2009, Lemma 2.1]).

One main reason why we almost always assume that A is approximately unital in this paper is that $\mathfrak{F}_A$ and $\mathfrak{r}_A$ are well-defined as above. However, as we said in the introduction, if A is not approximately unital, it is easy to see how to proceed in a large number of our results (namely in almost all of Sections 3, 4, and 7), and this is discussed briefly in Section 9.

The following is no doubt in the literature, but we do not know of a reference that proves all that is claimed. It follows from it that mixed identities in A^{**} are just the weak* limits of bais for A, when these limits exist.

Lemma 2.1. *If A is a Banach algebra, and if a bounded net $x_t \in A$ converges weak* to a mixed identity $e \in A^{**}$, then a bai for A can be found with weak* limit e, and formed from convex combinations of the x_t.*

Proof. Given $\epsilon > 0$ and a finite set $F \subset A^*$, there exists $t_{F,\epsilon}$ such that

$$|\varphi(x_t) - e(\varphi)| < \epsilon, \quad t \geq t_{F,\epsilon}, \varphi \in F.$$

Given a finite set $E = \{a_1, \ldots, a_n\} \subset A$, we have that $x_t a_k \to a_k$ and $a_k x_t \to a_k$ weakly. So there is a convex combination y of the x_t for $t \geq t_{F,\epsilon}$ with

$$\|y a_k - a_k\| + \|a_k y - a_k\| \leq \epsilon.$$

We also have $|\varphi(y) - e(\varphi)| \leq \epsilon$ for $\varphi \in F$. Write this y as y_λ, where $\lambda = (E, F, \epsilon)$. Given $\epsilon_0 > 0$ and $a \in A$, if $\epsilon \leq \epsilon_0$ and $\{a\} \subset E$, then $\|y_\lambda a - a\| + \|a y_\lambda - a\| \leq \epsilon \leq \epsilon_0$ for $\lambda = (E, F, \epsilon)$ with any F. So (y_λ) is a bai. Also if $\varphi \in F$ then $|\varphi(y_\lambda) - e(\varphi)| < \epsilon$. So $y_\lambda \to e$ weak*. □

Remark. The "sequential version" of the last result is false. For example, consider the usual cai $(n\chi_{[-1/(2n),1/(2n)]})$ of $L^1(\mathbb{R})$ with convolution product. A subnet of this converges weak* to a mixed identity $e \in L^1(\mathbb{R})^{**}$. However, there can be no weak* convergent sequential bai for $L^1(\mathbb{R})$, since $L^1(\mathbb{R})$ is weakly sequentially complete.

For a general approximately unital nonunital Banach algebra A with cai (e_t), the definition of "state" is problematic. There are many natural notions, for example: (i) a contractive functional φ on A with $\varphi(e_t) \to 1$ for some fixed cai (e_t) for A, (ii) a contractive functional φ on A with $\varphi(e_t) \to 1$ for all cai (e_t) for A, and (iii) a norm 1 functional on A that extends to a state on A^1, where A^1 is the multiplier unitization above. If A is not Arens regular then (i) and (ii) can differ; that is, whether $\varphi(e_t) \to 1$ depends on which cai for A we use. And if e is a mixed identity then the statement $\varphi(e) = 1$ may depend on which mixed identity one considers. In this paper, for simplicity, and because of its connections with the usual theory of numerical range and accretive operators, we will take (iii) above as the definition of a *state* of A. We shall also often consider states in the sense of (i), and will usually ignore (ii) since in some sense it may be treated as a "special case" of (i) (that is, almost all computations in the paper involving the class (i) are easily tweaked to give the "(ii) version"). We define $S(A)$ to be the set of states in the sense of (iii) above. This is easily seen to be norm closed, but will not be weak* closed if A is nonunital. We define

$$\mathfrak{c}_{A^*} = \{\varphi \in A^* : \operatorname{Re} \varphi(a) \geq 0 \text{ for all } a \in \mathfrak{r}_A\},$$

and note that this is a weak* closed cone containing $S(A)$. These are called the *real positive functionals* on A. If $\mathfrak{e} = (e_t)$ is a fixed cai for A, define

$$S_{\mathfrak{e}}(A) = \{\varphi \in \operatorname{Ball}(A^*) : \lim_t \varphi(e_t) = 1\}$$

(this corresponds to (i) above). Note that $S_{\mathfrak{e}}(A)$ is convex but $S(A)$ may not be (as in, e.g., Example 3.16). An argument in the next proof shows that $S_{\mathfrak{e}}(A) \subset S(A)$. Finally we remark that for any $y \in A$ of norm 1, if $\varphi \in \operatorname{Ball}(A^*)$ satisfies $\varphi(y) = 1$, then $x \mapsto \varphi(yx)$ is in $S_{\mathfrak{e}}(A)$ for all cais $\mathfrak{e}$ of A.

We recall that a subspace E of a Banach space X is called Hahn–Banach smooth in X if every functional on E has a unique Hahn–Banach extension to X. Any M-ideal in X is Hahn–Banach smooth in X. See [Harmand et al. 1993] and references therein for more on this topic.

Lemma 2.2. *For approximately unital Banach algebras A which are Hahn–Banach smooth in A^1, and therefore for M-approximately unital Banach algebras, and $\varphi \in A^*$ with norm* 1, *the following are equivalent*:

(i) *φ is a state on A (that is, extends to a state on A^1).*

(ii) *$\varphi(e_t) \to 1$ for every cai (e_t) for A.*

(iii) *$\varphi(e_t) \to 1$ for some cai (e_t) for A.*

(iv) *$\varphi(e) = 1$ whenever $e \in A^{**}$ is a weak* limit point of a cai for A (that is, whenever e is a mixed identity of norm 1 for A^{**}).*

Proof. Clearly (ii) implies (iii). If $\varphi \in \mathrm{Ball}(A^*)$, write $\tilde{\varphi}$ for its canonical weak* continuous extension to A^{**}. If (e_t) is a cai for A with weak* limit point e and $\varphi(e_t) \to 1$, then $\tilde{\varphi}(e) = 1$. It follows that $\tilde{\varphi}_{|A^1}$ is a state on A^1. So (iii) implies (i). To see that (i) implies (iv), suppose that A is Hahn–Banach smooth in A^1, and that φ is a norm 1 functional on A that extends to a state ψ on A^1. If (e_t) is a cai for A with weak* limit point e, then also $\tilde{\varphi}_{|A+\mathbb{C}e}$ is a norm-1 functional extending φ so that $\tilde{\varphi}_{|A+\mathbb{C}e} = \psi$, and for some subnet,

$$\varphi(e) = \lim_t \varphi(e_{t_\mu}) = \tilde{\varphi}(e) = \psi(1) = 1.$$

We leave the remaining implication as an exercise. □

Under certain conditions on an approximately unital Banach algebra A, we shall see in Corollary 2.8 that $S(A^1)$ is the convex hull of the trivial character χ_0 and the set of states on A^1 extending states of A, and that the weak* closure of $S(A)$ equals $\{\varphi_{|A} : \varphi \in S(A^1)\}$.

The numerical range $W(a)$ (or $W_A(a)$) of $a \in A$, if A is an approximately unital Banach algebra, will be defined to be $\{\varphi(a) : \varphi \in S(A)\}$. If A is Hahn–Banach smooth in A^1 then it follows from Lemma 2.2 that $S(A)$ is convex, and hence so is $W(a)$. We shall see in Corollary 2.8 that under the condition mentioned in the last paragraph, we have $\overline{W_A(a)} = \mathrm{conv}\{0, W_A(a)\} = W_{A^1}(a)$.

The following is related to results from [Smith and Ward 1979] or [Harmand et al. 1993, Section V.3] or [Arias and Rosenthal 2000; Davidson and Power 1986].

Lemma 2.3. *If A is an approximately unital Banach algebra, if A^1 is the unitization above, and if e is a weak* limit of a cai (resp. bai in $\mathfrak{F}_A$) for A then $\|1-2e\|_{(A^1)^{**}} \leq 1$ if and only if there is a cai (resp. bai in $\mathfrak{F}_A$) (e_i) with weak* limit e and* $\limsup_i \|1-2e_i\|_{A^1} \leq 1$.

Proof. One direction follows from Alaoglu's theorem. Suppose $\|1-2e\|_{(A^1)^{**}} \leq 1$ and there is a net (x_t) which is a cai (resp. bai in $\mathfrak{F}_A$) for A with $x_t \to e$ weak*. Then $1-2x_t \to 1-2e$ weak* in $(A^1)^{**}$. By Lemma 1.1, for any $n \in \mathbb{N}$, there exists a t_n such that for every $t \geq t_n$,

$$\inf\bigl\{\|1-2y\| : y \in \mathrm{conv}\{x_j : j \geq t\}\bigr\} < 1 + \frac{1}{2n}.$$

For every $t \geq t_n$, choose such a $y_t^n \in \mathrm{conv}\{x_j : j \geq t\}$ with $\|1-2y_t^n\| < 1+1/n$. If t does not dominate t_n, define $y_t^n = y_{t_n}^n$. So for all t, we have $\|1-2y_t^n\| < 1+1/n$. Writing (n,t) as i, we may view (y_t^n) as a net (e_i) indexed by i, with $\|1-2y_t^n\| \to 1$. Given $\epsilon > 0$ and $a_1, \ldots, a_m \in A$, there exists a t_1 such that $\|x_t a_k - a_k\| < \epsilon$ and $\|a_k x_t - a_k\| < \epsilon$ for all $t \geq t_1$ and all $k = 1, \ldots, m$. Hence the same assertion is true with x_t replaced by y_t^n. Thus $(y_t^n) = (e_i)$ is a bai for A with the desired property. □

We recall from the introduction that if A is an approximately unital Banach algebra which is an M-ideal in the particular unitization A^1 above, then A is an

M-approximately unital Banach algebra. Any unital Banach algebra is an M-approximately unital Banach algebra (here $A^1 = A$). By [Harmand et al. 1993, Proposition I.1.17(b)], examples of M-approximately unital Banach algebras include any Banach algebra that is an M-ideal in its bidual, and which is approximately unital (or whose bidual has an identity). Several examples of such are given in [loc. cit.], for example, the compact operators on ℓ^p for $1 < p < \infty$. We also recall that the property of being an M-ideal in its bidual is inherited by subspaces, and hence by subalgebras. Not every Banach algebra with a cai is M-approximately unital. By [loc. cit., Proposition II.3.5], $L^1(\mathbb{R})$ with convolution multiplication cannot be an M-ideal in any proper superspace.

We just said that any unital Banach algebra A is M-approximately unital; hence, any finite dimensional unital Banach algebra is Arens regular and M-approximately unital (if one wishes to avoid the redundancy of $A = A^1$ in the discussion below, take the direct sum of A with any Arens regular M-approximately unital Banach algebra, such as c_0). Thus any kind of bad behavior occurring in finite-dimensional unital Banach algebras (resp. unital Banach algebras) will appear in the class of Arens regular M-approximately unital Banach algebras (resp. M-approximately unital Banach algebras). This will have the consequence that several aspects of the Blecher–Read papers will not generalize, for instance, conclusions involving "near positivity". This can also be seen in the examples scattered through our paper, for instance, Examples 3.13–3.16 below.

Suppose that (e_t) is a cai for a Banach algebra A with weak* limit point $e \in A^{**}$. Then left multiplication by e (in the second Arens product) is a contractive projection from $(A^1)^{**}$ onto the ideal $A^{\perp\perp}$ of $(A^1)^{**}$ (note that $(A^1)^{**} = A^{\perp\perp} + \mathbb{C}1 = A^{\perp\perp} + \mathbb{C}(1-e)$). Thus by the theory of M-ideals [loc. cit.], A is an M-ideal in A^1 if and only if left multiplication by e is an M-projection.

Lemma 2.4. *A nonunital approximately unital Banach algebra A is M-approximately unital if and only if for all $x \in A^{**}$, we have*

$$\|1-x\|_{(A^1)^{**}} = \max\{\|e-x\|_{A^{**}}, 1\}.$$

Here e is a mixed identity for A^{**} of norm 1. *If these conditions hold then there is a unique mixed identity for A^{**} of norm* 1, *it belongs in $\frac{1}{2}\mathfrak{F}_{A^{**}}$, and*

$$\|1-\eta\| = 1 \quad \Longleftrightarrow \quad \|e-\eta\| \le 1, \quad \eta \in A^{**}.$$

Proof. By the statement immediately above the lemma, and by the theory of M-ideals [Harmand et al. 1993], A is an M-ideal in A^1 if and only if left multiplication by e is an M-projection, that is, if and only if

$$\|\eta + \lambda 1\|_{(A^1)^{**}} = \max\bigl\{\|\eta + \lambda e\|_{A^{**}}, |\lambda|\,\|1-e\|\bigr\}, \quad \eta \in A^{**}, \lambda \in \mathbb{C}.$$

If this holds then setting $\lambda = 1$ and $\eta = 0$ shows that $\|1-e\| \leq 1$. However, by the Neumann lemma we cannot have $\|1-e\| < 1$. Thus $\|1-e\| = 1$ if these hold. The statement is tautological if $\lambda = 0$, so we may assume the contrary. Dividing by $|\lambda|$ and setting $x = -\eta/|\lambda|$, one sees that A is M-approximately unital if and only if

$$\|1-x\|_{(A^1)^{**}} = \max\{\|e-x\|_{A^{**}}, 1\}, \quad x \in A^{**}.$$

In particular, $\|1-2e\|_{(A^1)^{**}} = \max\{\|e\|, 1\} = 1$. The final assertion is now clear too. The uniqueness of the mixed identity follows from the next result. $\square$

Remark. Indeed if B is any unitization of a nonunital approximately unital Banach algebra A, and if A is an M-ideal in B, then the first few lines of the last proof, with A^1 replaced by B, show that $B = A^1$, the multiplier unitization of A.

Thus A is M-approximately unital if and only if $\|1-x\|_{(A^1)^{**}} = \|e-x\|_{A^{**}}$ for all $x \in A^{**}$, unless the last quantity is less than 1, in which case $\|1-x\|_{(A^1)^{**}} = 1$.

We will show later that for M-approximately unital Banach algebras, there is a cai (e_t) for A with $\|1-2e_t\|_{A^1} \leq 1$ for all t.

Lemma 2.5. *Let A be a closed ideal, and also an M-ideal, in a unital Banach algebra B. If e and f are two weak* limit points in A^{**} of two cai for A, then $e = f$. Thus A^{**} has a unique mixed identity of norm* 1. *In particular, if A is M-approximately unital then A^{**} has a unique mixed identity of norm* 1.

Proof. As in the discussion above Lemma 2.4, left multiplications by e or f, in the second Arens product, are contractive projections onto the ideal $A^{\perp\perp}$ of $(A^1)^{**}$. So these maps equal the M-projection [Harmand et al. 1993], and hence are equal. So $e = f$. Thus every cai for A converges weak* to e, so that A^{**} has a unique mixed identity. $\square$

If A is an approximately unital Banach algebra, but A^{**} has no identity then we define $\mathfrak{r}_{A^{**}} = A^{**} \cap \mathfrak{r}_{(A^1)^{**}}$. If A is an approximately unital Banach algebra then $\mathfrak{F}_{A^{**}}$ and $\mathfrak{r}_{A^{**}}$ are weak* closed. Indeed the $\mathfrak{F}_{A^{**}}$ case of this is obvious. By [Magajna 2009], $\mathfrak{r}_{(A^1)^{**}}$ is weak* closed, hence so is $\mathfrak{r}_{A^{**}} = A^{**} \cap \mathfrak{r}_{(A^1)^{**}}$.

Remark. Note that if A^{**} has a mixed identity of norm 1 then we can define states of A^{**} to be norm-1 functionals φ with $\varphi(e) = 1$ for all mixed identities e of A^{**} of norm 1. Then one could define $\mathfrak{r}_{A^{**}}$ to be the elements $x \in A^{**}$ with $\operatorname{Re} \varphi(x) \geq 0$ for all such states of A^{**}. This coincides with the definition of $\mathfrak{r}_{A^{**}}$ above the remark if A is M-approximately unital. Indeed such states φ on A^{**} extend to states $\varphi(e\,\cdot)$ of $(A^1)^{**}$. Conversely if A is an M-approximately unital Banach algebra, then given a state φ of $(A^1)^{**}$, we have

$$1 = \|\varphi\| = \|\varphi \cdot e\| + \|\varphi \cdot (1-e)\| \geq |\varphi(e)| + |\varphi(1-e)| \geq \varphi(1) = 1 = \varphi(e) + \varphi(1-e).$$

It follows from this that $\|\varphi e\| = |\varphi(e)| = \varphi(e)$. Hence if $\eta \in \mathrm{Ball}(A^{**})$ then

$$|\varphi(\eta)| = |\varphi e(\eta)| \leq \|\varphi e\| = \varphi(e),$$

so that the restriction of φ to A^{**} is either zero or is a positive multiple of a state on A^{**}. Thus for M-approximately unital Banach algebras, the two notions of $\mathfrak{r}_{A^{**}}$ under discussion coincide.

Let $Q(A)$ be the quasistate space of A, namely $Q(A) = \{t\varphi : t \in [0,1], \varphi \in S(A)\}$. Similarly, $Q_{\mathfrak{e}}(A) = \{t\varphi : t \in [0,1], \varphi \in S_{\mathfrak{e}}(A)\}$. We set

$$\mathfrak{r}_A^{\mathfrak{e}} = \{x \in A : \operatorname{Re}\varphi(x) \geq 0 \text{ for all } \varphi \in S_{\mathfrak{e}}(A)\},$$
$$\mathfrak{c}_{A^*}^{\mathfrak{e}} = \{\varphi \in A^* : \operatorname{Re}\varphi(x) \geq 0 \text{ for all } x \in \mathfrak{r}_A^{\mathfrak{e}}\}.$$

Note that $\mathfrak{r}_A \subset \mathfrak{r}_A^{\mathfrak{e}}$ since $S_{\mathfrak{e}}(A) \subset S(A)$.

Lemma 2.6. *Let A be a nonunital Banach algebra with a cai $\mathfrak{e}$.*

(1) *Then 0 is in the weak* closure of $S_{\mathfrak{e}}(A)$. Hence 0 is in the weak* closure of $S(A)$. Thus $Q(A)$ is a subset of the weak* closure of $S(A)$, and similarly $Q_{\mathfrak{e}}(A) \subset \overline{S_{\mathfrak{e}}(A)}^{\mathrm{w}*}$.*

(2) *The weak* closure of $S_{\mathfrak{e}}(A)$ is contained in $\mathfrak{c}_{A^*}^{\mathfrak{e}} \cap \mathrm{Ball}(A^*)$. It is also contained in $S(A^1)_{|A}$, and both of the latter two sets are subsets of $\mathfrak{c}_{A^*} \cap \mathrm{Ball}(A^*)$.*

Proof. (1) For every t, there exists $s(t) \geq t$ such that $\|e_{s(t)} - e_t\| \geq 1/2$ (or else taking the limit over $s > t$, we get the contradiction $\|1 - e_t\| < 1$, which is impossible by the Neumann lemma, or since the trivial character χ_0 is contractive). Take a norm-1 $\psi_t \in A^*$ such that $\psi_t(e_{s(t)} - e_t) = \|e_{s(t)} - e_t\|$. Let $\Phi_t(x) = \psi_t((e_{s(t)} - e_t)x)/\|e_{s(t)} - e_t\|$. Then $\Phi_t \in S_{\mathfrak{e}}(A)$ because it has norm 1 and $\lim_s \Phi_t(e_s) = 1$. One has $\lim_t \Phi_t(x) = 0$ for all $x \in A$. To see this, given $\epsilon > 0$, choose t_0 such that $\|e_t x - x\| < \epsilon$ for all $t \geq t_0$. For such t, we have

$$\frac{|\psi_t((e_{s(t)} - e_t)x)|}{\|e_{s(t)} - e_t\|} \leq 2\|\psi_t\|\|(e_{s(t)} - e_t)x\| < 4\epsilon.$$

Thus $\Phi_t \to 0$ weak*. The rest is obvious.

(2) The first assertion is clear by the definitions and since $\mathfrak{c}_{A^*}^{\mathfrak{e}} \cap \mathrm{Ball}(A^*)$ is weak* closed. Similarly, that the weak* closure is contained in $S(A^1)_{|A}$ follows since $S_{\mathfrak{e}}(A) \subset S(A)$ as we saw above, and because $S(A^1)$ and hence $S(A^1)_{|A}$, are weak* closed. We leave the rest as an exercise using $\mathfrak{r}_A \subset \mathfrak{r}_A^{\mathfrak{e}}$. ☐

We will say that an approximately unital Banach algebra A is *scaled* (resp. $\mathfrak{e}$-*scaled*) if every f in $\mathfrak{c}_{A^*}$ (resp. in $\mathfrak{c}_{A^*}^{\mathfrak{e}}$) is a nonnegative multiple of a state, that is, if and only if $\mathfrak{c}_{A^*} = \mathbb{R}^+ S(A)$ (resp. $\mathfrak{c}_{A^*}^{\mathfrak{e}} = \mathbb{R}^+ S_{\mathfrak{e}}(A)$), equivalently, if and only if $\mathfrak{c}_{A^*} \cap \mathrm{Ball}(A^*) = Q(A)$ (resp. $\mathfrak{c}_{A^*}^{\mathfrak{e}} \cap \mathrm{Ball}(A^*) = Q_{\mathfrak{e}}(A)$). Examples of scaled Banach algebras include M-approximately unital Banach algebras (see Proposition 6.2) and $L^1(\mathbb{R})$ with convolution product. One can show that $L^1(\mathbb{R})$ is not $\mathfrak{e}$-scaled if $\mathfrak{e}$ is the usual cai (see the remark after Lemma 2.1 and Example 3.16).

Lemma 2.7. *Let A be an approximately unital Banach algebra.*

(1) *Suppose that $\mathfrak{e} = (e_t)$ is a cai for A. Then $Q_{\mathfrak{e}}(A)$ is weak* closed in A^* if and only if A is $\mathfrak{e}$-scaled. If these hold then $Q_{\mathfrak{e}}(A)$ is a weak* compact convex set in* $\mathrm{Ball}(A^*)$, *and $S_{\mathfrak{e}}(A)$ is weak* dense in $Q_{\mathfrak{e}}(A)$.*

(2) *If $S(A)$ or $Q(A)$ is convex then $Q(A)$ is weak* closed in A^* if and only if A is scaled.*

Proof. (1) By the bipolar theorem, $\mathfrak{c}^{\mathfrak{e}}_{A^*} = \overline{\mathbb{R}^+ s_{\mathfrak{e}}(a)}^{w^*}$. So $\mathbb{R}^+ S_{\mathfrak{e}}(A)$ is weak* closed if and only if $\mathfrak{c}^{\mathfrak{e}}_{A^*} = \mathbb{R}^+ S_{\mathfrak{e}}(A)$, that is, if and only if A is $\mathfrak{e}$-scaled. By the Krein–Smulian theorem this happens if and only if $\mathrm{Ball}(\mathbb{R}^+ S_{\mathfrak{e}}(A)) = Q_{\mathfrak{e}}(A)$ is weak* closed. The weak* density assertion follows from Lemma 2.6.

(2) This follows by a similar argument to (1) if $Q(A)$ is convex (and this is implied by $S(A)$ being convex). ☐

Corollary 2.8. *If A is a nonunital approximately unital Banach algebra*, *then the following are equivalent*:

(i) *A is scaled.*

(ii) *$S(A^1)$ is the convex hull of the trivial character χ_0 and the set of states on A^1 extending states of A.*

(iii) $Q(A) = \{\varphi_{|A} : \varphi \in S(A^1)\}$.

(iv) *$Q(A)$ is convex and weak* compact.*

If these hold then $Q(A) = \overline{S(A)}^{\mathrm{w}*}$, and the numerical range satisfies

$$\overline{W_A(a)} = \mathrm{conv}\{0, W_A(a)\} = W_{A^1}(a), \quad a \in A.$$

Proof. (i) ⇒ (ii): Clearly the convex hull in (ii) is a subset of $S(A^1)$. Conversely, if $\varphi \in S(A^1)$ then $\varphi_{|A}$ is real positive, so that by (i) we have $\varphi_{|A} = t\psi$ for $t \in (0,1]$ and $\psi \in S(A)$. Then $\varphi = t\widehat{\psi} + (1-t)\chi_0$, where $\widehat{\psi}$ is the state extending ψ.

(ii) ⇒ (iii): We leave this as an exercise.

(iii) ⇒ (iv): Suppose that (φ_t) is a net in $S(A^1)$ whose restrictions to A converge weak* to $\psi \in A^*$. A subnet (φ_{t_λ}) converges weak* to $\varphi \in S(A^1)$, and $\psi = \varphi_{|A}$, clearly. This gives the weak* compactness in (iv), and the convexity is easier.

(iv) ⇒ (i): This follows from (2) of the previous lemma.

Assume that these hold. Since $S(A) \subset Q(A)$, that $Q(A) = \overline{S(A)}^{\mathrm{w}*}$ is now clear from the fact from Lemma 2.6 that $Q(A) \subset \overline{S(A)}^{\mathrm{w}*}$. Since A is nonunital, we have $0 \in W_{A^1}(a)$. Clearly $W_A(a) \subset W_{A^1}(a)$, so that $\mathrm{conv}\{0, W_A(a)\} \subset W_{A^1}(a)$. The converse inclusion follows easily from the above, so $\mathrm{conv}\{0, W_A(a)\} = W_{A^1}(a)$.

Also, clearly $\overline{W_A(a)} \subset W_{A^1}(a)$, and the converse inclusion follows since $S(A^1)_{|A} = Q(A) = \overline{S(A)}^{\mathrm{w}*}$. $\square$

Remark. (1) Thus if $S(A) = S_{\mathfrak{e}}(A)$ for some cai $\mathfrak{e}$ of A, then A is scaled if and only if $Q(A)$ is weak* closed.

(2) In particular, if A is unital then conditions (i) and (iv) in the previous result are automatically true. Indeed $S(A)$ is weak* closed, and hence $Q(A)$ is too, and the rest follows from Lemma 2.7. Item (i) also follows from the proof of [Magajna 2009, Theorem 2.2].

Theorem 2.9. *Let $\mathfrak{e} = (e_n)$ be a sequential cai for a Banach algebra A. If $Q_{\mathfrak{e}}(A)$ is weak* closed, then A possesses a sequential cai in $\mathfrak{r}_A^{\mathfrak{e}}$. Moreover, for every $a \in A$ with* $\inf\{\mathrm{Re}\, \varphi(a) : \varphi \in S_{\mathfrak{e}}(A)\} > -1$, *there is a sequential cai (f_n) in $\mathfrak{r}_A^{\mathfrak{e}}$ such that $f_n + a \in \mathfrak{r}_A^{\mathfrak{e}}$ for all n.*

Proof. We first state a general fact about compact spaces K. If (f_n) is a bounded sequence in $C(K, \mathbb{R})$ such that $\lim_n f_n(x)$ exists for every $x \in K$ and is nonnegative, then for every $\epsilon > 0$, there is a function $f \in \mathrm{conv}\{f_n\}$ such that $f \geq -\epsilon$ on K. Indeed if this were not true, then $\overline{\mathrm{conv}}\{f_n\}$ and $C(K)_+$ would be disjoint. By a Hahn–Banach separation argument and the Riesz–Markov theorem, there is a probability measure m such that $\sup_n \int_K f_n\, dm < 0$. This is a contradiction since $\lim_n \int_K f_n\, dm \geq 0$ by Lebesgue's dominated convergence theorem.

Set K to be the weak* closure of $S_{\mathfrak{e}}(A)$ in A^* (so that $K = Q_{\mathfrak{e}}(A)$ by Lemma 2.6), and let $f_n(\varphi) = \mathrm{Re}\, \varphi(e_n)$ for $\varphi \in K$. Since $\lim_n \mathrm{Re}\, \varphi(e_n) \geq 0$ for all $\varphi \in Q_{\mathfrak{e}}(A)$, we can apply the previous paragraph to find an $x \in \mathrm{conv}\{e_n\}$ such that $\inf_{\varphi \in K} \varphi(x) > -\epsilon$. Similarly, choose $y_1 \in \mathrm{conv}\{e_n\}$ such that $\inf_{\varphi \in K} \varphi(x + \epsilon y_1) > -\epsilon/2$. Continue in this way, choosing $y_n \in \mathrm{conv}\{e_n\}$ such that

$$\inf_{\varphi \in K} \varphi\left(x + \epsilon \sum_{k=1}^{n} 2^{1-k} y_k\right) > -\epsilon/2^n.$$

Set $u = \sum_{k=1}^{\infty} 2^{-k} y_k \in \overline{\mathrm{conv}}\{e_n\}$, and $z = x + 2\epsilon u$. This is in $\mathfrak{r}_A^{\mathfrak{e}}$, and $\|z - x\| < 2\epsilon$.

Choose a subsequence (e_{k_n}) of (e_n) such that

$$\|e_{k_n} e_n - e_n\| + \|e_n e_{k_n} - e_n\| < 2^{-n}.$$

For each $m \in \mathbb{N}$, apply the last paragraph to $(e_{k_n})_{n \geq m}$, with ϵ replaced by 2^{-m}, to find $x_m, u_m \in \overline{\mathrm{conv}}\{e_{k_n} : n \geq m\}$ with $z_m = x_m + 2^{1-m} u_m \in \mathfrak{r}_A^{\mathfrak{e}}$. Then

$$\|x_m e_m - e_m\| + \|e_m x_m - e_m\| < 2^{-m}.$$

From this it is easy to see that (x_m) is a cai for A. It is also easy to see now that $e'_m = (1/\|z_m\|) z_m$ is a bai (hence also a cai) for A in $\mathfrak{r}_A^{\mathfrak{e}}$.

The case for the "moreover" is similar. Suppose that

$$\inf\{\operatorname{Re}\varphi(a) : \varphi \in S_{\mathfrak{e}}(A)\} > -1.$$

We may assume the infimum is negative, and choose $t > 1$ so that the infimum is still greater than -1 with a replaced by ta. We now begin to follow the argument in previous paragraphs, with the same K, but starting from a cai (e'_n) in $\mathfrak{r}^{\mathfrak{e}}_A$. Since $\lim_n \operatorname{Re}\varphi(ta+e'_n) \geq 0$ for all $\varphi \in Q_{\mathfrak{e}}(A)$, we can apply the above to find an element $x \in \operatorname{conv}\{e'_n\} \subset \mathfrak{r}^{\mathfrak{e}}_A$ such that $\inf_{\varphi\in K} \varphi(ta + x) > -\epsilon$. Continue as above to find $u \in \overline{\operatorname{conv}}\{e'_n\} \subset \mathfrak{r}^{\mathfrak{e}}_A$ so that $z = ta + x + 2\epsilon u$ is in $\mathfrak{r}^{\mathfrak{e}}_A$, with $\|z - x - ta\| < 2\epsilon$. For each $m \in \mathbb{N}$, there exists such $x_m, u_m \in \mathfrak{r}^{\mathfrak{e}}_A$ so that $z_m = ta + x_m + 2^{1-m}u_m$ is in $\mathfrak{r}^{\mathfrak{e}}_A$, with $\|z_m - x_m - ta\| \leq 2^{1-m}$, and such that (x_m) is a cai for A. Note that $z_m - ta \in \mathfrak{r}^{\mathfrak{e}}_A$, and hence $f_m = (1/\|z_m - ta\|)(z_m - ta) \in \mathfrak{r}^{\mathfrak{e}}_A$. Also (f_m) is a bai (hence a cai) for A in $\mathfrak{r}^{\mathfrak{e}}_A$. There exists an N such that $t/\|z_m - ta\| > 1$ for $m \geq N$. Thus $f_m + a \in \mathfrak{r}^{\mathfrak{e}}_A$ for $m \geq N$, since this is a convex combination of f_m and $f_m + ta/\|z_m - ta\| = z_m/\|z_m - ta\|$. □

Corollary 2.10. *Let $\mathfrak{e} = (e_n)$ be a sequential cai for a Banach algebra A. Assume that $S(A) = S_{\mathfrak{e}}(A)$ (which is the case, for example, if A is Hahn–Banach smooth). If $Q(A)$ is weak* closed, then A possesses a sequential cai in $\mathfrak{r}_A$. Moreover, for every $a \in A$ with* $\inf\{\operatorname{Re}\varphi(a) : \varphi \in S(A)\} > -1$, *there is a sequential cai (f_n) in $\mathfrak{r}_A$ such that $f_n \succeq -a$ for all n. If, in addition, A has a sequential cai in $\mathfrak{F}_A$ then the sequential cai (f_n) in the last line can also be chosen to be in $\mathfrak{F}_A$.*

Proof. By the last result, A has a sequential cai in $\mathfrak{r}_A$ satisfying the first two assertions. Suppose that A has a sequential cai, (e'_n) say, in $\mathfrak{F}_A$. One then follows the last paragraph of the last proof. Now $x_m, u_m \in \mathfrak{F}_A$. Define f_m as before, but the desired cai is

$$\frac{\|x_m + 2^{1-m}u_m\|}{1 + 2^{1-m}} f_m,$$

which is easy to see is a convex combination of x_m and u_m, and hence is in $\mathfrak{F}_A$. Moreover a tiny modification of the argument above shows that the sum of this cai and a is in $\mathfrak{r}_A$ for m large enough. □

Remark. Under the conditions of Corollary 2.10, and if A has a sequential approximate identity in $\frac{1}{2}\mathfrak{F}_A$ (resp. $\mathfrak{F}_A$), then a slight variant of the last proof shows that for any $a \in A$ with $\inf\{\operatorname{Re}\varphi(a) : \varphi \in S(A)\} > -1$, there is a sequential bai (f_n) in $\frac{1}{2}\mathfrak{F}_A$ (resp. $\mathfrak{F}_A$) such that $f_n \succeq -a$ for all n. By Corollary 3.9 (and the remark after it) below, if A has a sequential bai in $\mathfrak{r}_A$ then A does have a sequential bai in $\mathfrak{F}_A$.

We also remark that Corollary 3.4 of [Blecher 2015] generalizes the first assertion of Corollary 2.10 above to nonsequential cais.

Proposition 2.11. *If A is a scaled approximately unital Banach algebra then the weak* closure of $\mathfrak{r}_A$ is $\mathfrak{r}_{A^{**}}$.*

Proof. It is easy to see from the definitions that $\mathfrak{r}_A \subset \mathfrak{r}_{A^{**}}$. Clearly $\mathfrak{r}_A^\circ = \mathfrak{c}_{A^*}$, so the result will follow from the bipolar theorem if we can show that

$$(\mathfrak{c}_{A^*})^\circ = \mathfrak{r}_{A^{**}} = \mathfrak{r}_{(A^1)^{**}} \cap A^{**}.$$

Since $\mathfrak{r}_A \subset \mathfrak{r}_{A^{**}}$, it is clear that $(\mathfrak{r}_{A^{**}})_\circ \subset \mathfrak{c}_{A^*}$. If $\varphi \in \mathfrak{c}_{A^*}$ then $\varphi = t\psi$ for $t > 0$, $\psi \in S(A)$. Then ψ extends to a state $\hat{\psi}$ on A^1, and to a weak* continuous state ρ on $(A^1)^{**}$. If $\eta \in \mathfrak{r}_{A^{**}}$, we have

$$\operatorname{Re} \eta(\psi) = \operatorname{Re} \eta(\hat{\psi}) = \operatorname{Re} \rho(\eta) \geq 0.$$

That is, $\varphi \in (\mathfrak{r}_{A^{**}})_\circ$. Then $(\mathfrak{r}_{A^{**}})_\circ = \mathfrak{c}_{A^*}$, and hence by the bipolar theorem, $(\mathfrak{c}_{A^*})^\circ = \mathfrak{r}_{A^{**}}$. □

We remark that if an approximately unital Banach algebra A is scaled then any mixed identity e for A^{**} of norm 1 is lower semicontinuous on $Q(A)$. For if $\varphi_t \to \varphi$ weak* and $\varphi_t(e) = \|\varphi_t\| \leq r$ for all t, then $\|\varphi\| = \varphi(e) \leq r$. A similar assertion holds in the $\mathfrak{e}$-scaled case.

3. Positivity and roots in Banach algebras

Proposition 3.1. *If B is a closed subalgebra of a nonunital Banach algebra A, and if A and B have a common cai, then $B^1 \subset A^1$ isometrically and unitally, $S(B^1) = \{f_{|B^1} : f \in S(A^1)\}$, and $\mathfrak{F}_B = B \cap \mathfrak{F}_A$ and $\mathfrak{r}_B = B \cap \mathfrak{r}_A$. Moreover, in this case, if A is M-approximately unital then so is B.*

Proof. We leave the first part of this as an exercise. The last assertion follows using [Harmand et al. 1993, Proposition I.1.16], since in this case multiplying by e leaves $(B^1)^\perp$ invariant inside $(A^1)^{**}$. □

Remark. Similarly, in the situation of Proposition 3.1 we have $\mathfrak{r}_B^{\mathfrak{e}} = B \cap \mathfrak{r}_A^{\mathfrak{e}}$ if $\mathfrak{e}$ is the common cai.

Proposition 3.2. *Suppose that J is a closed approximately unital ideal in an approximately unital Banach algebra A, and that J is also an M-ideal in A. Then:*

(1) $\mathfrak{F}_J = J \cap \mathfrak{F}_A$ *and* $\mathfrak{r}_J = J \cap \mathfrak{r}_A$, *and states on J extend to states on A.*

(2) *If J is nonunital then $J^1 \subset A^1$ isometrically and unitally, and*

$$S(J^1) = \{f_{|J^1} : f \in S(A^1)\}.$$

(3) *If A is M-approximately unital, then so is J.*

(4) *If $\mathfrak{e} = (e_i)$ is a cai of A, then there is a cai $\mathfrak{h} = (h_j)$ of J such that $\varphi_{|J} \in Q_{\mathfrak{h}}(J)$ whenever $\varphi \in S_{\mathfrak{e}}(A)$.*

Proof. (2) For $a \in J$ and $\lambda \in \mathbb{C}$, we have

$$\|a+\lambda 1\|_{A^1} = \sup\{\|ax+\lambda x\|_A : x \in \mathrm{Ball}(A)\} \geq \|a+\lambda 1\|_{J^1}.$$

Let f be a mixed identity of J^{**} of norm 1, which is the limit of a cai (f_i). For every $x \in \mathrm{Ball}(A)$, one has

$$\|ax+\lambda x\|_A = \max\{\|fax+\lambda fx\|, \|\lambda(1-f)x\|\}.$$

Setting $a=0$ temporarily, we see that $\|\lambda(1-f)x\| \leq |\lambda| \leq \|a+\lambda 1\|_{J^1}$. For any $a \in J$, we have $fax = ax$ and $ax + \lambda fx = \mathrm{w}^* \lim_i af_i x + \lambda f_i x$, so that

$$\|fax+\lambda fx\| \leq \liminf_i \|af_i x+\lambda f_i x\| \leq \|a+\lambda 1\|_{J^1}.$$

Thus $\|a+\lambda 1\|_{A^1} = \|a+\lambda 1\|_{J^1}$.

(1) If J is nonunital then by (2) and the Hahn–Banach theorem, we have $S(J^1) = \{f_{|J^1} : f \in S(A^1)\}$, and so states on J extend to states on A. If J is unital, an extension of states is given by $\varphi \mapsto \varphi(1_J \cdot)$. It also is clear from (1) that $\mathfrak{F}_J = J \cap \mathfrak{F}_A$ in the nonunital case, and we leave the unital case as an exercise (using the fact that multiplication by the identity of J is an M-projection). The identity $\mathfrak{r}_J = J \cap \mathfrak{r}_A$ is handled similarly. Indeed, clearly $J \cap \mathfrak{r}_A \subset \mathfrak{r}_J$ since states on J extend to states on A^1. We leave the converse inclusion as an exercise (for example, it follows from $\mathfrak{F}_J = J \cap \mathfrak{F}_A \subset J \cap \mathfrak{r}_A$, and Proposition 3.5 below).

(3) We can assume J is nonunital. It follows from [Harmand et al. 1993, Proposition 1.17(b)] that if J is an M-ideal in A, and A is an M-ideal in A^1, then J is an M-ideal in A^1. By the same result, J is an M-ideal in J^1.

(4) Let e denote a weak* limit point in A^{**} of (e_i). Let (g_k) be any cai for J, with weak* limit point g in $J^{\perp\perp}$. Then $(h_j) = (g_k e_i)$ (indexed first by i and then j) is a cai for J. Then $h = ge$ is a weak* limit point of (h_j). We have $(1-g)e = e-h$. Since left multiplication by g is the M-projection of A^{**} onto $J^{\perp\perp}$, as we have seen several times above, one has $\|e-h\| \leq 1$. Let $\varphi \in S_{\mathfrak{e}}(A)$ be given. We claim that if $\varphi(h)=0$ then $\varphi_{|J}=0$; and if $\varphi(h)\neq 0$ then $\varphi(h\,\cdot)/\varphi(h)$ is a state on J^1. Note that if $\varphi(h) \neq 0$ then

$$1=\varphi(e)=\varphi(h)+\varphi((1-g)e)\leq|\varphi(h)|+|\varphi((1-g)e)|\leq\|\varphi(g\,\cdot)\|+\|\varphi((1-g)\,\cdot)\|,$$

which equals 1 due to the L-decomposition in A^*. Thus we must have $\varphi(h) \geq 0$. Let $a+\lambda 1 \in \mathrm{Ball}(J^1)$ be given. Then for any unimodular scalar γ, one has

$$\|\gamma(ha+\lambda h)+e-h\|_{A^{**}} = \max\{\|ha+\lambda h\|, \|e-h\|\} \leq 1.$$

Therefore,

$$\left|\varphi(\gamma(ha+\lambda h)+e-h)\right| = \left|\gamma\varphi(ha+\lambda h)+1-\varphi(h)\right| \leq 1$$

for all such γ. So for some such γ,

$$|\varphi(ha+\lambda h)|+1-\varphi(h)=\varphi(\gamma(ha+\lambda h)+e-h)\leq 1,$$

so that $|\varphi(ha+\lambda h)|\leq\varphi(h)$. $\square$

Proposition 3.3 (Esterle). *If A is a unital Banach algebra then $\mathfrak{F}_A$ is closed under (principal) t-th powers for any $t\in[0,1]$. Thus if A is an approximately unital Banach algebra then $\mathfrak{F}_A$ and $\mathbb{R}^+\mathfrak{F}_A$ are closed under t-th powers for any $t\in(0,1]$.*

Proof. This is in [Esterle 1978, Proposition 2.4] (see also [Blecher and Read 2011, Proposition 2.3]), but for convenience we repeat the construction. If $\|1-x\|\leq 1$, define

$$x^t=\sum_{k=0}^{\infty}\binom{t}{k}(-1)^k(1-x)^k,\quad t>0.$$

For $k\geq 1$, the sign of $\binom{t}{k}(-1)^k$ is always negative, and $\sum_{k=1}^{\infty}\binom{t}{k}(-1)^k=-1$. It follows that the series for x^t above is a norm-limit of polynomials in x with no constant term. Also, $1-x^t=\sum_{k=1}^{\infty}\binom{t}{k}(-1)^k(1-x)^k$, which is a convex combination in $\operatorname{Ball}(A^1)$. So $x^t\in\mathfrak{F}_A$.

Using the Cauchy product formula in Banach algebras in a standard way, one deduces that $(x^{1/n})^n=x$ for any positive integer n. $\square$

From [Esterle 1978, Proposition 2.4], if $x\in\mathfrak{F}_A$ then we also have $(x^t)^r=x^{tr}$ for $t\in[0,1]$ and any real r, and that if $ax_n\to a$, where $a\in A$ and (x_n) is a sequence with $\|x_n-1\|<1$, then $ax_n^t\to a$ with n for all real t.

If A is a unital Banach algebra then we define the $\mathfrak{F}$-transform to be $\mathfrak{F}(x)=x(1+x)^{-1}=1-(1+x)^{-1}$ for $x\in\mathfrak{r}_A$. Then $\mathfrak{F}(x)\in\operatorname{ba}(x)$. The inverse transform takes y to $y(1-y)^{-1}$.

Lemma 3.4. *If A is an approximately unital Banach algebra then $\mathfrak{F}(\mathfrak{r}_A)\subset\mathfrak{F}_A$.*

Proof. This is because by a result of Stampfli and Williams [1968, Lemma 1],

$$\|1-x(1+x)^{-1}\|=\|(1+x)^{-1}\|\leq d^{-1}\leq 1,$$

where d is the distance from -1 to the numerical range of x. $\square$

If A is also an operator algebra then we have shown elsewhere [Blecher and Read 2014, Lemma 2.5] that the range of the $\mathfrak{F}$-transform is exactly the set of strict contractions in $\frac{1}{2}\mathfrak{F}_A$.

Proposition 3.5. *If A is an approximately unital Banach algebra then $\overline{\mathbb{R}^+\mathfrak{F}_A}=\mathfrak{r}_A$.*

Proof. As in [Blecher and Read 2013a, Theorem 3.3], it follows that if $x\in\mathfrak{r}_A$ then $x=\lim_{t\to 0^+}(1/t)tx(1+tx)^{-1}$. By Lemma 3.4, $tx(1+tx)^{-1}\in\mathfrak{F}_A$. So $\mathbb{R}^+\mathfrak{F}_A$ is dense in $\mathfrak{r}_A$. $\square$

In the following results we will use the fact that if A is an approximately unital Banach algebra, then the "regular representation" $A \to B(A)$ is isometric. Thus we can view an accretive $x \in A$ and its principal roots as operators in $B(A)$. These are sectorial of angle $\pi/2$, and so we can use the theory of roots (fractional powers) from, e.g., [Haase 2006, Section 3.1] or [Li et al. 2003; Sz.-Nagy et al. 2010]. Basic properties of such (principal) powers include: $x^s x^t = x^{s+t}$, $(cx)^t = c^t x^t$ for positive scalars c, s, t, and $t \to x^t$ is continuous. See also, for example, [Yosida 1965, Chapter IX, Section 11], [Blecher 2015], [Blecher and Read 2014, Lemma 1.1(1)] or [Esterle 1978, p. 64]. Also $x^t = \lim_{t \to 0^+} (x + \epsilon 1)^t$ for $t > 0$, and the latter can be taken to be with respect to the usual Riesz functional calculus (see [Haase 2006, Proposition 3.1.9]). Principal n-th roots of accretive elements are unique for any positive integer n (see [Li et al. 2003]).

Remark. It is easy to see from the last fact that the definitions of x^t given in [Haase 2006] and [Li et al. 2003, Theorem 1.2] coincide. A similar argument shows that if $x \in \mathfrak{F}_A$ then the definitions of x^t given in [Haase 2006] and Proposition 3.3 coincide, if $t > 0$. Indeed for the latter we may assume that $0 < t \leq 1$ and work in $B(A)$ as above (and we may assume A unital). Then the two definitions of y^t coincide if $y = (1/(1+\epsilon))(x + \epsilon I)$, since both equal the t-th power of y as given by the Riesz functional calculus. However $\sum_{k=0}^{\infty} \binom{t}{k} (-1)^k (1-y)^k$ converges uniformly to $\sum_{k=0}^{\infty} \binom{t}{k} (-1)^k (1-x)^k$, as $\epsilon \to 0^+$, since the norm of the difference of these two series is dominated by

$$\sum_{k=1}^{\infty} \binom{t}{k} (-1)^k \left(\frac{1}{1+\epsilon} - 1 \right) \|(1-x)^k\| \leq \frac{\epsilon}{1+\epsilon} \to 0.$$

See [Blecher 2015] for more details concerning the last remark, and also for a better estimate in the next result in the operator algebra case.

Lemma 3.6. *Let A be an approximately unital Banach algebra. If $\|x\| \leq 1$ and $x \in \mathfrak{r}_A$, then*

$$\|x^{1/m}\| \leq \frac{2m^2}{(m-1)\pi} \sin\left(\frac{\pi}{m}\right) \leq \frac{2m}{m-1}$$

for $m \geq 2$. More generally,

$$\|x^{\alpha}\| \leq \frac{2\sin(\alpha\pi)}{\pi\alpha(1-\alpha)} \|x\|^{\alpha}$$

if $0 < \alpha < 1$ and $x \in \mathfrak{r}_A$. If A is also an operator algebra then one may remove the 2s in these estimates.

Proof. This follows from the well-known A. V. Balakrishnan representation of powers,

$$x^{\alpha} = \frac{\sin(\alpha\pi)}{\pi} \int_0^{\infty} t^{\alpha-1} (t+x)^{-1} x \, dt$$

(see, e.g., [Haase 2006]). We use the simple fact that $\|(t+x)^{-1}\| \le 1/t$ for accretive x and $t > 0$, and so

$$\|(t+x)^{-1}x\| = \left\|\left(1+\frac{x}{t}\right)^{-1}\frac{x}{t}\right\| = \left\|\mathfrak{F}\left(\frac{x}{t}\right)\right\| \le 2,$$

and is even less than or equal to 1 in the operator algebra case by the observation after Lemma 3.4. Then the norm of x^α is dominated by

$$\frac{2\sin(\alpha\pi)}{\pi}\left(\int_0^1 t^{\alpha-1}\cdot 1\,dt + \int_1^\infty t^{\alpha-1}\frac{1}{t}\,dt\right) = \frac{2\sin(\alpha\pi)}{\pi\alpha(1-\alpha)}.$$

The rest is clear from this. $\square$

We will sometimes use the fact from [Li et al. 2003, Corollary 1.3] that the n-th root function is continuous on $\mathfrak{r}_A$.

Lemma 3.7. *There is a nonnegative sequence (c_n) in c_0 such that for any unital Banach algebra A, and $x \in \mathfrak{F}_A$ or $x \in \mathrm{Ball}(A) \cap \mathfrak{r}_A$, we have $\|x^{1/n}x - x\| \le c_n$ for all $n \in \mathbb{N}$.*

Proof. We follow the proof of [Blecher and Read 2013a, Theorem 3.1], taking $R = 3$ there. This is based on the Banach algebra construction from [Li et al. 2003], so it will be valid in the present generality. There an estimate $\|x^{1/n}x - x\| \le Dc_n$ is given, for a nonnegative sequence (c_n) in c_0. We need to know that D does not depend on A or x. This follows if $\|\lambda(\lambda 1 - x)^{-1}\|$ is bounded independently of A or x on the curve Γ there. On the piece of the curve Γ_2, this follows by using [Stampfli and Williams 1968, Lemma 1] that $\|(\lambda 1 - x)^{-1}\| \le d^{-1}$, where d is the distance from λ to $W(x)$. On the other part of Γ, we have $\lambda = te^{i\theta}$ for $0 \le t \le R$, and for a fixed θ with $\pi/2 < |\theta| < \pi$. However, by the same result of Stampfli and Williams, $\|(\lambda 1 - x)^{-1}\| \le d^{-1}$ if $\lambda \ne 0$, where d is the distance from λ to the y-axis. Thus the quantity will be bounded since $|\lambda|/d = \csc(\theta - \pi/2)$. $\square$

The following (essentially from [Macaev and Palant 1962]) is a related result:

Lemma 3.8. *Let A be a unital Banach algebra. If $\alpha \in (0,1)$ then there exists a constant K such that if $a, b \in \mathfrak{r}_A$, and $ab = ba$, then $\|(a^\alpha - b^\alpha)c\| \le K\|(a-b)c\|^\alpha$ for any $c \in \mathrm{Ball}(A)$.*

Proof. By the Balakrishnan representation in the proof of Lemma 3.6, if $c \in \mathrm{Ball}(A)$, we have

$$(a^\alpha - b^\alpha)c = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty t^{\alpha-1}\big((t+a)^{-1}a - (t+b)^{-1}b\big)c\,dt.$$

By the inequality $\|(t+x)^{-1}\| \le 1/t$ for accretive x, we have

$$\big\|\big((t+a)^{-1}a - (t+b)^{-1}b\big)c\big\| = \big\|(t+a)^{-1}(t+b)^{-1}(a-b)tc\big\| \le \tfrac{1}{t}\|(a-b)c\|,$$

and so as in the proof of Lemma 3.6, $\left\|\int_0^\infty t^{\alpha-1}\big((t+a)^{-1}a-(t+b)^{-1}b\big)c\,dt\right\|$ is dominated by

$$4\int_0^\delta t^{\alpha-1}\,dt+\int_\delta^\infty t^{\alpha-2}\,dt\,\|(a-b)c\|=\frac{4}{\alpha}\delta^\alpha+\frac{\delta^{\alpha-1}}{1-\alpha}\|(a-b)c\|$$

for any $\delta>0$. We may now set $\delta=\|(a-b)c\|$ to obtain our inequality. ☐

Corollary 3.9. *An approximately unital Banach algebra with a left bai* (*resp. right bai*, *bai*) *in* $\mathfrak{r}_A$ *has a left bai* (*resp. right bai*, *bai*) *in* $\mathfrak{F}_A$.

Proof. If (e_t) is a left bai in $\mathfrak{r}_A$, let $b_t=\mathfrak{F}(e_t)\in\mathfrak{F}_A$. If $a\in A$ then

$$b_t^{1/n}a=b_t^{1/n}(a-e_ta)+(b_t^{1/n}e_t-e_t)a+e_ta.$$

The first term here converges to 0 with t since $(b_t^{1/n})$ is in $\mathfrak{F}_A$, and hence is bounded. Similarly, the middle term can be seen to converge to 0 with n by rewriting it as $(b_t^{1/n}b_t-b_t)(1+e_t)a$. Working in A^1 and applying Lemma 3.7, we have

$$\|(b_t^{1/n}b_t-b_t)(1+e_t)a\|\le c_n\|1+e_t\|\|a\|\le Kc_n\to 0$$

for a constant K independent of t. The third term converges to a with t. So $(b_t^{1/n})$ is a left bai. Similarly in the right and two-sided cases. ☐

Remark. If the bai in the last result is sequential, then so is the one constructed in $\mathfrak{F}_A$.

Corollary 3.10. *If A is an approximately unital Banach algebra then $\mathfrak{r}_A$ is closed under n-th roots for any positive integer n.*

Proof. From the proof of Proposition 3.5, we know that if $x\in\mathfrak{r}_A$, then $x=\lim_{t\to 0^+}(1/t)tx(1+tx)^{-1}$ and $tx(1+tx)^{-1}\in\mathfrak{F}_A$. Thus by [Li et al. 2003, Corollary 1.3], we have that $x^r=\lim_{t\to 0^+}1/t^r(tx(1+tx)^{-1})^r$ for $0<r<1$. By Proposition 3.3, the latter powers are in $\mathbb{R}^+\mathfrak{F}_A$, so that $x^r\in\overline{\mathbb{R}^+\mathfrak{F}_A}=\mathfrak{r}_A$. ☐

Proposition 3.11. *If A is an approximately unital Banach algebra and $x\in\mathfrak{r}_A$ then* $\mathrm{ba}(x)=\mathrm{ba}(\mathfrak{F}(x))$, *and so* $\overline{xA}=\overline{\mathfrak{F}(x)A}$.

Proof. This follows from the elementary spectral theory of unital Banach algebras, applied in A^1. Below we compute the spectrum in $\mathrm{ba}(x)^1$. Since $0\notin\mathrm{Sp}(1+x)$, we have $(1+x)^{-1}\in\mathrm{ba}(1,x)$, so that $\mathfrak{F}(x)\in\mathrm{ba}(x)$. Any character of $\mathrm{ba}(x)^1$ applied to $\mathfrak{F}(x)$ gives a number of the form $z=w(1+w)^{-1}$ in the open unit disk, and in fact also inside the circle $|z-\frac{1}{2}|\le\frac{1}{2}$ if $\mathrm{Re}(w)\ge 0$. Since $1\notin\mathrm{Sp}(\mathfrak{F}(x))$, we have $(1-\mathfrak{F}(x))^{-1}\in\mathrm{ba}(1,\mathfrak{F}(x))$, so that $x=-\mathfrak{F}(x)(1-\mathfrak{F}(x))^{-1}\in\mathrm{ba}(\mathfrak{F}(x))$. The rest is clear. ☐

Lemma 3.12. *If p is an idempotent in a unital Banach algebra A then $p\in\mathfrak{F}_A$ if and only if $p\in\mathfrak{r}_A$. If p is an idempotent in A^{**} for an approximately unital Banach algebra A then $p\in\mathfrak{F}_{A^{**}}$ if and only if $p\in\mathfrak{r}_{A^{**}}$.*

Proof. The first follows from the well-known Lumer–Phillips characterization of accretiveness in terms of $\|\exp(-tp)\| \leq 1$ for all $t > 0$ (see, e.g., [Bonsall and Duncan 1971, Theorem 6, p. 30]). If p is idempotent then $\exp(-tp) = 1 - (1 - e^{-t})p$, and if this is contractive for all $t > 0$ then $\|1 - p\| \leq 1$. For the second, work in $(A^1)^{**}$ and use facts above. $\square$

However, one cannot say that the idempotents in the last result are also in $\frac{1}{2}\mathfrak{F}_A$, as is the case for operator algebras. The following examples illustrate this, and other "bad behavior" not seen in the class of operator algebras.

Example 3.13. Let ℓ^1_4 be identified with the l^1-semigroup algebra of the abelian semigroup $\{1, a, b, c\}$ with relations making a, b, c idempotent, and $ab = ac = bc = c$. Then $p = 1 - a, q = 1 - b \in \mathfrak{F}_A \setminus \frac{1}{2}\mathfrak{F}_A \subset \mathfrak{r}_A$. For such p, set $x = \frac{1}{2}p \in \frac{1}{2}\mathfrak{F}_A$, and notice that $x^{1/n} = 2^{-1/n}p$ which is not always in $\frac{1}{2}\mathfrak{F}_A$ (if it were, then we get the contradiction that its limit p is in $\frac{1}{2}\mathfrak{F}_A$). So we see that $\frac{1}{2}\mathfrak{F}_A$ is not closed under n-th roots. We also see that if $x \in \frac{1}{2}\mathfrak{F}_A$ then $\overline{xA}$ need not have a left cai (even if A is commutative). It does have a left bai of norm at most 2, and indeed a left bai in $\mathfrak{F}_A$ by Corollary 3.18.

In this example, $pq = p^{\frac{1}{2}}q^{\frac{1}{2}} = 1 - a - b + c \notin \mathfrak{r}_A$ (as can be seen by considering states $f(\alpha a + \beta b + \gamma c + \lambda 1) = \gamma z + \lambda + \alpha + \beta$ for $|z| \leq 1$). So $x^{1/2}y^{1/2}$ need not be in $\mathfrak{r}_A$ even if $x, y \in \frac{1}{2}\mathfrak{F}_A$. This shows that the main results about roots in [Bearden et al. 2014] fail in more general M-approximately unital Arens regular Banach algebras. Note too that if $J_1 = pA$ and $J_2 = qA$, then $J_1 \cap J_2 = \mathbb{C}d = dA$, where $d = pq$, but dA has no identity or bai in $\mathfrak{r}_A$. This shows that, unlike in the operator algebra case, finite intersections of extremely nice closed ideals need not be "nice" in the sense of the theory developed in this paper. See, however, Section 8 for a context in which finite intersections will behave well.

Example 3.14. In the Banach algebra $A = l^1(\mathbb{Z}_2)$ with convolution multiplication, we know that $p = \left(\frac{1}{2}, \frac{1}{2}\right)$ is a contractive idempotent in $\frac{1}{2}\mathfrak{F}_A$ with numerical range $\overline{B\left(\frac{1}{2}, \frac{1}{2}\right)}$. The states in this example are the functionals $(a, b) \mapsto a + bz$ for $|z| \leq 1$. All of the principal n-th roots of p obviously have the same numerical range. So the numerical range of $p^{1/n}$ does not "converge" to the x-axis. Thus we cannot expect statements in the Blecher–Read papers involving "near positivity" to generalize (unless A is a Hermitian Banach $*$-algebra satisfying the conditions in the latter part of [Li et al. 2003], in which case the numerical ranges of $x^{1/n}$ do "converge" to the x-axis if x is accretive). Note also in this example that p is not an M-projection in A. Thus we cannot expect support projections to be associated with M-projections in general. In this example, it is easy to see that $x = (a, b) \in \mathfrak{r}_A$ if and only if $|b| \leq \operatorname{Re} a$, whereas $x \in \frac{1}{2}\mathfrak{F}_A$ if and only if $|b|^2 - |b| \leq \operatorname{Re} a - |a|^2$. In this example, the Cayley transform does not take $\mathfrak{r}_A$ into the set of contractions, so that $x(1+x)^{-1}$ need not be in $\frac{1}{2}\mathfrak{F}_A$.

This example also serves to show that if B is an approximately unital closed ideal in a commutative finite-dimensional approximately unital Banach algebra, then $\mathfrak{r}_B$ and $\mathfrak{F}_B$ need not be related to $\mathfrak{r}_A$ and $\mathfrak{F}_A$, unlike the setting of operator algebras (where there is a very strong relationship between these, even in the case that B is a subalgebra). Indeed let $B = \mathbb{C}(1,1)$ inside the last example. Then we have $1_B = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $\mathfrak{r}_B = \{(a,a) : \operatorname{Re} a \geq 0\}$ and $\mathfrak{F}_B = \left\{(a,a) : a \in \overline{B\left(\frac{1}{2}, \frac{1}{2}\right)}\right\}$.

For a state φ on an operator algebra A and $x \in \mathfrak{F}_A$, it is the case that $\varphi(s(x)) = 0$ if and only if $\varphi(x) = 0$ if and only if $\varphi \in \operatorname{ba}(x)^{\perp}$. Here $s(x)$ is the support projection of x from [Blecher and Read 2011]. In Example 3.14, if $x = \left(\frac{1}{2}, \frac{i}{2}\right)$ and $\varphi((a,b)) = a + ib$ then $x \in \operatorname{Ker}\varphi$ but x^2 and $s(x) = 1$ are not in $\operatorname{Ker}\varphi$. Thus much of the theory of "strictly real positive" elements from [loc. cit.] and its sequels breaks down.

A slight variant of this example is the same algebra, but with norm $|||(a,b)||| = |a| + 2|b|$. Here $J = \mathbb{C}\left(\frac{1}{2}, \frac{1}{2}\right)$ is an ideal equal to xA for $x \in \mathfrak{F}_A$, but this ideal has no cai.

Example 3.15. The unital Banach algebra $l^1(\mathbb{N})$, with convolution product, is easily seen to be equal to $\operatorname{ba}(x)$ where $x = 1 + \frac{1}{2}\vec{e}_2 \in \mathfrak{F}_A$. However $l^1(\mathbb{N})$ is not Arens regular; thus its second dual is not commutative in either one of the Arens products [Palmer 1994, §1.4.9]. Thus $\operatorname{ba}(x)^{**}$ need not be commutative if $x \in \mathfrak{F}_A$. In this example, it is easy to compute $\mathfrak{F}_A$ and $\mathfrak{r}_A$. C. A. Bearden has verified that in this example, unlike the operator algebra case [Bearden et al. 2014], $(x^{1/n})$ need not increase in the "real positive ordering" with n for $x \in \frac{1}{2}\mathfrak{F}_A$.

Example 3.16. The approximately unital Banach algebra $A = L^1(\mathbb{R})$ with convolution product has multiplier unitization $A^1 = A \oplus^1 \mathbb{C}$. This can be seen from Wendel's result that the measure algebra $M(\mathbb{R})$ embeds canonically in $B(L^1(\mathbb{R}))$ isometrically [Dales 2000], so that $L^1(\mathbb{R})^1$ can be identified with $L^1(\mathbb{R}) + \mathbb{C}\delta_0$, where δ_0 is the point mass at 0. Thus $S(A)$ corresponds to the set of $f \in L^\infty(\mathbb{R})$ of norm 1. It follows immediately that $\mathfrak{F}_A = \mathfrak{r}_A = (0)$ in this case. This algebra is not Arens regular. Note that any norm-1 functional on $L^1(\mathbb{R})$ extends to a state on $L^1(\mathbb{R})^1$ clearly. However, there are many norm-1 functions $g \in L^\infty(\mathbb{R})$ with $1 \neq \lim_{t \to 0+} \int_{\mathbb{R}} g e_t$ for the usual positive cai $\mathfrak{e} = (e_t)$ of $L^1(\mathbb{R})$ (the one in the remark after Lemma 2.1), for example, if g takes only negative values. This shows that Lemma 2.2 fails for more general Banach algebras. For this same cai $\mathfrak{e}$, we remark that $S_{\mathfrak{e}}(A)$ corresponds to the set of $f \in \operatorname{Ball}(L^\infty(\mathbb{R}))$ for which the mean value of f at 0 (this mean value is the limit with n of the (integral) average of f over the interval of width $1/n$ centered at 0) exists and equals 1. From this it is easy to see that $\mathfrak{r}_A^{\mathfrak{e}} = (0)$ and $\mathfrak{c}_{A^*}^{\mathfrak{e}} = A^*$.

Because of the above examples and the considerations mentioned after Lemma 2.3 above, the following result cannot be improved, even for M-approximately unital Arens regular Banach algebras:

Proposition 3.17. *If $x \in \mathfrak{r}_A$ then* $\mathrm{ba}(x)$ *has a bai in $\mathfrak{F}_A$, and hence any weak* limit point of this bai is a mixed identity residing in $\mathfrak{F}_{A^{**}}$. Indeed $(x^{1/n})$ is a bai for* $\mathrm{ba}(x)$ *in $\mathfrak{r}_A$, and $(\mathfrak{F}(x)^{1/n})$ is a bai for* $\mathrm{ba}(x)$ *in $\mathfrak{F}_A$.*

Proof. Note that $x^{1/n}x \to x$ by Lemma 3.7. That $(x^{1/n})$ is bounded follows from Lemma 3.6. Thus $(x^{1/n})$ is a bai for $\mathrm{ba}(x)$ in $\mathfrak{r}_A$.

In the case that $x \in \mathfrak{F}_A$, we have $(x^{1/n})$ is in $\mathfrak{F}_A$ (using Proposition 3.3). We remark that the proof of [Blecher and Read 2011, Lemma 2.1] (see also [Blecher et al. 2008]) displays a different, and often useful, bai in $\mathfrak{F}_A$. In the general case, note that if $x \in \mathfrak{r}_A$ then $\mathrm{ba}(x) = \mathrm{ba}(\mathfrak{F}(x))$ by Proposition 3.11, and so $(\mathfrak{F}(x)^{1/n})$ is a bai for $\mathrm{ba}(x)$. ☐

For an approximately unital Banach algebra A and $x \in \mathfrak{r}_A$, by Proposition 3.11 we have $\mathrm{ba}(x) = \mathrm{ba}(\mathfrak{F}(x))$ and $\overline{xA} = \overline{\mathfrak{F}(x)A}$. If A is not Arens regular then Example 3.15 shows that $\mathrm{ba}(x)$ need not be Arens regular if $x \in \mathfrak{F}_A$. (However, it is Arens semiregular as is any commutative Banach algebra [Palmer 1994].) Thus $\mathrm{ba}(x)^{**}$ need not be commutative. We write $s(x)$ for the weak* Banach limit of $(x^{1/n})$ in A^{**}. That is $s(x)(f) = \mathrm{LIM}_n\, f(x^{1/n})$ for $f \in A^*$, where LIM is a Banach limit. It is easy to see that $xs(x) = s(x)x = x$, by applying these to $f \in A^*$. Hence $s(x)$ is a mixed identity of $\mathrm{ba}(x)^{**}$ and is idempotent. By the Hahn–Banach theorem, it is easy to see that $s(x) \in \overline{\mathrm{conv}(\{x^{1/n} : n \in \mathbb{N}\})}^{w*}$. By Corollary 3.10 and Lemma 3.12, and the fact below Lemma 2.5 that $\mathfrak{F}_{A^{**}}$ is weak* closed, we see that $s(x)$ resides in $\mathfrak{F}_{A^{**}}$. If $\mathrm{ba}(x)$ is Arens regular then $s(x)$ will be the identity of $\mathrm{ba}(x)^{**}$. Therefore in this case, or more generally if $\mathrm{ba}(x)^{**}$ has a unique left identity in the second Arens product, $s(x)$ is also the weak* limit of $(\mathfrak{F}(x)^{1/n})$. Indeed in this case we can set $s(x)$ to be the weak* limit of any bai for $\mathrm{ba}(x)$. This is the case, for example, if $\mathrm{ba}(x)$ is M-approximately unital (that is, if it is an M-ideal in $\mathrm{ba}(x)^1$), by Lemma 2.5.

Remark. Note that if $x \in \mathfrak{r}_A$ then $\mathrm{ba}(x)$ is M-approximately unital if A is M-approximately unital and $\mathrm{ba}(x)^1 \subset A^1$ isometrically (by the argument in Proposition 3.1). It is claimed in [Smith 1979] that the support projection of an M-ideal in a commutative Banach algebra is central. We did not follow this proof (and its author confirmed that at present there seemed to him to be a gap), but this would imply that if $\mathrm{ba}(x)$ is M-approximately unital then $s(x)$ is central in $\mathrm{ba}(x)^{**}$, and thus is actually a (unique) two-sided identity for $\mathrm{ba}(x)^{**}$.

We call $s(x)$ above a *support* idempotent of x, or a (left) support idempotent of $\overline{xA}$ (or a (right) support idempotent of $\overline{Ax}$). The reason for this name is the following result.

Corollary 3.18. *If A is an approximately unital Banach algebra, and $x \in \mathfrak{r}_A$ then $\overline{xA}$ has a left bai in $\mathfrak{F}_A$ and $x \in \overline{xA} = s(x)A^{**} \cap A$ and $(xA)^{\perp\perp} = s(x)A^{**}$. (These products are with respect to the second Arens product.)*

Proof. Indeed if $J = \overline{xA}$ then $J = \overline{\mathfrak{F}(x)A}$ by Proposition 3.5. So we may assume that $x \in \mathfrak{F}_A$. Since $\overline{xA}$ contains $\overline{x\,\mathrm{ba}(x)}$, which in turn contains (actually, is equal to) $\mathrm{ba}(x)$, it contains x and $x^{1/n}$. So $(x^{1/n})$ is a left bai in $\mathfrak{F}_A$ for $\overline{xA}$. We have $s(x) \in J^{\perp\perp}$, and $J^{\perp\perp} \subset s(x)A^{**} \subset J^{\perp\perp}$, since $J^{\perp\perp}$ is a right ideal in A^{**}. Hence $J^{\perp\perp} = s(x)A^{**}$, so that $J = s(x)A^{**} \cap A$. □

As in [Blecher and Read 2011, Lemma 2.10] we have:

Corollary 3.19. *If A is an approximately unital Banach algebra, and $x, y \in \mathfrak{r}_A$, then $\overline{xA} \subset \overline{yA}$ if and only if $s(y)s(x) = s(x)$. In this case, $\overline{xA} = A$ if and only if $s(x)$ is a left identity for A^{**}. (These products are with respect to the second Arens product.)*

Proof. This is essentially just as in the proof of Lemma 2.10 (and Corollary 2.6) of [loc. cit.]. For example, if $\overline{xA} \subset \overline{yA}$ then, since $x \in \overline{xA}$, we have $s(y)x = x$. Hence $s(y)z = z$ for all $z \in \mathrm{ba}(x)$, and so $s(y)s(x) = s(x)$, since as we said earlier $s(x) \in \overline{\mathrm{ba}(x)}^{w*}$. □

As in [loc. cit., Corollary 2.7] we have:

Corollary 3.20. *Suppose that A is a closed approximately unital subalgebra of an approximately unital Banach algebra B, and that $\mathfrak{r}_A \subset \mathfrak{r}_B$. If $x \in \mathfrak{r}_A$, then the support projection of x computed in A^{**} is the same, via the canonical embedding $A^{**} \cong A^{\perp\perp} \subset B^{**}$, as the support projection of x computed in B^{**}.*

We recall that x is pseudo-invertible in A if there exists $y \in A$ with $xyx = x$. The following result (and several of its corollaries below) should be compared with the C^*-algebraic version of the result due to Harte and Mbekhta [1992; 1993], and to the earlier version of the result in the operator algebra case (see particularly [Blecher and Read 2011, Section 3; 2014, Subsection 2.4]).

Theorem 3.21. *Let A be an approximately unital Banach algebra A, and $x \in \mathfrak{r}_A$. The following are equivalent:*

(i) $s(x) \in A$.

(ii) *xA is closed.*

(iii) *Ax is closed.*

(iv) *x is pseudo-invertible in A.*

(v) *x is invertible in* $\mathrm{ba}(x)$.

Moreover, these conditions imply that

(vi) 0 *is isolated in, or absent from,* $\mathrm{Sp}_A(x)$.

Finally, if $\mathrm{ba}(x)$ *is semisimple then* (i)–(vi) *are equivalent.*

Proof. We recall that $(x^{1/m})_{m\in\mathbb{N}}$ is a bai for $\mathrm{ba}(x)$, by Proposition 3.17, and it has weak* limit point $s(x)\in \mathrm{ba}(x)^{\perp\perp}\subset A^{**}$.

(ii) $\Rightarrow$ (i): Suppose xA is closed. Then

$$x^{1/2}\in \mathrm{ba}(x)\subset \overline{x\,\mathrm{ba}(x)}\subset \overline{xA}=xA,$$

so $x^{1/2}=xy$ for some $y\in A$. Thus if $z=x^{1/2}y\in A$ then $x=x^{1/2}xy=xz$, and so $a=az$ for every $a\in \mathrm{ba}(x)$. Now $s(x)z=z$ since $x^{1/2}\in \mathrm{ba}(x)$, for example. On the other hand, $s(x)z=s(x)$ since $x^{1/n}z=x^{1/n}$ so that

$$(s(x)z)(f)=fs(x)(z)=\mathrm{LIM}_n\, f(x^{1/n}z)=\mathrm{LIM}_n\, f(x^{1/n})=s(x)(f),\quad f\in A^*.$$

Thus $s(x)=z\in A$. (Of course, in this case $x^{1/n}\to s(x)$ in norm.)

(i) $\Rightarrow$ (iv): Recall $s(x)$ is a left identity of $\mathrm{ba}(x)^{**}$ in the second Arens product, and if (i) holds, it is an identity, and $\mathrm{ba}(x)$ is unital. This implies, by the Neumann lemma, that x is invertible in $\mathrm{ba}(x)$, and hence that x is pseudo-invertible in A.

(iv) $\Rightarrow$ (ii): Item (iv) implies that $xA=xyA$ is closed since xy is idempotent.

That (iii) is equivalent to the others follows from (ii) and the symmetry in (i) or (iv). That (v) is equivalent to (i) is now obvious from the above.

For the equivalences with (vi), by the definition of spectrum, and because of the form of (v), we may assume A is unital. That (iv) implies (vi) may be proved similarly to the analogous argument in [Blecher and Read 2011, Theorem 3.2], but replacing $B(H)$ and $B(K)$ with $B(A)$ and $B(xA)$. We can assume that $0\in \mathrm{Sp}_A(x)$, so that x is not invertible. Then $xA\neq A$, for if $xA=A$ then $s(x)$ is a left identity for A. It is also a right identity since if (e_t) is a cai for A then $s(x)e_t=e_t\to s(x)$. Then the inverse of x in $\mathrm{ba}(x)$ is an inverse in A, contradicting the fact that x is not invertible in A^1. It may be simpler to prove the equivalent fact that 0 is isolated in the spectrum of $x^{1/2}$. By the argument in [loc. cit., Theorem 3.2] it is enough to prove that 0 is isolated in the spectrum of L in $B(A)$, where L is left multiplication by $x^{1/2}$. We note that

$$x^{1/2}A\subset xA\subset eA\subset x^{1/2}A,$$

where $e=x^{1/2}y=s(x)$ and y is the pseudo-inverse of x. So these subspaces coincide; call this space K. It follows that K is an invariant subspace for L, indeed $R=L_{|K}$ is continuous, surjective and one-to-one (since $x^{1/2}x^{1/2}a=0$ implies that $x^{1/2}a=0$, since $x^{1/2}$ is a limit of polynomials in x with no constant term). Thus $0\notin \mathrm{Sp}_{B(K)}(R)$; hence $R+zI_K$ is invertible for z in a small disk centered at 0. Since $A=eA\oplus(1-e)A$, it is easy to argue that $L+zI_A=(L+zI)e\oplus z(1-e)$ is invertible in $B(A)$ for such z if $z\neq 0$. So 0 is isolated in the spectrum of L in $B(A)$.

The last assertion follows just as in [loc. cit., Theorem 3.2]. $\square$

Remark. We have been informed by Matthias Neufang that he and M. Mbekhta have also generalized the analogous result from [Blecher and Read 2011; 2013b], or a variant of it, to the class of Banach algebras that are ideals in their bidual.

The next result is an analogue of [Blecher and Read 2011, Theorem 2.12]:

Proposition 3.22. *If A is an approximately unital Banach algebra, a subalgebra of a unital Banach algebra B with $\mathfrak{r}_A \subset \mathfrak{r}_B$, and $x \in \mathfrak{r}_A$, then x is invertible in B if and only if $1_B \in A$ and x is invertible in A, and if and only if* $\mathrm{ba}(x)$ *contains 1_B; and in this case $s(x) = 1_B$.*

Proof. It is clear by the Neumann lemma that if $\mathrm{ba}(x)$ contains 1_B then x is invertible in $\mathrm{ba}(x)$, and hence in A. Conversely, if x is invertible in B (or in A) then by the equivalences (i)–(iv) proved in the last theorem, we have $s(x) \in B$, and this is the identity of $\mathrm{ba}(x)$. If $xy = 1_B$, then $1_B = xy = s(x)xy = s(x) \in \mathrm{ba}(x) \subset A$. □

Corollary 3.23. *Let A be an approximately unital Banach algebra. A closed right ideal J of A is of the form xA for some $x \in \mathfrak{r}_A$ if and only if $J = qA$ for an idempotent $q \in \mathfrak{F}_A$.*

Proof. If xA is closed for a nonzero $x \in \mathfrak{r}_A$ then by Theorem 3.21, $q = s(x) \in \mathfrak{F}_A$. Hence it is easy to see that $xA = qA$. The other direction is trivial. □

Corollary 3.24. *If a nonunital approximately unital Banach algebra A contains a nonzero $x \in \mathfrak{r}_A$ with xA closed, then A contains a nontrivial idempotent in $\mathfrak{F}_A$.*

Proof. By the above, $xA = qA$ for a nontrivial idempotent q in $\mathfrak{F}_A$. □

Corollary 3.25. *If an approximately unital Banach algebra A has no left identity, then $xA \neq A$ for all $x \in \mathfrak{r}_A$.*

Remark. If A is a Banach algebra such that $\frac{1}{2}\mathfrak{F}_A$ is closed under n-th roots then one may also generalize other parts of the theory in [Blecher and Read 2011]. For example, in this case, if $x \in \mathfrak{F}_A$ then the support projection $s(x)$ is a bicontractive projection, and $\mathrm{ba}(x)$ has a cai in $\frac{1}{2}\mathfrak{F}_A$.

4. One-sided ideals and hereditary subalgebras

At the outset it should be said there seems to be no completely satisfactory theory of hereditary subalgebras. This can already be seen in finite-dimensional unital examples where one may have $pA = qA$ for projections $p, q \in \mathfrak{F}_A$, but no good relation between pAp and qAq. For example, one could take the opposite algebra to the one in Example 4.3. Another example arises when one considers various mixed identities in the second dual A^{**}, with the second Arens product, inside $(A^1)^{**}$. In this section we will investigate what initial parts of the theory do work. We shall see that things work considerably better if A is separable.

We define an *inner ideal* in A to be a closed subalgebra D with $DAD \subset D$. To see what kinds of results one might hope for, note that in the unital example in the last paragraph, given an idempotent $p \in A$, the right ideal $J = pA$ contains a unital inner ideal $D = pAp$ of A. Conversely, if $D = pAp$ then $J = DA = pA$ is a right ideal with a left identity.

In nonunital examples things become more complicated. One may define a hereditary subalgebra to be an inner ideal D of A which has a bai. This then induces a right ideal $J = DA$ with a left bai, and a left ideal $K = AD$ with a right bai. We shall call these the *induced* one-sided ideals. We have $JK = J \cap K = D$ just as in [Blecher et al. 2008, Corollary 2.6]. However, unlike the previous paragraph, without further conditions one cannot in general obtain a hereditary subalgebra from a right ideal with a left bai. The following example illustrates some of what can go wrong.

Example 4.1. One of the main results in [Blecher et al. 2008] is that if J is a closed right ideal with a left cai in an operator algebra A, then there exists an associated hereditary subalgebra D of A, in particular, a closed approximately unital subalgebra $D \subset J$ with $J = DA$. This is false without further conditions in more general Banach algebras. Indeed, suppose that $J = A$ is a separable Banach algebra with a sequential left cai, but no commuting bounded left approximate identity. See [Dixon 1978] for such an example. By way of contradiction, suppose that there is a closed subalgebra $D \subset J$ with a bai, such that $J = DA$. By [Sinclair 1978], D has a commuting bounded approximate identity, and this will be a commuting bounded left approximate identity for J, a contradiction.

This example also shows that if J is a closed right ideal with a left cai, we cannot rechoose another left cai (e_t) with $e_s e_t \to e_s$ with t for all s. This is critical in the operator algebra theory in, e.g., [Blecher et al. 2008, Section 2].

In order to obtain a working theory, we now impose the condition that the bais considered are in $\mathfrak{r}_A$. Thus we define a *right $\mathfrak{F}$-ideal* (resp. left $\mathfrak{F}$-ideal) in an approximately unital Banach algebra A to be a closed right (resp. left) ideal with a left (resp. right) bai in $\mathfrak{F}_A$ (or equivalently, by Corollary 3.9, in $\mathfrak{r}_A$). Henceforth in this section, by a *hereditary subalgebra* (HSA) of A we will mean an inner ideal D with a two-sided bai in $\mathfrak{F}_A$ (or equivalently, by Corollary 3.9, in $\mathfrak{r}_A$). Perhaps these should be called $\mathfrak{F}$-HSAs to avoid confusion with the notation in [Blecher et al. 2008; Blecher and Read 2011] where one uses cais instead of bais, but for brevity we shall use the shorter term. Also it is shown in [Blecher 2015] that in an operator algebra A these two notions coincide, and that right $\mathfrak{F}$-ideals in A are just the r-ideals of [Blecher et al. 2008] (and similarly in the left case).

Note that an HSA D induces a pair of right and left $\mathfrak{F}$-ideals $J = DA$ and $K = AD$. As we pointed out a few paragraphs back, it is not clear that the converse holds, namely that every right $\mathfrak{F}$-ideal comes from an HSA in this way. In fact, the main results of this section are, firstly, that if A is separable then this is true, and indeed all HSAs and $\mathfrak{F}$-ideals are of the form in the next lemma. Secondly, we shall prove (see Corollaries 4.6 and 4.11) that if A is not necessarily separable then the HSAs and $\mathfrak{F}$-ideals in A are just the closures of increasing unions of ones of the form in this lemma:

Lemma 4.2. *If A is an approximately unital Banach algebra, and $z \in \mathfrak{F}_A$, set $J = \overline{zA}$, $D = \overline{zAz}$, and $K = \overline{Az}$. Then D is an HSA in A and J and K are the induced right and left $\mathfrak{F}$-ideals mentioned above.*

Proof. By Cohen factorization, $D = D^4 \subset JK \subset J \cap K$, and if $x \in J \cap K$ then $x = \lim_n z^{1/n} x z^{1/n} \in D$. So $z \in D = JK = J \cap K$. Also $J = pA^{**} \cap A$ by Corollary 3.18, and $D = pA^{**} p \cap A$ is an HSA in A, and $K = A^{**} p \cap A$, where $p = s(z)$. To see this, note that $pz = z = zp$, so that $K \subset A^{**} p \cap A$. If $a \in A^{**} p \cap A$, then $az^{1/n}$ has weak* limit point $ap = a$. Hence a convex combination converges in norm, so that $a \in K$, and then $K = A^{**} p \cap A$. A similar argument works for D. Finally, $DA = J$, since $zA \subset DA \subset J$, and similarly $AD = K$. □

Remark. (1) In general D and K are determined by the particular z used above, and not by J alone.

(2) We note that if $z \in \mathfrak{F}_A$ then with the notation in the last proof, $K^{\perp\perp} = \overline{A^{**} p}^{\text{w*}}$ and $D^{\perp\perp} = \overline{pA^{**} p}^{\text{w*}}$. (The weak* closure here is not necessary if A is Arens regular.) Indeed $K^{\perp\perp} \subset \overline{A^{**} p}^{\text{w*}}$. Also $p \in \text{ba}(z)^{\perp\perp} \subset D^{\perp\perp} \subset K^{\perp\perp}$, so that $A^{**} p \subset K^{\perp\perp}$. Thus $K^{\perp\perp} = \overline{A^{**} p}^{\text{w*}}$. It is well known that $J + K$ is closed, which implies, as in the proof of [Blecher and Zarikian 2006, Lemma 5.29], that $(J \cap K)^{\perp} = \overline{J^{\perp} + K^{\perp}}$, so that $D^{\perp\perp} = J^{\perp\perp} \cap K^{\perp\perp} = \overline{pA^{**} p}^{\text{w*}}$.

Example 4.3. The following example illustrates some other issues that arise for left ideals in general Banach algebras, which obstruct following the r-ideal and hereditary subalgebra theory of operator algebras [Blecher et al. 2008; Blecher and Read 2011]. First, for $E \subset \mathfrak{F}_A$, it may be that $\overline{EA}$ has no left cai. Even if E has two elements this may fail, and, in this case, $\overline{EA}$ may not even equal $\overline{aA}$ for any $a \in A$. Thus, in general, the class of right $\mathfrak{F}$-ideals in noncommutative algebras is not closed under either finite sums or finite intersections (see Example 3.13). Also, it need not be the case that EAE has a bai if $E \subset \mathfrak{F}_A$. A simple three-dimensional example illustrating all of these points is the set of lower triangular 2×2 matrices with its norm as an operator on ℓ^1_2 (see [Smith and Ward 1978, Example 4.1]), and $E = \{E_{11} \pm E_{21}\}$.

Theorem 4.4. *Suppose that J is a right $\mathfrak{F}$-ideal in an approximately unital Banach algebra A. For every compact subset $K \subset J$, there exists $z \in J \cap \mathfrak{F}_A$ with $K \subset zJ \subset zA$.*

Proof. We may assume that A is unital, and follow the idea in the proof of Cohen's factorization theorem (see, e.g., [Pedersen 1998, Theorem 4.1] or [Dales 2000]).

For any $f_1, f_2, \ldots \in J \cap \mathfrak{F}_A$, define $z_n = \sum_{k=1}^n 2^{-k} f_k + 2^{-n} \in J + \mathbb{C}1$. We have

$$\|1 - z_n\| = \left\| \sum_{k=1}^n 2^{-k}(1 - f_k) \right\| \leq \sum_{k=1}^n 2^{-k} = 1 - 2^{-n},$$

and so by the Neumann lemma, $z_n^{-1} \in J + \mathbb{C}1$ and $\|z_n^{-1}\| \leq 2^n$.

Let (e_t) be a left cai for J in $\mathfrak{F}_A$, set $z_0 = 1$, and choose $\epsilon > 0$. For each $x \in K$, we have $\lim_t \|(1-e_t)z_n^{-1}x\| = 0$. Thus by the Arzelà–Ascoli theorem, and passing repeatedly to subnets, we can inductively choose a subsequence (f_n) of (e_t), and use these to inductively define z_n by the formula above, so that

$$\max_{x \in K} \left\| (1 - f_{n+1}) z_n^{-1} x \right\| \leq 2^{-n}\epsilon, \quad n \geq 0.$$

Set $z = \sum_{k=1}^\infty 2^{-k} f_k \in \overline{\mathrm{conv}}(e_n) \subset J \cap \mathfrak{F}_A$. If $x \in K$, set $x_n = z_n^{-1}x$. Then

$$\|x_{n+1} - x_n\| = \|z_{n+1}^{-1}(z_n - z_{n+1})z_n^{-1}x\| = \|2^{-n-1}z_{n+1}^{-1}(1 - f_{n+1})z_n^{-1}x\| \leq 2^{-n}\epsilon.$$

Hence $w = \lim_n x_n$ exists and $zw = x$. Note also that

$$\|x_n - x\| \leq \sum_{k=1}^n \|x_k - x_{k-1}\| \leq 2\epsilon,$$

so that $\|w - x\| \leq 2\epsilon$ if one wishes for that (so that $\|w\| \leq \|x\| + \epsilon$). ☐

Remark. In the case of operator algebras, or in the commutative case considered in Section 7, one can choose the z in the last result in $\mathrm{conv}(K)$, if K is, for example, a finite set in $J \cap \mathfrak{F}_A$. If A is noncommutative, this fails as we saw in Example 4.3.

Corollary 4.5. *Let A be an approximately unital Banach algebra. The closed right ideals with a countable left bai in $\mathfrak{r}_A$ are precisely the "principal right ideals" $\overline{zA}$ for some $z \in \mathfrak{F}_A$. Every separable right $\mathfrak{F}$-ideal is of this form.*

Proof. The one direction is easy since $(z^{1/n})$ is a left bai for $\overline{zA}$ (see the proof of Corollary 3.18). Conversely, if (e_n) is a countable left bai in $\mathfrak{r}_A$ for right ideal J, set $K = \{1/n e_n\}$ and apply Theorem 4.4.

For the last assertion, if $\{d_n\}$ is a countable dense set in a right $\mathfrak{F}$-ideal J, apply Theorem 4.4, with $K = \{d_n/(n\|d_n\|)\}$. There exists $z \in J \cap \mathfrak{F}_A$ with $K \subset \overline{zA}$. Hence $J \subset \overline{zA} \subset J$. ☐

Corollary 4.6. *The right $\mathfrak{F}$-ideals in an approximately unital Banach algebra A are precisely the closures of increasing unions of closed right $\mathfrak{F}$-ideals of the form $\overline{zA}$ for some $z \in \mathfrak{F}_A$.*

Proof. Suppose that J is an arbitrary right $\mathfrak{F}$-ideal in A. Let $\epsilon > 0$ be given (this is not needed for the proof but will be useful elsewhere). Let E be the left bai in $\mathfrak{F}_A$ considered as a set, and let Λ be the set of finite subsets of E ordered by inclusion. Define $z_G = x$ if $G = \{x\}$ for $x \in E$. For any two element set $G = \{x_1, x_2\}$ in Λ, one can apply Theorem 4.4 to obtain an element $z_G \in \mathfrak{F}_A$ with $GA \subset z_G A$, and, moreover, such that $x_k = z_G w_k$ with $\|w_k - x_k\| < \epsilon$ for each k, if one wishes for that. For any three element set $G = \{x_1, x_2, x_3\}$ in Λ, we can similarly choose $z_G \in \mathfrak{F}_A$ with $z_H A \subset z_G A$ for all proper subsets H of G (and with the "moreover" above too). Proceeding in this way, we can inductively choose for any n element set G in Λ an element $z_G \in \mathfrak{F}_A$ with $z_H A \subset z_G A$ for all proper subsets H of G (and, moreover, such that each such z_H can be written as $z_G w$ for some w with $\|w - z_H\| < \epsilon$, if one wishes for that). Thus $(\overline{z_G A})$ is increasing (as sets) with $G \in \Lambda$, and $\overline{\bigcup_{G \in \Lambda} z_G A} = J$.

Conversely, suppose that Λ is a directed set and that $J = \overline{\bigcup_t J_t}$, where $(J_t)_{t \in \Lambda}$ is an increasing net of subspaces of A, and $J_t = \overline{z_t A}$ for $z_t \in \mathfrak{F}_A$. Thus if $t_1 \le t_2$ then $J_{t_1} \subset J_{t_2}$, so that $s(z_{t_2}) z_{t_1} = z_{t_1}$. Hence $s(z_t)x \to x$ with t for all $x \in J$. Thus a weak* limit point p of $(s(z_t))_{t \in \Lambda}$ acts as a left identity for J, and hence is a left identity for $J^{\perp\perp}$. Thus $J^{\perp\perp} = pA^{**}$. Since this left identity p is in the weak* closure of the convex set $\mathfrak{F}_A \cap J$, the usual argument (see, e.g., p. 81 of [Blecher and Le Merdy 2004]) shows that J has a left bai in $\mathfrak{F}_A \cap J$. So J is a right $\mathfrak{F}$-ideal in A. ☐

Remark. (1) Note that $(z_G^{1/n})$ in the last proof is a left bai for the right ideal J there. This net is indexed by $n \in \mathbb{N}$ and $G \in \Lambda$. To see this, suppose $x \in J$ is given, and that $\|z_{G_1} a - x\| < \epsilon$, where $a \in A$. If $G_1 \subset G$ then $z_{G_1} \in z_G A$. By the proof of Corollary 4.6, we can choose w with $z_{G_1} = z_G w$ and $\|w\| \le 3$. Choose N such that $c_n < \epsilon/3$ for $n \ge N$, where c_n is as in Lemma 3.7. Then by that result, $\|z_G^{1/n} z_{G_1} - z_{G_1}\| = \|z_G^{1/n} z_G w - z_G w\| \le 3c_n < \epsilon$. Thus

$$\|z_G^{1/n} x - x\| \le \|z_G^{1/n} x - z_G^{1/n} z_{G_1} a\| + \|z_G^{1/n} z_{G_1} a - z_{G_1} a\| + \|z_{G_1} a - x\| < (3 + \|a\|)\epsilon$$

for all G containing G_1, and $n \ge N$. So $(z_G^{1/n})$ is a left bai for J.

(2) If $(z_G)_{G \in \Lambda}$ is as above, it is tempting to define $D = \overline{\bigcup_{G \in \Lambda} z_G A z_G}$. However, we do not see that this can be adjusted to make it an HSA.

In the operator algebra case, most of the following result and its proof were first in the preprint [Blecher and Read 2013b] (which, as we said on the first page, has now morphed into several papers). We thank Charles Read for discussions on that result in May 2013, and thank Garth Dales and Tomek Kania for conversations in the same period on algebraically finitely generated ideals in Banach algebras, and in particular, for drawing our attention to the results in [Sinclair and Tullo 1974] (these will not be used in the present proof below, but were used in an earlier version). We say that a right module Z over A is *algebraically countably generated* (resp. *algebraically finitely generated*) over A if there exists a countable (resp. finite) set $\{x_k\}$ in Z such that every $z \in Z$ may be written as a finite sum $\sum_{k=1}^n x_k a_k$ for some $a_k \in A$.

Corollary 4.7. *Let A be an approximately unital Banach algebra. A right $\mathfrak{F}$-ideal J in A is algebraically countably generated as a right module over A if and only if $J = qA$ for an idempotent $q \in \mathfrak{F}_A$. This is also equivalent to J being algebraically countably generated as a right module over A^1.*

Proof. Let J be a right $\mathfrak{F}$-ideal which is algebraically countably generated over A by elements $x_1, x_2, \ldots$ in A. We can assume that $\|x_k\| \to 0$, and so $\{x_k : k \in \mathbb{N}\}$ is compact. By Theorem 4.4, there exists $z \in J$ such that $\{x_k\} \subset zA$. Thus $x_k A \subset zA^2 = zA$ for all k, and so $J \subset zA \subset J$, and $J = zA$. By Corollary 3.23, $J = qA$ for an idempotent $q \in \mathfrak{F}_A$.

If J is algebraically countably generated over A^1 then by the above $J = qA^1$. Clearly $q \in A$, and so $J = \{x \in A : qx = x\} = qA$. ☐

Lemma 4.8. *Let A be an approximately unital Banach algebra, with a closed subalgebra D. If D has a bai from $\mathfrak{F}_A$, then for every compact subset $K \subset D$, there is $x \in D \cap \mathfrak{F}_A$ such that $K \subset xDx \subset xAx$.*

Proof. This can be done by adapting the proof of Theorem 4.4 as follows. We can inductively choose a subsequence (f_n) of the bai (e_n) with

$$\max_{x \in K}\big(\|(1 - f_{n+1})z_n^{-1}x\| + \|xz_n^{-1}(1 - f_{n+1})\|\big) \le 2^{-2n}\epsilon$$

for each n. Choose z as before. If $x \in K$, set $x_n = z_n^{-1}xz_n^{-1} \in D$. Then

$$\|x_{n+1} - x_n\| \le \|(z_{n+1}^{-1}x - z_n^{-1}x)z_{n+1}^{-1}\| + \|z_n^{-1}(xz_{n+1}^{-1} - xz_n^{-1})\|,$$

which is dominated by $2^{n+1}\|z_{n+1}^{-1}x - z_n^{-1}x\| + 2^n\|xz_{n+1}^{-1} - xz_n^{-1}\|$. Again we have $\|z_{n+1}^{-1}x - z_n^{-1}x\| \le 2^{-2n}\epsilon$, and similarly $\|xz_{n+1}^{-1} - xz_n^{-1}\| \le 2^{-2n}\epsilon$. So $\|x_{n+1} - x_n\| \le (2^{1-n} + 2^{-n})\epsilon < \epsilon/2^{n-2}$. Thus $w = \lim_n x_n$ exists in D, and $zwz = \lim_n z_n x_n z_n = x$ as desired. We also have $\|w - x\| \le 2\epsilon$ as before, if we wish for this. ☐

Remark. The above, and the next couple of results, are closely related to the results of Sinclair [1978], Esterle, and others on the Cohen factorization method, which also shows there is a commuting cai or bai under certain hypotheses. However the result above does not follow from Sinclair's results, and the latter do not directly connect to "positivity" in our sense.

Applying Lemma 4.8 to a suitable scaling of a countable bai in $\mathfrak{F}_A$, as in the proof of Corollary 4.5, we obtain:

Theorem 4.9. *Let A be an approximately unital Banach algebra, and let D be an inner ideal in A. Then D has a countable bai from $\mathfrak{F}_A$ (or equivalently, from $\mathfrak{r}_A$) if and only if there exists an element $z \in D \cap \mathfrak{F}_A$ with $D = \overline{zAz}$. Thus such D has a countable commuting bai from $\mathfrak{F}_A$. Any separable inner ideal in A with a bai from $\mathfrak{r}_A$ is of this form.*

The following is an Aarnes–Kadison-type theorem for Banach algebras. For another result of this type, see [Sinclair 1978].

Corollary 4.10. *If A is a subalgebra of a unital Banach algebra B, and we set $\mathfrak{r}_A = A \cap \mathfrak{r}_B$, then the following are equivalent*:

(i) *A has a sequential* (*commuting*) *bai from $\mathfrak{r}_A$.*

(ii) *There exists an $x \in \mathfrak{r}_A$ with $A = \overline{xAx}$.*

(iii) *There exists an $x \in \mathfrak{r}_A$ with $A = \overline{xA} = \overline{Ax}$.*

(iv) *There exists an $x \in \mathfrak{r}_A$ with $s(x)$, a mixed identity for A^{**}.*

Any separable Banach algebra with a bai from $\mathfrak{r}_A$ satisfies all of the above, as does any M-approximately unital Banach algebra which is separable or has a countable bai.

This is clear from earlier results. Indeed the last theorem gives the equivalence of (i) and (ii) above and the separability assertion, and that (ii) implies (iii) follows from Lemma 4.2, for example. Also (iii) implies (i) by considering $(x^{1/n})$, and (iii) is equivalent to (iv) by Corollary 3.19. Again, $\mathfrak{r}_A$ can be replaced by $\mathfrak{F}_A = A \cap \mathfrak{F}_B$ throughout this result, or in any of the items (i) to (iv).

As a consequence of the last results, if D is an HSA in an approximately unital Banach algebra A, and if D has a countable bai from $\mathfrak{F}_A$, then D is of the form in Lemma 4.2. We leave it to the reader to check that doing an "HSA variant" of the proof of Corollary 4.6, using Lemma 4.8 and mixed identities rather than left identities, yields:

Corollary 4.11. *The HSAs in an approximately unital Banach algebra A are exactly the closures of increasing unions of HSAs of the form $\overline{zAz}$ for $z \in \mathfrak{F}_A$.*

Proof. We just sketch the more difficult direction of this since this is so close to the proof of Corollary 4.6. Indeed we proceed as in the proof of Corollary 4.6, taking E to be the bai (e_t). Define Λ and $z_G \in D \cap \mathfrak{F}_A$ for $G \in \Lambda$ as before, but using Lemma 4.8. Note that each e_t is in some $\overline{z_G A z_G}$, which in turn is contained in the closed inner ideal $D' = \overline{\bigcup_{G \in \Lambda} z_G A z_G}$. Since for $x \in D$, we have $x = \lim_t e_t x e_t \in D' \subset D$, the result is now clear. □

Remark. As in the remark after Corollary 4.6, if one takes care with the choice of the z in the last corollary, the n-th roots of these z can be a bai for the HSA.

5. Better cai for M-approximately unital algebras

In this section we consider the better behaved class of M-approximately unital Banach algebras. We will use the fact that M-ideals in Banach spaces are *strongly proximinal*. (Actually the only "proximinality-type" condition we use here is "the strongly proximinal at 1 property" mentioned in the introduction.)

Lemma 5.1. *Let X be a Banach space, and suppose that J is an M-ideal in X, and $x \in X$, $y \in J$, and $\epsilon > 0$, with $\|x - y\| < d(x, J) + \epsilon$. Then there exists a $z \in J$ with $\|y - z\| < 3\epsilon$ and $\|x - z\| = d(x, J)$.*

Proof. This follows from the proof of [Harmand et al. 1993, Proposition II.1.1]. □

Theorem 5.2. *Let A be an M-approximately unital Banach algebra. Then $\mathfrak{F}_A$ is weak* dense in $\mathfrak{F}_{A^{**}}$, and $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$. Thus A has a cai in $\frac{1}{2}\mathfrak{F}_A$.*

Proof. This is easy if A is unital, so we will focus on the nonunital case. Suppose that $\eta \in A^{**}$ with $\|1 - \eta\| \le 1$. Suppose that (x_t) is a bounded net in A with weak* limit η in A^{**}, so that $1 - x_t \to 1 - \eta$ weak* in $(A^1)^{**}$. By Lemma 1.1, for any $n \in \mathbb{N}$, there exists a t_n such that for every $t \ge t_n$,

$$\inf\{\|1 - y\| : y \in \operatorname{conv}\{x_j : j \ge t\}\} < 1 + \frac{1}{2n}.$$

For every $t \ge t_n$, choose such a $y_t^n \in \operatorname{conv}\{x_j : j \ge t\}$ with $\|1 - y_t^n\| < 1 + 1/n$. If t does not dominate t_n, define $y_t^n = y_{t_n}^n$. So for all t, we have $\|1 - y_t^n\| < 1 + 1/n$. Writing (n, t) as i, we may view (y_t^n) as a net indexed by i, with $\|1 - y_t^n\| \to 1$. Given $\epsilon > 0$ and $\varphi \in A^*$, there exists a t_1 such that $|\varphi(x_t) - \eta(\varphi)| < \epsilon$ for all $t \ge t_1$. Hence $|\varphi(y_t^n) - \eta(\varphi)| \le \epsilon$ for all $t \ge t_1$ and all n. Thus $y_t^n \to \eta$ weak* with t. By Lemma 5.1, since $d(1, A) = 1$, we can choose $w_t^n \in A$ with $\|w_t^n - y_t^n\| < 3/n$ and $\|1 - w_t^n\| = 1$. Clearly $w_t^n \to \eta$ weak*.

That $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$ follows from this, and the idea in Proposition 3.5. We omit the details, since this also follows from Propositions 2.11 and 6.2.

Next, let e be the identity of A^{**}. By Lemma 2.4, we have that $e \in \frac{1}{2}\mathfrak{F}_{A^{**}}$. Suppose that (z_t) is a net in $\frac{1}{2}\mathfrak{F}_A$ with weak* limit e in A^{**}. Standard arguments (see, e.g., [Dales 2000, Proposition 2.9.16]) show that convex combinations w_t of the z_t have the property that $a w_t$ and $w_t a$ converge weakly to a for all $a \in A$. The usual argument (see, e.g., the proof of [Blecher et al. 2008, Theorem 6.1]) shows that further convex combinations are a cai in $\frac{1}{2}\mathfrak{F}_A$. □

Remark. For the first statements of Theorem 5.2, we do not need the full strength of the "M-approximately unital" condition, just strong proximinality at 1. For the existence of a cai in $\frac{1}{2}\mathfrak{F}_A$, the argument only uses strong proximinality at 1 and $\|1 - 2e\| \le 1$. Similarly, the existence of a bai in $\mathfrak{F}_A$ will follow from strong proximinality at 1 and $\|1 - e\| \le 1$.

Applied to operator algebras, the latter gives short proofs of a recent theorem of Read [2011] (see also [Blecher 2013]), as well as [Blecher and Read 2011, Lemma 8.1; 2013a, Theorem 3.3]. (We remark though that the proof of Read's theorem in [Blecher 2013] does contain useful extra information that does not seem to follow from the methods of the present paper, as is pointed out, for example, in Remark 2 after Theorem 2.1 in [Blecher and Read 2014].) Several other results from [Blecher and Read 2011] now follow from the last result, and with otherwise unchanged proofs, for M-approximately unital Banach algebras. For example:

Corollary 5.3 (cf. [Blecher and Read 2011, Corollary 1.5; Smith and Ward 1979, Theorem 2.8]). *If J is a closed two-sided ideal in a unital Arens regular Banach algebra A, and if J is M-approximately unital, and if the support projection of J in A^{**} is central there, then J has a cai (e_t) with $\|1-2e_t\| \leq 1$ for all t, which is also quasicentral (that is, $e_t a - a e_t \to 0$ for all $a \in A$).*

Corollary 5.4 (cf. [Blecher and Read 2011, Corollary 1.6]). *Let A be an M-approximately unital Banach algebra. Then A has a countable bai (f_n) if and only if A has a countable cai in $\frac{1}{2}\mathfrak{F}_A$. This is also equivalent (by Theorem 4.9) to $A = \overline{xAx}$ for some $x \in \mathfrak{F}_A$.*

Remark. We can also use the results in this section to develop a slightly different approach to hereditary subalgebras than the one taken in Section 4. For example, the following is a generalization of the phenomenon in the first example in [Blecher et al. 2008, Section 2], which can be interpreted as saying that for any contractive projection p in the multiplier algebra $M(A)$, pAp is an HSA in the sense of that paper. Suppose that A is an M-approximately unital Banach algebra, and that p is an idempotent in $M(A)$ with $\|1-2p\| \leq 1$. For simplicity, suppose that A is Arens regular. Define $D = pAp$. Note that D is an inner ideal in A. We claim that D has a bai in $\frac{1}{2}\mathfrak{F}_D$. To see this, note that by the usual arguments, $D^{\perp\perp} = pA^{**}p$. By Theorem 5.2, there is a net w_λ in $\frac{1}{2}\mathfrak{F}_A$ with $w_\lambda \to p$ weak*. Set $d_\lambda = pw_\lambda p$; then $d_\lambda \in \frac{1}{2}\mathfrak{F}_D$, and $d_\lambda \to p$ weak*. By the usual arguments, convex combinations of the d_λ give a cai for D in $\frac{1}{2}\mathfrak{F}_D$. It is easy to see that $\overline{DA} = pA$ and $\overline{AD} = Ap$ are the induced one-sided ideals, and (d_λ) is a one-sided cai for these.

6. Banach algebras and order theory

As we said earlier, $\mathfrak{r}_A$ and $\mathfrak{r}_A^{\mathfrak{e}}$ are closed cones in A, but are not proper in general (and hence are what are sometimes called *wedges*). By the argument at the start of Section 2 in [Blecher and Read 2014], $\mathfrak{c}_A = \mathbb{R}^+\mathfrak{F}_A$ is a proper cone. These cones naturally induce orderings: we write $a \preceq b$ (resp. $a \preceq_{\mathfrak{e}} b$) if $b-a \in \mathfrak{r}_A$ (resp. $b-a \in \mathfrak{r}_A^{\mathfrak{e}}$). These are preorderings, but are not in general antisymmetric. Because of this, some aspects of the classical theory of ordered linear spaces will not generalize. Certainly many books on ordered linear spaces assume that their cones are proper. However, other books (such as [Asimow and Ellis 1980] or [Jameson 1970]) do not make this assumption in large segments of the text, and it turns out that the ensuing theory interacts in a remarkable way with our recent notion of positivity, as we point out in this section and in [Blecher and Read 2014; 2013a]. For example, in the ordered space theory, the cone $\mathfrak{d} = \{x \in X : x \geq 0\}$ in an ordered space X is said to be *generating* if $X = \mathfrak{d} - \mathfrak{d}$. This is sometimes called *positively generating* or *directed* or *conormal*. If it is not generating, one often looks at the subspace $\mathfrak{d} - \mathfrak{d}$. In this language, we shall see next that $\mathfrak{r}_A$ and $\mathfrak{c}_A = \mathbb{R}^+ \mathfrak{F}_A$ are generating cones if A is M-approximately unital, or has a sequential cai and satisfies some further conditions of the type met in Section 2. We first discuss the order theory of M-approximately unital algebras.

Theorem 6.1. *Let A be an M-approximately unital Banach algebra. Any $x \in A$ with $\|x\| < 1$ may be written as $x = a - b$ with $a, b \in \mathfrak{r}_A$ and $\|a\| < 1$ and $\|b\| < 1$. In fact, one may choose such a, b to also be in $\frac{1}{2}\mathfrak{F}_A$.*

Proof. Assume that $\|x\| = 1$. Since $\mathfrak{F}_{A^{**}} = e + \mathrm{Ball}(A^{**})$ by Lemma 2.4, $x = \eta - \xi$ for $\eta, \xi \in \frac{1}{2}\mathfrak{F}_{A^{**}}$. We may assume that A is nonunital (the unital case follows from the last line with A^{**} replaced by A). By [Blecher and Read 2011, Lemma 8.1], we deduce that x is in the weak closure of the convex set $\frac{1}{2}\mathfrak{F}_A - \frac{1}{2}\mathfrak{F}_A$. Therefore it is in the norm closure, so given $\epsilon > 0$, there exists $a_0, b_0 \in \frac{1}{2}\mathfrak{F}_A$ with $\|x - (a_0 - b_0)\| < \epsilon/2$. Similarly, there exists $a_1, b_1 \in \frac{1}{2}\mathfrak{F}_A$ with $\|x - (a_0 - b_0) - \epsilon/2(a_1 - b_1)\| < \epsilon/2^2$. Continuing in this manner, one produces sequences $(a_k), (b_k)$ in $\frac{1}{2}\mathfrak{F}_A$. Setting $a' = \sum_{k=1}^{\infty}(1/2^k)a_k$ and $b' = \sum_{k=1}^{\infty}(1/2^k)b_k$, which are in $\frac{1}{2}\mathfrak{F}_A$ since the latter is a closed convex set, we have $x = (a_0 - b_0) + \epsilon(a' - b')$. Let $a = a_0 + \epsilon a'$ and $b = b_0 + \epsilon b'$. By convexity, $(1/(1+\epsilon))a \in \frac{1}{2}\mathfrak{F}_A$ and $(1/(1+\epsilon))b \in \frac{1}{2}\mathfrak{F}_A$.

If $\|x\| < 1$, choose $\epsilon > 0$ with $\|x\|(1+\epsilon) < 1$. Then $x/\|x\| = a - b$ as above, so that $x = \|x\|a - \|x\|b$. We have

$$\|x\|a = \big(\|x\|(1+\epsilon)\big) \cdot \Big(\frac{1}{1+\epsilon}a\Big) \in [0, 1) \cdot \tfrac{1}{2}\mathfrak{F}_A \subset \tfrac{1}{2}\mathfrak{F}_A,$$

and similarly $\|x\|b \in \frac{1}{2}\mathfrak{F}_A$. □

Remark. (1) If A is M-approximately unital then can every $x \in \mathrm{Ball}(A)$ be written as $x = a - b$ with $a, b \in \mathfrak{r}_A \cap \mathrm{Ball}(A)$? As we said above, this is true if A is unital. We are particularly interested in this question when A is an operator algebra (or uniform algebra). We can show that in general $x \in \mathrm{Ball}(A)$ cannot be written as $x = a - b$ with $a, b \in \frac{1}{2}\mathfrak{F}_A$. To see this let A be the set of functions in the disk algebra vanishing at -1, an approximately unital function algebra. Let W be the closed connected set obtained from the unit disk by removing the "slice" consisting of all complex numbers with negative real part and argument in a small open interval containing π. By the Riemann mapping theorem, it is easy to see that there is a conformal map h of the disk onto W taking -1 to 0, so that $h \in \mathrm{Ball}(A)$. By way of contradiction, suppose that $h = a - b$ with $a, b \in \frac{1}{2}\mathfrak{F}_A$. We use the geometry of circles in the plane: if $z, w \in \overline{B(\frac{1}{2}, \frac{1}{2})}$ with $|z - w| = 1$ then $z + w = 1$. It follows that $a + b = 1$ on a nontrivial arc of the unit circle, and hence everywhere (by [Hoffman 1962, p. 52]). However, $a(-1) + b(-1) = 0$, which is the desired contradiction.

(2) Applying Theorem 6.1 to ix for $x \in A$, one gets a similar decomposition $x = a - b$ with the "imaginary parts" of a and b positive. One might ask if, as is suggested by the C^*-algebra case, one may write for each ϵ, any $x \in A$ with $\|x\| < 1$ as $a_1 - a_2 + i(a_3 - a_4)$ for a_k with numerical range in a thin horizontal "cigar" of height less than ϵ centered on the line segment $[0, 1]$ in the x-axis. In fact this is false, as one can see in the case that A is the set of upper triangular 2×2 matrices with constant diagonal entries.

A bounded $\mathbb{R}$-linear $\varphi : A \to \mathbb{R}$ (resp. $\mathbb{C}$-linear $\varphi : A \to \mathbb{C}$) is called real positive if $\varphi(\mathfrak{r}_A) \subset [0, \infty)$ (resp. $\operatorname{Re} \varphi(\mathfrak{r}_A) \geq 0$). The set of real positive functionals on A is the *real dual cone*, and we write it as $\mathfrak{c}^{\mathbb{R}}_{A^*}$. Similarly, the "real version" of $\mathfrak{c}^{\mathfrak{e}}_{A^*}$ will be written as $\mathfrak{c}^{\mathfrak{e},\mathbb{R}}_{A^*}$. By the usual trick, for any $\mathbb{R}$-linear $\varphi : A \to \mathbb{R}$, there is a unique $\mathbb{C}$-linear $\tilde{\varphi} : A \to \mathbb{C}$ with $\operatorname{Re} \tilde{\varphi} = \varphi$, and clearly φ is real positive if and only if $\tilde{\varphi}$ is real positive.

Proposition 6.2. *Let A be an M-approximately unital Banach algebra. An $\mathbb{R}$-linear $f : A \to \mathbb{R}$ (resp. $\mathbb{C}$-linear $f : A \to \mathbb{C}$) is real positive if and only if f is a nonnegative multiple of the real part of a state (resp. nonnegative multiple of a state). Thus M-approximately unital algebras are scaled Banach algebras.*

Proof. The one direction is obvious. For the other, by the observation above the proposition, we can assume that $f : A \to \mathbb{C}$ is $\mathbb{C}$-linear and real positive. If A is unital then the result follows from the proof of [Magajna 2009, Theorem 2.2]. Otherwise by Proposition 3.2(4) applied to the inclusion $A \subset A^1$, we see that the condition in Corollary 2.8(iii) holds. So A is scaled by Corollary 2.8. (We remark that we had a different proof in an earlier draft.) ☐

We now turn to other classes of algebras (although we will obtain another couple of results for M-approximately unital algebras later in this section in parts (2) of Corollaries 6.7 and 6.8).

The following is a variant and simplification of [Blecher and Read 2013b, Lemma 2.7 and Corollary 2.9] and [Blecher and Read 2013a, Corollary 3.6].

Proposition 6.3. *Let A be an scaled approximately unital Banach algebra. Then the real dual cone $\mathfrak{c}^{\mathbb{R}}_{A^*}$ equals $\{t \operatorname{Re}(\psi) : \psi \in S(A), t \in [0, \infty)\}$. The prepolar of $\mathfrak{c}^{\mathbb{R}}_{A^*}$, which equals its real predual cone, is $\mathfrak{r}_A$, and the polar of $\mathfrak{c}^{\mathbb{R}}_{A^*}$, which equals its real dual cone, is $\mathfrak{r}_{A^{**}}$.*

Proof. It follows as in Proposition 6.2 that

$$\mathfrak{c}^{\mathbb{R}}_{A^*} = \big\{t \operatorname{Re}(\psi) : \psi \in S(A), t \in [0, \infty)\big\}.$$

The prepolar of $\mathfrak{c}^{\mathbb{R}}_{A^*}$, which equals its real predual cone, is $\mathfrak{r}_A$ by the bipolar theorem. We proved in Proposition 2.11 that $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$. This together with the bipolar theorem gives the last assertion. ☐

The following is a "Kaplansky density" result for $\mathfrak{r}_{A^{**}}$:

Proposition 6.4. *Let A be an approximately unital Banach algebra such that $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$ (as we saw in Proposition 2.11 was the case for scaled approximately unital algebras). Then the set of contractions in $\mathfrak{r}_A$ is weak* dense in the set of contractions in $\mathfrak{r}_{A^{**}}$. If, in addition, there exists a mixed identity of norm 1 in $\mathfrak{r}_{A^{**}}$, then A has a cai in $\mathfrak{r}_A$.*

Proof. We use a standard kind of bipolar argument from the theory of ordered spaces. If E and F are closed sets in a TVS with E compact, then $E+F$ is closed. By this principle, and by Alaoglu's theorem, $\mathrm{Ball}(A^*)+\mathfrak{c}_{A^*}$ is weak* closed. Its prepolar (resp. polar) certainly is contained in $\mathrm{Ball}(A)\cap\mathfrak{r}_A$ (resp. $\mathrm{Ball}(A^{**})\cap\mathfrak{r}_{A^{**}}$). This uses the fact that

$$(\mathfrak{c}_{A^*})^\circ = \mathfrak{r}_A^{\circ\circ} = \overline{\mathfrak{r}_A}^{\mathrm{w}*} = \mathfrak{r}_{A^{**}}$$

by the bipolar theorem. However, if $a\in\mathrm{Ball}(A)\cap\mathfrak{r}_A$ and $f\in\mathrm{Ball}(A^*)$ and $g\in\mathfrak{c}_{A^*}$, then $\mathrm{Re}(f(a)+g(a))\geq -1+0=-1$. So the prepolar of $\mathrm{Ball}(A^*)+\mathfrak{c}_{A^*}$ is $\mathrm{Ball}(A)\cap\mathfrak{r}_A$, and similarly its polar is $\mathrm{Ball}(A^{**})\cap\mathfrak{r}_{A^{**}}$. Thus $\mathrm{Ball}(A)\cap\mathfrak{r}_A$ is weak* dense in $\mathrm{Ball}(A^{**})\cap\mathfrak{r}_{A^{**}}$ by the bipolar theorem. The last assertion clearly follows from this and Lemma 2.1. $\square$

The condition in the next result that A^{**} is unital is a bit restrictive (it holds, for example, if A is Arens regular and approximately unital), but the result illustrates some of what one might like to be true in more general situations:

Theorem 6.5. *Let A be a Banach algebra such that A^{**} is unital, and suppose that $\mathfrak{e}$ is a cai for A. Then $\mathfrak{r}_A^{\mathfrak{e}}\subset\mathfrak{r}_{A^{**}}$ if and only if $\mathfrak{r}_A^{\mathfrak{e}}=\mathfrak{r}_A$. Suppose that the latter is true, and that $Q_{\mathfrak{e}}(A)$ is weak* closed. Then A is scaled, $S(A)=S_{\mathfrak{e}}(A)$, and A has a cai in $\mathfrak{r}_A$. Also in this case, $A=\mathfrak{r}_A-\mathfrak{r}_A$. Indeed, any $x\in A$ with $\|x\|<1$ may be written as $x=a-b$ for $a,b\in\mathfrak{r}_A\cap\mathrm{Ball}(A)$.*

Proof. If $f\in S(A)$ then by viewing $A^1=A+\mathbb{C}e$, we may extend f to a state $\hat{f}$ of A^{**}. If $x\in\mathfrak{r}_A^{\mathfrak{e}}\subset\mathfrak{r}_{A^{**}}$ then $\mathrm{Re}\, f(x)=\mathrm{Re}\,\hat{f}(x)\geq 0$. Thus $\mathfrak{r}_A^{\mathfrak{e}}\subset\mathfrak{r}_A$, and so these sets are equal. We also see that $\mathfrak{c}_{A^*}=\mathfrak{c}_{A^*}^{\mathfrak{e}}$. If $Q_{\mathfrak{e}}(A)$ is weak* closed then A is $\mathfrak{e}$-scaled by Lemma 2.7, so that $f=tg$ for some $g\in S_{\mathfrak{e}}(A)$ and for some t which must equal 1. It follows that $S(A)=S_{\mathfrak{e}}(A)$. Hence A is scaled, so that the weak* closure of $\mathfrak{r}_A\cap\mathrm{Ball}(A)$ is $\mathfrak{r}_{A^{**}}\cap\mathrm{Ball}(A^{**})$ by Proposition 6.4. Since the latter contains an identity, A has a cai in $\mathfrak{r}_A$ by the observation after that result. The assertion concerning $\|x\|<1$ follows by a slight variant of the proof of Theorem 6.1. $\square$

In fact it is not too hard to see, as we shall show in another paper, that if A^{**} is unital (or if it has a unique mixed identity), and A has a cai in $\mathfrak{r}_A$ then A has a cai in $\mathfrak{F}_A$ (and the latter cai can be chosen to be sequential if the first cai is sequential).

We now attempt to prove parts of the last theorem, and some other order theoretic results, in the case that A^{**} is not unital. We will mostly be using the class of states $S_{\mathfrak{e}}(A)$ with respect to a fixed cai $\mathfrak{e}$, and the matching cones $\mathfrak{r}_A^{\mathfrak{e}}$ and $\mathfrak{c}_{A^*}^{\mathfrak{e}}$, as opposed to $S(A)$ and its matching cones. The reason for this is that we will want norm additivity

$$\|c_1\varphi_1 + \cdots + c_n\varphi_n\| = c_1 + \cdots + c_n, \quad \varphi_k \in S(A), c_k \geq 0.$$

In many interesting examples, $S(A)$ satisfies this additivity property (for example, if A is Hahn–Banach smooth, by Lemma 2.2), and in this case almost all the rest of the results in this section will be true for the $S(A)$ variants, and with all the subscripts and superscripts and every hyphenated $\mathfrak{e}$ dropped.

Lemma 6.6. *Suppose that $\mathfrak{e} = (e_t)$ is a fixed cai for a Banach algebra A, and suppose that $Q_{\mathfrak{e}}(A)$ is weak* closed in A^*.*

(1) *The cones $\mathfrak{c}_{A^*}^{\mathfrak{e}}$ and $\mathfrak{c}_{A^*}^{\mathfrak{e},\mathbb{R}}$ are* additive (*that is, the norm on the dual space of A is additive on these cones*).

(2) *If (φ_t) is an increasing net in $\mathfrak{c}_{A^*}^{\mathfrak{e},\mathbb{R}}$ which is bounded in norm, then the net converges in norm, and its limit is the least upper bound of the net.*

Proof. (1) If $\psi = c\varphi$ for $\varphi \in S_{\mathfrak{e}}(A)$ and $c \geq 0$, then

$$\|\psi\| = c\|\varphi\| = \lim_t \psi(e_t).$$

Indeed, for an appropriate mixed identity e of A^{**} of norm 1, we have $\|\varphi\| = \langle e, \varphi\rangle$ for all $\varphi \in \mathfrak{c}_{A^*}^{\mathfrak{e},\mathbb{R}}$. It follows that the norm on $B(A,\mathbb{R})$ is additive on $\mathfrak{c}_{A^*}^{\mathfrak{e},\mathbb{R}}$. The complex scalar case is similar.

(2) It follows from (1) and [Asimow and Ellis 1980, Proposition 3.2, Chapter 2]. ☐

We recall that the positive part of the open unit ball of a C^*-algebra is a directed set. The following is a Banach algebra version of this:

Corollary 6.7. (1) *Let $\mathfrak{e}$ be a cai for a Banach algebra A, and suppose that $Q_{\mathfrak{e}}(A)$ is weak* closed in A^*. Then the open unit ball of A is a directed set with respect to the $\preceq_{\mathfrak{e}}$ ordering. That is, if $x, y \in A$ with $\|x\|, \|y\| < 1$, then there exists $z \in A$ with $\|z\| < 1$ and $z \in \mathfrak{r}_A^{\mathfrak{e}}$, and also $x \preceq_{\mathfrak{e}} z$ and $y \preceq_{\mathfrak{e}} z$.*

(2) *If A is an M-approximately unital Banach algebra, then given $x, y \in A$ with $\|x\|, \|y\| < 1$, a majorant z can be chosen as in* (1), *but also with $z \in \frac{1}{2}\mathfrak{F}_A$.*

Proof. (1) By Lemma 6.6(1) together with [Asimow and Ellis 1980, Corollary 3.6, Chapter 2], for any $x, y \in A$ with $\|x\| < 1$ and $\|y\| < 1$, there exists a $w \in A$ with $\|w\| < 1$ and $w - x, w - y \in \mathfrak{r}_A^{\mathfrak{e}}$. By the last assertion of Theorem 2.9 (setting the

a there to be $-tw$ for some appropriate $t > 1$), we have $w \preceq_{\mathfrak{e}} z$ for some $z \in \mathfrak{r}_A^{\mathfrak{e}}$ with $\|z\| < 1$. So

$$-z \preceq_{\mathfrak{e}} -w \preceq_{\mathfrak{e}} x \preceq_{\mathfrak{e}} w \preceq_{\mathfrak{e}} z.$$

Similarly, y "lies between" z and $-z$, from which it is easy to see that z is in $\mathfrak{r}_A^{\mathfrak{e}}$.

(2) This is similar to (1), but uses the fact that $S(A) = S_{\mathfrak{e}}(A)$ by Lemma 2.2, so every $\mathfrak{e}$ can be dropped. We also use the following principle twice in place of the cited results in the proof above: if $\|z\| < 1$ then by Theorem 6.1 we may write $z = a - b$ for $a, b \in \frac{1}{2}\mathfrak{F}_A$, and then $-b \preceq z \preceq a$. □

For a C^*-algebra B, a natural ordering on the positive part of the open unit ball of B turns the latter into a net which is a positive cai for B (see [Pedersen 1979]). A similar result holds for operator algebras [Blecher and Read 2014, Proposition 2.6]. We are not sure if there is an analogue of this for the classes of algebras in the last result.

Corollary 6.8. (1) *Let $\mathfrak{e}$ be a cai for a Banach algebra A, and suppose that $Q_{\mathfrak{e}}(A)$ is weak* closed in A^*. For all $x \in A$, there exists an element $z \in A$ with z in $\mathfrak{r}_A^{\mathfrak{e}}$ and $-z \preceq_{\mathfrak{e}} x \preceq_{\mathfrak{e}} z$. Thus $x = a - b$, where $a, b \in \mathfrak{r}_A^{\mathfrak{e}}$. Moreover, if $\|x\| < 1$ then z, a, b can all be chosen in* $\mathrm{Ball}(A)$.

(2) *If A is an M-approximately unital Banach algebra, then given $x \in A$ with $\|x\| < 1$, an element z can be chosen satisfying the inequalities in* (1), *but also with $z \in \frac{1}{2}\mathfrak{F}_A$.*

Proof. Apply Corollary 6.7 to x and $-x$. Clearly, $a = (z+x)/2$ and $b = (z-x)/2$. □

In the language of [Messerschmidt 2015], item (1) implies that the associated preorder on A is *approximately* 1-*absolutely conormal*, and from the theory of ordered Banach spaces in that reference, this is equivalent to $B(A, \mathbb{R})$ being "absolutely monotone". That is, with respect to the natural induced ordering on $B(A, \mathbb{R})$, if $-\psi \leq \varphi \leq \psi$ then $\|\varphi\| \leq \|\psi\|$.

Corollary 6.9. *Let $\mathfrak{e}$ be a cai for a Banach algebra A, and suppose that $Q_{\mathfrak{e}}(A)$ is weak* closed in A^*. If $f \leq g \leq h$ in $B(A, \mathbb{R})$ in the natural $\mathfrak{c}_{A^*}^{\mathfrak{e}}$-ordering, then $\|g\| \leq \|f\| + \|h\|$.*

Proof. This follows from Corollary 6.8 by [Batty and Robinson 1984, Theorem 1.1.4]. □

Corollary 6.10. *If A is an approximately unital Banach algebra then the last four results are true with all the subscripts and superscripts and every hyphenated $\mathfrak{e}$ dropped if also $S(A) = S_{\mathfrak{e}}(A)$ for the cai $\mathfrak{e}$ appearing in those results* (*which holds, for example, if A is Hahn–Banach smooth in A^1*).

Proof. Indeed, in the Hahn–Banach smooth case, $S(A) = S_{\mathfrak{e}}(A)$ by Lemma 2.2, and if the latter holds then all $\mathfrak{e}$ may be dropped. □

In the part of Corollary 6.10 dealing with Corollary 6.7(1), and with Corollary 6.8 in the $\|x\| < 1$ case, one may often get the majorants z appearing in those corollaries to also be in $\mathfrak{F}_A$ (and even get a sequential cai for A in $\mathfrak{F}_A$ consisting of such majorants z). We will discuss this in another paper, but briefly this follows from the ideas in Corollary 2.10 and the paragraphs after that, and the idea in the paragraph after Theorem 6.5.

Remark. (1) Above we saw that under various hypotheses, a Banach algebra A had a cai in $\mathfrak{r}_A$, and the latter was a generating cone, that is $A = \mathfrak{r}_A - \mathfrak{r}_A$. Conversely, we shall see in Corollary 7.6 that if A is commutative, approximately unital, and $A = \mathfrak{r}_A - \mathfrak{r}_A$, then A has a bai in $\mathfrak{F}_A$.

(2) It is probably never true for an approximately unital operator algebra A that $B(A, \mathbb{R}) = \mathfrak{c}^{\mathbb{R}}_{A^*} - \mathfrak{c}^{\mathbb{R}}_{A^*}$. Indeed, in the case $A = \mathbb{C}$, the latter space has real dimension 1. However, the complex span of the (usual) states of an approximately unital operator algebra A is A^* (the complex dual space). Indeed, by a result of Moore [1971] (see also [Asimow and Ellis 1972]), the complex span of the states of any unital Banach algebra A is A^*. In the approximately unital Banach algebra case, at least if A is scaled, the same fact follows by using a Hahn–Banach extension and Corollary 2.8(iii).

(3) Every element $x \in \frac{1}{2}\mathfrak{F}_A$ need not achieve its norm at a state, even in M_2 (consider $x = (I + E_{12})/2$, for example).

(4) We thank Miek Messerschmidt for calling our attention to the result in [Batty and Robinson 1984] used in Corollary 6.9. Previously we had a cruder inequality in that result.

(5) Note that A is not usually "order-cofinal" in A^1, in the sense of the ordered space literature, even for A any C^*-algebra with no countable cai (and hence no strictly real positive element).

7. Ideals in commutative Banach algebras

Throughout this section, A will be a commutative approximately unital Banach algebra. We will use ideas from [Blecher et al. 2008; Blecher and Read 2011; 2013a] (see [Esterle 1978; Kaniuth et al. 2010] for some other Banach algebra variants of some of these ideas). In the following statement, the "respectively"s are placed correctly, despite first impressions.

Theorem 7.1. *Let A be a commutative approximately unital Banach algebra. The closed ideals in A with a bai in $\mathfrak{r}_A$ (resp. $\mathfrak{F}_A$) are precisely the ideals of the form $\overline{EA}$ for some subset $E \subset \mathfrak{F}_A$ (resp. $E \subset \mathfrak{r}_A$). They are also the closures of increasing unions of ideals of the form $\overline{xA}$ for $x \in \mathfrak{F}_A$ (resp. $x \in \mathfrak{r}_A$).*

Proof. Suppose that $E \subset \mathfrak{r}_A$, and we will prove that $\overline{EA}$ has a bai in $\mathfrak{F}_A$. We may assume that $E \subset \mathfrak{F}_A$ since $\overline{EA} = \overline{\mathfrak{F}(E)A}$, as may be seen using Proposition 3.11. We will first suppose that E has two elements, and here we will include a separate argument if A is Arens regular since the computations are interesting. Then we will discuss the case where E has n elements, and then the general case.

If $x, y \in \mathfrak{r}_A$ then $\overline{xA}$ and $\overline{yA}$ are ideals with bais in $\mathfrak{F}_A$ by Corollary 3.18. Their support idempotents $s(x)$ and $s(y)$ are in $\mathfrak{F}_{A^{**}}$. Indeed if $J = \overline{xA}$ then by Corollary 3.18, we have $J^{\perp\perp} = s(x)A^{**}$, and $J = s(x)A^{**} \cap A$. (In the non-Arens regular case we are using the second Arens product here.) In the rest of this paragraph, we assume that A is Arens regular. Set

$$s(x, y) = s(x) + s(y) - s(x)s(y) = 1 - (1 - s(x))(1 - s(y)),$$

where $s(x, y)$ is an idempotent dominating both $s(x)$ and $s(y)$ in the sense that $s(x, y)s(x) = s(x)$ and $s(x, y)s(y) = s(y)$. If f is another idempotent dominating both $s(x)$ and $s(y)$ then $fs(x, y) = s(x, y)$, so that $s(x, y)$ is the “supremum” of $s(x)$ and $s(y)$ in this ordering. Then notice that $\|(1-x^{1/n})(1-y^{1/m})\| \leq 1$, and also

$$\big\|(1 - s(x))(1 - s(y))\big\| = \|1 - s(x, y)\| \leq 1.$$

Notice too that $\overline{xA + yA}$ has a bai in $\mathfrak{F}_A$ with terms of form

$$x^{1/n} + y^{1/m} - x^{1/n}y^{1/m} = 1 - (1 - x^{1/n})(1 - y^{1/m}),$$

which has bound 2. A double weak* limit point of this bai from $\mathfrak{F}_A \cap \overline{EA}$ is $s(x, y)$. So as usual $\overline{xA + yA} = \{a \in A : s(x, y)a = a\}$.

In the non-Arens regular case we use the second Arens product below. We show that $\overline{xA + yA} = \overline{((x + y)/2)A} = \overline{aA}$, where $a = (x + y)/2 \in \mathfrak{F}_A$. By the proof of [Blecher and Read 2011, Lemma 2.1], we know that $(1 - 1/n \sum_{k=1}^{n}(1 - a)^k) \in \mathfrak{F}_A$ is a bai for ba(a), and for $\overline{aA}$. Write $x = 1 - z, y = 1 - w$ for contractions $z, w \in A^1$, and let $b = (z + w)/2$. Then $a = 1 - b$. Let r be a weak* limit point of the bai above, which is a mixed identity for ba$(a)^{**}$. Then $ra = a$, so that $(1 - r)b = (1 - r)$. Note that $s = 1 - r$ is a contractive idempotent, and is an identity for $s(A^1)^{**}s$. Since the identity in a Banach algebra is an extreme point, and since $(sz + sw)/2 = s$, we deduce that $sz = zs = s$. Similarly $sw = ws = s$. Thus $rx = x$, so that $x \in rA^{**} \cap A = \overline{aA}$ (as in Corollary 3.18). This works similarly for y, and thus $\overline{xA + yA} = \overline{((x + y)/2)A}$. Thus if $x, y \in \mathfrak{F}_A$ then the support idempotent $s((x + y)/2)$ for a can be taken to be a “support idempotent” for $\overline{xA + yA}$.

A very similar argument works for three elements $x, y, z \in \mathfrak{F}_A$, using, for example, the fact that $\|(1 - x^{1/n})(1 - y^{1/n})(1 - z^{1/n})\| \leq 1$. Indeed, a similar argument works for any finite collection $G = \{x_1, \ldots, x_m\} \in \mathfrak{F}_A$. We have $\overline{GA} = \overline{x_G A}$, where

$$x_G = \tfrac{1}{m}(x_1 + \cdots + x_m) \in \mathfrak{F}_A \cap \overline{EA}.$$

Let us write $s(G)$ for $s((1/m)(x_1+\cdots+x_m))$. Then $s(G)$ is the support idempotent of $\overline{GA}$, and $s(G)A^{**} = (GA)^{\perp\perp}$, and thus $\overline{GA} = s(G)A^{**} \cap A$. This has a bai in $\mathfrak{F}_A \cap \overline{EA}$, namely $(1-[(1-x_1^{1/n})\cdots(1-x_m^{1/n})])$, or $(1-[(1-x_1^{1/n_1})\cdots(1-x_m^{1/n_m})])$.

If E is a subset of $\mathfrak{F}_A$, let $J = \overline{EA}$, and let Λ be the collection of finite subsets G of E ordered by inclusion. Writing Λ as a net $(G_i)_{i\in\Lambda}$, we have

$$J = \overline{EA} = \overline{\bigcup_{i\in\Lambda} G_i A} = \overline{\bigcup_{i\in\Lambda} x_{G_i} A},$$

where $x_{G_i} \in \mathfrak{F}_A \cap \overline{EA}$. To see that J has a bai in $\mathfrak{F}_A$, as in [Palmer 1994, Theorem 5.1.2(a)], it is enough to show that given $G \in \Lambda$ and $\epsilon > 0$, there exists $a \in \mathfrak{F}_A \cap J$ with $\|ax - x\| < \epsilon$ for all $x \in G$. However, this is clear since, as we saw above, $\overline{GA}$ has a bai in $\mathfrak{F}_A$.

Conversely, suppose that J is an ideal in A with a bai (x_t) in $\mathfrak{r}_A$. Then $J = \overline{\sum_t x_t A} = \overline{EA}$, where $E = \{\mathfrak{F}(x_t)\} \subset \mathfrak{F}_A$ by Proposition 3.11. The remaining results are clear from what we have proved. ☐

Remark. (1) See [Lau and Ülger 2014] for a recent characterization of ideals with bais.

(2) We saw in Example 4.3 that several of the methods used in the last proof fail for noncommutative algebras. First, it is not true there that if $x, y \in \mathfrak{F}_A$ then $\overline{xA + yA} = \overline{((x+y)/2)A}$. Also $\overline{xA + yA}$ may have no left cai. Also, it need not be the case that EAE has a bai if $E \subset \mathfrak{F}_A$.

If E is any subset of $\mathfrak{F}_A$ and $J = \overline{EA}$, and if $s = s_E$ is a weak* limit point of any bai in $\mathfrak{F}_A$ for J, then we call s a *support idempotent* for J. Note that $sA^{**} = J^{\perp\perp}$ as usual, and so $J = sA^{**} \cap A$.

Remark. Suppose that I is a directed set, and that $\{E_i : i \in I\}$ is a family of subsets of $\mathfrak{F}_A$ with $E_i \subset E_j$ if $i \leq j$. Then $\overline{\sum_i E_i A} = \overline{EA}$, where $E = \bigcup_i E_i$. Moreover, if s_i is a support idempotent for $\overline{E_i A}$, and if s_i has weak* limit point s' in A^{**} then we claim that s' is a support idempotent for $J = \overline{EA}$. Indeed, clearly $s' \in (J \cap \mathfrak{F}_A)^{\perp\perp}$, since each s_i resides here. Conversely, if $x \in E_i$ then $s_j x = x$ if $j \geq i$, so that $s'x = x$. Thus $s_i x \to x$ in norm for all $x \in J$, so that $s'x = x$ for all $x \in J$. Hence $s'x = x$ for all $x \in J^{\perp\perp}$. Therefore s' is idempotent, and $J^{\perp\perp} \subset s'A^{**}$, and so $J^{\perp\perp} = s'A^{**}$. As usual, $J = s'A^{**} \cap A$. This concludes the proof of the claim. If (x_t) is a net in $J \cap \mathfrak{F}_A$ with weak* limit s' then we leave it as an exercise that one can choose a net of convex combinations of the x_t, which is a bai for J in $\mathfrak{F}_A$ with weak* limit s'. In particular, if $(G_i)_{i\in\Lambda}$ is as in the proof of Theorem 7.1, then the net $s_i = s(G_i)$ has a weak* limit point which is a support projection for $J = \overline{EA}$.

Let us define an $\mathfrak{F}$-*ideal* to be an ideal of the kind characterized in Theorem 7.1, namely a closed ideal in A with a bai in $\mathfrak{r}_A$.

Theorem 7.2. *Let A be a commutative approximately unital Banach algebra. Any separable $\mathfrak{F}$-ideal in A is of the form $\overline{xA}$ for $x \in \mathfrak{F}_A$. Also, the closure of the sum of a countable set of ideals $\overline{x_k A}$ for $x_k \in \mathfrak{F}_A$, equals $\overline{zA}$, where $z = \sum_{k=1}^{\infty}(1/2^k)x_k$.*

Proof. The first assertion follows from the matching result Corollary 4.5, or from the second assertion as in [Blecher and Read 2011, Theorem 2.16]. For the second assertion, let x_k, z be as in the statement. Inductively one can prove that $x_k \in \overline{zA}$, which is what is needed. One begins by setting $x = x_1$ and $y = \sum_{k=2}^{\infty}(1/2^{k-1})x_k \in \mathfrak{F}_A$. Then $z = (x+y)/2$, and the third paragraph of the proof of Theorem 7.1 shows that $x = x_1 \in \overline{zA}$, and $y \in \overline{zA}$. One then repeats the argument to show all $x_k \in \overline{zA}$. ☐

As in Section 4, we obtain again that, for example:

Corollary 7.3. *Let A be a commutative M-approximately unital Banach algebra. Then A has a countable cai if and only if there exists $x \in \mathfrak{F}_A$ with $A = \overline{xA}$ (or equivalently, if and only if $s(x)$ is the unique mixed identity of A^{**} of norm 1).*

With this in hand, one can generalize some part of the theory of left ideals and cais in [Blecher et al. 2008; Blecher and Read 2011; 2013a] to the class of ideals in the last theorem, in the commutative case. This class is not closed under finite intersections. In fact, this fails rather badly (see Example 3.13). One may define an $\mathfrak{F}$-*open idempotent* in A^{**} to be an idempotent $p \in A^{**}$ for which there exists a net (x_t) in $\mathfrak{F}_A$ (or equivalently, as we shall see, in $\mathfrak{r}_A$) with $x_t = px_t \to p$ weak*. Thus a left identity for the second Arens product in A^{**} is $\mathfrak{F}$-open if and only if it is in the weak* closure of $\mathfrak{F}_A$. See [Akemann 1970; Pedersen 1979] for the notion of open projection in a C^*-algebra.

Lemma 7.4. *If A is a commutative approximately unital Banach algebra then the $\mathfrak{F}$-open idempotents in A^{**} are precisely the support idempotents for $\mathfrak{F}$-ideals.*

Proof. If p is an $\mathfrak{F}$-open idempotent then it follows that $p \in \mathfrak{F}_{A^{**}}$, and that $J = \overline{EA}$ is an $\mathfrak{F}$-ideal, where $E = \{x_t\}$ (using Theorem 7.1). Also $px = x$ if $x \in J$, and $p \in J^{\perp\perp}$. So $pA^{**} = J^{\perp\perp}$, from which it is easy to see that p is a support idempotent of J.

The converse is obvious by the definition of support idempotent above, and the fact that $\overline{EA} = s_E A^{**} \cap A$. ☐

Corollary 7.5. *If A is a commutative approximately unital Banach algebra, and $E \subset \mathfrak{r}_A$, then the closed subalgebra generated by E has a bai in $\mathfrak{F}_A$.*

Proof. In Theorem 7.1 we constructed a bai in $\mathfrak{F}_A$ for $\overline{EA}$, and this bai is clearly in the closed subalgebra generated by E, and is a bai for that subalgebra. ☐

If A is any approximately unital commutative Banach algebra, define $A_H = \overline{\mathfrak{F}_A A}$. This is an ideal of the type in Theorem 7.1, and is the largest such (by that result).

If A is an operator algebra, it is proved in [Blecher and Read 2013a] that $A = \mathfrak{r}_A - \mathfrak{r}_A$ if and only if A has a cai. In our setting we at least have:

Corollary 7.6. *If A is a commutative approximately unital Banach algebra which is generated by $\mathfrak{r}_A$ as a Banach algebra* (*and certainly if $A = \mathfrak{r}_A - \mathfrak{r}_A$*), *then A has a bai in $\mathfrak{F}_A$.*

Proof. This follows from Corollary 7.5 because A is generated by $\mathfrak{r}_A$ in this case, and hence is generated by $\mathfrak{F}_A$ since $\mathfrak{r}_A = \overline{\mathbb{R}^+ \mathfrak{F}_A}$. ☐

Conversely, if A is M-approximately unital or has a sequential cai satisfying certain conditions discussed in Section 6, then we saw in Section 6 that $A = \mathfrak{r}_A - \mathfrak{r}_A$. Indeed, we saw in the M-approximately unital case in Theorem 6.1 that

$$A = \mathbb{R}^+(\mathfrak{F}_A - \mathfrak{F}_A) \subset \mathfrak{r}_A - \mathfrak{r}_A \subset A.$$

We do not know if it is always true if, as in the operator algebra case, for any approximately unital commutative Banach algebra we have $A_H = \mathfrak{r}_A - \mathfrak{r}_A = \mathbb{R}^+(\mathfrak{F}_A - \mathfrak{F}_A)$.

8. M-ideals which are ideals

We now turn to an interesting class of closed approximately unital ideals in a general approximately unital Banach algebra that generalizes the class of approximately unital closed two-sided ideals in operator algebras. (Unfortunately, we see no way yet to apply the theory in [Blecher and Zarikian 2006] to generalize the results in this section to one-sided ideals.) The study of this class was initiated in [Smith and Ward 1978; 1979; Smith 1979]. We will use basic ideas from these papers (see also Werner's theory of inner ideals in the sense of [Harmand et al. 1993, Section V.3]).

First, let A be a unital Banach algebra. We define an *M-ideal ideal* in A to be a subspace J of A which is an M-ideal in A, such that if P is the M-projection then $z = P1$ is central in A^{**} (the latter is automatic, for example, if A is commutative and Arens regular). Actually it suffices in all the arguments below that simply $za = az$ for $a \in A$, but for convenience we will stick to the "central" hypothesis. By [Smith and Ward 1978, Proposition 3.1], z is a hermitian projection of norm 1 (or 0). It is then a consequence of Sinclair's theorem on hermitians [Sinclair 1971] that z is accretive, indeed $W(z) \subset [0, 1]$. The proof of [Smith and Ward 1978, Proposition 3.4] shows that $(1 - z)J^{\perp\perp} = (0)$ (it is shown there that $zJ^{\perp\perp}z \subset J^{\perp\perp} = J_1$ in the notation there, and that $(1-z)J \subset J_2$, but clearly $zJ \subset J_1$ so that $(1 - z)J \subset (J - J_1) \cap J_2 \subset J_1 \cap J_2 = (0)$). It also shows that $z(I - P)A^{**} = 0$, so that P is simply left multiplication by z, and

$J^{\perp\perp} = zA^{**}$. Since the latter is an ideal, $J = J^{\perp\perp} \cap A$ is an ideal in A. Moreover, J is approximately unital since z is a mixed identity for $J^{\perp\perp}$ of norm 1. We call z the support projection of J, and write it as s_J. The correspondence $J \mapsto s_J$ is bijective on the class of M-ideal ideals.

Proposition 8.1. *An M-ideal ideal J in a unital Banach algebra A is M-approximately unital, indeed J has a cai in $\frac{1}{2}\mathfrak{F}_A$. Also J is a two-sided $\mathfrak{F}$-ideal in A, and $J = \overline{EA} = \overline{AE}$ for some subset $E \in J \cap \mathfrak{F}_A$.*

Proof. By Proposition 3.2, J is M-approximately unital, so by Theorem 5.2 it has a cai in $\frac{1}{2}\mathfrak{F}_J = J \cap \frac{1}{2}\mathfrak{F}_A$. (The latter equality follows from Proposition 3.2 applied in A^1.) Thus J is a two-sided $\mathfrak{F}$-ideal. We also deduce from Proposition 3.2 that $J^1 \cong J + \mathbb{C}1_A$. Hence $J = \overline{EA} = \overline{AE}$ for some $E \subset J \cap \mathfrak{F}_A$; for example, take E to be the cai above. □

The converse of the last result fails. Indeed even in a commutative algebra, not every ideal $\overline{EA}$ for a subset $E \in \mathfrak{F}_A$, is an M-ideal ideal, nor need have a cai in $\frac{1}{2}\mathfrak{F}_A$ (see Example 3.14).

Suppose that J_1 and J_2 are M-ideal ideals in A, and that P_1, P_2 are the corresponding M-projections on A^{**} with $z_k = P_k 1$ central in A^{**}. As in Corollary 3.19, $J_1 \subset J_2$ if and only if $z_2 z_1 = z_1$, and the latter equals $z_1 z_2$. So the correspondence $J \mapsto s_J$ is an order embedding with respect to the usual ordering of projections in A^{**}. Then by facts above, $P_1 P_2(1) = P_1(z_2) = z_1 z_2$, and this is central in A^{**}. Similarly, $(P_1 + P_2 - P_1 P_2)1 = z_1 + z_2 - z_1 z_2$, and this is central in A^{**}. Hence $J_1 \cap J_2$ and $J_1 + J_2$ are M-ideal ideals in A.

To describe the matching fact about "joins" of an infinite family of ideals, we introduce some notation. Set N to be A^{**}. We will use the fact that N contains a commutative von Neumann algebra. We recall that the *centralizer* $Z(X)$ of a dual Banach space X is a weak* closed subalgebra of $B(X)$, and it is densely spanned in the norm topology by its contractive projections, which are the M-projections (see, e.g., [Harmand et al. 1993] and [Blecher and Zarikian 2006, Section 7.1]). It is also a commutative W^*-algebra in the weak* topology from $B(X)$. By [Harmand et al. 1993, Theorem V.2.1]), the map $\theta : Z(N) \to N$ taking $T \in Z(N)$ to $T(1)$ is an isometric homomorphism, and it is weak* continuous by the definition of the weak* topology on $B(N)$ and hence on $Z(N)$. Therefore by the Krein–Smulian theorem, the range of θ is weak* closed, and θ is a weak* homeomorphism onto its range. Thus $Z(N)$ is identifiable with a weak* closed subalgebra Δ of N, which is a commutative W^*-algebra, via the map $T \mapsto T(1)$. All computations can be done inside this commutative von Neumann algebra. Indeed the ordering of support projections z_1, z_2, and their "meet" and "join", which we met a couple of paragraphs above, are simply the standard operations $z_1 \leq z_2, z_1 \vee z_2, z_1 \wedge z_2$ with projections, computed in the W^*-algebra Δ. Of course, we are specifically interested in the weak* closed subalgebra consisting of elements in Δ that commute with A. The projections in this subalgebra densely span a commutative von Neumann algebra inside Δ.

Lemma 8.2. *The closure of the span of a family $\{J_i : i \in I\}$ of M-ideal ideals in a unital Banach algebra A is an M-ideal ideal in A.*

Proof. Let $\{P_i : i \in I\}$ be the corresponding family of M-projections on A^{**} with $z_i = P_i 1$ central in A^{**}. Let Λ be the collection of finite subsets of I ordered by inclusion. For $F \in \Lambda$, let $J_F = \sum_{i \in F} J_i$; by the above, this will be an M-ideal ideal in A whose support projection s_{J_F} corresponds to $P_F(1)$, where P_F is the M-projection for J_F. Next suppose that (P_F) has weak* limit P in $Z(N)$; by the theory of M-projections, P is the M-projection corresponding to the M-ideal $J = \overline{\sum_i J_i} = \overline{\sum_{F \in \Lambda} J_F}$. We have $P(1) = z$ is the weak* limit of the (z_i); this is a contractive hermitian projection in the ideal $J^{\perp\perp}$. For $\eta \in N$, we have $z\eta \in J^{\perp\perp}$ so that

$$z\eta = P(z\eta) = \lim_i P_i(z\eta) = \lim_i z_i z\eta = \lim_i z_i \eta = \lim_i \eta z_i = \eta z.$$

Thus z is central in N, and so J is an M-ideal ideal with support projection z, and z is the supremum $\vee_i z_i$ in Δ. ☐

Next assume that A is an approximately unital Banach algebra. We define an *M-ideal ideal* in A to be a subspace J of A which is an M-ideal in A^1 such that $z = P1$ is central in A^{**} (or, as we said above, simply that $za = az$ for $a \in A$, which will then allow an M-approximately unital A to always be an M-ideal ideal in itself). We may then apply the theory in the last several paragraphs to A^1; thus $N = (A^1)^{**}$ there. Set Δ' to be the weak* closure in Δ of the span of those projections that happen to be in A^{**}. This is also a commutative W^*-algebra.

Theorem 8.3. *If A is an approximately unital Banach algebra then the class of M-ideal ideals in A forms a lattice*; *indeed*, *the intersection of a finite number*, *or the closure of the sum of any collection*, *of M-ideal ideals is again an M-ideal ideal. The correspondence between M-ideal ideals J in A and their support projections s_J in $\Delta' \subset A^{**}$ is bijective and preserves order*, *and preserves finite meets and arbitrary joins . That is*, *$s_{J_1 \cap J_2} = s_{J_1} s_{J_2}$ for M-ideal ideals J_1, J_2 in A*; *and if $\{J_i : i \in I\}$ is any collection of M-ideal ideals in A and J is the closure of their span*, *then s_J is the supremum in $\Delta' \subset A^{**}$ of $\{s_{J_i} : i \in I\}$.*

Proof. This result is essentially a summary of some facts above with these facts applied to A^1 instead of A, and with $N = (A^1)^{**}$. ☐

Clearly any M-ideal ideal in A is Hahn–Banach smooth in A^1 [Harmand et al. 1993], and hence in A.

If J is an M-ideal ideal then we call s_J above a *central open projection* in A^{**}. Clearly such open projections p are weak* limits of nets $x_t \in \frac{1}{2}\mathfrak{F}_A$ with $px_t =$

$x_t p = x_t$. However, not every projection in A^{**} which is such a weak* limit is the support idempotent of an M-ideal ideal (again, see Example 3.14). Nonetheless we expect to generalize more of the theory in [Blecher et al. 2008; Blecher and Read 2011; 2013a] of open projections and r-ideals to this setting. For a start, it is now clear that suprema of any collection, and infima of finite collections, of central open projections are central open projections. If A is an M-approximately unital Banach algebra then the mixed identity e for A^{**} of norm 1 is a central open projection.

Proposition 8.4. *If A is an approximately unital Banach algebra then any central open projection is lower semicontinuous on $Q(A)$.*

Proof. If A is unital then this result is in [Smith and Ward 1979], and we use this below. Let $\varphi_t \to \varphi$ weak* in $Q(A)$, and suppose that $\varphi_t(p) \leq r$ for all t. Write $\varphi_t = c_t \psi_t$ for $\psi_t \in S(A)$, and let $\widehat{\psi}_t \in S(A^1)$ be a state extending ψ_t. By replacing by a subnet, we can assume that $c_t \to s \in [0, 1]$. A further subnet $\widehat{\psi}_{t_\nu}$ converges to $\rho \in S(A^1)$ weak*. Thus $\varphi = s\rho_{|A}$, since

$$\varphi_{t_\nu}(a) = c_{t_\nu} \psi_{t_\nu}(a) = c_{t_\nu} \widehat{\psi}_{t_\nu}(a) \to s\rho(a), \quad a \in A.$$

By the result from [loc. cit.] mentioned above,

$$\rho(p) \leq \liminf_\nu \widehat{\psi}_{t_\nu}(p) = \liminf_\nu \psi_{t_\nu}(p).$$

Hence

$$\varphi(p) = s\rho(p) \leq \liminf_\nu s\psi_{t_\nu}(p) = \liminf_\nu c_{t_\nu} \psi_{t_\nu}(p) \leq r,$$

as desired. □

Given a central open projection $p \in A^{**}$, we set $F_p = \{\varphi \in Q(A) : \varphi(p) = 0\}$.

Theorem 8.5. *Suppose that A is a scaled approximately unital Banach algebra, and p is a central open projection in A^{**}, and $J = pA^{**} \cap A$ is the corresponding ideal. Then $F_p = Q(A) \cap J^\perp$, and this is a weak* closed face of $Q(A)$. Moreover, the assignment Θ taking $p \mapsto F_p$ (resp. $J \mapsto F_p$) from the set of central open projections (resp. M-ideal ideals of A) into the set of weak* closed faces of $Q(A)$, is one-to-one and is a (reverse) order embedding. Moreover, "suprema" (that is, joins of arbitrary families) are taken by Θ to intersections of the corresponding faces.*

Proof. If $J = pA^{**} \cap A$ and $\varphi \in Q(A) \cap J^\perp$ then $\varphi \in F_p$ since $p \in J^{\perp\perp}$. Conversely, if $\varphi \in F_p$ has norm 1 then we have

$$1 = \|\varphi\| = \|\varphi \cdot p\| + \|\varphi \cdot (1 - p)\| \geq |\varphi(1 - p)| = 1.$$

Thus $\varphi \cdot p = 0$, and so $\varphi \in Q(A) \cap J^\perp$.

If $\varphi \in F_p$ and $\varphi = t\psi_1 + (1 - t)\psi_2$ for $\psi_1, \psi_2 \in Q(A)$ and $t \in [0, 1]$, then it is clear that $\psi_1, \psi_2 \in F_p$. So F_p is a face of $Q(A)$. Since $F_p = Q(A) \cap J^\perp$, it is weak* closed.

Write $F_p^1 = \{\varphi \in S(A^1) : \varphi(p) = 0\}$. Suppose that $\varphi_t \to \varphi \in Q(A)$ weak*, with $\varphi_t \in F_p$ and $\varphi \neq 0$. Suppose that $\varphi_t = c_t \psi_t$ with $\psi_t \in S(A)$. We may assume that $\psi_t \in S(A^1)$, and then $\psi_t \in F_p^1$. By [Smith and Ward 1978; 1979], F_p^1 is weak* closed, so we have a weak* convergent subnet $\varphi_{t_\mu} \to \psi \in F_p^1$. A further subnet of the c_{t_μ} converges to $c \in [0,1]$ say. In fact, $c \neq 0$ or else φ_{t_μ} has a norm null subnet, so that $\varphi = 0$. Now it is clear that $c\psi_{|A} = \varphi \in F_p$. So F_p is weak* closed.

If we have two central open projections $p_1 \leq p_2$ then $w = p_2 - p_1$ is a hermitian projection in $(A^1)^{**}$, so that as we said above $W(z) \subset [0,1]$. Thus it is clear that $\varphi(p_1) \leq \varphi(p_2)$ for states $\varphi \in S(A)$. Hence $F_{p_2} \subset F_{p_1}$.

Conversely, suppose that $F_{p_2} \subset F_{p_1}$. If $\varphi \in F_{p_2}^1$ and φ is nonzero on A then, since it is real positive on A, it will be a positive multiple of a state ψ on A. We have $\psi \in F_{p_2} \subset F_{p_1}$, so that $\varphi \in F_{p_1}^1$. That is, $F_{p_2}^1 \subset F_{p_1}^1$. We are now in the setting of [Smith and Ward 1978; 1979], from where we see that these are split faces of $S(A^1)$ and are weak* closed. Let $N_1 \subset N_2$ be the complementary split faces. We may view p_1, p_2 as affine lower semicontinuous functions f_1, f_2 on $S(A^1)$. As in those references, we have $f_k = 0$ on $F_{p_k}^1$, and $f_k = 1$ on N_k. From this and the theory of split faces [Alfsen 1971, Section II.6], it is easy to see that $f_1 \leq f_2$. That is, $\varphi(p_2 - p_1) \geq 0$ for all $\varphi \in S(A^1)$. By [Magajna 2009], this is also true if $\varphi \in S((A^1)^{**})$, and hence if $\varphi \in S(\Delta)$. Therefore $p_1 \leq p_2$ in Δ, so that indeed $p_1 \leq p_2$ in the usual ordering of projections in A^{**}.

The last assertion follows from the identity

$$Q(A) \cap \Big(\sum_i J_i\Big)^{\perp} = \bigcap_i (Q(A) \cap J_i^{\perp}). \qquad \square$$

Note that the support projection $s(x) \notin \Delta$ in general if $x \in \mathfrak{F}_A$. This can be overcome by restricting to the class where this is true — but unfortunately this class seems often only to be interesting if A is commutative. Thus if A is an approximately unital Banach algebra, write $\mathfrak{F}'_A$ for the set of $x \in \mathfrak{F}_A$ such that multiplying on the left by $s(x)$ in the second Arens product is an M-projection on $N = (A^1)^{**}$, and $s(x)$ is commutes with A^1 (again the latter is automatic if A is commutative and Arens regular). (Note that if A is M-approximately unital then multiplying on the left by $s(x)$ is an M-projection on A^{**} if and only if it is an M-projection on $(A^1)^{**}$.) Define an m-ideal in A to be an ideal of form $\overline{EA}$ for a subset $E \subset \mathfrak{F}'_A$. If A is also a commutative operator algebra then the m-ideals in A are exactly the closed ideals with a cai, by the characterization of r-ideals in [Blecher and Read 2011] (see also [Effros and Ruan 1990]), since in this case $\mathfrak{F}'_A = \mathfrak{F}_A$.

Proposition 8.6. *If A is an approximately unital Banach algebra then any m-ideal in A is an M-ideal ideal in A.*

Proof. Suppose that $x \in \mathfrak{F}'_A$. Setting $J_x = \overline{xA} \subset s(x)A^{**} \cap A$, we have $J_x^{\perp\perp} = s(x)A^{**} = s(x)N$, as in the proof of Corollary 3.18. So $J_x = s(x)A^{**} \cap A$ is an M-ideal ideal. Then $\overline{EA} = \overline{\sum_{x \in E} xA}$ is also an M-ideal ideal by Theorem 8.3. $\square$

The above class is perhaps also a context to which there is a natural generalization of some of the results in [Blecher et al. 2008; Blecher and Read 2011; 2013a; Hay 2007] related to noncommutative peak interpolation, and noncommutative peak and p-sets (see [Blecher 2013] for a short survey of this topic). However, one should not expect the ensuing theory to be particularly useful for noncommutative algebras since the projections in this section are all "central".

Indeed it is unlikely that one could generalize to general Banach algebras the main noncommutative peak interpolation results surveyed in [Blecher 2013], or see [Hay 2007; Blecher et al. 2008; Blecher and Read 2013a; 2014]. However, we end with one nice noncommutative peak interpolation result concerning M-ideal ideals in general Banach algebras, which can also be viewed as a "noncommutative Tietze theorem". In particular, it also solves a problem that arose at the time of [Blecher and Read 2013a], and was mentioned in [Blecher and Read 2013b], namely whether $\mathfrak{r}_{A/J} = q_J(\mathfrak{r}_A)$ when J is an approximately unital ideal in an operator algebra A, and $q_J : A \to A/J$ is the quotient map. In [Blecher and Read 2011], it was shown that $\mathfrak{F}_{A/J} = q_J(\mathfrak{F}_A)$, and it is easy to see that $q_J(\mathfrak{r}_A) \subset \mathfrak{r}_{A/J}$. In fact a much more general fact is true. The main new ingredient needed is [Chui et al. 1977, Theorem 3.1]. Their proof of this result, while remarkable and deep, clearly contains misstatements. However, we were able to confirm that (a small modification of) their proof works at least in the case of unital Banach algebras. For the reader's interest, we will give a rather different, and more direct, proof of their full result.

Let (X, e) be a pair consisting of a Banach space X and an element $e \in X$ such that $\|e\| \le 1$. Let

$$S_e(X) = \{\varphi \in X^* : \|\varphi\| = 1 = \varphi(e)\} \quad \text{and} \quad W(x) = W_X^e(x) = \{\varphi(x) : \varphi \in S_e(X)\}$$

denote respectively the state space and the numerical range of $x \in X$, relative to e. Of course, these are empty if $\|e\| < 1$. Below we write $B(\lambda, r)$ for the closed disk centered at λ of radius r. The following formula in the Banach algebra case is attributed to Williams in [Bonsall and Duncan 1973], and it may be proved by a tiny modification of the proof at the end of page 1 there.

Lemma 8.7 (Williams formula). *For every $x \in X$, one has*

$$W(x) = \bigcap_{\lambda \in \mathbb{C}} B(\lambda, \|x - \lambda e\|).$$

In particular, $W_X^e(x) = W_{X^{**}}^e(x)$ for every $x \in X$.

Theorem 8.8 (Chui, Smith, Smith, and Ward). *Let (X, e) be as above. Suppose that J is an M-ideal in X and $x \in X$ is such that $W_{X/J}^{Q(e)}(Q(x)))$ has nonempty interior, where $Q: X \to X/J$ is the quotient map. Then there exists $y \in J$ such that*

$$\|x - y\|_X = \|Q(x)\|_{X/J} \quad \text{and} \quad W_X^e(x - y) = W_{X/J}^{Q(e)}(Q(x)).$$

Proof. For a bounded convex subset $C \subset \mathbb{C}$, $\alpha \in C$, and $\epsilon > 0$, we define

$$N(C, \alpha, \epsilon) = \{\alpha + (1+\epsilon)(\gamma - \alpha) : \gamma \in C\}.$$

It is an exercise to show that the $N(C, \alpha, \epsilon)$ are open convex neighborhoods of C if $\alpha \in \operatorname{int}(C)$, and they shrink as ϵ decreases.

Let $x \in X$ be given, and fix $\alpha \in \operatorname{int}(W_{X/J}^{Q(e)}(Q(x)))$. Then $|\alpha| < \|Q(x)\|$. Now

$$N(W_{X/J}^{Q(e)}(Q(x)), \alpha, 1)$$

is an open neighborhood of the compact subset $W_{X/J}^{Q(e)}(Q(x))$. By Lemma 8.7, the latter equals $\bigcap_{\lambda \in \mathbb{C}} B(\lambda, \|Q(x - \lambda e)\|_{X/J})$, and so we can find $0 = \lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{C}$, and $\delta > 0$, such that

$$\bigcap_i B\big(\lambda_i, \|Q(x - \lambda_i e)\|_{X/J} + \delta\big) \subset N\big(W_{X/J}^{Q(e)}(Q(x)), \alpha, 1\big).$$

Let $z_0 = P(x - \alpha e) \in J^{\perp\perp}$ and $\lambda \in \mathbb{C}$. Since P is an M-projection,

$$\|x - z_0 - \lambda e\| = \max\{\|P((\alpha - \lambda)e)\|, \|(I - P)(x - \lambda e + y)\|\}, \quad y \in J,$$

which is dominated by

$$\max\big\{|\lambda - \alpha|, \|Q(x - \lambda e)\|_{X/J}\big\} = \|Q(x - \lambda e)\|_{X/J}$$

since $\alpha \in \bigcap_{\lambda \in \mathbb{C}} B(\lambda, \|Q(x - \lambda e)\|_{X/J})$. Thus $\|x - z_0 - \lambda_i e\| < r_i$ for each i, where $r_i = \|Q(x - \lambda_i e)\|_{X/J} + \delta$. Hence by Lemma 1.1, there exists $y_0 \in J$ such that $\|x - y_0 - \lambda_i e\| < r_i$ for all i. Indeed using that lemma similarly to some other proofs in our paper, if $x' \in X$ and $z \in J^{\perp\perp}$ are such that $\|z + x'\|_{X^{**}} < r$, and if $\{y_i\}$ is a net in J which converges to z weak*, one can find a net $\{y_j'\}$ of convex combinations of the y_j such that $y_j' \to z$ and $\|y_j' + x'\|_X < r$. One can iterate this procedure and obtain the same conclusion for any finite sequence $x_1', \dots, x_m' \in X$ such that $\|z + x_i'\|_{X^{**}} < r_i$ for all $i = 1, \dots, m$.

It follows that $x_0 = x - y_0$ satisfies $\|x_0\| < \|Q(x)\|_{X/J} + \delta$, and

$$|\varphi(x_0) - \lambda_i| = |\varphi(x - y_0 - \lambda_i e)| \leq \|Q(x - \lambda_i e)\|_{X/J} + \delta, \quad \varphi \in S_e(X).$$

This implies

$$W_X(x_0) \subset \bigcap_i B\big(\lambda_i, \|Q(x - \lambda_i e)\|_{X/J} + \delta\big) \subset N\big(W_{X/J}^{Q(e)}(Q(x)), \alpha, 1\big).$$

Now we iterate the above process, controlling the increments. If $\epsilon > 0$, let $N(\epsilon)$ denote the set of those $x' \in x + J \subset X$ such that

$$\|x'\|_X \leq \|Q(x)\|_{X/J} + \frac{\epsilon}{1-\epsilon}\big(\|Q(x)\|_{X/J} - |\alpha|\big),$$

and such that $W_X(x') \subset N(W_{X/J}^{Q(e)}(Q(x)), \alpha, \epsilon)$. Note that $x_0 \in N(1)$ (the first condition in the definition of $N(1)$ we treat as being vacuous).

Claim. For any $n = 0, 1, 2, \ldots$ and $x_n \in N(2^{-n})$, there is $x_{n+1} \in N(2^{-(n+1)})$ such that $\|x_{n+1} - x_n\| \leq 3 \cdot 2^{-n}\|Q(x)\|$ when $n \geq 1$.

Before we prove the claim, we finish the proof of the theorem. Note that if $n \geq 1$ then $\|x_n\| \leq 2\|Q(x)\|_{X/J}$ by the first clause in the definition of $N(\epsilon)$. It follows from this and the inequality in the claim that the norm-limit $v = \lim x_n$ exists in $x + J$. It satisfies $\|v\| \leq \|Q(x)\|_{X/J}$ by the first clause in the definition of $N(2^{-n})$, and $W_X(v) \subset W_{X/J}(Q(x))$ since by the second clause in that definition,

$$\varphi(v) = \lim \varphi(x_n) \in \bigcap_n N\big(W_{X/J}^{Q(e)}(Q(x)), \alpha, 2^{-n}\big) = W_{X/J}(Q(x)), \quad \varphi \in S_e(X).$$

That $W_{X/J}(Q(x)) \subset W_X(v)$ is an easy exercise. This completes the proof of the theorem.

To prove the claim, let $z = 2^{-n}P(x_n - \alpha e) \in J^{\perp\perp}$. Using the first clause in the definition of $x_n \in N(2^{-n})$, we have

$$\|z\| \leq 2^{-n}(\|x_n\| + |\alpha|) < 3 \cdot 2^{-n}\|Q(x)\|.$$

Also, $P(x_n - z) = (1 - 2^{-n})x_n + 2^{-n}\alpha$, so by an argument similar to the M-projection argument in the second paragraph of the proof, we have

$$\|x_n - z\| \leq \max\big\{(1 - 2^{-n})\|x_n\| + 2^{-n}|\alpha|, \|Q(x)\|_{X/J}\big\}.$$

The latter equals $\|Q(x)\|_{X/J}$, using the first clause in the definition of $x_n \in N(2^{-n})$.

Suppose that $\varphi_1 \in S_e(X^{**})$ with $\varphi_1 \circ P = \varphi_1$. There exists $\gamma \in W_{X/J}^{Q(e)}(Q(x))$ such that $\varphi_1(x_n) = \alpha + (1 + 2^{-n})(\gamma - \alpha)$, by the second clause in the definition of $x_n \in N(2^{-n})$. Hence, one has

$$\varphi_1(x_n - z) = \alpha + (1 - 2^{-n})(\varphi_1(x_n) - \alpha) = \alpha + (1 - 2^{-2n})(\gamma - \alpha),$$

and the latter is in $W_{X/J}^{Q(e)}(Q(x))$ since it is a convex combination of α and γ. Next, suppose that $\varphi_2 \in S_e(X^{**})$ with $\varphi_2 \circ P = 0$. Then φ_2 induces a "state" on $(X/J)^{**} \cong X^{**}/J^{\perp\perp}$, so that

$$\varphi_2(x_n - z) = \varphi_2(x_n) \in W_{(X/J)^{**}}^{Q(e)}(Q(x)) = W_{X/J}^{Q(e)}(Q(x)).$$

Thus $W_{X^{**}}^{e}(x_n - z) \subset W_{X/J}^{Q(e)}(Q(x))$, since any $\varphi \in S_e(X^{**})$ is a convex combination of $\varphi_1 = \varphi \circ P$ and $\varphi_2 = \varphi \circ (I - P)$ as above. Here we are using the

L-projection argument we have seen several times, relying on

$$1 = \varphi(e) = \varphi_1(e) + \varphi_2(e) \le \|\varphi_1\| + \|\varphi_2\| = 1.$$

By the Williams formula (Lemma 8.7),

$$\bigcap_{\lambda \in \mathbb{C}} B\big(\lambda, \|x_n - z - \lambda e\|_{X^{**}}\big) = W^e_{X^{**}}(x_n - z) \subset W^{Q(e)}_{X/J}(Q(x)).$$

Let $\delta = 2^{-(n+1)}$. By the argument at the start of the proof, one can choose a finite sequence $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ such that

$$\bigcap_i B\big(\lambda_i, \|x_n - z - \lambda_i e\|\big) \subset N\big(W^{Q(e)}_{X/J}(Q(x)), \alpha, \delta\big).$$

Choose $r_i > \|x_n - z - \lambda_i e\|$ with $\bigcap_i B(\lambda_i, r_i) \subset N(W^{Q(e)}_{X/J}(Q(x)), \alpha, \delta)$. By the argument using Lemma 1.1 in the second paragraph of the proof, we can replace z in these inequalities by an element in J. Thus there exists $y \in J$ such that $\|y\| < 3 \cdot 2^{-n} \|Q(x)\|$,

$$\|x_n - y\| \le \|Q(x)\|_{X/J} + \frac{\delta}{1-\delta}(\|Q(x)\|_{X/J} - |\alpha|),$$

and

$$W(x_n - y) \subset \bigcap_i B\big(\lambda_i, \|x_n - y - \lambda_i e\|\big) \subset \bigcap_i B(\lambda_i, r_i) \subset N\big(W^{Q(e)}_{X/J}(Q(x)), \alpha, \delta\big).$$

Hence $x_{n+1} = x_n - y \in N(\delta)$, which completes the proof of the claim. ☐

We next deal with the exceptional case when $W^{Q(e)}_{X/J}(Q(x)))$ has empty interior, which by convexity happens exactly when it is a line segment or point.

Corollary 8.9. *Suppose that J is an M-ideal ideal* (*or simply an ideal which is an M-ideal*) *in a unital Banach algebra A. Let $x \in A/J$ with $K = W_{A/J}(x)$. Then*

(1) *If K is a point, then there exists $a \in A$ with $\|a\| = \|x\|$ and with $W_A(a) = W_{A/J}(x)$.*
(2) *If $K = W_{A/J}(x)$ is a nontrivial line segment then* (1) *is true "within epsilon". More precisely, in this case, let $\widehat{K}$ be any thin triangle with K as one of the sides* (*so contained in a thin rectangle with side K*)*. Then there exists $a \in A$ with $\|a\| = \|x\|$ and with $K \subset W_A(a) \subset \widehat{K}$.*

Proof. If K is a point, then x is a scalar multiple of 1, so this case is obvious. For (2), if K is a nontrivial line segment, choose λ within a small distance ϵ of the midpoint of the line. Then replace A by $B = A \oplus^\infty \mathbb{C}$, replace J by $I = J \oplus (0)$, and consider $(x, \lambda) \in B/I$. It is easy to see that $W_{B/I}((x, \lambda))$ is the convex hull $\widehat{K}$ of K and λ. By Theorem 8.8 there exists $(a, \lambda) \in B$ with $W_B((a, \lambda)) = \widehat{K}$. If ϵ is small enough, we also have $\|a\| = \|x\|$ (since then $|\lambda|$ is dominated by the maximum of the moduli of two numbers in the numerical range, which is dominated by $\|x\| \le \|a\|$).

However, similarly $W_B((a,\lambda))$ is the convex hull of $W_A(a)$ and λ, which makes the rest of the proof of (2) an easy exercise in the geometry of triangles. $\square$

We remark that in a previous version of our paper, the last result (and Theorem 8.8 in the unital Banach algebra case) was stated as a claim , not as a theorem. Thus it is referred to in [Blecher and Read 2014] as "the Claim at the end of" the present paper.

We can now answer the open question referred to above Theorem 8.8.

Corollary 8.10. *If A is an approximately unital Banach algebra, and if J is an M-ideal ideal in A, then $\mathfrak{r}_{A/J} = q_J(\mathfrak{r}_A)$. In particular, $\mathfrak{r}_{A/J} = q_J(\mathfrak{r}_A)$ for approximately unital closed two-sided ideals J in any* (*not necessarily approximately unital*) *operator algebra A.*

Proof. First suppose that A is unital. We leave it as an exercise that $q_J(\mathfrak{r}_A) \subset \mathfrak{r}_{A/J}$. The converse inclusion follows from Theorem 8.8 and Corollary 8.9 (in the line situation take the triangle above and/or to the right of K). Next suppose that A is a nonunital approximately unital Banach algebra, and that A/J is also nonunital. Then by the last paragraph of A.4.3 in [Blecher and Le Merdy 2004], the inclusion $A/J \subset A^1/J$ induces an isometric isomorphism $A^1/J \cong (A/J)^1$. The result then follows by applying the unital case to the canonical map from A^1 onto $(A/J)^1$. If A/J is unital then one can reduce to the previous case where it is not, by considering the ideal $J \oplus^\infty K$ in $A \oplus^\infty B$, where K is an approximately unital ideal in (e.g., a commutative C^*-algebra) B such that B/J is not unital. For this latter trick, one needs to know that $\mathfrak{r}_{A\oplus^\infty B} = \{(x,y) \in A \oplus^\infty B : x \in \mathfrak{r}_A, y \in \mathfrak{r}_B\}$ for approximately unital Banach algebras, but this is an easy exercise (and a similar relation holds for $\mathfrak{F}_{A\oplus^\infty B}$).

Finally, suppose that A is any nonunital operator algebra and J is an approximately unital closed ideal in A. Then J is an M-ideal in A^1 by [Effros and Ruan 1990]. Also, by the uniqueness of the unitization of an operator algebra mentioned in the introduction, we have $A^1/J \cong (A/J)^1$ completely isometrically if A/J is nonunital (see also [Blecher and Read 2014, Lemma 4.11]). Then the result follows again by applying the unital case to the canonical map from A^1 onto $(A/J)^1$. If A/J is unital, we can reduce to the case where it is not by the trick in the last paragraph. $\square$

By the assertion about the norms in Theorem 8.8 and Corollary 8.9, we can lift elements in $\mathfrak{r}_{A/J}$ to elements in $\mathfrak{r}_A$, keeping the same norm, in the situations considered in the corollary.

As we said, these results may be viewed as noncommutative peak interpolation or noncommutative Tietze theorems. For in the case that A is a uniform algebra on a compact Hausdorff set Ω, the M-ideals J are well known to be the closed ideals with a cai, and are exactly the functions in A vanishing on some p-set $E \subset \Omega$ (see [Smith 1979] and [Harmand et al. 1993, Theorem V.4.2]). Then q_J is identifiable with the restriction map $f \mapsto f_{|E}$, and $A/J \cong \{f_{|E} : f \in A\} \subset C(E)$. The lifting result in Theorems 8.8 and 8.9 in this case say that if $f \in A$ with $f(E) \subset C$ for a compact convex set C in the plane, then there exists a function $g \in A$ which agrees with f on E, which has norm $\|g\|_\Omega = \|f_{|E}\|_E$, and which has range $g(\Omega) \subset C$ (or $g(\Omega) \subset \widehat{K}$ if $\mathrm{conv}(f(E))$ is a line segment K, where $\widehat{K}$ is a thin triangle given in advance, one of whose sides is K).

9. Banach algebras without cai

If A is a Banach algebra without a cai, or without any kind of bai, we briefly indicate here how to obtain nearly all the results from Sections 3, 4, and 7. We give more details in a forthcoming conference proceedings survey article [Blecher 2015]; however, the interested reader will have no trouble reconstructing this independently from the discussion below. Namely, if B is any unital Banach algebra containing A, for example, any unitization of A, one can define $\mathfrak{F}_A^B = \{a \in A : \|1_B - a\| \le 1\}$ and $\mathfrak{r}_A^B$ to be the set of $a \in A$ whose numerical range in B is contained in the right half-plane. Also one can define $\mathfrak{F}_A$ (resp. $\mathfrak{r}_A$) to be the union of the $\mathfrak{F}_A^B$ (resp. $\mathfrak{r}_A^B$) over all B as above. Unfortunately it is not clear to us that $\mathfrak{F}_A$ and $\mathfrak{r}_A$ are always convex, which is needed in Sections 4 and 7 (indeed we often need them closed too there). Of course, $\mathfrak{F}_A$ and $\mathfrak{r}_A$ are convex and closed if there is an "extremal" unitization B of A such that $\mathfrak{F}_A^B = \mathfrak{F}_A$ (resp. $\mathfrak{r}_A^B = \mathfrak{r}_A$). This is the case with B equal to the multiplier unitization if A is approximately unital, or more generally if the left regular representation embeds A isometrically in $B(A)$.

Most of the results in Sections 3, 4, and 7 of our paper then work without the approximately unital hypothesis if $\mathfrak{F}_A^B$ and $\mathfrak{r}_A^B$ are used. In particular, we mention the results 3.3–3.6, 3.9–3.11, 3.17–3.19, 3.21, 3.23–3.25, and all lemmas, theorems, and corollaries in Sections 4 and 7 not concerning M-approximately unital algebras. Every one of the statements of these results is still correct if one drops the approximately unital hypothesis, but uses $\mathfrak{F}_A^B$ and $\mathfrak{r}_A^B$ in place of $\mathfrak{F}_A$ and $\mathfrak{r}_A$. Indeed the results just mentioned in Section 3 (and also the first lemma in Section 4) are also correct for general Banach algebras if one uses $\mathfrak{F}_A$ or $\mathfrak{r}_A$ as defined in the last paragraph (the other results in Sections 4 and 7 would seem to need $\mathfrak{F}_A$ and $\mathfrak{r}_A$ (as defined in the last paragraph) being closed and convex).

Some of the results asserted in the last paragraph are obvious from the unital case of the result, and some follow by the obvious modification of the given proof of the result. However, in some of these results, one also needs to know that $\overline{EA} = \overline{EB}$, where B is a unitization of A and E is a subset of $\mathfrak{F}_A^B$ or $\mathfrak{r}_A^B$. This follows from the following fact: if $x \in \mathfrak{r}_A$ as defined in the last paragraph then

$$x \in \overline{xA} = \overline{\mathrm{ba}(x)A} = \overline{xB}$$

for any unitization B of A. Indeed this is clear since by Cohen factorization, $x \in \mathrm{ba}(x) = \mathrm{ba}(x)^2 \subset \overline{xA}$. We also need to know that the $\mathfrak{F}$-transform, and n-th roots, are independent of the particular unitization used, but this is easy to see using the fact that all unitization norms are equivalent.

Acknowledgments

We thank Charles Read for useful discussions, and for allowing us to take out some of the material in [Blecher and Read 2013b] for inclusion here. We thank the referee for his careful reading of the manuscript, which was plagued by innumerable typos, and for his suggestions. We were also supported as participants in the Thematic Program on Abstract Harmonic Analysis, Banach and Operator Algebras 2014 at the Fields Institute, for which we thank the Institute and the organizers of that program. As we said earlier, the survey article [Blecher 2015] contains a few additional details on some of the material in the present paper, as well as some small improvements found while this paper was in press.

References

[Akemann 1970] C. A. Akemann, "Left ideal structure of C^*-algebras", *J. Functional Analysis* **6** (1970), 305–317. MR 43 #934 Zbl 0199.45901

[Alfsen 1971] E. M. Alfsen, *Compact convex sets and boundary integrals*, Ergebnisse der Mathematik und ihrer Grenzgebiete **57**, Springer, New York-Heidelberg, 1971. MR 56 #3615 Zbl 0209.42601

[Arias and Rosenthal 2000] A. Arias and H. P. Rosenthal, "M-complete approximate identities in operator spaces", *Studia Math.* **141**:2 (2000), 143–200. MR 2001m:46125 Zbl 0983.46045

[Asimow and Ellis 1972] L. A. Asimow and A. J. Ellis, "On Hermitian functionals on unital Banach algebras", *Bull. London Math. Soc.* **4** (1972), 333–336. MR 48 #2763 Zbl 0267.46037

[Asimow and Ellis 1980] L. Asimow and A. J. Ellis, *Convexity theory and its applications in functional analysis*, London Mathematical Society Monographs **16**, Academic Press, London-New York, 1980. MR 82m:46009 Zbl 0453.46013

[Batty and Robinson 1984] C. J. K. Batty and D. W. Robinson, "Positive one-parameter semigroups on ordered Banach spaces", *Acta Appl. Math.* **2**:3-4 (1984), 221–296. MR 86b:47068 Zbl 0554.47022

[Bearden et al. 2014] C. A. Bearden, D. P. Blecher, and S. Sharma, "On positivity and roots in operator algebras", *Integral Equations Operator Theory* **79**:4 (2014), 555–566. MR 3231244 Zbl 06358021

[Blecher 2013] D. P. Blecher, "Noncommutative peak interpolation revisited", *Bull. Lond. Math. Soc.* **45**:5 (2013), 1100–1106. MR 3105002 Zbl 1284.46053

[Blecher 2015] D. P. Blecher, "Generalization of C^*-algebra methods via real positivity for operator and Banach algebras", preprint, 2015.

[Blecher and Le Merdy 2004] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules—an operator space approach*, London Mathematical Society Monographs. New Series **30**, Oxford Univ. Press, 2004. MR 2006a:46070 Zbl 1061.47002

[Blecher and Neal 2012a] D. P. Blecher and M. Neal, "Open projections in operator algebras, I: Comparison theory", *Studia Math.* **208**:2 (2012), 117–150. MR 2910983 Zbl 1259.46045

[Blecher and Neal 2012b] D. P. Blecher and M. Neal, "Open projections in operator algebras, II: Compact projections", *Studia Math.* **209**:3 (2012), 203–224. MR 2944468 Zbl 1259.46046

[Blecher and Read 2011] D. P. Blecher and C. J. Read, "Operator algebras with contractive approximate identities", *J. Funct. Anal.* **261**:1 (2011), 188–217. MR 2012e:47226 Zbl 1235.47087

[Blecher and Read 2013a] D. P. Blecher and C. J. Read, "Operator algebras with contractive approximate identities, II", *J. Funct. Anal.* **264**:4 (2013), 1049–1067. MR 3004957 Zbl 1270.47067

[Blecher and Read 2013b] D. P. Blecher and C. J. Read, "Operator algebras with contractive approximate identities, III", preprint, 2013. arXiv 1308.2723

[Blecher and Read 2014] D. P. Blecher and C. J. Read, "Order theory and interpolation in operator algebras", *Studia Math.* **225**:1 (2014), 61–95. MR 3299396 Zbl 06390240

[Blecher and Zarikian 2006] D. P. Blecher and V. Zarikian, *The calculus of one-sided M-ideals and multipliers in operator spaces*, vol. 179, Mem. Amer. Math. Soc. **842**, Amer. Math. Soc., Providence, RI, 2006. MR 2006j:46063 Zbl 1108.46043

[Blecher et al. 2008] D. P. Blecher, D. M. Hay, and M. Neal, "Hereditary subalgebras of operator algebras", *J. Operator Theory* **59**:2 (2008), 333–357. MR 2009m:46074 Zbl 1164.46018

[Bonsall and Duncan 1971] F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, London Mathematical Society Lecture Note Series **2**, Cambridge Univ. Press, 1971. MR 44 #5779 Zbl 0207.44802

[Bonsall and Duncan 1973] F. F. Bonsall and J. Duncan, *Numerical ranges, II*, London Mathematical Society Lecture Notes Series **10**, Cambridge Univ. Press, 1973. MR 56 #1063 Zbl 0262.47001

[Chui et al. 1977] C. K. Chui, P. W. Smith, R. R. Smith, and J. D. Ward, "L-ideals and numerical range preservation", *Illinois J. Math.* **21**:2 (1977), 365–373. MR 55 #3822 Zbl 0343.46040

[Dales 2000] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs. New Series **24**, Oxford Univ. Press, 2000. MR 2002e:46001 Zbl 0981.46043

[Davidson and Power 1986] K. R. Davidson and S. C. Power, "Best approximation in C^*-algebras", *J. Reine Angew. Math.* **368** (1986), 43–62. MR 87k:47100 Zbl 0579.46038

[Dixon 1978] P. G. Dixon, "Approximate identities in normed algebras, II", *J. London Math. Soc.* (2) **17**:1 (1978), 141–151. MR 80b:46055 Zbl 0384.46029

[Doran and Wichmann 1979] R. S. Doran and J. Wichmann, *Approximate identities and factorization in Banach modules*, Lecture Notes in Mathematics **768**, Springer, Berlin-New York, 1979. MR 83e:46044 Zbl 0418.46039

[Effros and Ruan 1990] E. G. Effros and Z.-J. Ruan, "On nonselfadjoint operator algebras", *Proc. Amer. Math. Soc.* **110**:4 (1990), 915–922. MR 91c:47086 Zbl 0718.46020

[Esterle 1978] J. Esterle, "Injection de semi-groupes divisibles dans des algèbres de convolution et construction d'homomorphismes discontinus de $\mathscr{C}(K)$", *Proc. London Math. Soc.* (3) **36**:1 (1978), 59–85. MR 58 #2300 Zbl 0411.46039

[Haase 2006] M. Haase, *The functional calculus for sectorial operators*, Operator theory: Advances and applications **169**, Birkhäuser, Basel, 2006. MR 2007j:47030 Zbl 1101.47010

[Harmand et al. 1993] P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Mathematics **1547**, Springer, Berlin, 1993. MR 94k:46022 Zbl 0789.46011

[Harte and Mbekhta 1992] R. Harte and M. Mbekhta, "On generalized inverses in C^*-algebras", *Studia Math.* **103**:1 (1992), 71–77. MR 93i:46097 Zbl 0810.46062

[Harte and Mbekhta 1993] R. Harte and M. Mbekhta, "Generalized inverses in C^*-algebras, II", *Studia Math.* **106**:2 (1993), 129–138. MR 94k:46113 Zbl 0810.46063

[Hay 2007] D. M. Hay, "Closed projections and peak interpolation for operator algebras", *Integral Equations Operator Theory* **57**:4 (2007), 491–512. MR 2008f:46064 Zbl 1127.46033

[Hoffman 1962] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, NJ, 1962. Reprinted Dover, New York, 1988. MR 24 #A2844 Zbl 0117.34001

[Jameson 1970] G. Jameson, *Ordered linear spaces*, Lecture Notes in Mathematics **141**, Springer, Berlin-New York, 1970. MR 55 #10996 Zbl 0196.13401

[Kaniuth et al. 2010] E. Kaniuth, A. T. Lau, and A. Ülger, "Multipliers of commutative Banach algebras, power boundedness and Fourier–Stieltjes algebras", *J. Lond. Math. Soc.* (2) **81**:1 (2010), 255–275. MR 2011c:46107 Zbl 1196.46037

[Kelley and Vaught 1953] J. L. Kelley and R. L. Vaught, "The positive cone in Banach algebras", *Trans. Amer. Math. Soc.* **74** (1953), 44–55. MR 14,883e Zbl 0050.11004

[Lau and Ülger 2014] A. T.-M. Lau and A. Ülger, "Characterization of closed ideals with bounded approximate identities in commutative Banach algebras, complemented subspaces of the group von Neumann algebras and applications", *Trans. Amer. Math. Soc.* **366**:8 (2014), 4151–4171. MR 3206455 Zbl 06345416

[Li et al. 2003] C.-K. Li, L. Rodman, and I. M. Spitkovsky, "On numerical ranges and roots", *J. Math. Anal. Appl.* **282**:1 (2003), 329–340. MR 2004g:47009 Zbl 1039.47003

[Macaev and Palant 1962] V. I. Macaev and J. A. Palant, "On the powers of a bounded dissipative operator", *Ukrain. Mat. Ž.* **14** (1962), 329–337. In Russian. MR 26 #4184 Zbl 0199.45001

[Magajna 2009] B. Magajna, "Weak* continuous states on Banach algebras", *J. Math. Anal. Appl.* **350**:1 (2009), 252–255. MR 2010b:46110 Zbl 1165.46023

[Messerschmidt 2015] M. Messerschmidt, "Normality of spaces of operators and quasi-lattices", *Positivity* (online publication February 2015).

[Moore 1971] R. T. Moore, "Hermitian functionals on B-algebras and duality characterizations of C^*-algebras", *Trans. Amer. Math. Soc.* **162** (1971), 253–265. MR 44 #803 Zbl 0213.14002

[Palmer 1994] T. W. Palmer, *Banach algebras and the general theory of *-algebras, I: Algebras and Banach algebras*, Encyclopedia of Mathematics and its Applications **49**, Cambridge Univ. Press, 1994. MR 95c:46002 Zbl 0809.46052

[Pedersen 1979] G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs **14**, Academic Press, London-New York, 1979. MR 81e:46037 Zbl 0416.46043

[Pedersen 1998] G. K. Pedersen, "Factorization in C^*-algebras", *Exposition. Math.* **16**:2 (1998), 145–156. MR 99f:46086 Zbl 0912.46054

[Read 2011] C. J. Read, "On the quest for positivity in operator algebras", *J. Math. Anal. Appl.* **381**:1 (2011), 202–214. MR 2012g:47217 Zbl 1235.47090

[Sinclair 1971] A. M. Sinclair, "The norm of a hermitian element in a Banach algebra", *Proc. Amer. Math. Soc.* **28** (1971), 446–450. MR 43 #921 Zbl 0242.46035

[Sinclair 1978] A. M. Sinclair, "Bounded approximate identities, factorization, and a convolution algebra", *J. Funct. Anal.* **29**:3 (1978), 308–318. MR 80c:46054 Zbl 0385.46030

[Sinclair and Tullo 1974] A. M. Sinclair and A. W. Tullo, "Noetherian Banach algebras are finite dimensional", *Math. Ann.* **211** (1974), 151–153. MR 50 #8081 Zbl 0275.46037

[Smith 1979] R. R. Smith, "An addendum to: "M-ideal structure in Banach algebras"", *J. Funct. Anal.* **32**:3 (1979), 269–271. MR 80j:46087 Zbl 0409.46057

[Smith and Ward 1978] R. R. Smith and J. D. Ward, "M-ideal structure in Banach algebras", *J. Functional Analysis* **27**:3 (1978), 337–349. MR 57 #7175 Zbl 0369.46044

[Smith and Ward 1979] R. R. Smith and J. D. Ward, "Applications of convexity and M-ideal theory to quotient Banach algebras", *Quart. J. Math. Oxford Ser.* (2) **30**:119 (1979), 365–384. MR 80h:46071 Zbl 0412.46042

[Stampfli and Williams 1968] J. G. Stampfli and J. P. Williams, "Growth conditions and the numerical range in a Banach algebra", *Tôhoku Math. J.* (2) **20**:4 (1968), 417–424. MR 39 #4674 Zbl 0175.43902

[Sz.-Nagy et al. 2010] B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy, *Harmonic analysis of operators on Hilbert space*, 2nd ed., Springer, New York, 2010. MR 2012b:47001 Zbl 1234.47001

[Yosida 1965] K. Yosida, *Functional analysis*, Grundlehren der Math. Wiss. **123**, Springer, Berlin, 1965. MR 31 #5054

Received May 21, 2014. Revised January 6, 2015.

DAVID P. BLECHER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HOUSTON
HOUSTON, TX 77204-3008
UNITED STATES
dblecher@math.uh.edu

NARUTAKA OZAWA
RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
KYOTO UNIVERSITY
KYOTO 606-8502
JAPAN
narutaka@kurims.kyoto-u.ac.jp

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.61

ON SHRINKING GRADIENT RICCI SOLITONS WITH NONNEGATIVE SECTIONAL CURVATURE

MINGLIANG CAI

Perelman proved that an open 3-dimensional shrinking gradient Ricci soliton with bounded nonnegative sectional curvature is a quotient of $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$. We extend this result to higher dimensions with a decay condition on the Ricci tensor.

1. Introduction

A gradient Ricci soliton is a Riemannian manifold (M, g) together with a smooth function f such that

$$\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g,$$

where λ is a constant. It is called shrinking, steady and expanding when $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

Gradient Ricci solitons are self-similar solutions of Hamilton's Ricci flow and play a vital role in the analysis of singularities of the flow. In dimension 2, Hamilton [1988] completely classified shrinking gradient Ricci solitons with bounded curvature and proved that they are the sphere, the projective space and the Euclidean space with constant curvature. In dimension 3, Ivey [1993] proved that compact shrinking gradient Ricci solitons have positive sectional curvature, and Perelman [2003] proved that shrinking gradient Ricci solitons with bounded nonnegative sectional curvature are quotients of S^3, $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$.

In higher dimensions, there have been many results in the last several years. Chen [2009] showed that a complete shrinking gradient Ricci soliton has nonnegative scalar curvature. Ni and Wallach [2008] gave the classification of shrinking gradient Ricci solitons with nonnegative Ricci curvature and zero Weyl tensor. Petersen and Wylie [2010] and independently, Cao, Wang and Zhu [Cao et al. 2011], classified the shrinking gradient Ricci solitons with zero Weyl tensor. Fernández-López and García-Río [2011] considered solitons with harmonic Weyl tensor. In [Petersen and Wylie 2009], several natural curvature conditions are given that characterize gradient Ricci solitons of the flat vector bundle $N \times_\Gamma \mathbb{R}^m$, where N is an Einstein manifold,

MSC2010: primary 53C25; secondary 53C20, 53C24.
Keywords: shrinking gradient Ricci soliton, rigidity, nonnegative sectional curvature.

Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^m$, and $f = \frac{1}{4}d^2$ with d being the distance on the flat fiber to the base. In particular, it is shown in [Petersen and Wylie 2009] that a shrinking gradient Ricci soliton is rigid, i.e., of the form $N \times_\Gamma \mathbb{R}^m$, if the scalar curvature is constant and the sectional curvature of the plane containing ∇f is nonnegative. As a consequence of a theorem of Böhm and Wilking [2008], the gradient Ricci solitons with positive curvature operators are trivial. In view of this and the aforementioned result of Perelman, one naturally asks to what extend shrinking gradient Ricci solitons with nonnegative sectional curvature are rigid. Our first result in this paper is the rigidity under a decay condition on $|D\,\mathrm{Ric}|$, extending Perelman's result to higher dimensions. In all theorems we scale the metric so that $\lambda = \frac{1}{2}$.

Theorem 1.1. *Let (M, g, f) be a complete noncompact shrinking gradient Ricci soliton with bounded nonnegative sectional curvature. Assume that there exists $\delta > 0$ such that*

$$\int_M e^{\delta f} |D\,\mathrm{Ric}|\, d\,\mathrm{vol}_g < \infty.$$

Then (M^n, g) is isometric to $N \times_\Gamma \mathbb{R}^m$, where N is a compact Einstein manifold.

This is, to our knowledge, the first rigidity result in high dimensions without assumptions on the Weyl tensor. The potential function f is known to grow quadratically with respect to the distance from a fixed point, so our condition on $D\,\mathrm{Ric}$ says that it decays exponentially. Our proof also works under the assumption that $D\,\mathrm{Ric}$ decays polynomially with a degree depending on other geometric quantities.

The Cheeger–Gromoll soul theorem states that an open manifold with nonnegative sectional curvature is diffeomorphic to a vector bundle over a compact submanifold called a soul. The pull-back metric on the bundle can be highly twisted. However, if there exists a gradient soliton structure on such a bundle, then, by Theorem 1.1, the metric has to be locally trivial, provided that the decay condition is satisfied. The decay condition on $D\,\mathrm{Ric}$ in Theorem 1.1 is imposed in the region where f is large. Our next result deals with the rigidity under a condition on $D\,\mathrm{Ric}$ imposed in the region where f is small.

Theorem 1.2. *Let (M^n, g, f) be a complete shrinking gradient Ricci soliton with bounded nonnegative sectional curvature. Assume that the minima of f is a smooth compact nondegenerate critical submanifold and $D\,\mathrm{Ric}$ and $D^2\,\mathrm{Ric}$ vanish on the minima. Then (M^n, g) is noncompact and isometric to $N \times_\Gamma \mathbb{R}^m$, where N is a compact Einstein manifold.*

We derive some basic formulas in Section 2, and prove Theorems 1.1 and 1.2 in Sections 3 and 4 respectively.

2. Basic formulas

There are different conventions for the curvature tensor in the literature, so to avoid the confusion, we state ours as follows. The (3, 1) tensor $\mathrm{Rm}(X, Y, Z) = \mathrm{Rm}(X, Y)Z$ is defined as

$$\mathrm{Rm}(X, Y)Z = D_X D_Y Z - D_Y D_X X - D_{[X,Y]} Z$$

and the (4, 0) tensor as

$$\mathrm{Rm}(X, Y, Z, W) = \langle \mathrm{Rm}(X, Y)Z, W\rangle.$$

We use Ric to denote the Ricci tensor and R the scalar curvature. For a tangent vector X at p, we use $\mathrm{Ric}(X)$ to denote the vector such that

$$\langle \mathrm{Ric}(X), Y\rangle = \mathrm{Ric}(X, Y)$$

for any vector Y at p. For any smooth vector field V and any smooth function ϕ on manifold M, by $V(\phi)$, we mean $V(\phi) = d\phi(V) = \langle V, \nabla\phi\rangle$. In the remainder of the paper, we will rescale the metric and assume that our gradient Ricci soliton satisfies

$$\mathrm{Ric} + \mathrm{Hess}\, f = \tfrac{1}{2} g.$$

Since the curvature of (M, g) is assumed to be bounded, there exists a flow $\Phi_t : M \to M$ defined for all time with $\Phi_0 = \mathrm{Id}$ and $\partial\Phi/\partial t = \nabla f$ [Morgan and Tian 2007, p. 207]. For $t \in (\infty, 0)$, define $G(t) = |t|\Phi^*_{-\ln|t|} g$. Then $G(-1) = g$ and $G(t)$ satisfies

$$\mathrm{Ric}(G(t)) + \mathrm{Hess}\, f = \frac{1}{2\tau} G(t),$$

where Hess is taken with respect to the metric $G(t)$ and $\tau = |t| = -t$.

In the next lemma, we collect some well-known formulae.

Lemma 2.1. *On* $(M, G(t))$, *we have*

(1) $dR = 2\mathrm{Ric}(\nabla f, \cdot)$,

(2) $|\nabla f|^2 = f/\tau - R + \textit{constant}$,

(3) $R/\tau + \langle \nabla f, \nabla R\rangle = \Delta R + 2|\mathrm{Ric}|^2$,

(4) $\operatorname{div} \mathrm{Rm}(X, Y, Z) = \mathrm{Rm}(\nabla f, X, Y, Z)$,

(5) $D_X \mathrm{Ric}(Y, Z) = D_Y \mathrm{Ric}(X, Z) - \mathrm{Rm}(X, Y, \nabla f, Z)$,

where $\operatorname{div} \mathrm{Rm}(X, Y, Z) = \mathrm{trace}_{1,2}\, D\, \mathrm{Rm}(\cdot, \cdot, X, Y, Z)$.

Proof. The derivations of (1)–(3) can be found in [Hamilton 1995] and (4)–(5) in [Petersen and Wylie 2010]. □

Lemma 2.2. *On (M, g), we have*

$$\Delta|\mathrm{Ric}|^2 = 2|D\mathrm{Ric}|^2 + 2|\mathrm{Ric}|^2 + \nabla f(|\mathrm{Ric}|^2) - 4K_{ij}\lambda_i\lambda_j,$$

where λ_i are the eigenvalues of the Ricci tensor and K_{ij} is the sectional curvature of the plane spanned by the eigenvectors belonging to λ_i and λ_j respectively.

Proof. This follows from the formula derived in Lemma 2.1 in [Petersen and Wylie 2010]:

$$\Delta\mathrm{Ric} = D_{\nabla f}\mathrm{Ric} + \mathrm{Ric} - 2\sum_{k=1}^{n}\mathrm{Rm}(\,\cdot\,, e_k, \mathrm{Ric}(e_k), \cdot\,). \qquad \square$$

Throughout the computations in the paper, we assume $\{e_1, \dots, e_n\}$ is an orthonormal basis in a neighborhood of a fixed point x with $D_{e_i}e_j(x) = 0$ and further assume that each e_i is an eigenvector of Ric at x corresponding to the eigenvalue λ_i. Such a basis always exists. We also use the Einstein summation convention (unless otherwise specified).

Lemma 2.3. *On (M, g), we have*

$$\mathrm{div}(\mathrm{Ric}(\nabla R)) = \nabla f(|\mathrm{Ric}|^2) + \tfrac{1}{2}|\nabla R|^2 - 2\langle Z, \nabla f\rangle + |\mathrm{Ric}|^2 - 2\sum_i \lambda_i^3,$$

where $Z = \mathrm{Ric}(e_i, e_j)\,\mathrm{Rm}(\nabla f, e_i, e_j)$.

Proof. The following computations are done at x. From Lemma 2.1, we have

$$\begin{aligned} D_{e_i}\mathrm{Ric}(\nabla R, e_i) &= D_{\nabla R}\mathrm{Ric}(e_i, e_i) - \mathrm{Rm}(e_i, \nabla R, \nabla f, e_i) \\ &= |\nabla R|^2 - \mathrm{Ric}(\nabla R, \nabla f) = \tfrac{1}{2}|\nabla R|^2. \end{aligned}$$

We then obtain

$$\begin{aligned} \mathrm{div}(\mathrm{Ric}(\nabla R)) &= \langle D_{e_i}\mathrm{Ric}(\nabla R), e_i\rangle = e_i\mathrm{Ric}(\nabla R, e_i) \\ &= D_{e_i}\mathrm{Ric}(\nabla R, e_i) + \mathrm{Ric}(D_{e_i}\nabla R, e_i) \\ &= \tfrac{1}{2}|\nabla R|^2 + \mathrm{Ric}(e_i, e_j)\langle D_{e_i}\nabla R, e_j\rangle \\ &= \tfrac{1}{2}|\nabla R|^2 + 2\mathrm{Ric}(e_i, e_j)\langle D_{e_i}\mathrm{Ric}(\nabla f), e_j\rangle \\ &= \tfrac{1}{2}|\nabla R|^2 + 2\mathrm{Ric}(e_i, e_j)e_i\mathrm{Ric}(\nabla f, e_j) \\ &= \tfrac{1}{2}|\nabla R|^2 + 2\mathrm{Ric}(e_i, e_j)\big(D_{e_i}\mathrm{Ric}(\nabla f, e_j) + \mathrm{Ric}(D_{e_i}\nabla f, e_j)\big). \end{aligned}$$

That is,

$$\mathrm{div}(\mathrm{Ric}(\nabla R)) = \tfrac{1}{2}|\nabla R|^2 + 2\mathrm{Ric}(e_i, e_j)\big(D_{e_i}\mathrm{Ric}(\nabla f, e_j) + \mathrm{Ric}(D_{e_i}\nabla f, e_j)\big). \tag{2-1}$$

From the soliton equation

$$\mathrm{Ric} + \mathrm{Hess}\, f = \tfrac{1}{2}g,$$

it follows that

$$D_{e_i}\nabla f = \tfrac{1}{2}e_i - \mathrm{Ric}(e_i) = \tfrac{1}{2}e_i - \lambda_i e_i,$$

where we have used the assumption that e_i is an eigenvector of Ric at x belonging to the eigenvalue λ_i. Hence,

$$2\operatorname{Ric}(e_i, e_j)\operatorname{Ric}(D_{e_i}\nabla f, e_j) = 2\big(\tfrac{1}{2} - \lambda_i\big)(\operatorname{Ric}(e_i, e_j))^2 = 2\lambda_i^2\big(\tfrac{1}{2} - \lambda_i\big). \tag{2-2}$$

Lemma 2.1(5) implies that

$$D_{e_i}\operatorname{Ric}(\nabla f, e_j) = D_{\nabla f}\operatorname{Ric}(e_i, e_j) - \operatorname{Rm}(e_i, \nabla f, \nabla f, e_j).$$

It follows that

$$\begin{aligned} 2\operatorname{Ric}(e_i, e_j) D_{e_i}\operatorname{Ric}(\nabla f, e_j) &= 2\operatorname{Ric}(e_i, e_j)(D_{\nabla f}\operatorname{Ric}(e_i, e_j) - \operatorname{Rm}(e_i, \nabla f, \nabla f, e_j)) \\ &= 2\operatorname{Ric}(e_i, e_j) D_{\nabla f}\operatorname{Ric}(e_i, e_j) - 2\langle Z, \nabla f\rangle \\ &= \nabla f(|\operatorname{Ric}|^2) - 2\langle Z, \nabla f\rangle. \end{aligned} \tag{2-3}$$

Combining (2-2) and (2-3), we obtain that

$$\begin{aligned} 2\operatorname{Ric}(e_i, e_j)(D_{e_i}\operatorname{Ric}(\nabla f, e_j) + \operatorname{Ric}(D_{e_i}\nabla f, e_j)) \\ = \nabla f(|\operatorname{Ric}|^2) - 2\langle Z, \nabla f\rangle + 2\lambda_i^2(\tfrac{1}{2} - \lambda_i). \end{aligned}$$

Substituting the above into (2-1) gives

$$\begin{aligned} \operatorname{div}(\operatorname{Ric}(\nabla R)) &= \tfrac{1}{2}|\nabla R|^2 + \nabla f(|\operatorname{Ric}|^2) - 2\langle Z, \nabla f\rangle + 2\lambda_i^2(\tfrac{1}{2} - \lambda_i) \\ &= \tfrac{1}{2}|\nabla R|^2 + \nabla f(|\operatorname{Ric}|^2) - 2\langle Z, \nabla f\rangle + |\operatorname{Ric}|^2 - 2\sum_i \lambda_i^3. \qquad \square \end{aligned}$$

Remark 2.4. We have $\langle Z, \nabla f\rangle \geq 0$ when the sectional curvature of (M, g) is nonnegative. In fact, at x, $\langle Z, \nabla f\rangle = \lambda_i \operatorname{Rm}(\nabla f, e_i, e_i, \nabla f)$.

The next lemma is a slight variation of Lemma 2.3.

Lemma 2.5. *On* (M, g), *we have*

$$\nabla f(|\operatorname{Ric}|^2) = 2\bigg(\langle Z, \nabla f\rangle + \sum_{i=1}^n \lambda_i\big(\lambda_i - \tfrac{1}{2}\big)^2\bigg) + \tfrac{1}{2}\langle\nabla f, \nabla R\rangle - \tfrac{1}{2}|\nabla R|^2 - \operatorname{div}(D_{\nabla R}\nabla f).$$

Proof. It follows from Lemma 2.3 that

$$\operatorname{div}(\operatorname{Ric}(\nabla R)) = \tfrac{1}{2}|\nabla R|^2 + \nabla f(|\operatorname{Ric}|^2) - 2\langle Z, \nabla f\rangle + |\operatorname{Ric}|^2 - 2\sum_i \lambda_i^3.$$

Using $\operatorname{Ric}(\nabla R) = \frac{1}{2}\nabla R - D_{\nabla R}\nabla f$ and Lemma 2.1(3), we have

$$\begin{aligned} \nabla f(|\operatorname{Ric}|^2) = \frac{R}{2} - 2|\operatorname{Ric}|^2 + 2\sum_i \lambda_i^3 + 2\langle Z, \nabla f\rangle \\ + \tfrac{1}{2}\langle\nabla f, \nabla R\rangle - \tfrac{1}{2}|\nabla R|^2 - \operatorname{div}(D_{\nabla R}\nabla f). \end{aligned}$$

The lemma now follows as $R/2 - 2|\operatorname{Ric}|^2 + 2\sum_i \lambda_i^3 = 2\sum_{i=1}^n \lambda_i\big(\lambda_i - \frac{1}{2}\big)^2$. $\square$

Combining Lemmas 2.2 and 2.3 gives the following proposition.

Proposition 2.6. *On* (M, g),

$$P = \tfrac{1}{2}\nabla f(|\mathrm{Ric}|^2) + \tfrac{1}{2}|\nabla R|^2 + \mathrm{div}\big(\tfrac{1}{2}\nabla|\mathrm{Ric}|^2 - \mathrm{Ric}(\nabla R)\big),$$

where $P = K_{ij}(\lambda_i - \lambda_j)^2 + |D\mathrm{Ric}|^2 + 2\langle Z, \nabla f\rangle$.

Proof. Lemma 2.2 implies that

$$-2K_{ij}\lambda_i\lambda_j + |D\mathrm{Ric}|^2 = -\tfrac{1}{2}\nabla f(|\mathrm{Ric}|^2) - |\mathrm{Ric}|^2 + \mathrm{div}\big(\tfrac{1}{2}\nabla|\mathrm{Ric}|^2\big),$$

while Lemma 2.3 implies that

$$2\sum_i \lambda_i^3 + 2\langle Z, \nabla f\rangle = \nabla f(|\mathrm{Ric}|^2) + |\mathrm{Ric}|^2 + \tfrac{1}{2}|\nabla R|^2 - \mathrm{div}(\mathrm{Ric}(\nabla R)).$$

Adding the corresponding sides of the last two equations and noting that $2\sum_i \lambda_i^3 - 2\sum_{i,j} K_{ij}\lambda_i\lambda_j = \sum_{i,j} K_{ij}(\lambda_i - \lambda_j)^2$, we obtain Proposition 2.6. □

Remark 2.7. Clearly, $P \geq 0$ when the sectional curvature of (M, g) is nonnegative.

The proof of Theorem 1.1 will use an alternative form of Proposition 2.6 in which the term $|D\mathrm{Ric}|^2$ is replaced by $|\operatorname{div}\mathrm{Rm}|^2$. An integral from of the next lemma is proved in [Cao 2007].

Lemma 2.8. *On* (M, g),

$$|D\mathrm{Ric}|^2 = |\operatorname{div}\mathrm{Rm}|^2 + 2\langle Z, \nabla f\rangle - \tfrac{1}{2}\nabla f(|\mathrm{Ric}|^2) + \mathrm{div}\big(\tfrac{1}{2}\nabla|\mathrm{Ric}|^2 - 2Z\big).$$

Proof. As before, we fix an orthonormal basis, $\{e_1, \dots, e_n\}$, in a neighborhood of a fixed point x and assume that $D_{e_i}e_j(x) = 0$ and that each e_i is an eigenvector of Ric at x corresponding to the eigenvalue λ_i. Recall that $Z = \mathrm{Ric}(e_i, e_j)\,\mathrm{Rm}(\nabla f, e_i, e_j)$, so at x,

$$\begin{aligned}
\mathrm{div}(Z) &= \langle D_{e_k}Z, e_k\rangle = \langle D_{e_k}(\mathrm{Ric}(e_i, e_j)\,\mathrm{Rm}(\nabla f, e_i, e_j)), e_k\rangle \\
&= e_k(\mathrm{Ric}(e_i, e_j))\,\mathrm{Rm}(\nabla f, e_i, e_j, e_k) + \mathrm{Ric}(e_i, e_j)\langle D_{e_k}(\mathrm{Rm}(\nabla f, e_i, e_j)), e_k\rangle \\
&= D_{e_k}\mathrm{Ric}(e_i, e_j)\,\mathrm{Rm}(\nabla f, e_i, e_j, e_k) + \mathrm{Ric}(e_i, e_j)e_k(\mathrm{Rm}(\nabla f, e_i, e_j, e_k)) \\
&= D_{e_k}\mathrm{Ric}(e_i, e_j)\operatorname{div}\mathrm{Rm}(e_i, e_j, e_k) \\
&\qquad + \mathrm{Ric}(e_i, e_j)\big(D_{e_k}\mathrm{Rm}(\nabla f, e_i, e_j, e_k) + \mathrm{Rm}(D_{e_k}\nabla f, e_i, e_j, e_k)\big) \\
&= (D_{e_i}\mathrm{Ric}(e_j, e_k) - \mathrm{Rm}(e_k, e_i, \nabla f, e_j))\operatorname{div}\mathrm{Rm}(e_i, e_j, e_k) \\
&\qquad + \mathrm{Ric}(e_i, e_j)\operatorname{div}\mathrm{Rm}(e_j, e_i, \nabla f) + \lambda_i\,\mathrm{Rm}\big(\big(\tfrac{1}{2} - \lambda_k\big)e_k, e_i, e_i, e_k\big) \\
&= D_{e_i}\mathrm{Ric}(e_j, e_k)\operatorname{div}\mathrm{Rm}(e_i, e_j, e_k) + \operatorname{div}\mathrm{Rm}(e_j, e_i, e_k)\operatorname{div}\mathrm{Rm}(e_i, e_j, e_k) \\
&\qquad + \mathrm{Ric}(e_i, e_j)\,\mathrm{Rm}(\nabla f, e_j, e_i, \nabla f) + K_{ij}\lambda_i\big(\tfrac{1}{2} - \lambda_j\big).
\end{aligned}$$

In the above calculation, we have repeatedly used Lemma 2.1. The lemma now follows from Lemma 2.2 and the following two identities, whose proofs are easy:

$$D_{e_i}\mathrm{Ric}(e_j, e_k)\,\mathrm{div}\,\mathrm{Rm}(e_i, e_j, e_k) = 0,$$
$$\mathrm{div}\,\mathrm{Rm}(e_j, e_i, e_k)\,\mathrm{div}\,\mathrm{Rm}(e_i, e_j, e_k) = \tfrac{1}{2}|\mathrm{div}\,\mathrm{Rm}|^2. \qquad \square$$

Lemma 2.8, together with Proposition 2.6, implies the following:

Lemma 2.9. *On* (M, g),

$$Q = \nabla f(|\mathrm{Ric}|^2) + \tfrac{1}{2}|\nabla R|^2 + \mathrm{div}(2Z - \mathrm{Ric}(\nabla R)),$$

where $Q = K_{ij}(\lambda_i - \lambda_j)^2 + |\mathrm{div}\,\mathrm{Rm}|^2 + 4\langle Z, \nabla f\rangle$.

Remark 2.10. We note that $Q \geq 0$ when the sectional curvature of (M, g) is nonnegative.

The next lemma deals with the term $\nabla f(|\mathrm{Ric}|^2)$ in Lemma 2.9.

Lemma 2.11. *On* (M, g),

$$\nabla f(|\mathrm{Ric}|^2) = \tfrac{1}{2}|\nabla R|^2 + \tfrac{1}{2}\langle \nabla f, \nabla R\rangle + \tfrac{1}{2}\nabla f(\langle \nabla f, \nabla R\rangle) + \mathrm{div}\big(D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle \nabla f, \nabla R\rangle\big). \tag{2-4}$$

Proof. It follows from Lemma 2.1(1) and (3) that

$$\tfrac{1}{2}\nabla f(\Delta R) = -\nabla f(|\mathrm{Ric}|^2) + \tfrac{1}{2}\langle \nabla f, \nabla R\rangle + \tfrac{1}{2}\nabla f(\langle \nabla f, \nabla R\rangle).$$

The Bochner–Weitzenböck formula implies that

$$\begin{aligned}
\mathrm{div}\big(\tfrac{1}{2}\nabla\langle \nabla f, \nabla R\rangle\big) &= \tfrac{1}{2}\Delta\langle \nabla f, \nabla R\rangle \\
&= \langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\rangle + \tfrac{1}{2}\nabla f(\Delta R) + \tfrac{1}{2}\nabla R(\Delta f) + \mathrm{Ric}(\nabla f, \nabla R) \\
&= \langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\rangle + \tfrac{1}{2}\nabla f(\Delta R) + \tfrac{1}{2}\nabla R\big(\tfrac{n}{2} - R\big) + \tfrac{1}{2}|\nabla R|^2 \\
&= \langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\rangle + \tfrac{1}{2}\nabla f(\Delta R).
\end{aligned}$$

But,

$$\begin{aligned}
\mathrm{div}(D_{\nabla R}\nabla f) &= \langle D_{e_i}D_{\nabla R}\nabla f, e_i\rangle = e_i\langle D_{\nabla R}\nabla f, e_i\rangle = e_i\langle D_{e_i}\nabla f, \nabla R\rangle \\
&= \big\langle D_{e_i}\big(\tfrac{1}{2}e_i - \mathrm{Ric}(e_i)\big), \nabla R\big\rangle + \big\langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\big\rangle \\
&= -D_{e_i}\mathrm{Ric}(e_i, \nabla R) + \langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\rangle \\
&= -\tfrac{1}{2}|\nabla R|^2 + \langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\rangle.
\end{aligned}$$

The lemma follows. $\square$

We now have the following proposition which will be used in the proof of Theorem 1.1.

Proposition 2.12. *On* (M, g),

$$Q = |\nabla R|^2 + \tfrac{1}{2}\langle \nabla f, \nabla R\rangle + \tfrac{1}{2}\nabla f(\langle \nabla f, \nabla R\rangle) + \operatorname{div}\bigl(2Z - \operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle \nabla f, \nabla R\rangle\bigr).$$

Proof. This is merely a consequence of Lemmas 2.9 and 2.11. □

3. Proof of Theorem 1.1

We will use ϕ to denote a real-valued nonnegative C^4 function on $\mathbb{R}$ and write $\phi \circ f$ as $\phi(f)$. We will show that R is a constant function and then appeal to [Petersen and Wylie 2009] to complete the proof. We begin with the following proposition.

Proposition 3.1. *On* (M, g),

$$\phi(f)Q = \tfrac{1}{2}\langle \nabla f, \nabla R\rangle\bigl((\phi-\phi')(f) - (\phi+\phi')(f)\Delta f - (\phi''+\phi')(f)|\nabla f|^2\bigr) + (\phi+\phi')(f)|\nabla R|^2 - 2\phi'\langle Z, \nabla f\rangle + \operatorname{div}(X), \tag{3-1}$$

where

$$X = \tfrac{1}{2}\langle \nabla f, \nabla R\rangle(\phi'+\phi)(f)\nabla f + \phi(f)\bigl(2Z - \operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle \nabla f, \nabla R\rangle\bigr).$$

Proof. We multiply each side of the equation in Proposition 2.12 by $\phi(f)$ to get

$$\begin{aligned}\phi(f)Q = \phi(f)|\nabla R|^2 &+ \frac{\phi(f)}{2}\langle \nabla f, \nabla R\rangle + \frac{\phi(f)}{2}\nabla f(\langle \nabla f, \nabla R\rangle) \\ &- \phi'(f)\langle 2Z - \operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle \nabla f, \nabla R\rangle, \nabla f\rangle \\ &+ \operatorname{div}\bigl(\phi(f)\bigl(2Z - \operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle \nabla f, \nabla R\rangle\bigr)\bigr).\end{aligned}$$

It follows from the soliton equation and Lemma 2.1(1) that

$$\begin{aligned}\langle -\operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f, \nabla f\rangle &= \bigl\langle \tfrac{1}{2}\nabla R - 2\operatorname{Ric}(\nabla R), \nabla f\bigr\rangle \\ &= \tfrac{1}{2}\langle \nabla f, \nabla R\rangle - |\nabla R|^2.\end{aligned}$$

We thus obtain

$$\begin{aligned}\phi(f)Q = (\phi+\phi')(f)|\nabla R|^2 &+ \frac{\phi-\phi'}{2}(f)\langle \nabla f, \nabla R\rangle \\ &- 2\phi'\langle Z, \nabla f\rangle + \frac{\phi+\phi'}{2}(f)\nabla f(\langle \nabla f, \nabla R\rangle) \\ &+ \operatorname{div}\bigl(\phi(f)\bigl(2Z - \operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle \nabla f, \nabla R\rangle\bigr)\bigr).\end{aligned}\tag{3-2}$$

Now, we observe that

$$\begin{aligned}(\phi+\phi')(f)\nabla f(\langle \nabla f, \nabla R\rangle) &= \langle \nabla\langle \nabla f, \nabla R\rangle, (\phi'+\phi)(f)\nabla f\rangle \\ &= -\langle \nabla f, \nabla R\rangle\bigl((\phi'+\phi)(f)\Delta f + (\phi''+\phi')(f)|\nabla f|^2\bigr) \\ &\quad + \operatorname{div}\bigl(\langle \nabla f, \nabla R\rangle(\phi'+\phi)(f)\nabla f\bigr).\end{aligned}$$

Substituting the above into (3-2), we obtain (3-1). Proposition 3.1 is thus proved. $\square$

The idea now is to choose an appropriate function ϕ and integrate (3-1) over M. The divergence term, after integration, vanishes because of the fall-off condition we impose. The right-hand side will then be nonpositive while the left is always nonnegative, and consequently, R is a constant. Theorem 1.1 follows from [Petersen and Wylie 2009].

Proof of Theorem 1.1. We normalize f by adding a constant so that Lemma 2.1(2) takes the form $|\nabla f|^2 = f - R$. Since $R \geq 0$, we always have $|\nabla f|^2 \leq f$. On the other hand, since R is assumed to be bounded and f grows quadratically with respect to the distance from a fixed point [Cao and Zhou 2010; Naber 2006], we have $|\nabla f|^2 \geq \frac{1}{2} f$, when f is sufficiently large. Thus, there exists $T > 2$ so that when $f \geq T$,

$$\tfrac{1}{2} f \leq |\nabla f|^2 \leq f. \tag{3-3}$$

Fix $0 < \eta < \delta$ and define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(t) = 0$ for $t \leq T$, and $\phi(t) = (t - T)^k e^{\eta t}$ for $t \geq T$, where k is a sufficiently large number to be determined. Throughout this section, we will use this ϕ in (3-1). By our fall-off assumption, there exists a sequence $t_i \to \infty$ such that

$$\int_{f=t_i} e^{\delta f} \frac{1}{|\nabla f|} |D\,\mathrm{Ric}| \to 0, \quad \text{as } i \to \infty.$$

From this, we now deduce that

$$\int_{f \leq t_i} \mathrm{div}(X) = \int_{f=t_i} \frac{\langle X, \nabla f\rangle}{|\nabla f|} \to 0, \quad \text{as } i \to \infty. \tag{3-4}$$

To this end, we look at each of the five terms in X and denote by X_i the i-th term. Then, when $f > T$,

$$\frac{|\langle X_1, \nabla f\rangle|}{|\nabla f|} = \tfrac{1}{2} |\langle \nabla f, \nabla R\rangle| (\phi' + \phi)(f) |\nabla f| \leq C_1 f^{k+1} e^{\eta f} |\nabla R|,$$

where C_1 is a constant depending only on k and η. Now by the Cauchy–Schwarz inequality,

$$|D\,\mathrm{Ric}|^2 = \sum_{i,j,k} (D_{e_i} \mathrm{Ric}(e_j, e_k))^2 \geq \frac{1}{n} \sum_i \Bigl(\sum_j D_{e_i} \mathrm{Ric}(e_j, e_j) \Bigr)^2 = \frac{1}{n} |\nabla R|^2.$$

Thus,

$$|\nabla R| \leq \sqrt{n}\, |D\,\mathrm{Ric}|.$$

Hence,

$$\frac{|\langle X_1, \nabla f\rangle|}{|\nabla f|} \leq C_1 \sqrt{n} f^{k+1} e^{\eta f} |D\,\mathrm{Ric}|.$$

Integrating the above over $\{f = t_i\}$ and noting that

$$C_1\sqrt{n}\, f^{k+1} e^{\eta f} |D\,\mathrm{Ric}| \le e^{\delta f} \frac{|D\,\mathrm{Ric}|}{|\nabla f|},$$

when f is sufficiently large, we conclude that

$$\int_{f=t_i} \frac{|\langle X_1, \nabla f\rangle|}{|\nabla f|} \to 0, \quad \text{as } i \to \infty.$$

Now note that $\langle X_2, \nabla f\rangle = 2\phi\langle Z, \nabla f\rangle = 2\phi \sum_i \lambda_i \,\mathrm{Rm}(\nabla f, e_i, e_i, \nabla f)$. Since Ric is assumed to be bounded and since the sectional curvature is nonnegative,

$$\frac{|\langle X_2, \nabla f\rangle|}{|\nabla f|} \le C_2 f^{k-1/2} e^{\eta f} \mathrm{Ric}(\nabla f, \nabla f) = C_2 f^{k-1/2} e^{\eta f} \tfrac{1}{2}\langle \nabla f, \nabla R\rangle,$$

where C_2 is a constant dependent only on the bound of Ric, and the last equality follows from Lemma 2.1. Hence, when f is sufficiently large,

$$\frac{|\langle X_2, \nabla f\rangle|}{|\nabla f|} \le \tfrac{1}{2} C_2 f^k e^{\eta f} |\nabla R| \le e^{\delta f} \frac{|D\,\mathrm{Ric}|}{|\nabla f|}.$$

It then follows that

$$\int_{f=t_i} \frac{|\langle X_2, \nabla f\rangle|}{|\nabla f|} \to 0, \quad \text{as } i \to \infty.$$

The arguments for the other X_i are similar; we will skip X_3 and X_4. Now look at X_5. Repeatedly using Lemma 2.1(2), we see that

$$\begin{aligned}\langle X_5, \nabla f\rangle &= -\tfrac{1}{2}\phi \nabla f(\langle \nabla f, \nabla R\rangle) = -\phi\nabla f(\mathrm{Ric}(\nabla f, \nabla f))\\ &= -\phi\big(D_{\nabla f}\mathrm{Ric}(\nabla f, \nabla f) + 2\mathrm{Ric}(D_{\nabla f}\nabla f, \nabla f)\big)\\ &= -\phi\big(D_{\nabla f}\mathrm{Ric}(\nabla f, \nabla f) + \mathrm{Ric}(\nabla f - \nabla R, \nabla f)\big)\\ &= -\phi\big(D_{\nabla f}\mathrm{Ric}(\nabla f, \nabla f) + \tfrac{1}{2}\langle \nabla f, \nabla R\rangle - \mathrm{Ric}(\nabla R, \nabla f)\big).\end{aligned}$$

Since $|\nabla R|$ can be bounded by $|D\,\mathrm{Ric}|$, we have $|\langle X_5, \nabla f\rangle| \le C_5 e^{\eta f} f^{k+3} |D\,\mathrm{Ric}|$. Equation (3-4) then follows.

To simplify notations, we put

$$\begin{aligned}F = \tfrac{1}{2}\langle \nabla f, \nabla R\rangle\big((\phi - \phi')(f) - (\phi + \phi')(f)\Delta f - (\phi'' + \phi')(f)|\nabla f|^2\big)\\ + (\phi + \phi')(f)|\nabla R|^2 - 2\phi'\langle Z, \nabla f\rangle.\end{aligned}$$

Then,

$$\phi(f) Q = F + \mathrm{div}(X).$$

It follows easily from the arguments in the proof of (3-4) that $\int_M F\, d\mathrm{vol}_g < \infty$. We thus have

$$\int_M \phi(f) Q = \int_M F. \tag{3-5}$$

We now show that $\int_M F\, d\mathrm{vol}_g \le 0$. First, we note that $-\Delta f = R - n/2 \le \Lambda$, where Λ is an upper bound of R; hence $-(\phi+\phi')(f)\Delta f \le \Lambda(\phi+\phi')$, as ϕ and ϕ' are both nonnegative. Next, we observe that, by Lemma 2.1,

$$|\nabla R|^2 = 2\mathrm{Ric}(\nabla f, \nabla R) = 2\sum_i \lambda_i e_i(f) e_i(R)$$

and $e_i(R) = \langle \nabla R, e_i\rangle = 2\mathrm{Ric}(\nabla f, e_i) = 2\lambda_i e_i(f)$. So for each i, $e_i(f)e_i(R) \ge 0$. Hence $|\nabla R|^2 \le 2\Lambda\langle \nabla f, \nabla R\rangle$. Finally, we recall that $\langle Z, \nabla f\rangle \ge 0$ (Remark 2.4). We thus conclude, from (3-3), that

$$F \le \tfrac{1}{2}\langle \nabla f, \nabla R\rangle F_1, \tag{3-6}$$

where

$$F_1 = (\phi-\phi')(f) + \Lambda(\phi+\phi')(f) + 4\Lambda(\phi+\phi') - \tfrac{1}{2} f(\phi''+\phi')(f).$$

It follows from (3-5) and (3-6) that

$$\int_M \phi(f) Q \le \frac{1}{2}\int_M \langle \nabla f, \nabla R\rangle F_1. \tag{3-7}$$

A direct computation leads to

$$\begin{aligned} F_1 &= (\phi-\phi')(t)+\Lambda(\phi+\phi')(t)+4\Lambda(\phi+\phi')(t)-\tfrac{1}{2}t(\phi''+\phi')(t) \\ &= -\tfrac{1}{2}\delta(1+\delta)(t-T)^{k+1}e^{\delta t} - \Big(\tfrac{1}{2}(1+2\delta)k - 5(1+\delta)\Lambda - 1 + \frac{T-2}{2}\delta\Big)(t-T)^k e^{\delta t} \\ &\qquad -k\big(\tfrac{1}{2}(k-1) - 5\Lambda + \tfrac{1}{2}T + \big)1(t-T)^{k-1}e^{\delta t} - \tfrac{1}{2}T\phi''. \end{aligned}$$

If we choose $k > 10\Lambda + 2$, the above expression will clearly be negative for $t > T$. We have therefore shown that $F_1 \le 0$ everywhere and $F_1 < 0$ where $f > T$. Since $Q \ge 0$ (Remark 2.10) and $\langle \nabla f, \nabla R\rangle = 2\mathrm{Ric}(\nabla f, \nabla f) \ge 0$ (Lemma 2.1), we conclude from (3-7) that $\langle \nabla f, \nabla R\rangle = 0$ in the region $\{f > T\}$. But as we noted earlier in the proof, $|\nabla R|^2 \le 2\Lambda\langle \nabla f, \nabla R\rangle$. Hence $\nabla R = 0$ in the region $\{f > T\}$. The analyticity of the metric [Bando 1987; Kotschwar 2013] then implies that R is a constant function. Theorem 1.1 then follows from [Petersen and Wylie 2009]. □

4. Proof of Theorem 1.2

We first show that the Ricci tensor has a zero eigenvalue at any point p in C, then show that the soliton splits in a neighborhood of p, which, in turn, implies that the scalar curvature is a constant.

Let C be the critical manifold of minima of f. Since C is assumed to be nondegenerate, the Morse–Bott lemma implies that for any point $p \in C$, there exists an open neighborhood U of p and a diffeomorphism $\phi : U \to \mathbb{R}^n$ such that $\phi(U \cap C) = \{(0, \dots, 0, x_{m+1}, \dots, x_n)\}$, $\phi(p) = 0$ and $f \circ \phi^{-1}(x_1, \dots, x_n) = c + \frac{1}{4}(x_1^2 + \cdots + x_m^2)$.

In what follows in this section, unless specified otherwise, the range for the Greek letters $\alpha, \beta, \dots$ is 1 to m while that for the Latin letters $i, j, \dots$ is $m+1$ to n.

We observe that we may assume that for all α and i, $g^{\alpha i}(p) = 0$. In fact, by making a change of variables, $y_\alpha = x_\alpha$ and $y_i = x_i - \sum_{\beta=1}^m g^{i\beta}(p)x_\beta$, we see that in the new coordinates, at p, $g^{\alpha i} = \langle \nabla y_\alpha, \nabla y_i\rangle = 0$ for α and i. Moreover, $f(y_1, \dots, y_m, y_{m+1}, \dots, y_n) = c + \frac{1}{4}(y_1^2 + \cdots + y_m^2)$. From now on, we assume in the original coordinates $(x_1, \dots, x_n)$ that $g^{\alpha i}(p) = 0$ for all α and i. As a consequence, we also have $g_{\alpha i}(p) = 0$.

Next lemma computes the Ricci tensor at p.

Lemma 4.1. *At p, we have*

$$\mathrm{Ric}(p)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = \tfrac{1}{2}(g_{\alpha\beta}(p) - \delta_{\alpha\beta}), \quad \mathrm{Ric}(p)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \tfrac{1}{2} g_{ij},$$
$$\mathrm{Ric}(p)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_i}\right) = 0.$$

Proof. Since

$$\nabla f = \tfrac{1}{2} g^{\alpha\beta} x_\alpha \frac{\partial}{\partial x_\beta} + \tfrac{1}{2} g^{\alpha i} x_\alpha \frac{\partial}{\partial x_i},$$

we have at p,

$$\mathrm{Hess}(f)(p)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = \tfrac{1}{2}\delta_{\alpha\beta},$$
$$\mathrm{Hess}(f)(p)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_i}\right) = \mathrm{Hess}(f)(p)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = 0.$$

The lemma follows from the soliton equation. □

Let μ_γ^{-1} ($\gamma = 1, \dots, m$) denote the eigenvalues of the positive definite symmetric matrix $g_{\alpha\beta}(p)$. Then there exists $(v_{1\gamma}, \dots, v_{m\gamma}) \neq 0$ such that $\sum_\beta g_{\alpha\beta}(p) v_{\beta\gamma} = \mu_\gamma^{-1} v_{\alpha\gamma}$. Let $v_\gamma = \sum_\alpha v_{\alpha\gamma}(\partial/\partial x_\alpha)$. The first part of Lemma 4.1 implies that

$$\begin{aligned}
\mathrm{Ric}(p)(v_\gamma, v_\gamma) &= \sum_{\alpha,\beta} v_{\alpha\gamma} v_{\beta\gamma} \mathrm{Ric}(p)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) \\
&= \tfrac{1}{2}(\mu_\gamma^{-1} - 1) \sum_\alpha (v_{\alpha\gamma})^2 \\
&= \tfrac{1}{2}(\mu_\gamma^{-1} - 1)\mu_\gamma g(p)(v_\gamma, v_\gamma) \\
&= \tfrac{1}{2}(1 - \mu_\gamma) g(p)(v_\gamma, v_\gamma).
\end{aligned}$$

We conclude from this and the rest of Lemma 4.1 that the eigenvalues of the Ricci tensor at p are $(1-\mu_\alpha)/2$, with $\alpha = 1, \dots, m$, and $\frac{1}{2}$ with multiplicity $n - m$. Since the Ricci tensor is assumed to be semipositive definite, $\mu_\alpha \leq 1$ for each α. Of course, $\mu_\alpha > 0$. Our goal is to show that $\mu_\alpha = 1$.

Now assume $\{e_1, \dots, e_n\}$ is an orthonormal basis in a neighborhood of a fixed point $p \in C$ with $D_{e_i} e_j(p) = 0$ for $1 \le i, j \le n$. We may assume that each e_α is an eigenvector of Ric at p corresponding to the eigenvalue $(1-\mu_\alpha)/2$ for $1 \le \alpha \le m$ and e_i an eigenvector corresponding to $\frac{1}{2}$ for $m+1 \le i \le n$.

By our assumption, $D\mathrm{Ric} = D^2\mathrm{Ric} = 0$ at p. Hence, for each $1 \le s \le n$, in the neighborhood of p,

$$\mathrm{Ric}(e_s, e_s) = r_s + \sum_{i,j,k=1}^{n} r_{sijk} x_i x_j x_k + \text{higher-order terms},$$

where r_s and r_{sijk} are constants. We make the following observation.

Lemma 4.2. *Given that $K_{s\alpha}$ is the sectional curvature of the section spanned by e_s and e_α, we have*

$$r_\alpha = \frac{1-\mu_\alpha}{2}, \quad \alpha = 1, \dots, m, \qquad r_i = \tfrac{1}{2}, \quad i = m+1, \dots, n, \qquad \sum_{\alpha=1}^{m} K_{s\alpha}\mu_\alpha = 0,$$

Proof. We only need to prove the second line. At p,

$$(\Delta\mathrm{Ric})(e_s, e_s) = \Delta(\mathrm{Ric}(e_s, e_s)) = 0.$$

On the other hand, we have $\Delta\mathrm{Ric} = D_{\nabla f}\mathrm{Ric} + \mathrm{Ric} - 2\sum_{l=1}^{n} \mathrm{Rm}(\,\cdot\,, e_l, \mathrm{Ric}(e_l), \cdot\,)$ (Lemma 2.1 in [Petersen and Wylie 2010], see also the proof of Lemma 2.2). Hence,

$$\begin{aligned}
0 &= \mathrm{Ric}(e_s, e_s) - 2\sum_{l=1}^{n} \mathrm{Rm}(e_s, e_l, \mathrm{Ric}(e_l), e_s) \\
&= r_s - 2\sum_{\alpha=1}^{m} \mathrm{Rm}(e_s, e_\alpha, \mathrm{Ric}(e_\alpha), e_s) - 2\sum_{i=m+1}^{n} \mathrm{Rm}(e_s, e_i, \mathrm{Ric}(e_i), e_s) \\
&= r_s - \sum_{\alpha=1}^{m} (1-\mu_\alpha)\,\mathrm{Rm}(e_s, e_\alpha, e_\alpha, e_s) - \sum_{i=m+1}^{n} \mathrm{Rm}(e_s, e_i, e_i, e_s) \\
&= \sum_{\alpha=1}^{m} K_{s\alpha}\mu_\alpha. \qquad \square
\end{aligned}$$

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. It follows from Lemma 4.2 and the assumption of nonnegative sectional curvature that $K_{s\alpha}(p) = 0$ for all $1 \le s \le n$. So, $\mathrm{Ric}(p)$ vanishes on the subspace spanned by $\{\partial/\partial x_\alpha | \alpha = 1, \dots, m\}$.

We first prove that a neighborhood of p splits isometrically as $U \times V$, where U is at least m-dimensional and $\mathrm{Ric} \equiv 0$ on U. We have shown that $\mathrm{Ric}_{\alpha\beta}(p) = 0$. The rest of the argument is along the lines of the proof of Lemma 8.2 in [Hamilton

1986] and that of Corollary 2.1 in [Ni and Tam 2003]. Denote by $K(x,t)$ the null space of $\mathrm{Ric}(x,t)$, i.e.,

$$K(x,t)=\{w\in T_xM \mid \mathrm{Ric}(x,t)(w)=0\}.$$

Let $w_0\in K(p,-1)$ and $\gamma(s)$ a smooth curve starting from p. Parallel translating w_0 along γ gives a vector field w along γ. Denote the extension of w to a neighborhood of γ still by w. Now we project w onto $K(x,t)$ to get a vector field $v(x,t)$. Then $v(\gamma(s),t)\in K(\gamma(s),t)$. We first show that $D_{\gamma'}v$ is also in $K(\gamma(s),t)$. We fix an orthonormal basis in $g(t)$, $\{e_1,\dots,e_n\}$, in a neighborhood of a fixed point $\gamma(s)$ and assume that $e_i(\gamma(s))$ are the eigenvectors of Ric. For simplicity, we denote $e_i(\gamma(s))$ by $e_i(s)$. Since $\mathrm{Ric}(v)=0$, we have $((\partial/\partial t)\mathrm{Ric})(v,v)=0$. The evolution equation for Ricci tensor then implies that at $\gamma(s)$,

$$(\Delta\mathrm{Ric})(v,v)-2\langle\mathrm{Ric}(v),\mathrm{Ric}(v)\rangle+2\mathrm{Ric}(e_i,e_i)K(e_i,v)=0,$$

where the repeated indices are being summed over. Since the sectional curvature $K(e_i,v)$ is nonnegative and since $\mathrm{Ric}(v)=0$, we deduce that $(\Delta\mathrm{Ric})(v,v)\le 0$. Direct computations give

$$\begin{aligned}(\Delta\mathrm{Ric})(v,v)=\Delta(\mathrm{Ric}(v,v))-4e_i(\mathrm{Ric}(v,D_{e_i}v))+2\mathrm{Ric}(v,D_{e_i}D_{e_i}v)\\ +2\mathrm{Ric}(v,D_{D_{e_i}e_i}v)+2\mathrm{Ric}(D_{e_i}v,D_{e_i}v).\end{aligned}$$

Using $(\Delta\mathrm{Ric})(v,v)\le 0$ and $\mathrm{Ric}(v)=0$, we obtain $\mathrm{Ric}(D_{e_i}v,D_{e_i}v)\le 0$. Since Ric is positive semidefinite, we conclude that $\mathrm{Ric}(D_{e_i}v)=0$ for each i, and hence $D_{\gamma'}v\in K(\gamma(s),t)$. As in the proof of Corollary 2.1 in [Ni and Tam 2003], we conclude that $w\in K(x,t)$. Since parallel translation preserves inner product, for each fixed t, the dimension of $K(x,t)$ is independent of x. We then use the de Rham decomposition theorem to conclude that a neighborhood of p splits.

Note that $|\nabla f|^2\ge f$ on $U\times V$. In fact, for any $q\in V$, the restriction of g and f on $U\times\{q\}$ gives a soliton on $U\times\{q\}$ with zero Ric tensor. Lemma 2.1(2) implies that $|\nabla_{U\times\{q\}}f|^2=f|_{U\times\{q\}}$, where $\nabla_{U\times\{q\}}f$ is the gradient of $f|_{U\times\{q\}}$ with respect to the metric $g|_{U\times\{q\}}$. Since $|\nabla f|^2\ge|\nabla_{U\times\{q\}}f|^2$, we infer that $|\nabla f|^2(x,q)\ge f(x,q)$ for all $x\in U$. Since q is an arbitrary point in V, it follows that $|\nabla f|^2\ge f$ on $U\times V$.

We now prove that $|\nabla f|^2\le f$ on $U\times V$. Given any point $y\in U\times V$, denote by $\gamma(s)$ the integral curve of $\nabla f/|\nabla f|^2$ such that $\gamma(0)=y$. Then $f(\gamma(s))=s+f(\gamma(0))$. On the other hand, using Lemma 2.1(1) and (2), we have

$$\begin{aligned}\frac{d}{ds}|\nabla f|^2(\gamma(s))&=\frac{1}{|\nabla f|^2}\nabla f(|\nabla f|^2)=\frac{1}{|\nabla f|^2}(|\nabla f|^2-\langle\nabla f,\nabla R\rangle)\\&=\frac{1}{|\nabla f|^2}(|\nabla f|^2-2\mathrm{Ric}(\nabla f,\nabla f)).\end{aligned}$$

Since $\mathrm{Ric}(\nabla f, \nabla f) \geq 0$, we obtain $(d/ds)|\nabla f|^2(\gamma(s)) \leq 1$. Integrating this inequality from $-f(\gamma(0))$ to s and noting that $\nabla f(\gamma(s)) = 0$ at $s = -f(\gamma(0))$ give us the desired inequality $|\nabla f|^2 \leq f$.

We have thus proved that $|\nabla f|^2 = f$, which, when combined with Lemma 2.1(2), implies that R is constant in a neighborhood of p. Hence R is constant on the entire M. The proof of Theorem 1.2 is therefore completed. □

Acknowledgements

I thank Professors Peter Petersen and DaGang Yang for their interests in this work and for helpful discussions. I thank Professor Ovidiu Munteanu for pointing out an error in an earlier version of the paper. I also thank the referee for the thorough review and helpful suggestions.

References

[Bando 1987] S. Bando, "Real analyticity of solutions of Hamilton's equation", *Math. Z.* **195**:1 (1987), 93–97. MR 88i:53073 Zbl 0606.58051

[Böhm and Wilking 2008] C. Böhm and B. Wilking, "Manifolds with positive curvature operators are space forms", *Ann. of Math.* (2) **167**:3 (2008), 1079–1097. MR 2009h:53146 Zbl 1185.53073

[Cao 2007] X. Cao, "Compact gradient shrinking Ricci solitons with positive curvature operator", *J. Geom. Anal.* **17**:3 (2007), 425–433. MR 2008h:53119 Zbl 1135.53044

[Cao and Zhou 2010] H.-D. Cao and D. Zhou, "On complete gradient shrinking Ricci solitons", *J. Differential Geom.* **85**:2 (2010), 175–186. MR 2011k:53040 Zbl 1246.53051

[Cao et al. 2011] X. Cao, B. Wang, and Z. Zhang, "On locally conformally flat gradient shrinking Ricci solitons", *Commun. Contemp. Math.* **13**:2 (2011), 269–282. MR 2012e:53074 Zbl 1215.53061

[Chen 2009] B.-L. Chen, "Strong uniqueness of the Ricci flow", *J. Differential Geom.* **82**:2 (2009), 363–382. MR 2010h:53095 Zbl 1177.53036

[Fernández-López and García-Río 2011] M. Fernández-López and E. García-Río, "Rigidity of shrinking Ricci solitons", *Math. Z.* **269**:1-2 (2011), 461–466. MR 2012g:53072 Zbl 1226.53047

[Hamilton 1986] R. S. Hamilton, "Four-manifolds with positive curvature operator", *J. Differential Geom.* **24**:2 (1986), 153–179. MR 87m:53055 Zbl 0628.53042

[Hamilton 1988] R. S. Hamilton, "The Ricci flow on surfaces", pp. 237–262 in *Mathematics and general relativity* (Santa Cruz, CA, 1986), edited by J. A. Isenberg, Contemp. Math. **71**, Amer. Math. Soc., Providence, RI, 1988. MR 89i:53029 Zbl 0663.53031

[Hamilton 1995] R. S. Hamilton, "The formation of singularities in the Ricci flow", pp. 7–136 in *Surveys in differential geometry, II* (Cambridge, MA, 1993), edited by C. C. Hsiung and S.-T. Yau, Int. Press, Cambridge, MA, 1995. MR 97e:53075 Zbl 0867.53030

[Ivey 1993] T. Ivey, "Ricci solitons on compact three-manifolds", *Differential Geom. Appl.* **3**:4 (1993), 301–307. MR 94j:53048 Zbl 0788.53034

[Kotschwar 2013] B. L. Kotschwar, "A local version of Bando's theorem on the real-analyticity of solutions to the Ricci flow", *Bull. Lond. Math. Soc.* **45**:1 (2013), 153–158. MR 3033963 Zbl 1259.53065

[Morgan and Tian 2007] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs **3**, Amer. Math. Soc., Providence, RI, 2007. MR 2008d:57020 Zbl 1179.57045

[Naber 2006] A. Naber, “Some geometry and analysis on Ricci solitons”, preprint, 2006. arXiv math/0612532

[Ni and Tam 2003] L. Ni and L.-F. Tam, “Plurisubharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature”, *J. Differential Geom.* **64**:3 (2003), 457–524. MR 2005a:32023 Zbl 1088.32013

[Ni and Wallach 2008] L. Ni and N. Wallach, “On a classification of gradient shrinking solitons”, *Math. Res. Lett.* **15**:5 (2008), 941–955. MR 2010i:53127 Zbl 1158.53052

[Perelman 2003] G. Perelman, “Ricci flow with surgery on three manifolds”, preprint, 2003. arXiv math/0303109

[Petersen and Wylie 2009] P. Petersen and W. Wylie, “Rigidity of gradient Ricci solitons”, *Pacific J. Math.* **241**:2 (2009), 329–345. MR 2010j:53071 Zbl 1176.53048

[Petersen and Wylie 2010] P. Petersen and W. Wylie, “On the classification of gradient Ricci solitons”, *Geom. Topol.* **14**:4 (2010), 2277–2300. MR 2012a:53060 Zbl 1202.53049

Received March 21, 2014. Revised December 23, 2014.

MINGLIANG CAI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MIAMI
CORAL GABLES, FL 33124
UNITED STATES

mcai@math.miami.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.77

FROM QUASIMODES TO RESONANCES: EXPONENTIALLY DECAYING PERTURBATIONS

ORAN GANNOT

We consider self-adjoint operators of black-box type which are exponentially close to the free Laplacian near infinity, and prove an exponential bound for the resolvent in a strip away from resonances. Here the resonances are defined as poles of the meromorphic continuation of the resolvent between appropriate exponentially weighted spaces. We then use a local version of the maximum principle to prove that any cluster of real quasimodes generates at least as many resonances, with multiplicity, rapidly converging to the quasimodes.

1. Introduction

It is expected that for open systems, trapping of classical trajectories produces scattering resonances close to the real axis; this is often referred to as the Lax–Phillips conjecture [1989, Section V.3]. When trapping is weak, for instance in the sense of hyperbolicity, the general conjecture is not true, as shown by Ikawa [1982]. For an account of recent results about resonances near the real axis under weak trapping; see the review by Wunsch [2012]. On the other hand, when the trapping is sufficiently strong so that a construction of real quasimodes is possible, there exist resonances close to the quasimodes [Stefanov and Vodev 1996; Tang and Zworski 1998; Stefanov 1999]. These results were established in the setting of compactly supported perturbations, or more generally for perturbations which are dilation analytic near infinity [Sjöstrand and Zworski 1991; Sjöstrand 1997].

Complementary to the aforementioned results, in this note we prove analogues for "black box" operators which are exponentially close to the free Laplacian at infinity. More precisely, we allow both metric and potential perturbations of the Laplacian outside a compact set (the black box), but require only minimal assumptions on the operator in the black box. Standard techniques give a meromorphic continuation of the exponentially weighted resolvent through the real axis to a strip whose width is of size $O(h)$; the choice of exponential weight and the width of the strip depend on the decay rate of the perturbation. We then apply a complex analytic

MSC2010: primary 35P25; secondary 47F05, 47A40.

Keywords: scattering resonances, quasimodes, exponentially decaying potentials.

framework — summarized, for example, in [Petkov and Zworski 2001] — to deduce an exponential a priori bound on the weighted resolvent away from resonances.

A typical application of such an exponential bound — well-established in [Stefanov and Vodev 1996; Tang and Zworski 1998; Stefanov 1999; 2005] — is to show that any family of sufficiently independent quasimodes generates at least as many resonances, counting multiplicity; these resonances converge rapidly not only to the real axis, but to the quasimodes; see Theorem B for a precise statement. The general assumptions are presented beginning in Section 1B.

One motivation for this work comes from a recent investigation of resonances for Schwarzschild–AdS black holes, where quasimodes have been constructed [Gannot 2014; Holzegel and Smulevici 2014]. Due to the spherical symmetry in that setting, the stationary wave operator P decomposes as a sum of one-dimensional operators P_ℓ on a half-line, which are just restrictions to spaces of spherical harmonics with angular momentum ℓ. Each P_ℓ is a self-adjoint perturbation of the Laplacian by an exponentially decaying potential which is singular near the origin — the results of this paper imply that the resolvent $R_\ell(\sigma)$ of P_ℓ has a meromorphic continuation through the real axis. Although meromorphy of each one-dimensional resolvent does not imply meromorphy for the global resolvent (this requires uniform control as $\ell \to \infty$ and was recently established in the Schwarzschild–AdS setting; see [Warnick 2015]), the results of this paper do imply the existence of a sequence of poles σ_ℓ for $R_\ell(\sigma)$ satisfying

$$0 < -\operatorname{Im}\sigma_\ell < Ce^{-\ell/C} \quad \text{for } \ell \text{ sufficiently large.}$$

We also remark that in the Schwarzschild–AdS case the effective potential is dilation analytic, so the results of [Sjöstrand 1997] indeed apply. One advantage to the approach taken here is that the exponential decay of the potential remains stable under small (radial, static) perturbations of the Schwarzschild–AdS metric.

1A. ***Free resolvent.*** We begin by gathering several results about the free resolvent. The Laplacian $-\Delta$ on $\mathbb{R}^n$ with domain $H^2(\mathbb{R}^n)$ is self-adjoint and we denote by $R_0(\sigma)$ the free resolvent

$$R_0(\sigma) = (-\Delta - \sigma^2)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n), \quad \operatorname{Im}\sigma > 0.$$

Choose $\varphi \in C^\infty(\mathbb{R}^n)$ with the property that $\varphi(x) = |x|$ for $|x|$ large enough. If $\mathcal{A}$ denotes some function space, we will use the notation $\mathcal{A}_\gamma = e^{-\gamma\varphi}\mathcal{A}$ for its weighted counterpart. We will also freely move between the equivalent notions

$$T : \mathcal{A}_\alpha \to \mathcal{B}_\beta \iff e^{\beta\varphi}Te^{-\alpha\varphi} : \mathcal{A} \to \mathcal{B},$$

depending on convenience.

Our starting point is the well known fact [McLeod 1967] that for each $\gamma > 0$ the weighted resolvent

$$e^{-\gamma\varphi} R_0(\sigma) e^{-\gamma\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

extends holomorphically across $\operatorname{Re}\sigma > 0$ as a bounded operator to the strip $\operatorname{Im}\sigma > -\gamma$, with the usual caveats in even dimensions when winding around the origin. We also have the standard representation,

$$e^{-\gamma\varphi} R_0(\sigma) e^{-\gamma\varphi} = e^{-\gamma\varphi} R_0(-\sigma) e^{-\gamma\varphi} + \sigma^{n-2} e^{-\gamma\varphi} M(\sigma) e^{-\gamma\varphi} \tag{1-1}$$

whenever $\operatorname{Re}\sigma > 0$ and $-\gamma < \operatorname{Im}\sigma < 0$. Here $M(\sigma)$ is the operator with kernel

$$M(\sigma, x, y) = (i/2)(2\pi)^{-n+1} \int_{\mathbb{S}^{n-1}} e^{i\sigma\langle\omega, x-y\rangle}\, d\omega.$$

We can also write

$$M(\sigma) = (i/2)(2\pi)^{-n+1} \Phi^t(\sigma)\Phi(-\sigma), \tag{1-2}$$

where $\Phi(\sigma) : L^2(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$ has kernel $\Phi(\sigma, \omega, x) = e^{i\sigma\langle\omega, x\rangle}$ and $\Phi^t : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{R}^n)$ has the transposed kernel.

The following two lemmas establish standard polynomial bounds for the free resolvent in the case of exponential weights.

Lemma 1.1. *For each $\epsilon > 0$ there exists a constant $C = C(\epsilon) > 0$ such that whenever $|\operatorname{Im}\sigma| < \gamma - \epsilon$ and $\operatorname{Re}\sigma \geq 1$,*

$$\|e^{-\gamma\varphi} M(\sigma) e^{-\gamma\varphi}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} < C|\sigma|^{1-n}.$$

Proof. The proof is adapted from [Burq 2002]. First note that the Fourier transform $\mathcal{F}(e^{-\gamma\varphi})(\xi)$ extends holomorphically to the strip $\{\xi \in \mathbb{C}^n : |\operatorname{Im}\xi| < \gamma - \epsilon\}$ and

$$|\mathcal{F}(e^{-\gamma\varphi})(\xi)| < C_N \langle\xi\rangle^{-N} \tag{1-3}$$

in the strip for each N. In light of (1-1) and (1-2), it suffices to prove that

$$\|\Phi(\sigma) e^{-\gamma\varphi}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{S}^{n-1})} < C|\sigma|^{(1-n)/2},$$

which by Plancherel's theorem is equivalent to the same estimate for the composition $(\Phi(\sigma)e^{-\gamma\varphi}) \circ \mathcal{F}$. The operator $(\Phi(\sigma)e^{-\gamma\varphi}) \circ \mathcal{F}$ has kernel $\mathcal{F}(e^{-\gamma\varphi})(\sigma\omega - \xi)$. By Schur's lemma it suffices to obtain an estimate of the form

$$\sup_{\xi\in\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |\mathcal{F}(e^{-\gamma\varphi})(\sigma\omega - \xi)|\, d\omega < C|\sigma|^{1-n},$$

since in the other direction we may use (1-3) to obtain the trivial estimate

$$\sup_{\omega\in\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} |\mathcal{F}(e^{-\gamma\varphi})(\sigma\omega - \xi)|\, d\xi < C.$$

Write ξ as $\xi = \langle \xi, \omega \rangle \omega + \xi^\perp(\omega)$ where $\langle \xi^\perp(\omega), \omega \rangle = 0$. Then, by (1-3), we are left estimating

$$\int_{\mathbb{S}^{n-1}} \bigl(1 + |\langle \xi, \omega \rangle - \operatorname{Re}\sigma| + |\xi^\perp(\omega)|\bigr)^{-N} d\omega.$$

Fix $\xi \in \mathbb{R}^n$ and $\delta > 0$, and decompose the sphere into two sets,

$$U = \{\omega \in \mathbb{S}^{n-1} : |\langle \xi, \omega \rangle - \operatorname{Re}\sigma| < \delta \operatorname{Re}\sigma,\ |\xi^\perp(\omega)| < \delta \operatorname{Re}\sigma\}$$

and its complement U^c. The integral over U^c is of the order $O(|\operatorname{Re}\sigma|^{-\infty})$, so it suffices to examine the integral over U.

Observe that unless $\operatorname{Re}\sigma$ is comparable to $|\xi|$, the set U is empty. Indeed, if $\omega \in U$ then $(1-\delta)\operatorname{Re}\sigma < \langle \xi, \omega \rangle < (1+\delta)\operatorname{Re}\sigma$. Hence,

$$\frac{\operatorname{Re}\sigma}{2} < (1-\delta)\operatorname{Re}\sigma < \langle \xi, \omega \rangle \le |\xi|,$$

while on the other hand,

$$|\xi|^2 = |\langle \xi, \omega \rangle|^2 + |\xi^\perp(\omega)|^2 < 3(\operatorname{Re}\sigma)^2$$

for δ sufficiently small.

Write a typical point of $\mathbb{R}^n$ as (y, y') where $y \in \mathbb{R}^{n-1}$ and $y' \in \mathbb{R}$. By a rotation we may assume that $\xi = (0, |\xi|)$. In that case U is contained in the upper hemisphere, in a cap around $|\xi|^{-1}\xi = (0, 1)$ whose size is independent of ξ. This is true since $\omega \in U$ implies

$$\langle |\xi|^{-1}\xi, \omega \rangle > \frac{1}{2\sqrt{3}} > 0.$$

We then parametrize the upper hemisphere $\mathbb{S}^{n-1}_+$ (which contains ξ) by the diffeomorphism

$$p : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1}_+, \quad y \mapsto \frac{(y, |\xi|)}{|(y, |\xi|)|}.$$

Whenever $y \in p^{-1}(U)$ we have

$$|\xi^\perp(p(y))| \ge |y|.$$

To see this, compute

$$|\xi^\perp(p(y))|^2 = |\xi|^2 - |\langle \xi, p(y) \rangle|^2 = |\xi|^2 - \frac{|\xi|^4}{|y|^2 + |\xi|^2} = |y|^2 \frac{|\xi|^2}{|y|^2 + |\xi|^2} \ge |y|^2.$$

Furthermore, the Jacobian satisfies

$$|\partial p / \partial y| = O(|\xi|^{1-n}).$$

We can now bound the integral over U by

$$\int_U \bigl(1+|\xi^\perp(\omega)|\bigr)^{-N}\,d\omega = \int_{p^{-1}(U)} \bigl(1+|\xi^\perp(p(y))|\bigr)^{-N}\,|\partial p/\partial y|\,dy$$
$$\le C_1\,|\mathrm{Re}\,\sigma|^{1-n}\int_{\mathbb{R}^{n-1}}(1+|y|)^{-N}\,dy \le C_2\,|\mathrm{Re}\,\sigma|^{1-n}$$

for N large enough. In the second step, we used the fact that $|\xi|$ and $\mathrm{Re}\,\sigma$ were comparable. □

Lemma 1.2. *For each $\epsilon > 0$ and $|\alpha| \le 2$ there exists $C_\alpha = C_\alpha(\gamma,\epsilon)$ such that whenever* $\mathrm{Im}\,\sigma > -\gamma+\epsilon$ *and* $\mathrm{Re}\,\sigma \ge 1$,

$$\|D^\alpha(e^{-\gamma\varphi}R_0(\sigma)e^{-\gamma\varphi})\|_{L^2\to L^2} \le C_\alpha\,|\sigma|^{|\alpha|-1}.$$

Proof. (1) First we handle the case $|\alpha| = 0$ and $n > 1$; see [Rauch 1978; Vodev 1994] for similar arguments. Let $U(t) = \cos(t\sqrt{-\Delta})$ denote the propagator for the Cauchy problem

$$\begin{cases}(\partial_t^2-\Delta)U(t)f(x) = 0, & (t,x)\in\mathbb{R}\times\mathbb{R}^n,\\ U(0)f(x) = f(x), & \partial_t U(0)f(x) = 0.\end{cases}$$

For $\mathrm{Im}\,\sigma > 0$, write the resolvent

$$e^{-\gamma\varphi}R_0(\sigma)e^{-\gamma\varphi} = \frac{i}{\sigma}\int_0^\infty e^{i\sigma t}e^{-\gamma\varphi}U(t)e^{-\gamma\varphi}\,dt. \tag{1-4}$$

Let r_0 be such that $\varphi(x) = |x|$ for $|x| \ge r_0$. Notice that $\|U(t)\|_{L^2\to L^2} \le 1$ and

$$\|1_{\{|x|\ge t/4\}}e^{-\gamma\varphi}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le e^{-\gamma t/4},\qquad t \ge 4r_0.$$

Writing

$$U(t) = 1_{\{|x|<t/4\}}U(t)1_{\{|x|<t/4\}} + 1_{\{|x|\ge t/4\}}U(t)1_{\{|x|<t/4\}}$$
$$+\,1_{\{|x|<t/4\}}U(t)1_{\{|x|\ge t/4\}} + 1_{\{|x|\ge t/4\}}U(t)1_{\{|x|\ge t/4\}},$$

we see the norms of the latter three terms are of size $O(e^{-\gamma t/4})$ after multiplication by $e^{-\gamma\varphi}$ on the left and right. Hence, we only need to estimate the norm of the operator with kernel

$$1_{\{|x|<t/4\}}(x)\,e^{-\gamma\varphi(x)}U(t,x,y)\,e^{-\gamma\varphi(y)}1_{\{|x|<t/4\}}(y),$$

using explicit knowledge of the kernel $U(t,x,y)$.

In odd dimensions, the kernel vanishes identically by the strong Huygens principle. In even dimensions, the kernel vanishes unless $|x|, |y| < t/4$, which implies that $|x-y| < t/2$ and thus

$$|1_{\{|x|<t/4\}}(x)\,U(t,x,y)\,1_{\{|x|<t/4\}}(y)| \le Ct^{-n},$$

again from explicit formulas for $U(t,x,y)$. Schur's lemma then gives

$$\|1_{\{|x|<t/4\}}e^{-\gamma\varphi}U(t)e^{-\gamma\varphi}1_{\{|x|<t/4\}}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le Ct^{-n}.$$

Therefore we see that the integral in (1-4) actually converges for $\operatorname{Im}\sigma \ge 0$ with the uniform estimate

$$\|e^{-\gamma\varphi}R_0(\sigma)e^{-\gamma\varphi}\|_{L^2\to L^2} \le C|\sigma|^{-1}, \quad \operatorname{Im}\sigma \ge 0 \text{ and } \operatorname{Re}\sigma \ge 1.$$

The result for $-\gamma+\epsilon < \operatorname{Im}\sigma < 0$ follows immediately by reflection from (1-1) and Lemma 1.1.

(2) In the case $\alpha = 0$ and $n = 1$, one can simply apply Schur's lemma to the Schwartz kernel

$$e^{-\gamma\varphi}R_0(x,y,\sigma)e^{-\gamma\varphi} = e^{-\gamma\varphi(x)}\frac{ie^{i\sigma|x-y|}}{\sigma}e^{-\gamma\varphi(y)}.$$

The $|\alpha| = 1, 2$ cases follow from the $|\alpha| = 0$ case by interpolation, as in [Zworski 1989, Lemma 3]; we supply a proof for the reader's convenience. Consider first the case $|\alpha| = 2$. By analytic continuation, if $u \in L^2(\mathbb{R}^n)$, then

$$\Delta R_0(\sigma)e^{-\gamma\varphi}u = -e^{-\gamma\varphi}u - \sigma^2 R_0(\sigma)e^{-\gamma\varphi}u \tag{1-5}$$

and hence $R_0(\sigma) : L^2_\gamma \to H^2_{-\gamma}$ is bounded for $\operatorname{Im}\sigma > -\gamma$. Now, choose $u \in L^2(\mathbb{R}^n)$ and set $f = R_0(\sigma)e^{-\gamma\varphi}u$. Then

$$\begin{aligned}\Delta(e^{-\gamma\varphi}R_0(\sigma)e^{-\gamma\varphi}u)&\\ = (\gamma^2|\nabla\varphi|^2 - \gamma\Delta\varphi)e^{-\gamma\varphi}f &- 2\gamma\nabla\varphi\cdot(e^{-\gamma\varphi}\nabla f) + e^{-\gamma\varphi}\Delta f\end{aligned} \tag{1-6}$$

In light of (1-5) it suffices to estimate the L^2 norm of $-\gamma\nabla\varphi\cdot(e^{-\gamma\varphi}\nabla f)$. But since φ has uniformly bounded derivatives,

$$\|\nabla\varphi\cdot(e^{-\gamma\varphi}\nabla f)\|^2_{L^2} \le C\|e^{-\gamma\varphi}\nabla f\|^2_{L^2}.$$

We now integrate by parts and estimate

$$\begin{aligned}&\|e^{-\gamma\varphi}\nabla f\|^2_{L^2}\\ &\le 2\int|\gamma\nabla\varphi|\,|e^{-\gamma\varphi}\nabla f|\,|e^{-\gamma\varphi}f|\,dx + \int|e^{-\gamma\varphi}\Delta f|\,|e^{-\gamma\varphi}f|\,dx.\end{aligned} \tag{1-7}$$

Applying the inequality $2ab \le 2a^2 + \frac{1}{2}b^2$ to the integrand, the first term on the right hand side is bounded by

$$\int 2|\gamma\nabla\varphi|^2\,|e^{-\gamma\varphi}f|^2\,dx + \int\tfrac{1}{2}|e^{-\gamma\varphi}\nabla f|^2\,dx,$$

while for the second term we use (1-5). We conclude that

$$\|e^{-\gamma\varphi}\nabla f\|^2_{L^2} \le C(1+|\sigma|^2)\|e^{-\gamma\varphi}f\|^2_{L^2} + \|e^{-2\gamma\varphi}u\|^2_{L^2}.$$

Returning to (1-6), it follows that

$$\|\Delta(e^{-\gamma\varphi}R_0(\sigma)e^{-\gamma\varphi}u)\|_{L^2} \le C(1+|\sigma|^2)\|u\|_{L^2}.$$

Moreover, (1-7) actually shows

$$\|\nabla(e^{-\gamma\varphi}R_0(\sigma)e^{-\gamma\varphi}u)\|_{L^2} \le C\|u\|_{L^2}. \qquad \square$$

We now introduce the semiclassical rescaling by setting $\lambda = h\sigma$. Let $R_0(\sigma, h)$ denote $(-h^2\Delta-\lambda^2)^{-1}$ and its corresponding analytic continuation. We are interested in λ lying in a set of the form

$$(a,b)+i((-\gamma+\epsilon)h, 1),$$

where $0 < a < b$. For the remainder of the paper, equip $H^k(\mathbb{R}^n)$ with the h-dependent norm $\|u\|_{H^k}^2 = \sum_{|\alpha|\le k}\|(hD)^\alpha u\|_{L^2}^2$. Since $R_0(\lambda, h) = h^{-2}R_0(\lambda/h)$, we have uniform estimates

$$\|R_0(\lambda,h)\|_{L^2_\gamma \to H^s_{-\gamma}} = O(h^{-1}), \quad s = 0, 1, 2,$$

for $\lambda \in (a,b)+i((-\gamma+\epsilon)h, 1)$.

1B. ***Black box model.*** As our scattering problem, we consider exponentially decaying perturbations of the Laplacian outside a compact set, formulated in the black box setting as follows. Suppose $\mathcal{H}$ is a Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0))$$

where $B(x,R) = \{y \in \mathbb{R}^n : |x-y| < R\}$ and R_0 is fixed. The orthogonal projections onto $\mathcal{H}_{R_0}$ and $L^2(\mathbb{R}^n \setminus B(0,R_0))$ will be denoted $1_{B(0,R_0)}u = u|_{B(0,R_0)}$ and $1_{\mathbb{R}^n\setminus B(0,R)}u = u|_{\mathbb{R}^n\setminus B(0,R_0)}$ for $u \in \mathcal{H}$. Note that any bounded continuous function $\chi \in C_b(\mathbb{R}^n)$ which is constant near $B(0,R_0)$ acts naturally on $\mathcal{H}$ by

$$\chi u = C_0 u + (\chi - C_0)1_{\mathbb{R}^n\setminus B(0,R_0)}u,$$

where $\chi \equiv C_0$ near $B(0,R_0)$.

Now consider an unbounded self-adjoint operator $P(h)$ on $\mathcal{H}$ with domain $\mathcal{D} \subset \mathcal{H}$ (independent of h for simplicity) with the following properties:

- If $u \in \mathcal{D}$, then $1_{\mathbb{R}^n\setminus B(0,R_0)}u \in H^2(\mathbb{R}^n \setminus B(0,R_0))$.
- If $u \in H^2(\mathbb{R}^n \setminus B(0,R_0))$ vanishes near $B(0,R_0)$, then $u \in \mathcal{D}$.

We assume there exists a real-valued and uniformly positive-definite matrix (a_{ij}) and a real-valued function V (which are allowed to be h-dependent) such that for $u \in \mathcal{D}$,

$$(P(h)u)|_{\mathbb{R}^n\setminus B(0,R_0)} = \Big(-\sum_{i,j}(h\partial_i)a_{ij}(h\partial_j)+V\Big)(u|_{\mathbb{R}^n\setminus B(0,R_0)}). \tag{1-8}$$

Furthermore, we require that

$$a_{ij}(x;h) \in C_b^\infty(\mathbb{R}^n \setminus B(0,R_0)) \quad \text{and} \quad V(x;h) \in C_b^\infty(\mathbb{R}^n \setminus B(0,R_0)),$$

with all derivatives uniformly bounded in h.

The perturbation is assumed to decay exponentially to the Laplacian in the sense that there exists $\gamma > 0,\ \delta > 0$ so that for $x \in \mathbb{R}^n \setminus B(0,R_0)$,

$$|a_{ij}(x;h) - \delta_{ij}| \le Ce^{-(2\gamma+\delta)|x|} \quad \text{and} \quad |V(x;h)| \le Ce^{-(2\gamma+\delta)|x|}. \tag{1-9}$$

Finally, assume that the mapping

$$1_{B(0,R_0)}(P(h)+i)^{-1} : \mathcal{H} \to \mathcal{H}_{R_0} \tag{1-10}$$

is compact.

Under these hypotheses, we show that

$$R(\lambda,h) = (P(h)-\lambda^2)^{-1}, \quad \operatorname{Re}\lambda > 0 \text{ and } \operatorname{Im}\lambda > 0,$$

admits a meromorphic continuation to the strip $\operatorname{Im}\lambda > (-\gamma+\epsilon)h$ as an operator $\mathcal{H}_\gamma \to \mathcal{H}_{-\gamma}$. In order that the associated weighted space $\mathcal{H}_\gamma$ makes sense, we choose $\varphi \in C^\infty(\mathbb{R}^n)$, as above, satisfying $\varphi \equiv 0$ near $B(0,R_0)$.

Remark. All of the results in this note also apply to black box operators on the half-line $(0,\infty)$. For the most part this amounts to replacing the Laplacian on $\mathbb{R}^n$ with the Dirichlet Laplacian on $(0,\infty)$, and replacing $H^s(\mathbb{R}^n)$ with $H^s(0,\infty) \cap H_0^1(0,\infty)$. Estimates for the free resolvent on $(0,\infty)$ follow from those on $\mathbb{R}$ by the method of odd reflection; all other necessary modifications should be clear.

1C. ***Meromorphic continuation.*** As a preliminary, arbitrarily extend a_{ij} and V to functions defined on all of $\mathbb{R}^n$ with the same properties as their original counterparts. Since the choice of extension has no bearing on the final result, we denote them by the same letters. Now define

$$\tilde{P}(h) = -\sum_{i,j} (h\partial_i) a_{ij} (h\partial_j) + V,$$

$$\tilde{R}(\lambda,h) = (\tilde{P}(h) - \lambda)^{-1}, \quad \lambda^2 \notin \sigma(\tilde{P}(h)).$$

Since $\tilde{P}(h)$ is uniformly elliptic, it is self-adjoint with domain $H^2(\mathbb{R}^n)$. We will also write $A(h)$ for the difference

$$A(h) = \tilde{P}(h) - (-h^2\Delta).$$

The important fact about $A(h)$ is that it is bounded as a map $H_\alpha^s \to H_{\alpha+2\gamma+\delta}^{s-2}$ for each $s,\alpha \in \mathbb{R}$.

We will need information about the $L_\gamma^2 \to H_\gamma^s$ mapping properties of $\tilde{R}(\lambda,h)$ for $\lambda^2 \notin \sigma(\tilde{P}(h))$.

Lemma 1.3. *Fix an interval* $(a, b) \Subset \mathbb{R}_+$. *For each* $\gamma > 0$ *there exists* $T_0 > 0$ *such that*

$$\|e^{\gamma\varphi}\tilde{R}(\lambda, h)e^{-\gamma\varphi}\|_{L^2\to H^s} = O(|\operatorname{Im}\lambda|^{-1}), \quad s = 0, 1, 2$$

uniformly for $\lambda \in (a, b) + i(T_0 h, 1)$.

Proof. Conjugating $\tilde{P}(h)$ by $e^{\gamma\varphi}$ yields

$$e^{\gamma\varphi}\tilde{P}(h)e^{-\gamma\varphi} = \tilde{P}(h) + h^2 B,$$

where

$$B = \sum_{i,j}(2\gamma a_{ij}\partial_i\varphi)\partial_j - \gamma^2 a_{ij}\partial_i\varphi\,\partial_j\varphi + \gamma\partial_i(a_{ij}\partial_j\varphi)$$

is a first order operator with uniformly bounded coefficients. It follows that for $\lambda^2 \notin \sigma(\tilde{P}(h))$ (in particular for $\operatorname{Im}\lambda > 0$ and $\operatorname{Re}\lambda > 0$) we can write

$$e^{\gamma\varphi}\tilde{P}(h)e^{-\gamma\varphi} - \lambda^2 = (I + h^2 B\tilde{R}(\lambda, h))(\tilde{P}(h) - \lambda^2).$$

Since $\tilde{P}(h) : H^2 \to L^2$ is self-adjoint,

$$\|u\|_{H^2} < C\,\|(\tilde{P}(h) + i)u\|_{L^2}.$$

It follows that for $\lambda \in (a, b) + i(0, 1)$,

$$\|\tilde{R}(\lambda, h)\|_{L^2\to H^s} = O(|\operatorname{Im}\lambda|^{-1}), \quad s = 0, 1, 2. \tag{1-11}$$

We immediately deduce that

$$\|h^2 B\tilde{R}(\lambda, h)\|_{L^2\to L^2} = O(h|\operatorname{Im}\lambda|^{-1}) \le \tfrac{1}{2}, \quad \lambda \in (a, b) + i[T_0 h, 1),$$

for $T_0 > 0$ large enough. In particular, $I + h^2 B\tilde{R}(\lambda, h) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is invertible for $\lambda \in (a, b) + i[T_0 h, 1)$ and

$$e^{\gamma\varphi}\tilde{R}(\lambda, h)e^{-\gamma\varphi} = \tilde{R}(\lambda, h)(I + h^2 B\tilde{R}(\lambda, h))^{-1}.$$

This also shows that

$$\|e^{\gamma\varphi}\tilde{R}(\lambda, h)e^{-\gamma\varphi}\|_{L^2\to H^s} = O(|\operatorname{Im}\lambda|^{-1}), \quad s = 0, 1, 2$$

for $\lambda \in (a, b) + i[T_0 h, 1)$. □

The following lemma is useful in the proof of the meromorphic continuation. Equip $\mathcal{D}$ with the h-dependent norm

$$\|u\|_{\mathcal{D}} = \|(P(h) + i)u\|_{\mathcal{H}}.$$

Then it is easy to see that under the uniform boundedness conditions on the derivatives of a_{ij} and V, the analog of [Sjöstrand and Zworski 1991, Proposition 4.1] remains true:

Lemma 1.4. *Suppose $\chi \in C_b^\infty(\mathbb{R}^n)$ has support disjoint from $\overline{B(0, R_0)}$. Then multiplication by χ is bounded $\mathcal{D} \to H^2(\mathbb{R}^n)$ and $H^2(\mathbb{R}^n) \to \mathcal{D}$ with a norm bounded independently of h.*

Proof. Consider first the map $\chi : \mathcal{D} \to H^2(\mathbb{R}^n)$. Since $\tilde{P}(h)$ is elliptic, we have the a priori estimate

$$\begin{aligned}\|\chi u\|^2_{H^2(\mathbb{R}^n)} &\le C_1\big(\|\chi_1 \tilde{P}(h) 1_{\mathbb{R}^n \setminus B(0,R_0)} u\|^2_{L^2(\mathbb{R}^n \setminus B(0,R_0))} \\ &\qquad + \|\chi_1 1_{\mathbb{R}^n \setminus B(0,R_0)} u\|^2_{L^2(\mathbb{R}^n \setminus B(0,R_0))}\big) \\ &\le C_2 \|(P(h)+i)u\|^2_{\mathcal{H}},\end{aligned}$$

where $\chi_1 \equiv 1$ on $\operatorname{supp}\chi$ and χ_1 also has support disjoint from $\overline{B(0, R_0)}$. All the constants are independent of h. For the case $\chi : H^2(\mathbb{R}^n) \to \mathcal{D}$ this is equivalent to the uniform boundedness of $\tilde{P}(h)$ on $H^2(\mathbb{R}^n)$, namely

$$\|\chi u\|_{\mathcal{D}} = \|(\tilde{P}(h)+i)(\chi u)\|_{L^2(\mathbb{R}^n)} \le C\|u\|_{H^2(\mathbb{R}^n)}. \qquad \square$$

In what follows, we will always be concerned with λ ranging in a precompact neighborhood of $\mathbb{R}_+$. So fix $0 < a_0 < b_0$ and $\epsilon_0 > 0$, and define

$$\Omega(h) = (a_0, b_0) + i((-\gamma + \epsilon_0)h, 1).$$

For each $\epsilon > 0$, we also define a shrunken neighborhood,

$$\Omega_\epsilon(h) = (a_0 + \epsilon, b_0 - \epsilon) + i((-\gamma + \epsilon_0 + \epsilon)h, 1).$$

Proposition 1.5. *The function $R(\lambda, h)$ has a meromorphic continuation in $\Omega(h)$ as a family of bounded operators $\mathcal{H}_\gamma \to \mathcal{H}_{-\gamma}$.*

Proof. Choose cutoff functions $\chi, \chi_i \in C_c^\infty(\mathbb{R}^n)$, $i = 0, 1, 2$, so that $\chi_0 \equiv 1$ near $B(0, R_0)$ with $\chi_i \equiv 1$ on $\operatorname{supp}\chi_{i-1}$ and $\chi \equiv 1$ on $\operatorname{supp}\chi_2$. We can always choose these so that $\chi\varphi = 0$ and $\chi_i\varphi \equiv 0$. Approximate $R(\lambda, h)$ by a parametrix of the form $Q_0(\lambda, \lambda_0, h) + Q_1(\lambda_0, h)$ where

$$\begin{aligned}Q_0(\lambda, \lambda_0, h) &= (1-\chi_0)(R_0(\lambda, h) - \tilde{R}(\lambda_0, h)A(h)R_0(\lambda, h))(1-\chi_1), \\ Q_1(\lambda_0, h) &= \chi_2 R(\lambda_0, h)\chi_1;\end{aligned}$$

see also [Sá Barreto and Zworski 1995]. Here, $\lambda_0 = \lambda_0(h)$ denotes a point in $\Omega(h)$ with $\operatorname{Im}\lambda_0 \ge T_0 h$. We now compute

$$(P(h) - \lambda^2)Q_0(\lambda, \lambda_0, h) = (1-\chi_1) + K_0(\lambda, \lambda_0, h) + K_1(\lambda, \lambda_0, h)$$

where

$$\begin{aligned}K_0(\lambda, \lambda_0, h) &= -[\tilde{P}(h), \chi_0]\big(R_0(\lambda, h) - \tilde{R}(\lambda_0, h)A(h)R_0(\lambda, h)\big)(1-\chi_1), \\ K_1(\lambda, \lambda_0, h) &= (1-\chi_0)(\lambda^2 - \lambda_0^2)\tilde{R}(\lambda_0, h)A(h)R_0(\lambda, h)(1-\chi_1),\end{aligned}$$

and

$$(P(h)-\lambda^2)Q_1(\lambda_0,h)=\chi_1+K_2(\lambda_0,h)+K_3(\lambda,\lambda_0,h),$$

where

$$K_2(\lambda_0,h)=-[\tilde{P}(h),\chi_2]R(\lambda_0,h)\chi_1,$$
$$K_3(\lambda,\lambda_0,h)=\chi_2(\lambda_0^2-\lambda^2)R(\lambda_0,h)\chi_1.$$

If we let $K=K_0+K_1+K_2+K_3$, then

$$(P(h)-\lambda^2)(Q_0(\lambda,\lambda_0,h)+Q_1(\lambda_0,h))=I+K(\lambda,\lambda_0,h).$$

Note that if $\psi\in C_c^\infty(\mathbb{R}^n)$ then $[\tilde{P}(h),\psi]$ is a first order operator with compactly supported coefficients and $\|[\tilde{P}(h),\psi]\|_{H^1(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}=O(h)$.

It is easy to see that $Q_0+Q_1:\mathcal{H}_\gamma\to\mathcal{H}_{-\gamma}$. For Q_0 this follows from the mapping properties of $R_0(\lambda,h)$, $A(h)$, and $\tilde{R}(\lambda_0,h)$. For Q_1, this fact is trivial since Q_1 contains compactly supported cutoffs. We also remark that by the resolvent identity,

$$K_0(\lambda_0,\lambda_0,h)=-[\tilde{P}(h),\chi_0]\tilde{R}(\lambda_0,h)(1-\chi_1).$$

To apply the Fredholm theory, we begin by showing that $K:\mathcal{H}_\gamma\to\mathcal{H}_\gamma$ is compact. First note that

$$K_0(\lambda,\lambda_0,h)=-[\tilde{P}(h),\chi_0]\big(R_0(\lambda,h)-\tilde{R}(\lambda_0,h)A(h)R_0(\lambda,h)\big)(1-\chi_1):\mathcal{H}_\gamma\to\mathcal{H}_\gamma$$

is compact: we see that $R_0(\lambda,h):L^2_\gamma(\mathbb{R}^n)\to H^2_{-\gamma}(\mathbb{R}^n)$ and $\tilde{R}(\lambda_0,h)A(h)R_0(\lambda,h):L^2_\gamma(\mathbb{R}^n)\to H^2_{\gamma+\delta}(\mathbb{R}^n)$. On the other hand $[\tilde{P}(h),\chi_0]$ is compactly supported and hence maps $H^2_\alpha(\mathbb{R}^n)\to L^2(\mathbb{R}^n\setminus B(0,R_0))$ compactly for each $\alpha\in\mathbb{R}$.

Similarly, we can write

$$K_2(\lambda_0,h)=[\tilde{P}(h),\chi_2](1-\chi_0)R(\lambda_0,h)\chi_1$$

which is compact since $(1-\chi_0)R(\lambda_0,h)\chi_1:\mathcal{H}_\gamma\to H^2(\mathbb{R}^n\setminus B(0,R_0))$ and $[\tilde{P}(h),\chi_2]$ is compactly supported. To see that K_1 is compact, again use that $\tilde{R}(\lambda_0,h)A(h)R_0(\lambda,h):L^2_\gamma(\mathbb{R}^n)\to H^2_{\gamma+\delta}(\mathbb{R}^n)$ and now appeal to the fact that the inclusion

$$H^2_{\gamma+\delta}(\mathbb{R}^n)\hookrightarrow L^2_\gamma(\mathbb{R}^n)$$

is compact. Finally, the compactness of $K_3(\lambda,\lambda_0,h)$ follows from (1-10).

Next, we need to verify the invertibility of $I+K(\lambda,\lambda_0,h)$ for at least one value of $\lambda\in\Omega(h)$. Recall that multiplication by $(1-\chi_0):H^2(\mathbb{R}^n)\to\mathcal{D}$ is uniformly bounded in h. It follows that for $\lambda_0\in\Omega(h)$ in the upper half-plane, for $u\in\mathcal{H}$,

$$\|(1-\chi_0)R(\lambda_0,h)u\|_{H^2(\mathbb{R}^n)}\le C_1\|(P(h)+i)R(\lambda_0,h)u\|_{\mathcal{H}}\le C_2|\operatorname{Im}\lambda_0|^{-1}\|u\|_{\mathcal{H}},$$

and hence

$$\|(1-\chi)R(\lambda_0,h)\|_{\mathcal{H}\to H^2(\mathbb{R}^n)} = O(|\operatorname{Im}\lambda_0|^{-1}), \quad \lambda_0 \in \Omega(h),\ \operatorname{Im}\lambda_0 > 0.$$

Here, we used

$$(P(h)+i)R(\lambda_0,h) = I + (\lambda_0^2+i)R(\lambda_0,h)$$

and $R(\lambda_0,h) = O_{\mathcal{H}\to\mathcal{H}}(|\operatorname{Im}\lambda_0|^{-1})$. Combining this with (1-11), we see there exists $T_1 > T_0$ such that if $\lambda_0 \in \Omega(h)$ satisfies $\operatorname{Im}\lambda_0 \geq T_1 h$, then

$$\|K(\lambda_0,\lambda_0,h)\|_{\mathcal{H}_\gamma\to\mathcal{H}_\gamma} = O(h|\operatorname{Im}\lambda_0|^{-1}) \leq \tfrac{1}{2},$$

and hence $I + K(\lambda_0,\lambda_0,h)$ will be invertible. □

Remark. The poles and their multiplicities of the extension obtained above do not depend on the particular choice of φ. Indeed, if φ_1 and φ_2 both vanish near $\mathbb{R}^n \setminus B(0,R_0)$ and equal $|x|$ for large $|x|$, then

$$e^{-\gamma\varphi_1}R(\lambda,h)e^{-\gamma\varphi_1} = e^{-\gamma(\varphi_1-\varphi_2)}e^{-\gamma\varphi_2}R(\lambda,h)e^{-\gamma\varphi_2}e^{-\gamma(\varphi_1-\varphi_2)}$$

and vice versa. Hence the poles and multiplicities of one such extension agree with those of any other.

Remark. As pointed out by the anonymous referee, an interesting question is whether $R(\lambda,h)$ can be continued to a larger region in the lower half plane when the perturbations are smooth functions of $\exp((-2\gamma-\delta)|x|)$ for large $|x|$ (and also whether the corresponding resolvent estimates hold). Such hypotheses are satisfied for stationary wave operators arising from black hole metrics with nondegenerate event horizons; see [Dyatlov 2011; Gannot 2014] for two examples.

At this point we need to introduce a new assumption on a reference operator $P^\sharp(h)$, defined as follows: choose $R_1 > R_0$ and $R_2 > 2R_1$ and let $\mathbb{T}$ denote the torus $\mathbb{T} = (\mathbb{R}/R_2\mathbb{Z})^n$. Let

$$\mathcal{H}^\sharp = \mathcal{H}_{R_0} \oplus L^2(\mathbb{T}\setminus B(0,R_0)),$$

where $B(0,R_1)$ is considered a subset of $\mathbb{T}$. Define the dense subspace

$$\mathcal{D}^\sharp = \{u \in \mathcal{H}^\sharp : \psi u \in \mathcal{D},\ (1-\psi)u \in H^2(\mathbb{T})\},$$

where $\psi \in C_c^\infty(B(0,R_1))$ satisfies $\psi \equiv 1$ near $B(0,R_0)$. Now set

$$P^\sharp(h)u = P(h)\psi u + \Bigl(-\sum_{i,j}(h\partial_i)a_{ij}(h\partial_j) + V\Bigr)(1-\psi)u, \quad u \in \mathcal{D}^\sharp.$$

Then $P^\sharp(h)$ is self-adjoint on $\mathcal{D}^\sharp$ with discrete spectrum. We require that

$$\#\bigl\{z \in \sigma(P^\sharp(h)) : z \in [-L,L]\bigr\} \leq C(L/h^2)^{n^\sharp/2} \tag{1-12}$$

for some $n^\sharp \geq n$ and each $L \geq 1$. Here the eigenvalues are counted with multiplicity. If $z_1, z_2, z_3, \ldots$ are the eigenvalues of $P^\sharp(h)$ ordered so $|z_1| \leq |z_2| \leq |z_3| \leq \cdots$, then the singular values of $(P^\sharp(h) - \lambda_0^2)^{-1}$ are $\mu_j((P^\sharp(h) - \lambda_0^2)^{-1}) = |z_j - \lambda_0^2|^{-1}$. If $\operatorname{Im} \lambda_0 = T_1 h$, then (1-12) implies that there exists a constant $C > 0$ so that

$$\mu_j((P^\sharp(h) - \lambda_0^2)^{-1}) \leq C h^{-2} j^{-2/n^\sharp}, \quad j > C h^{-n^\sharp}.$$

2. Resolvent estimates

To estimate $R(\lambda, h)$, we make use of the following general fact [Gohberg and Kreĭn 1969, Chapter V, Theorem 5.1]: Suppose A is a compact operator lying in some p-class. If $(I + A)$ is invertible, then

$$\|(I + A)^{-1}\| \leq \frac{\det(I + |A|^p)}{|\det(I + A^p)|}.$$

We wish to apply this inequality to $(I+K)$, but first we need to verify that a suitable power of K is of trace class. Under our hypotheses we cannot estimate the singular values of K_2; nevertheless, the proof of Proposition 1.5 shows that $I + K_2(\lambda_0, h)$ is invertible on $\mathcal{H}_\gamma$ for $\operatorname{Im} \lambda_0 > T_1 h$, so we use the decomposition

$$(I + K(\lambda, \lambda_0, h)) = (I + K_2(\lambda_0, h))(I + \tilde{K}(\lambda, \lambda_0, h)),$$

where $\tilde{K} = (I + K_2)^{-1}(K_0 + K_1 + K_3)$. Note that $I + K$ and $I + \tilde{K}$ have the same poles.

2A. *Singular values.* From now on we will always choose $\lambda_0 \in \Omega(h)$ with fixed imaginary part $\operatorname{Im} \lambda_0 = T_1 h$. Throughout, it will be clear that whenever an estimate depends on $\lambda_0 \in \Omega(h)$, it really only depends on $\operatorname{Im} \lambda_0$.

Proposition 2.1. *The operator $\tilde{K}(\lambda, \lambda_0, h)^{n^\sharp+1} : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ is of trace class for $\lambda \in \Omega(h)$.*

Proof. We estimate the singular values of each summand in $\tilde{K}$. Since the weighted resolvent only continues to a narrow strip in the lower half-plane, in such a region it is particularly simple to estimate $\mu_j(K_0)$: choose an open ball $B \subseteq \mathbb{R}^n$ containing $\operatorname{supp} \nabla \chi_0$ and let $-\Delta_B$ denote the Dirichlet Laplacian on B. Again using that the inclusion $1_{\mathbb{R}^n \setminus B(0,R_0)} : \mathcal{D}_\gamma \to H^2_\gamma$ is uniformly bounded in h, we consider K_0 as a map $\mathcal{H}_\gamma \to H^1(B)$. By Weyl asymptotics,

$$\mu_j((-h^2 \Delta_B)^{-1}) \leq C h^{-2} j^{-2/n}, \quad j = 1, 2, 3, \ldots$$

Thus, we estimate

$$\begin{aligned} \mu_j(K_0(\lambda, \lambda_0, h)) &\leq C \mu_j((-h^2 \Delta_B)^{-1/2}) \|(-h^2 \Delta_B)^{1/2} K_0(\lambda, \lambda_0, h)\|_{\mathcal{H}_\gamma \to L^2(B)} \\ &\leq C h^{-3} j^{-1/n}, \quad \lambda \in \Omega(h). \end{aligned}$$

By the same reasoning we estimate $\mu_j(K_1)$, writing

$$\mu_j(K_1(\lambda,\lambda_0,h)) \\ \le C\mu_j(e^{\gamma\varphi}\tilde{R}(\lambda_0,h)e^{-(\gamma+\delta)\varphi})\,\|e^{(\gamma+\delta)\varphi}A(h)e^{\gamma\varphi}e^{-\gamma\varphi}R_0(\lambda)\|_{L^2_\gamma(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}.$$

In order to bound $\mu_j(e^{\gamma\varphi}\tilde{R}(\lambda_0,h)e^{-(\gamma+\delta)\varphi})$, let $P_0(h)=-h^2\Delta+x^2$ denote the harmonic oscillator. The inequality $\mu_j(P_0(h)^{-1})\le Ch^{-1}j^{-1/n}$ follows, in this case by explicit knowledge of the spectrum. Since $P_0(h)e^{-\delta\varphi}:H^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ is bounded,

$$\begin{aligned}\mu_j(e^{\gamma\varphi}\tilde{R}(\lambda_0,h)e^{-(\gamma+\delta)\varphi}) \\ &\le \mu_j(P_0(h)^{-1})\,\|P_0(h)e^{-\delta\varphi}e^{(\gamma+\delta)\varphi}\tilde{R}(\lambda_0,h)e^{-(\gamma+\delta)\varphi}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}\\ &\le Ch^{-2}j^{-1/n}.\end{aligned}$$

Combined with the previous estimate we obtain

$$\mu_j(K_1)\le Ch^{-3}j^{-1/n},\qquad \lambda\in\Omega(h).$$

Next we estimate the singular values of K_3 using (1-12). Recall that $(P(h)-\lambda^2)\chi=(P^\sharp(h)-\lambda^2)\chi$, which implies that

$$(P(h)-\lambda_0^2)^{-1}\chi_1=\chi(P^\sharp(h)-\lambda_0^2)^{-1}\chi_1-(P(h)-\lambda_0^2)^{-1}[P^\sharp(h),\chi](P^\sharp(h)-\lambda_0^2)^{-1}\chi_1.$$

Multiply this equation on the left by χ_2 and apply Fan's inequality, $\mu_{2k-1}(A+B)\le\mu_k(A)+\mu_k(B)$. Using the fact that $(P(h)-\lambda_0^2)^{-1}[P^\sharp(h),\chi]$ has norm $O(1)$,

$$\mu_j(K_3(\lambda,\lambda_0,h))\le Ch^{-2}j^{-2/n^\sharp},\qquad j>Fh^{-n^\sharp}$$

for some constant $F>0$. For $j\le Fh^{-n^\sharp}$, we simply bound $\mu_j(K_3)\le Ch^{-1}$ using the trivial norm estimate.

It is now clear that $\mu_j(K_i)^{n^\sharp}$ is summable for $j=0,1,3$. $\square$

Applying the resolvent estimate as above, we obtain

$$\begin{aligned}\|R(\lambda,h)\|_{\mathcal{H}_\gamma\to\mathcal{H}_{-\gamma}} \\ &\le \|Q_0+Q_1\|_{\mathcal{H}_\gamma\to\mathcal{H}_{-\gamma}}\,\|(I+K_2)^{-1}\|_{\mathcal{H}_\gamma\to\mathcal{H}_\gamma}\,\|(I+\tilde{K})^{-1}\|_{\mathcal{H}_\gamma\to\mathcal{H}_\gamma}\\ &\le C\|Q_0+Q_1\|_{\mathcal{H}_\gamma\to\mathcal{H}_{-\gamma}}\,\frac{\det\big(I+(\tilde{K}^*\tilde{K})^{\frac{n^\sharp+1}{2}}\big)}{|\det(I+\tilde{K}^{n^\sharp+1})|}.\end{aligned}\tag{2-1}$$

Since

$$\|Q_0+Q_1\|_{\mathcal{H}_\gamma\to\mathcal{H}_{-\gamma}}=O(h^{-2}),\qquad \lambda\in\Omega(h),$$

it remains only to estimate the determinants. Define

$$f(\lambda,h)=\det(I+\tilde{K}^{n^\sharp+1}(\lambda,\lambda_0,h))$$

in $\Omega(h)$. By Weyl convexity inequalities, it follows that $|f(\lambda, h)| \leq M(h),\ \lambda \in \Omega(h)$, where

$$M(h) = \sup_{\lambda \in \Omega(h)} \det\bigl(I + (\tilde{K}^* \tilde{K})^{\frac{n^\sharp+1}{2}}\bigr).$$

We therefore need to bound $M(h)$ from above and $|f(\lambda, h)|$ from below.

2B. ***Estimating the determinant from above.*** Here we obtain an upper bound for $M(h)$ of the form $M(h) \leq e^{Ch^{-p}}$. For the application in mind, the value of p is unimportant and we do not attempt to optimize the exponent. In fact h^{-p} also represents a polynomial upper bound for the number of resonances in a disk of radius h, but again obtaining an optimal value is unimportant in this context.

Proposition 2.2. *There exists $C > 0$ depending only on* $\operatorname{Im}\lambda_0$, *and $p > 0$ such that*

$$M(h) \leq e^{Ch^{-p}}.$$

Proof. We estimate $M(h)$ using Fan's inequalities:

$$\begin{aligned}
\prod_{j\geq 1}\bigl(1 + \mu_j(\tilde{K}^{n^\sharp+1})\bigr) &= \prod_{j\geq 1}\bigl(1 + \mu_j(\tilde{K})^{n^\sharp+1}\bigr) \leq \prod_{j\geq 1}\bigl(1 + \mu_{3j-2}(\tilde{K})^{n^\sharp+1}\bigr)^3 \\
&\leq \prod_{j\geq 1}\bigl(1 + C_0\bigl(\mu_j(K_0)^{n^\sharp+1} + \mu_j(K_1)^{n^\sharp+1} + \mu_j(K_3)^{n^\sharp+1}\bigr)\bigr)^3 \\
&\leq \prod_{i=0,1,3}\prod_{j\geq 1}\bigl(1 + C_0\mu_j(K_i)^{n^\sharp+1}\bigr)^3.
\end{aligned}$$

For $i = 0, 1$, the singular values occurring in this product are bounded above by $\mu_j(K_i) \leq Ch^{-3} j^{-1/n^\sharp}$, and so we bound the product by the trace,

$$\prod_{j\geq 1}\bigl(1 + C_0\mu_j(K_i)^{n^\sharp+1}\bigr) \leq \exp\Bigl(C_1 h^{-3n^\sharp-3} \sum_{j\geq 1} j^{-1+1/n^\sharp}\Bigr) \leq e^{Ch^{-3n^\sharp-3}}.$$

On the other hand for K_3,

$$\begin{aligned}
&\prod_{j\geq 1}\bigl(1 + C_0\mu_j(K_3)^{n^\sharp+1}\bigr) \\
&\qquad\leq \prod_{1\leq j\leq Fh^{-n^\sharp}}\bigl(1 + C_0\mu_j(K_3)^{n^\sharp+1}\bigr) \prod_{j > Fh^{-n^\sharp}}\bigl(1 + C_0\mu_j(K_3)^{n^\sharp+1}\bigr) \\
&\qquad\leq \Bigl(e^{Ch^{-n^\sharp}\log(1/h)}\Bigr)\Bigl(e^{Ch^{-n^\sharp}}\Bigr).
\end{aligned}$$

Thus,

$$M(h) \leq e^{Ch^{-p}}$$

for some $p > 0$, where the constant C only depends on $\operatorname{Im}\lambda_0$. □

2C. *Estimating the determinant from below.* Next we need to estimate $|f(\lambda,h)|$ from below. Note that λ_0 is not a zero of $f(\lambda,h)$ and that we have

$$(I+\tilde{K}(\lambda_0,\lambda_0,h)^{n^\sharp+1})^{-1} = I-\tilde{K}(\lambda_0,\lambda_0,h)^{n^\sharp+1}(I+\tilde{K}(\lambda_0,\lambda_0,h)^{n^\sharp+1})^{-1}.$$

By taking determinants and arguing as in the previous section, we obtain a lower bound at λ_0,

$$|f(\lambda_0,h)| \geq e^{-Ch^{-p}},$$

where the constant again depends only on $\operatorname{Im}\lambda_0$. Since we can bound $|f(\lambda,h)|$ from above by $M(h)$ and from below at a chosen point, we are in a position to employ Cartan's principle [Levin 1972, Theorem 11] to obtain a lower bound away from resonances.

Proposition 2.3. *For each $\epsilon > 0$ there exists $C = C(\epsilon)$ such that*

$$|f(\lambda,h)| \geq e^{-Ah^{-p}\log(1/S(h))}, \qquad \lambda \in \Omega_\epsilon(h) \setminus \bigcup_j D(r_j(h), S(h)),$$

where $S(h) \ll 1$ and $\{r_j(h)\}$ denote the resonances of $P(h)$ in $\Omega_\epsilon(h)$.

Proof. Rather than applying [Levin 1972, Theorem 11] directly, we prefer to control the set where the lower bound holds at the expense of the quality of the lower bound, just as in [Petkov and Zworski 2001]. For the reader's convenience we reproduce the proof, making the necessary adjustments.

Choose λ_0 with fixed real part. Define radii and disks

$$\rho_s(h) = T_1+\gamma-\epsilon_0-s\epsilon, \quad D_s(h) = D(\lambda_0,\rho_s(h)), \quad s=1,2,3.$$

We see that $f(\lambda,h)$ is analytic in the disk $D_1(h)$. Let $r_j(h)$, $j=1,\dots,N(h)$ denote the zeros of $f(\lambda,h)$ in $D_2(h)$, including multiplicity, and define the Blaschke product

$$\phi(\lambda,h) = \frac{(-\rho_2(h))^{N(h)}}{(r_1(h)-\lambda_0)\cdots(r_{N(h)}(h)-\lambda_0)} \prod_j \frac{\rho_2(h)(\lambda-r_j(h))}{\rho_2(h)^2-(\overline{r_j(h)-\lambda_0})(\lambda-\lambda_0)}.$$

Then ϕ has the same zeros as $f(\lambda,h)$, no poles in $D_2(h)$, and satisfies $\phi(\lambda_0,h)=1$. Moreover, on the boundary of $D_2(h)$,

$$|\phi(\lambda,h)| = \frac{\rho_2^{N(h)}(h)}{|(\lambda_0-r_1(h))\cdots(\lambda_0-r_{N(h)})|} \geq 1. \tag{2-2}$$

Since the function defined by

$$\psi(\lambda,h) = \frac{f(\lambda,h)}{\phi(\lambda,h)}$$

has no zeros in $D_2(h)$, we may apply (2-2) and Carathéodory's estimate [Levin 1972, Theorem 8] to conclude that in $D_3(h)$ we have the lower bound

$$\begin{aligned}\log|\psi(\lambda,h)| &\geq -\frac{2\rho_3(h)}{\epsilon}\log\sup_{\lambda\in D_1(h)}|\psi(\lambda,h)| + \frac{\rho_2(h)+\rho_3(h)}{\epsilon}\log|\psi(\lambda_0,h)| \\ &\geq -\frac{2\rho_3(h)}{\epsilon}\log\sup_{\lambda\in D_2(h)}|f(\lambda,h)| + \frac{\rho_2(h)+\rho_3(h)}{\epsilon}\log|f(\lambda_0,h)|.\end{aligned}$$

It therefore suffices to bound $|\phi(\lambda,h)|$ from below in $D_3(h)$.

Outside the set $\bigcup_j D(r_j(h),S(h))$, the polynomial appearing in the numerator of $\phi(\lambda,h)$ is bounded below by $S(h)^{N(h)}$. On the other hand, the polynomial in the denominator of $\phi(\lambda,h)$ is bounded above in $D_3(h)$ by $\rho_2(h)^{N(h)}(\rho_2(h)+\rho_3(h))^{N(h)}$. Therefore

$$|\phi(\lambda,h)| \geq \left(\frac{S(h)}{\rho_2(h)(\rho_2(h)+\rho_3(h))}\right)^{N(h)}, \quad \lambda\in D_3(h)\setminus\bigcup_j D(r_j(h),S(h)).$$

Moreover, we can apply Jensen's formula to estimate the number of zeros $N(h)$ in $D_2(h)$ by

$$\begin{aligned}N(h) &\leq \frac{1}{\log\frac{\rho_1(h)}{\rho_2(h)}}\bigl(\log\sup_{\lambda\in D_1}|f(\lambda,h)| - \log|f(\lambda_0,h)|\bigr) \\ &\leq \frac{1}{\log\frac{\rho_1(h)}{\rho_2(h)}}\bigl(\log M(h) - \log|f(\lambda_0,h)|\bigr) \\ &= O(h^{-p}).\end{aligned}$$

Combining all the contributions, we obtain

$$|f(\lambda,h)| \geq e^{-Ch^{-p}\log(1/S(h))}, \quad \lambda\in D_3(h)\setminus\bigcup_j D(r_j(h),S(h)).$$

Since all the constants appearing are uniform in $\operatorname{Re}\lambda_0$, we can vary the real part in $\Omega_\epsilon(h)$ and obtain the necessary lower bound. Of course, ϵ is arbitrary and the result follows. □

We can now establish our main theorem on resolvent estimates.

Theorem A. *For each* $\epsilon>0$, *there exists* $A=A(\epsilon)$ *such that*

$$\|R(\lambda,h)\|_{\mathcal{H}_\gamma\to\mathcal{H}_{-\gamma}} < e^{Ah^{-p}\log(1/S(h))}, \quad \lambda\in\Omega_\epsilon(h)\setminus\bigcup_j D(r_j(h),S(h)),$$

where $S(h)\ll 1$ *and* $\{r_j(h)\}$ *denote the resonances of* $P(h)$ *in* $\Omega_\epsilon(h)$.

Proof. Apply Propositions 2.2 and 2.3 to (2-1). □

3. From quasimodes to resonances

The passage from quasimodes to resonances is essentially an argument by contradiction. In the absence of resonances, the exponential bound appearing in Theorem A would hold throughout $\Omega_\epsilon(h)$; combined with the self-adjoint bound in the upper half-plane, an application of the "semiclassical maximum principle" implies a resolvent estimate on the real axis that contradicts the existence of a real quasimode. First results in this direction are due to Stefanov and Vodev [1996] who used the Phragmén–Lindelöf principle to show that having high energy real quasimodes implies the existence of resonances converging to the real axis. Bounds on the resolvent play a central role in that argument which go back to the work of Carleman [1936] on the completeness of sets of eigenfunctions. Tang and Zworski [1998] replaced the Phragmén–Lindelöf principle with a local version of the maximum principle which showed that there exists a resonance close to each quasimode. Stefanov further refined these method by dealing with multiplicities [1999], and modifying the maximum principle [2005] to allow the localization of resonances exponentially close to the real axis.

3A. ***Quasimodes.*** Suppose that $u(h) \in \mathcal{D}$ satisfies $\|u(h)\| = 1$ and

$$\operatorname{supp} u(h) \subset K \quad \text{for a compact set } K \text{ independent of } h.$$

Suppose further that there exists $\lambda(h)^2 \in (a_0, b_0)$ such that

$$\left\|(P(h) - \lambda(h)^2)u(h)\right\| \leq R(h)$$

for a function $R(h) \geq 0$. We refer to such functions as quasimodes with accuracy $R(h)$. For the resolvent, choose a weight φ so that $\varphi \equiv 0$ on K. Also choose χ_1 with $\varphi \equiv 0$ on $\operatorname{supp} \chi_1$ and $\chi_1 \equiv 1$ on K. Notice that for λ in the upper half-plane,

$$e^{-\gamma\varphi} R(\lambda, h) e^{-\gamma\varphi}(P(h) - \lambda^2)u = e^{-\gamma\varphi} R(\lambda, h) e^{-\gamma\varphi}(P(h) - \lambda^2)\chi_1 u = u,$$

and hence this equation holds away from poles by analytic continuation. We also recall the following standard fact: consider the Laurent expansion of $e^{-\gamma\varphi} R(\lambda, h) e^{-\gamma\varphi}$ near a resonance $r(h)$:

$$e^{-\gamma\varphi} R(\lambda, h) e^{-\gamma\varphi} = \text{holomorphic}(\lambda) + \sum_{j=1}^{N} A_j (\lambda^2 - r(h)^2)^{-j}.$$

Then, $\operatorname{range}(A_j) \subseteq \operatorname{range}(A_1)$ for $j = 1, \ldots, N$. For a very general discussion of these types of results, see [Agmon 1998]. Consider the resonances $r_i(h)$ for $i = 1, \ldots, N(h)$ contained in the set $\Omega_\epsilon(h)$, each with the associated residue $A_1^{(i)}$. If Π denotes the projection onto $\bigoplus_i \operatorname{range}(A_1^{(i)})$, then $(I - \Pi)A_j^{(i)} = 0$ for each i, j.

Hence,

$$(I-\Pi)e^{-\gamma\varphi}R(\lambda,h)e^{-\gamma\varphi}$$

is holomorphic in $\Omega_\epsilon(h)$. By the maximum principle, this operator satisfies the bound given by Theorem A in a set slightly smaller than $\Omega_\epsilon(h)$ — see the proof of [Stefanov 1999, Theorem 1] or [Stefanov 2005, Theorem 3] for a precise statement.

3B. ***Semiclassical maximum principle.*** We now review the semiclassical maximum principle, as presented in [Stefanov 2005].

Lemma 3.1. *Let $a(h) < b(h)$ and suppose that $S_\pm(h), \alpha(h), w(h)$ are functions satisfying*

$$0 < S_+(h) \le S_-(h), \quad 1 \le \alpha(h), \quad S_-(h)\alpha(h)\log\alpha(h) \le w(h).$$

Also, suppose $F(\lambda,h)$ is a holomorphic function defined in a neighborhood of

$$[a(h)-w(h), b(h)+w(h)] + i[-\alpha(h)S_-(h), S_+(h)].$$

If

$$\begin{cases} |F(\lambda,h)| \le e^{\alpha(h)}, & \lambda \in [a(h)-w(h), b(h)+w(h)] + i[-\alpha(h)S_-(h), S_+(h)], \\ |F(\lambda,h)| \le M(h), & \lambda \in [a(h)-w(h), b(h)+w(h)] + iS_+(h), \end{cases}$$

with $M(h) \ge 1$, then there exists $h_1 = h_1(S_-, S_+, \alpha) > 0$ such that

$$|F(\lambda,h)| \le e^3 M(h), \quad \lambda \in [a(h), b(h)] + i[S_-(h), S_+(h)]$$

for $h \le h_1$.

For our application, we will apply this lemma with

- $S_-(h) = S_+(h) = S(h)$,
- $F(\lambda,h) = (I-\Pi)e^{-\gamma\varphi}R(\lambda,h)e^{-\gamma\varphi}$,
- $\alpha(h) = Ch^{-p}\log(1/S(h))$,
- $M(h) = 1/S(h)$.

The choice of $S(h)$ and $w(h)$ is made as in [Stefanov 2005] according to the accuracy $R(h)$ of the quasimodes.

3C. ***Lower bounds on the number of resonances.*** Here we state the main theorem on the existence of resonances rapidly converging to the real axis. We refer to [Stefanov 2005, Theorem 3] for the proof; the only modification is that instead of a compactly truncated resolvent $(I-\Pi)\chi R(\lambda,h)\chi$, we use $(I-\Pi)e^{-\gamma\varphi}R(\lambda,h)e^{-\gamma\varphi}$.

Theorem B. *Let $P(h)$ satisfy the black box hypotheses. Let $0 < a_0 < a(h) < b(h) < b_0 < \infty$. Assume there is an h_0 such that for $h < h_0$ there exists $m(h) \in \{1, 2, \dots\}$, $\lambda_n(h)^2 \in [a(h), b(h)]$, and $u_n(h) \in \mathcal{D}$ with $\|u_n(h)\| = 1$ for $1 \le n \le m(h)$ such that* supp $u_n(h) \subset K$ *for a compact set K, independent of h. Suppose further that*

(1) $\left\|(P(h)-\lambda_n(h)^2)u_n(h)\right\| \leq R(h)$,

(2) *whenever a collection* $\{v_n(h)\}_{n=1}^{m(h)} \subset \mathcal{H}$ *satisfies* $\|u_n(h)-v_n(h)\| < h^N/M$, $\{v_n(h)\}_{n=1}^{m(h)}$ *is linearly independent*,

where $R(h) \leq h^{p+N+1}/C\log(1/h)$ *and* $C \gg 1$, $N \geq 0$, $M > 0$. *Then there exists* $C_0 > 0$ *depending on* a_0, b_0 *and the operator* $P(h)$ *such that for* $B > 0$ *there exists* $h_1 < h_0$ *depending on* A, B, M, N *so that the following holds*: *whenever* $h \in (0, h_1)$, *the operator* $P(h)$ *has at least* $m(h)$ *resonances in the strip*

$$\left[a(h)-c(h)\log\tfrac{1}{h}, b(h)+c(h)\log\tfrac{1}{h}\right]-i[0, c(h)]$$

where $c(h)=\max(C_0BMR(h)h^{-p-N-1}, e^{-B/h})$.

Acknowledgments

I would like to thank Maciej Zworski for suggesting the problem, along with many valuable conversations. Thanks to the anonymous referee for suggesting improvements in the exposition. I am also grateful to Jeffrey Galkowski for his interest in the problem and some helpful discussions.

References

[Agmon 1998] S. Agmon, "A perturbation theory of resonances", *Comm. Pure Appl. Math.* **51**:11-12 (1998), 1255–1309. MR 99i:47023 Zbl 0941.47008

[Burq 2002] N. Burq, "Semi-classical estimates for the resolvent in nontrapping geometries", *Int. Math. Res. Not.* **2002**:5 (2002), 221–241. MR 2002k:81069 Zbl 1161.81368

[Carleman 1936] T. Carleman, "Über die asymptotische Verteilung der Eigenwerte partieller Differentialgleichungen", *Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl.* **88** (1936), 119–134. Zbl 0017.11402

[Dyatlov 2011] S. Dyatlov, "Quasi-normal modes and exponential energy decay for the Kerr–de Sitter black hole", *Comm. Math. Phys.* **306**:1 (2011), 119–163. MR 2012g:58055 Zbl 1223.83029

[Gannot 2014] O. Gannot, "Quasinormal modes for Schwarzschild–AdS black holes: exponential convergence to the real axis", *Comm. Math. Phys.* **330**:2 (2014), 771–799. MR 3223487 Zbl 1295.85001

[Gohberg and Kreĭn 1969] I. C. Gohberg and M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators*, Translations of Mathematical Monographs **18**, American Mathematical Society, Providence, RI, 1969. MR 39 #7447 Zbl 0181.13504

[Holzegel and Smulevici 2014] G. Holzegel and J. Smulevici, "Quasimodes and a lower bound on the uniform energy decay rate for Kerr–AdS spacetimes", *Anal. PDE* **7**:5 (2014), 1057–1090. MR 3265959 Zbl 1300.83030 arXiv 1303.5944

[Ikawa 1982] M. Ikawa, "Decay of solutions of the wave equation in the exterior of two convex obstacles", *Osaka J. Math.* **19**:3 (1982), 459–509. MR 84e:35018 Zbl 0498.35008

[Lax and Phillips 1989] P. D. Lax and R. S. Phillips, *Scattering theory*, 2nd ed., Pure and Applied Mathematics **26**, Academic Press, Boston, 1989. MR 90k:35005 Zbl 0697.35004

[Levin 1972] B. Y. Levin, *Distribution of zeros of entire functions*, American Mathematical Society, Providence, RI, 1972. MR 28 #217 Zbl 0152.06703

[McLeod 1967] J. B. McLeod, "The analytic continuation to the unphysical sheet of the resolvent kernel associated with the Schroedinger operator", *Quart. J. Math. Oxford Ser.* (2) **18** (1967), 219–231. MR 37 #608a Zbl 0152.11401

[Petkov and Zworski 2001] V. Petkov and M. Zworski, "Semi-classical estimates on the scattering determinant", *Ann. Henri Poincaré* **2**:4 (2001), 675–711. MR 2002h:35222 Zbl 1041.81041

[Rauch 1978] J. Rauch, "Local decay of scattering solutions to Schrödinger's equation", *Comm. Math. Phys.* **61**:2 (1978), 149–168. MR 58 #14590 Zbl 0381.35023

[Sá Barreto and Zworski 1995] A. Sá Barreto and M. Zworski, "Existence of resonances in three dimensions", *Comm. Math. Phys.* **173**:2 (1995), 401–415. MR 96j:35183 Zbl 0835.35099

[Sjöstrand 1997] J. Sjöstrand, "A trace formula and review of some estimates for resonances", pp. 377–437 in *Microlocal analysis and spectral theory* (Lucca, 1996), edited by L. Rodino, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **490**, Kluwer Academic, Dordrecht, 1997. MR 99e:47064 Zbl 0877.35090

[Sjöstrand and Zworski 1991] J. Sjöstrand and M. Zworski, "Complex scaling and the distribution of scattering poles", *J. Amer. Math. Soc.* **4**:4 (1991), 729–769. MR 92g:35166 Zbl 0752.35046

[Stefanov 1999] P. Stefanov, "Quasimodes and resonances: sharp lower bounds", *Duke Math. J.* **99**:1 (1999), 75–92. MR 2000k:35230 Zbl 0952.47013

[Stefanov 2005] P. Stefanov, "Approximating resonances with the complex absorbing potential method", *Comm. Partial Differential Equations* **30**:10-12 (2005), 1843–1862. MR 2006i:35266 Zbl 1095.35017

[Stefanov and Vodev 1996] P. Stefanov and G. Vodev, "Neumann resonances in linear elasticity for an arbitrary body", *Comm. Math. Phys.* **176**:3 (1996), 645–659. MR 96k:35122 Zbl 0851.35032

[Tang and Zworski 1998] S.-H. Tang and M. Zworski, "From quasimodes to resonances", *Math. Res. Lett.* **5**:3 (1998), 261–272. MR 99i:47088 Zbl 0913.35101

[Vodev 1994] G. Vodev, "Sharp bounds on the number of scattering poles in the two-dimensional case", *Math. Nachr.* **170** (1994), 287–297. MR 95i:35209 Zbl 0829.35091

[Warnick 2015] C. M. Warnick, "On quasinormal modes of asymptotically anti-de Sitter black holes", *Comm. Math. Phys.* **333**:2 (2015), 959–1035. MR 3296168 Zbl 06394860

[Wunsch 2012] J. Wunsch, "Resolvent estimates with mild trapping", preprint, 2012. arXiv 1209.0843

[Zworski 1989] M. Zworski, "Sharp polynomial bounds on the number of scattering poles", *Duke Math. J.* **59**:2 (1989), 311–323. MR 90h:35190 Zbl 0705.35099

Received August 6, 2013. Revised December 10, 2014.

ORAN GANNOT
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, BERKELEY
EVANS HALL
BERKELEY, CA 94720
UNITED STATES

ogannot@math.berkeley.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.99

A GENERAL SIMPLE RELATIVE TRACE FORMULA

JAYCE R. GETZ AND HEEKYOUNG HAHN

In this paper we prove a relative trace formula for all pairs of connected algebraic groups $H \leq G \times G$, with G a reductive group and H the direct product of a reductive group and a unipotent group, given that the test function satisfies simplifying hypotheses. As an application, we prove a relative analogue of the Weyl law, giving an asymptotic formula for the number of eigenfunctions of the Laplacian on a locally symmetric space associated to G weighted by their L^2-restriction norm over a locally symmetric subspace associated to $H_0 \leq G$.

1. Introduction

Let G be a connected reductive algebraic group over a number field F and let A_G be the neutral component of the real points of the greatest $\mathbb{Q}$-split torus in the center of $\mathrm{Res}_{F/\mathbb{Q}} G$. Throughout this paper, we let

$$H \leq G \times G$$

be a connected algebraic subgroup such that H is the direct product of a reductive group and a unipotent group; both of these groups are necessarily connected. We do not assume that the decomposition of H into a reductive and unipotent group is compatible with the embedding $H \hookrightarrow G \times G$.

Getz is thankful for partial support provided by NSF grants DMS 1405708. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.
MSC2010: primary 11F70; secondary 35P20.
Keywords: relative trace formula, Weyl law.

Let $\chi : H(\mathbb{A}_F) \to \mathbb{C}^\times$ be a quasi-character trivial on $A_{G,H}H(F)$ (see Section 2B for the definition of $A_{G,H}$ and the other $A_?$ groups; they are all central subgroups). Let

$$\varphi \in L^2_{\mathrm{cusp}}\big(A_G G(F)\backslash G(\mathbb{A}_F) \times A_G G(F)\backslash G(\mathbb{A}_F)\big)$$

be a smooth cusp form, and let

$$\mathcal{P}_\chi(\varphi) := \int_{A_{G,H}H(F)\backslash H(\mathbb{A}_F)} \chi(h_\ell, h_r)\varphi(h_\ell, h_r)\, d(h_\ell, h_r) \tag{1.1}$$

whenever this period is well-defined (for a criterion see Corollary 3.2 below). Here $d(h_\ell, h_r)$ is a Haar measure; we will set our conventions on Haar measures in Section 2C below. The relative trace formula is a tool for studying the period integrals $\mathcal{P}_\chi(\varphi)$. Many particular instances of the relative trace formula have been developed, but the development has not been systematic.

In this paper we establish the formula in what we view as the natural level of generality in terms of the subgroup H for test functions satisfying the usual "simple trace formulae" hypotheses. In particular, we only make the assumption that H is connected and a direct product of a reductive and unipotent group. In contrast, in all references known to the authors the subgroup H is assumed to be "large", e.g., spherical and satisfy other simplifying hypotheses. We also note that this greater generality is not vacuous in that it leads to new applications, for example, Theorem 1.2 below. It is also used in constructing the four-variable automorphic kernel functions of [Getz 2014].

For $f \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ let

$$\begin{aligned} R(f) : L^2(A_G G(F)\backslash G(\mathbb{A}_F)) &\to L^2(A_G G(F)\backslash G(\mathbb{A}_F)) \\ \varphi &\mapsto \left(x \mapsto \int_{A_G\backslash G(\mathbb{A}_F)} f(g)\varphi(xg)\, dg\right) \end{aligned}$$

denote the operator defined by the right regular action and f. We prove the following theorem:

Theorem 1.1. *Let $f \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ be a function such that $R(f)$ has cuspidal image and such that if the $H(\mathbb{A}_F)$-orbit of $\gamma \in G(F)$ intersects the support of f then γ is elliptic, unimodular and closed. Then*

$$\sum_\gamma \tau(H_\gamma)\, \mathrm{RO}^\chi_\gamma(f) = \sum_\pi \mathrm{rtr}\, \pi(f),$$

where the sum on γ is over elliptic unimodular closed relevant classes and the sum on π is over isomorphism classes of cuspidal automorphic representations of $A_G\backslash G(\mathbb{A}_F)$.

Here elliptic, unimodular and closed are defined as in Section 2A, the action of H on G is given in (2A.1), and relevant is defined as in Section 4A. Moreover, $\tau(H_\gamma)$ is a volume term that can be viewed as a Tamagawa number if normalized appropriately, $\mathrm{RO}_\gamma^\chi(f)$ is a relative orbital integral (see Section 4 for both of these notions) and $\mathrm{rtr}\,\pi(f)$ is the relative trace of $\pi(f)$, defined in (3.2) (it is a period integral of the form (1.1)). A cuspidal automorphic representation π of $A_G\backslash G(\mathbb{A}_F)$, by convention, is an automorphic representation of $G(\mathbb{A}_F)$ trivial on A_G that can be realized in $L^2_{\mathrm{cusp}}(A_G G(F)\backslash G(\mathbb{A}_F))$. In particular, we do not fix an embedding; the definition of $\mathrm{rtr}\,\pi(f)$ involves the entire π-isotypic subspace of $L^2_{\mathrm{cusp}}(A_G G(F)\backslash G(\mathbb{A}_F))$.

Remarks. (1) Given the work of Lindenstrauss and Venkatesh [2007], henceforth abbreviated [LV], the assumption that $R(f)$ has purely cuspidal image may not be as severe a restriction as one might think (see also the proof of Theorem 5.1).

(2) Though the method of proof is the usual one (take a kernel and compute the integral over $A_{G,H}H(F)\backslash H(\mathbb{A}_F)$ two ways) there are many points in the proof of Theorem 1.1 that are not obvious. On the spectral side we check that $\mathrm{rtr}\,\pi(f)$ is well-defined for all f, not just K_∞-finite f. On the geometric side we define a notion of elliptic elements and the relative analogue of semisimple elements (which we call unimodular and closed). These have only appeared in special cases in the literature. We also use Galois cohomology to deal with nonconnected stabilizers in a way that we have never seen in the literature in the context of the relative trace formula.

The formula in Theorem 1.1 is called *simple* because we have imposed conditions on the test function f to ensure that various analytic difficulties disappear. Theorem 1.1 is *general* because the geometric set-up includes all trace formulae that the authors have seen as special cases. For example, the simple twisted relative trace formula of the second author [Hahn 2009] is a special case of this formula, as is the usual simple trace formula of Deligne and Kazhdan [Bernstein et al. 1984] (see also [Rogawski 1983]), as one can see by taking χ to be trivial and H to be the diagonal copy of G inside $G\times G$. As another example, let E/F be a quadratic extension, let $G=\mathrm{Res}_{E/F}\,\mathrm{GL}_n$, let $U_n\le G$ be a unitary group, let $N\le G$ be the unipotent radical of the Borel subgroup of upper triangular matrices, let $\psi : N(F)\backslash N(\mathbb{A}_F)\to\mathbb{C}^\times$ be a character, and set

$$H=U_n\times N \quad\text{and}\quad \chi=1\times\psi.$$

In this case the trace formula above is a simple version of one introduced by Jacquet and Ye [1996]. We also note that the formula does not hold for a general connected algebraic subgroup $H\le G\times G$ without serious modification (see the remark after Proposition 3.4), so in some sense it is as general as possible.

As an application of these ideas, we prove a relative analogue of the Weyl law in Theorem 1.2 below. It gives an asymptotic formula for the number of eigenfunctions of the Laplacian on a locally symmetric space associated to G weighted by the L^2-restriction norm over a locally symmetric subspace associated to $H_0 \le G$.

To state it, assume that G is split and adjoint over $\mathbb{Q}$. Note that $G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})$ is of finite volume but noncompact. Let $H_0 \le G$ be the direct product of a reductive group and a unipotent group and let

$$K := K_\infty \times K^\infty \le G(\mathbb{A}_\mathbb{Q}),$$

where $K_\infty \le G(\mathbb{R})$ is a maximal compact subgroup and $K^\infty \le G(\mathbb{A}_\mathbb{Q}^\infty)$ is a compact open subgroup satisfying the torsion-freeness assumption (TF) of Section 5 below.

In the setting above, using a technique developed in [LV], we prove Theorem 1.2 below. We remark that since $G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})$ is noncompact, even if $H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_\mathbb{Q})$ is compact the theorem does not follow in any obvious way from the classical Weyl law or its local variants.

Theorem 1.2. *Assume that $H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_\mathbb{Q})$ is compact. As $X \to \infty$ one has*

$$\sum_{\pi:\,\pi(\Delta)\le X} \sum_{\varphi\in\mathcal{B}(\pi)^K} \int_{H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_\mathbb{Q})} |\varphi(h)|^2\, dh \sim \alpha(G)\,\mathrm{meas}_{dh}(H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_\mathbb{Q}))X^{d/2},$$

where the sum is over isomorphism classes of cuspidal automorphic representations π of $G(\mathbb{A}_\mathbb{Q})$, $\mathcal{B}(\pi)$ is an orthonormal basis of the π-isotypic subspace of $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q}))$, $\pi(\Delta)$ is the eigenvalue of the Casimir operator Δ acting on the space of K_∞-fixed vectors in π, $\alpha(G) > 0$ is a constant related to the Plancherel measure defined in [LV], and $d = \dim(G(\mathbb{R})/K_\infty)$.

We refer to the asymptotic in Theorem 1.2 as a relative Weyl law. We can in fact weaken the assumption that $H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_\mathbb{Q})$ is compact. Specifically, in Proposition 5.2 we prove that if $H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_\mathbb{Q})$ is of finite volume but noncompact, then the relative Weyl law holds provided that one assumes the upper bound of the relative Weyl law (in the setting of the usual Weyl law this was proven in [Donnelly 1982]). Interestingly, this is not known in the relative case.

We now outline the sections of this paper. In the following section we recall the notion of relative classes and relative analogues of definitions often used in the context of the absolute trace formula. The proof of Theorem 1.1 comes down to evaluating an integral of a kernel function in two ways. The spectral evaluation is given in Section 3 and the geometric evaluation is given in Section 4. Finally, in Section 5 we prove Theorem 1.2.

2. Preliminaries and notation

2A. ***Relative classes.*** Let G be a connected reductive algebraic group over a characteristic zero field F with algebraic closure $\overline{F}$ and let

$$H \leq G \times G$$

be a connected algebraic subgroup that is the direct product of a reductive and a unipotent group. We let

$$\mathrm{diag} : G \to G \times G$$

denote the diagonal embedding. The letter R will denote an F-algebra. There is an action of H on G given at the level of points by

$$\begin{aligned} \cdot : H(R) \times G(R) &\to G(R) \\ ((h_\ell, h_r), g) &\mapsto h_\ell g h_r^{-1}. \end{aligned} \tag{2A.1}$$

The stabilizer of a $\gamma \in G(R)$ will be denoted by H_γ. By assumption, we can write

$$H = H^r \times H^u$$

where H^r is reductive and H^u is unipotent.

Definition 2.1. Let k/F be a field. An element $\gamma \in G(k)$ is

- *closed* if the orbits of γ under H and H^r are both closed.
- *unimodular* if H_γ is the direct product of a reductive and a unipotent group.
- *elliptic* if the maximal reductive quotient of $H_\gamma / \mathrm{diag}(Z_G) \cap H$ has anisotropic center.

Remark. If H is reductive, then a closed element has reductive stabilizer and hence is unimodular.

If R is an F-algebra, then an element of

$$\Gamma(R) := H(R) \backslash G(R) \tag{2A.2}$$

is called a *relative class*, or simply a class. Note that here the quotient is taken with respect to the action (2A.1). All of the conditions mentioned in the previous definition depend only on the relative class of an element of $\Gamma(R)$, and not on the particular element. If k is a field with algebraic closure $\bar{k}$ we say that $\gamma, \gamma' \in G(k)$ are in the same *geometric class* if there is an $h \in H(\bar{k})$ such that $h \cdot \gamma = \gamma'$. We denote the set of geometric classes by

$$\Gamma^{\mathrm{geo}}(k) := \mathrm{Im}(G(k) \to H \backslash G(k)). \tag{2A.3}$$

2B. ***The A groups.*** If H is a connected algebraic group over a number field F, we let A_H be the neutral component (in the real topology) of the real points of the maximal $\mathbb{Q}$-split torus in $\mathrm{Res}_{F/\mathbb{Q}} H$. We let

$$\begin{aligned} A_{G,H} &:= A_H \cap (A_G \times A_G) \\ A &:= A_H \cap \mathrm{diag}(A_G). \end{aligned}$$

We choose Haar measures da_G on A_G, $d(a_\ell, a_r)$ on $A_{G,H}$ and da on A.

2C. ***Haar measures.*** Throughout this work we fix a Haar measure dg on $G(\mathbb{A}_F)$ and use it and da to obtain a Haar measure, also denoted by dg, on $A_G\backslash G(\mathbb{A}_F)$. We also fix a Haar measure $d(h_\ell, h_r)$ on $H(\mathbb{A}_F)$ and also denote by $d(h_\ell, h_r)$ the induced measure on $A_{G,H}\backslash H(\mathbb{A}_F)$. For each unimodular $\gamma \in H(F)$ we let $d(h_\ell, h_r)_\gamma$ be a Haar measure on $H_\gamma(\mathbb{A}_F)$ and let

$$\dot{d}(h_\ell, h_r)$$

denote the induced right-invariant Radon measure on $H_\gamma(\mathbb{A}_F)\backslash H(\mathbb{A}_F)$.

3. Relative traces

As in the introduction, let

$$\chi : H(\mathbb{A}_F) \to \mathbb{C}^\times$$

be a quasi-character trivial on $A_{G,H}H(F)$. Let $f \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$, and let π be a cuspidal automorphic representation of $A_G\backslash G(\mathbb{A}_F)$. We let $\mathcal{B}(\pi)$ be an orthonormal basis of the π-isotypic subspace of $L^2_{\text{cusp}}(A_G G(F)\backslash G(\mathbb{A}_F))$ consisting of smooth vectors and let

$$K_{\pi(f)}(x, y) := \sum_{\varphi \in \mathcal{B}(\pi)} R(f)\varphi(x)\overline{\varphi}(y). \tag{3.1}$$

A priori this expression only converges in $L^2(A_G G(F)\backslash G(\mathbb{A}_F) \times A_G G(F)\backslash G(\mathbb{A}_F))$. However, it follows from the Dixmier–Malliavin lemma [1978] that there is a unique smooth (jointly in (x, y)) square-integrable function that represents $K_{\pi(f)}$ (compare the proof of Theorem 3.1). From now on we use the notation $K_{\pi(f)}$ to refer to this function, and whenever $R(f)$ has cuspidal image we let

$$K_f(x, y) := \sum_{\pi} \sum_{\varphi \in \mathcal{B}(\pi)} R(f)\varphi(x)\overline{\varphi}(y),$$

where the sum is over isomorphism classes of cuspidal automorphic representations π of $A_G\backslash G(\mathbb{A}_F)$.

We refer to the integral

$$\operatorname{rtr} \pi(f) := \mathcal{P}_\chi(K_{\pi(f)}) \tag{3.2}$$

as the *relative trace* of $\pi(f)$, where $\mathcal{P}_\chi$ is the period integral defined in (1.1) above. We will show in the course of the proof of Theorem 3.1 that the integral in the definition of $\mathcal{P}_\chi(K_{\pi(f)})$ is well-defined.

The following theorem amounts to the computation of the spectral side of our relative trace formula:

Theorem 3.1. *Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$, and assume that $R(f)$ has cuspidal image. Then*

$$\int_{A_{G,H} H(F)\backslash H(\mathbb{A}_F)} \chi(h_\ell, h_r) K_f(h_\ell, h_r) d(h_\ell, h_r) = \sum_\pi \operatorname{rtr} \pi(f).$$

Moreover, the integral on the left and the sum on the right are absolutely convergent.

This is the main result of this section. A similar result is proven in [Hahn 2009] in a special case, but we give a simpler proof here.

Fix a maximal compact subgroup K_∞ of $G(F_\infty)$, where $F_\infty := \prod_{v|\infty} F_v$ is the product of the archimedean completions of F. As mentioned above, in the course of the proof of the theorem we will prove that the integral in the definition of $\operatorname{rtr} \pi(f)$ is absolutely convergent. Assuming this for the moment, we obtain the following corollary:

Corollary 3.2. *Assume that $\varphi \in L^2_{\text{cusp}}(A_G G(F)\backslash G(\mathbb{A}_F))$ is a cuspidal automorphic form, that is, φ is cuspidal, K_∞-finite and finite under the center of the universal enveloping algebra of* $\operatorname{Lie}(\operatorname{Res}_{F/\mathbb{Q}} G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}$. *Then the integral defining $\mathcal{P}_\chi(\varphi \times \overline{\varphi})$ is absolutely convergent.*

Proof. It suffices to verify the corollary when φ lies in the π-isotypic subspace $L^2_{\text{cusp}}(\pi)$ of the cuspidal subspace of $L^2(A_G G(F)\backslash G(\mathbb{A}_F))$ for a cuspidal automorphic representation π. By a standard argument one can choose an $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ such that $R(f)\varphi = \varphi$ and $R(f)$ acts by zero on the orthogonal complement of φ in $L^2_{\text{cusp}}(\pi)$. Hence

$$\mathcal{P}_\chi(\varphi \times \overline{\varphi}) = \mathcal{P}_\chi(K_{\pi(f)}) = \operatorname{rtr} \pi(f). \qquad \square$$

3A. *Integrals of rapidly decreasing functions.* Let $Z \leq \operatorname{Res}_{F/\mathbb{Q}} G$ be the maximal split torus in the center of G. Let $T \leq \operatorname{Res}_{F/\mathbb{Q}} G \times \operatorname{Res}_{F/\mathbb{Q}} G$ be a maximal split torus and let Δ be a choice of simple roots of $T/(Z \times Z)$ in $\operatorname{Res}_{F/\mathbb{Q}} G \times \operatorname{Res}_{F/\mathbb{Q}} G$. Set

$$A^G := T(\mathbb{R})^+ / A_G \times A_G$$

where the $+$ denotes the neutral component in the real topology. For any positive real number r we set

$$A^G_r := \{t \in A^G : t^\alpha > r \text{ for all } \alpha \in \Delta\}. \tag{3A.1}$$

For concreteness, we record the following definition:

Definition 3.3. A function

$$\varphi : A_G G(F)\backslash G(\mathbb{A}_F) \times A_G G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$$

is *rapidly decreasing* if it is smooth and, for all compact subsets Ω of the domain and all $r \in \mathbb{R}_{>0}$ and $p \in \mathbb{Z}$, there is a constant $C = C_{\Omega,r,p}$ such that

$$|\varphi(tx)| \leq Ct^{\alpha p}$$

for all $t \in A_r^G$, $x \in \Omega$, and $\alpha \in \Delta$.

Proposition 3.4. *For all rapidly decreasing* (*smooth*) *functions* φ *belonging to* $L^2((A_G G(F)\backslash G(\mathbb{A}_F))^{\times 2})$, *the period integral*

$$\mathcal{P}_\chi(\varphi) := \int_{A_{G,H}H(F)\backslash H(\mathbb{A}_F)} \chi(h_\ell, h_r)\varphi(h_\ell, h_r)d(h_\ell, h_r)$$

is absolutely convergent.

Proof. Since H is the direct product of a unipotent group and a reductive group, and $U(F)\backslash U(\mathbb{A}_F)$ is compact for any unipotent group U, it suffices to prove the proposition in the special case where H is reductive. In this case, the argument proving [Ash et al. 1993, Proposition 1] implies the proposition. □

Remark. This proposition depends crucially on the fact that H is assumed to be a direct, not a semidirect, product of a reductive group and a unipotent group. It is false for a general connected algebraic group. Examples of this occur already in low-rank applications of the Rankin–Selberg method (see [Getz and Goresky 2012, Lemma 10.3] for an example).

We also recall the following basic theorem.

Theorem 3.5 [Godement 1966]. *Let* $r \in \mathbb{R}_{>0}$, $p \in \mathbb{Z}$ *and let* Ω *be a compact subset of* $(A_G G(F)\backslash G(\mathbb{A}_F))^{\times 2}$. *If* $\Phi \in C_c^\infty((A_G\backslash G(\mathbb{A}_F))^{\times 2})$ *then one has an estimate*

$$|R(\Phi)\varphi(tx)| \leq Ct^{\alpha p} \parallel \varphi \parallel$$

for all $\varphi \in L^2_{\text{cusp}}((A_G G(F)\backslash G(\mathbb{A}_F))^{\times 2})$, $t \in A_r^G$, $\alpha \in \Delta$ *and* $x \in \Omega$, *where the constant* $C := C_{r,p,\Omega,\Phi}$ *is independent of* φ. *In particular*, $R(\Phi)\varphi$ *is rapidly decreasing.* □

3B. ***Proof of Theorem 3.1.*** By assumption, $R(f)$ has image in the cuspidal spectrum. Thus the operator $R(f)$ is trace class and hence is Hilbert–Schmidt. We therefore have the convergent L^2-expansion

$$K_f(x, y) = \sum_\pi K_{\pi(f)}(x, y) = \sum_\pi \sum_{\varphi \in \mathcal{B}(\pi)} R(f)\varphi(x)\overline{\varphi}(y) \tag{3B.1}$$

where the sum is over isomorphism classes of cuspidal automorphic representations of $A_G\backslash G(\mathbb{A}_F)$. By the Dixmier–Malliavin lemma [1978] we can write f as a finite sum of functions of the form

$$f_1 * f_2 * f_3$$

for $f_1, f_2, f_3 \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$. It clearly suffices to prove the theorem for f of this special form, so for the moment we assume that $f = f_1 * f_2 * f_3$. For $f \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ let

$$(f)^\vee(g) := f(g^{-1}).$$

We note that

$$\sum_{\varphi\in\mathcal{B}(\pi)} R(f)\varphi(x)\overline{\varphi}(y) = \sum_{\varphi\in\mathcal{B}(\pi)} \varphi(x)R((f)^\vee)\overline{\varphi}(y)$$

because they both represent the same kernel. Thus

$$\begin{aligned} K_{\pi(f)}(x,y) &= \sum_{\varphi\in\mathcal{B}(\pi)} R(f_1 * f_2 * f_3)\varphi(x)\overline{\varphi}(y) \\ &= \sum_{\varphi\in\mathcal{B}(\pi)} R(f_2 * f_3)\varphi(x)R(f_1^\vee)\overline{\varphi}(y) \\ &= (R(f_2)\times R(f_1^\vee)) \sum_{\varphi\in\mathcal{B}(\pi)} R(f_3)\varphi(x)\overline{\varphi}(y). \end{aligned} \tag{3B.2}$$

The latter function is smooth as a function of (x, y) (jointly) and this is the unique smooth function representing $K_{\pi(f)}(x, y)$ as mentioned earlier (to prove convergence one can invoke Theorem 3.5). Thus we can view $K_{\pi(f)}(x, y)$ as an honest function. The same is true of $K_f(x, y)$ and (3B.1) holds pointwise.

Thus in view of Proposition 3.4, to complete the proof of the theorem it suffices to show that for any $f \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ one has

$$\sum_{\pi} |K_{\pi(f)}(x,y)| \tag{3B.3}$$

is rapidly decreasing as a function of $(x, y) \in (A_G G(F)\backslash G(\mathbb{A}_F))^{\times 2}$. To see this, we use a trick going back to Selberg. Using the Dixmier–Malliavin lemma we reduce to the case where $f = f_1 * f_2$. For $f \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ we set $f^*(g) := \overline{f(g^{-1})}$. Applying the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} |K_{\pi(f)}(x,y)|^2 &= \Big|\sum_{\varphi\in\mathcal{B}(\pi)} \pi(f_1)\varphi(x)\overline{\pi(f_2^*)\varphi(y)}\Big|^2 \\ &\leq \sum_{\varphi\in\mathcal{B}(\pi)} |\pi(f_1)\varphi(x)|^2 \sum_{\varphi\in\mathcal{B}(\pi)} |\overline{\pi(f_2^*)\varphi}(y)|^2 \\ &= K_{\pi(f_1^* * f_1)}(x,x)K_{\pi(f_2 * f_2^*)}(y,y). \end{aligned}$$

We note that originally the first identity is an identity of L^2-functions, but using the Dixmier–Malliavin lemma and Theorem 3.5 as above we can regard it as a pointwise identity of continuous functions. The same is true of the rest of the functions appearing in the inequalities above, and in particular the application of Cauchy–Schwarz makes sense. The point of all of this is that the kernels $K_{\pi(f_1 * f_1^*)}(x, x)$, $K_{\pi(f_2^* * f_2)}(y, y)$ are positive as functions of x and y.

By Hölder's inequality one has

$$\sum_{\pi}\big(K_{\pi(f_1^* * f_1)}(x, x)K_{\pi(f_2 * f_2^*)}(y, y)\big)^{1/2} \leq \Big(\sum_{\pi} K_{\pi(f_1^* * f_1)}(x, x)\Big)^{1/2}\Big(\sum_{\pi} K_{\pi(f_2 * f_2^*)}(y, y)\Big)^{1/2}.$$

Thus it is enough to prove that for all $h \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ the sum

$$\sum_{\pi} K_{\pi(h)}(x, x) \tag{3B.4}$$

is rapidly decreasing as a function of x. Using the Dixmier–Malliavin lemma again we reduce to the case that $h = h_1 * h_2 * h_3$, and arguing as in the beginning of the proof we obtain

$$\sum_{\pi} K_{\pi(h)}(x, y) = R(h_2) \times R(h_1^\vee)\sum_{\pi} K_{\pi(h_3)}(x, y). \tag{3B.5}$$

In the notation of Definition 3.3, Theorem 3.5 implies that for all compact subsets $\Omega \subset (A_G G(F)\backslash G(\mathbb{A}_F))^{\times 2}$, $x \in \Omega$, $r \in \mathbb{R}_{>0}$ and $p \in \mathbb{Z}$ one has

$$\Big|\sum_{\pi} K_{\pi(h)}(tx, tx)\Big| \ll_{h_1,h_2,\Omega,r,p} t^{\alpha p}\Big(\sum_{\pi} \operatorname{tr}\pi(h_3^* * h_3)\Big)^{1/2}$$

for all $t \in A_r^G$ and $\alpha \in \Delta$. Note that $\sum_\pi \operatorname{tr}\pi(h_3^* * h_3) < \infty$ since the restriction of the operator $R(h_3)$ to the cuspidal spectrum is of trace class (and hence Hilbert–Schmidt). This implies the desired rapid decrease of (3B.4) and hence the theorem. □

4. The geometric side

4A. ***Relative orbital integrals.*** Let H and G be connected algebraic F-groups with $H \leq G \times G$, where G is reductive, and H is the direct product of a reductive and a unipotent group. Let $\chi : H(\mathbb{A}_F) \to \mathbb{C}^\times$ be a quasi-character trivial on $A_{G,H}H(F)$.

Definition 4.1. An element $\gamma_v \in G(F_v)$ is *relevant* if χ_v is trivial on $H_{\gamma_v}(F_v)$. An element $\gamma \in G(F)$ is *relevant* if γ_v is relevant for all v.

The point of this definition is that irrelevant elements will not end up contributing to the trace formula. We note that if χ is trivial then all elements are relevant.

Definition 4.2. Let v be a place of F. For $f_v \in C_c^\infty(G(F_v))$ and $\gamma_v \in G(F_v)$ relevant, unimodular and closed we define the *local relative orbital integral*:

$$\mathrm{RO}_{\gamma_v}^{\chi_v}(f_v) = \int_{H_{\gamma_v}(F_v)\backslash H(F_v)} \chi_v(h_\ell, h_r) f_v(h_\ell^{-1}\gamma_v h_r)\, \dot{d}(h_\ell, h_r).$$

Remark. The assumption of unimodularity is used to define the right-invariant Radon measure on $H_{\gamma_v}(F_v)\backslash H(F_v)$.

Proposition 4.3. *If $\gamma_v \in G(F_v)$ is relevant, unimodular and closed then the integral $\mathrm{RO}_{\gamma_v}^{\chi_v}(f_v)$ is absolutely convergent.*

Proof. Since the measure $\dot{d}(h_\ell, h_r)$ is a Radon measure on $H_{\gamma_v}(F_v)\backslash H(F_v)$, to show the integral is well-defined and absolutely convergent it is enough to construct a pull-back map

$$C_c^\infty(G(F_v)) \to C_c^\infty(H_{\gamma_v}\backslash H(F_v)) \tag{4A.1}$$

attached to the natural map $H_{\gamma_v}\backslash H(F_v) \to G(F_v)$. But this map is a closed embedding (since the underlying map of schemes is a closed embedding) and is therefore proper. Thus the pull-back map in (4A.1) exists. □

4B. ***Global relative orbital integrals.***

Definition 4.4. For $f \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ and for relevant, unimodular and closed $\gamma \in G(F)$ we define the *global relative orbital integral*:

$$\mathrm{RO}_\gamma^\chi(f) = \int_{A_{G,H} H_\gamma(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \chi(h_\ell, h_r) f(h_\ell^{-1}\gamma h_r)\, \dot{d}(h_\ell, h_r).$$

Proposition 4.5. *If $\gamma \in G(F)$ is relevant unimodular closed then the integral defining $\mathrm{RO}_\gamma^\chi(f)$ converges absolutely.*

Proof. As in the proof of Proposition 4.3 it suffices to show that the map

$$H_\gamma\backslash H(\mathbb{A}_F) \to G(\mathbb{A}_F)$$

is proper, but this is obvious since it is a closed embedding. □

4C. ***The geometric side of the general simple relative trace formula.*** Let

$$F_\infty := \prod_{v|\infty} F_v$$

be the product of the archimedean completions of F. We note that $A \le H_\gamma(F_\infty)$ for all $\gamma \in G(F)$, and

$$\tau(H_\gamma) := \mathrm{meas}_{d(h_\ell,h_r)_\gamma}(AH_\gamma(F)\backslash H_\gamma(\mathbb{A}_F)) \tag{4C.1}$$

is finite if γ is elliptic. Let

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y). \tag{4C.2}$$

This kernel is equal to the earlier kernel of (3B.1) under the additional assumption that $R(f)$ has cuspidal image. With this in mind, combining Theorem 3.1 and the following theorem immediately implies Theorem 1.1:

Theorem 4.6. *Assume that if the $H(\mathbb{A}_F)$-orbit of $\gamma \in G(F)$ meets the support of f then γ is elliptic, unimodular and closed. Then*

$$\sum_{[\gamma] \in \Gamma(F)} \tau(H_\gamma)\,\mathrm{RO}_\gamma^\chi(f) = \int_{A_{G,H} H(F) \backslash H(\mathbb{A}_F)} \chi(h_\ell, h_r) K_f(h_\ell, h_r)\, d(h_\ell, h_r).$$

Moreover, the sum on the left and the integral on the right are absolutely convergent.

In the theorem we use the notation $[\gamma]$ for the class of γ; we will continue to use this convention. We will also denote by $[\gamma]^{\mathrm{geo}}$ the geometric class of γ. To prove Theorem 4.6, it is convenient to first prove the following proposition:

Proposition 4.7. *Let $C \subset G(\mathbb{A}_F)$ be a compact subset. Then, if H is reductive, there exist only finitely many closed classes $[\gamma] \in \Gamma(F)$ such that $H(\mathbb{A}_F) \cdot \gamma' \cap C \neq \varnothing$ for some $\gamma' \in [\gamma]$.* (Here the $\cdot$ refers to the action (2A.1).)

We will prove this in several steps.

Lemma 4.8. *Let $C \subset G(\mathbb{A}_F)$ be a compact subset. Then, if H is reductive, there exist only finitely many closed classes $[\gamma]^{\mathrm{geo}} \in \Gamma^{\mathrm{geo}}(F)$ such that $H(\mathbb{A}_F) \cdot \gamma' \cap C \neq \varnothing$ for some $\gamma' \in [\gamma]^{\mathrm{geo}}$.*

Proof. Since H is reductive there exists a categorical quotient X of G by the action (2A.1) of H; it is an affine scheme of finite type over F. Let

$$B : G \to X$$

be the canonical quotient map. Note that if $\gamma, \gamma' \in G(F)$ are closed then $B(\gamma) = B(\gamma')$ if and only if γ and γ' define the same element of $\Gamma^{\mathrm{geo}}(F)$. Moreover, assuming γ' is closed, if $H(\mathbb{A}_F) \cdot \gamma' \cap C \neq \varnothing$ then $B(C)$ contains the geometric class of γ'. On the other hand $B(C) \cap X(F)$ is finite because $B(C)$ is compact and $X(F) \subseteq X(\mathbb{A}_F)$ is discrete and closed. $\square$

We now show that for each closed γ there are only finitely many classes in $[\gamma]^{\mathrm{geo}}$ that intersect C. To do this it is convenient to review some Galois cohomology.

Let S_0 be a finite set of places of F including the infinite places. For a smooth linear algebraic group L over $\mathcal{O}_F^{S_0}$ let $H^1(\mathbb{A}_F, L)$ denote the adelic cohomology of L:

$$H^1(\mathbb{A}_F, L) := \Big\{(\sigma_v) \in \prod_v H^1(F_v, L) : \sigma_v \in H^1_{\mathrm{nr}}(F_v, L) \text{ for a.e. } v \notin S_0\Big\}.$$

Here

$$H^1_{\mathrm{nr}}(F_v, L) := \mathrm{Im}\big(H^1(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), L(\mathbb{O}^{\mathrm{nr}}_{F_v})) \to H^1(F_v, L)\big),$$

where F_v^{nr} is the maximal unramified extension of F_v and $\mathbb{O}^{\mathrm{nr}}_{F_v}$ is its ring of integers. We endow $H^1(F_v, L)$ with the discrete topology for all v and endow $H^1(\mathbb{A}_F, L)$ with the restricted direct product topology with respect to the subgroups $H^1_{\mathrm{nr}}(F_v, L)$ for $v \notin S_0$ (again given the discrete topology).

Lemma 4.9. *The image of the diagonal map $H^1(F, L) \to \prod_v H^1(F_v, L)$ lies in $H^1(\mathbb{A}_F, L)$ and the induced map*

$$H^1(F, L) \to H^1(\mathbb{A}_F, L)$$

is proper if we give $H^1(F, L)$ the discrete topology.

Let $S \supseteq S_0$ be a finite set of places of F. It is convenient to say that an element $\sigma = (\sigma_v) \in H^1(\mathbb{A}_F, L)$ is *unramified outside of S* if $\sigma_v \in H^1_{\mathrm{nr}}(F_v, L)$ for all $v \notin S$ and that $\sigma \in H^1(F, L)$ is *unramified outside of S* if σ maps to an element of $H^1(\mathbb{A}_F, L)$ unramified outside of S under the diagonal map (i.e., the map of Lemma 4.9).

Proof. It is not hard to see that $H^1(F, L)$ has image in $H^1(\mathbb{A}_F, L)$. We now prove the properness statement. For this we follow the proof of [Harari and Skorobogatov 2002, Proposition 4.4]. Since $H^1(F_v, L)$ is finite for all v it is enough to show that for all sufficiently large $S \supseteq S_0$, the inverse image of $\prod_{v \notin S} H^1_{\mathrm{nr}}(F_v, L)$ in $H^1(F, L)$ is finite, in other words, there are only finitely many classes in $H^1(F, L)$ unramified outside of S. We denote by L° the schematic closure in L of the neutral component of L_F. By enlarging S if necessary we can assume that L, L°, $\pi_0(L) := L/L^\circ$ and $\mathrm{Aut}(\pi_0(L))$ are all smooth over $\mathbb{O}_F^S$ and that the sequence

$$1 \longrightarrow L^\circ \longrightarrow L \longrightarrow \pi_0(L) \longrightarrow 1$$

is exact, which in turn yields a cartesian diagram

$$\begin{tikzcd}
H^1(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), L^\circ(\mathbb{O}^{\mathrm{nr}}_{F_v})) \arrow[r] \arrow[d] & H^1(F_v, L^\circ) \arrow[d] \\
H^1(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), L(\mathbb{O}^{\mathrm{nr}}_{F_v})) \arrow[r] \arrow[d, "\alpha"] & H^1(F_v, L) \arrow[d] \\
H^1(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), \pi_0(L)(\mathbb{O}^{\mathrm{nr}}_{F_v})) \arrow[r, "\beta"'] & H^1(F_v, \pi_0(L))
\end{tikzcd} \tag{4C.3}$$

with exact columns for all $v \notin S$. All of the maps are the natural ones; we have just labeled two of them α and β. We now use this diagram to prove that the map

$$H^1_{\mathrm{nr}}(F_v, L) \to H^1_{\mathrm{nr}}(F_v, \pi_0(L)) \tag{4C.4}$$

is injective.

We first claim that $H^1(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), L^\circ(\mathbb{O}_{F_v}^{\mathrm{nr}}))$ is trivial for all $v \notin S$. Indeed, let X be an $L^\circ_{\mathbb{O}_{F_v}}$-torsor representing an element. Then, denoting by ϖ_v a uniformizer for $\mathbb{O}_{F_v}$ one has

$$X(\mathbb{O}_{F_v}/\varpi_v) \neq \varnothing$$

by Lang's theorem [Serre 2002, §III.2.3]. Since X is smooth, Hensel's lemma implies that the map $X(\mathbb{O}_{F_v}) \to X(\mathbb{O}_{F_v}/\varpi_v)$ is surjective. In particular $X(\mathbb{O}_{F_v}) \neq \varnothing$, proving our claim. This implies that the map α in (4C.3) is injective.

We now claim that the map

$$\beta : H^1(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), \pi_0(L)(\mathbb{O}_{F_v}^{\mathrm{nr}})) \longrightarrow H^1(F_v, \pi_0(L)(F_v)), \tag{4C.5}$$

of (4C.3) is injective. Assuming this, it follows that (4C.4) is injective as asserted above. To prove that β is injective, let X_1, X_2 be two $\pi_0(L)_{\mathbb{O}_{F_v}}$-torsors isomorphic over $\mathbb{O}_{F_v}^{\mathrm{nr}}$ such that $X_{1F_v} \cong X_{2F_v}$, which is to say that the classes of these torsors map to the same element of $H^1(F_v, \pi_0(L)(F_v))$ under β. The $\mathbb{O}_{F_v}^{\mathrm{nr}}$-isomorphisms between $X_{1\mathbb{O}_{F_v}^{\mathrm{nr}}}$ and $X_{2\mathbb{O}_{F_v}^{\mathrm{nr}}}$ form an $\mathrm{Aut}(\pi_0(L))_{\mathbb{O}_{F_v}}$-torsor Y such that $Y(F_v) \neq \varnothing$ (since $X_{1F_v} \cong X_{2F_v}$), and $Y(\mathbb{O}_{F_v}) \neq \varnothing$ if and only if $X_1 \cong X_2$ (over $\mathbb{O}_{F_v}$), i.e., if and only if X_1 and X_2 represent the same class in $H^1(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), \pi_0(L)(\mathbb{O}_{F_v}^{\mathrm{nr}}))$. But $\mathrm{Aut}(\pi_0(L))$ is proper over $\mathbb{O}_{F_v}$ (even finite), and hence so is Y. By the valuative criterion of properness, $Y(F_v) \neq \varnothing$ implies $Y(\mathbb{O}_{F_v}) \neq \varnothing$, implying that $X_1 \cong X_2$ (over $\mathbb{O}_{F_v}$). As already remarked, this completes our proof that (4C.4) is injective as asserted above.

Suppose that $\sigma \in H^1(F, L)$ is unramified outside of S. Then the image of σ in

$$\mathrm{Im}\big(H^1(F, \pi_0(L)) \longrightarrow H^1(\mathbb{A}_F, \pi_0(L))\big),$$

say ξ, is also unramified outside of S. The cocycle ξ is attached to the spectrum of an étale F-algebra (i.e., direct sum of finite extension fields) of degree at most $\pi_0(L)(\overline{F})$ that is unramified outside of S. There are only finitely many such étale F-algebras, so to complete the proof of the lemma it suffices to fix a cocycle ξ and show that there are only finitely many $\sigma \in H^1(F, L)$ unramified outside of S that map to it. For this, we combine the fact that $H^1(F_v, L)$ is finite for all v and the injection (4C.4) to conclude that there are only finitely many elements of $H^1(\mathbb{A}_F, L)$ unramified outside of S that map to ξ. We now employ the Borel–Serre theorem [Serre 2002, §III.4.6], which states that the fibers of the diagonal map

$H^1(F, L) \to \prod_v H^1(F_v, L)$ are finite, to deduce that there are only finitely many $\sigma \in H^1(F, L)$ mapping to ξ that are unramified outside of S. □

Now assume that $L \leq M$ are smooth linear algebraic groups over $\mathbb{O}_F^S$ such that M has connected fibers. Then the map $M \to L\backslash M$ is smooth and surjective. We obtain a characteristic map

Lemma 4.10. *The characteristic map* cl *maps compact sets to compact sets.*

Remark. We do not know whether cl is continuous.

Proof. Any cocycle $\sigma \in \mathrm{cl}(L\backslash M(F_v)) \subseteq H^1(F_v, L)$ gives rise to forms ${}_\sigma L, {}_\sigma M$ of L_{F_v} and M_{F_v} equipped with a map

$$ {}_\sigma L(F_v) \setminus {}_\sigma M(F_v) \longrightarrow L\backslash M(F_v) \tag{4C.6}$$

with the property that the inverse image of σ under cl is the image of (4C.6) (compare [Serre 2002, §I.5.4, Corollary 2]). Moreover, ${}_\sigma M(F_v) \to L\backslash M(F_v)$ is open (see above the proof of [Conrad 2012, Theorem 4.5]). Thus the maps $\mathrm{cl} : L\backslash M(F_v) \to H^1(F_v, L)$ are continuous for each v if we give $H^1(F_v, L)$ the discrete topology.

The map $M(\mathbb{O}_{F_v}^{\mathrm{nr}}) \to L\backslash M(\mathbb{O}_{F_v}^{\mathrm{nr}})$ is surjective by Hensel's lemma, and it follows that $\mathrm{cl}(L\backslash M(\mathbb{O}_{F_v})) \subseteq H^1_{\mathrm{nr}}(F_v, L)$, which completes the proof of the lemma. □

Proof of Proposition 4.7. For a large enough set S_0 of places of F including the infinite places we can and do choose models of $H_\gamma \leq H$ over $\mathbb{O}_F^{S_0}$ that are smooth linear algebraic groups. We use the same letters to denote these models and use the models to define adelic cohomology as above.

In view of Lemma 4.8 it suffices to check that for a given closed $\gamma \in G(F)$ there are finitely many γ' in the geometric class of γ such that $H(\mathbb{A}_F) \cdot \gamma' \cap C \neq \varnothing$.

One has a commutative diagram with exact rows

$$\begin{tikzcd}
H_\gamma(F) \arrow[r] \arrow[d] & H(F) \arrow[r] \arrow[d] & H_\gamma\backslash H(F) \arrow[r, "\mathrm{cl}"] \arrow[d] & H^1(F, H_\gamma) \arrow[d, "a"'] \\
H_\gamma(\mathbb{A}_F) \arrow[r] & H(\mathbb{A}_F) \arrow[r] & H_\gamma\backslash H(\mathbb{A}_F) \arrow[r, "\mathrm{cl}"] & H^1(\mathbb{A}_F, H_\gamma)
\end{tikzcd}$$

and the image of the map cl on the upper line can be identified with the set of classes in the geometric class of γ. We give the first three sets on the bottom row their natural topologies and give $H^1(\mathbb{A}_F, H_\gamma)$ the topology described above Lemma 4.9.

Identifying $H_\gamma\backslash H(\mathbb{A}_F)$ with a subset of $G(\mathbb{A}_F)$ via the action of $H(\mathbb{A}_F)$ on γ, the set of γ' in the geometric class of γ such that $H(\mathbb{A}_F) \cdot \gamma' \cap C \neq \varnothing$ injects into the subset of $\mathrm{cl}(H_\gamma\backslash H(F))$ mapping to

$$\mathrm{cl}\big(C\cap H_\gamma\backslash H(\mathbb{A}_F)\big) \tag{4C.7}$$

under a. Since a is proper by Lemma 4.9, it suffices to show (4C.7) is compact. Since $C\cap H_\gamma\backslash H(\mathbb{A}_F)$ is compact by the fact γ is closed, the compactness of (4C.7) follows from Lemma 4.10. □

Remark. One can prove Proposition 4.7 in a simpler manner as follows. Let $C\subset G(\mathbb{A}_F)$ be a compact set. Observe that the $\gamma'\in G(F)$ in the geometric class of a given closed $\gamma\in G(F)$ such that $H(\mathbb{A}_F)\cdot\gamma'\cap C\neq\varnothing$ are in the intersection of C and the image of the topological embeddings

$$H_\gamma\backslash H(F)\longrightarrow H_\gamma\backslash H(\mathbb{A}_F)\longrightarrow G(\mathbb{A}_F).$$

Since $H_\gamma\backslash H(\mathbb{A}_F)\cap C$ is compact and $H_\gamma\backslash H(F)$ is discrete and closed in $H_\gamma\backslash H(\mathbb{A}_F)$, we can deduce Proposition 4.7 from Lemma 4.8. However, the more refined information presented in the discussion above ought to be useful as a starting point towards future work on the stabilization of the relative trace formula.

Proof of Theorem 4.6. Proceeding formally for the moment, we have

$$\begin{aligned}\sum_{\substack{[\gamma]\in\Gamma(F)\\ \gamma\text{ relevant}}}&\tau(H_\gamma)\,\mathrm{RO}^\chi_\gamma(f)\\ &=\sum_{\substack{[\gamma]\in\Gamma(F)\\ \gamma\text{ relevant}}}\tau(H_\gamma)\int_{(A\backslash A_{G,H})H_\gamma(\mathbb{A}_F)\backslash H(\mathbb{A}_F)}\chi(h_\ell,h_r)f(h_\ell^{-1}\gamma h_r)\,\dot{d}(h_\ell,h_r).\end{aligned} \tag{4C.8}$$

Notice that

$$\int_{A_{G,H}H_\gamma(F)\backslash H(\mathbb{A}_F)}\chi(h_\ell,h_r)f(h_\ell^{-1}\gamma h_r)\,d(h_\ell,h_r)=0$$

if γ is not relevant, because in this case

$$\int_{AH_\gamma(F)\backslash H_\gamma(\mathbb{A}_F)}\chi(h_\ell,h_r)\,d(h_\ell,h_r)_\gamma=0.$$

Thus (4C.8) is equal to

$$\begin{aligned}\sum_{[\gamma]\in\Gamma(F)}&\int_{A_{G,H}H_\gamma(F)\backslash H(\mathbb{A}_F)}\chi(h_\ell,h_r)f(h_\ell^{-1}\gamma h_r)\,d(h_\ell,h_r)\\ &=\int_{A_{G,H}H(F)\backslash H(\mathbb{A}_F)}\chi(h_\ell,h_r^{-1})\sum_{\gamma\in G(F)}f(h_\ell^{-1}\gamma h_r)\,d(h_\ell,h_r)\\ &=\int_{A_{G,H}H(F)\backslash H(\mathbb{A}_F)}\chi(h_\ell,h_r)K_f(h_\ell,h_r)\,d(h_\ell,h_r).\end{aligned}$$

We now justify these formal manipulations. By dominated convergence, it suffices to consider the case where $\chi = |\chi|$ and f is nonnegative; we henceforth assume this. Suppose that $\gamma \in G(F)$ is relevant, unimodular and closed. Then by Proposition 4.5 one has

$$|\,\mathrm{RO}_\gamma^\chi(f)| < \infty.$$

If γ is unimodular, closed and elliptic we have

$$|\tau(H_\gamma)| < \infty.$$

If H is also reductive then the sum over γ in (4C.8) is finite by Proposition 4.7 so in this case our formal manipulations are justified.

In the general case, write

$$H = M_H \times N_H$$

where M_H (resp. N_H) is reductive (resp. unipotent).

Decompose the measure $d(h_\ell, h_r)$ on $A_{G,H}H(F)\backslash H(\mathbb{A}_F)$ as the product of a measure $d(m_\ell, m_r)$ on $A_{G,H}M_H(F)\backslash M_H(\mathbb{A}_F)$, induced by a Haar measure on $A_{G,H}\backslash M_H(\mathbb{A}_F)$, with a measure $d(n_\ell, n_r)$ on $N_H(F)\backslash N_H(\mathbb{A}_F)$ induced by a Haar measure on $N_H(\mathbb{A}_F)$. Since $N_H(F)\backslash N_H(\mathbb{A}_F)$ is compact, we can choose a compact subset $\Omega \subseteq N(\mathbb{A}_F)$ such that

$$\begin{aligned}
&\int_{A_{G,H}H(F)\backslash H(\mathbb{A}_F)} |\chi|(h_\ell, h_r)K_f(h_\ell, h_r)d(h_\ell, h_r)\\
&\quad= \int_{A_{G,H}M_H(F)\backslash M_H(\mathbb{A}_F)\times\Omega} |\chi|(m_\ell n_\ell, m_r n_r)K_f(m_\ell n_\ell, m_r n_r)d(m_\ell, m_r)d(n_\ell, n_r)\\
&\quad= \int_{A_{G,H}M_H(F)\backslash M_H(\mathbb{A}_F)} |\chi|(m_\ell, m_r)K_{\tilde{f}}(m_\ell, m_r)d(m_\ell, m_r)
\end{aligned}$$

where

$$\tilde{f}(x) := \int_\Omega |\chi|(n_\ell, n_r)f(n_\ell^{-1}xn_r)d(n_\ell, n_r) \in C_c^\infty(A\backslash G(\mathbb{A}_F)).$$

This allows us to reduce to the reductive case with which we have already dealt. □

5. A relative Weyl law

Let G be a split adjoint semisimple group over $\mathbb{Q}$. Note that $G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})$ is of finite volume but noncompact. We also let G denote the Chevalley group over $\mathbb{Z}$ whose generic fiber is G. Fix a maximal compact subgroup $K_\infty \le G(\mathbb{R})$ and a compact open subgroup $K^\infty \le G(\mathbb{A}_\mathbb{Q}^\infty)$ and let

$$K := K_\infty \times K^\infty.$$

We assume that $K^S = G(\widehat{\mathbb{Z}}^S)$ for any sufficiently large finite set of places S of $\mathbb{Q}$ containing infinity. For our later use we fix a maximal split torus $T \le G$ and assume that the Cartan involution fixing K_∞ acts as inversion on the identity component $T(\mathbb{R})^+$ of $T(\mathbb{R})$ in the real topology. We impose the following torsion-freeness assumption:

(TF) For all $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ the group $g^{-1}K^{\infty}g \cap G(\mathbb{Q})$ is torsion-free.

This can always be arranged by taking K^∞ to be contained in a sufficiently small principal congruence subgroup.

To deduce the relative Weyl law of Theorem 1.2, we investigate the following special case of the setting of the previous sections of the paper:

Let $H_0 \le G$ be a subgroup that is a direct product of a reductive group and a unipotent group and let $H \le G \times G$ be the image of the diagonal embedding $H_0 \hookrightarrow G \times G$. We point out that though $H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_{\mathbb{Q}})$ is compact, we make no such assumption on $G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})$, so Theorem 1.2 does not follow in any obvious way from the usual Weyl law and its local variants. Moreover, we will also show in Proposition 5.2 how the same asymptotic would follow for noncompact $H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_{\mathbb{Q}})$ of finite volume provided that we knew the upper bound of the relative Weyl law (in the setting of the usual Weyl law this was proven in [Donnelly 1982]).

We restate Theorem 1.2 for convenience:

Theorem 5.1. *Assume that* $[H_0] := H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_{\mathbb{Q}})$ *is compact. As* $X \to \infty$ *one has*

$$\sum_{\pi:\,\pi(\Delta)\le X}\ \sum_{\varphi\in\mathcal{B}(\pi)^K} \int_{[H_0]} |\varphi(h)|^2 dh \sim \alpha(G)\,\mathrm{meas}_{dh}([H_0])X^{d/2}, \tag{5.1}$$

where the sum is over isomorphism classes of cuspidal automorphic representations π *of* $G(\mathbb{A}_{\mathbb{Q}})$, $\mathcal{B}(\pi)$ *is an orthonormal basis of the* π*-isotypic subspace of* $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}}))$, $\pi(\Delta)$ *is the eigenvalue of the Casimir operator* Δ *acting on the space of* K_∞*-fixed vectors in* π, *and* $d = \dim(G(\mathbb{R})/K_\infty)$.

Here $\alpha(G) > 0$ is the same constant appearing in [LV], and the Casimir operator and the Haar measure on $G(\mathbb{R})$ are normalized as in [LV]. The Haar measure on $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is normalized to give K^∞ volume 1.

The proof of Theorem 5.1 follows from the observation that if we replace the diagonal embedding $G \hookrightarrow G \times G$ considered in Lindenstrauss and Venkatesh's work [LV] by the diagonal embedding $H_0 \hookrightarrow G \times G$, the argument of [LV] can be followed line by line to deduce the result. In particular, one can use the same test functions that were constructed in that reference. We will give a few more details but will be quite brief.

With a view towards future generalizations, until otherwise stated we merely assume that $H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_\mathbb{Q})$ has finite volume (which is not implied by the fact that $G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})$ has finite volume).

Arguing exactly as in [LV] one proves the following theorem:

Proposition 5.2. *Let* $[H_0] := H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_\mathbb{Q})$ *be of finite volume* (*not necessarily compact*) *and let* $0 < \varepsilon < 1$. *If we assume the upper bound of the relative Weyl law*, *namely*, *if*

$$\sum_{\pi:\pi(\Delta)\leq X}\ \sum_{\varphi\in\mathcal{B}(\pi)^K}\int_{[H_0]}|\varphi(h)|^2\,dh \leq (\alpha(G)+O(\varepsilon))\,\mathrm{meas}_{dh}([H_0])X^{d/2}$$

for $X\to\infty$, *then* (5.1) *follows.* □

In [LV], the upper bound of Proposition 5.2 follows from work of Donnelly [1982]. Interestingly, the corresponding relative analogue is not known. However, in case where $H_0(F)\backslash H_0(\mathbb{A}_F)$ is compact one can establish the following result using standard techniques:

Proposition 5.3. *Suppose that* $[H_0] := H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_\mathbb{Q})$ *is compact and that* $0 < \varepsilon < 1$. *With notation as in Theorem 5.1*, *for* $X\in\mathbb{R}_{>0}$ *one has the upper bound*:

$$\sum_{\pi:\pi(\Delta)\leq X}\ \sum_{\varphi\in\mathcal{B}(\pi)^K}\int_{[H_0]}|\varphi(h)|^2\,dh \leq (\alpha(G)+O(\varepsilon))\,\mathrm{meas}_{dh}([H_0])X^{d/2}.$$

Proof. One can mimic the argument in [LV; §5]. There are only two minor differences between the argument there and the argument proving the proposition above. First, in [LV; Lemma 2(4)] one replaces $1-\varepsilon$ with $1+\varepsilon$, since we are interested in upper bounds. Second, one has to include Eisenstein series in the expansion of the spectral kernel. However, unlike in the usual trace formula, their contribution is absolutely convergent in the setting above because we have assumed $H_0(\mathbb{Q})\backslash H_0(\mathbb{A}_\mathbb{Q})$ is compact. This contribution is also positive by the choice of test function in [LV]. □

Combining Proposition 5.3 and Proposition 5.2 yields Theorem 5.1.

Acknowledgements

The authors thank M. Stern for answering questions on the Weyl law in the context of differential geometry. We also thank the referee for remarks that improved the exposition.

References

[Ash et al. 1993] A. Ash, D. Ginzburg, and S. Rallis, "Vanishing periods of cusp forms over modular symbols", *Math. Ann.* **296**:4 (1993), 709–723. MR 94f:11044 Zbl 0786.11028

[Bernstein et al. 1984] J. Bernstein, P. Deligne, D. Kazhdan, and M.-F. Vignéras, *Représentations des groupes réductifs sur un corps local*, Travaux en Cours **8**, Hermann, Paris, 1984. MR 85h:22001

[Conrad 2012] B. Conrad, "Weil and Grothendieck approaches to adelic points", *Enseign. Math.* (2) **58**:1-2 (2012), 61–97. MR 2985010 Zbl 06187657

[Dixmier and Malliavin 1978] J. Dixmier and P. Malliavin, "Factorisations de fonctions et de vecteurs indéfiniment différentiables", *Bull. Sci. Math.* (2) **102**:4 (1978), 307–330. MR 80f:22005 Zbl 0392.43013

[Donnelly 1982] H. Donnelly, "On the cuspidal spectrum for finite volume symmetric spaces", *J. Differential Geom.* **17**:2 (1982), 239–253. MR 83m:58079 Zbl 0494.58029

[Getz 2014] J. R. Getz, "Automorphic kernel functions in four variables", preprint, 2014. arXiv 1409.2360

[Getz and Goresky 2012] J. R. Getz and M. Goresky, *Hilbert modular forms with coefficients in intersection homology and quadratic base change*, Progress in Mathematics **298**, Birkhäuser/Springer Basel AG, Basel, 2012. MR 2918131 Zbl 1285.11073

[Godement 1966] R. Godement, "The spectral decomposition of cusp-forms", pp. 225–234 in *Algebraic Groups and Discontinuous Subgroups* (Boulder, CO, 1965), edited by A. Borel and G. D. Mostow, Proc. Sympos. Pure Math. **9**, Amer. Math. Soc., Providence, R.I., 1966. MR 35 #1713 Zbl 0172.18503

[Hahn 2009] H. Hahn, "A simple twisted relative trace formula", *Int. Math. Res. Not.* **2009**:21 (2009), 3957–3978. MR 2010k:22024 Zbl 1210.22013

[Harari and Skorobogatov 2002] D. Harari and A. N. Skorobogatov, "Non-abelian cohomology and rational points", *Compositio Math.* **130**:3 (2002), 241–273. MR 2003b:11056 Zbl 1019.14012

[Jacquet and Ye 1996] H. Jacquet and Y. Ye, "Distinguished representations and quadratic base change for GL(3)", *Trans. Amer. Math. Soc.* **348**:3 (1996), 913–939. MR 96h:11041 Zbl 0861.11033

[Lindenstrauss and Venkatesh 2007] E. Lindenstrauss and A. Venkatesh, "Existence and Weyl's law for spherical cusp forms", *Geom. Funct. Anal.* **17**:1 (2007), 220–251. MR 2008c:22016 Zbl 1137.22011

[Rogawski 1983] J. D. Rogawski, "Representations of $\mathrm{GL}(n)$ and division algebras over a p-adic field", *Duke Math. J.* **50**:1 (1983), 161–196. MR 84j:12018 Zbl 0523.22015

[Serre 2002] J.-P. Serre, *Galois cohomology*, Springer, Berlin, 2002. MR 2002i:12004 Zbl 1004.12003

Received October 30, 2014. Revised February 17, 2015.

JAYCE R. GETZ
DEPARTMENT OF MATHEMATICS
DUKE UNIVERSITY
DURHAM, NC 27708-0320
UNITED STATES

jgetz@math.duke.edu

HEEKYOUNG HAHN
DEPARTMENT OF MATHEMATICS
DUKE UNIVERSITY
DURHAM, NC 27708-0320
UNITED STATES

hahn@math.duke.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.119

CHERN–SIMONS FUNCTIONS ON TORIC CALABI–YAU THREEFOLDS AND DONALDSON–THOMAS THEORY

ZHENG HUA

We use the notion of strong exceptional collections to give a construction of the global Chern–Simons functions for toric Calabi–Yau stacks of dimension three. Moduli spaces of sheaves on such stacks can be identified with critical loci of these functions. We give two applications of these functions. First, we prove Joyce's integrality conjecture of generalized DT invariants on local surfaces. Second, we prove a dimension reduction formula for virtual motives, which leads to a recursion formula for motivic Donaldson–Thomas invariants.

1. Introduction

Moduli spaces of sheaves (more generally, complexes of sheaves) on Calabi–Yau threefolds are examples of moduli problems with symmetric obstruction theories [Behrend 2009]. It is expected that such a moduli space is locally the critical set of a holomorphic function. Such functions are called Chern–Simons (CS) functions. Chern–Simons functions play an important role in Calabi–Yau (CY) geometry because Behrend proved that the Milnor number of a CS function is the microlocal version of the Donaldson–Thomas invariant [loc. cit.].

In a seminal work, Joyce and Song [2012] proved the existence of CS functions for moduli spaces of stable sheaves on compact CY 3-folds using analytic techniques in gauge theory. In this paper, we give a different construction of the CS functions on toric CY 3-folds. Our construction has a few new ingredients. First, the functions we construct are algebraic. Second, the moduli spaces of stable sheaves are, in fact, globally critical sets of these functions. Third, the construction is explicit; i.e., there is an algorithm to write down such functions starting with a toric CY 3-fold together with some extra data; see the end of Section 5.

The construction of CS function consists of three steps:

(1) Let Y be a complex CY 3-fold. Find a good t-structure in the derived category $\mathrm{D}^b(Y)$. The heart of this t-structure is the abelian category of representations of

MSC2010: 14F05, 14N35.
Keywords: algebraic geometry, derived category, Donaldson–Thomas theory.

a quiver with relations. Such an abelian category is good in the sense that it has enough projective modules and has finite projective dimension.

(2) On a moduli space of representations with fixed dimension vector, we find a *maximally degenerate* point, which corresponds to the semisimple representation. The tangent complex of the moduli space at this point is given by the well studied L_∞ (A_∞) Yoneda algebra in representation theory. We compute the L_∞ (A_∞) products and prove they are bounded. The Calabi–Yau condition defines a cyclic pairing on this L_∞ algebra, which together with the L_∞ products determines the CS function.

(3) Embed the moduli spaces of sheaves into the moduli spaces of representations as open substacks.

Step one is based on the existence of full, strong, exceptional collections of line bundles on toric Fano stacks of dimension two; see Theorem 3.3. This was proved in [Borisov and Hua 2009]. Passing from a strong exceptional collection to the associated quiver is a consequence of derived Morita equivalence. We will study this in Section 3.

Step two is based on the cyclic completion (see Theorem 4.2) and boundedness of L_∞ products (see Theorem 4.4). Theorem 4.2 was first proved by Aspinwall and Fidkowski [2006] and later reproved in a much more general setting by Segal [2008]. The terminology *cyclic completion* is due to Segal. The proofs of these two theorems are given in Section 4 just for our convenience.

In Section 5, we construct the CS functions and show that the moduli spaces of sheaves are open substacks of the critical sets modulo gauge groups. Several examples of CS functions are discussed in Section 6.

The language of L_∞ algebras and derived schemes (stacks) — developed in [Kontsevich and Soibelman 2009] — is extensively used in the paper. Each of the moduli spaces mentioned above is the zero locus of an odd vector field on a differential graded (dg) symplectic manifold and the CS functions we construct are essentially Hamiltonian functions associated to it. In Section 2, we give a short introduction to L_∞ algebras and dg schemes.

In the last three sections, we give two applications of the CS function. In Theorem 7.4, we prove that the L_∞ products vanish at semistable points of moduli space of sheaves on local surfaces, which leads to a proof of a special case of the integrality conjecture of Joyce and Song [2012]. In Theorem 8.3, we prove a dimension reduction formula of virtual motives for CS functions, which generalizes some results in [Behrend et al. 2013]. By manipulating this dimension reduction formula, we compute the generating series of moduli spaces of noncommutative Hilbert schemes on toric CY stacks; this is done in Section 9.

Notation. Three dimensional smooth toric Calabi–Yau stacks are in one to one correspondence with the set of 3-dimensional cones over convex lattice polygons Δ contained in an affine hyperplane, together with a triangulation of Δ. When the polygon Δ has at least one interior lattice point, we can consider the barycentric triangulation. (This means the triangulation has only one interior lattice point.) This gives a fan Σ on the affine hyperplane such that its supporting polygon is Δ. The fan Σ determines a 2-dimensional toric Fano stack X_Σ (X, for short). The cone over Σ determines a 3-dimensional toric CY stack Y_Σ (Y, for short), which is the total space of the canonical bundle over X_Σ. We call such a toric CY 3-stack a *local surface*. The CY 3-stacks associated to other triangulations of Δ are related to Y_Σ by a sequence of flops.

- $\pi : Y \to X$ is the projection and $\iota : X \to Y$ is the inclusion of zero section;
- $\mathrm{D}^b(X)$ is the bounded derived category of coherent sheaves on X;
- $\mathrm{D}^b(Y)$ is the bounded derived category of coherent sheaves on Y;
- D_ω is the full subcategory of $\mathrm{D}^b(Y)$ of objects with cohomology sheaves supported on X.

2. L_∞ algebras and differential graded schemes

This is a short introduction to L_∞ algebras and differential graded schemes. A standard reference for this topic is [Kontsevich and Soibelman 2009]. The reader who is familiar with ∞-algebras can skip this section.

2A. L_∞ *algebras.* Let $\boldsymbol{k}$ be a field.

Definition 2.1. An L_∞ *algebra* is a graded $\boldsymbol{k}$-vector space L with a sequence $\mu_1, \ldots, \mu_k, \ldots$ of graded antisymmetric operations of degree 2, or equivalently, homogeneous multilinear maps

$$\mu_k : \textstyle\bigwedge^k L \to L[2-k]$$

such that for each $n > 0$, the n-Jacobi rule holds:

$$\sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1<\cdots<i_k;\, j_1<\cdots<j_{n-k} \\ \{i_1,\ldots,i_k\}\cup\{j_1,\ldots,j_{n-k}\}=\{1,\ldots,n\}}} (-1)^\epsilon\, \mu_n(\mu_k(x_{i_1}, \ldots, x_{i_k}), x_{j_1}, \ldots, x_{j_{n-k}}) = 0.$$

Here, the sign $(-1)^\epsilon$ equals the product of the sign $(-1)^\pi$ associated to the permutation

$$\pi = \left(\begin{smallmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ i_1 & \cdots & i_k & j_1 & \cdots & j_{n-k} \end{smallmatrix}\right)$$

with the sign associated by the Koszul sign convention to the action of π on the elements $(x_1, \ldots, x_n)$ of L.

Definition 2.2. Let (L, μ_k) be an L_∞ algebra. An element $x \in L^1$ is called a *Maurer–Cartan element* if x satisfies the formal *Maurer–Cartan equation*:

$$\sum_{k=1}^{\infty} \frac{1}{k!}\mu_k(x, \dots, x) = 0.$$

If the above formal sum is convergent, then there is a map $Q : L^1 \to L^2$, defined by

$$x \mapsto \sum_{k=1}^{\infty} \frac{1}{k!}\mu_k(x, \dots, x).$$

called the *curvature map*. The set of elements in L^1 satisfying the Maurer–Cartan equation is denoted by $\mathrm{MC}(L)$.

Definition 2.3. Let L be an L_∞ algebra. We write δ for the first L_∞ product $\mu_1 : L \to L[1]$. It follows from the L_∞ relations that $\delta^2 = 0$. Let x be a Maurer–Cartan element of L. We define the *twisted* differential δ^x by the formula

$$\delta^x(y) = \delta(y) + \sum_{k=2}^{\infty} \frac{1}{(k-1)!}\mu_k(x, \dots, x, y).$$

By manipulating the Maurer–Cartan equation and the L_∞ relations, one can check that $(\delta^x)^2 = 0$.

Given a homogeneous element $a \in L$, we denote its grading by $|a|$.

Definition 2.4. A finite dimensional L_∞ algebra (L, μ_k) is called *cyclic* if there exists a homogeneous bilinear map

$$\kappa : L \otimes L \longrightarrow \boldsymbol{k}[-3]$$

satisfies:

(1) $\kappa(a, b) = (-1)^{|a||b|}\kappa(b, a)$;

(2) $\kappa(\mu_k(a_1, \dots, a_k), a_{k+1}) = (-1)^{|a_1|(|a_2|+\dots+|a_{k+1}|)}\kappa(\mu_k(a_2, \dots, a_{k+1}), a_1)$;

(3) κ is nondegenerate on $H^\bullet(L, \delta)$.

We call such a κ a *cyclic pairing* on L.

Definition 2.5. Let (L, μ_k, κ) be a cyclic L_∞ algebra. The *Chern–Simons function* associated to L is the formal function

$$f(z) = \sum_{k=1}^{\infty} \frac{(-1)^{\frac{k(k+1)}{2}}}{(k+1)!}\kappa(\mu_k(z, \dots, z), z).$$

2B. *Differential graded schemes.*

Definition 2.6. A *differential graded scheme* X is a pair $(X^0, \mathcal{O}_X^\bullet)$, where X^0 is an ordinary scheme and $\mathcal{O}_X^\bullet$ is a sheaf of $\mathbb{Z}^-$-graded commutative dg algebras on X^0 such that:

(1) $\mathcal{O}_X^0 = \mathcal{O}_{X^0}$;

(2) $\mathcal{O}_X^i$ are quasicoherent $\mathcal{O}_{X^0}$ modules.

The cohomology sheaves of $\mathcal{O}_X^\bullet$, denoted by $\underline{H}^i(\mathcal{O}_X^\bullet)$ are $\mathcal{O}_{X^0}$ modules. In particular, $\underline{H}^0(\mathcal{O}_X^\bullet)$ is a quotient ring of $\mathcal{O}_X^0 = \mathcal{O}_{X^0}$. We define the *"0-truncation"* of X to be the ordinary scheme

$$\pi_0(X) = \operatorname{Spec} \underline{H}^0(\mathcal{O}_X^\bullet).$$

It is a subscheme of X^0.

Definition 2.7. A *morphism of dg schemes* $f : X \to Y$ is a morphism of ordinary schemes $f_0 : X^0 \to Y^0$ together with a morphism of dg algebras $f_0^* \mathcal{O}_Y^\bullet \to \mathcal{O}_X^\bullet$. A morphism f is called a *quasi-isomorphism* if f induces isomorphisms between $\underline{H}^i(\mathcal{O}_X^\bullet)$ and $\underline{H}^i(\mathcal{O}_Y^\bullet)$ for all i.

Definition 2.8. A dg scheme X is called *smooth* (or a *dg manifold*) if the following conditions hold:

(a) X^0 is a smooth algebraic variety.

(b) Locally over the Zariski topology on X^0, we have an isomorphism of graded algebras $\mathcal{O}_X^\bullet \simeq \operatorname{Sym}_{\mathcal{O}_{X^0}} Q^{-1} \oplus Q^{-2} \oplus \cdots$, where Q^{-i} are vector bundles (of finite rank) on X^0.

Every L_∞ algebra defines a dg manifold.

Example 2.9. Let $L = L^{-k} \oplus \cdots \oplus L^0 \oplus L^1 \oplus \cdots$ be a finite dimensional L_∞ algebra and $\tau^{>0}L$ be the truncation of L in positive degrees. Let X^0 be the linear manifold L^1 and $\mathcal{O}_X^\bullet$ be the completed symmetric algebra $(\operatorname{Sym} \tau^{>0} L[1]^*)\widehat{\ }$, considered as a sheaf over L^1. It has the structure of differential graded algebra (dga). The L_∞ structure comprises the multilinear maps $\mu_k : \operatorname{Sym}^k L[1] \to L[2]$. The dual map of $\sum 1/k!\, \mu_k$ defines a derivation from $q : \mathcal{O}_X^\bullet \to \mathcal{O}_X^\bullet$ of degree one. The L_∞ relations are equivalent to the condition that $q^2 = 0$. It can be interpreted as an odd vector field on the dg manifold. The "0-truncation" $\pi_0(X)$ can be identified with the Maurer–Cartan locus $\mathrm{MC}(L)$. We call the dg manifold constructed in this way the *formal dg manifold associated to L*.

Given a cyclic L_∞ algebra (L, μ_k, κ), the formal dg manifold constructed in Example 2.9 is a formal symplectic dg manifold in the sense of [Kontsevich and Soibelman 2009]. The pairing κ can be viewed as an odd symplectic form.

On a formal dg manifold, we can define the analogue of the usual Cartan calculus [loc. cit.]. The CS function f is the Hamiltonian function of the odd vector field q on X with respect to the odd symplectic form κ. In particular, $\mathrm{crit}(f)$ coincides with the Maurer–Cartan locus of L.

Comments on A_∞ and L_∞ algebras. Given an A_∞ algebra (R, m_k), we can construct, in a canonical way, an L_∞ algebra (L, μ_k). This is done by replacing m_k by its antisymmetrizer. A lazy way to do that is to first construct a dg algebra quasi-isomorphic to R. Antisymmetrize it to form a dg Lie algebra and then take the cohomology. The Maurer–Cartan sets of R_ω and L_ω agree as sets. In the process of antisymmetrization, a cyclic A_∞ algebra goes to a cyclic L_∞ algebra. We will skip the formal definition of A_∞ algebra (it can be found in [loc. cit.]) although it is implicitly used in the later sections. Using L_∞ algebras has the advantage that one can make sense of the Maurer–Cartan set as a scheme instead of as a noncommutative scheme.

3. Derived categories of toric stacks and Morita equivalence

Definition 3.1. Let $\boldsymbol{k}$ be a field. Given a $\boldsymbol{k}$-linear triangulated category $\mathcal{T}$, an object $E \in \mathcal{T}$ is called *exceptional*, if $\mathrm{Ext}^i(E, E) = 0$ for all $i \neq 0$ and $\mathrm{Ext}^0(E, E) = \boldsymbol{k}$.

- A sequence of exceptional objects $E_1, \dots, E_n$ is called an *exceptional collection* if $\mathrm{Ext}^i(E_j, E_k) = 0$ for arbitrary i when $j > k$.
- An exceptional collection is called *strong* if $\mathrm{Ext}^i(E_j, E_k) = 0$ for any j and k unless $i = 0$.
- We say an exceptional collection is *full* if it generates $\mathcal{T}$.

Let E, F be an exceptional collection of length 2 in $\mathcal{T}$. We define the left and right mutation, $L_E F$ and $R_F E$ respectively, using the distinguished triangles.

$$L_E F \longrightarrow \mathbf{R}\mathrm{Hom}(E, F) \otimes E \longrightarrow F$$
$$E \longrightarrow \mathbf{R}\mathrm{Hom}(E, F)^* \otimes F \longrightarrow R_F E$$

Mutations of exceptional collection are exceptional [Bondal 1990]. But mutations of strong exceptional collections are not necessary strong.

Given an exceptional collection $E_0, \dots, E_n$, we can define another exceptional collection $F_{-n}, F_{-n+1}, \dots, F_0$, called the *dual* exceptional collection to $E_0, \dots, E_n$. First let F_0 equal to E_0. Second, make $F_{-1} = L_{E_0} E_1$. Then define F_{-i} inductively by $L_{F_{-i+1}} L_{F_{-i+2}} \cdots L_{F_0} E_i$.

In our application, $\mathcal{T}$ will be the bounded derived category $\mathrm{D}^b(X)$ of a smooth algebraic variety (stack) X. The exceptional objects are always assumed to belong to the heart of a certain t-structure.

Given a full strong exceptional collection $E_0, \dots, E_n$, we denote the direct sum $\bigoplus_{i=0}^{n} E_i$ by T. It is called a tilting object.

Theorem 3.2 [Bondal 1990]. *The exact functor* $\mathbf{R}\mathrm{Hom}(T, -)$ *induces an equivalence between triangulated categories* $\mathrm{D}^b(X)$ *and* $\mathrm{D}^b(\text{mod-}A)$, *where* $A = \mathrm{End}(T)$. *This equivalence is usually referred to as derived Morita equivalence.*

Let $\mathcal{E}$ be an object in $\mathrm{D}^b(X)$, the right A-module structure on $\mathbf{R}\mathrm{Hom}(T, \mathcal{E})$ is given by precomposition. The quasi-inverse functor of $\mathbf{R}\mathrm{Hom}(T, -)$ is $- \otimes_A^{\boldsymbol{L}} T$.

We can define a quiver with relations from a strong exceptional collection by the following recipe. First, define the set of nodes of $\mathcal{Q}$, denoted by $\mathcal{Q}_0$ to be the ordered set $\{0, 1, \dots, n\}$. The i-th node corresponds to the generator of $\mathrm{Hom}(E_i, E_i)$. The set of arrows of $\mathcal{Q}$, denoted by $\mathcal{Q}_1$ is double graded by source and target. The graded piece $\mathcal{Q}_1^{i,j}$ is a set with cardinality $\dim_{\mathbb{C}} \mathrm{Hom}(E_i, E_j)$. With a choice of basis on $\mathrm{Hom}(E_i, E_j)$, the elements of $\mathcal{Q}_1^{i,j}$ are in one-to-one correspondence with such a basis. The exceptional condition guarantees that there is no arrow that decreases the indices of nodes. The relations of $\mathcal{Q}$ are determined by the commutativity of composition of morphisms. The nodes and arrows generate the free path algebra $\mathbb{C}\mathcal{Q}$, which is spanned as a vector space by all the possible paths. Multiplication in $\mathbb{C}\mathcal{Q}$ is defined by concatenation of paths. The relations in $\mathcal{Q}$ form a two-side ideal $\mathcal{I}$ of $\mathbb{C}\mathcal{Q}$. We call $\mathbb{C}\mathcal{Q}/\mathcal{I}$ the path algebra of $(\mathcal{Q}, \mathcal{I})$. In some situations, we omit $\mathcal{I}$ and write just $\mathcal{Q}$. It follows from the construction that $\mathbb{C}\mathcal{Q}/\mathcal{I} \simeq A$.

A representation of $(\mathcal{Q}, \mathcal{I})$ is given by the following pieces of data:

- a finite dimensional vector space V_i associated to each node i;
- a matrix $a^{i,j}$ associated to each arrow from nodes i to j such that the matrix associated to any element in $\mathcal{I}$ is zero.

Denote the category of finite dimensional representations of $(\mathcal{Q}, \mathcal{I})$ by $\mathrm{Rep}_{\boldsymbol{k}}(\mathcal{Q}, \mathcal{I})$. There are equivalences of abelian categories:

$$\mathrm{Rep}_{\boldsymbol{k}}(\mathcal{Q}, \mathcal{I}) \cong \mathbb{C}\mathcal{Q}/\mathcal{I}\text{-mod} \cong A\text{-mod}.$$

The abelian category mod-A is Noetherian and Artinian. Its simple objects are exactly those representations S_i that have a one-dimensional vector space over node i and 0 over all other nodes. Under the functor $\mathbf{R}\mathrm{Hom}(T, -)$, the exceptional objects E_i are mapped to projective right A-modules, and the objects F_{-i} are mapped to shifts of simple modules $S_i[-i]$.

The *Yoneda algebra* R of A is defined to be $\mathrm{Ext}_A^{\bullet}(\bigoplus_{i=0}^{n} S_i, \bigoplus_{i=0}^{n} S_i)$. It has a canonical A_∞ algebra structure.

Theorem 3.2 builds up a link between the geometry and the representation theory of a quiver, assuming that one can find a full strong exceptional collection in $\mathrm{D}^b(X)$. In general, there is no reason why such a collection (even a single exceptional object) should exist. However, the existence result can be proved for toric Fano stacks of dimension two.

Recall that a two dimensional convex lattice polygon Δ with a distinguished interior lattice point determines a fan Σ associated to the barycentric triangulation. This uniquely determines a toric stack, which is denoted by X_Σ. The Fano condition is equivalent to the convexity of Δ. We refer the reader to [Borisov and Hua 2009, Section 3] for an introduction to toric Deligne–Mumford (DM) stacks.

Theorem 3.3 [Borisov and Hua 2009]. *Let X_Σ be a complete toric Fano DM stack of dimension two. The bounded derived category of coherent sheaves $\mathrm{D}^b(X_\Sigma)$ has a full strong exceptional collection consisting of line bundles. The length of the strong exceptional collection is always equal to the integral volume of Δ, which is also equal to the Euler characteristic $\chi(X_\Sigma)$.*

We will try to extend the derived Morita equivalence to the study of the CY stack Y. Consider the exact functor $\mathbf{R}\mathrm{Hom}(\pi^*T, -)$ from $\mathrm{D}^b(Y)$ to $\mathrm{D}^b(\text{mod-}B)$, where $B = \mathrm{Hom}^\bullet(\pi^*T, \pi^*T)$. It turns out that this is still an equivalence of triangulated categories if we define the right-hand side appropriately. The algebra B (called the roll-up helix algebra by Bridgeland), in general, carries a nontrivial dg algebra structure. However, in order to apply the quiver techniques, we need to find a strong exceptional collection such that the differential of B vanishes; this is an additional condition on a strong exceptional collection.

The following proposition generalizes [Bridgeland 2005, Proposition 4.1], which was originally proved for $\mathbb{P}^2$.

Proposition 3.4. *Let $\mathcal{L}_0, \dots, \mathcal{L}_n$ be a full strong exceptional collection of line bundles on a toric Fano stack of dimension two. The roll-up (dg)-helix algebra B is in fact an algebra, i.e., $\mathrm{Ext}^{>0}(\pi^*T, \pi^*T) = 0$. Therefore, the exact functor $\mathbf{R}\mathrm{Hom}(\pi^*T, -)$ induces an equivalence from $\mathrm{D}^b(Y)$ to $\mathrm{D}^b(\text{mod-}B)$.*

Proof. We need a technical lemma from [Borisov and Hua 2009] about cohomology of line bundles on toric stacks.

For every $\boldsymbol{r} = (r_i)_{i=1}^n \in \mathbb{Z}^n$ we denote by $\mathrm{Supp}(\boldsymbol{r})$ the simplicial complex on the vertices $\{1, \dots, n\}$ which consists of all subsets $J \subseteq \{1, \dots, n\}$ such that $r_i \geq 0$ for all $i \in J$ and there exists a cone of Σ that contains all v_i, $i \in J$. For example, if all coordinates r_i are negative then the simplicial complex $\mathrm{Supp}(\boldsymbol{r})$ consists of the empty set only, and its geometric realization is the zero cone of Σ. In the other extreme case, if all r_i are nonnegative then the simplicial complex $\mathrm{Supp}(\boldsymbol{r})$ encodes the fan Σ, which is its geometric realization.

Lemma 3.5 [Borisov and Hua 2009, Proposition 4.1]. *Let N be an integral lattice, Σ a fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$, and X_Σ the toric stack associated to Σ. The cohomology*

$H^p(X_\Sigma, \mathcal{L})$ is isomorphic to the direct sum over all $\boldsymbol{r} = (r_i)_{i=1}^n$ such that

$$\bigoplus\Big(\sum_{i=1}^n r_i E_i\Big) \cong \mathcal{L}$$

with E_i being toric invariant divisors of the $(\mathrm{rk}(N) - p)$*-th reduced homology of the simplicial complex* $\mathrm{Supp}(\boldsymbol{r})$.

By adjunction,

$$\mathrm{Hom}^d(\pi^*T, \pi^*T) = \bigoplus_{k\geq 0} \mathrm{Hom}_X^d(T, T\otimes\omega_X^{-k}).$$

In order to prove the proposition, it suffices to show that $H^d(X, \mathcal{L}_i^{-1}\otimes\mathcal{L}_j\otimes\omega_X^{-1}) = 0$ for $d = 1, 2$. Since $\mathcal{L}_0, \dots, \mathcal{L}_j$ is strong exceptional, we have $H^d(X, \mathcal{L}_i^{-1}\otimes\mathcal{L}_j) = 0$ for $d = 1, 2$. Consider all the possible integral linear combinations $\sum_{i=1}^m r_i E_i$ such that $\mathcal{O}\big(\sum_{i=1}^m r_i E_i\big) = \mathcal{L}_i^{-1}\otimes\mathcal{L}_j$. By Lemma 3.5, $H^d(X, \mathcal{L}_i^{-1}\otimes\mathcal{L}_j) = 0$ for $d = 1, 2$ means $\mathrm{Supp}(\boldsymbol{r})$ is contractible. Notice that if $\mathrm{Supp}(\boldsymbol{r})$ is contractible then $\mathrm{Supp}(\boldsymbol{r}+1)$ is also contractible. Again by Lemma 3.5, $H^d(X, \mathcal{L}_i^{-1}\otimes\mathcal{L}_j\otimes\omega_X^{-1}) = 0$ for $d = 1, 2$. $\square$

Now we can write B simply by $\mathrm{End}(\pi^*T)$. It is also the path algebra of a quiver with relations. This quiver can be constructed by the same recipe as in the previous section. Let's denote it by $\mathcal{Q}_\omega$. Notice that $\mathcal{Q}_\omega$ will have cyclic paths because the pull back of exceptional objects will have homomorphisms in both directions. Again, we have an equivalence of abelian categories

$$\mathrm{Rep}_{\boldsymbol{k}}(\mathcal{Q}_\omega, \mathcal{I}) \cong \text{mod-}B.$$

The path algebra B is naturally graded by path length. A B-module M is called *nilpotent* if there exists $k \gg 0$ such that $B_k M = 0$. The exact functor $\mathbf{R}\mathrm{Hom}(\pi^*T, -)$ maps D_ω to the derived category of nilpotent B-modules $D^b(\mathrm{mod}_0\text{-}B)$.

The pushforward ι_* defines an exact functor from $\mathrm{D}^b(X)$ to D_ω. Under Morita equivalence, the modules $\iota_*(F_{-i}[i])$ are the simple modules in $\mathrm{D}^b(\mathrm{mod}_0\text{-}B)$ corresponding to those one dimensional representations associated to each of the vertices of $\mathcal{Q}_\omega$.

Similarly, we call the self-extension algebra

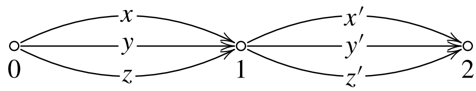
$$\mathrm{Ext}_B^\bullet\Big(\bigoplus_{i=0}^n \iota_* S_i, \bigoplus_{i=0}^n \iota_* S_i\Big)$$

the Yoneda algebra, denoted by R_ω. It carries a natural A_∞ structure as well.

We now give the example of derived Morita equivalence on $\mathbb{P}^2$ and local $\mathbb{P}^2$.

Example 3.6. Let X be $\mathbb{P}^2$. The line bundles $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ form a full strong exceptional collection. Take the tilting bundle to be $T = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$. The quiver Q is

(1)



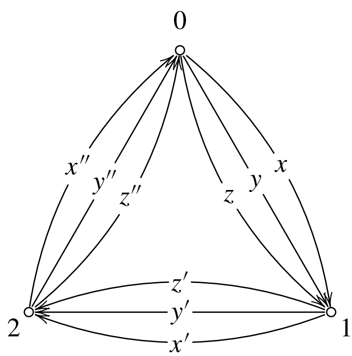
with the ideal of relations generated by

$$x'y - y'x, \quad y'z - z'y, \quad z'x - x'z.$$

The dual collection to $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ is $\Omega^2(2), \Omega^1(1), \mathcal{O}$. They map to simple modules $S_2[-2], S_1[-1], S_0$ under $\mathbf{R}\mathrm{Hom}(T, -)$.

The roll-up helix algebra $B = \mathrm{End}(\pi^* T)$ is the path algebra of the quiver Q_ω given by

(2)



with relations

$$\begin{aligned} &x'y - y'x, && y'z - z'y, && z'x - x'z; \\ &x''y' - y''x', && y''z' - z''y', && z''x' - x''z'; \\ &xy'' - yx'', && yz'' - zy'', && zx'' - xz''. \end{aligned}$$

4. The cyclic completion of the Yoneda algebra

Two technical results are proved in this section.

- We show the Yoneda algebra L_ω is the cyclic completion of the Yoneda algebra L. This is the algebraic counterpart of the cotangent bundle construction.
- We show that the operations μ_k on L vanish when $k > \chi(X)$. Then, by the cyclic completion construction, the same is true for L_ω.

Theorem 4.2 was proved first by Aspinwall and Fidkowski [2006, Section 4.3] and reproved in a more general form by Segal [2008, Theorem 4.2]. For our own convenience, we give a slightly different proof here since some techniques in the proof are used in the later sections. But the ideas are quite similar to the ones given in those two references.

These two results, together with the existence theorem of strong exceptional collections (Definition 3.1) and Proposition 3.4, guarantee the existence of global algebraic CS functions. In fact, they provide a recipe to construct CS functions, starting from a strong exceptional collection satisfying Proposition 3.4.

Definition 4.1. [Segal 2008] Let $L = \bigoplus_{i=0}^{d} L^i$ be a finite dimensional L_∞ algebra over $\boldsymbol{k}$, with its L_∞ products denoted by μ_k. Define $\bar{L}$ to be the graded vector space $L \oplus L[-d-1]$, i.e., $\bar{L}^i = L^i \oplus (L^{d+1-i})^*$. Define the cyclic pairing and L_∞ products $\bar{\mu}_k : \bigwedge^k \bar{L} \to \bar{L}[2-k]$ according to the following rules:

(1) Define the bilinear form κ on $\bar{L}$ by the natural pairing between L and L^*.

(2) If the inputs of $\bar{\mu}_k$ all belong to L, then define $\bar{\mu}_k = \mu_k$.

(3) If more than one input belongs to L^*, then define $\bar{\mu}_k = 0$.

(4) If there is exactly one input $a_i^* \in L^*$, then define $\bar{\mu}_k$ by

$$\kappa(\bar{\mu}_k(a_1, \ldots, a_i^*, \ldots, a_k), b) = (-1)^{\epsilon} \kappa(\mu_k(a_{i+1}, \ldots, a_k, b, a_1, \ldots, a_{i-1}), a_i^*)$$

for arbitrary $b \in L$, where $\epsilon = |a_1|(|a_2|+\cdots+|b|)+\cdots+|a_i^*|(|a_{i+1}|+\cdots+|b|)$;

It is easy to check that $(\bar{L}, \bar{\mu}_k, \kappa)$ forms a cyclic L_∞ algebra. We call $\bar{L}$ the *cyclic completion* of L.

We have defined the Yoneda algebras $R = \mathrm{Ext}_A^\bullet(\bigoplus_{i=0}^n S_i, \bigoplus_{i=0}^n S_i)$ and $R_\omega = \mathrm{Ext}^\bullet(\bigoplus_{i=0}^n \iota_* S_i, \bigoplus_{i=0}^n \iota_* S_i)$ in previous section. Take the associated L_∞ algebras and denote them by L and L_ω. Since X is a surface, $d = 2$ in Definition 4.1.

The following theorem will play a central role in this paper.

Theorem 4.2 [Aspinwall and Fidkowski 2006; Segal 2008]. *The Yoneda algebra L_ω is the cyclic completion of the Yoneda algebra L.*

Proof. This can be done in three steps. First, we need to verify that L_ω and $\bar{L}$ coincide as graded vector spaces. Second, we will show the pairing on $\bar{L}$ defined by (1) of Definition 4.1 coincides with the Serre pairing on L_ω. Finally, we need to check that the L_∞ products on L_ω satisfy properties (2)–(4) in Definition 4.1.

Given an object $E \in \mathrm{D}^b(\text{mod-}B) \simeq \mathrm{D}^b(Y)$ that is scheme theoretically supported on X, one can view E as a complex of finitely generated A-modules. There is a projective A resolution $P^\bullet$ for E:

$$P^\bullet \longrightarrow E \longrightarrow 0$$

such that each P^i is a direct sum of copies of $E_0, \ldots, E_n$.

Because Y is the total space of canonical bundle over X, there is a tautological short exact sequence of sheaves:

$$0 \longrightarrow \pi^*(\omega_X^{-1}) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Tensor it with π^*E to obtain

$$0 \longrightarrow \pi^*E(\omega_X^{-1}) \longrightarrow \pi^*E \longrightarrow \iota_*E \longrightarrow 0.$$

Since π^* preserves the projective modules, by replacing E with $P^\bullet$ we obtain a projective B resolutions of ι_*E as total complex of the following double complex

$$\begin{tikzcd}
\cdots \arrow[r] & \pi^*P^{-2} \arrow[r] & \pi^*P^{-1} \arrow[r] & \pi^*P^{0} \arrow[r] & 0 \\
\cdots \arrow[r] & \pi^*P^{-2}\otimes\pi^*\omega_X^{-1} \arrow[r] \arrow[u] & \pi^*P^{-1}\otimes\pi^*\omega_X^{-1} \arrow[r] \arrow[u] & \pi^*P^{0}\otimes\pi^*\omega_X^{-1} \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{3}$$

We denote this resolution of ι_*E by $P^\bullet_\omega$.

As a graded vector space, L_ω is computed as the cohomology of $\mathrm{Hom}^\bullet_Y(P^\bullet_\omega, \iota_*E)$. Because $P^\bullet_\omega$ is the total complex of the above double complex, $\mathrm{Hom}^\bullet_Y(P^\bullet_\omega, \iota_*E)$ is quasi-isomorphic with the total complex of the following double complex:

$$\begin{tikzcd}
\cdots & \mathrm{Hom}(\pi^*P^{-1}, \iota_*E) \arrow[l] \arrow[d] & \mathrm{Hom}(\pi^*P^{0}, \iota_*E) \arrow[l] \arrow[d] & 0 \arrow[l] \\
\cdots & \mathrm{Hom}(\pi^*P^{-1}\otimes\pi^*\omega_X^{-1}, \iota_*E) \arrow[l] & \mathrm{Hom}(\pi^*P^{0}\otimes\pi^*\omega_X^{-1}, \iota_*E) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

The spectral sequence associated to this double complex degenerates at E_1 page. Using adjunction together with Serre duality, we obtain

$$\begin{aligned}
\mathrm{Hom}(\iota_*E, \iota_*E) &= \mathrm{Hom}_X(E, E),\\
\mathrm{Ext}^1(\iota_*E, \iota_*E) &= \mathrm{Ext}^1_X(E, E) \oplus \mathrm{Ext}^2_X(E, E)^*,\\
\mathrm{Ext}^2(\iota_*E, \iota_*E) &= \mathrm{Ext}^2_X(E, E) \oplus \mathrm{Ext}^1_X(E, E)^*,\\
\mathrm{Ext}^3(\iota_*E, \iota_*E) &= \mathrm{Hom}_X(E, E)^*.
\end{aligned}$$

The above fact holds for any object E with scheme theoretic support on X. We are particularly interested in the case when E is $\oplus_{i=0}^n F_{-i}[i]$, i.e., the direct sum of the simple objects in mod-A. This identifies L_ω and $\bar{L}$ as graded vector spaces since both will be equal to $L \oplus L[-3]^*$.

In order to verify property (1), we need to write down a bilinear pairing κ on $\mathrm{Hom}^\bullet(P^\bullet_\omega, P^\bullet_\omega)$ such that its restriction on cohomology gives the obvious duality between L and L^*. By adjunction, $\mathrm{Hom}^3(P^\bullet_\omega, P^\bullet_\omega)$ has a direct summand $\mathrm{Hom}^2(\pi^*P^\bullet \otimes \omega_X^{-1}, \pi^*P^\bullet)$, which is isomorphic to $\mathrm{Hom}^2_X(P^\bullet, P^\bullet \otimes (\bigoplus_{k\le 1} \omega_X^k))$. It contains the finite dimensional graded piece $\mathrm{Hom}^2_X(P^\bullet, P^\bullet \otimes \omega)$, which has a trace map to $H^2(X, \omega_X) \simeq \mathbb{C}$. Given any two elements x and y in $\mathrm{Hom}^\bullet(P^\bullet_\omega, P^\bullet_\omega)$, we define the bilinear pairing $\kappa(x, y)$ to be the projection of $x \circ y$ to the graded piece $\mathrm{Hom}^2_X(P^\bullet, P^\bullet \otimes \omega)$ followed by the trace map. Clearly, the restriction of κ on cohomology satisfies property (1).

Now we need to verify properties (2) to (4) for L_ω. For dimension reasons, it suffices to check the case when all the inputs of the L_∞ products μ_k lie in L^1_ω. Since L_ω is constructed as the cohomology of $\mathrm{Hom}^\bullet(P^\bullet_\omega, P^\bullet_\omega)$, the element in L^1_ω can be represented by either the vertical or horizontal arrows in diagram (3). More specifically, a class in $\mathrm{Ext}^1_X(E, E)$ is represented by a horizontal arrow and a class in $\mathrm{Ext}^2_X(E, E)^*$ is represented by a vertical arrow. Then property (2) follows immediately since the rows of the double complex are simply the pullback of $P^\bullet$ (up to $\otimes\, \omega_X^{-1}$), which is the projective resolution of E.

If we write $\mathrm{Ext}^2(E, E)^*$ as $\mathrm{Ext}^0(E, E \otimes \omega_X)$, then we can see that

$$\bar{\mu}_2 : \mathrm{Ext}^1(E, E) \otimes \mathrm{Ext}^0(E, E \otimes \omega_X) \longrightarrow \mathrm{Ext}^1(E, E \otimes \omega_X) \simeq \mathrm{Ext}^1(E, E)^*$$

is the only nonzero term that can involve $\mathrm{Ext}^2(E, E)^*$. For example, if both inputs of μ_2 belong to $\mathrm{Ext}^0(E, E \otimes \omega_X)$, then the output is $\mathrm{Ext}^0(E, E \otimes \omega_X^2)$, which is not in L^2_ω. Similarly, this argument shows that any nonzero term of μ_k of L_ω can involve at most one $\mathrm{Ext}^2(E, E)^*$ term. This proves property (3).

Property (4) is essentially the cyclic symmetry of μ_k. Since the κ on cohomology is a restriction of a bilinear form (also denoted by κ) on the dga $\mathrm{Hom}^\bullet(P^\bullet_\omega, P^\bullet_\omega)$ with differential d, property (4) will follow from the following cyclic symmetry properties on $\mathrm{Hom}^\bullet(P^\bullet_\omega, P^\bullet_\omega)$. For arbitrary elements x, y, and z:

- ◇ $\kappa(x, y) = \pm\kappa(y, x)$
- ◇ $\kappa(dx, y) = \pm\kappa(dy, x)$;
- ◇ $\kappa(x \circ y, z) = \pm\kappa(y \circ z, x)$.

The first property is clear since the commutator is trace-free. The trace map will factor through the morphism

$$\mathrm{Hom}^2(P^\bullet, P^\bullet \otimes \omega) \longrightarrow L^3_\omega = \mathrm{Ext}^2(E, E \otimes \omega) \simeq \mathrm{Hom}(E, E)^*.$$

Therefore, the trace of a coboundary is zero, so the second property follows from the Leibniz rule. The third property follows from the first and associativity of the product. □

Remark 4.3 (the geometric meaning of cyclic completion). From Example 2.9 recall that the completion of the truncated symmetric algebra $(\mathrm{Sym}\, L[1]^*)^\frown$ (we omit $\tau^{>0}$ for simplicity) can be interpreted as the structure sheaf of the graded linear manifold $M = L[1]$.

The odd cotangent bundle of the graded manifold M, denoted as $T^*[-1]M$, is defined to be the graded manifold $L[1] \oplus (L[1]^*[-1])$. As graded vector spaces,

$T^*[-1]M$ is the same as $L_\omega[1]$. Then, $\mathcal{O}_{T^*[-1]M}$ coincides with $(\operatorname{Sym} L_\omega[1]^*)^\frown$ as graded algebras. The L_∞ products $\bar{\mu}_k$ defines a derivation on $\mathcal{O}_{T^*[-1]M}$ and the cyclic pairing κ defines an odd two-form on $T^*[-1]M$. In fact, this process is functorial. Hence, passing to the cyclic completion of an L_∞ algebra is an algebraic counterpart for taking the odd cotangent bundle of a dg manifold.

The L_∞ (or A_∞) structure of the Yoneda algebra L has been studied for a long time in the representation theory of finite dimensional algebras. The following boundedness theorem turns out to be very important for the purpose of this paper.

Theorem 4.4. *The L_∞ products (higher brackets) μ_k on L vanish when $k > \chi(X)$.*

Proof. Let A be a finite dimensional algebra and $\{S_i\}$ be the collection of simple A-modules. It is well known that the Yoneda algebra $R = \operatorname{Ext}^\bullet_A(\bigoplus_i S_i, \bigoplus_i S_i)$ controls the deformation of A. If A is presented as a path algebra of a quiver with relations, then the A_∞ products m_k on R can be interpreted as relations of the path algebra; see [Keller 2006, Section 7.8].

Since in our situation the quiver is constructed from a strong exceptional collection of line bundles on X (recall the construction in Section 3), the elements in the path algebra A carry an extra grading given by the ordering on the strong exceptional collection. The A_∞ products preserve this extra grading. Therefore, the length of the strong exceptional collection, which is equal to the Euler characteristic $\chi(X)$, gives an upper bound for number of nonvanishing m_k. This is intuitively clear since, on a directed quiver generated by, say, a length 4 strong exceptional collection, there cannot be a relation involving length 5 paths.

Finally, we pass from an A_∞ algebra to an L_∞ algebra. Since L is the antisymmetrization of R, we get $\mu_k = 0$ when $m_k = 0$. □

5. Moduli spaces and Chern–Simons functions

We fix the ground field $\boldsymbol{k} = \mathbb{C}$. Let Γ be the Grothendieck group of D_ω. By derived Morita equivalence, Γ also equals the Grothendieck group of the derived category of nilpotent representations of $\mathcal{Q}_\omega$. It is a free abelian group of rank $n+1$ generated by the collection of simple modules $\iota_* S_0, \dots, \iota_* S_n$. If we fix these simple modules as a $\mathbb{Z}$-basis of Γ, every effective class can be written as a vector $\boldsymbol{d} = (d_0, \dots, d_n)$ with nonnegative entries. We call such a choice of $\boldsymbol{d}$ a *dimension vector*.

Theorem 5.1. *Let X be a toric Fano stack of dimension two and Y the total space of its canonical bundle. Pick a strong exceptional collection constructed in [Borisov and Hua 2009] and denote the corresponding quiver of Y by $\mathcal{Q}_\omega$. Let $\mathfrak{M}_\gamma$ be a bounded family of sheaves on Y support on X with class $\gamma \in \Gamma$. There exists a dimension vector $\boldsymbol{d}$ and an open immersion of Artin stacks from $\mathfrak{M}_\gamma$ to the quotient stack $[\mathrm{MC}(L_{\omega,\boldsymbol{d}})/G_{\boldsymbol{d}}]$, where $\mathrm{MC}(L_{\omega,\boldsymbol{d}})$ is the space of representations of $\mathcal{Q}_\omega$ with* *dimension vector* $\boldsymbol{d}$ *and* $G_{\boldsymbol{d}}$ (*defined later in this section*) *is the gauge group acting by changing of basis.*

Theorem 5.2. *Given a class* $\gamma \in \Gamma$, *a bounded family of sheaves on* Y *supported on* X *with class* γ *is the critical set of an algebraic function* $f_{\boldsymbol{d}}$.

We call such a function a *Chern–Simons (CS) function*. The infinitesimal deformation of representations is controlled by the following L_∞ algebras. Fix a dimension vector $\boldsymbol{d}$, define

$$L_{\boldsymbol{d}} := \operatorname{Ext}^\bullet\left(\bigoplus_{i=0}^{n} S_i \otimes V_i,\ \bigoplus_{i=0}^{n} S_i \otimes V_i\right)$$

and

$$L_{\omega,\boldsymbol{d}} := \operatorname{Ext}^\bullet\left(\bigoplus_{i=0}^{n} \iota_* S_i \otimes V_i,\ \bigoplus_{i=0}^{n} \iota_* S_i \otimes V_i\right),$$

where each V_i is a vector space of dimension d_i. They are generalizations of the Yoneda algebras: if we take $\boldsymbol{d} = (1, \dots, 1)$ we obtain the Yoneda algebras. All the results in Section 4 clearly generalize to $L_{\boldsymbol{d}}$ and $L_{\omega,\boldsymbol{d}}$.

The space $L^1_{\boldsymbol{d}}$ can be identified with the space $\bigoplus_{a\in\mathcal{Q}_1} \operatorname{Hom}(V_i, V_j)$ of matrices, summing over all the arrows, and similarly for $L^1_{\omega,\boldsymbol{d}}$ with $a \in \mathcal{Q}_{\omega 1}$. It carries a natural bigrading by the source and target of each arrow. The space $L^0_{\boldsymbol{d}}$ can be identified with the space $\bigoplus_{i\in\mathcal{Q}_0} \operatorname{End}(V_i)$, which is the Lie algebra associated to the group $\prod_{i\in\mathcal{Q}_0} \operatorname{GL}(V_i)$. We denote this group by $G_{\boldsymbol{d}}$ for simplicity. It acts on $L_{\boldsymbol{d}}$ by conjugation. Analogously, the space $L^0_{\omega,\boldsymbol{d}}$ can be identified with the Lie algebra associated to the same group, which acts on $L_{\omega,\boldsymbol{d}}$.

The following lemma is well known in representation theory of quivers.

Lemma 5.3. *The elements of* $\operatorname{MC}(L_{\boldsymbol{d}})$ *are in one to one correspondence with the representations of* $\mathcal{Q}$ *of dimension vector* $\boldsymbol{d}$, *and analogously for the elements of* $\operatorname{MC}(L_{\omega,\boldsymbol{d}})$ *and the representations of* $\mathcal{Q}_\omega$. *Two representations are isomorphic if and only if they belong to the same orbits of* $G_{\boldsymbol{d}}$.

Proof. See [Keller 2006, Section 7.8] or [Segal 2008, Proposition 3.8]. □

The L_∞ algebra L controls the infinitesimal deformation of representations in the following sense. Let M be an A-module with dimension vector $\boldsymbol{d}$. We denote its corresponding Maurer–Cartan element by x. The homology groups $H^i(L_{\boldsymbol{d}}, \delta^x)$ are isomorphic to $\operatorname{Ext}^i_A(M, M)$. In general, $L_{\boldsymbol{d}}$ is just the formal tangent space at the point $\bigoplus_i S_i \otimes V_i$. However, in our situation because of the boundedness of μ_k (Theorem 4.4), the Maurer–Cartan equation actually converges. An analogous argument holds for the L_∞ algebra $L_{\omega,\boldsymbol{d}}$, with M a B-module with dimension vector $\boldsymbol{d}$, in which case the homology groups $H^i(L_{\omega,\boldsymbol{d}}, \delta^x)$ are isomorphic to

$\mathrm{Ext}^i_B(M, M)$. Therefore the moduli space can be constructed globally as mentioned in the previous Lemma.

Proof of Theorem 5.1. Given Lemma 5.3, it suffices to show the existence of an open immersion of $\mathfrak{M}_\gamma$ into $[\mathrm{MC}(L_{\omega,\boldsymbol{d}})/G_{\boldsymbol{d}}]$.

First, we need to construct a monomorphism of stacks. Let's pick an ample line bundle L on X. If T is a tilting bundle on X then $T \otimes L^{-N}$ is again a tilting bundle for any integer N. Therefore, the functor $\mathbf{R}\mathrm{Hom}(\pi^*(T \otimes L^{-N}), -)$ induces an equivalence from $\mathrm{D}^b(Y)$ to $\mathrm{D}^b(\text{mod-}B)$. Because T is direct sum of line bundles, we can choose $N \gg 0$ such that for any sheaf $\mathcal{E} \in \mathfrak{M}_\gamma$, $\mathbf{R}\mathrm{Hom}(\pi^*(T \otimes L^{-N}), \mathcal{E})$ is concentrated in degree zero, i.e., is a module over B.[1] Let $\boldsymbol{d}$ be its dimension vector, which depends on both γ and N. Then we obtain a morphism between stacks. Because of the derived Morita equivalence, this is clearly an injection.

Next we need to argue this morphism is étale. Let $A' \to A \to \mathbb{C}$ be a small extension of pointed $\mathbb{C}$-algebras, and let $T = \mathrm{Spec}\, A$ and $T' = \mathrm{Spec}\, A'$. Consider the 2-commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, hook] & \mathfrak{M}_\gamma \arrow[d, hook, "\mathbf{R}\mathrm{Hom}(\pi^*(T\otimes L^{-N}),-)"] \\ T' \arrow[r] \arrow[ur, dashed] & {[\mathrm{MC}(L_{\omega,\boldsymbol{d}})/G_{\boldsymbol{d}}]} \end{tikzcd}$$

of solid arrows. We have to prove that the dotted arrow exists, uniquely, up to a unique 2-isomorphism. This follows from standard deformation theory. We need that $\mathbf{R}\mathrm{Hom}(\pi^*(T \otimes L^{-N}), -)$ induces a bijection on deformation spaces and an injection on obstruction spaces (associated to the above diagram). They follow immediately for the equivalence between $\mathrm{D}^b(Y)$ and $\mathrm{D}^b(\text{mod-}B)$. In fact, all the obstruction groups are isomorphic. $\square$

Proof of Theorem 5.2. As we have seen in Definition 2.5, there is always a formal function

$$f_{\boldsymbol{d}}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k(k+1)/2}}{(k+1)!} \kappa(\bar{\mu}_k(z, \dots, z), z)$$

associated to the cyclic L_∞ algebra $L_{\omega,\boldsymbol{d}}$, where $z \in L^1_{\omega,\boldsymbol{d}}$. The critical set of $f_{\boldsymbol{d}}$ coincides with $\mathrm{MC}(L_{\omega,\boldsymbol{d}})$.

By the boundedness in Theorem 4.4, such a formal function is, in fact, a polynomial function of degree at most $\chi(X)$. Therefore, $\mathrm{MC}(L_{\omega,\boldsymbol{d}})$, as a subvariety of $L^1_{\omega,\boldsymbol{d}}$, is the critical scheme of $f_{\boldsymbol{d}}$. Since the $G_{\boldsymbol{d}}$ action is induced from the action of the Lie subalgebra $L^0_{\omega,\boldsymbol{d}}$, it is clear that $f_{\boldsymbol{d}}$ is invariant under this action.

[1] This is *not* true when T contains torsion.

By Theorem 5.1, $\mathfrak{M}_\gamma$ is an open substack of $[\mathrm{MC}(L_{\omega,\boldsymbol{d}})/G_{\boldsymbol{d}}]$ for an appropriate choice of $\boldsymbol{d}$. The theorem follows since we can restrict the function $f_{\boldsymbol{d}}$. □

Remark 5.4. Recall that by Theorem 4.2, $L^1_{\omega,\boldsymbol{d}}$ decomposes into $L^1_{\boldsymbol{d}} \oplus (L^2_{\boldsymbol{d}})^*$. The CS function $f_{\boldsymbol{d}}$ has a nice property coming from this decomposition:

If we write the cyclic pairing $\kappa(x, y)$ as $\mathrm{tr}(x \circ y)$, then the function $f_{\boldsymbol{d}}$ can be written as the trace of the cyclic invariant polynomial of matrices. Definition 4.1 tells us that the variables in $(L^2_{\boldsymbol{d}})^*$ appear exactly once (in degree one) in all the monomials. This means that we can always write $f_{\boldsymbol{d}}$ as an inner product of a polynomials of elements in $L^1_{\boldsymbol{d}}$ and elements of $(L^2_{\boldsymbol{d}})^*$. This property plays a central role in Section 8.

As a summary of Sections 4 and 5, we give an algorithm to compute CS functions on local toric Fano surfaces.

STEP 1 Choose a strong exceptional collection of line bundles on X. By results in Section 3, this completely determines the quiver $\mathcal{Q}$, together with its relations.

STEP 2 Compute the A_∞ structures on the Yoneda algebra R using the correspondence between m_k and the relations on $\mathcal{Q}$.

STEP 3 Apply Theorem 4.2 to compute $\overline{m}_k$ for R_ω.

STEP 4 Plug in specific dimension vector $\boldsymbol{d}$, antisymmetrize $R_{\omega,\boldsymbol{d}}$ to $L_{\omega,\boldsymbol{d}}$, and apply Definition 2.5 to compute $f_{\boldsymbol{d}}$.

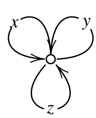
6. Examples of CS functions

In these section, we discuss some examples of CS functions.

6A. *Complex affine* 3*-space* $\mathbb{C}^3$. The easiest example of a Calabi–Yau 3-fold is the three dimensional affine space. Rigorously speaking, it is not a local surface but still the CS function can be computed using the same philosophy.

Let B be the polynomial algebra with three variables. The category $\mathrm{Coh}(\mathbb{C}^3)$ equals mod-B. Consider the quiver $\mathcal{Q}_\omega$:

(4)



with relations $xy - yx$, $yz - zy$, $zx - xz$. Its path algebra is equal to B.

Given a positive integer n, let $L_{\omega,n}$ be the Yoneda algebra $\mathrm{Ext}^\bullet_{\mathbb{C}^3}(\mathcal{O}_{\{0\}}, \mathcal{O}_{\{0\}}) \otimes \mathfrak{gl}_n$. Since the only nonvanishing product is $\bar{\mu}_2$, $L_{\omega,n}$ is a graded Lie algebra. Now, let A, B, C be $n \times n$ matrices associated to x, y, z. The CS function f_n is equal to $\mathrm{tr}((AB - BA)C)$.

The Morita equivalence in this case is the classical Koszul duality between symmetric and exterior algebras

$$\mathrm{D}^b(\mathrm{Coh}(V)) = D^b\big(\text{mod-}{\textstyle\bigwedge^\bullet}(V)\big).$$

The quiver $\mathcal{Q}_\omega$ gives combinatorial description for both $\mathbb{C}^3$ and the cotangent bundle of the three dimensional torus. The first is clear since the path algebra of $\mathcal{Q}_\omega$ is the algebra of functions on $\mathbb{C}^3$. For the second, we can think of the quiver as the 1-skeleton of T^3 and the relations as the gluing conditions of two cells.

The stack $[\mathrm{crit}(f_{\boldsymbol{d}})/G_{\boldsymbol{d}}]$ is related to two interesting moduli spaces. The first one[2] is the moduli space of length n sheaves on $\mathbb{C}^3$ and the second one[3] is the moduli space of flat GL_n vector bundles on T^3. These two moduli spaces are related by homological mirror symmetry.

6B. ***The local projective plane $\boldsymbol{\omega_{\mathbb{P}^2}}$.*** Using the calculations done in Example 3.6, the CS function for the local projective plane is

$$\mathrm{tr}\big(C''(A'B - B'A) + A''(B'C - C'B) + B''(C'A - A'C)\big)$$

where A, B, C, A', B', C', A'', B'', C'' are matrices associated, respectively, to the arrows x, y, z, x', y', z', x'', y'', z''.

6C. ***The Calabi–Yau 3-folds $\boldsymbol{\omega_{\mathbb{P}(1:3:1)}}$ and $\boldsymbol{\omega_{\mathbb{P}(2:1:2)}}$.*** In this subsection, we will compute the CS functions of $\omega_{\mathbb{P}(1:3:1)}$ and $\omega_{\mathbb{P}(2:1:2)}$. These two Calabi–Yau 3-folds are K-equivalent; consequently, there is some interesting symmetry between these two CS functions.

For simplicity, we set $X_1 := \mathbb{P}(1:3:1)$ and $X_2 := \mathbb{P}(2:1:2)$. The stacky fan Σ_1 of X_1 has rays $(0, 1)$, $(1, -1)$, $(-1, -2)$; the stacky fan Σ_2 of X_2 has rays $(0, 2)$, $(1, 0)$, $(-1, -1)$. Denote their canonical bundles by Y_1 and Y_2.

The Picard groups of X_1 and X_2 both equal $\mathbb{Z}$. We denote the positive generator by $\mathcal{O}(1)$. On X_1, $\mathcal{O}(1)$ can be written as $\mathcal{O}(D_2)$, with D_2 being the toric invariant divisor for $(1, -1)$. On X_2, $\mathcal{O}(1)$ can be written as $\mathcal{O}(D_1)$ with D_1 being the toric invariant divisor for $(0, 2)$. For both $\mathrm{D}^b(X_1)$ and $\mathrm{D}^b(X_2)$,

$$\mathcal{O}, \quad \mathcal{O}(1), \quad \mathcal{O}(2), \quad \mathcal{O}(3), \quad \mathcal{O}(4)$$

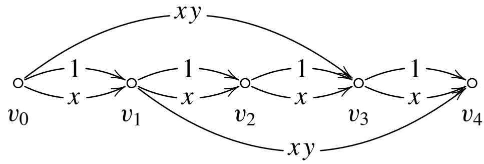
form a full strong exceptional collection. The quivers associated to these two collections are denoted by $\mathcal{Q}_1$ and $\mathcal{Q}_2$. The sets of vertices $\{0, 1, 2, 3, 4\}$ correspond

[2]One can modify the construction slightly to include the Hilbert scheme of points; see [Behrend et al. 2013].

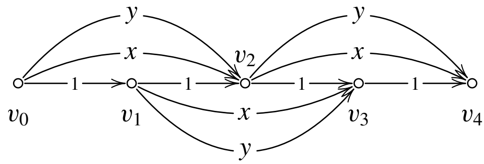
[3]One needs to include a stability condition to make it hold rigorously.

to $\mathcal{O}$, $\mathcal{O}(1)$, $\mathcal{O}(2)$, $\mathcal{O}(3)$, $\mathcal{O}(4)$.

(5) $\mathcal{Q}_1$



(6) $\mathcal{Q}_2$



Notice that Σ_1 and Σ_2 are related by a shift of origin. This shift changes the stack completely. But surprisingly, the full strong exceptional collections on X_{Σ_1} and X_{Σ_2} are related [Borisov and Hua 2009]. We denote the arrows from the i-th node to the j-th node by A_{ij}, B_{ij} or C_{ij} and the relations from the i-th node to the j-th node by R_{ji}. Because the quivers are directed, i is strictly less than j.

Using the algorithm at the end of last section, the CS function for $\omega_{\mathbb{P}(1:3:1)}$ is

$$
\begin{aligned}
(7)\quad f = \operatorname{tr}\bigl(R_{20}(B_{12}A_{01} - A_{12}B_{01}) + R_{31}(B_{23}A_{12} - A_{23}B_{12})\\
+ R_{42}(B_{34}A_{23} - A_{34}B_{23}) + R_{40}(A_{34}C_{03} - C_{14}A_{01})\\
+ S_{40}(B_{34}C_{03} - C_{14}B_{01})\bigr).
\end{aligned}
$$

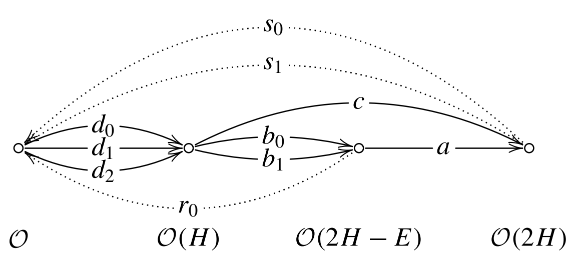
The CS function for $\omega_{\mathbb{P}(2:1:2)}$ is

$$
\begin{aligned}
(8)\quad f = \operatorname{tr}\bigl(R_{30}(A_{23}B_{02} - B_{13}A_{01}) + R_{41}(B_{24}A_{12} - A_{34}B_{13})\\
+ S_{30}(A_{23}C_{02} - C_{13}A_{01}) + S_{41}(C_{24}A_{12} - A_{34}C_{13})\\
+ R_{40}(B_{24}C_{02} - C_{24}B_{02})\bigr).
\end{aligned}
$$

6D. ***Blow-up of the projective plane $\mathbb{P}^2$ at one point.*** The first example which involves μ_k terms with $k > 2$ is the local DelPezzo surface of degree one. It was first computed in [Aspinwall and Fidkowski 2006].

Let X be the blow-up of $\mathbb{P}^2$ at one point. Denote the pull back of a hyperplane by H and the exceptional divisor by E. The derived category $\mathrm{D}^b(X)$ has a strong exceptional collection, consisting of $\mathcal{O}$, $\mathcal{O}(H)$, $\mathcal{O}(2H-E)$, $\mathcal{O}(2H)$, and the corresponding quiver is

(9) $\mathcal{Q}_3$



The graded piece L^2 of the Yoneda algebra has dimension three. We denote the basis by r_0, s_0, s_1. If we denote the matrices associated to each arrow by uppercase letters, then the CS function is

$$f = \mathrm{tr}\big(R_0(B_0D_1 - B_1D_0) + S_0(AB_0D_2 - CD_0) + S_1(AB_1D_2 - CD_1)\big).$$

7. Integrality of generalized DT invariants

In this section, we give the first geometric application of CS functions. The main result is Theorem 7.4, where we show that the L_∞ products vanish at semistable points of the moduli space of sheaves of local surfaces. As a consequence, the generalized Donaldson–Thomas invariants defined by Joyce and Song [2012] are integral on local surfaces.

We only consider sheaves on Y that belong to the category D_ω, i.e., set theoretically supported on X. Furthermore, we assume they are supported in dimension bigger than zero. The integrality of the zero dimensional sheaves has been proved in [Joyce and Song 2012, Section 6.3].

Let L be an ample line bundle on X. The Hilbert polynomial of a coherent sheaf $\mathcal{E}$ on Y is defined to be $\chi(\mathcal{E}\otimes\pi^*L^k)$ for $k \gg 0$. The slope of $\mathcal{E}$, denoted by $\mu(\mathcal{E})$ is defined to be the quotient of the second nonzero coefficient of its Hilbert polynomial by the first. We will adopt the notation of Joyce and Song [loc. cit.]. A sheaf $\mathcal{E}$ is called *τ-stable* if for any proper subsheaf $\mathcal{F}$, the slopes satisfy $\mu(\mathcal{F}) < \mu(\mathcal{E})$. Similarly, $\mathcal{E}$ is called *τ-semistable* if $\mu(\mathcal{F}) \le \mu(\mathcal{E})$. The moduli space of τ semistable sheaves on Y with class $\gamma \in \Gamma$ is denoted by $\mathfrak{M}^\tau(Y,\gamma)$.

Lemma 7.1. *Assume X is a Gorenstein toric Fano stack of dimension two. If $\mathcal{E}$ is a τ-stable sheaf on Y, then $\mathcal{E}$ is supported on X scheme theoretically.*

Proof. Let Z be the scheme theoretical support of a τ-stable sheaf $\mathcal{E}$. There is a trace map $\mathrm{tr}_\mathcal{E} : \underline{\mathrm{Hom}}^\bullet(\mathcal{E},\mathcal{E}) \to \mathcal{O}_Z$ and a map $i_\mathcal{E} : \mathcal{O}_Z \to \underline{\mathrm{Hom}}^0(\mathcal{E},\mathcal{E})$ such that $\mathrm{tr}_\mathcal{E}\circ i_\mathcal{E} = \mathrm{rk}_Z(\mathcal{E})$. (We refer the reader to [Huybrechts and Lehn 1997, §10.1] for the precise definitions of these maps.) Since the rank of $\mathcal{E}$ (over Z) is positive, $i_\mathcal{E}$ must be an injection. By local-to-global spectral sequence, there is an injection $H^0(Z,\mathcal{O}_Z) \to \mathrm{Ext}^0_Z(\mathcal{E},\mathcal{E})$.

By stability, $\mathcal{E}$ must be pure. We first assume $\mathcal{E}$ is supported in dimension two. Then Z is an order n thickening of X in the normal direction. The cohomology group $H^0(Z,\mathcal{O}_Z)$ is equal to $\bigoplus_{i=0}^n H^0(X,\omega_X^{-i})$. The dimension of $H^0(X,\omega_X^{-1})$ can be identified with number of lattice points in $\Delta^\vee$ in $M_\mathbb{R} := \mathrm{Hom}(M,\mathbb{R})$, where M is the dual lattice of N. Recall that the polytope supporting the fan Σ lives in $N_\mathbb{R}$. In general, $\Delta^\vee$ is only a rational polytope. However, since the origin is always in the interior of $\Delta^\vee$, the dimension of $H^0(X,\omega_X^{-1})$ is at least one. Therefore, the dimension of $H^0(Z, \mathcal{O}_Z)$ is strictly bigger than one. We get a contradiction since a stable sheaf can only have one dimensional endomorphisms.

Now, let Z be a thickening of a divisor C in X. Similarly, it suffices to show that $H^0(C, \omega_X^{-1})$ is nonzero. There is a morphism $H^0(X, \omega_X^{-1}) \to H^0(C, \omega_X^{-1})$. Let us denote the toric divisors of X by E_i. Because C is an effective divisor, it can be written as a linear combination $\sum_i a_i E_i$ where all a_i are nonnegative integers and at least one of them is positive. Consider the short exact sequence

$$0 \longrightarrow I_C \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

The cohomology group $H^0(C, \omega_X^{-1})$ vanishes only if the morphism

$$H^0(X, I_C \otimes \omega_X^{-1}) \to H^0(X, \omega_X^{-1})$$

is a bijection. The first group can be written as $H^0\big(X, \mathcal{O}\big(\sum_i (1-a_i)E_i\big)\big)$. The Gorenstein condition implies that Δ and $\Delta^\vee$ are both lattice polytopes. The dimension of $H^0\big(X, \mathcal{O}\big(\sum_i (1-a_i)E_i\big)\big)$ is equal to the number of lattice points inside the polytope that is obtained from $\Delta^\vee$ by translating its faces towards origin. Because at least one a_i is positive and $\Delta^\vee$ is a lattice polytope to begin with, the number of lattice points will decrease when one face is pushed. As a consequence, $H^0(C, \omega_X^{-1})$ is nonzero. $\square$

Lemma 7.2. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be τ-semistable sheaves on X such that $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$. Then*, $\mathrm{Ext}^2(\mathcal{E}_1, \mathcal{E}_2) = 0$.

Proof. By Serre duality, $\mathrm{Ext}^2_X(\mathcal{E}_1, \mathcal{E}_2) = \mathrm{Hom}_X(\mathcal{E}_2, \mathcal{E}_1 \otimes \omega_X)^*$. Because ω_X^{-1} is ample and $\mathcal{E}_1$, $\mathcal{E}_2$ have dimension bigger than zero, $\mu(\mathcal{E}_1 \otimes \omega_X) < \mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$. Hence, $\mathrm{Ext}^2(\mathcal{E}_1, \mathcal{E}_2)$ vanishes by stability. $\square$

Lemma 7.1 doesn't hold for semistable sheaves. For example, if we take a proper but nonreduced curve in Y, then its structure sheaf can be semistable but not stable.

Lemma 7.3. *Let $\mathcal{E}$ be a τ-semistable sheaf on Y. Then the restriction $\mathcal{E}|_X$ is a semistable sheaf on X.*

Proof. Because $\mathcal{E}$ is set theoretically supported on X, it can be written as consequent extensions of stable sheaves on X with the same slope (the Jordan–Holder filtration). Furthermore, the natural morphism $\mathcal{E} \to \mathcal{E}|_X$ is always a surjection of sheaves. Since $\mu(\mathcal{E}|_X) = \mu(\mathcal{E})$, any quotient sheaf that destabilizes $\mathcal{E}|_X$ will destabilize $\mathcal{E}$ as well. $\square$

From now on, we will assume X is Gorenstein.

Theorem 7.4. *The L_∞ products $\bar{\mu}_k$ of L_ω vanish at semistable points.*

Proof. Let $\mathcal{E}$ be a τ-semistable sheaf on Y. It follows from Theorem 5.1 that we can define a cyclic L_∞ algebra L_ω such that $\mathcal{E}$ is mapped to a Maurer–Cartan element $\bar{x}$.

Moreover, $\mathrm{Ext}^i_Y(\mathcal{E},\mathcal{E})$ coincides with $H^i(L_\omega,\delta^{\bar{x}})$. The L_∞ products $\bar{\mu}_k$ uniquely defines L_∞ products on $H^\bullet(L_\omega,\delta^{\bar{x}})$ up to L_∞ isomorphisms. We say that $\bar{\mu}_k$ vanish at x if they vanish after passing to $H^\bullet(L_\omega,\delta^{\bar{x}})$.

An MC element $\bar{x}$ of L_ω decomposes into (x,ϵ), with respect to the decomposition $L^1_\omega = L^1 \oplus (L^2)^*$. It follows from Theorem 4.2 that x is an MC element of L. The cohomology $H^\bullet(L_\omega,\delta^{\bar{x}})$ can be computed as the cohomology of the total complex of

$$\begin{tikzcd} (L^2)^* \arrow[r] & (L^1)^* \arrow[r] & (L^0)^* \\ L^0 \arrow[r] \arrow[u] & L^1 \arrow[r] \arrow[u] & L^2 \arrow[u] \end{tikzcd}$$

where the horizontal differential is δ^x and the vertical differential is induced by $[\epsilon,-]$.

If $\bar{x}$ is the image of a sheaf of the form $\iota_*\mathcal{E}$ for some sheaf $\mathcal{E}$ on X then $\bar{x}=(x,0)$. In that case, the associated spectral sequence will degenerate at E_1 page.

If $\epsilon \neq 0$, we need to pass to the E_2 page of

$$\begin{tikzcd} H^2(L,\delta^x)^* \arrow[r, "0"] & H^1(L,\delta^x)^* \arrow[r] & H^0(L,\delta^x)^* \\ H^0(L,\delta^x) \arrow[r] \arrow[u, "{[\epsilon,-]}"] & H^1(L,\delta^x) \arrow[r] \arrow[u, "{[\epsilon,-]}"] & H^2(L,\delta^x) \arrow[u, "{[\epsilon,-]}"] \end{tikzcd}$$

The MC element $(x,0)$ is exactly the one corresponding to $\mathcal{E}|_X$. So $H^i(L,\delta^x) = \mathrm{Ext}^i_X(\mathcal{E}|_X,\mathcal{E}|_X)$. Now by Lemmas 7.3 and 7.2, $H^2(L,\delta^x)$ vanishes. By the previous commutative diagram, $H^1(L_\omega,\delta^{\bar{x}})$ and $H^2(L_\omega,\delta^{\bar{x}})$ are equal to the kernel and cokernel of the map

$$H^1(L,\delta^x) \xrightarrow{[\epsilon,-]} H^1(L,\delta^x)^*.$$

The L_∞ structure $\bar{\mu}_k$ on $H^\bullet(L_\omega,\delta^{\bar{x}})$ is obtained from μ_k by transferring. The vanishing of $H^2(L,\delta^x)$ and $H^2(L,\delta^x)^*$ together with Theorem 4.2 implies $\mu_k=0$. Therefore $\bar{\mu}_k$ must vanish after transferring to cohomology with respect to $[\epsilon,-]$. □

Remark 7.5. A corollary of Theorem 7.4 is that the moduli space of τ-semistable sheaves on Y is smooth as an Artin stack since the images of $\bar{\mu}_k$ are nothing but obstructions to smoothness of moduli space.

We are not going to define Joyce's generalized DT invariants and state the general form of the integrality conjecture since it requires too much work. The interested reader can refer to [Joyce and Song 2012] for the full story.

Corollary 7.6. *The generalized Donaldson–Thomas invariants $\widehat{\mathrm{DT}}(\tau)$ for τ-semistable sheaves are integers on local surfaces.*

Proof. The integrality has been proved for the DT invariants of a quiver without relations. The proof can be found in [Joyce and Song 2012] or [Reineke 2011]. By [Joyce and Song 2012, Proposition 7.28], the formal neighborhood of a point of the moduli space of sheaves is isomorphic as formal schemes to a formal neighborhood of the origin of the moduli space of representation of the Ext-quiver (see the proposition for the definition). By Theorems 7.4 and 4.4 the relations of the Ext-quiver vanish when the point is taken to be semistable. Jiang [2010] proved that Behrend function only depends on the formal neighborhood of a moduli space. Therefore, the integrality of the moduli space of semistable sheaves is equivalent to the integrality of the moduli space of representations of quivers without relations. □

8. A dimension reduction formula for virtual motives

In this section, we give the second application of CS functions. We prove a decomposition theorem for virtual motives of $f_{\boldsymbol{d}}$, which partially generalizes [Behrend et al. 2013, Section 3]. If we could identify geometric stability with the appropriate quiver stability condition, then we would obtain a decomposition theorem of virtual motives of Hilbert schemes, which generalizes the most interesting part of [loc. cit.]. However, so far we have no idea how to deal with geometric stability.

Let $\mathbb{L}$ be the motive of the affine line. Given a scheme X, we will denote its motive by $[X]$.

Consider a smooth scheme M with an action of a special algebraic group G, together with a G-invariant regular function $f : M \to \mathbb{C}$. Denef and Loeser [2001] defined the motivic vanishing cycle $[\phi_f]$ in a suitable augmented Grothendieck ring of varieties (called the ring of motivic weights). Since our result is not going to involve the precise definition of this ring, we refer to [Behrend et al. 2013, Section 1] for the precise definition of the ring of motivic weights.

Definition 8.1. [Behrend et al. 2013] In the appropriate ring of motivic weights, we define the *virtual motive of a degeneracy locus* by

$$[\mathrm{crit}^G(f)]_{\mathrm{vir}} := -\mathbb{L}^{-\frac{\dim M-\dim G}{2}} \cdot \frac{[\phi_f]}{[G]}.$$

We will try to get some property of the virtual motive of the CS function $f_{\boldsymbol{d}}$. The following lemma guarantees that the main technical result [Behrend et al. 2013, Proposition 1.11] applies.

Lemma 8.2. *Let $f_{\boldsymbol{d}} : L^1_{\omega,\boldsymbol{d}} \to \mathbb{C}$ be the CS function constructed in Section 5. There is a $\mathbb{C}^*$ action on $L^1_{\omega,\boldsymbol{d}}$ such that*:

(1) *For $\lambda \in \mathbb{C}^*$, $f_{\boldsymbol{d}}(\lambda \cdot z) = \lambda f_{\boldsymbol{d}}(z)$.*

(2) *The limit $\lim_{\lambda\to 0} \lambda \cdot z$ exists in $L^1_{\omega,\boldsymbol{d}}$.*

Proof. Let us choose coordinate $z = (y_1, \dots, y_j, \dots, w_1^*, \dots, w_i^*, \dots)$ on $L^1_{\omega,\boldsymbol{d}}$ with respect to the decomposition $L^1_{\omega,\boldsymbol{d}} = L^1_{\boldsymbol{d}} \oplus (L^2_{\boldsymbol{d}})^*$. As mentioned in Remark 5.4,

$$f_{\boldsymbol{d}} = \sum_{i=1}^{\dim L^2_{\boldsymbol{d}}} a_i(\dots, y_j, \dots)\, w_i^*$$

where the a_i are polynomials in y_j. We define the $\mathbb{C}^*$ action by scaling w_i^*. The limit of the orbits of this one parameter subgroup is $L^1_{\boldsymbol{d}}$. □

Theorem 8.3. *Take X, Y and $L_{\boldsymbol{d}}$, $L_{\omega,\boldsymbol{d}}$ as before. We have the dimension reduction formula*

$$[\phi_{f_d}] = -[(L^2_{\boldsymbol{d}})^*] \cdot [\mathrm{MC}(L_{\boldsymbol{d}})].$$

Proof. The existence of the $\mathbb{C}^*$ action in Lemma 8.2 implies that the Milnor fibration given by $f_{\boldsymbol{d}}$ is Zariski trivial outside the central fiber. Hence

$$[f_{\boldsymbol{d}}^{-1}(1)] = \frac{[L^1_{\omega,\boldsymbol{d}}] - [f_{\boldsymbol{d}}^{-1}(0)]}{(\mathbb{L}-1)}.$$

Furthermore, Lemma 8.2 together with [Behrend et al. 2013, Proposition 1.11] implies that

$$[\phi_{f_d}] = [f_{\boldsymbol{d}}^{-1}(1)] - [f_{\boldsymbol{d}}^{-1}(0)].$$

Recall that

$$f_{\boldsymbol{d}} = \sum_{i=1}^{r} a_i(y_1, \dots, y_j, \dots) \cdot w_i^*,$$

where $r = \dim L^2_{\boldsymbol{d}}$. We can stratify $L^1_{\boldsymbol{d}}$ by the union of $\{a_i = 0 \mid i = 1, \dots, r\}$ and its complement. The first subscheme is nothing but $\mathrm{MC}(L_{\boldsymbol{d}})$. Using this stratification, we obtain

$$\begin{aligned}
[f_{\boldsymbol{d}}^{-1}(0)] &= [(L^2_{\boldsymbol{d}})^*][\mathrm{MC}(L_{\boldsymbol{d}})] + ([L^1_{\boldsymbol{d}}] - [\mathrm{MC}(L_{\boldsymbol{d}})])[(L^2_{\boldsymbol{d}})^*]\mathbb{L}^{-1} \\
&= (1 - \mathbb{L}^{-1})[(L^2_{\boldsymbol{d}})^*][\mathrm{MC}(L_{\boldsymbol{d}})] + \mathbb{L}^{-1}[L^1_{\omega,\boldsymbol{d}}].
\end{aligned}$$

Then we obtain the formula for $[\phi_{f_d}]$:

$$\begin{aligned}
(10)\qquad [\phi_{f_d}] &= [f_{\boldsymbol{d}}^{-1}(1)] - [f_{\boldsymbol{d}}^{-1}(0)] = -[f^{-1}(0)]\frac{\mathbb{L}}{\mathbb{L}-1} + \frac{[L^1_{\omega,\boldsymbol{d}}]}{\mathbb{L}-1} \\
&= -\left(\frac{\mathbb{L}-1}{\mathbb{L}}[(L^2_{\boldsymbol{d}})^*][\mathrm{MC}(L_{\boldsymbol{d}})] + \frac{L^1_{\omega,\boldsymbol{d}}}{\mathbb{L}}\right)\frac{\mathbb{L}}{\mathbb{L}-1} + \frac{[L^1_{\omega,\boldsymbol{d}}]}{\mathbb{L}-1} \\
&= -[(L^2_{\boldsymbol{d}})^*][\mathrm{MC}(L_{\boldsymbol{d}})].
\end{aligned}$$

□

9. Virtual motives of the moduli space of framed representations

In this section, we will compute the virtual motive of the moduli space of framed representations, which is a noncommutative analogue of Hilbert schemes. The main result is the formula

$$Z(t) = \frac{C(\mathbb{L}^{\frac{1}{2}}t)}{C(\mathbb{L}^{-\frac{1}{2}}t)}.$$

where $C(\mathbb{L}^{\frac{1}{2}}t)$ is a generating series defined in (16).

Using the Chern–Simons function we obtained, this is a straightforward generalization of the work in [Behrend et al. 2013] in the case of $\mathbb{C}^3$. The same calculation is also obtained independently by Morrison [2012].

We fix the following notations for motives:

$$[d]_{\mathbb{L}}! := (\mathbb{L}^d - 1)(\mathbb{L}^{d-1} - 1)\cdots(\mathbb{L} - 1), \qquad [\boldsymbol{d}]_{\mathbb{L}}! := \prod_{i=0}^{n} [d_i]_{\mathbb{L}}!,$$

$$\begin{bmatrix} d \\ d' \end{bmatrix} := \frac{[d]_{\mathbb{L}}!}{[d-d']_{\mathbb{L}}!\,[d']_{\mathbb{L}}!}, \qquad \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} := \prod_{i=0}^{n} \begin{bmatrix} d_i \\ d'_i \end{bmatrix}.$$

Let $\mathrm{GL}_{\boldsymbol{d}} = \prod_{i=0}^{n} \mathrm{GL}_{d_i}$ and $\mathrm{Gr}_{\boldsymbol{d}',\boldsymbol{d}} = \prod_{i=0}^{n} \mathrm{Gr}(d'_i, d_i)$. It is easy to show that

$$[\mathrm{GL}_{\boldsymbol{d}}] = \mathbb{L}^{\sum_{i=0}^{n} \binom{d_i}{2}} [\boldsymbol{d}]_{\mathbb{L}}! \quad \text{and} \quad [\mathrm{Gr}_{\boldsymbol{d}',\boldsymbol{d}}] = \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}}.$$

Definition 9.1. Consider the quiver $\mathcal{Q}_\omega$ defined in the previous section. Given a dimension vector $\boldsymbol{d}$, let $V_0, \ldots, V_n$ be the sequence of vector spaces of dimensions $d_0, \ldots, d_n$ over the nodes. A *framed representation* V of $\mathcal{Q}_\omega$ with dimension vector $\boldsymbol{d}$ is a representation of $\mathcal{Q}_\omega$ together with a vector $v = (v_0, \ldots, v_n)$ such that $v_i \in V_i$. A framed representation V is called *cyclic* if $v_0, \ldots, v_n$ generate V.

Denote the submodule generated by v by M_v, and let

$$Y_{\boldsymbol{d}} = \{(A, v) \in L^1_{\omega,\boldsymbol{d}} \times V_0 \times \ldots \times V_n \mid f_{\boldsymbol{d}} = 0\},$$
$$Z_{\boldsymbol{d}} = \{(A, v) \in L^1_{\omega,\boldsymbol{d}} \times V_0 \times \ldots \times V_n \mid f_{\boldsymbol{d}} = 1\}.$$

Then $Y_{\boldsymbol{d}} = \bigsqcup_{\boldsymbol{d}'<\boldsymbol{d}} Y_{\boldsymbol{d}}^{\boldsymbol{d}'}$ and $Z_{\boldsymbol{d}} = \bigsqcup_{\boldsymbol{d}'<\boldsymbol{d}} Z_{\boldsymbol{d}}^{\boldsymbol{d}'}$, where

$$Y_{\boldsymbol{d}}^{\boldsymbol{d}'} = \{(A, v) \in L^1_{\omega,\boldsymbol{d}} \times V_0 \times \ldots \times V_n \mid f_{\boldsymbol{d}} = 0, \mathrm{cl}(M_v) = \boldsymbol{d}'\},$$
$$Z_{\boldsymbol{d}}^{\boldsymbol{d}'} = \{(A, v) \in L^1_{\omega,\boldsymbol{d}} \times V_0 \times \ldots \times V_n \mid f_{\boldsymbol{d}} = 1, \mathrm{cl}(M_v) = \boldsymbol{d}'\}.$$

Now, write $w_{\boldsymbol{d}} = [Y_{\boldsymbol{d}}] - [Z_{\boldsymbol{d}}]$ and $w_{\boldsymbol{d}}^{\boldsymbol{d}'} = [Y_{\boldsymbol{d}}^{\boldsymbol{d}'}] - [Z_{\boldsymbol{d}}^{\boldsymbol{d}'}]$.

Let $|\boldsymbol{d}| = \sum_{i=0}^{n} d_i$. By Theorem 8.3, we have

$$w_{\boldsymbol{d}} = \mathbb{L}^{|\boldsymbol{d}|}[(L^2_{\boldsymbol{d}})^*][\mathrm{MC}(L_{\boldsymbol{d}})]. \tag{11}$$

There is a projection from $Y_{\boldsymbol{d}}^{\boldsymbol{d}'}$ to the Grassmannian $\mathrm{Gr}_{\boldsymbol{d}',\boldsymbol{d}}$, whose fiber is the set

$$\left\{\left(\left(\begin{smallmatrix} A^0 & A' \\ 0 & A^1 \end{smallmatrix}\right), v\right) \;\middle|\; f_{\boldsymbol{d}} = 0\right\},$$

where A^0 are matrices of size $\boldsymbol{d}' \times \boldsymbol{d}'$ (depending on the source and target vertices), A^1 are matrices of size $(\boldsymbol{d}-\boldsymbol{d}') \times (\boldsymbol{d}-\boldsymbol{d}')$ and A' are matrices of size $\boldsymbol{d}' \times (\boldsymbol{d}-\boldsymbol{d}')$. There is an embedding of $L^1_{\omega,\boldsymbol{d}'} \times L^1_{\omega,\boldsymbol{d}-\boldsymbol{d}'}$ into $L^1_{\omega,\boldsymbol{d}}$ by mapping to block diagonal matrices.

The CS function $f_{\boldsymbol{d}}$ satisfies

$$f_{\boldsymbol{d}}\left(\left(\begin{smallmatrix} A^0 & A' \\ 0 & A^1 \end{smallmatrix}\right), v\right) = f_{\boldsymbol{d}'}(A^0, v) + f_{\boldsymbol{d}-\boldsymbol{d}'}(A^1, v).$$

Denote the subgroup of $\mathrm{GL}_{\boldsymbol{d}}$ that preserves these Borel matrices by $B_{\boldsymbol{d},\boldsymbol{d}'}$ and the Euler form of $\mathcal{Q}_\omega$ by χ.

$$\begin{aligned}[][Y_{\boldsymbol{d}}^{\boldsymbol{d}'}] = \frac{[B_{\boldsymbol{d},\boldsymbol{d}'}]}{[\mathrm{GL}_{\boldsymbol{d}'}][\mathrm{GL}_{\boldsymbol{d}-\boldsymbol{d}'}]} \cdot \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} &([Y_{\boldsymbol{d}'}^{\boldsymbol{d}'}] \cdot [Y_{\boldsymbol{d}-\boldsymbol{d}'}] \\ &+ (\mathbb{L}-1) \cdot [Z_{\boldsymbol{d}'}^{\boldsymbol{d}'}] \cdot [Z_{\boldsymbol{d}-\boldsymbol{d}'}]) \cdot \mathbb{L}^{-|\boldsymbol{d}-\boldsymbol{d}'|}.\end{aligned}$$

A similar analysis yields

$$\begin{aligned}[][Z_{\boldsymbol{d}}^{\boldsymbol{d}'}] = \frac{[B_{\boldsymbol{d},\boldsymbol{d}'}]}{[\mathrm{GL}_{\boldsymbol{d}'}][\mathrm{GL}_{\boldsymbol{d}-\boldsymbol{d}'}]} \cdot \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} &([Y_{\boldsymbol{d}'}^{\boldsymbol{d}'}] \cdot [Z_{\boldsymbol{d}-\boldsymbol{d}'}] \\ &+ (\mathbb{L}-2) \cdot [Z_{\boldsymbol{d}'}^{\boldsymbol{d}'}] \cdot [Z_{\boldsymbol{d}-\boldsymbol{d}'}] + [Z_{\boldsymbol{d}'}^{\boldsymbol{d}'}] \cdot [Y_{\boldsymbol{d}-\boldsymbol{d}'}]) \cdot \mathbb{L}^{-|\boldsymbol{d}-\boldsymbol{d}'|}.\end{aligned}$$

The above formulas, combined with (11), yield

$$\begin{aligned} w_{\boldsymbol{d}}^{\boldsymbol{d}'} &= \frac{[B_{\boldsymbol{d},\boldsymbol{d}'}]}{[\mathrm{GL}_{\boldsymbol{d}'}][\mathrm{GL}_{\boldsymbol{d}-\boldsymbol{d}'}]} \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \mathbb{L}^{-|\boldsymbol{d}-\boldsymbol{d}'|} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} (w_{\boldsymbol{d}'}^{\boldsymbol{d}'} \cdot w_{\boldsymbol{d}-\boldsymbol{d}'}) \\ &= \frac{[B_{\boldsymbol{d},\boldsymbol{d}'}][(L^2_{\boldsymbol{d}-\boldsymbol{d}'})^*]}{[\mathrm{GL}_{\boldsymbol{d}'}][\mathrm{GL}_{\boldsymbol{d}-\boldsymbol{d}'}]} \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} [\mathrm{MC}(L_{\boldsymbol{d}-\boldsymbol{d}'})] \cdot w_{\boldsymbol{d}'}^{\boldsymbol{d}'}. \end{aligned} \tag{12}$$

Because $Y_{\boldsymbol{d}} = \bigsqcup_{\boldsymbol{d}'<\boldsymbol{d}} Y_{\boldsymbol{d}}^{\boldsymbol{d}'}$ and $Z_{\boldsymbol{d}} = \bigsqcup_{\boldsymbol{d}'<\boldsymbol{d}} Z_{\boldsymbol{d}}^{\boldsymbol{d}'}$, we get

$$w_{\boldsymbol{d}}^{\boldsymbol{d}} = w_{\boldsymbol{d}} - \sum_{\boldsymbol{d}'<\boldsymbol{d}} w_{\boldsymbol{d}}^{\boldsymbol{d}'}.$$

Let $\tilde{c}_{\boldsymbol{d}} = [\mathrm{MC}(L_{\boldsymbol{d}})]/[\mathrm{GL}_{\boldsymbol{d}}]$. Applying (11) and (12), we obtain the recursion formula

$$
\begin{aligned}
(13)\quad w_{\boldsymbol{d}}^{\boldsymbol{d}} &= \mathbb{L}^{|\boldsymbol{d}|}[(L_{\boldsymbol{d}}^2)^*][\mathrm{MC}(L_{\boldsymbol{d}})] \\
&\quad - \sum_{\boldsymbol{d}'<\boldsymbol{d}} \frac{[B_{\boldsymbol{d},\boldsymbol{d}'}][(L_{\boldsymbol{d}-\boldsymbol{d}'}^2)^*]}{[\mathrm{GL}_{\boldsymbol{d}'}][\mathrm{GL}_{\boldsymbol{d}-\boldsymbol{d}'}]} \cdot \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} [\mathrm{MC}(L_{\boldsymbol{d}-\boldsymbol{d}'})] \cdot w_{\boldsymbol{d}'}^{\boldsymbol{d}'} - \frac{[\phi_{f_{\boldsymbol{d}}^{\mathrm{ss}}}]}{[\mathrm{GL}_{\boldsymbol{d}}]} \\
&= \mathbb{L}^{|\boldsymbol{d}|}[(L_{\boldsymbol{d}}^2)^*]\tilde{c}_{\boldsymbol{d}} + \sum_{\boldsymbol{d}'<\boldsymbol{d}} [(L_{\boldsymbol{d}-\boldsymbol{d}'}^2)^*] \cdot \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \tilde{c}_{\boldsymbol{d}-\boldsymbol{d}'} \cdot \frac{[\phi_{f_{\boldsymbol{d}'}^{\mathrm{ss}}}]}{[\mathrm{GL}_{\boldsymbol{d}'}]}
\end{aligned}
$$

Here $f_{\boldsymbol{d}}^{\mathrm{ss}}$ is the restriction of $f_{\boldsymbol{d}}$ to the semistable loci.

Define the virtual motive of the noncommutative Hilbert scheme $\mathrm{Hilb}^{\boldsymbol{d}}$ by

$$
(14)\qquad [\mathrm{Hilb}^{\boldsymbol{d}}]_{\mathrm{vir}} := -\mathbb{L}^{\frac{\chi(\boldsymbol{d},\boldsymbol{d})-|\boldsymbol{d}|}{2}} \frac{[\phi_{f_{\boldsymbol{d}}^{\mathrm{ss}}}]}{[\mathrm{GL}_{\boldsymbol{d}}]}.
$$

After replacing $\phi_{f_{\boldsymbol{d}}^{\mathrm{ss}}}$ by $[\mathrm{Hilb}^{\boldsymbol{d}}]_{\mathrm{vir}}$, subject to the above formula, we obtain

$$
\begin{aligned}
(15)\quad \mathbb{L}^{|\boldsymbol{d}|}[(L_{\boldsymbol{d}}^2)^*]\tilde{c}_{\boldsymbol{d}} &= \sum_{\boldsymbol{d}'\le\boldsymbol{d}} \mathbb{L}^{-\frac{2\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')+\chi(\boldsymbol{d}',\boldsymbol{d}')}{2}} \mathbb{L}^{\frac{|\boldsymbol{d}'|}{2}} [(L_{\boldsymbol{d}-\boldsymbol{d}'}^2)^*]\tilde{c}_{\boldsymbol{d}-\boldsymbol{d}'} \cdot [\mathrm{Hilb}^{\boldsymbol{d}'}]_{\mathrm{vir}} \\
\mathbb{L}^{\frac{|\boldsymbol{d}|}{2}}[(L_{\boldsymbol{d}}^2)^*]\tilde{c}_{\boldsymbol{d}} &= \sum_{\boldsymbol{d}'\le\boldsymbol{d}} \mathbb{L}^{-\frac{\chi(\boldsymbol{d},\boldsymbol{d})-\chi(\boldsymbol{d}-\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')}{2}} \mathbb{L}^{-\frac{|\boldsymbol{d}-\boldsymbol{d}'|}{2}} [(L_{\boldsymbol{d}-\boldsymbol{d}'}^2)^*]\tilde{c}_{\boldsymbol{d}-\boldsymbol{d}'} \cdot [\mathrm{Hilb}^{\boldsymbol{d}'}]_{\mathrm{vir}} \\
\mathbb{L}^{\frac{\chi(\boldsymbol{d},\boldsymbol{d})}{2}}\mathbb{L}^{\frac{|\boldsymbol{d}|}{2}}[(L_{\boldsymbol{d}}^2)^*]\tilde{c}_{\boldsymbol{d}} &= \sum_{\boldsymbol{d}'\le\boldsymbol{d}} \mathbb{L}^{\frac{\chi(\boldsymbol{d}-\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')}{2}} \mathbb{L}^{-\frac{|\boldsymbol{d}-\boldsymbol{d}'|}{2}} [(L_{\boldsymbol{d}-\boldsymbol{d}'}^2)^*]\tilde{c}_{\boldsymbol{d}-\boldsymbol{d}'} \cdot [\mathrm{Hilb}^{\boldsymbol{d}'}]_{\mathrm{vir}}
\end{aligned}
$$

Define the generating series for $\tilde{c}_{\boldsymbol{d}}$ by

$$
(16)\qquad C(t) = \sum_{\boldsymbol{d}\in\mathbb{Z}_{\ge 0}^{n+1}} \mathbb{L}^{\frac{\chi(\boldsymbol{d},\boldsymbol{d})}{2}} [(L_{\boldsymbol{d}}^2)^*]\tilde{c}_{\boldsymbol{d}} \cdot t^{\boldsymbol{d}}
$$

and the generating series of noncommutative Hilbert schemes by

$$
Z(t) = \sum_{\boldsymbol{d}\in\mathbb{Z}_{\ge 0}^{n+1}} [\mathrm{Hilb}^{\boldsymbol{d}}]_{\mathrm{vir}} \cdot t^{\boldsymbol{d}}.
$$

Then the generating series of Hilbert schemes can be written as

$$
(17)\qquad Z(t) = \frac{C(\mathbb{L}^{\frac{1}{2}}t)}{C(\mathbb{L}^{-\frac{1}{2}}t)}.
$$

Finally, notice that $\mathbb{L}^{\chi(\boldsymbol{d},\boldsymbol{d})/2}[(L_{\boldsymbol{d}}^2)^*]$ is nothing but $\mathbb{L}^{\chi_{\mathcal{Q}}(\boldsymbol{d},\boldsymbol{d})/2}$ for the Euler form of the quiver $\mathcal{Q}$. So $C(t)$ is the generating series of the moduli space of representations of $\mathcal{Q}$ (without stability).

Acknowledgments

I would like to thank Jim Bryan, Kai Behrend and Kentaro Nagao for their interest in my work, comments, conversations, and helpful correspondence. The computations in Section 9 were also obtained independently by Andrew Morrison [2012] — thanks to Andrew Morrison for kindly sharing his preprint. Section 7 was finished during my visit to Imperial college London. It is motivated by a discussion with Martijn Kool. I would like to thank Martijn Kool and Richard Thomas for their very helpful comments. I would also like to thank Dominic Joyce for pointing out a mistake in the earlier draft. This work was partially funded by RGC Early Career Grant No. 27300214.

References

[Aspinwall and Fidkowski 2006] P. S. Aspinwall and L. M. Fidkowski, "Superpotentials for quiver gauge theories", *J. High Energy Phys.* 10 (2006), 047. MR 2007g:81091

[Behrend 2009] K. Behrend, "Donaldson–Thomas type invariants via microlocal geometry", *Ann. of Math.* (2) **170**:3 (2009), 1307–1338. MR 2011d:14098 Zbl 1191.14050

[Behrend et al. 2013] K. Behrend, J. Bryan, and B. Szendrői, "Motivic degree zero Donaldson–Thomas invariants", *Invent. Math.* **192**:1 (2013), 111–160. MR 3032328 Zbl 1267.14008

[Bondal 1990] A. I. Bondal, "Helices, representations of quivers and Koszul algebras", pp. 75–95 in *Helices and vector bundles*, edited by A. N. Rudakov, London Math. Soc. Lecture Note Ser. **148**, Cambridge Univ. Press, 1990. MR 92i:14016 Zbl 0742.14010

[Borisov and Hua 2009] L. Borisov and Z. Hua, "On the conjecture of King for smooth toric Deligne–Mumford stacks", *Adv. Math.* **221**:1 (2009), 277–301. MR 2010m:14064 Zbl 1210.14006

[Bridgeland 2005] T. Bridgeland, "T-structures on some local Calabi–Yau varieties", *J. Algebra* **289**:2 (2005), 453–483. MR 2006a:14067 Zbl 1069.14044

[Denef and Loeser 2001] J. Denef and F. Loeser, "Geometry on arc spaces of algebraic varieties", pp. 327–348 in *European Congress of Mathematics, I* (Barcelona, 2000), edited by C. Casacuberta et al., Progr. Math. **201**, Birkhäuser, Basel, 2001. MR 2004c:14037 Zbl 1079.14003

[Huybrechts and Lehn 1997] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics **E31**, Friedr. Vieweg & Sohn, Braunschweig, 1997. MR 98g:14012 Zbl 0872.14002

[Jiang 2010] Y. Jiang, "Motivic Milnor fiber of cyclic L-infinity algebras", preprint, 2010. arXiv 0909.2858

[Joyce and Song 2012] D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants*, vol. 217, Mem. Amer. Math. Soc. **1020**, Amer. Math. Soc., Providence, RI, 2012. MR 2951762 Zbl 1259.14054

[Keller 2006] B. Keller, "A_∞-algebras, modules and functor categories", preprint, 2006. arXiv math/0510508

[Kontsevich and Soibelman 2009] M. Kontsevich and Y. Soibelman, "Notes on A_∞-algebras, A_∞-categories and non-commutative geometry", pp. 153–219 in *Homological mirror symmetry*, edited by A. Kapustin et al., Lecture Notes in Phys. **757**, Springer, Berlin, 2009. MR 2011f:53183 Zbl 1202.81120

[Morrison 2012] A. Morrison, "Motivic invariants of quivers via dimensional reduction", *Selecta Math.* (*N.S.*) **18**:4 (2012), 779–797. MR 3000467 Zbl 1256.14058

[Reineke 2011] M. Reineke, "Cohomology of quiver moduli, functional equations, and integrality of Donaldson–Thomas type invariants", *Compos. Math.* **147**:3 (2011), 943–964. MR 2012i:16031 Zbl 1266.16013

[Segal 2008] E. Segal, "The A_∞ deformation theory of a point and the derived categories of local Calabi–Yaus", *J. Algebra* **320**:8 (2008), 3232–3268. MR 2009k:16016 Zbl 1168.18005

Received May 23, 2014. Revised January 27, 2015.

ZHENG HUA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HONG KONG
POKFULAM
HONG KONG
zheng.hua.zju@gmail.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.149

ON THE FLAG CURVATURE OF A CLASS OF FINSLER METRICS PRODUCED BY THE NAVIGATION PROBLEM

LIBING HUANG AND XIAOHUAN MO

One of the important approaches in discussing Finsler geometry is the navigation problem. In this paper, we determine the flag curvature of a Finsler metric produced from *any* Finsler metric and *any* conformal field in terms of the navigation problem, and therefore we provide a unifying framework for the fundamental equations due to Bao, Robles, and Shen, Cheng and Shen, Foulon, and Mo and Huang.

1. Introduction

The navigation problem (or shortest time problem [Shen 2003]) was first studied by E. Zermelo [1931]. Bao, Robles, and Shen [Bao et al. 2004] classified Randers metrics of constant flag curvature via the navigation problem on Riemannian manifolds. Flag curvature is an important quantity in Finsler geometry because it takes the place of sectional curvature in the Riemannian case [Bao and Chern 1993]. The complete classification of Randers metrics of constant flag curvature, due to Bao, Robles, and Shen, is motivated by the following result [Bao et al. 2004; Chern and Shen 2005]:

Theorem. *A Randers metric F is of constant flag curvature $K = \lambda$ if and only if* (i) *h has constant sectional curvature $\mu = \lambda + c^2$ and* (ii) *V is a homothetic field of h with dilation c, where (h, V) is the navigation data of F.*

Condition (ii) is equivalent to F having constant S-curvature [Shen and Xing 2008; Xing 2005]. Recently, Cheng and Shen [2009] established a relationship between the flag curvature of F and h for a Randers metric F of isotropic S-curvature (see also [Chern and Shen 2005]), generalizing the flag curvature nonincreasing equation of [Bao et al. 2004]. More generally, they obtained a relationship between the Riemann curvature of F and h. Using this, they locally classified Randers metrics of scalar flag curvature with isotropic S-curvature [Cheng and Shen 2009;

Xiaohuan Mo is the corresponding author. Huang was supported by the National Natural Science Foundation of China 11301283. Mo was supported by the National Natural Science Foundation of China 11371032.
MSC2010: 58E20.
Keywords: Finsler metric, flag curvature, conformal field, navigation problem.

Chern and Shen 2005; Shen 2004]. Mo [2008] gave a global classification for these metrics on compact manifolds by using a formula of Cheng and Shen [2009]. It is worth mentioning the recent result by Xing [Shen and Xing 2008] that a Randers metric F is of isotropic S-curvature $S=(n+1)c(x)F$ if and only if V is conformal with respect to h. The theorems of Cheng and Shen [2003] and Mo generalize results previously only known in the case of locally projectively flat Randers metrics with isotropic S-curvature. Recall that all locally projectively flat Finsler metrics are of scalar curvature [Chern and Shen 2005, Proposition 6.1.3].

Recall that a vector field V on a Finsler manifold (M, F) is called *conformal with dilation* $c(x)$ if its flow Φ_t satisfies

$$F(\Phi_t(x), \Phi_{t*}(y)) = e^{2\sigma_t(x)} F(x, y), \quad \forall x \in M,\ y \in T_x M,$$

where $c(x) = [d\sigma_t(x)/dt]_{t=0}$ [Shen and Xia 2012; Huang and Mo 2013]. In particular, V is called a *homothetic* field if c is constant, and V is called a *Killing* field if $c=0$ [Huang and Mo 2011; Mo and Hang 2007].

At the 2004 International Conference on Riemann–Finsler Geometry at Nankai University, P. Foulon announced that if F is a Finsler metric and V is a Killing field, then F and $\widetilde{F}$ have the same flag curvature. Mo and Huang [2007] studied the navigation problem for any Finsler metric F and any homothetic field (for instance, the Funk metric on a strongly convex domain) in the spirit of the flag curvature nonincreasing equation of Bao, Robles, and Shen and the announcement of P. Foulon. They showed that for a homothetic field, the navigation representation satisfies the flag curvature nonincreasing equation. In particular, the navigation problem has the flag curvature preserving property for a Killing field. Applying this result, Hu and Deng [2012] established a principle to classify homogeneous Randers spaces with (almost) isotropic S-curvature and positive flag curvature, and then they gave a complete classification of these homogeneous Randers spaces.

In this paper, we provide a unifying framework for [Bao et al. 2004; Cheng and Shen 2009; Mo and Hang 2007]. We study the Finsler metric $\widetilde{F}$ produced from *any* Finsler metric F and *any* conformal field V in terms of the shortest time problem and give the relation between the flag curvatures of F and $\widetilde{F}$. Precisely we show the following:

Theorem 1.1. *Let $F=F(x,y)$ be a Finsler metric on a manifold M with Cartan torsion A and V be a vector field on M with $F(x, V_x) < 1$. Let $\widetilde{F}=\widetilde{F}(x,y)$ denote the Finsler metric on M defined in (2-2). Suppose that V is conformal with dilation $c(x)$. Then the flag curvatures of $\widetilde{F}$ and F are related by*

$$K_{\widetilde{F}}(y, y\wedge u) - \left[3\frac{y^i c_{x^i}}{\widetilde{F}(x,y)} - c^2 + 2V(c)\right] = K_F(\tilde{y}, \tilde{y}\wedge u) - 2\frac{A_{(x,[\tilde{y}])}(u, \nabla c, u)}{h_{(x,[\tilde{y}])}(u,u)},$$

where $\tilde{y} = y + F(x,\tilde{y})V$ and h is the angular metric of F.

For the definition of a conformal field V with dilation $c(x)$, see Section 2. In Theorem 1.1, we denote the partial derivative $\partial c/\partial x^i$ by c_{x^i}. The case where F is a Riemannian manifold implies a formula of Cheng and Shen [2009], whilst V is homothetic implies the curvature nonincreasing equation of Mo and Huang [2007]. In particular, if $\widetilde{F}$ has constant flag curvature and is of Randers type, our formula has been obtained by Bao, Robles, and Shen [2004].

Our approach to proving Theorem 1.1 is partially in the contact geometry [Blair 2002]. Recall that a Finsler metric is Riemannian if and only if its Cartan torsion vanishes [Chern and Shen 2005].

As an application of Theorem 1.1, we determine the flag curvature of a Finsler metric produced by a generalized Poincaré metric and its nonhomothetic conformal field via the navigation problem (see Section 5).

Finally, we should point out that very recently [Shen and Xia 2012; Xia 2013] established the relationship between the flag curvatures of $\widetilde{F}$ and F, where F is a Randers metric with some special curvature properties and $\widetilde{F}$ is produced from (F, V) via the navigation problem, where V is a conformal field.

2. Preliminaries

Let (M, F) be a Finsler manifold with Hilbert form ω. Let SM be the projective sphere bundle of M, obtained from TM by identifying nonzero vectors which differ from each other by a positive multiplicative factor. It is easy to verify that

$$\omega \wedge (d\omega)^{n-1} \neq 0, \quad n = \dim M.$$

That is, ω defines a contact structure on SM [Chern 1996]. Hence there is a unique vector field X on SM that satisfies $\omega(X) = 1$ and $X \lrcorner (d\omega) = 0$. This vector field X is known as the *Reeb vector field* [Blair 2002; Bryant 2002; Huang and Mo 2011].

Every vector $y \in T_xM\backslash\{0\}$ uniquely determines a covector $p \in T_x^*M\backslash\{0\}$ by

$$p(u) := \frac{1}{2}\frac{d}{dt}\big(F^2(x, y+tu)\big)\Big|_{t=0}, \quad u \in T_xM.$$

The resulting map $L_x^F : y \in T_xM \to p \in T_x^*M$ is called the *Legendre transformation* at x.

Define a nonnegative scalar function $H = H(x, p)$ by

$$H(x, p) := \max_{y\in T_xM\backslash\{0\}} \frac{p(y)}{F(x, y)}. \tag{2-1}$$

The function H is C^∞ on $T^*M\backslash\{0\}$ and $H_x := H|_{T_x^*M}$ is a Minkowski norm on T_x^*M for $x \in M$. Such a function is called a *Cartan metric* [Miron et al. 2001; Mo and Hang 2007] (or co-Finsler metric [Shen 2004; 2002]). The pair (M, H) is called a *Cartan manifold*.

Every covector $p \in T_x^*M\backslash\{0\}$ uniquely determines a vector $y \in T_xM\backslash\{0\}$ by

$$q(y) := \frac{1}{2}\frac{d}{dt}\big(H^2(x, p+tq)\big)\Big|_{t=0}, \quad q \in T_x^*M.$$

The resulting map $L_x^{F*} : p \in T_x^*M \to y \in T_xM$ is called the *inverse Legendre transformation* at x. Indeed L_x^F and L_x^{F*} are inverses of each other. Moreover, they preserve the Minkowski norms $H(x, p) = F(x, L_x^{F*}p)$.

Recently, one of the important approaches in discussing Finsler metrics is the (Zermelo) navigation problem [Bao et al. 2004; Hu and Deng 2012; Huang and Mo 2011; Shen 2003; Zermelo 1931; Xia 2013]. The main technique of the navigation problem is described as follows. Given a Finsler metric F and a vector field V with $F(x, V_x) < 1$, define a new Finsler metric $\widetilde{F}$ by

$$F\left(x, \frac{y}{\widetilde{F}(x,y)} + V_x\right) = 1, \quad \forall x \in M,\ y \in T_xM. \tag{2-2}$$

A *(local) flow* (or *local one-parameter group*) on a manifold M is a map $\Phi : (-\epsilon, \epsilon) \times M \to M$, also denoted by $\Phi_t := \Phi(t, \cdot)$, satisfying

- $\Phi_0 = \mathrm{id} : M \to M$;
- $\Phi_s \circ \Phi_t = \Phi_{s+t}$ for any $s, t \in (-\epsilon, \epsilon)$ with $s + t \in (-\epsilon, \epsilon)$.

Hence, the lift of a flow Φ_t on M is a flow $\hat{\Phi}_t$ on T^*M,

$$\hat{\Phi}_t(x, p) := \big(\Phi_t(x), (\Phi_t^*)^{-1}(p)\big). \tag{2-3}$$

By the relationship between vector fields and flows, (2-3) induces a natural way a lift of a vector field V on M to a vector field X_V^* on T^*M.

A vector field V on a Finsler manifold (M, F) is called *conformal with dilation* $c(x)$ if its flow Φ_t satisfies

$$F\big(\Phi_t(x), \Phi_{t*}(y)\big) = e^{2\sigma_t(x)}F(x, y), \quad \forall x \in M,\ y \in T_xM, \tag{2-4}$$

where $c(x) = [d\sigma_t(x)/dt]_{t=0}$ [Shen and Xia 2012]. In particular, V is called a *homothetic* field if c is constant.

Similarly, a vector field V on a Cartan manifold (M, H) is called *conformal with dilation* $c(x)$ if its flow Φ_t is a conformal transformation on (M, H), i.e.,

$$H\big(\Phi_t(x), (\Phi_t^*)^{-1}(p)\big) = e^{-2\sigma_t(x)}H(x, p), \quad \forall x \in M,\ p \in T_x^*M, \tag{2-5}$$

where $c(x) = [d\sigma_t(x)/dt]_{t=0}$.

Lemma 2.1. *Let V be a conformal field on a Finsler manifold (M, F) with dilation $c(x)$ and H its Cartan metric defined by (2-1). Then V is a conformal field of H with dilation $c(x)$.*

Proof. By using (2-1) and (2-4) we have

$$
\begin{aligned}
H\big(\Phi_t(x), (\Phi_t^*)^{-1}(p)\big) &= \max_{\tilde{y}\in T_{\Phi_t(x)}M\setminus\{0\}} \frac{[(\Phi_t^*)^{-1}(p)](\tilde{y})}{F(\Phi_t(x), \tilde{y})} \\
&= \max_{\tilde{y}\in T_{\Phi_t(x)}M\setminus\{0\}} \frac{p((\Phi_{t*})^{-1}(\tilde{y}))}{F(\Phi_t(x), \tilde{y})} \\
&= \max_{y\in T_xM\setminus\{0\}} \frac{p(y)}{F(\Phi_t(x), \Phi_{t*}(y))} \\
&= \max_{y\in T_xM\setminus\{0\}} \frac{p(y)}{e^{2\sigma_t(x)}F(x, y)} \\
&= e^{-2\sigma_t(x)} \max_{y\in T_xM\setminus\{0\}} \frac{p(y)}{F(x, y)} = e^{-2\sigma_t(x)} H(x, p),
\end{aligned}
$$

where $y := (\Phi_{t*})^{-1}(\tilde{y})$. The lemma follows. □

The Hilbert form $\omega^\flat$ of the co-Finsler metric H is given by

$$\omega^\flat = \frac{p}{H} \tag{2-6}$$

[Mo and Hang 2007]. Let S^*M be the cosphere bundle of M and $\pi : S^*M \to M$ the natural projection. We call $\operatorname{Ker}\pi_*$ the *vertical distribution* of S^*M, denoted by VS^*M.

Lemma 2.2. *For an arbitrary function $f \in C^\infty(S^*M)$, there is a unique vector field X_f on S^*M satisfying*

$$\omega^\flat(X_f) = f, \quad X_f \lrcorner (d\omega^\flat) = -df + X^\flat(f)\omega^\flat. \tag{2-7}$$

This vector field X_f is called the Reeb field associated with f.

Proof. The Hilbert form $\omega^\flat$ defines a contact structure on S^*M. By using [Blair 2002, Theorem 4.4], there exists an almost contact metric structure $(\Phi, X^\flat, \omega^\flat, g)$ such that $g(X, \Phi Y) = d\omega^\flat(X, Y)$. A direct computation tells us that the second equation of (2-7) is equivalent to $\mathcal{L}_{X_f}\omega^\flat = X^\flat(f)\omega^\flat$. Together with [loc. cit., Theorem 5.7], we have $X_f = -\Phi Df + fX^\flat$, where $g(Df, Y) = Y(f)$. □

Remark. (i) It is easy to see that $X_1 = X^\flat$ is known as the Reeb vector field.

(ii) Let $\{e_\alpha, X^\flat, e_{\bar\alpha}\}$ be a locally orthonormal frame on S^*M such that $e_\alpha \in HS^*M$ (see (2-10) below) and $e_{\bar\alpha} \in VS^*M$. By using (2-7), we have

$$X_f = fX^\flat + \Sigma_\alpha e_{\bar\alpha}(f)e_\alpha - \Sigma_\alpha e_\alpha(f)e_{\bar\alpha}.$$

By the definition of VS^*M, we have $e_{\bar\alpha}(f) = 0$ for $f \in C^\infty(M)$. It follows that

$$Y_f := X_f - fX^\flat = -\Phi Df \in VS^*M \quad \text{for } f \in C^\infty(M). \tag{2-8}$$

(iii) Note that the $d\omega^\flat$ adopted here differs from that of D. E. Blair [2002], where $d\omega^\flat$ is defined by

$$d\omega^\flat(X,Y) = \tfrac{1}{2}\big(X(\omega^\flat(Y)) - Y(\omega^\flat(X)) - \omega^\flat([X,Y])\big).$$

In the same work, X_f is called an *infinitesimal contact transformation*.

Let F be a Finsler metric and $\widetilde{F}$ denote the Finsler metric defined in (2-2). With the help of the inverse Legendre transformation at x, we obtain co-Finsler metrics $H(x,p)$ and $\widetilde{H}(x,p)$ respectively. Then H and $\widetilde{H}$ are related by

$$\widetilde{H}(x,p) = H(x,p) - p(V) \tag{2-9}$$

[Mo and Hang 2007]. Furthermore the Hilbert form $\tilde{\omega}^\flat$ of the co-Finsler metric $\widetilde{H}$ satisfies $\tilde{\omega}^\flat = p/\widetilde{H}$. Taking this together with (2-6), we obtain $\operatorname{Ker}\omega^\flat = \operatorname{Ker}\tilde{\omega}^\flat$. The vertical endomorphism $\mathcal{V}^\flat$ is characterized by

$$\mathcal{V}^\flat(v) = 0, \quad \mathcal{V}^\flat(X^\flat) = 0, \quad \mathcal{V}^\flat[X^\flat, v] = -v, \qquad \forall\, v \in VS^*M.$$

The horizontal endomorphism $\mathcal{H}^\flat$ is given by

$$\mathcal{H}^\flat(v) = -[X^\flat, v] - \tfrac{1}{2}\mathcal{V}^\flat[X^\flat,[X^\flat,v]], \quad \mathcal{H}^\flat(X^\flat) = 0, \quad \mathcal{H}^\flat(\mathcal{H}^\flat(v)) = 0$$

for $v \in VS^*M$. The horizontal distribution of S^*M is defined by

$$HS^*M = \mathcal{H}^\flat(VS^*M). \tag{2-10}$$

It is easy to see that

$$TS^*M = HS^*M \oplus VS^*M \oplus \operatorname{Span}\{X^\flat\} = \operatorname{Ker}\omega^\flat \oplus \operatorname{Span}\{X^\flat\}.$$

We denote the projection to VS^*M (resp. HS^*M) by $P^\flat_{\mathcal{V}} := \mathcal{V}^\flat \circ \mathcal{H}^\flat$ (resp. $P^\flat_{\mathcal{H}} := \mathcal{H}^\flat \circ \mathcal{V}^\flat$). Define the Riemann tensor of $\mathcal{R}^\flat$ by

$$\mathcal{R}^\flat(v) = \mathcal{V}^\flat \circ \mathcal{H}^\flat[X^\flat, \mathcal{H}^\flat(v)], \quad v \in VS^*M. \tag{2-11}$$

Then the flag curvature $K^\flat$ is given by

$$K^\flat(v) = \frac{h^\flat(\mathcal{R}^\flat(v), v)}{h^\flat(v,v)}, \quad v \in VS^*M\backslash\{0\}, \tag{2-12}$$

where $h^\flat$ is the angular metric on VS^*M which satisfies

$$h^\flat(v,v) = d\omega^\flat([X^\flat,u],v) = d\omega^\flat(u, \mathcal{H}^\flat(v)).$$

The Cartan torsion $A^\flat$ is characterized by

$$2A^\flat(u,v,w) = u(d\omega^\flat([X^\flat,v],w) + d\omega^\flat([u,[X^\flat,v]],w) + d\omega^\flat([u,[X^\flat,w]],v))$$

for $u, v, w \in VS^*M$. We require the following result in Lemma 3.5, the proof of which is omitted.

Lemma 2.3. *There is a unique affine connection $\nabla : VS^*M \times VS^*M \to VS^*M$ satisfying*

$$\nabla_u v = \mathcal{V}^\flat[u, \mathcal{H}^\flat(v)], \quad \nabla_u v - \nabla_v u = [u, v], \quad (\nabla_u h^\flat)(v, w) = 2A^\flat(u, v, w)$$

for $u, v, w \in VS^*M$.

The following lemma will be used in Section 4.

Lemma 2.4 [Mo and Hang 2007]. *Assume that Cartan metrics H and $\widetilde{H}$ are related by (2-9). Then vertical endomorphisms $\mathcal{V}^\flat$ and $\widetilde{\mathcal{V}}^\flat$ are related by $\mathcal{V}^\flat = \widetilde{\mathcal{V}}^\flat - \widetilde{\mathcal{V}}^\flat(X_V^*) \otimes \omega^\flat$, where X_V^* is the left of V on T^*M.*

3. Conformal transformations

In this section, we establish some properties for a conformal transformation on a Cartan manifold required in next section. For the definition of conformal transformation, see (2-5) above.

Lemma 3.1. *Let φ be a conformal transformation on a Cartan manifold (M, H), i.e., $\hat{\varphi}^* H = e^{-2\sigma(x)} H$, where $\hat{\varphi}(x, p) = (\varphi(x), (\varphi^*)^{-1}(p))$. Then*

$$\hat{\varphi}_* X^\flat = X_{e^{2\sigma(x)}},$$

where $X^\flat$ denotes the Reeb field of H.

Proof. By (2-5) and (2-6), we have

$$\hat{\varphi}^* \omega^\flat = e^{2\sigma(x)} \omega^\flat. \tag{3-1}$$

Hence $\hat{\varphi} : S^*M \to S^*M$ is a contact transformation [Blair 2002]. It follows that

$$\omega^\flat(\hat{\varphi}_* X^\flat) = (\hat{\varphi}^* \omega^\flat) X^\flat = e^{2\sigma(x)} \omega^\flat(X^\flat) = e^{2\sigma(x)}$$

and

$$\begin{aligned}
\hat{\varphi}_* X^\flat \lrcorner (d\omega^\flat) &= X^\flat \lrcorner (\hat{\varphi}^* d\omega^\flat) \\
&= X^\flat \lrcorner [d(\hat{\varphi}^* \omega^\flat)] \\
&= X^\flat \lrcorner [d(e^{2\sigma(x)} \omega^\flat)] \\
&= X^\flat \lrcorner [de^{2\sigma(x)} \wedge \omega^\flat + e^{2\sigma(x)} d\omega^\flat] \\
&= de^{2\sigma(x)}(X^\flat)\omega^\flat - \omega^\flat(X^\flat) de^{2\sigma(x)} + e^{2\sigma(x)} X^\flat \lrcorner (d\omega^\flat) \\
&= -de^{2\sigma(x)} + X^\flat(e^{2\sigma(x)})\omega^\flat.
\end{aligned}$$

The lemma follows from the uniqueness of the Reeb field associated with $e^{2\sigma(x)}$. $\square$

Proposition 3.2. *Let φ be a conformal transformation on a Cartan manifold (M, H), i.e., $\hat{\varphi}^* H = e^{-2\sigma(x)} H$. Then $\hat{\varphi}_* X^\flat = e^{2\sigma(x)}(X^\flat + 2Y_{\sigma(x)})$, where $Y_{\sigma(x)}$ is defined in (2-8).*

Proof. By virtue of (2-8), we conclude that

$$Y_{e^{2\sigma(x)}} = -\Phi D e^{2\sigma(x)} = 2e^{2\sigma(x)}(-\Phi D\sigma(x)) = 2e^{2\sigma(x)} Y_{\sigma(x)}.$$

It follows that

$$\begin{aligned}\hat{\varphi}_* X^\flat &= X_{e^{2\sigma(x)}}\\ &= Y_{e^{2\sigma(x)}} + e^{2\sigma(x)} X^\flat\\ &= 2e^{2\sigma(x)} Y_{\sigma(x)} + e^{2\sigma(x)} X^\flat = e^{2\sigma(x)}(X^\flat + 2Y_{\sigma(x)}).\end{aligned}$$ □

Lemma 3.3. *For a conformal transformation φ on a Cartan manifold (M, H), we have*

$$\hat{\varphi}_* \circ \mathcal{V}^\flat = e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*.$$

Proof. For $v \in VS^*M$ and $\hat{\varphi}_* v \in VS^*M$, it follows that

$$\hat{\varphi}_* \circ \mathcal{V}^\flat(v) = 0 = e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*(v).$$

Similarly, from (i) we have $\hat{\varphi}_* \circ \mathcal{V}^\flat(X^\flat) = e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*(X^\flat)$. For $u \in HS^*M$, we write $u = \mathcal{H}^\flat(v)$, where $v \in VS^*M$. Then

$$\begin{aligned}\hat{\varphi}_* \circ \mathcal{V}^\flat(u) &= \hat{\varphi}_* \circ \mathcal{V}^\flat(-[X^\flat, v] - \tfrac{1}{2}\mathcal{V}^\flat[X^\flat, [X^\flat, v]])\\ &= -\hat{\varphi}_* \circ \mathcal{V}^\flat[X^\flat, v] - \tfrac{1}{2}\hat{\varphi}_* \circ \mathcal{V}^\flat \circ \mathcal{V}^\flat[X^\flat, [X^\flat, v]] = \hat{\varphi}_* v,\end{aligned}$$

and

$$\begin{aligned}e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*(u) &= e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*(\mathcal{H}^\flat(v))\\ &= e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*(-[X^\flat, v] - \tfrac{1}{2}\mathcal{V}^\flat[X^\flat, [X^\flat, v]])\\ &= -e^{-2\sigma(x)} \mathcal{V}^\flat[\hat{\varphi}_* X^\flat, \hat{\varphi}_* v]\\ &= -e^{-2\sigma(x)} \mathcal{V}^\flat[e^{2\sigma(x)} X^\flat, \hat{\varphi}_* v]\\ &= -e^{-2\sigma(x)} e^{2\sigma(x)} \mathcal{V}^\flat[X^\flat, \hat{\varphi}_* v] = \hat{\varphi}_* v.\end{aligned}$$ □

Lemma 3.4. *Write $X^\flat(f) = \dot{f}$ for an arbitrary function $f \in C^\infty(M)$. Then*

$$[X^\flat, X_f] = X_{\dot{f}}.$$

Proof. Simple calculations give

$$\omega^\flat([X^\flat, X_f]) = \dot{f}, \quad [X^\flat, X_f] \lrcorner (d\omega^\flat) = -d\dot{f} + \ddot{f}\omega^\flat.$$

The lemma now follows from the uniqueness of the Reeb field associated with $\dot{f}$. □

Lemma 3.5. *If $f \in C^\infty(M)$ and $v \in VS^*M$, then*

$$\mathcal{V}^\flat[X_{\dot f}, v] = -2A^\flat(Y_f, v), \tag{3-2}$$

where $h^\flat(A^\flat(Y_f, v), u) := A^\flat(v, Y_f, u)$.

Proof. By (2-8) and Lemma 3.4, we have

$$\begin{aligned}\mathcal{V}^\flat X_{\dot f} &= \mathcal{V}^\flat[X^\flat, X_f] \\ &= \mathcal{V}^\flat[X^\flat, Y_f + fX^\flat] \\ &= \mathcal{V}^\flat[X^\flat, Y_f] + \mathcal{V}^\flat[X^\flat(f)X^\flat] \\ &= -Y_f + X^\flat(f)\mathcal{V}^\flat(X^\flat) = -Y_f.\end{aligned} \tag{3-3}$$

Note that $[P^\flat_{\mathcal{V}} X_{\dot f}, v] \in VS^*M$. It follows that

$$\mathcal{V}^\flat[P_{\mathcal{V}} X_{\dot f}, v] = 0. \tag{3-4}$$

Together with (2-7) and (3-3), we obtain

$$\begin{aligned}\mathcal{V}^\flat[X_{\dot f}, v] &= \mathcal{V}^\flat[\dot f X^\flat + P^\flat_{\mathcal{H}} X_{\dot f} + P^\flat_{\mathcal{V}} X_{\dot f}, v] \\ &= \mathcal{V}^\flat[\dot f X^\flat + \mathcal{H}^\flat \circ \mathcal{V}^\flat X_{\dot f}, v] \\ &= \mathcal{V}^\flat[\dot f X^\flat - \mathcal{H}^\flat Y_f, v] = \mathcal{V}^\flat[\dot f X^\flat, v] - \mathcal{V}^\flat[\mathcal{H}^\flat Y_f, v].\end{aligned} \tag{3-5}$$

On the other hand,

$$\mathcal{V}^\flat[\dot f X^\flat, v] = -v(\dot f)\mathcal{V}^\flat(X^\flat) + \dot f \mathcal{V}^\flat[X^\flat, v] = -\dot f v.$$

Plugging this into (3-5) yields $\mathcal{V}^\flat[X_{\dot f}, v] = -\dot f v + \mathcal{V}^\flat[v, \mathcal{H}^\flat Y_f]$. It follows that

$$\begin{aligned}h^\flat(\mathcal{V}^\flat[X_{\dot f}, v], u) &= -\dot f h^\flat(v, u) + h^\flat(\mathcal{V}^\flat[v, \mathcal{H}^\flat Y_f], u) \\ &= -\dot f h^\flat(v, u) + h^\flat(\nabla_v Y_f, u).\end{aligned} \tag{3-6}$$

By Lemma 2.3, we have

$$\begin{aligned}h^\flat(\nabla_v Y_f, u) &= -(\nabla_v h^\flat)(Y_f, u) - h^\flat(Y_f, \nabla_v u) + v(h^\flat(Y_f, u)) \\ &= -2A^\flat(v, Y_f, u) - h^\flat(Y_f, \nabla_v u) + v(h^\flat(Y_f, u)).\end{aligned} \tag{3-7}$$

By a straightforward computation, one obtains

$$h^\flat(Y_f, v) = -\mathcal{H}^\flat(v)(f) = -df(\mathcal{H}^\flat(v)), \quad v \in VS^*M.$$

It follows that

$$h^\flat(Y_f, \nabla_v u) = -(P^\flat_{\mathcal{H}}[v, \mathcal{H}^\flat(u)])(f) \tag{3-8}$$

and

$$v(h^\flat(Y_f, u)) = -(P^\flat_{\mathcal{H}}[v, \mathcal{H}^\flat(u)])(f) + \dot f h^\flat(u, v), \tag{3-9}$$

where we have used $h^\flat(u,v) = -\omega^\flat[u, \mathcal{H}^\flat(v)]$. Substituting (3-8) and (3-9) into (3-7) and then combining it with (3-6), we have (3-2). □

Proposition 3.6. *For a conformal transformation φ on a Cartan manifold (M, H), we have*

$$\hat{\varphi}_*\mathcal{H}^\flat(v) = e^{2\sigma(x)}[\mathcal{H}^\flat(\hat{\varphi}_* v) + 2\dot{\sigma}\hat{\varphi}_* v - 2A^\flat(Y_\sigma, \hat{\varphi}_* v)]. \tag{3-10}$$

Proof. By Lemma 3.3, we have

$$\begin{aligned}\hat{\varphi}_*\mathcal{H}^\flat(v) &= -\hat{\varphi}_*[X^\flat, v] - \tfrac{1}{2}\hat{\varphi}_* \circ \mathcal{V}^\flat[X^\flat, [X^\flat, v]] \\ &= -[\hat{\varphi}_* X^\flat, \hat{\varphi}_* v] - \tfrac{1}{2}e^{-2\sigma(x)}\mathcal{V}^\flat \circ \hat{\varphi}_*[X^\flat, [X^\flat, v]] \\ &= -[e^{2\sigma(x)}(X^\flat + 2Y_{\sigma(x)}), \hat{\varphi}_* v] + (I),\end{aligned} \tag{3-11}$$

where

$$\begin{aligned}(I) &= -\tfrac{1}{2}e^{-2\sigma(x)}\mathcal{V}^\flat[\hat{\varphi}_* X^\flat, \hat{\varphi}_*[X^\flat, v]] \\ &= -\tfrac{1}{2}e^{-2\sigma(x)}\mathcal{V}^\flat[e^{2\sigma(x)}(X^\flat + 2Y_{\sigma(x)}), \hat{\varphi}_*[X^\flat, v]] \\ &= -\tfrac{1}{2}e^{-2\sigma(x)}\mathcal{V}^\flat\big(-\hat{\varphi}_*[X^\flat, v](e^{2\sigma(x)})(X^\flat + 2Y_{\sigma(x)})\big) \\ &\qquad\qquad - \tfrac{1}{2}\mathcal{V}^\flat[X^\flat + 2Y_{\sigma(x)}, [\hat{\varphi}_* X^\flat, \hat{\varphi}_* v]] \\ &= -\tfrac{1}{2}e^{-2\sigma(x)}\big(-\hat{\varphi}_*[X^\flat, v](e^{2\sigma(x)})\mathcal{V}^\flat(X^\flat + 2Y_{\sigma(x)})\big) \\ &\qquad\qquad - \tfrac{1}{2}\mathcal{V}^\flat[X^\flat + 2Y_{\sigma(x)}, e^{2\sigma(x)}[X^\flat + 2Y_{\sigma(x)}, \hat{\varphi}_* v]] \\ &= -\tfrac{1}{2}\big((II) + e^{2\sigma(x)}\mathcal{V}^\flat[X^\flat + 2Y_{\sigma(x)}, [X^\flat + 2Y_{\sigma(x)}, \hat{\varphi}_* v]]\big),\end{aligned} \tag{3-12}$$

and

$$\begin{aligned}(II) &= \mathcal{V}^\flat(X^\flat + 2Y_{\sigma(x)})(e^{2\sigma(x)})[X^\flat + 2Y_{\sigma(x)}, \hat{\varphi}_* v] \\ &= X^\flat(e^{2\sigma(x)})\big(\mathcal{V}^\flat[X^\flat, \hat{\varphi}_* v] + 2\mathcal{V}^\flat[Y_{\sigma(x)}, \hat{\varphi}_* v]\big) = -X^\flat(e^{2\sigma(x)})\hat{\varphi}_* v.\end{aligned}$$

Plugging this into (3-12) and combining with (3-11), we obtain

$$\hat{\varphi}_*\mathcal{H}^\flat(v) = e^{2\sigma(x)}\big(\mathcal{H}^\flat(\hat{\varphi}_* v) - [Y_{\sigma(x)}, \hat{\varphi}_* v] + X^\flat(v)\hat{\varphi}_* v - \mathcal{V}^\flat[Y_{\sigma(x)}, [X^\flat, \hat{\varphi}_* v]]\big). \tag{3-13}$$

By using the Jacobi identity and Lemma 3.4, we have

$$\begin{aligned}-\mathcal{V}^\flat[Y_{\sigma(x)}, [X^\flat, \hat{\varphi}_* v]] &= \mathcal{V}^\flat[X^\flat, [\hat{\varphi}_* v, Y_{\sigma(x)}]] - \mathcal{V}^\flat[\hat{\varphi}_* v, [X^\flat, Y_{\sigma(x)}]] \\ &= -[\hat{\varphi}_* v, Y_{\sigma(x)}] - \mathcal{V}^\flat[\hat{\varphi}_* v, X_{\dot{\sigma}}].\end{aligned}$$

Plugging this into (3-13) and using Lemma 3.5, we get (3-10). □

4. Conformal navigation problems

We call the navigation problem (2-2) *conformal* if V is a conformal field. In this section, we explore some properties of conformal navigation problems and prove Theorem 1.1.

Lemma 4.1. *Let V be a conformal field on a Cartan manifold (M, H) with dilation $c(x)$. Let $\widetilde{H}$ be the Cartan metric given in (2-9). Then for $v \in VS^*M$*

$$\mathcal{H}^\flat(v) = \widetilde{\mathcal{H}}^\flat(v) - cv, \tag{4-1}$$

where $\mathcal{H}^\flat$ (resp. $\widetilde{\mathcal{H}}^\flat$) is the horizontal endomorphism of H (resp. $\widetilde{H}$).

Proof. By [Mo and Hang 2007, Lemma 4.10], we have

$$[X^\flat, v] \in \operatorname{Ker} \omega^\flat = HS^*M \oplus VS^*M, \quad [X^\flat, [X^\flat, v]] \in \operatorname{Ker} \omega^\flat. \tag{4-2}$$

Together with Lemma 2.4 we get

$$-\mathcal{H}^\flat(v) = [X^\flat, v] + \tfrac{1}{2}\widetilde{\mathcal{V}}^\flat[X^\flat, [X^\flat, v]]. \tag{4-3}$$

According to [loc. cit., Lemma 6.2], the Reeb fields of $X^\flat$ and $\widetilde{X}^\flat$ satisfy

$$X^\flat = \widetilde{X}^\flat + X_V^*, \tag{4-4}$$

where

$$X_V^* = v^i \frac{\partial}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial p_i},$$

with $V = v^i(\partial/\partial x^i)$. It follows that

$$\begin{aligned} \widetilde{\mathcal{V}}^\flat[X^\flat, [X^\flat, v]] &= \widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [X^\flat, v]] + \widetilde{\mathcal{V}}^\flat[X_V^*, [X^\flat, v]] \\ &= \widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [\widetilde{X}^\flat, v]] + \widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [X_V^*, v]] + \widetilde{\mathcal{V}}^\flat[X_V^*, [X^\flat, v]]. \end{aligned} \tag{4-5}$$

Let $\hat{\varphi}_t$ be flow of X_V^*. Then $(\hat{\varphi}_t)_* v$ is vertical for $v \in VS^*M$. Hence,

$$[X_V^*, v] := \lim_{t \to 0} \frac{v - (\hat{\varphi}_t)_* v}{t} \tag{4-6}$$

is also vertical. It follows that

$$\widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [X_V^*, v]] = -[X_V^*, v]. \tag{4-7}$$

By using the Jacobi identity, we have

$$[X_V^*, [X^\flat, v]] = -[X^\flat, [v, X_V^*]] - [v, [X_V^*, X^\flat]]. \tag{4-8}$$

Now we assume that V is a conformal field of Cartan metric H with dilation $c(x)$; that is, the flow φ_t of V satisfies

$$\hat{\varphi}_t^* H = e^{-2\sigma_t(x)} H, \quad c(x) = \left[\frac{d\sigma_t(x)}{dt}\right]_{t=0}. \tag{4-9}$$

Differentiating the first of these equations with respect to t at $t = 0$ yields

$$
\begin{aligned}
-2c(x)H &= \frac{\partial}{\partial t}(e^{-2\sigma_t(x)}H)\Big|_{t=0} \\
&= \frac{\partial}{\partial t}(\hat{\varphi}_t^* H)\Big|_{t=0} \\
&= \frac{\partial}{\partial t}(H \circ \hat{\varphi}_t)\Big|_{t=0} \\
&= \frac{\partial \hat{\varphi}_t}{\partial t}\Big|_{t=0} H = X_V^*(H).
\end{aligned}
$$

Recall that $VS^*M = \operatorname{Ker} \pi_* = \{v \in TSM \mid v(f) = 0,\ \forall f \in C^\infty(M) \subset C^\infty(S^*M)\}$. Together with (4-2), we have

$$[v, 2cX^\flat] = v(2c)X^\flat - 2c[X^\flat, v] = -2c[X^\flat, v] \in \operatorname{Ker} \omega^\flat.$$

Note that the vertical distribution is involutive. We obtain

$$\widetilde{\mathcal{V}}^\flat\left[v, \frac{\partial c}{\partial x^i} H \frac{\partial}{\partial p_i}\right] = 0.$$

A direct calculation (see [Huang and Mo 2011, Lemma 3.2]) gives the formula

$$[X^\flat, X_V^*] = 2cX^\flat - 2\frac{\partial c}{\partial x^i} H \frac{\partial}{\partial p_i}.$$

By Lemma 2.4, we obtain

$$\widetilde{\mathcal{V}}^\flat[v, [X_V^*, X^\flat]] = 2\widetilde{\mathcal{V}}^\flat\left[v, \frac{\partial c}{\partial x^i} H \frac{\partial}{\partial p_i}\right] - \widetilde{\mathcal{V}}^\flat[v, 2cX^\flat] = -2cv.$$

Together with (4-6) and (4-8), we have

$$
\begin{aligned}
\widetilde{\mathcal{V}}^\flat[X_V^*, [X^\flat, v]] &= -\widetilde{\mathcal{V}}^\flat[X^\flat, [v, X_V^*]] - \widetilde{\mathcal{V}}^\flat[v, [X_V^*, X^\flat]] \\
&= -\mathcal{V}^\flat[X^\flat, [v, X_V^*]] + 2cv = [v, X_V^*] + 2cv.
\end{aligned}
$$

Plugging this and (4-7) into (4-5) yields

$$
\begin{aligned}
\widetilde{\mathcal{V}}^\flat[X^\flat, [X^\flat, v]] &= \widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [\widetilde{X}^\flat, v]] - [X_V^*, v] + [v, X_V^*] + 2cv \\
&= \widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [\widetilde{X}^\flat, v]] - 2[X_V^*, v] + 2cv.
\end{aligned}
$$

Substituting this into (4-3) and using (4-4), we deduce that

$$
\begin{aligned}
-\mathcal{H}^\flat(v) &= [X^\flat, v] + \tfrac{1}{2}\big(\widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [\widetilde{X}^\flat, v]] - 2[X_V^*, v] + 2cv\big) \\
&= [\widetilde{X}^\flat, v] + \tfrac{1}{2}\widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [\widetilde{X}^\flat, v]] + cv = -\widetilde{\mathcal{H}}^\flat(v) + cv.
\end{aligned}
$$

This gives (4-1). □

Lemma 4.2. *Let V be a conformal field on a Cartan manifold (M, H) with dilation $c(x)$. Let $\widetilde{H}$ be the Cartan metric given in (2-9). Then on $HS^*M \oplus VS^*M$, we have $P^\flat_{\widetilde{\mathcal{V}}} = P^\flat_{\mathcal{V}} - c\widetilde{\mathcal{V}}^\flat = P^\flat_{\mathcal{V}} - c\mathcal{V}^\flat$, where $P^\flat_{\mathcal{V}}$ (resp. $P^\flat_{\widetilde{\mathcal{V}}}$) is the projection of H (resp. $\widetilde{H}$).*

Proof. The second equality follows from Lemma 2.4. For $v \in VS^*M$,

$$\widetilde{\mathcal{V}}^\flat(v) = 0, \qquad P^\flat_{\widetilde{\mathcal{V}}}(v) = P^\flat_{\mathcal{V}}.$$

It follows that

$$P^\flat_{\widetilde{\mathcal{V}}}(v) = (P^\flat_{\mathcal{V}} - c\widetilde{\mathcal{V}}^\flat)(v), \qquad \forall\, v \in VS^*M.$$

For $u \in HS^*M$, we write $u \in \mathcal{H}^\flat(v)$, where $v \in VS^*M$. By the definition of $\widetilde{\mathcal{H}}^\flat$ and Lemma 4.1, we obtain

$$\begin{aligned}
P^\flat_{\widetilde{\mathcal{V}}}(u) + c\widetilde{\mathcal{V}}^\flat(u) &= \widetilde{\mathcal{V}}^\flat \circ \widetilde{\mathcal{H}}^\flat(\mathcal{H}^\flat(v)) + c\widetilde{\mathcal{V}}^\flat(\mathcal{H}^\flat(v)) \\
&= \widetilde{\mathcal{V}}^\flat \circ \widetilde{\mathcal{H}}^\flat(\widetilde{\mathcal{H}}^\flat(v) - cv) + c\widetilde{\mathcal{V}}^\flat(\widetilde{\mathcal{H}}^\flat(v) - cv) \\
&= -c\widetilde{\mathcal{V}}^\flat \circ \widetilde{\mathcal{H}}^\flat(v) + \widetilde{\mathcal{V}}^\flat \circ \widetilde{\mathcal{H}}^\flat(v) - c^2\widetilde{\mathcal{V}}^\flat(v) \\
&= 0 = P^\flat_{\mathcal{V}}(u).
\end{aligned}$$ □

Proposition 4.3. *Let V be a conformal field of H with dilation $c(x)$. Then*

$$[X^*_V, \mathcal{H}^\flat(v)] = -2c\mathcal{H}^\flat(v) + \mathcal{H}^\flat[X^*_V, v] - 2\dot{c}v + 2A^\flat(Y_c, v). \tag{4-10}$$

Proof. By using Proposition 3.6, we have

$$\begin{aligned}
[X^*_V, \mathcal{H}^\flat(v)] &= -\frac{d}{dt}\Big|_{t=0} \hat{\varphi}_{t*}\mathcal{H}^\flat(v) \\
&= -\frac{d}{dt}\Big|_{t=0} \big(e^{2\sigma_t(x)}\big[\mathcal{H}^\flat(\hat{\varphi}_{t*}v) + 2\dot{\sigma}_t\hat{\varphi}_{t*}v - 2A^\flat(Y_{\sigma_t}, \hat{\varphi}_{t*}v)\big]\big),
\end{aligned} \tag{4-11}$$

where φ_t is the flow of V. By direct calculations, we have

$$-\frac{d}{dt}\Big|_{t=0} \mathcal{H}^\flat(\hat{\varphi}_{t*}v) = \mathcal{H}^\flat[X^*_V, v], \qquad -\frac{d}{dt}\Big|_{t=0} A^\flat(Y_{\sigma_t}, \hat{\varphi}_{t*}v) = A^\flat(Y_c, v),$$
$$-\frac{d}{dt}\Big|_{t=0} (\dot{\sigma}_t\hat{\varphi}_{t*}v) = \frac{d\dot{\sigma}_t}{dt}\Big|_{t=0} \hat{\varphi}_{0*}v + \dot{\sigma}_t\Big|_{t=0} \frac{d}{dt}\hat{\varphi}_{t*}v = X^\flat(c)v = \dot{c}v.$$

Plugging them into (4-11), we have (4-10). □

Proposition 4.4. *Let V be a conformal field on a Cartan manifold (M, H) with dilation $c(x)$. Let $\widetilde{H}$ be the Cartan metric given in (2-9). Then*

$$\tilde{\mathcal{R}}^\flat(v) = \mathcal{R}^\flat(v) + [3\widetilde{X}^\flat(c) - c^2 + 2X^*_V(c)]v - 2A^\flat(Y_c, v), \tag{4-12}$$

where $\mathcal{R}^\flat$ (resp. $\tilde{\mathcal{R}}^\flat$) is the Riemann tensor of H (resp. $\widetilde{H}$).

Proof. From [Mo and Hang 2007, Lemma 4.9], we have

$$P^{\flat}_{\mathcal{V}}[X^{\flat}, v] = \mathcal{V}^{\flat}[X^{\flat}, \mathcal{H}^{\flat}(v)], \quad v \in VS^*M. \tag{4-13}$$

By (2-11), (4-1) and (4-4),

$$\begin{aligned}
\tilde{\mathcal{R}}^{\flat}(v) &= P^{\flat}_{\tilde{\mathcal{V}}}[\widetilde{X}^{\flat}, \tilde{\mathcal{H}}^{\flat}(v)]\\
&= P^{\flat}_{\tilde{\mathcal{V}}}[\widetilde{X}^{\flat}, \mathcal{H}^{\flat}(v) + cv]\\
&= P^{\flat}_{\tilde{\mathcal{V}}}[\widetilde{X}^{\flat}, \mathcal{H}^{\flat}(v)] + P^{\flat}_{\tilde{\mathcal{V}}}[\widetilde{X}^{\flat}, cv]\\
&= P^{\flat}_{\tilde{\mathcal{V}}}[X^{\flat} - X^*_V, \mathcal{H}^{\flat}(v)] + P^{\flat}_{\tilde{\mathcal{V}}}\big(\widetilde{X}^{\flat}(c)v + c[\widetilde{X}^{\flat}, v]\big)\\
&= (I) - P^{\flat}_{\tilde{\mathcal{V}}}[X^*_V, \mathcal{H}^{\flat}(v)] + \widetilde{X}^{\flat}(c)v,
\end{aligned}$$

where

$$\begin{aligned}
(I) :&= P^{\flat}_{\tilde{\mathcal{V}}}[X^{\flat}, \mathcal{H}^{\flat}(v)] + cP^{\flat}_{\tilde{\mathcal{V}}}[\widetilde{X}^{\flat}, v]\\
&= (P^{\flat}_{\mathcal{V}} - c\mathcal{V}^{\flat})[X^{\flat}, \mathcal{H}^{\flat}(v)] + c(P^{\flat}_{\mathcal{V}} - c\mathcal{V}^{\flat})[X^{\flat} - X^*_V, v]\\
&= P^{\flat}_{\mathcal{V}}[X^{\flat}, \mathcal{H}^{\flat}(v)] - c\mathcal{V}^{\flat}[X^{\flat}, \mathcal{H}^{\flat}(v)] + cP^{\flat}_{\mathcal{V}}[X^{\flat}, v]\\
&\qquad - c^2\mathcal{V}^{\flat}[X^{\flat}, v] - cP^{\flat}_{\mathcal{V}}[X^*_V, v] + c^2\mathcal{V}^{\flat}[X^*_V, v]\\
&= \mathcal{R}^{\flat}(v) + c^2 v - c[X^*_V, v],
\end{aligned}$$

where we have used (4-13). It follows that

$$\tilde{\mathcal{R}}^{\flat}(v) = \mathcal{R}^{\flat}(v) - P^{\flat}_{\tilde{\mathcal{V}}}[X^*_V, \mathcal{H}^{\flat}(v)] - c[X^*_V, v] + [\widetilde{X}^{\flat}(c) + c^2]v. \tag{4-14}$$

From (4-4), we have

$$\widetilde{X}^{\flat}(c) = \dot{c} - X^*_V(c). \tag{4-15}$$

By using (4-11) and Lemma 4.2, we obtain

$$\begin{aligned}
P^{\flat}_{\tilde{\mathcal{V}}}[X^*_V, \mathcal{H}^{\flat}(v)] &= 2c^2 P^{\flat}_{\mathcal{V}} - cP^{\flat}_{\mathcal{V}}[X^*_V, v] - 2\dot{c}P^{\flat}_{\mathcal{V}}v\\
&\qquad + 2c\dot{c}\mathcal{V}^{\flat}(v) + 2P^{\flat}_{\mathcal{V}}A^{\flat}(Y_c, v) - 2c\mathcal{V}^{\flat}A^{\flat}(Y_c, v)\\
&= 2c^2 v - c[X^*_V, v] - 2\dot{c}v + 2A^{\flat}(Y_c, v).
\end{aligned}$$

Plugging this and (4-15) into (4-14) yields (4-12). $\square$

Proposition 4.5. *Let V be a conformal field on a Cartan manifold (M, H) with dilation $c(x)$. Let $\widetilde{H}$ be the Cartan metric given in (2-9). Then*

$$\widetilde{K}^{\flat}(v) - [3\widetilde{X}^{\flat}(c) - c^2 + 2V(c)] = K^{\flat}(v) - 2\frac{A^{\flat}(v, Y_c, v)}{h^{\flat}(v, v)}, \tag{4-16}$$

where $K^{\flat}$ (resp. $\widetilde{K}^{\flat}$) is the flag curvature of H (resp. $\widetilde{H}$).

Proof. By [Mo and Hang 2007, Lemma 6.2], we have $h^\flat(v_1,v_2)=(\widetilde{H}/H)\tilde{h}^\flat(v_1,v_2)$. Together with (4-12) and (2-12), this yields

$$\widetilde{K}^\flat(v)=K^\flat(v)+3\widetilde{X}^\flat(c)-c^2+2X_V^*(c)-2\frac{A^\flat(v,Y_c,v)}{h^\flat(v,v)}. \tag{4-17}$$

On the other hand,

$$X_V^*(c)=\left(v^i\frac{\partial}{\partial x^i}-p_j\frac{\partial v^j}{\partial x^i}\frac{\partial}{\partial p_i}\right)c(x)=v^i\frac{\partial c}{\partial x^i}=V(c).$$

Together with (4-17), we have (4-16). □

Proof of Theorem 1.1. Let F be a Finsler metric with flag curvature K, Cartan torsion A and angular metric h. Let V be a conformal field on M with $F(x,V_x)<1$. Let $\widetilde{F}$ be the Finsler metric given in (2-2) with flag curvature $\widetilde{K}$. Then their Cartan metrics are related by (2-9). From Lemma 2.1, we obtain that V is a conformal field of H with dilation $c(x)$. Hence K and $\widetilde{K}$ satisfy (4-16). By (2-8), we have $A^\flat(v,Y_c,v)=-A^\flat(v,\Phi Dc,v)$. Plugging this into (4-16) yields

$$\begin{aligned}[\widetilde{K}^\flat(v)]_{(x,[p])}-[3\widetilde{X}^\flat(c)-c^2+2V(c)]_{(x,[p])}&\\ =[K^\flat(v)]_{(x,[p])}&+2\frac{A^\flat(v,\Phi Dc,v)_{(x,[p])}}{h^\flat(v,v)_{(x,[p])}}.\end{aligned}$$

Pulling back to the sphere bundle, we have

$$[\widetilde{K}(u)]_{(x,[y])}-\left[3\frac{y^ic_{x^i}}{\widetilde{F}}-c^2+2V(c)\right]=[K(u)]_{(x,[\tilde{y}])}-2\frac{A(u,\nabla c,u)_{(x,[\tilde{y}])}}{h(u,u)_{(x,[\tilde{y}])}},$$

where $u:=(L_x^{F*})_*v$, $\nabla c:=(L_x^{F*})_*\Phi Dc$ and where we have used $\partial\widetilde{H}/\partial p_i=y^i/\widetilde{F}$. By [Mo and Hang 2007, Lemma 3.9], we get the desired result. □

Remark. (i) The reader should note that the navigation problem adopted here differs from that of [Shen and Xia 2012; Shen 2003], where the navigation problem is defined by $F\big(x,y/\widetilde{F}(x,y)-V\big)=1$; i.e., the $\widetilde{F}$ that we define with (F,V) is precisely the $\widetilde{F}$ that Shen defines with $(F,-V)$.

(ii) We have two special cases of Theorem 1.1:

(1) If V is homothetic, i.e., its dilation $c(x)$ is constant, then $\nabla c=0$ and our formula is reduced to that of Mo and Huang [2007].

(2) If F is Riemannian and has sectional curvature $K=K(x)$, then our formula is reduced to that of Cheng and Shen [2009] (see also [Chern and Shen 2005]).

5. An example

In this section, we determine the flag curvature of a nontrivial example using Theorem 1.1.

Consider the case $\dim M = 2$; so $x = (x^1, x^2)$ and $y = (y^1, y^2)$. In order to avoid the excessive use of parentheses, we shall abbreviate x^1, x^2 as s, t and y^1, y^2 as p, q. Let

$$M := \{(s,t) \in \mathbb{R}^2 \mid t > 1\}.$$

Define $F : TM \to \mathbb{R}$ by

$$F(s,t;\ p,q) := \frac{1}{t}\Phi(p,q), \tag{5-1}$$

where

$$\Phi(p,q) := (p^4 + 2\epsilon p^2 q^2 + q^4)^{1/4}, \quad \epsilon \in (0,3), \tag{5-2}$$

is a Minkowski norm on $\mathbb{R}^2$ (see [Shen 2001, Example 1.1.3]) and F is a Finsler metric on M.

For the Finsler surface (M, F), its Gaussian curvature K takes the place of the flag curvature in general case. A direct calculation shows that the Gaussian curvature of F is given by

$$K_F(s,t;\ p,q) = \frac{[\Phi(p,q)]^2 Q(p,q)}{[\Delta(p,q)]^4}, \tag{5-3}$$

where

(5-4)
$$\begin{aligned}Q(p,q) := {}& \epsilon(2\epsilon^2-3)p^{14} + (17\epsilon^4-42\epsilon^3+18)p^{12}q^2 + \epsilon(8\epsilon^4-50\epsilon^2+21)p^{10}q^4 \\ &+ (9\epsilon^6-89\epsilon^4+81\epsilon^2-36)p^8q^6 - 5\epsilon(5\epsilon^4-4\epsilon^2+6)p^6q^8 \\ &+ \epsilon^2(5\epsilon^4-5\epsilon^2-21)p^4q^{10} + \epsilon^3(5\epsilon^2-12)p^2q^{12} - \epsilon^4 q^{14}\end{aligned}$$

and

$$\Delta(p,q) := \epsilon p^4 + (3-\epsilon^2)p^2q^2 + \epsilon q^4. \tag{5-5}$$

We denote the determinant of the fundamental tensor by g. Then

$$g = \frac{\Delta(p,q)}{t^4[\Phi(p,q)]^4}, \tag{5-6}$$

where we have used (5-1), (5-2) and (5-5). The Cartan form η is given by

$$\eta = \left(F \frac{\partial}{\partial y^j} \log \sqrt{g}\right) dx^j. \tag{5-7}$$

Then the main scalar I of F is given by

$$I(x,y) = \eta(e_1) = \frac{-1}{\sqrt{g}}\left(\left(\frac{\partial}{\partial p}\log\sqrt{g}\right)\left(\frac{F^2}{2}\right)_q - \left(\frac{\partial}{\partial q}\log\sqrt{g}\right)\left(\frac{F^2}{2}\right)_p\right) = \frac{3(1-\epsilon^2)pq}{[\Delta(p,q)]^{3/2}}(p^4-q^4), \tag{5-8}$$

where $\{e_1, e_2\}$ is the Berwald frame with $\omega(e_1) = 0$. Let V denote a vector field on M defined by

$$V := \frac{\partial}{\partial t}. \tag{5-9}$$

By using the isomorphism $T_xM \simeq \mathbb{R}^2$, we have $F(x, V_x) < 1$ on M. Denote the lift of V by X_V. Then

$$X_V = V + y^j \frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial y^i} = V$$

[Huang and Mo 2011]. It follows that

$$X_V(F) = \frac{\partial F}{\partial t} = -\frac{1}{t}F,$$

where we have made use of (5-1). Thus V is conformal with dilation $c = -1/(2t)$ (see [Huang and Mo 2013, Lemma 3.1]). In particular, V is not homothetic.

Now we calculate the following scalar function on SM.

$$\xi(x,y) := \frac{A_{(x,[y])}(u, \nabla c, u)}{h_{(x,[y])}(u,u)}, \tag{5-10}$$

where $u \wedge y \neq 0$. Taking $u = e_1$ we obtain

$$h_{(x,[y])}(e_1,e_1) = 1, \quad A_{(x,[y])}(e_1,e_1,e_1) = I(x,y), \quad A_{(x,[y])}(e_1,e_2,e_1) = 0. \tag{5-11}$$

Define ∇c by

$$\nabla c = \lambda e_1 + \mu e_2, \tag{5-12}$$

where $\{e_1, e_2\}$ is the Berwald frame on M. Then

$$\lambda(x,y) = \boldsymbol{g}_{(x,[y])}(\nabla c, e_1) = \frac{\partial c}{\partial s}\left(-\frac{F_q}{\sqrt{g}}\right) + \frac{\partial c}{\partial t}\frac{F_p}{\sqrt{g}} = \frac{p(p^2+\epsilon q^2)}{2F\sqrt{\Delta(p,q)}t^2}, \tag{5-13}$$

where $\boldsymbol{g}$ denotes the fundamental tensor. From (5-10), (5-11) and (5-12), it follows that

$$\begin{aligned}\xi(x, y) &= \frac{A_{(x,[y])}(e_1, \lambda e_1 + \mu e_2, e_1)}{h_{(x,[y])}(e_1, e_1)} \\ &= \lambda(x, y) A_{(x,[y])}(e_1, e_1, e_1) = \lambda(x, y) I(x, y),\end{aligned}$$

where λ and I are given in (5-13) and (5-8) respectively.

Now we consider the navigation data (F, V), where F and V are defined in (5-1) and (5-9) respectively. (F, V) produces a new Finsler metric $\widetilde{F}$ by

$$F\left(x, \frac{y}{\widetilde{F}(x, y)} + V_x\right) = 1, \quad \forall x \in M,\ y \in T_x M. \tag{5-14}$$

By (5-1), (5-2) and (5-9), (5-14) holds if and only if

$$p^4 + 2\epsilon p^2 (q + \widetilde{F})^2 + (q + \widetilde{F})^4 = t^4 \widetilde{F}^4, \tag{5-15}$$

that is, $\widetilde{F}$ is the unique nonnegative solution of (5-15). By direct calculation we have

$$\frac{y^i c_{x^i}}{\widetilde{F}(x, y)} = \frac{q}{2t^2 \widetilde{F}(x, y)}, \quad -c^2 + 2V(c) = \frac{3}{4t^2}.$$

For the Finsler surface (M, F), F is of scalar flag curvature. Using Theorem 1.1, we obtain that the Gaussian curvature $K_{\widetilde{F}}$ is given by

$$\begin{aligned}K_{\widetilde{F}}(x, y) &= K_F(x, \tilde{y}) + \left[3 \frac{y^i c_{x^i}}{\widetilde{F}(x, y)} - c^2 + 2V(c)\right] - 2 \frac{A_{(x,[\tilde{y}])}(u, \nabla c, u)}{h_{(x,[\tilde{y}])}(u, u)} \\ &= K_F(x, \tilde{y}) + \frac{3q}{2t^2 \widetilde{F}(x, y)} + \frac{3}{4t^2} - 2\lambda(x, \tilde{y}) I(x, \tilde{y}),\end{aligned}$$

where

$$\tilde{y} = y + F(x, y) V = \left(p, q + \frac{(p^4 + 2\epsilon p^2 q^2 + q^4)^{1/4}}{t}\right)$$

and K_F, λ, I are given in (5-3), (5-13) and (5-8) respectively.

Let us take a look at the special case when $\epsilon = 1$,

$$F(s, t;\ p, q) := \frac{(p^2 + q^2)^{1/2}}{t}.$$

F is the famous Poincaré metric of constant sectional curvature $K_F = -1$. In this case, $\widetilde{F}$ is of Randers type and its Gaussian curvature is given by

$$K_{\widetilde{F}}(x, y) = \frac{3}{4t^2}\left(\frac{2q}{\widetilde{F}(x, y)} + 1\right) - 1.$$

References

[Bao and Chern 1993] D. Bao and S.-S. Chern, "On a notable connection in Finsler geometry", *Houston J. Math.* **19**:1 (1993), 135–180. MR 94g:53049 Zbl 0787.53018

[Bao et al. 2004] D. Bao, C. Robles, and Z. Shen, "Zermelo navigation on Riemannian manifolds", *J. Differential Geom.* **66**:3 (2004), 377–435. MR 2005k:58023 Zbl 1078.53073

[Blair 2002] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics **203**, Birkhäuser, Boston, 2002. MR 2002m:53120 Zbl 1011.53001

[Bryant 2002] R. L. Bryant, "Some remarks on Finsler manifolds with constant flag curvature", *Houston J. Math.* **28**:2 (2002), 221–262. MR 2003h:53102 Zbl 1027.53086

[Chen et al. 2003] X. Chen, X. Mo, and Z. Shen, "On the flag curvature of Finsler metrics of scalar curvature", *J. London Math. Soc.* (2) **68**:3 (2003), 762–780. MR 2004h:53103 Zbl 1063.53078

[Cheng and Shen 2009] X. Cheng and Z. Shen, "Randers metrics of scalar flag curvature", *J. Aust. Math. Soc.* **87**:3 (2009), 359–370. MR 2011e:53121 Zbl 1182.53022

[Chern 1996] S.-S. Chern, "Riemannian geometry as a special case of Finsler geometry", pp. 51–58 in *Finsler geometry* (Seattle, WA, 1995), edited by D. Bao et al., Contemp. Math. **196**, Amer. Math. Soc., Providence, RI, 1996. MR 98e:53026 Zbl 0868.53051

[Chern and Shen 2005] S.-S. Chern and Z. Shen, *Riemann–Finsler geometry*, Nankai Tracts in Mathematics **6**, World Scientific, Hackensack, NJ, 2005. MR 2006d:53094 Zbl 1085.53066

[Hu and Deng 2012] Z. Hu and S. Deng, "Homogeneous Randers spaces with isotropic S-curvature and positive flag curvature", *Math. Z.* **270**:3-4 (2012), 989–1009. MR 2892934 Zbl 1239.53095

[Huang and Mo 2011] L. Huang and X. Mo, "On geodesics of Finsler metrics via navigation problem", *Proc. Amer. Math. Soc.* **139**:8 (2011), 3015–3024. MR 2012e:53141 Zbl 1261.53037

[Huang and Mo 2013] L. Huang and X. Mo, "On conformal fields of a Randers metric with isotropic S-curvature", *Illinois J. Math.* **57**:3 (2013), 685–696. MR 3275734 Zbl 1303.53033

[Miron et al. 2001] R. Miron, D. Hrimiuc, H. Shimada, and S. V. Sabau, *The geometry of Hamilton and Lagrange spaces*, Fundamental Theories of Physics **118**, Kluwer, Dordrecht, 2001. MR 2002e:53111 Zbl 1001.53053

[Mo 2008] X. Mo, "A global classification result for Randers metrics of scalar curvature on closed manifolds", *Nonlinear Anal.* **69**:9 (2008), 2996–3004. MR 2009k:53195 Zbl 1237.53029

[Mo and Hang 2007] X. Mo and L. Hang, "On curvature decreasing property of a class of navigation problems", *Publ. Math. Debrecen* **71**:1-2 (2007), 141–163. MR 2008g:53092 Zbl 1136.53022

[Shen 2001] Z. Shen, *Differential geometry of spray and Finsler spaces*, Kluwer, Dordrecht, 2001. MR 2003k:53090 Zbl 1009.53004

[Shen 2002] Z. Shen, "Two-dimensional Finsler metrics with constant flag curvature", *Manuscripta Math.* **109**:3 (2002), 349–366. MR 2003k:53091 Zbl 1027.53093

[Shen 2003] Z. Shen, "Finsler metrics with $\boldsymbol{K}=0$ and $\boldsymbol{S}=0$", *Canad. J. Math.* **55**:1 (2003), 112–132. MR 2004e:53112 Zbl 1035.53104

[Shen 2004] Z. Shen, "Landsberg curvature, S-curvature and Riemann curvature", pp. 303–355 in *A sampler of Riemann–Finsler geometry*, edited by D. Bao et al., Math. Sci. Res. Inst. Publ. **50**, Cambridge Univ. Press, 2004. MR 2005k:53132 Zbl 1074.53063

[Shen and Xia 2012] Z. Shen and Q. Xia, "On conformal vector fields on Randers manifolds", *Sci. China Math.* **55**:9 (2012), 1869–1882. MR 2960866 Zbl 1267.53027

[Shen and Xing 2008] Z. M. Shen and H. Xing, "On Randers metrics with isotropic S-curvature", *Acta Math. Sin.* (*Engl. Ser.*) **24**:5 (2008), 789–796. MR 2010c:53109 Zbl 1151.53065

[Xia 2013] Q. Xia, "On the flag curvature of a class of Randers metric generated from the navigation problem", *J. Math. Anal. Appl.* **397**:1 (2013), 415–427. MR 2969001 Zbl 1254.53045

[Xing 2005] H. Xing, "The geometric meaning of Randers metrics with isotropic S-curvature", *Adv. Math.* (*China*) **34**:6 (2005), 717–730. MR 2006m:53117

[Zermelo 1931] E. Zermelo, "Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung", *Z. Angew. Math. Mech.* **11**:2 (1931), 114–124. Zbl 0001.34101

Received December 15, 2013.

LIBING HUANG
SCHOOL OF MATHEMATICAL SCIENCES
NANKAI UNIVERSITY
TIANJIN, 300071
CHINA
huanglb@nankai.edu.cn

XIAOHUAN MO
KEY LABORATORY OF PURE AND APPLIED MATHEMATICS
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING, 100871
CHINA
moxh@pku.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.169

ANGULAR DISTRIBUTION OF DIAMETERS FOR SPHERES AND RAYS FOR PLANES

NOBUHIRO INNAMI AND YUYA UNEME

Grove and Shiohama used the critical point theory of a distance function to prove the diameter sphere theorem. In light of the angular distribution of minimizing geodesics, we examine and develop the techniques in its proof to make some diameter sphere theorems and study complete noncompact manifolds, using a generalized Toponogov comparison theorem.

1. Introduction

Let M be a compact Riemannian n-manifold with distance $d(\cdot,\cdot)$ induced from its Riemannian metric. Let $\operatorname{diam}(M)=\max\{d(x,y)\mid x,y\in M\}$ denote its diameter. Grove and Shiohama [1977] have proved that if the sectional curvature of M is greater than or equal to 1 and $\operatorname{diam}(M)>\pi/2$, then M is homeomorphic to an n-sphere, using the critical point theory of a distance function. From this point of view, the unit sphere has nice properties as a reference surface. We examine those properties to make some other diameter sphere theorems and show some conditions under which M is diffeomorphic to an n-plane. In order to do this, we introduce the angular distribution of minimizing geodesic segments and the reference map from M into a reference surface. The angular distribution measures how the minimizing geodesics are distributed in M. The reference map will be used to compare the geometry on M with the geometry on a reference surface $\widetilde{M}$ through the generalized Toponogov comparison theorem.

In Section 2, we define the angular distribution of minimizing geodesic segments connecting two points and the reference map $\Phi_{p,q}$ for q in (M,p) with a base point at p into a reference surface $(\widetilde{M},\tilde{p})$ of revolution with vertex $\tilde{p}$. We propose a domain $D(\tilde{p},\tilde{q})\subset\widetilde{M}$ such that the generalized Toponogov comparison theorem is valid if $\Phi_{p,q}(M)\subset D(\tilde{p},\tilde{q})$. Using this terminology we state some theorems.

In Section 3, we summarize some properties of geodesics in a surface of revolution and present the generalized Toponogov comparison theorem of the form used in

Innami's research was partially supported by Grant-in-Aid for Scientific Research (C), 22540072.
MSC2010: 53C20, 53C22.
Keywords: sphere, diameter, ray, plane.

this note. In Section 4, we show some properties of the domain $D(\tilde{p}, \tilde{q})$ and give proofs of the theorems stated in Section 2. In Section 5, we study the case that $\widetilde{M}$ is a *κ-plane M_κ* — which is, by definition, a complete simply connected Riemannian surface with constant Gaussian curvature κ. We have some sphere theorems depending on the relation among the angular distribution of minimizing geodesic segments, the distance between two points, and the Gaussian curvature of a model surface. In Section 6, we discuss the case of noncompact manifolds referred to a κ-plane with $\kappa < 0$.

Klingenberg [1963] was first interested in radial sectional curvature. Some roles of critical point theory have been introduced in [Abresch and Meyer 1997]. A general introduction to the techniques used in this note is found in [Cheeger and Ebin 1975]. There are some generalized Toponogov comparison theorems for radial curvature. But the version used in this note was first proved in [Itokawa et al. 2001; 2003] and developed in [Kondo and Tanaka 2010; Innami et al. 2013a]. As its application, some diameter sphere theorems have been proved in [Kondo 2007; Kondo and Ohta 2007; Lee 2005; Innami et al. 2013b]. The geometry of geodesics on surfaces of revolution has been developed in [Belegradek et al. 2012; Sinclair and Tanaka 2007; Tanaka 1992].

2. Definitions and statements

Let M be a complete Riemannian manifold. We introduce a function $\alpha_p(x)$ that measures the angular distribution of minimizing geodesic segments from x to p. For $p \in M$ let $d_p(x) = d(p, x)$ for all $x \in M$. Let $T_x M$ denote the tangent space of M at x. Let $A_p(x)$ be the set of tangent vectors $T(x, p)^\bullet(0)$ at $x \neq p$ of all minimizing geodesic segments $T(x, p)$ from x to p. The geodesics are supposed to be parameterized by arclength. Let $\beta_x(v) = \min\{\angle(v, w) \mid w \in A_p(x)\}$ for $v \in T_x M$ and

$$\alpha_p(x) = \max\{\beta_x(v) \mid v \in T_x M\}.$$

Obviously, $\alpha_p(x) \leq \pi$ for all $x \in M$, $x \neq p$. If x is not a cut point of p, then $\alpha_p(x) = \pi$. We call $\alpha_p(x)$ the *angular distribution* of $A_p(x)$ in the unit sphere $S_x M$ in $T_x M$. We call $x \in M$ a *critical point* of d_p if $\alpha_p(x) \leq \pi/2$. If $p, q \in M$ satisfy $d(p, q) = \operatorname{diam}(M)$, then q is a critical point of d_p, and p is a critical point of d_q.

The distribution of critical points of d_p depends on the topological and metric structure of M. The diameter sphere theorem is based on the following lemma due to Grove and Shiohama [1977].

Lemma 2.1 (basic lemma). *Let M be a complete Riemannian manifold and $p \in M$. If there exists no critical point of d_p in $M \setminus \{p\}$, then M is diffeomorphic to the Euclidean space $\mathbb{E}^n$. If there exists only one critical point $q \in M \setminus \{p\}$ of d_p and* *if* $\alpha_p(q) < \pi/2$ *or* $d_p(q) = \max\{d_p(x) \mid x \in M\}$, *then* M *is homeomorphic to an* n*-sphere.*

In this note, using the angular distribution, we propose some conditions under which the assumption of Lemma 2.1 is satisfied. In order to do this we use the generalized Toponogov comparison theorem for radial curvature proved in [Itokawa et al. 2003; Innami et al. 2013a; Kondo and Tanaka 2010].

Let $(\widetilde{M}, \tilde{p})$ be a surface of revolution homeomorphic to a sphere or a plane with a geodesic polar coordinate system (r, θ) around $\tilde{p}$. Its metric is of class C^2 and given by

$$ds^2 = dr^2 + m(r)^2 d\theta^2,$$

where $m(r) > 0$, $0 < r < \ell \le \infty$, $\theta \in S^1$, and $m : [0, \ell) \to \mathbb{R}$ satisfies the Jacobi equation

$$m'' + \widetilde{K} m = 0, \quad m(0) = 0, \quad m'(0) = 1,$$

and if $\ell < \infty$,

$$m(\ell) = 0, \quad m'(\ell) = -1.$$

The function $\widetilde{K}$ is called the *radial curvature function* of $\widetilde{M}$.

Let (M, p) be a complete Riemannian manifold with a base point at p. A *radial plane* $\Pi \subset T_x M$ at a point $x \in M$ is a plane containing a vector tangent to a minimizing geodesic segment emanating from p. A *radial sectional curvature* $K_M(\Pi)$ is a sectional curvature with respect to a radial plane Π. We say that (M, p) is *referred* to $(\widetilde{M}, \tilde{p})$ if every radial sectional curvature at $x \in M$ is bounded below by $\widetilde{K}(d(p, x))$, namely, $K_M(\Pi) \ge \widetilde{K}(d(p, x))$.

Let (M, p) be referred to $(\widetilde{M}, \tilde{p})$. If $\ell < \infty$, we then have $d_p(x) \le \ell$ for all $x \in M$, equality holding if and only if M is isometric to the warped product $S^{n-1} \times_m [0, \ell]$, where $n = \dim M$ and S^{n-1} is a sphere; see [Itokawa et al. 2001]. From this fact, we may assume that $\max\{d_p(x) \mid x \in M\} < \ell$ if $\ell < \infty$, because our purpose is to study some conditions on M being homeomorphic to a sphere. Thus, we have the point $\tilde{q} = (d(p, q), 0) \in \widetilde{M}$ for any point $q \in M$.

Let $\Phi_{p,q}$ denote the reference map from M to the east side $\widetilde{M}^+$ of the meridian containing $T(\tilde{p}, \tilde{q})$ in $\widetilde{M}$, namely $\widetilde{M}^+ = \{(r, \theta) \mid 0 \le r, 0 \le \theta \le \pi\}$. By definition, for a point $x \in M$,

$$d(\tilde{p}, \Phi_{p,q}(x)) = d(p, x) \quad \text{and} \quad d(\tilde{q}, \Phi_{p,q}(x)) = d(q, x).$$

It is not certain whether or not every point $x \in M$ has a reference point and every geodesic triangle $\triangle(pqx)$, $q, x \in M$, admits the corresponding geodesic triangle $\triangle(\tilde{p}\tilde{q}\tilde{x})$, $\tilde{q}, \tilde{x} \in \widetilde{M}$. This question has been answered affirmatively under a certain condition in [Innami et al. 2013a]. However, we use only a quarter of $\widetilde{M}$ in the critical point theory. More precisely, as the image space of the reference map $\Phi_{p,q}$, we define a special domain $D(\tilde{p},\tilde{q})$ in $\widetilde{M}^+$ for $\tilde{q}=(r_0,0)\in\widetilde{M}$, $0<r_0<\ell$. For $\theta\in[0,\pi/2]$ let

$$\lambda_{\tilde{q}}(\theta)=\sup\Big\{r>0 \,\Big|\, \angle\Big(v_s,-\frac{\partial}{\partial r}\Big)>\frac{\pi}{2},\ v_s\in A_{\tilde{q}}(z_s),\\ \angle\Big(w_s,-\frac{\partial}{\partial r}\Big)<\frac{\pi}{2},\ w_s\in A_{z_s}(\tilde{q}),\ 0\le s<r\Big\}$$

where $z_s=(s,\theta)$, and set

$$D(\tilde{p},\tilde{q})=\{(r,\theta)\in\widetilde{M}\mid 0\le r<\lambda_{\tilde{q}}(\theta),\ 0\le\theta<\pi/2\}\cup\{\tilde{p},\tilde{q}\}.$$

Obviously, $D(\tilde{p},\tilde{q})\supset T(\tilde{p},\tilde{q})$, since $\angle(\tilde{p}z\tilde{q})=\pi$ for all $z\in T(\tilde{p},\tilde{q})\setminus\{\tilde{p},\tilde{q}\}$. Moreover, as will be shown in Lemma 4.1, there exists no cut point of $\tilde{q}$ in $D(\tilde{p},\tilde{q})$. Hence, if $\Phi_{p,q}(M)\subset D(\tilde{p},\tilde{q})$, then the generalized Toponogov comparison theorem is valid for all geodesic triangles $\Delta(pqx)$ and for all $x\in M$.

We define a dominant triangle for M with respect to p and q. Let $z\in\widetilde{M}$ and T a minimizing geodesic segment with $z\in T$. For an angle ω let $S=S(z,T,\omega)$ denote the geodesic such that the angle of S with T at z is ω. We make a trilateral with three geodesic segments:

$$S_0=T(\tilde{p},\tilde{q}),\quad S_1=S(\tilde{p},T(\tilde{p},\tilde{q}),\alpha_q(p)),\quad S_2=S(\tilde{q},T(\tilde{p},\tilde{q}),\alpha_p(q)).$$

We call the domain D_M bounded by S_0, S_1 and S_2 a *dominant domain* for M if it exists. The dominant domain D_M becomes a triangle if S_1 and S_2 intersect. Otherwise, it may not become a triangle. If S_0, S_1 and S_2 make a triangle, we call it the *dominant triangle* for M, and it is denoted by $\Delta_M=\triangle(T(\tilde{p},\tilde{q}),\alpha_q(p),\alpha_p(q))$.

For a triangle Δ, the triangle domain bounded by Δ in $\widetilde{M}^+$ is also denoted by Δ. If the dominant triangle Δ_M exists and the generalized Toponogov comparison theorem is valid for (M,p) referred to $(\widetilde{M},\tilde{p})$, then $\Phi_{p,q}(M)\subset\Delta_M$ because of the Alexandrov convexity. The vertex of the dominant triangle Δ_M other than $\tilde{p}$ and $\tilde{q}$ is denoted by $z(\Delta_M)$.

Theorem 2.2. *Let (M,p) be a complete Riemannian manifold referred to $(\widetilde{M},\tilde{p})$. Assume that there exists a point q in M such that the dominant triangle $\Delta_M=\triangle(T(\tilde{p},\tilde{q}),\alpha_q(p),\alpha_p(q))$ for M can be made from p and q. If $z(\Delta_M)\in D(\tilde{p},\tilde{q})$, then M is topologically an n-sphere.*

We have a generalization of the diameter sphere theorem if we impose a certain condition on $\widetilde{M}$; see Lemma 4.3. We say that $\widetilde{M}$ is *without conjugate points in a half* if any point $z\in\mathrm{Int}(\widetilde{M}^+)$ has no point conjugate to z along any geodesic segment from z contained in $\mathrm{Int}(\widetilde{M}^+)$. Here $\mathrm{Int}(\widetilde{M}^+)$ is the interior of $\widetilde{M}^+$. Any point in $\mathrm{Int}(\widetilde{M}^+)$ has no cut point in $\mathrm{Int}(\widetilde{M}^+)$ if and only if $\widetilde{M}$ is without conjugate points in a half. Tanaka [1992] proved that $\widetilde{M}$ is without conjugate points in a half if $\widetilde{M}$ is a von Mangoldt surface of revolution.

We say that $\widetilde{M}$ is *without meridian focal points in a quarter* if there exists no focal point of the meridian $\{(r, 0) \mid 0 \le r \le \ell\}$ in a quarter $\{(r, \theta) \mid 0 \le r < \ell,\ 0 < \theta < \pi/2\}$ of $\widetilde{M}$. If $\widetilde{M}$ is without conjugate points in a half, then it is without meridian focal points in a quarter; see Proposition 3.1. If $\widetilde{M}$ is without meridian focal points in a quarter, then it is without conjugate points in a quarter; see Proposition 3.2.

If $\widetilde{M}$ is without meridian focal points in a quarter and $m'(r(\tilde{q})) < 0$, then $\triangle(T(\tilde{p}, \tilde{q}), \pi/2, \pi/2) \subset D(\tilde{p}, \tilde{q})$; see Lemma 4.3. Kondo and Ohta [2007] have proved the following corollary, assuming that $\widetilde{M}$ is a von Mangoldt surface of revolution.

Corollary 2.3. *Let $(\widetilde{M}, \tilde{p})$ be a reference surface homeomorphic to a sphere such that $\widetilde{M}$ is without meridian focal points in a quarter. Let (M, p) be a complete Riemannian manifold referred to $(\widetilde{M}, \tilde{p})$. If there exists a point $q \in M$ such that q and p are critical points of d_p and d_q, respectively, and if $m'(d_p(q)) < 0$, then M is homeomorphic to an n-sphere.*

When $\ell = \infty$, let $\tilde{\gamma}(t) = (t, 0)$ for $t \in [0, \infty)$. For $\theta \in [0, \pi]$, let $\lambda_{\tilde{\gamma}}(\theta)$ denote the supremum of those $r > 0$ such that there exists a unique coray from (s, θ), $0 < s < r$, to $\tilde{\gamma}$ whose initial tangent vector v satisfies $\angle(v, -\partial/\partial r) > \pi/2$. Using this function $\lambda_{\tilde{\gamma}}(\theta)$, we define a special domain $D(\tilde{\gamma})$ in a reference surface of revolution $\widetilde{M}$. Namely, we set

$$D(\tilde{\gamma}) = \{(r, \theta) \in \widetilde{M} \mid 0 \le r < \lambda_{\tilde{\gamma}}(\theta),\ 0 \le \theta \le \pi\}.$$

Obviously, $\lambda_{\tilde{\gamma}}(0) = \infty$. Let $\rho_{\tilde{p}}(\tilde{\gamma}) = \sup\{\theta_0 \mid \lambda_{\tilde{\gamma}}(\theta) = \infty \text{ for } \theta \in [0, \theta_0)\}$. When $\widetilde{M}$ is a κ-plane with $\kappa \le 0$, we have $\rho_{\tilde{p}}(\tilde{\gamma}) = 0$ if $\kappa < 0$ and $\rho_{\tilde{p}}(\tilde{\gamma}) = \pi/2$ if $\kappa = 0$. If $\widetilde{M}$ is a paraboloid of revolution, then $\rho_{\tilde{p}}(\tilde{\gamma}) = \pi$.

Let Γ_p denote the set of all rays from p in (M, p). Let

$$\eta_p(v) = \min\{\angle(v, \dot{\gamma}(0)) \mid \gamma \in \Gamma_p\}$$

for any $v \in T_pM$, and set

$$\zeta_p = \max\{\eta_p(v) \mid v \in T_pM\}.$$

Obviously, $\zeta_p \le \pi$ for all $p \in M$. We call ζ_p the *angular distribution of rays* from p. We call $\widetilde{M}^+(\theta_0) = \{(r, \theta) \mid 0 \le r < \ell,\ 0 \le \theta \le \theta_0\}$ a *sector* of $\widetilde{M}$ for $\theta_0 \in [0, \pi]$.

Theorem 2.4. *Let (M, p) be a complete noncompact Riemannian n-manifold referred to $(\widetilde{M}, \tilde{p})$ such that $\rho_{\tilde{p}}(\tilde{\gamma}) > 0$. Assume that the sector $\mathrm{Int}(\widetilde{M}^+(\rho_{\tilde{p}}(\tilde{\gamma})))$ is without conjugate points. If $\zeta_p < \rho_{\tilde{p}}(\tilde{\gamma})$, then M is diffeomorphic to an n-plane.*

Since $\rho_{\tilde{p}}(\tilde{\gamma}) = 0$ for M_κ with $\kappa < 0$, the theorem shows an advantage of using a surface of revolution as a reference surface.

3. Preliminaries

Let $(\widetilde{M}, \tilde{p})$ be a surface of revolution with vertex $\tilde{p}$ and let $\gamma : (-\infty, \infty) \to \widetilde{M}$ be a geodesic with unit speed. We write $\gamma(s) = (r(s), \theta(s))$ for all $s \in (-\infty, \infty)$. Let $\{E_1(s) = \dot{\gamma}(s), E_2(s)\}$ denote a set of parallel orthonormal vector fields along γ. Since the vector field $Y(s) = \partial/\partial\theta$ along γ is generated from a variation through geodesics $\gamma_u(s) = (r(s), \theta(s)+u)$, it is a Jacobi vector field along γ. If $\varphi(s)$ denotes the angle of $Y(s)$ with $\dot{\gamma}(s)$, we then have $\langle E_1(s), Y(s)\rangle = m(r(s)) \cos\varphi(s) = \nu$ which is called the Clairaut relation. Note that $-m(r(0)) \le \nu \le m(r(0))$. The orthogonal complement of $Y(s)$ to $\dot{\gamma}(s)$ is $\sqrt{m(r(s))^2 - \nu^2}E_2(s)$. Therefore,

$$y(s) = \sqrt{m(r(s))^2 - \nu^2}$$

satisfies the Jacobi equation,

$$y''(s) + \widetilde{K}(r(s))y(s) = 0.$$

If $C(\gamma) = \{s \mid r'(s) = 0\}$, then the number of elements of $C(\gamma)$ is 1 or ∞. The Sturm separation theorem states that if $C(\gamma) = \{s_0\}$, then for every $s < s_0$ there exists at most one point $\gamma(s_1)$, $s_1 > s_0$, conjugate to $\gamma(s)$. The Clairaut relation states that if $\cdots < s_{-1} < s_0 < s_1 < \cdots$ are the solutions of the equation $y(s) = 0$, then γ is tangent to the parallel circle $r = r(s_i)$ with $m(r(s_i)) = \nu$ and $\gamma(s_i)$ are conjugate to one another for $i \in \mathbb{Z}$. From the Sturm separation theorem, if $\bar{y}(s)$ is the length of a perpendicular Jacobi vector field along γ such that $\bar{y}(t_0) = 0$, $s_0 < t_0 < s_1$, then the zeros of $\bar{y}(s)$ appear in each interval (s_i, s_{i+1}) once for every $i \in \mathbb{Z}$.

Proposition 3.1. *Let $(\widetilde{M}, \tilde{p})$ be a surface of revolution with vertex $\tilde{p}$. If $\widetilde{M}$ is without conjugate points in a half, then $\widetilde{M}$ is without meridian focal points in a quarter.*

Proof. Suppose that $\widetilde{M}$ is not without meridian focal points in a quarter. Then there exists a geodesic $\gamma : [0, a] \to \mathrm{Int}(\widetilde{M}^+)$ normal to the meridian $\theta = \pi/2$ such that $\theta(\gamma(a)) = \pi/2$ and $\gamma(0)$ is a focal point of $\theta = \pi/2$ along γ. Since $\widetilde{M}$ is a surface of revolution, $\widetilde{M}$ is symmetric with respect to $\theta = \pi/2$. From this symmetry, if $\gamma_e : [0, \infty) \to \widetilde{M}$ denotes the extension of γ, we see that $\gamma_e(2a) \in \mathrm{Int}(\widetilde{M}^+)$ is a point conjugate to $\gamma_e(0)$. Namely, $\widetilde{M}$ is not without conjugate points in a half. □

Proposition 3.2. *Let $(\widetilde{M}, \tilde{p})$ be a surface of revolution with vertex $\tilde{p}$. Assume that $\widetilde{M}$ is without meridian focal points in a quarter. Then, $\widetilde{M}$ is without conjugate points in a quarter. In particular, there exists a unique geodesic segment in $\widetilde{M}^+(\pi/2)$ connecting any two points in $\widetilde{M}^+(\pi/2)$.*

Proof. Suppose that there exists a geodesic segment $\omega : [0, L] \to \widetilde{M}^+(\pi/2)$ such that $\omega(L)$ is the first point conjugate to $\omega(0)$ along ω. Then, $r(s) = r(\omega(s))$, $s \in [0, L]$, is not monotone because $\widetilde{M}$ is a surface of revolution without meridian focal points in a quarter. Assume that $r'(s_0) = 0$ at s_0 with $0 < s_0 < L$.

The complete extension of ω is denoted by the same symbol and its parametrization is changed by $\bar{\omega}(s)=\omega(s+s_0)$, $s\in(-\infty,\infty)$. By the symmetry of $\widetilde{M}$ with respect to the meridian through $\bar{\omega}(0)$, $\bar{\omega}(s_0)$ is a point conjugate to $\bar{\omega}(s_0-L)$. From the Sturm separation theorem, there exists a number $L_1>0$ such that $s_0-L<-L_1<0$ and $\bar{\omega}(L_1)$ is a point conjugate to $\bar{\omega}(-L_1)$ along $\bar{\omega}$. Then, $\bar{\omega}(L_1)$ is a focal point of the meridian through $\bar{\omega}(0)$ along $\bar{\omega}$ and $\left|\theta(\bar{\omega}(0))-\theta(\bar{\omega}(L_1))\right|<\pi/2$. This contradicts that $\widetilde{M}$ is without meridian focal points in a quarter.

We prove the second part. If there exist two geodesic segments connecting the same endpoints in $\widetilde{M}^+(\pi/2)$, then they may bounds a biangle domain in $\widetilde{M}^+(\pi/2)$. There exists a minimizing geodesic segment in the biangle domain such that the endpoints are conjugate to each other. This contradicts the first part. □

Lemma 3.3. *Let $(\widetilde{M},\tilde{p})$ be a surface of revolution with vertex $\tilde{p}$. If $\widetilde{M}$ is without meridian focal points in a quarter, then* $\mathrm{Int}(\widetilde{M}^+)$ *is foliated by geodesics perpendicular to the meridian $\theta=\pi/2$. In particular, if $\widetilde{M}$ is compact, then those geodesics cross the meridian $\theta=0$ at points between the focal points along the meridian $\theta=0$.*

Proof. Let $z\in\mathrm{Int}(\widetilde{M}^+)$. Since $\widetilde{M}$ is without meridian focal points in a quarter, there exists a unique foot w of z on $\theta=\pi/2$, namely $z\in X=\theta^{-1}(\pi/2)$ and $d(z,w)=d(z,X)$. This proves the first part of the lemma.

If $\widetilde{M}$ is compact, then $\tilde{q}=(\ell,0)$ is the unique point conjugate to $\tilde{p}=(0,0)$. Hence, there exist focal points to $\theta=\pi/2$ along $\theta=0$ from $\tilde{p}$ and $\tilde{q}$. Let $(a,0)$ and $(b,0)$ be focal points of $\theta=\pi/2$ along $\theta=0$ from $\tilde{p}$ and $(\ell,0)$, respectively. We then have $a\le b$. In fact, if $a>b$, then the geodesics normal to $\theta=\pi/2$ from points near $\tilde{p}$ and $(\ell,0)$ meet in $\mathrm{Int}(M^+)$, contradicting the first part. If $a=b$, then all geodesics normal to $\theta=\pi/2$ pass through $(a,0)$. If $a<b$, then they pass the interval $([a,b],0)$, keeping their order. □

We review the generalized Toponogov comparison theorem. Let (M,p) be a complete Riemannian manifold referred to $(\widetilde{M},\tilde{p})$. Let $q\in M$, $q\neq p$. For a point $x\in M$, let $\gamma:[0,a]\to M$ denote a minimizing geodesic segment such that $\gamma(0)=q$ and $\gamma(a)=x$. As was seen in [Itokawa et al. 2003], if $\Phi_{p,q}(\gamma(s))$, $s\in[0,a]$, do not intersect the cut locus $\mathrm{Cut}(\tilde{q})$ of $\tilde{q}$ in $\widetilde{M}$, then the generalized Toponogov comparison theorem for the base angles is valid. Namely, we have

$$\angle(\tilde{p}\tilde{q}\tilde{x})\le\angle(pqx)\quad\text{and}\quad\angle(\tilde{p}\tilde{x}\tilde{q})\le\angle(pxq). \tag{1}$$

Let $\alpha:[0,b]\to M$ be a minimizing geodesic segment such that $\alpha(0)=p$ and $\alpha(b)=x$. As was seen in [Innami et al. 2013a], the generalized Toponogov comparison theorem for the angle at p is valid, under the condition that if $\Phi_{p,q}(\alpha(s))$, $s\in[0,b]$, intersects $\mathrm{Cut}(\tilde{q})$ at $s=s_0$, then for any minimizing geodesic segment $T(\tilde{q},\Phi_{p,q}(\alpha(s_0)))$, there exists a minimizing geodesic segment from q to $\alpha(s_0)$

satisfying (1). Namely, we then have

$$\angle(\tilde{q}\tilde{p}\tilde{x}) \le \angle(qpx).$$

For $p, q, x \in M$, the minimum angle $\angle^i(pqx)$ and maximum one $\angle^s(pqx)$ are defined by

$$\angle^i(pqx) = \min\{\angle(v, w) \mid v \in A_p(q), w \in A_x(q)\},$$
$$\angle^s(pqx) = \max\{\angle(v, w) \mid v \in A_p(q), w \in A_x(q)\}.$$

It should be noted that there may not exist any triangle $\triangle(pqx)$ with three angles $\angle^s(pqx)$, $\angle^s(pxq)$, and $\angle^s(qpx)$.

In this note, we use the generalized Toponogov comparison theorem of the following form, which is a conclusion of the argument in [Itokawa et al. 2003].

Theorem 3.4. *Let (M, p) be a complete Riemannian manifold referred to a surface of revolution $(\widetilde{M}, \tilde{p})$. Let $q \in M$, $q \ne p$. If there exists a star-shaped domain D around $\tilde{q}$ contained in the dominant domain D_M such that $\Phi_{p,q}(M) \subset D$, then for all $x \in M$,*

$$\angle(\tilde{p}\tilde{q}\tilde{x}) \le \angle^i(pqx), \quad \angle(\tilde{p}\tilde{x}\tilde{q}) \le \angle^i(pxq), \quad \angle(\tilde{q}\tilde{p}\tilde{x}) \le \angle^i(qpx).$$

We say that a domain $D \subset \widetilde{M}^+$ is star-shaped around $\tilde{q}$ in $\widetilde{M}$ if there exists a unique minimizing geodesic segment from $\tilde{q}$ to any point $z \in D$ contained in D.

4. Dominant domains

Let $(\widetilde{M}, \tilde{p})$ be a surface of revolution homeomorphic to a sphere or a plane with a geodesic polar coordinate system (r, θ) around $\tilde{p}$. Let $\tilde{q} = (r_0, 0) \in \widetilde{M}$, $0 < r_0 < \ell$.

Lemma 4.1. *Let $D(\tilde{p}, \tilde{q})$ be the subset defined before. Then, there is no cut point of $\tilde{q}$ in $D(\tilde{p}, \tilde{q})$, and $D(\tilde{p}, \tilde{q})$ is star-shaped around $\tilde{p}$ and $\tilde{q}$.*

Proof. Let $z \in D(\tilde{p}, \tilde{q})$ and let $\gamma : [0, a] \to \widetilde{M}$, $a = d(\tilde{q}, z)$, a minimizing geodesic segment such that $\gamma(0) = \tilde{q}$, $\gamma(a) = z$, $\angle(\dot{\gamma}(0), -\partial/\partial r) < \pi/2$, and $\angle(\dot{\gamma}(a), -\partial/\partial r) < \pi/2$. If $r(s) = r(\gamma(s))$, $s \in [0, a]$, then $r'(0) < 0$ and $r'(a) < 0$.

We prove that $\gamma(a)$ is not conjugate to $\gamma(0)$ along it. In order to prove this, it is enough to prove that $r(s)$ is monotone decreasing in $s \in [0, a]$, since $\widetilde{M}$ is a surface of revolution. If $r'(s) \ge 0$ for some $s \in [0, a]$, then, from $r'(a) < 0$, there exist at least two parameters s_1 and s_2 such that $0 < s_1 < s_2 < a$ and $r'(s_1) = r'(s_2) = 0$. This implies that $\gamma(s_2)$ is a point conjugate to $\gamma(s_1)$ along γ, contradicting the fact that $\gamma([0, a])$ is minimizing.

Next, we prove that z is joined to $\tilde{q}$ by a unique minimizing geodesic. Suppose for indirect proof that $\gamma_1 : [0, a] \to \widetilde{M}$ is another minimizing geodesic segment satisfying the same condition as γ. Set $\varphi(s) = \angle(\dot{\gamma}(s), \partial/\partial\theta)$ and $\varphi_1(s) = \angle(\dot{\gamma}_1(s), \partial/\partial\theta)$ for $s \in [0, a]$. Without loss of generality, $0 > \varphi(0) > \varphi_1(0) > -\pi/2$, so

$$m(r(0)) \cos\varphi(0) > m(r(0)) \cos\varphi_1(0).$$

From this, the Clairaut relation states that

$$m(r(a)) \cos\varphi(a) > m(r(a)) \cos\varphi_1(a).$$

Therefore, we have $0 > \varphi(a) > \varphi_1(a) > -\pi/2$. On the other hand, since z is the first meeting point of γ and γ_1, the relation between $\varphi(a)$ and $\varphi_1(a)$ must be $\varphi(a) < \varphi_1(a)$, a contradiction. This implies that z is not a cut point of $\tilde{q}$.

We next prove that $\gamma([0, a]) \subset D(\tilde{p}, \tilde{q})$. If $z = (r_0, \theta)$, then we define $z_t = (t, \theta)$ for $t \in [0, r_0]$. We set

$$t_0 = \sup\{s \mid T(z_t, \tilde{q}) \subset D(\tilde{p}, \tilde{q}) \text{ for all } t \in [0, s)\}.$$

From the first variation formula, we see there exists a number $\varepsilon > 0$ such that there exists a unique minimizing geodesic segment $T(z_t, \tilde{q})$ and $z_t \in D(\tilde{p}, \tilde{q})$ for every $t \in [0, \varepsilon)$. As seen above, $T(z_t, \tilde{q}) \subset D(\tilde{p}, \tilde{q})$ for all $t \in [0, \varepsilon)$; hence $t_0 > 0$. If $T(z_{t_0}, \tilde{q})$ is tangent to the parallel circle at $\tilde{q}$, then $t_0 = \lambda_{\tilde{q}}(\theta)$, contradicting $r_0 < \lambda_{\tilde{q}}(\theta)$. This is not the case. Otherwise, from the facts seen above, there exists a neighborhood of $T(z_{t_0}, \tilde{q})$ contained in $D(\tilde{p}, \tilde{q})$. This implies that $t_0 = r_0$. $\square$

This lemma makes it possible to use the generalized Toponogov comparison theorem if $\Phi_{p,q}(M) \subset D(\tilde{p}, \tilde{q})$.

Lemma 4.2. *Let (M, p) be a complete Riemannian manifold referred to $(\widetilde{M}, \tilde{p})$. Assume that there exists a point q in M such that the dominant triangle $\Delta_M = \triangle(T(\tilde{p}, \tilde{q}), \alpha_q(p), \alpha_p(q))$ for M can be made from p and q. If $z(\Delta_M) \in D(\tilde{p}, \tilde{q})$, then $\Phi_{p,q}(M) \subset \Delta_M \subset D(\tilde{p}, \tilde{q})$. In particular, the generalized Toponogov comparison theorem by $\Phi_{p,q}$ for (M, p) referred to $(\widetilde{M}, \tilde{p})$ is valid.*

Proof. From Lemma 4.1, $D(\tilde{p}, \tilde{q})$ is star-shaped around $\tilde{p}$ and $\tilde{q}$. Therefore, the triangle domain Δ_M satisfies $\Delta_M \subset D(\tilde{p}, \tilde{q})$.

We prove that $\Phi_{p,q}(M) \subset \Delta_M$. For a sufficiently small $\varepsilon > 0$, the generalized Toponogov comparison theorem is valid for all triangles $\triangle(pqx)$ if

$$d(p, x) + d(q, x) < d(p, q) + \varepsilon;$$

see [Itokawa et al. 2003; Innami et al. 2013a; Kondo and Tanaka 2010]. Let $\tilde{x} = \Phi_{p,q}(x)$. Since $\angle(\tilde{p}\tilde{q}\tilde{x}) \le \angle(pqx) \le \alpha_p(q)$ and $\angle(\tilde{q}\tilde{p}\tilde{x}) \le \angle(qpx) \le \alpha_q(p)$, we have $\tilde{x} \in \Delta_M$.

Let $x \in M$ be any point and $\gamma : [0, a] \to M$, a minimizing geodesic segment such that $\gamma(0) = q$ and $\gamma(a) = x$. We define

$$t_0 = \sup\{t \mid \Phi_{p,q}(\gamma(s)) \text{ is defined and } \Phi_{p,q}(\gamma(s)) \in \Delta_M \text{ for } s \in [0, t)\}.$$

As is seen above, we have $t_0 > 0$. Suppose for indirect proof that $t_0 < a$. Then $\tilde{y} = \Phi_{p,q}(\gamma(t_0))$ is defined and $\tilde{y} \in T(\tilde{q}, z(\Delta_M))$ or $\tilde{y} \in T(\tilde{p}, z(\Delta_M))$. Let $\widetilde{U}$ be an open set such that $\Delta_M \setminus T(\tilde{p}, \tilde{q}) \subset \widetilde{U} \subset D(\tilde{p}, \tilde{q})$. Since $\tilde{y}$ is not a cut point of $\tilde{q}$, there exists a number t_1 with $t_1 > t_0$, such that the points $\Phi_{p,q}(\gamma(s))$ exist in $\widetilde{U}$ for all $s \in [t_0, t_1]$ and $\tilde{x}_1 = \Phi_{p,q}(\gamma(t_1)) \notin \Delta_M$. In fact, we find those reference points because of the method in [Itokawa et al. 2003]. Therefore, we have either $\angle(\tilde{p}\tilde{q}\tilde{x}_1) > \alpha_p(q)$ or $\angle(\tilde{q}\tilde{p}\tilde{x}_1) > \alpha_q(p)$.

On the other hand, since there is no cut point of $\tilde{q}$ in $\widetilde{U}$, the generalized Toponogov comparison theorem is valid in $\Phi_{p,q}^{-1}(\widetilde{U})$. Hence,

$$\angle(\tilde{p}\tilde{q}\tilde{x}_1) \le \angle(pq\gamma(t_1)) \le \alpha_p(q), \quad \angle(\tilde{q}\tilde{p}\tilde{x}_1) \le \angle(qp\gamma(t_1)) \le \alpha_q(p),$$

a contradiction. Therefore, $t_0 = a$ and $\tilde{x} \in \Delta_M$. □

Proof of Theorem 2.2. Since $z(\Delta_M) \in D(\tilde{p}, \tilde{q})$, we have both $\alpha_p(q) < \pi/2$ and $\alpha_q(p) < \pi/2$. In particular, q is a critical point of d_p. In order to apply Lemma 2.1, we have only to prove that there exists no critical point in $M \setminus \{p, q\}$. Let $x \in M$. From Lemma 4.2, the generalized Toponogov comparison theorem by $\Phi_{p,q}$ for (M, p) referred to $(\widetilde{M}, \tilde{p})$ is valid. Hence, we have $\pi/2 < \angle(\tilde{p}\tilde{x}\tilde{q}) \le \angle(pxq)$ since $\tilde{x} = \Phi_{p,q}(x) \in D(\tilde{p}, \tilde{q})$. Consequently, $\alpha_p(x) > \pi/2$, so x is not a critical point of d_p. □

A special case of the next lemma has been proved in [Kondo and Ohta 2007].

Lemma 4.3. *Let $(\widetilde{M}, \tilde{p})$ be a reference surface without meridian focal points in a quarter and $\tilde{q} = (r_0, 0)$. If $m'(r_0) < 0$, then $\Delta = \triangle(T(\tilde{p}, \tilde{q}), \pi/2, \pi/2) \subset D(\tilde{p}, \tilde{q})$.*

Proof. We first prove that the domain Ω — bounded by the minimizing geodesic segment $T(\tilde{p}, \tilde{q})$, the parallel circle $r = r_0 = r(\tilde{q})$, and the meridian $\theta = \pi/2$ — is foliated by geodesic segments which are either tangent to $r = r_0$ or perpendicular to the meridian $\theta = \pi/2$ and cross the meridian $\theta = 0$.

Let $r_1 < r_0$ satisfy $m(r_1) = m(r_0)$ and $m(r) > m(r_0)$ for all $r \in (r_1, r_0)$. Since $m'(r_0) < 0$, there exists at least one r_1. The Clairaut relation states that the strip between parallels $r = r_1$ and $r = r_0$ is foliated by the geodesic segments $T_\tau(t)$, $0 \le t \le t_0$, where $T_\tau(0) = (r_0, \tau)$, $\dot{T}_\tau(0) = -(1/m(r_0))\partial/\partial\theta$, and $r(T_\tau(t)) \in (r_1, r_0)$ for all $t \in (0, t_0)$. Hence the subset Ω_1 of Ω bounded by $T(\tilde{p}, \tilde{q})$, $r = r_0$, and $T_{\pi/2}$ is foliated by geodesic segments T_τ which are tangent to $r = r_0$.

Let $S_\sigma(t)$, $\sigma \in (0, r_0)$, denote the geodesic segments such that $S_\sigma(0) = (\sigma, \pi/2)$ and $\dot{S}_\sigma(0) = -(1/m(\sigma))\partial/\partial\theta$. Since there exists no point focal to $\theta = \pi/2$ in the sector $\{(r, \theta) \mid \theta \in (0, \pi/2)\}$, those geodesic segments give a foliation of the subset Ω_2 of Ω, bounded by $T(\tilde{p}, \tilde{q})$, $T_{\pi/2}$, and $\theta = \pi/2$; see Lemma 3.3. Since $\Omega = \Omega_1 \cup \Omega_2$, the first claim is proved.

Let $\gamma : [0, L] \to \widetilde{M}$ denote the geodesic segment which is the edge of Δ opposite to $\tilde{p}$. Hence, we have $\gamma(0) = \tilde{q}$, $\dot{\gamma}(0) = (1/m(r_0))\partial/\partial\theta$, and $\theta(\gamma(L)) = \pi/2$. Let

$z = (r, \pi/2)$ for $r \in (0, r(\gamma(L)))$. From Proposition 3.2, there exists a unique minimizing geodesic segment $\omega : [0, L_1] \to \widetilde{M}$ from $\tilde{q}$ to z in Δ.

We have only to prove that the r-coordinate of ω is monotone decreasing. We have $\angle(\dot{\omega}(0), -\partial/\partial r) < \pi/2$ and $\angle(\dot{\omega}(L_1), -\partial/\partial r) > \pi/2$ because of the foliation given in the first part. Therefore, if it is not monotone, then there exist two parameters s_1 and s_2 such that ω is tangent to the parallel circles at s_1 and s_2, since then $\omega(s_2)$ is a point conjugate to $\omega(s_1)$, contradicting the fact that ω is minimizing.

Since the r-coordinate of any geodesic segment from $\tilde{q}$ in Δ is monotone decreasing, $\triangle(T(\tilde{p}, \tilde{q}), \pi/2, \pi/2) \subset D(\tilde{p}, \tilde{q})$. □

Proof of Corollary 2.3. This corollary follows from Proposition 3.1, Lemma 4.3 and Theorem 2.2, since $\Delta_M \subset \triangle(T(\tilde{p}, \tilde{q}), \pi/2, \pi/2) \subset D(\tilde{p}, \tilde{q})$. □

We need two lemmas to prove Theorem 2.4. For $z \in D(\tilde{\gamma})$, let $z_t \in T(\tilde{p}, z)$ be the point such that $r(z_t) = t$.

Lemma 4.4. *Let $(\widetilde{M}, \tilde{p})$ be a surface of revolution with vertex $\tilde{p}$ such that $\ell = \infty$ and let $\tilde{\gamma} : [0, \infty) \to \widetilde{M}$ be a ray such that $\tilde{\gamma}(t) = (t, 0)$ for all $t \geq 0$. Let $z \in D(\tilde{\gamma})$. Then, there exists a number $R_0 > 0$ such that the angles of $T(z_t, \tilde{\gamma}(s))$ with $-\partial/\partial r$ at z_t are greater than $\pi/2$ for all $z_t \in T(\tilde{p}, z)$ and $s > R_0$.*

Proof. For any $s > 0$, let $\psi(s)$ be the supremum of the angles of $T(z_t, \tilde{\gamma}(s))$ with $-\partial/\partial r$ at z_t for all $z_t \in T(\tilde{p}, z)$. Then $\psi(s)$ is monotone and increasing in $s \in (0, \infty)$, since $(\widetilde{M}, \tilde{p})$ is a surface of revolution homeomorphic to a plane. Since $T(z_t, \tilde{\gamma}(s))$ converges to the corays from z_t to $\tilde{\gamma}$, $\psi(s)$ converges to a real number greater than $\pi/2$ as $s \to \infty$. □

Lemma 4.5. *Let (M, p) be a complete noncompact Riemannian n-manifold referred to $(\widetilde{M}, \tilde{p})$. Let $\gamma : [0, \infty) \to M$ be a ray such that $\gamma(0) = p$. Then, for any points $x \in M$ and $z \in \widetilde{M}$, there exists a sequence of parameters s_j such that $s_j \to \infty$ and the angles of $T(\gamma(s_j), x)$ with $-\dot{\gamma}(s_j)$ and $T(\tilde{\gamma}(s_j), z)$ with $-\dot{\tilde{\gamma}}(s_j)$ converge to zero as $j \to \infty$.*

Proof. This follows from the following inequality and the first variation formula.

$$\left|2s - d(\gamma(s), x) - d(\tilde{\gamma}(s), z)\right| \leq d(\gamma(0), x) + d(\tilde{\gamma}(0), z).$$

In fact, if this lemma is not true, then the left hand side of the inequality goes to ∞ as $s \to \infty$. □

Proof of Theorem 2.4. From Lemma 2.1, we have only to prove that there exists no critical point of d_p in $M \setminus \{p\}$. Let $x \in M \setminus \{p\}$ and $\alpha : [0, a] \to M$ a minimizing geodesic segment such that $\alpha(0) = p$ and $\alpha(a) = x$. From the assumption, there exists a ray $\gamma : [0, \infty) \to M$ from p such that $\angle(\dot{\gamma}(0), \dot{\alpha}(0)) \leq \zeta_p$. Let $z = (d(p, x), \xi)$, where $\zeta_p < \xi < \rho_{\tilde{p}}(\tilde{\gamma})$. For this point z, let $R_0 > 0$ denote the number given in Lemma 4.4. Furthermore, for this x and z, there exists a number $s_0 > R_0$

satisfying the property in Lemma 4.5. If Δ is the triangle domain bounded by $T(\tilde{p}, \tilde{\gamma}(s_0)) \cup T(\tilde{\gamma}(s_0), z) \cup T(\tilde{p}, z)$, as is seen in the proof of Lemma 4.1, then $\Delta \subset D(\tilde{p}, \tilde{\gamma}(s_0))$.

We have to prove that $\Phi_{p,\gamma(s_0)}(x) \in \Delta$. Since $\tilde{p}$ is not a cut point of $\tilde{\gamma}(s_0)$, there exists a number $\varepsilon > 0$ such that if $0 \leq t < \varepsilon$, then $y_t = \Phi_{p,\gamma(s_0)}(\alpha(t)) \in \Delta$. In fact, $r(y_t) = t$ and $\angle(\tilde{\gamma}(s_0)\tilde{p}y_t) \leq \angle(\gamma(s_0)px) < \xi$, since the generalized Toponogov comparison theorem is valid in some neighborhood of $\gamma([0, s_0])$. Set

$$t_0 = \sup\{t \in (0, a] \mid y_t \in \Delta\}.$$

As seen before, $t_0 > 0$ and $\alpha(t_0) \in \Delta$. If $t_0 \neq a$, we find a number $\varepsilon_1 > 0$ such that $y_t \in \Delta$ for all $t \in (t_0, t_0 + \varepsilon_1)$, since the sector $\mathrm{Int}(\widetilde{M}^+(\rho_{\tilde{p}}(\tilde{\gamma})))$ is without conjugate points and, hence, the generalized Toponogov comparison theorem is valid. This contradicts the choice of t_0. Thus, we have $y_a = \Phi_{p,\gamma(s_0)}(x) \in \Delta$.

Therefore, $\angle(\gamma(s_0)xp) \geq \angle(\tilde{\gamma}(s_0)y_a\tilde{p}) > \pi/2$, meaning that $\alpha_p(x) > \pi/2$. Thus, x is not a critical point of d_p. $\square$

5. The κ-plane as a reference surface for spheres

Let M_κ be the κ-plane, by definition isometric to the 2-sphere $S^2(1/\sqrt{\kappa})$ with radius $1/\sqrt{\kappa}$ if $\kappa > 0$, the Euclidean plane $\mathbb{E}^2$ if $\kappa = 0$, or the Poincaré disk with Gauss curvature κ if $\kappa < 0$. Notice that M_κ is without meridian focal points in a quarter. However, Lemma 4.3 is not applied if $\kappa \leq 0$, since no parameter r_0 exists such that $m'(r_0) < 0$. This means that the condition of being critical, namely $\alpha_p(q) \leq \pi/2$ and $\alpha_q(p) \leq \pi/2$, are not enough for a sphere theorem if the reference surface is M_κ, $\kappa \leq 0$. We need a restricted condition on $\alpha_p(q)$ and $\alpha_q(p)$ which depends on the distance $d(p, q)$ and κ.

Let M be a complete Riemannian n-manifold with sectional curvature bounded below by a constant κ. For points $p, q \in M$ we have points $\tilde{p}, \tilde{q} \in M_\kappa$ such that $d(p, q) = d(\tilde{p}, \tilde{q})$. When $\kappa > 0$, we assume that $d(p, q) < \pi/\sqrt{\kappa}$. Because, in general, $d(p, q) \leq \pi/\sqrt{\kappa}$, with equality holding if and only if M is isometric to the sphere with radius $1/\sqrt{\kappa}$.

Obviously, $D(\tilde{p}, \tilde{q}) = \{z \in M_\kappa \mid \angle(\tilde{p}z\tilde{q}) > \pi/2\}$. More precisely, $z \in D(\tilde{p}, \tilde{q})$ if and only if z satisfies the inequalities:

(1) $\cos\sqrt{\kappa}\, d(\tilde{p}, \tilde{q}) < \cos\sqrt{\kappa}\, d(\tilde{p}, z) \cos\sqrt{\kappa}\, d(\tilde{q}, z)$ if $\kappa > 0$,

(2) $d(\tilde{p}, \tilde{q})^2 > d(\tilde{p}, z)^2 + d(\tilde{q}, z)^2$ if $\kappa = 0$,

(3) $\cosh\sqrt{-\kappa}\, d(\tilde{p}, \tilde{q}) > \cosh\sqrt{-\kappa}\, d(\tilde{p}, z) \cosh\sqrt{-\kappa}\, d(\tilde{q}, z)$ if $\kappa < 0$.

Example 5.1. In M_1, if $\tilde{p}$ and $\tilde{q}$ satisfy $\pi > d(\tilde{p}, \tilde{q}) > \pi/2$ and $z \in M_1$ is a meeting point of the perpendiculars to $T(\tilde{p}, \tilde{q})$ at $\tilde{p}$ and $\tilde{q}$, then the domain bounded by the geodesic triangle $\triangle(\tilde{p}z\tilde{q})$ is contained in $D(\tilde{p},\tilde{q})$. In $M_0=\mathbb{E}^2$, by elementary geometry, we see that $D(\tilde{p},\tilde{q})$ is the open disk with diameter $d(\tilde{p},\tilde{q})$.

Corollary 5.2. *Let M be a complete Riemannian manifold with sectional curvature bounded below by κ. Assume that there exist two points p and q such that a dominant triangle $\Delta_M=\triangle(T(\tilde{p},\tilde{q}),\alpha_q(p),\alpha_q(p))$ for M can be made from p and q. If its inner angle at $z(\Delta_M)$ is greater than $\pi/2$, then M is topologically an n-sphere.*

Proof. Since the dominant triangle Δ_M is contained in $D(\tilde{p},\tilde{q})$, this proposition follows from Theorem 2.2. □

Let $\tilde{p},\tilde{q}\in M_\kappa$ such that $\tilde{p}\neq\tilde{q}$. Let $E(\tilde{p},\tilde{q})=\{z\in M_\kappa \mid \angle(\tilde{p}z\tilde{q})=\pi/2\}$. Namely, $E(\tilde{p},\tilde{q})=\partial D(\tilde{p},\tilde{q})$. Set

$$\omega=\omega(\kappa,d(\tilde{p},\tilde{q}))=\min\{\angle(z\tilde{p}\tilde{q})+\angle(z\tilde{q}\tilde{p}) \mid z\in E(\tilde{p},\tilde{q})\}.$$

Obviously, $\omega>0$. From the Gauss–Bonnet formula, we have $\omega=\pi/2$ when $\kappa\geq 0$ and $\omega<\pi/2$ when $\kappa<0$. If $\alpha_p(q)+\alpha_q(p)<\omega$, then there exists a dominant triangle for M.

Corollary 5.3. *Let M be a complete Riemannian n-manifold with sectional curvature bounded below by κ. If there exist two points $p,q\in M$ such that*

$$\alpha_p(q)+\alpha_q(p)<\omega(\kappa,d(p,q)),$$

then M is homeomorphic to an n-sphere.

Proof. From the assumption, there exists a dominant triangle Δ_M for M which is contained in $D(\tilde{p},\tilde{q})$. This corollary follows from Theorem 2.2. □

Remark 5.4. Let $\mathbb{E}^2$ denote the Euclidean plane. Let G be the isometry group generated by two translations $\mu(x,y)=(x+a,y)$ and $\nu(x,y)=(x,y+b)$ where a and b are positive constants. The quotient space is a flat torus $T^2=\mathbb{E}^2/G$. The equivalence class containing (x,y) is written with $[(x,y)]$. Let $p=[(a/2,b/2)]$ and $q=[(0,0)]$. There exist four minimizing geodesic segments connecting p and q in T^2. We then have $d(p,q)=\operatorname{diam}(T^2)$ and $\alpha_p(q)+\alpha_q(p)=\pi/2$, meaning that Corollary 5.3 is optimal.

Let $\mathcal{C}=\mathcal{C}(p,q)$ be the set of all midpoints between p and q, namely

$$\mathcal{C}=\{x\in M \mid d(p,x)=d(x,q)=d(p,q)/2\}.$$

If $x\in\mathcal{C}$, then $T(p,x)\cup T(x,q)$ is the unique minimizing geodesic segment through x connecting p and q.

Corollary 5.5. *Let M be a complete Riemannian n-manifold of nonnegative sectional curvature and $p,q\in M$. If $d(x,\mathcal{C}(p,q))<d(p,q)/2$ for all $x\in M\setminus\{p,q\}$, then M is topologically an n-sphere.*

Proof. We have only to prove that any point $x \in M \setminus \{p, q\}$ is not a critical point of the distance function d_p. We use the Euclidean plane $\mathbb{E}^2$ as a model space for the Toponogov comparison theorem. Let $\widetilde{T} = T(\tilde{p}, \tilde{q})$ be a segment in $\mathbb{E}^2$ with length $d(p, q)$ and $\tilde{m}$ the midpoint of $\widetilde{T}$.

Let $x \in M \setminus \{p, q\}$. From the assumption, there exists a midpoint m between p and q such that $d(x, m) < d(p, q)/2$. Let $\triangle(\tilde{p}\tilde{q}\tilde{x})$ be the comparison triangle in $\mathbb{E}^2$ corresponding to $\triangle(pqx)$. Then it follows from the Alexandrov convexity that $d(x, m) \geq d(\tilde{x}, \tilde{m})$. Therefore, we have $d(\tilde{m}, \tilde{x}) < d(\tilde{p}, \tilde{q})/2$. Thus we have $\angle(\tilde{p}\tilde{x}\tilde{q}) > \pi/2$. From the Toponogov comparison theorem, we have $\angle(pxq) > \pi/2$. This implies that x is not a critical point of d_p. □

Remark 5.6. Let T^2, p, and q be as in Remark 5.4. Let $s = [(0, b/2)]$. We then have $d(s, x) = \operatorname{diam}(T^2)/2$ for all $x \in \mathcal{C}(p, q)$. From this example, Corollary 5.5 is optimal.

6. Noncompact manifolds referred to M_κ

Let M be a complete noncompact Riemannian n-manifold with sectional curvature bounded below by $\kappa \leq 0$ and M_κ the κ-plane. Let γ be a ray in M with $\gamma(0) = p$. The Busemann function f_γ for γ is defined by

$$f_\gamma(x) = \lim_{t\to\infty} \bigl(t - d(x, \gamma(t))\bigr), \quad x \in M.$$

Let $B_\gamma(x)$ be the *open horoball* of a ray γ given by $\{y \in M \mid f_\gamma(y) > f_\gamma(x)\}$.

Let Γ_p denote the set of all rays from p in M. The *super* Busemann function f_p is given by $f_p(x) = \sup_{\gamma\in\Gamma_p} f_\gamma(x)$ for all $x \in M$.

Let $\tilde{\gamma}$ be a fixed ray in M_κ with $\tilde{\gamma}(0) = \tilde{p}$. We call $B_{\tilde{\gamma}}(z)$ a *horoball* of $\tilde{\gamma}$ determined by $z \in M_\kappa$. Since $\kappa \leq 0$, all horoballs are convex in M_κ, meaning that if $w_1, w_2 \in B_{\tilde{\gamma}}(z)$, then the unique minimizing geodesic segment $T(z_1, z_2)$ is contained in $B_{\tilde{\gamma}}(z)$.

Let $v(z)$ be the unit tangent vector at $z \in M_\kappa$ of the coray to $\tilde{\gamma}$ and $w(z)$ the unit tangent vector of geodesic segment from z to $\tilde{p}$ at z, respectively. Set

$$D(\tilde{\gamma}) = \{z \in M_\kappa \mid \angle(v(z), w(z)) > \pi/2\}.$$

We have $D(\tilde{\gamma}) = \lim_{t\to\infty} B_{\tilde{\gamma}(t)}(t)$ if $\kappa = 0$. When $\kappa < 0$, the boundary $\partial D(\tilde{\gamma})$ of $D(\tilde{\gamma})$ is the trace of those points $z(t) \in M_\kappa$, $t \geq 0$, such that the straight line tangent to the horocircle ${f_{\tilde{\gamma}}}^{-1}(t)$ through $\tilde{\gamma}(t)$ at $z(t)$ passes through $\tilde{p}$.

Example 6.1. Let $M_{-1} = \{(x, y) \mid x^2 + y^2 < 1\}$ and

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$$

be the Poincaré disk model. Let $\tilde{p} = (0, 0)$ and $\tilde{\gamma}([0, \infty)) = \{(0, t) \mid 0 \le t < 1\}$. If $x = r\cos\theta$, $y = r\sin\theta$, then $\partial D(\tilde{\gamma})$ is the trace of the curve given by the equation $r = \tan(\theta/2)$, $0 < \theta < \pi/2$. In fact, since any horocircle of $\tilde{\gamma}$ is a subarc of a circle with center $(u\cos\theta, 1)$ and radius $u\cos\theta$ and any geodesic from $(0, 0)$ is a subsegment of a straight line through $(0, 0)$ with slope $\tan\theta$, they meet at points satisfying

$$r = u - u\cos\theta, \quad 1 = u\sin\theta.$$

Hence, we have

$$r = \frac{1-\cos\theta}{\sin\theta} = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \frac{\sin(\theta/2)}{\cos(\theta/2)}.$$

Here we assume that $\kappa < 0$. As before, let $z(t) = \partial D(\tilde{\gamma}) \cap {f_{\tilde{\gamma}}}^{-1}(t)$ in M_κ. Let $\rho_{\tilde{p}}(t)$ be the angle of $\tilde{\gamma}$ with $T(\tilde{p}, z(t))$ at $\tilde{p}$ for $t \ge 0$. Then $\rho_{\tilde{p}}(0) = \pi/2$ and $\lim_{t\to\infty} \rho_{\tilde{p}}(t) = 0$. Moreover, $\rho_{\tilde{p}}(t)$ is monotone decreasing in $t \ge 0$.

Let $\tilde{\gamma}$ be a fixed ray in $(M_\kappa, \tilde{p})$ with $\tilde{\gamma}(0) = \tilde{p}$. Let Ψ_p be the reference map from M to ${M_\kappa}^+$. By definition, we have, for all points $x \in M$,

$$d(\tilde{p}, \Psi_p(x)) = d(p, x), \quad f_{\tilde{\gamma}}(\Psi_p(x)) = f_p(x).$$

Corollary 6.2. *Let M be a complete noncompact Riemannian n-manifold with sectional curvature bounded below by κ. If there exists a point $p \in M$ such that $\Psi_p(M \setminus \{p\}) \subset D(\tilde{\gamma})$, then M is diffeomorphic to the Euclidean space $\mathbb{E}^n$.*

Proof. From the definition of $D(\tilde{\gamma})$, there exists no critical point of d_p in $M \setminus \{p\}$. Lemma 2.1 proves this corollary. □

Proposition 6.3. *Let M denote a complete noncompact Riemannian n-manifold with sectional curvature bounded below by $\kappa < 0$. Assume that $\zeta_p < \pi/2$. Then p is a minimum point of f_p in M. If t_0 satisfies $\rho_{\tilde{p}}(t_0) = \zeta_p$, then there exists no critical point of d_p in ${f_p}^{-1}((0, t_0))$.*

Proof. Since $\zeta_p < \pi/2$, it follows that $f_p(p) = 0$ is a minimum of f_p in M. Let $x \in M$ be such that $0 < f_p(x) < t_0$. Let v be the initial tangent vector of a minimizing geodesic segment from p to x. From the definition of ζ_p, there exists $\gamma \in \Gamma_p$ such that $\angle(v, \dot{\gamma}(0)) \le \zeta_p$. From the definition of f_p, we have $f_\gamma(x) \le f_p(x) < t_0$ and, hence, from the Toponogov comparison theorem,

$$\rho_{\tilde{p}}(f_\gamma(x)) > \rho_{\tilde{p}}(t_0) = \zeta_p \ge \angle(v, \dot{\gamma}(0)) \ge \angle(\tilde{v}, \dot{\tilde{\gamma}})$$

where $\tilde{v}$ is the initial tangent vector of the minimizing geodesic segment from $\tilde{p}$ to $\Psi_\gamma(x)$ in M_κ. This inequality shows $\Psi_\gamma(x) \in D(\tilde{\gamma})$. □

References

[Abresch and Meyer 1997] U. Abresch and W. T. Meyer, "Injectivity radius estimates and sphere theorems", pp. 1–47 in *Comparison geometry* (Berkeley, CA, 1993–1994), edited by K. Grove and P. Petersen, Mathematical Sciences Research Institute Publications **30**, Cambridge University Press, 1997. MR 98e:53052 Zbl 0888.53001

[Belegradek et al. 2012] I. Belegradek, E. Choi, and N. Innami, "Rays and souls in von Mangoldt planes", *Pacific J. Math.* **259**:2 (2012), 279–306. MR 2988492 Zbl 1268.53041

[Cheeger and Ebin 1975] J. Cheeger and D. G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Mathematical Library **9**, North-Holland, Amsterdam, 1975. Reprinted by AMS Chelsea, Providence, RI, 2008. MR 0458335 Zbl 0309.53035

[Grove and Shiohama 1977] K. Grove and K. Shiohama, "A generalized sphere theorem", *Ann. of Math.* (2) **106**:2 (1977), 201–211. MR 58 #18268 Zbl 0341.53029

[Innami et al. 2013a] N. Innami, K. Shiohama, and Y. Uneme, "The Alexandrov–Toponogov comparison theorem for radial curvature", *Nihonkai Math. J.* **24**:2 (2013), 57–91. MR 3178500 Zbl 1296.53067

[Innami et al. 2013b] N. Innami, K. Shiohama, and Y. Uneme, "A sphere theorem for radial curvature", *Nihonkai Math. J.* **24**:2 (2013), 93–102. MR 3178501 Zbl 1288.53027

[Itokawa et al. 2001] Y. Itokawa, Y. Machigashira, and K. Shiohama, "Maximal diameter theorems for manifolds with restricted radial curvature", pp. 61–68 in *Proceedings of the 5th Pacific Rim Geometry Conference* (Sendai, 2000), edited by S. Nishikawa, Tôhoku Mathematical Publications **20**, Tôhoku University, Sendai, 2001. MR 2002k:53055 Zbl 1065.53033

[Itokawa et al. 2003] Y. Itokawa, Y. Machigashira, and K. Shiohama, "Generalized Toponogov's theorem for manifolds with radial curvature bounded below", pp. 121–130 in *Explorations in complex and Riemannian geometry*, edited by J. Bland et al., Contemporary Mathematics **332**, American Mathematical Society, Providence, RI, 2003. MR 2004j:53051 Zbl 1046.53017

[Klingenberg 1963] W. Klingenberg, "Manifolds with restricted conjugate locus", *Ann. of Math.* (2) **78** (1963), 527–547. MR 28 #2506 Zbl 0117.38701

[Kondo 2007] K. Kondo, "Radius sphere theorems for compact manifolds with radial curvature bounded below", *Tokyo J. Math.* **30**:2 (2007), 465–475. MR 2009b:53061 Zbl 1145.53025

[Kondo and Ohta 2007] K. Kondo and S.-I. Ohta, "Topology of complete manifolds with radial curvature bounded from below", *Geom. Funct. Anal.* **17**:4 (2007), 1237–1247. MR 2008k:53069 Zbl 1144.53050

[Kondo and Tanaka 2010] K. Kondo and M. Tanaka, "Total curvatures of model surfaces control topology of complete open manifolds with radial curvature bounded below. II", *Trans. Amer. Math. Soc.* **362**:12 (2010), 6293–6324. MR 2011f:53061 Zbl 1225.53034

[Lee 2005] H. Lee, "Generalized Alexandrov–Toponogov theorems for radially curved manifolds and their applications", *Kyushu J. Math.* **59**:2 (2005), 365–373. MR 2006i:53036 Zbl 1098.53029

[Sinclair and Tanaka 2007] R. Sinclair and M. Tanaka, "The cut locus of a two-sphere of revolution and Toponogov's comparison theorem", *Tohoku Math. J.* (2) **59**:3 (2007), 379–399. MR 2008k:53075 Zbl 1158.53033

[Tanaka 1992] M. Tanaka, "On the cut loci of a von Mangoldt's surface of revolution", *J. Math. Soc. Japan* **44**:4 (1992), 631–641. MR 93h:53034 Zbl 0789.53023

Received March 30, 2014. Revised December 10, 2014.

NOBUHIRO INNAMI
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
NIIGATA UNIVERSITY
NIIGATA 950-2181
JAPAN
innami@math.sc.niigata-u.ac.jp

YUYA UNEME
GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY
NIIGATA UNIVERSITY
NIIGATA 950-2181
JAPAN
f12j004g@alumni.niigata-u.ac.jp

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.187

A NOTE ON AN L^p-BRUNN–MINKOWSKI INEQUALITY FOR CONVEX MEASURES IN THE UNCONDITIONAL CASE

ARNAUD MARSIGLIETTI

We consider a different L^p-Minkowski combination of compact sets in $\mathbb{R}^n$ than the one introduced by Firey and we prove an L^p-Brunn–Minkowski inequality, $p \in [0, 1]$, for a general class of measures called convex measures that includes log-concave measures, under unconditional assumptions. As a consequence, we derive concavity properties of the function $t \mapsto \mu(t^{1/p} A)$, $p \in (0, 1]$, for unconditional convex measures μ and unconditional convex body A in $\mathbb{R}^n$. We also prove that the (B)-conjecture for all uniform measures is equivalent to the (B)-conjecture for all log-concave measures, completing recent works by Saroglou.

1. Introduction

The Brunn–Minkowski inequality is a fundamental inequality which states that, for every convex subset $A, B \subset \mathbb{R}^n$ and for every $\lambda \in [0, 1]$, one has

$$|(1-\lambda)A + \lambda B|^{\frac{1}{n}} \geq (1-\lambda)|A|^{\frac{1}{n}} + \lambda|B|^{\frac{1}{n}}, \tag{1}$$

where

$$A + B = \{a + b : a \in A, b \in B\}$$

denotes the *Minkowski sum* of A and B and where $|\cdot|$ denotes Lebesgue measure. The inequality and its consequences are well covered in the book [Schneider 1993] and the survey [Gardner 2002].

Several extensions of the Brunn–Minkowski inequality have been developed during the last decades by establishing functional versions (see, e.g., [Henstock and Macbeath 1953; Dubuc 1977; Dancs and Uhrin 1980; Uhrin 1994]), by considering different measures (see, e.g., [Borell 1974; 1975]), by generalizing the Minkowski sum (see, e.g., [Firey 1961; 1962; 1964; Lutwak 1993; 1996]), among others.

This research was supported in part by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation.
MSC2010: 28A75, 52A40, 60B11.
Keywords: Brunn–Minkowski–Firey theory, L^p-Minkowski combination, convex body, convex measure, (B)-conjecture.

In this paper, we will combine these extensions to prove an L^p-Brunn–Minkowski inequality for a large class of measures, including the log-concave measures.

Firstly, let us consider measures other than Lebesgue measure. Following Borell [1974; 1975], we say that a Borel measure μ in $\mathbb{R}^n$ is *s-concave*, $s \in [-\infty, +\infty]$, if the inequality

$$\mu((1-\lambda)A + \lambda B) \geq M_s^\lambda(\mu(A), \mu(B))$$

holds for every $\lambda \in [0, 1]$ and for every compact subset $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$. Here $M_s^\lambda(a, b)$ denotes the s-mean of the nonnegative real numbers a, b with weight λ, defined as

$$M_s^\lambda(a, b) = ((1-\lambda)a^s + \lambda b^s)^{\frac{1}{s}} \quad \text{if } s \notin \{-\infty, 0, +\infty\},$$

$M_{-\infty}^\lambda(a, b) = \min(a, b)$, $M_0^\lambda(a, b) = a^{1-\lambda}b^\lambda$, $M_{+\infty}^\lambda(a, b) = \max(a, b)$. Hence the Brunn–Minkowski inequality tells us that Lebesgue measure in $\mathbb{R}^n$ is $\frac{1}{n}$-concave.

As a consequence of the Hölder inequality, one has $M_p^\lambda(a, b) \leq M_q^\lambda(a, b)$ for every $p \leq q$. Thus every s-concave measure is $-\infty$-concave. The $-\infty$-concave measures are also called *convex measures*.

For $s \leq \frac{1}{n}$, Borell showed that every measure μ which is absolutely continuous with respect to n-dimensional Lebesgue measure is s-concave if and only if its density is an α-concave function, with

$$\alpha = \frac{s}{1-sn} \in \left[-\frac{1}{n}, +\infty\right]. \tag{2}$$

A function $f : \mathbb{R}^n \to [0, +\infty)$ is said to be *α-concave*, with $\alpha \in [-\infty, +\infty]$, if the inequality

$$f((1-\lambda)x + \lambda y) \geq M_\alpha^\lambda(f(x), f(y))$$

holds for every $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$ and for every $\lambda \in [0, 1]$.

Secondly, let us consider a generalization of the notion of the Minkowski sum introduced by Firey, which leads to an *L^p-Brunn–Minkowski theory*. For *convex bodies* A and B in $\mathbb{R}^n$ (i.e., compact convex sets containing the origin in the interior), the *L^p-Minkowski combination*, $p \in [-\infty, +\infty]$, of A and B with weight $\lambda \in [0, 1]$ is defined by

$$(1-\lambda) \cdot A \oplus_p \lambda \cdot B = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq M_p^\lambda(h_A(u), h_B(u)) \text{ for all } u \in S^{n-1}\},$$

where h_A denotes the *support function* of A defined by

$$h_A(u) = \max_{x \in A}\langle x, u\rangle, \quad u \in S^{n-1}.$$

Notice that, for every $p \leq q$, one has

$$(1-\lambda) \cdot A \oplus_p \lambda \cdot B \subset (1-\lambda) \cdot A \oplus_q \lambda \cdot B.$$

The support function is an important tool in convex geometry: it has the property of determining the convex body, since

$$A = \big\{x \in \mathbb{R}^n : \langle x, u\rangle \le h_A(u) \text{ for all } u \in S^{n-1}\big\},$$

and it is linear with respect to Minkowski sum and dilation:

$$h_{A+B} = h_A + h_B, \quad h_{\mu A} = \mu h_A$$

($A, B \subset \mathbb{R}^n$ and $\mu \ge 0$). Thus,

$$(1-\lambda)\cdot A \oplus_1 \lambda \cdot B = (1-\lambda)A + \lambda B.$$

In this paper, we consider a different L^p-Minkowski combination. We denote by $\mathbb{R}_+$ the set of nonnegative real numbers. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *unconditional* if there exists a basis $(a_1, \dots, a_n)$ of $\mathbb{R}^n$ (the canonical basis in the sequel) such that, for every $x = \sum_{i=1}^n x_i a_i \in \mathbb{R}^n$ and for every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, one has $f\big(\sum_{i=1}^n \varepsilon_i x_i a_i\big) = f(x)$. A measure which is absolutely continuous with respect to n-dimensional Lebesgue measure is *unconditional* if its density function is unconditional. For $\boldsymbol{p} = (p_1, \dots, p_n) \in [-\infty, +\infty]^n$, $a = (a_1, \dots, a_n) \in (\mathbb{R}_+)^n$, $b = (b_1, \dots, b_n) \in (\mathbb{R}_+)^n$ and $\lambda \in [0, 1]$, let us denote

$$(1-\lambda)a +_{\boldsymbol{p}} \lambda b = (M_{p_1}^{\lambda}(a_1, b_1), \dots, M_{p_n}^{\lambda}(a_n, b_n)) \in (\mathbb{R}_+)^n.$$

For subsets $A, B \subset \mathbb{R}^n$ such that $A \cap (\mathbb{R}_+)^n$ and $B \cap (\mathbb{R}_+)^n$ are nonempty, for $\boldsymbol{p} \in [-\infty, +\infty]^n$ and for $\lambda \in [0, 1]$, we define the L^p-Minkowski combination of A and B with weight λ, denoted by $(1-\lambda)\cdot A +_{\boldsymbol{p}} \lambda \cdot B$, to be the unconditional subset (i.e., the indicator function is unconditional) such that

$$((1-\lambda)\cdot A +_{\boldsymbol{p}} \lambda \cdot B) \cap (\mathbb{R}_+)^n = \big\{(1-\lambda)a +_{\boldsymbol{p}} \lambda b : a \in A \cap (\mathbb{R}_+)^n, b \in B \cap (\mathbb{R}_+)^n\big\}.$$

This definition is consistent with the well known fact that an unconditional set (or function) is determined by its restriction to the positive octant $(\mathbb{R}_+)^n$. Moreover, this L^p-Minkowski combination coincides with the classical Minkowski sum when $\boldsymbol{p} = (1, \dots, 1)$ and A, B are unconditional convex subsets of $\mathbb{R}^n$ (see Proposition 2.1).

Using an extension of the Brunn–Minkowski inequality discovered by Uhrin [1994], we prove the following result:

Theorem 1.1. *Let $\boldsymbol{p} = (p_1, \dots, p_n) \in [0, 1]^n$ and $\alpha \in \mathbb{R}$ with $\alpha \ge -\big(\sum_{i=1}^n p_i^{-1}\big)^{-1}$. Let μ be an unconditional measure in $\mathbb{R}^n$ that has an α-concave density function with respect to Lebesgue measure. Then, for every unconditional convex body A, B in $\mathbb{R}^n$ and for every $\lambda \in [0, 1]$,*

$$\mu((1-\lambda)\cdot A +_{\boldsymbol{p}} \lambda \cdot B) \ge M_{\gamma}^{\lambda}(\mu(A), \mu(B)), \tag{3}$$

where $\gamma = \big(\sum_{i=1}^n p_i^{-1} + \alpha^{-1}\big)^{-1}$.

In Theorem 1.1, if α or one of the p_i is equal to 0, then $\left(\sum_{i=1}^n p_i^{-1}\right)^{-1}$ and γ are defined by continuity and are equal to 0.

The case of Lebesgue measure and $\boldsymbol{p} = (0, \dots, 0)$ is treated by Saroglou [2015], answering a conjecture by Böröczky, Lutwak, Yang and Zhang [Böröczky et al. 2012] in the unconditional case.

Conjecture 1.2 (log-Brunn–Minkowski inequality [Böröczky et al. 2012]). *Let A, B be symmetric convex bodies in $\mathbb{R}^n$ and let $\lambda \in [0, 1]$. Then*

$$|(1-\lambda)\cdot A \oplus_0 \lambda \cdot B| \geq |A|^{1-\lambda}|B|^{\lambda}. \tag{4}$$

Useful links between Conjecture 1.2 and the (B)-conjecture have been discovered by Saroglou [2014; 2015].

Conjecture 1.3 ((B)-conjecture [Latała 2002; Cordero-Erausquin et al. 2004]). *Let μ be a symmetric log-concave measure in $\mathbb{R}^n$ and let A be a symmetric convex subset of $\mathbb{R}^n$. Then the function $t \mapsto \mu(e^t A)$ is log-concave on $\mathbb{R}$.*

The (B)-conjecture was solved by Cordero-Erausquin, Fradelizi and Maurey [Cordero-Erausquin et al. 2004] for the Gaussian measure and for the unconditional case. As a variant of the (B)-conjecture, one may study concavity properties of the function $t \mapsto \mu(V(t)A)$ where $V : \mathbb{R} \to \mathbb{R}_+$ is a convex function. As a consequence of Theorem 1.1, we deduce concavity properties of the function $t \mapsto \mu(t^{\frac{1}{p}} A)$, $p \in (0, 1]$, for every unconditional s-concave measure μ and every unconditional convex body A in $\mathbb{R}^n$ (see Proposition 2.4).

Saroglou [2014] has also proved that the log-Brunn–Minkowski inequality for Lebesgue measure — which is to say, inequality (4) — is equivalent to the log-Brunn–Minkowski inequality for all log-concave measures. We continue these kinds of equivalences by proving that the (B)-conjecture for all uniform measures is equivalent to the (B)-conjecture for all log-concave measures (see Proposition 3.1).

We also investigate functional versions of the (B)-conjecture, which may be read as follows:

Conjecture 1.4 (functional version of the (B)-conjecture). *Let $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be even log-concave functions. Then the function*

$$t \mapsto \int_{\mathbb{R}^n} f(e^{-t}x)g(x)\,dx$$

is log-concave on $\mathbb{R}$.

We prove that Conjecture 1.4 is equivalent to Conjecture 1.3 (see Proposition 3.2).

Let us note that other developments in the use of the earlier mentioned extensions of the Brunn–Minkowski inequality have been recently made as well. See, e.g., [Bobkov et al. 2014; Caglar and Werner 2014; Caglar et al. 2015; Gardner et al. 2014].

The rest of the paper is organized as follows: in the next section, we prove Theorem 1.1 and we extend it to m sets, $m \geq 2$. We also compare our L^p-Minkowski combination to the Firey combination and derive an L^p-Brunn–Minkowski inequality for the Firey combination. We then discuss the consequences of a variant of the (B)-conjecture, namely we deduce concavity properties of the function $t \mapsto \mu(t^{\frac{1}{p}} A)$, $p \in (0, 1]$. In Section 3, we prove that the (B)-conjecture for all uniform measures is equivalent to the (B)-conjecture for all log-concave measures, and we also prove that the (B)-conjecture is equivalent to its functional version, Conjecture 1.4.

2. Proof of Theorem 1.1 and consequences

Before proving Theorem 1.1, let us show that our L^p-Minkowski combination coincides with the classical Minkowski sum when $\boldsymbol{p} = (1, \dots, 1)$, for unconditional convex sets.

Proposition 2.1. *Let A, B be unconditional convex subsets of $\mathbb{R}^n$ and let $\lambda \in [0, 1]$. Then*

$$(1-\lambda) \cdot A +_{\mathbf{1}} \lambda \cdot B = (1-\lambda)A + \lambda B,$$

where $\mathbf{1} = (1, \dots, 1)$.

Proof. Since the sets $(1-\lambda) \cdot A +_{\mathbf{1}} \lambda \cdot B$ and $(1-\lambda)A + \lambda B$ are unconditional, it is sufficient to prove that

$$((1-\lambda) \cdot A +_{\mathbf{1}} \lambda \cdot B) \cap (\mathbb{R}_+)^n = ((1-\lambda)A + \lambda B) \cap (\mathbb{R}_+)^n.$$

Let $x \in ((1-\lambda)A + \lambda B) \cap (\mathbb{R}_+)^n$. There exists $a = (a_1, \dots, a_n) \in A$ and $b = (b_1, \dots, b_n) \in B$ such that $x = (1-\lambda)a + \lambda b$ and, for every $i \in \{1, \dots, n\}$, $(1-\lambda)a_i + \lambda b_i \in \mathbb{R}_+$. Let $\varepsilon, \eta \in \{-1, 1\}^n$ such that $(\varepsilon_1 a_1, \dots, \varepsilon_n a_n) \in (\mathbb{R}_+)^n$ and $(\eta_1 b_1, \dots, \eta_n b_n) \in (\mathbb{R}_+)^n$. Notice that, for every $i \in \{1, \dots, n\}$, $0 \leq (1-\lambda)a_i + \lambda b_i \leq (1-\lambda)\varepsilon_i a_i + \lambda \eta_i b_i$. Since the sets A and B are convex and unconditional, it follows that $x \in (1-\lambda)(A \cap (\mathbb{R}_+)^n) + \lambda(B \cap (\mathbb{R}_+)^n) = ((1-\lambda) \cdot A +_{\mathbf{1}} \lambda \cdot B) \cap (\mathbb{R}_+)^n$.

The other inclusion is clear due to the definition of the set $(1-\lambda) \cdot A +_{\mathbf{1}} \lambda \cdot B$. □

Proof of Theorem 1.1. Let $\lambda \in [0, 1]$ and let A, B be unconditional convex bodies in $\mathbb{R}^n$.

It has been shown by Uhrin [1994] that if $f, g, h : (\mathbb{R}_+)^n \to \mathbb{R}_+$ are bounded measurable functions such that, for every $x, y \in (\mathbb{R}_+)^n$, $h((1-\lambda)x +_{\boldsymbol{p}} \lambda y) \geq M_\alpha^\lambda(f(x), g(y))$, then

$$\int_{(\mathbb{R}_+)^n} h(x)\, dx \geq M_\gamma^\lambda\biggl(\int_{(\mathbb{R}_+)^n} f(x)\, dx, \int_{(\mathbb{R}_+)^n} g(x)\, dx\biggr),$$

where $\gamma = \bigl(\sum_{i=1}^n p_i^{-1} + \alpha^{-1}\bigr)^{-1}$.

Let us denote by ϕ the density function of μ and let us set $h = 1_{(1-\lambda)\cdot A +_{p} \lambda \cdot B}\phi$, $f = 1_A\phi$ and $g = 1_B\phi$. By assumption, the function ϕ is unconditional and α-concave, hence ϕ is nonincreasing in each coordinate on the octant $(\mathbb{R}_+)^n$. Then for every $x, y \in (\mathbb{R}_+)^n$ one has

$$\phi((1-\lambda)x +_{p} \lambda y) \geq \phi((1-\lambda)x + \lambda y) \geq M_\alpha^\lambda(\phi(x), \phi(y)).$$

Hence,

$$h((1-\lambda)x +_{p} \lambda y) \geq M_\alpha^\lambda(f(x), g(y)).$$

Thus we may apply the result mentioned at the beginning of the proof to obtain that

$$\int_{(\mathbb{R}_+)^n} h(x)\,dx \geq M_\gamma^\lambda\left(\int_{(\mathbb{R}_+)^n} f(x)\,dx, \int_{(\mathbb{R}_+)^n} g(x)\,dx\right),$$

where $\gamma = \left(\sum_{i=1}^n p_i^{-1} + \alpha^{-1}\right)^{-1}$. In other words, one has

$$\mu\big(((1-\lambda)\cdot A +_{p} \lambda \cdot B) \cap (\mathbb{R}_+)^n\big) \geq M_\gamma^\lambda\big(\mu(A \cap (\mathbb{R}_+)^n), \mu(B \cap (\mathbb{R}_+)^n)\big).$$

Since the sets $(1-\lambda)\cdot A +_{p} \lambda \cdot B$, A and B are unconditional, it follows that

$$\mu((1-\lambda)\cdot A +_{p} \lambda \cdot B) \geq M_\gamma^\lambda(\mu(A), \mu(B)). \qquad \square$$

Remark. One may similarly define the L^p-Minkowski combination

$$\lambda_1 \cdot A_1 +_{p} \cdots +_{p} \lambda_m \cdot A_m$$

for m convex bodies $A_1, \ldots, A_m \subset \mathbb{R}^n$, $m \geq 2$, where $\lambda_1, \ldots, \lambda_m \in [0, 1]$ are such that $\sum_{i=1}^m \lambda_i = 1$, by extending the definition of the p-mean M_p^λ to m nonnegative numbers. By induction, one has under the same assumptions of Theorem 1.1 that

$$\mu(\lambda_1 \cdot A_1 +_{p} \cdots +_{p} \lambda_m \cdot A_m) \geq M_\gamma^\lambda(\mu(A_1), \ldots, \mu(A_m)), \tag{5}$$

where $\gamma = \left(\sum_{i=1}^n p_i^{-1} + \alpha^{-1}\right)^{-1}$. Indeed, let $m \geq 2$ and let us assume that inequality (5) holds. Notice that

$$\lambda_1 \cdot A_1 +_{p} \cdots +_{p} \lambda_m \cdot A_m +_{p} \lambda_{m+1} \cdot A_{m+1} = \left(\sum_{i=1}^m \lambda_i\right) \cdot \widetilde{A} +_{p} \lambda_{m+1} \cdot A_{m+1},$$

where

$$\widetilde{A} := \left(\frac{\lambda_1}{\sum_{i=1}^m \lambda_i} \cdot A_1 +_{p} \cdots +_{p} \frac{\lambda_m}{\sum_{i=1}^m \lambda_i} \cdot A_m\right).$$

Thus,

$$\mu\left(\left(\sum_{i=1}^{m}\lambda_i\right)\cdot\widetilde{A}+_{\boldsymbol{p}}\lambda_{m+1}\cdot A_{m+1}\right)\geq\left(\left(\sum_{i=1}^{m}\lambda_i\right)\mu(\widetilde{A})^{\gamma}+\lambda_{m+1}\mu(A_{m+1})^{\gamma}\right)^{\frac{1}{\gamma}}$$
$$\geq\left(\sum_{i=1}^{m+1}\lambda_i\mu(A_i)^{\gamma}\right)^{\frac{1}{\gamma}}.$$

Consequences. The following result compares the L^p-Minkowski combinations $\oplus_p$ and $+_{\boldsymbol{p}}$.

Lemma 2.2. *Let $p\in[0,1]$ and set $\boldsymbol{p}=(p,\dots,p)\in[0,1]^n$. For every unconditional convex body A, B in $\mathbb{R}^n$ and for every $\lambda\in[0,1]$, one has*

$$(1-\lambda)\cdot A\oplus_p\lambda\cdot B\supset(1-\lambda)\cdot A+_{\boldsymbol{p}}\lambda\cdot B.$$

Proof. The case $p=0$ is proved in [Saroglou 2015]. Let $p\neq 0$. Since the sets $(1-\lambda)\cdot A\oplus_p\lambda\cdot B$ and $(1-\lambda)\cdot A+_{\boldsymbol{p}}\lambda\cdot B$ are unconditional, it is sufficient to prove that

$$((1-\lambda)\cdot A\oplus_p\lambda\cdot B)\cap(\mathbb{R}_+)^n\supset((1-\lambda)\cdot A+_{\boldsymbol{p}}\lambda\cdot B)\cap(\mathbb{R}_+)^n.$$

Let $u\in S^{n-1}\cap(\mathbb{R}_+)^n$ and let $x\in((1-\lambda)\cdot A+_{\boldsymbol{p}}\lambda\cdot B)\cap(\mathbb{R}_+)^n$. One has

$$\langle x,u\rangle=\sum_{i=1}^{n}((1-\lambda)a_i^p+\lambda b_i^p)^{\frac{1}{p}}u_i=\sum_{i=1}^{n}((1-\lambda)(a_iu_i)^p+\lambda(b_iu_i)^p)^{\frac{1}{p}}$$
$$=\|(1-\lambda)X+\lambda Y\|_{\frac{1}{p}}^{\frac{1}{p}},$$

where $X=((a_1u_1)^p,\dots,(a_nu_n)^p)$ and $Y=((b_1u_1)^p,\dots,(b_nu_n)^p)$. Notice that $\|X\|_{\frac{1}{p}}\leq h_A(u)^p$, $\|Y\|_{\frac{1}{p}}\leq h_B(u)^p$ and that $\|\cdot\|_{\frac{1}{p}}$ is a norm. It follows that

$$\langle x,u\rangle\leq\big((1-\lambda)\|X\|_{\frac{1}{p}}+\lambda\|Y\|_{\frac{1}{p}}\big)^{\frac{1}{p}}\leq\big((1-\lambda)h_A(u)^p+\lambda h_B(u)^p\big)^{\frac{1}{p}}.$$

Hence, $x\in((1-\lambda)\cdot A\oplus_p\lambda\cdot B)\cap(\mathbb{R}_+)^n$. □

From Lemma 2.2 and Theorem 1.1, one obtains the following result:

Corollary 2.3. *Let $p\in[0,1]$. Let μ be an unconditional measure in $\mathbb{R}^n$ that has an α-concave density function, with $\alpha\geq-\frac{p}{n}$. Then, for every unconditional convex body A, B in $\mathbb{R}^n$ and for every $\lambda\in[0,1]$,*

$$\mu((1-\lambda)\cdot A\oplus_p\lambda\cdot B)\geq M_\gamma^\lambda(\mu(A),\mu(B)),\tag{6}$$

where $\gamma=\left(\frac{n}{p}+\frac{1}{\alpha}\right)^{-1}$.

In Corollary 2.3, if α or p is equal to 0, then γ is defined by continuity and is equal to 0.

Remarks. (1) By taking $\alpha = 0$ in Corollary 2.3 (corresponding to log-concave measures), one obtains

$$\mu((1-\lambda)\cdot A \oplus_0 \lambda \cdot B) \geq \mu(A)^{1-\lambda}\mu(B)^{\lambda}.$$

(2) By taking $\alpha = +\infty$ in Corollary 2.3 (corresponding to $\frac{1}{n}$-concave measures), one obtains that, for every $p \in [0, 1]$,

$$\mu((1-\lambda)\cdot A \oplus_p \lambda \cdot B)^{\frac{p}{n}} \geq (1-\lambda)\mu(A)^{\frac{p}{n}} + \lambda\mu(B)^{\frac{p}{n}}.$$

Equivalently, for every $p \in [0, 1]$, for every unconditional convex body A, B in $\mathbb{R}^n$ and for every unconditional convex set $K \subset \mathbb{R}^n$,

$$|((1-\lambda)\cdot A \oplus_p \lambda \cdot B) \cap K|^{\frac{p}{n}} \geq (1-\lambda)|A \cap K|^{\frac{p}{n}} + \lambda|B \cap K|^{\frac{p}{n}}.$$

Let us recall that the function $t \mapsto \mu(e^t A)$ is log-concave on $\mathbb{R}$ for every unconditional log-concave measure μ and every unconditional convex body A in $\mathbb{R}^n$ (see [Cordero-Erausquin et al. 2004]). By adapting the argument of [Marsiglietti 2015], Proof of Proposition 3.1 (see Proof of Corollary 2.5), it follows that the function $t \mapsto \mu(t^{\frac{1}{p}} A)$ is $\frac{p}{n}$-concave on $\mathbb{R}_+$, for every $p \in (0, 1]$, for every unconditional s-concave measure μ, with $s \geq 0$, and for every unconditional convex body A in $\mathbb{R}^n$. However, no concavity properties are known for the function $t \mapsto \mu(e^t A)$ when μ is an s-concave measure with $s < 0$. Instead, for these measures we prove concavity properties of the function $t \mapsto \mu(t^{\frac{1}{p}} A)$.

Proposition 2.4. *Let $p \in (0, 1]$ and $\alpha \in [-\frac{p}{n}, 0)$, let μ be an unconditional measure that has an α-concave density function, and let A be an unconditional convex body in $\mathbb{R}^n$. Then the function $t \mapsto \mu(t^{\frac{1}{p}} A)$ is $\left(\frac{n}{p} + \frac{1}{\alpha}\right)^{-1}$-concave on $\mathbb{R}_+$.*

Proof. Let $t_1, t_2 \in \mathbb{R}_+$. By applying Corollary 2.3 to the sets $t_1^{\frac{1}{p}} A$ and $t_2^{\frac{1}{p}} A$, one obtains

$$\begin{aligned}\mu\big(((1-\lambda)t_1 + \lambda t_2)^{\frac{1}{p}} A\big) &= \mu\big((1-\lambda)\cdot t_1^{\frac{1}{p}} A \oplus_p \lambda \cdot t_2^{\frac{1}{p}} A\big) \\ &\geq M_{\gamma}^{\lambda}\big(\mu(t_1^{\frac{1}{p}} A), \mu(t_2^{\frac{1}{p}} A)\big),\end{aligned}$$

where $\gamma = \left(\frac{n}{p} + \frac{1}{\alpha}\right)^{-1}$. Hence the function $t \mapsto \mu(t^{\frac{1}{p}} A)$ is γ-concave on $\mathbb{R}_+$. □

As a consequence, we derive concavity properties for the function $t \mapsto \mu(tA)$.

Corollary 2.5. *Let $p \in (0, 1]$, let μ be an unconditional measure that has an α-concave density function, with $\alpha \in [-\frac{p}{n}, 0)$, and let A be an unconditional convex body in $\mathbb{R}^n$. Then the function $t \mapsto \mu(tA)$ is $\left(\frac{1-p}{n} + \gamma\right)$-concave on $\mathbb{R}_+$, where $\gamma = \left(\frac{n}{p} + \frac{1}{\alpha}\right)^{-1}$.*

Proof. We adapt [Marsiglietti 2015], Proof of Proposition 3.1. Let us denote by ϕ the density function of the measure μ and let us denote by F the function $t \mapsto \mu(tA)$. From Proposition 2.4, the function $t \mapsto F(t^{\frac{1}{p}})$ is γ-concave, hence the right derivative of F, denoted by F'_+, exists everywhere and the function $t \mapsto \frac{1}{p} t^{\frac{1}{p}-1} F'_+(t^{\frac{1}{p}}) F(t^{\frac{1}{p}})^{\gamma-1}$ is nonincreasing. Notice that

$$F(t) = t^n \int_A \phi(tx)\, dx$$

and that $t \mapsto \phi(tx)$ is nonincreasing; thus the function $t \mapsto \dfrac{1}{t^{1-p}} F(t)^{\frac{1-p}{n}}$ is nonincreasing. Since

$$F'_+(t) F(t)^{\frac{1-p}{n}+\gamma-1} = t^{1-p} F'_+(t) F(t)^{\gamma-1} \cdot \frac{1}{t^{1-p}} F(t)^{\frac{1-p}{n}},$$

it follows that $F'_+(t) F(t)^{\frac{1-p}{n}+\gamma-1}$ is nonincreasing as the product of two nonnegative nonincreasing functions. Hence F is $\left(\frac{1-p}{n} + \gamma\right)$-concave. □

Remark. For every s-concave measure μ and for every convex subset $A \subset \mathbb{R}^n$, the function $t \mapsto \mu(tA)$ is s-concave. Hence Corollary 2.5 is of value only if $\frac{1-p}{n} + \gamma \geq \alpha/(1+\alpha n)$ (see relation (2)). Notice that this condition is satisfied if $\alpha \geq -p/(n(1+p))$. We thus obtain:

Corollary 2.6. *Let* $p \in (0, 1]$, *let* μ *be an unconditional measure that has an* α*-concave density function, with* $-p/(n(1+p)) \leq \alpha < 0$, *and let* K *be an unconditional convex body in* $\mathbb{R}^n$. *Then, for all subsets* $A, B \in \{\mu K : \mu > 0\}$ *and all* $\lambda \in [0, 1]$, *one has*

$$\mu((1-\lambda)A + \lambda B) \geq M^{\lambda}_{\frac{1-p}{n}+\gamma}(\mu(A), \mu(B)),$$

where $\gamma = \left(\frac{n}{p} + \frac{1}{\alpha}\right)^{-1}$.

In [Marsiglietti 2015] we investigated improvements of concavity properties of convex measures under additional assumptions, such as symmetries. Corollary 2.6 follows the same path and completes the results found there.

We conclude this section with a remark on the question of improving the concavity properties of convex measures.

Remark. Let μ be a Borel measure that has a density function with respect to Lebesgue measure in $\mathbb{R}^n$. One may write the density function of μ in the form e^{-V}, where $V : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. Let us assume that V is C^2. Let $\gamma > 0$. The function e^{-V} is γ-concave if $\mathrm{Hess}(\gamma e^{-\gamma V})$, the Hessian of $\gamma e^{-\gamma V}$, is nonpositive (in the sense of symmetric matrices). One has

$$\mathrm{Hess}(\gamma e^{-\gamma V}) = -\gamma^2 \nabla \cdot (\nabla V e^{-\gamma V}) = \gamma^2 e^{-V} (\gamma \nabla V \otimes \nabla V - \mathrm{Hess}\, V),$$

where

$$\nabla V \otimes \nabla V = \left(\frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} \right)_{1 \le i,j \le n}.$$

It follows that the matrix $\mathrm{Hess}(\gamma e^{-\gamma V})$ is nonpositive if and only if the matrix $\gamma \nabla V \otimes \nabla V - \mathrm{Hess}\, V$ is nonpositive.

Let us apply this remark to the Gaussian measure

$$d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}}\, dx, \quad x \in \mathbb{R}^n.$$

Here $V(x) = \frac{|x|^2}{2} + c_n$, where $c_n = \frac{n}{2}\log(2\pi)$. Thus $\nabla V \otimes \nabla V = (x_i x_j)_{1 \le i,j \le n}$ and $\mathrm{Hess}\, V = \mathrm{Id}$, the identity matrix. The eigenvalues of $\gamma \nabla V \otimes \nabla V - \mathrm{Hess}\, V$ are -1 (with multiplicity $(n-1)$) and $\gamma |x|^2 - 1$. Hence, if $\gamma |x|^2 - 1 \le 0$, then $\gamma \nabla V \otimes \nabla V - \mathrm{Hess}\, V$ is nonpositive. One deduces that, for every $\gamma > 0$, for all compact sets $A, B \subset \frac{1}{\sqrt{\gamma}} B_2^n$ and for every $\lambda \in [0, 1]$, one has

$$\gamma_n((1-\lambda)A + \lambda B) \ge M^{\lambda}_{\frac{\gamma}{1+\gamma n}}(\gamma_n(A), \gamma_n(B)), \tag{7}$$

where B_2^n denotes the Euclidean closed unit ball in $\mathbb{R}^n$.

Since the Gaussian measure is a log-concave measure, inequality (7) is an improvement of the concavity of the Gaussian measure when restricted to compact sets $A, B \subset \frac{1}{\sqrt{\gamma}} B_2^n$.

3. Equivalence between (B)-conjecture-type problems

The next proposition reduces the proof of the (B)-conjecture for all uniform measures in $\mathbb{R}^n$, for every $n \in \mathbb{N}^*$, to proving the (B)-conjecture for all symmetric log-concave measures in $\mathbb{R}^n$, for every $n \in \mathbb{N}^*$. This completes recent work by Saroglou [2014; 2015].

We will say that a measure μ satisfies the (B)-property if the function $t \mapsto \mu(e^t A)$ is log-concave on $\mathbb{R}$ for every symmetric convex set $A \subset \mathbb{R}^n$.

Proposition 3.1. *If every symmetric uniform measure in $\mathbb{R}^n$, for every $n \in \mathbb{N}^*$, satisfies the (B)-property, then every symmetric log-concave measure in $\mathbb{R}^n$, for every $n \in \mathbb{N}^*$, satisfies the (B)-property.*

Proof. The proof is inspired by [Artstein-Avidan et al. 2004, beginning of Section 3].

Step 1: Stability under orthogonal projection. Let us show that the (B)-property is stable under orthogonal projection onto an arbitrary subspace.

Let F be a k-dimensional subspace of $\mathbb{R}^n$. Let us define, for every compactly supported measure μ in $\mathbb{R}^n$ and every measurable subset $A \subset F$,

$$\Pi_F \mu(A) := \mu(\Pi_F^{-1}(A)),$$

where Π_F denotes the orthogonal projection onto F and

$$\Pi_F^{-1}(A) := \{x \in \mathbb{R}^n : \Pi_F(x) \in A\}.$$

We have $\Pi_F^{-1}(e^t A) = e^t(A \times F^{\perp})$, where $F^{\perp}$ denotes the orthogonal complement of F. Hence if μ satisfies the (B)-property, so does $\Pi_F \mu$.

Step 2: Approximation of log-concave measures. Let us show that for every compactly supported log-concave measure μ in $\mathbb{R}^n$ there exists a sequence $(K_p)_{p\in\mathbb{N}^*}$ of convex subsets of $\mathbb{R}^{n+p}$ such that $\lim_{p\to+\infty} \Pi_{\mathbb{R}^n}\mu_{K_p} = \mu$ in the sense that the density function of μ is the pointwise limit of the density functions of $(\mu_{K_p})_{p\in\mathbb{N}^*}$, where μ_{K_p} denotes the uniform measure on K_p (up to a constant).

Let μ be a compactly supported log-concave measure in $\mathbb{R}^n$ with density function $f = e^{-V}$, where $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function. To simplify notation, define

$$W(x) = \left(1 - \frac{V(x)}{p}\right)_+, \tag{8}$$

where $a_+ = \max(a, 0)$ for every $a \in \mathbb{R}$. Notice that $e^{-V(x)} = \lim\limits_{p\to+\infty} W(x)^p$ for every $x \in \mathbb{R}^n$. Let us define for every $p \in \mathbb{N}^*$

$$K_p = \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : |y| \le W(x)\big\}.$$

One has, for every $x \in \mathbb{R}^n$,

$$W(x)^p = \int_0^{W(x)} p r^{p-1}\,dr = p\int_0^{+\infty} 1_{[0,W(x)]}(r) r^{p-1}\,dr = \frac{1}{v_p}\int_{\mathbb{R}^p} 1_{K_p}(x, y)\,dy.$$

The last equality follows from an integration in polar coordinates, where v_p denotes the volume of the Euclidean closed unit ball in $\mathbb{R}^p$. By denoting μ_{K_p} the measure in $\mathbb{R}^{n+p}$ with density function

$$\frac{1}{v_p} 1_{K_p}(x, y), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^p,$$

it follows that, for every $p \in \mathbb{N}^*$, the measure $\Pi_{\mathbb{R}^n}\mu_{K_p}$ has density function $W(x)^p$, $x \in \mathbb{R}^n$. We conclude that $\lim_{p\to+\infty} \Pi_{\mathbb{R}^n}\mu_{K_p} = \mu$.

Step 3: Conclusion. Let $n \in \mathbb{N}^*$ and let μ be a symmetric log-concave measure in $\mathbb{R}^n$. By approximation, one can assume that μ is compactly supported. Since μ is symmetric, the sequence $(K_p)_{p\in\mathbb{N}^*}$ defined in Step 2 is a sequence of symmetric convex subsets of $\mathbb{R}^{n+p}$. If we assume that the (B)-property holds for all uniform measures in $\mathbb{R}^m$, for every $m \in \mathbb{N}^*$, then, for every $p \in \mathbb{N}^*$, μ_{K_p} satisfies the (B)-property. It follows from Step 1 that, for every $p \in \mathbb{N}^*$, $\Pi_{\mathbb{R}^n}\mu_{K_p}$ satisfies the (B)-property. Since $\lim_{p\to+\infty} \Pi_{\mathbb{R}^n}\mu_{K_p} = \mu$ (see Step 2) and since a pointwise limit of log-concave functions is log-concave, we conclude that μ satisfies the (B)-property. □

Similarly, let us now prove that the functional form of the (B)-conjecture (Conjecture 1.4) is equivalent to the classical (B)-conjecture (Conjecture 1.3).

Proposition 3.2. *One has equivalence between the following properties*:

(1) *For every $n \in \mathbb{N}^*$, for every symmetric log-concave measure μ in $\mathbb{R}^n$ and for every symmetric convex subset A of $\mathbb{R}^n$, the function $t \mapsto \mu(e^t A)$ is log-concave on $\mathbb{R}$.*

(2) *For every $n \in \mathbb{N}^*$ and for all even log-concave functions $f, g : \mathbb{R}^n \to \mathbb{R}_+$, the function $t \mapsto \int_{\mathbb{R}^n} f(e^{-t}x)g(x)\,dx$ is log-concave on $\mathbb{R}$.*

Proof. (2) $\Longrightarrow$ (1) This is clear by taking f to be 1_A, the indicator function of a symmetric convex set A, and by taking g to be the density function of a log-concave measure μ.

(1) $\Longrightarrow$ (2) Let $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be even log-concave functions. By approximation, one may assume that f and g are compactly supported. Let us write $g = e^{-V}$, where $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is an even convex function. One has

$$G(t) := \int_{\mathbb{R}^n} f(e^{-t}x)e^{-V(x)}\,dx = \lim_{p\to+\infty} \int_{\mathbb{R}^n} f(e^{-t}x)W(x)^p\,dx,$$

where $W(x)$ is as in (8). Let us denote, for $t \in \mathbb{R}$,

$$G_p(t) = \int_{\mathbb{R}^n} f(e^{-t}x)W(x)^p\,dx.$$

We have seen in the proof of Proposition 3.1 that

$$W(x)^p = \frac{1}{v_p} \int_{\mathbb{R}^p} 1_{K_p}(x, y)\,dy,$$

where $K_p := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : |y| \le W(x)\}$ and where v_p denotes the volume of the Euclidean closed unit ball in $\mathbb{R}^p$. Hence,

$$G_p(t) = \frac{1}{v_p} \int_{K_p} f(e^{-t}x)1_{\mathbb{R}^p}(y)\,dx\,dy.$$

Notice that K_p is a symmetric convex subset of $\mathbb{R}^{n+p}$. The change of variable $\tilde{x} = e^{-t}x$ and $\tilde{y} = e^{-t}y$ leads to

$$G_p(t) = \frac{e^{t(n+p)}}{v_p}\mu_p(e^{-t}K_p),$$

where μ_p is the measure with density function

$$h(x, y) = f(x)1_{\mathbb{R}^p}(y), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^p.$$

Since a pointwise limit of log-concave functions is log-concave, we conclude that the function G is log-concave on $\mathbb{R}$ as the pointwise limit of the log-concave functions G_p, $p \in \mathbb{N}^*$. $\square$

Recall that the (B)-conjecture holds true for the Gaussian measure and for the unconditional case (see [Cordero-Erausquin et al. 2004]). From the techniques of the proof of Proposition 3.2, it follows that Conjecture 1.4 holds true if one function is the density function of the Gaussian measure or if both functions are unconditional.

References

[Artstein-Avidan et al. 2004] S. Artstein-Avidan, B. Klartag, and V. Milman, "The Santaló point of a function, and a functional form of the Santaló inequality", *Mathematika* **51**:1-2 (2004), 33–48. MR 2007a:52008 Zbl 1121.52021

[Bobkov et al. 2014] S. G. Bobkov, A. Colesanti, and I. Fragalà, "Quermassintegrals of quasi-concave functions and generalized Prékopa–Leindler inequalities", *Manuscripta Math.* **143**:1-2 (2014), 131–169. MR 3147446 Zbl 1290.26019

[Borell 1974] C. Borell, "Convex measures on locally convex spaces", *Ark. Mat.* **12** (1974), 239–252. MR 52 #9311 Zbl 0297.60004

[Borell 1975] C. Borell, "Convex set functions in d-space", *Period. Math. Hungar.* **6**:2 (1975), 111–136. MR 53 #8359 Zbl 0307.28009

[Böröczky et al. 2012] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, "The log–Brunn–Minkowski inequality", *Adv. Math.* **231**:3-4 (2012), 1974–1997. MR 2964630 Zbl 1258.52005

[Caglar and Werner 2014] U. Caglar and E. M. Werner, "Divergence for s-concave and log concave functions", *Adv. Math.* **257** (2014), 219–247. MR 3187648 Zbl 1310.26016

[Caglar et al. 2015] U. Caglar, M. Fradelizi, O. Guédon, J. Lehec, C. Schütt, and E. M. Werner, "Functional versions of L_p-affine surface area and entropy inequalities", *International Mathematics Research Notices* (2015), 1–28.

[Cordero-Erausquin et al. 2004] D. Cordero-Erausquin, M. Fradelizi, and B. Maurey, "The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems", *J. Funct. Anal.* **214**:2 (2004), 410–427. MR 2005g:60064 Zbl 1073.60042

[Dancs and Uhrin 1980] S. Dancs and B. Uhrin, "On a class of integral inequalities and their measure-theoretic consequences", *J. Math. Anal. Appl.* **74**:2 (1980), 388–400. MR 81g:26009 Zbl 0442.26011

[Dubuc 1977] S. Dubuc, "Critères de convexité et inégalités intégrales", *Ann. Inst. Fourier* (*Grenoble*) **27**:1 (1977), 135–165. MR 56 #3210 Zbl 0331.26008

[Firey 1961] W. J. Firey, "Mean cross-section measures of harmonic means of convex bodies", *Pacific J. Math.* **11** (1961), 1263–1266. MR 25 #3427 Zbl 0122.41101

[Firey 1962] W. J. Firey, "p-means of convex bodies", *Math. Scand.* **10** (1962), 17–24. MR 25 #4416 Zbl 0188.27303

[Firey 1964] W. J. Firey, "Some applications of means of convex bodies", *Pacific J. Math.* **14** (1964), 53–60. MR 28 #4428 Zbl 0126.38405

[Gardner 2002] R. J. Gardner, "The Brunn–Minkowski inequality", *Bull. Amer. Math. Soc.* (*N.S.*) **39**:3 (2002), 355–405. MR 2003f:26035 Zbl 1019.26008

[Gardner et al. 2014] R. J. Gardner, D. Hug, and W. Weil, "The Orlicz–Brunn–Minkowski theory: A general framework, additions, and inequalities", *J. Differential Geom.* **97**:3 (2014), 427–476. MR 3263511 Zbl 1303.52002

[Henstock and Macbeath 1953] R. Henstock and A. M. Macbeath, "On the measure of sum-sets, I: The theorems of Brunn, Minkowski, and Lusternik", *Proc. London Math. Soc.* (3) **3** (1953), 182–194. MR 15,109g Zbl 0052.18302

[Latała 2002] R. Latała, "On some inequalities for Gaussian measures", pp. 813–822 in *Proceedings of the International Congress of Mathematicians, II* (Beijing, 2002), edited by T. Li, Higher Ed. Press, Beijing, 2002. MR 2004b:60055 Zbl 1015.60011

[Lutwak 1993] E. Lutwak, "The Brunn–Minkowski–Firey theory, I: Mixed volumes and the Minkowski problem", *J. Differential Geom.* **38**:1 (1993), 131–150. MR 94g:52008 Zbl 0788.52007

[Lutwak 1996] E. Lutwak, "The Brunn–Minkowski–Firey theory, II: Affine and geominimal surface areas", *Adv. Math.* **118**:2 (1996), 244–294. MR 97f:52014 Zbl 0853.52005

[Marsiglietti 2015] A. Marsiglietti, "On the improvement of concavity of convex measures", *Proceedings of the American Mathematical Society* (2015), 1–12.

[Saroglou 2014] C. Saroglou, "More on logarithmic sums of convex bodies", preprint, 2014. arXiv 1409.4346

[Saroglou 2015] C. Saroglou, "Remarks on the conjectured log-Brunn–Minkowski inequality", *Geom. Dedicata* **177**:1 (2015), 353–365. MR 3370038

[Schneider 1993] R. Schneider, *Convex bodies: The Brunn–Minkowski theory*, Encyclopedia of Mathematics and its Applications **44**, Cambridge Univ. Press, 1993. MR 94d:52007 Zbl 0798.52001

[Uhrin 1994] B. Uhrin, "Curvilinear extensions of the Brunn–Minkowski–Lusternik inequality", *Adv. Math.* **109**:2 (1994), 288–312. MR 95j:52017 Zbl 0847.52007

Received November 25, 2014. Revised March 13, 2015.

ARNAUD MARSIGLIETTI
INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
207 CHURCH STREET SE
306 LIND HALL
MINNEAPOLIS, MN 55455
UNITED STATES

arnaud.marsiglietti@ima.umn.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.201

STRUCTURE OF SEEDS IN GENERALIZED CLUSTER ALGEBRAS

TOMOKI NAKANISHI

We study generalized cluster algebras, introduced by Chekhov and Shapiro. When the coefficients satisfy the normalization and quasireciprocity conditions, one can naturally extend the structure theory of seeds in the ordinary cluster algebras by Fomin and Zelevinsky to generalized cluster algebras. As the main result, we obtain formulas expressing cluster variables and coefficients in terms of c-vectors, g-vectors, and F-polynomials.

1. Introduction

Chekhov and Shapiro [2014] introduced *generalized cluster algebras*, which naturally generalize the ordinary cluster algebras by Fomin and Zelevinsky [2002]. In generalized cluster algebras, the celebrated *binomial* exchange relation for cluster variables of ordinary cluster algebras

$$
\begin{aligned}
x_k' x_k &= p_k^- \prod_{j=1}^{n} x_j^{[-b_{jk}]_+} + p_k^+ \prod_{j=1}^{n} x_j^{[b_{jk}]_+} \\
&= \left(\prod_{j=1}^{n} x_j^{[-b_{jk}]_+} \right) (p_k^- + p_k^+ w_k), \qquad w_k = \prod_{j=1}^{n} x_j^{b_{jk}},
\end{aligned} \tag{1-1}
$$

is replaced by the *polynomial* one of arbitrary degree $d_k \geq 1$,

$$
x_k' x_k = \left(\prod_{j=1}^{n} x_j^{[-\beta_{jk}]_+} \right)^{d_k} \sum_{s=0}^{d_k} p_{k,s} w_k^s, \qquad w_k = \prod_{j=1}^{n} x_j^{\beta_{jk}}, \tag{1-2}
$$

where $\beta_{jk} = b_{jk}/d_k$ are assumed to be integers and the coefficients $p_{k,s}$ should also be mutated appropriately. This generalization is expected to be natural, since it originates in the transformations preserving the associated Poisson bracket [Gekhtman et al. 2005]. In fact, it was shown in [Chekhov and Shapiro 2014] that the generalized cluster algebras have the *Laurent property*, which is regarded as the most characteristic feature of the ordinary cluster algebras. It was also shown in

MSC2010: 13F60.
Keywords: cluster algebra.

the same paper that the *finite-type classification* of the generalized cluster algebras reduces to the one for the ordinary case. These results already imply that, despite the apparent complexity of their exchange relations (1-2), generalized cluster algebras may be well controlled like the ordinary ones. See also [Rupel 2013] for the result on greedy bases in rank 2 generalized cluster algebras.

Besides the above cluster-algebra-theoretic interest, the generalized cluster algebra structure naturally appears for the Teichmüller spaces of Riemann surfaces with orbifold points [Chekhov and Shapiro 2014]. More recently, it also appears in representation theory of quantum affine algebras [Gleitz 2014] and also in the study of WKB analysis [Iwaki and Nakanishi 2014]. In view of these developments, and also for potentially more versatility of polynomial exchange relations than the binomial one, it is not only natural but also necessary to develop a structure theory of seeds in generalized cluster algebras which is parallel to the one for the ordinary cluster algebras by [Fomin and Zelevinsky 2007]. The core notion of the theory of that paper is a cluster pattern with *principal coefficients*, from which other important notions such as *c-vectors*, *g-vectors*, and *F-polynomials* are also induced. Then, the main result of [Fomin and Zelevinsky 2007] is the formulas expressing cluster variables and coefficients in terms of c-vectors, g-vectors, and F-polynomials. These formulas are especially important in view of the categorification of cluster algebras by (generalized) cluster categories (see [Plamondon 2011] and references therein).

The purpose of this paper is to provide results parallel to the above ones for generalized cluster algebras. To be more precise, we consider a class of generalized cluster algebras whose coefficients satisfy the *normalization condition* and what we call the *quasireciprocity condition*. For this class of generalized cluster algebras, we introduce the notions of a cluster pattern with principal coefficients, c-vectors, g-vectors, and F-polynomials. Then, as a main result, we obtain the formulas expressing cluster variables and coefficients in terms of c-vectors, g-vectors, and F-polynomials, which are parallel to the ones in [Fomin and Zelevinsky 2007]. To summarize, *generalized cluster algebras preserve essentially every feature of the ordinary ones*, and this is the main message of the paper.

2. Generalized cluster algebras

In this section we recall basic notions of generalized cluster algebras following [Chekhov and Shapiro 2014]. However, we slightly modify the setting of Chekhov and Shapiro to match the setting of (ordinary) cluster algebras in [Fomin and Zelevinsky 2007].

2A. ***Generalized seed mutations.*** Throughout the paper we always assume that any matrix is an *integer* matrix.

Recall that a matrix $B = (b_{ij})_{i.j=1}^{n}$ is said to be *skew-symmetrizable* if there is an n-tuple of positive integers $\boldsymbol{d} = (d_1, \dots, d_n)$ such that $d_i b_{ij} = -d_j b_{ji}$.

We start by fixing a semifield $\mathbb{P}$, whose addition is denoted by $\oplus$. Let $\mathbb{ZP}$ be the group ring of $\mathbb{P}$, and let $\mathbb{QP}$ be the field of fractions of $\mathbb{ZP}$. Let $w_1, \dots, w_n$ be any algebraic independent variables, and let $\mathcal{F} = \mathbb{QP}(w)$ be the field of rational functions in $w = (w_1, \dots, w_n)$ with coefficients in $\mathbb{QP}$.

The following definition is the usual one [Fomin and Zelevinsky 2007].

Definition 2.1. A (labeled) seed in $\mathbb{P}$ is a triplet $(\boldsymbol{x}, \boldsymbol{y}, B)$ such that

- B is a skew-symmetrizable matrix, called an *exchange matrix*,
- $\boldsymbol{x} = (x_1, \dots, x_n)$ is an n-tuple of elements in $\mathcal{F}$, called *cluster variables* or *x-variables*,
- $\boldsymbol{y} = (y_1, \dots, y_n)$ is an n-tuple of elements in $\mathbb{P}$, called *coefficients* or *y-variables*.

Next we introduce a pair $(\boldsymbol{d}, \boldsymbol{z})$ of data for generalized seed mutations. Firstly, $\boldsymbol{d} = (d_1, \dots, d_n)$ is an n-tuple of positive integers, and we call these integers the *mutation degrees*. We stress that we do *not* impose the skew-symmetric condition $d_i b_{ij} = -d_j b_{ji}$. Secondly, $\boldsymbol{z}$ is a family of elements in $\mathbb{P}$,

$$\boldsymbol{z} = (z_{i,s})_{i=1,\dots,n;s=1,\dots,d_i-1}, \tag{2-1}$$

satisfying the *reciprocity* condition

$$z_{i,s} = z_{i,d_i-s} \quad (s = 1, \dots, d_i - 1). \tag{2-2}$$

We call them the *frozen coefficients*, since they are not "mutated", or simply the *z-variables*. We also set

$$z_{i,0} = z_{i,d_i} = 1. \tag{2-3}$$

For $\boldsymbol{d} = (1, \dots, 1)$, $\boldsymbol{z}$ is empty, and it reduces to the ordinary case. (Here and below, "ordinary" means the case of ordinary cluster algebras.)

Definition 2.2. Let $(\boldsymbol{d}, \boldsymbol{z})$ be given as above. For any seed $(\boldsymbol{x}, \boldsymbol{y}, B)$ in $\mathbb{P}$ and $k = 1, \dots, n$, the *$(\boldsymbol{d}, \boldsymbol{z})$-mutation of* $(\boldsymbol{x}, \boldsymbol{y}, B)$ *at* k is another seed $(\boldsymbol{x}', \boldsymbol{y}', B') = \mu_k(\boldsymbol{x}, \boldsymbol{y}, B)$ in $\mathbb{P}$ defined by the following rule:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + d_k\big([-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+\big) & \text{if } i, j \neq k, \end{cases} \tag{2-4}$$

$$y'_i = \begin{cases} y_k^{-1} & \text{if } i = k, \\ y_i \big(y_k^{[\varepsilon b_{ki}]_+}\big)^{d_k} \left(\bigoplus_{s=0}^{d_k} z_{k,s} y_k^{\varepsilon s}\right)^{-b_{ki}} & \text{if } i \neq k, \end{cases} \tag{2-5}$$

$$x'_i = \begin{cases} x_k^{-1}\left(\prod_{j=1}^{n} x_j^{[-\varepsilon b_{jk}]_+}\right)^{d_k} \dfrac{\sum_{s=0}^{d_k} z_{k,s}\hat{y}_k^{\varepsilon s}}{\bigoplus_{s=0}^{d_k} z_{k,s}y_k^{\varepsilon s}} & \text{if } i=k,\\ x_i & \text{if } i\neq k, \end{cases} \tag{2-6}$$

where $\varepsilon=\pm 1$, $[a]_+=\max(a,0)$, and we set

$$\hat{y}_i = y_i \prod_{j=1}^{n} x_j^{b_{ji}}. \tag{2-7}$$

When the data $(\boldsymbol{d}, \boldsymbol{z})$ is clearly assumed, we may drop the prefix and simply call it the (*generalized*) *mutation*.

Let $D=(d_i\delta_{ij})_{i,j=1}^n$ be the diagonal matrix with diagonal entries $\boldsymbol{d}$. It is important to note that the mutation (2-4) is equivalent to the *ordinary* mutation of exchange matrices between DB and DB', and also between BD and $B'D$ in [Fomin and Zelevinsky 2007].

The following properties are easy to confirm:

- The formulas (2-5) and (2-6) are *independent* of the choice of the sign ε due to (2-2).
- The mutation μ_k is involutive, i.e., $\mu_k(\mu_k(\boldsymbol{x}, \boldsymbol{y}, B)) = (\boldsymbol{x}, \boldsymbol{y}, B)$.

Remark 2.3. Here we transposed every matrix in [Chekhov and Shapiro 2014]. Also, the matrix B therein is the matrix DB^T here, and β_{ij} therein is b_{ji} here.

Remark 2.4. In this paper we do not use the freedom of the choice of sign ε in (2-5) and (2-6), and it can be safely set as $\varepsilon=1$ throughout. Nevertheless, we keep it in all formulas involved since it is useful for several purposes, for example, to consider *signed mutations*, which appeared in [Iwaki and Nakanishi 2014].

Proposition 2.5. *Under the mutation μ_k, the $\hat{y}$-variables (2-7) mutate in the same way as the y-variables, namely,*

$$\hat{y}'_i = \begin{cases} \hat{y}_k^{-1} & \text{if } i=k,\\ \hat{y}_i\left(\hat{y}_k^{[\varepsilon b_{ki}]_+}\right)^{d_k}\left(\sum_{s=0}^{d_k} z_{k,s}\hat{y}_k^{\varepsilon s}\right)^{-b_{ki}} & \text{if } i\neq k. \end{cases} \tag{2-8}$$

Proof. This is proved using the technique in [Fomin and Zelevinsky 2007, Proposition 3.9]. □

Next let us explain how our setting is regarded as a specialization of the setting of [Chekhov and Shapiro 2014]. In that paper a seed in $\mathbb{P}$ is defined as a triplet

$(\boldsymbol{x}, \boldsymbol{p}, B)$, where $\boldsymbol{x}$ and B are the same as in this paper (up to the identification of B as in Remark 2.3), but $\boldsymbol{p}$ is a family of elements in $\mathbb{P}$,

$$\boldsymbol{p} = (p_{i,s})_{i=1,\dots,n;\, s=0,\dots,d_i}. \tag{2-9}$$

Then, for the mutation $(\boldsymbol{x}', \boldsymbol{p}', B') = \mu_k(\boldsymbol{x}, \boldsymbol{p}, B)$, the following formulas replace (2-5) and (2-6):

$$\begin{aligned} p'_{k,s} &= p_{k,d_k-s}, \\ \frac{p'_{i,s}}{p'_{i,0}} &= \begin{cases} \dfrac{p_{i,s}}{p_{i,0}}(p_{k,d_k}^{b_{ki}})^s & \text{if } i \neq k,\ b_{ki} \geq 0, \\ \dfrac{p_{i,s}}{p_{i,0}}(p_{k,0}^{b_{ki}})^s & \text{if } i \neq k,\ b_{ki} \leq 0, \end{cases} \end{aligned} \tag{2-10}$$

$$x'_i = \begin{cases} x_k^{-1}\left(\displaystyle\prod_{j=1}^{n} x_j^{[-b_{jk}]_+}\right)^{d_k}\left(\displaystyle\sum_{s=0}^{d_k} p_{k,s} u_k^s\right) & \text{if } i = k, \\ x_i & \text{if } i \neq k, \end{cases} \tag{2-11}$$

where

$$u_i = \prod_{j=1}^{n} x_j^{b_{ji}}. \tag{2-12}$$

Now, let us start from a seed $(\boldsymbol{x}, \boldsymbol{y}, B)$ in our setting. Comparing (2-6) and (2-11), we naturally identify

$$p_{i,s} = \frac{z_{i,s} y_i^s}{\bigoplus_{r=0}^{d_i} z_{i,r} y_i^r}. \tag{2-13}$$

Then, it is easy to check that the mutation (2-10) follows from (2-2) and (2-5). Moreover, the specialization (2-13) satisfies the *normalization* property

$$\bigoplus_{s=0}^{d_i} p_{i,s} = 1 \tag{2-14}$$

and the *quasireciprocity* property that for each $i = 1, \dots, n$ there is some $y_i \in \mathbb{P}$ such that

$$\frac{p_{i,s}}{p_{i,0}} \frac{p_{i,d_i}}{p_{i,d_i-s}} = y_i^{2s}, \qquad s = 1, \dots, d_i. \tag{2-15}$$

Conversely, suppose that a family $\boldsymbol{p}$ in (2-9) satisfies properties (2-14) and (2-15). First we note that such a y_i is unique, since any semifield $\mathbb{P}$ is torsion-free [Fomin and Zelevinsky 2002, Section 5]. Next we define $z_{i,s} \in \mathbb{P}$ ($i = 1, \dots, n$; $s = 0, \dots, d_i$) by

$$\frac{p_{i,s}}{p_{i,0}} = y_i^s z_{i,s}. \tag{2-16}$$

In particular, we have $z_{i,0}=1$. Then, substituting (2-16) in (2-15), we obtain

(2-17) $$z_{i,s}z_{i,d_i}z_{i,d_i-s}^{-1}=1, \qquad s=1,\dots,d_i.$$

In particular, by setting $s=d_i$, we have $z_{i,d_i}^2=1$. Once again, since $\mathbb{P}$ is torsion-free, we have $z_{i,d_i}=1$. Then, again by (2-17), we have the reciprocity $z_{i,s}=z_{i,d_i-s}$ ($s=1,\dots,d_i-1$). Meanwhile, by (2-14) and (2-16), we have

(2-18) $$p_{i,0}=\frac{1}{\bigoplus_{s=0}^{d_i} z_{i,s}y_i^s}.$$

Then, by (2-16) again, we recover the specialization (2-13). Finally, it is straightforward to recover the mutation (2-5) from (2-10) and (2-15). Furthermore, by (2-16), one can also confirm that the coefficients $z_{i,s}$ do not mutate.

2B. ***Generalized cluster algebras and Laurent property.*** Let $\mathbb{T}_n$ be the n-regular tree whose edges are labeled by the numbers $1,\dots,n$. Following [Fomin and Zelevinsky 2002], let us write $t\overset{k}{-}t'$ if the vertices t and t' of $\mathbb{T}_n$ are connected by the edge labeled by k.

Definition 2.6. A $(\boldsymbol{d},\boldsymbol{z})$-*cluster pattern* Σ *in* $\mathbb{P}$ is an assignment of a seed Σ_t in $\mathbb{P}$ to each vertex t of $\mathbb{T}$ such that if $t\overset{k}{-}t'$ then the assigned seeds Σ_t and $\Sigma_{t'}$ are obtained from each other by the $(\boldsymbol{d},\boldsymbol{z})$-mutation at k.

We fix a vertex t_0 of $\mathbb{T}_n$ and call it the *initial vertex*. Accordingly, the assigned seed $\Sigma_{t_0}=(\boldsymbol{x}_{t_0},\boldsymbol{y}_{t_0},B_{t_0})$ at t_0 is called the *initial seed*. Let us write, for simplicity,

(2-19) $$\boldsymbol{x}_{t_0}=\boldsymbol{x}=(x_1,\dots,x_n), \quad \boldsymbol{y}_{t_0}=\boldsymbol{y}=(y_1,\dots,y_n), \quad B_{t_0}=B=(b_{ij})_{i,j=1}^n.$$

On the other hand, for the seed $\Sigma_t=(\boldsymbol{x}_t,\boldsymbol{y}_t,B_t)$ assigned to a general vertex t of $\mathbb{T}_n$, we write

(2-20) $$\boldsymbol{x}_t=(x_1^t,\dots,x_n^t), \quad \boldsymbol{y}_t=(y_1^t,\dots,y_n^t), \quad B_t=(b_{ij}^t)_{i,j=1}^n.$$

Definition 2.7. The *generalized cluster algebra* $\mathcal{A}$ *associated with a* $(\boldsymbol{d},\boldsymbol{z})$-*cluster pattern* Σ *in* $\mathbb{P}$ is a $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by all x-variables x_i^t ($t\in\mathbb{T}$, $i=1,\dots,n$) occurring in Σ. It is denoted by $\mathcal{A}=\mathcal{A}(\boldsymbol{x},\boldsymbol{y},B;\boldsymbol{d},\boldsymbol{z})$, where $(\boldsymbol{x},\boldsymbol{y},B)$ is the initial seed of Σ.

For any $(\boldsymbol{d},\boldsymbol{z})$-cluster pattern in $\mathbb{P}$, each x-variable x_i^t is expressed as a subtraction-free rational function of $\boldsymbol{x}$ with coefficients in $\mathbb{QP}$. The following stronger property due to [Chekhov and Shapiro 2014] is of fundamental importance.

Theorem 2.8 (Laurent property [Chekhov and Shapiro 2014, Theorem 2.5]). *For any* $(\boldsymbol{d},\boldsymbol{z})$-*cluster pattern in* $\mathbb{P}$, *each* x-*variable* x_i^t *is expressed as a Laurent polynomial of* $\boldsymbol{x}$ *with coefficients in* $\mathbb{ZP}$.

2C. *Example.* As the simplest nontrivial example, we consider $\boldsymbol{d} = (2, 1)$, $\boldsymbol{z} = (z_{1,1})$, and an initial seed $(\boldsymbol{x}, \boldsymbol{y}, B)$ in $\mathbb{P}$ such that

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2-21}$$

(This example also appears in [Chekhov and Shapiro 2014, proof of Theorem 2.7].) Accordingly,

$$\hat{y}_1 = y_1 x_2, \quad \hat{y}_2 = y_2 x_1^{-1}. \tag{2-22}$$

We note that

$$DB = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}, \quad BD = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \tag{2-23}$$

which are the initial exchange matrices for ordinary cluster algebras of type $B_2 = C_2$. Set $\Sigma(1) = (\boldsymbol{x}(1), \boldsymbol{y}(1), B(1))$ to be the initial seed $(\boldsymbol{x}, \boldsymbol{y}, B)$, and consider the seeds $\Sigma(t) = (\boldsymbol{x}(t), \boldsymbol{y}(t), B(t))$ $(t = 2, \dots, 7)$ obtained by the following sequence of alternative mutations of μ_1 and μ_2.

$$\Sigma(1) \overset{\mu_1}{\leftrightarrow} \Sigma(2) \overset{\mu_2}{\leftrightarrow} \Sigma(3) \overset{\mu_1}{\leftrightarrow} \Sigma(4) \overset{\mu_2}{\leftrightarrow} \Sigma(5) \overset{\mu_1}{\leftrightarrow} \Sigma(6) \overset{\mu_2}{\leftrightarrow} \Sigma(7). \tag{2-24}$$

By (2-4), we have

$$B(t) = (-1)^{t+1} B. \tag{2-25}$$

Then, using the exchange relations (2-5) and (2-6), we obtain the explicit expressions of x- and y-variables in Table 1, where we set $z_{1,1} = z$ for simplicity. We observe the same periodicity of mutations of seeds for the ordinary cluster algebras of type $B_2 = C_2$.

3. Structure of seeds in generalized cluster patterns

The goal of this section is to establish some basic structural results on seeds in a $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern which are parallel to the ones in [Fomin and Zelevinsky 2007].

3A. *X-functions and Y-functions.* Let us temporarily regard $\boldsymbol{y} = (y_i)_{i=1}^n$ and $\boldsymbol{z} = (z_{i,s})_{i=1,\dots,n; s=1,\dots,d_i-1}$ with $z_{i,s} = z_{i,d_i-s}$ as formal variables. Let $\mathbb{Q}_{\mathrm{sf}}(\boldsymbol{y}, \boldsymbol{z})$ be the *universal semifield* of $\boldsymbol{y}$ and $\boldsymbol{z}$, which consists of the rational functions in $\boldsymbol{y}$ and $\boldsymbol{z}$ with subtraction-free expressions [Fomin and Zelevinsky 2007]. Let $\mathrm{Trop}(\boldsymbol{y}, \boldsymbol{z})$ be the *tropical semifield* of $\boldsymbol{y}$ and $\boldsymbol{z}$, which is the multiplicative abelian group freely generated by $\boldsymbol{y}$ and $\boldsymbol{z}$ with *tropical sum* $\oplus$ defined by

$$\left(\prod_i y_i^{a_i} \prod_{i,s} z_{i,s}^{a_{i,s}}\right) \oplus \left(\prod_i y_i^{b_i} \prod_{i,s} z_{i,s}^{b_{i,s}}\right) = \prod_i y_i^{\min(a_i, b_i)} \prod_{i,s} z_{i,s}^{\min(a_{i,s}, b_{i,s})}. \tag{3-1}$$

$\begin{cases} x_1(1) = x_1 \\ x_2(1) = x_2 \end{cases}$	$\begin{cases} y_1(1) = y_1 \\ y_2(1) = y_2 \end{cases}$
$\begin{cases} x_1(2) = x_1^{-1} \dfrac{1+z\hat{y}_1+\hat{y}_1^2}{1\oplus zy_1\oplus y_1^2} \\ x_2(2) = x_2 \end{cases}$	$\begin{cases} y_1(2) = y_1^{-1} \\ y_2(2) = y_2(1\oplus zy_1\oplus y_1^2) \end{cases}$
$\begin{cases} x_1(3) = x_1^{-1} \dfrac{1+z\hat{y}_1+\hat{y}_1^2}{1\oplus zy_1\oplus y_1^2} \\ x_2(3) = x_2^{-1} \dfrac{1+\hat{y}_2+z\hat{y}_1\hat{y}_2+\hat{y}_1^2\hat{y}_2}{1\oplus y_2\oplus zy_1y_2\oplus y_1^2y_2} \end{cases}$	$\begin{cases} y_1(3) = y_1^{-1}(1\oplus y_2\oplus zy_1y_2\oplus y_1^2y_2) \\ y_2(3) = y_2^{-1}(1\oplus zy_1\oplus y_1^2)^{-1} \end{cases}$
$\begin{cases} x_1(4) = x_1x_2^{-2} \dfrac{1+2\hat{y}_2+\hat{y}_2^2+z\hat{y}_1\hat{y}_2+z\hat{y}_1\hat{y}_2^2+\hat{y}_1^2\hat{y}_2^2}{1\oplus 2y_2\oplus y_2^2\oplus zy_1y_2\oplus zy_1y_2^2\oplus y_1^2y_2^2} \\ x_2(4) = x_2^{-1} \dfrac{1+\hat{y}_2+z\hat{y}_1\hat{y}_2+\hat{y}_1^2\hat{y}_2}{1\oplus y_2\oplus zy_1y_2\oplus y_1^2y_2} \end{cases}$	$\begin{cases} y_1(4) = y_1(1\oplus y_2\oplus zy_1y_2\oplus y_1^2y_2)^{-1} \\ y_2(4) = y_1^{-2}y_2^{-1}(1\oplus 2y_2\oplus y_2^2 \\ \qquad\qquad \oplus zy_1y_2\oplus zy_1y_2^2\oplus y_1^2y_2^2) \end{cases}$
$\begin{cases} x_1(5) = x_1x_2^{-2} \dfrac{1+2\hat{y}_2+\hat{y}_2^2+z\hat{y}_1\hat{y}_2+z\hat{y}_1\hat{y}_2^2+\hat{y}_1^2\hat{y}_2^2}{1\oplus 2y_2\oplus y_2^2\oplus zy_1y_2\oplus zy_1y_2^2\oplus y_1^2y_2^2} \\ x_2(5) = x_1x_2^{-1} \dfrac{1+\hat{y}_2}{1\oplus y_2} \end{cases}$	$\begin{cases} y_1(5) = y_1^{-1}y_2^{-1}(1\oplus y_2) \\ y_2(5) = y_1^2y_2(1\oplus 2y_2\oplus y_2^2 \\ \qquad\qquad \oplus zy_1y_2\oplus zy_1y_2^2\oplus y_1^2y_2^2)^{-1} \end{cases}$
$\begin{cases} x_1(6) = x_1 \\ x_2(6) = x_1x_2^{-1} \dfrac{1+\hat{y}_2}{1\oplus y_2} \end{cases}$	$\begin{cases} y_1(6) = y_1y_2(1\oplus y_2)^{-1} \\ y_2(6) = y_2^{-1} \end{cases}$
$\begin{cases} x_1(7) = x_1 \\ x_2(7) = x_2 \end{cases}$	$\begin{cases} y_1(7) = y_1 \\ y_2(7) = y_2 \end{cases}$

Table 1. x- and y-variables for sequence (2-24).

Definition 3.1. A $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with *principal coefficients* is a $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern in $\mathbb{P} = \mathrm{Trop}(\boldsymbol{y}, \boldsymbol{z})$ with initial seed $(\boldsymbol{x}, \boldsymbol{y}, B)$, where $\boldsymbol{x}$ and B are arbitrary.

Definition 3.2. Let Σ be the $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, B)$. By the Laurent property in Theorem 2.8, each x-variable x_i^t in Σ is expressed as $X_i^t(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{ZP}[\boldsymbol{x}^{\pm 1}]$ with $\mathbb{P} = \mathrm{Trop}(\boldsymbol{y}, \boldsymbol{z})$. We call them the *$X$-functions* of Σ.

For principal coefficients, we actually have the following result, which is stronger than Theorem 2.8 and which is parallel to [Fomin and Zelevinsky 2003, Proposition 11.2; 2007, Proposition 3.6].

Proposition 3.3. *We have*

$$X_i^t(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{Z}[\boldsymbol{x}^{\pm 1}, \boldsymbol{y}, \boldsymbol{z}]. \tag{3-2}$$

Proof. We follow the argument in the proof of [Fomin and Zelevinsky 2003, Proposition 11.2]. Let p be any variable in $\boldsymbol{y}$ or $\boldsymbol{z}$. Let us view $X_i^t(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ as a Laurent polynomial in p, say $h(p)$, whose coefficients are Laurent polynomials in the rest of the variables in $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$. We show that $h(p)$ is a polynomial in p with nonzero constant term having subtraction-free rational expression by induction on the distance between t and t_0 in $\mathbb{T}_n$. The crucial point is that the coefficients $p_{k,s} = z_{k,s} y_k^s / \bigoplus_{r=0}^{d_k} z_{k,r} y_k^r$ in the mutation (2-6) are normalized as (2-14). Since $\mathbb{P} = \mathrm{Trop}(\boldsymbol{y}, \boldsymbol{z})$, this means that $p_{k,s}$ $(s = 0, \dots, d_r)$ are polynomials in p, and there is no common factor in p. Thus, the right-hand side of (2-6) is a polynomial in p with nonzero constant term having subtraction-free rational expression by the induction hypothesis and the "trivial lemma" (Lemma 5.2) in [Fomin and Zelevinsky 2003]. □

Definition 3.4. We denote by Σ the $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern in the universal semifield $\mathbb{Q}_{\mathrm{sf}}(\boldsymbol{y}, \boldsymbol{z})$ with initial seed $(\boldsymbol{x}, \boldsymbol{y}, B)$. Each y-variable y_i^t in Σ is expressed as a subtraction-free rational function $Y_i^t(\boldsymbol{y}, \boldsymbol{z}) \in \mathbb{Q}_{\mathrm{sf}}(\boldsymbol{y}, \boldsymbol{z})$. We call them the *$Y$-functions* of Σ.

Due to the universal property of the semifield $\mathbb{Q}_{\mathrm{sf}}(\boldsymbol{y}, \boldsymbol{z})$ [Fomin and Zelevinsky 2007, Definition 2.1], the following fact holds.

Lemma 3.5. *For any $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern in $\mathbb{P}$ with the same initial exchange matrix B as above, we have*

$$y_i^t = Y_i^t|_{\mathbb{P}}(\boldsymbol{y}, \boldsymbol{z}), \tag{3-3}$$

where the right-hand side stands for the evaluation of $Y_i^t(\boldsymbol{y}, \boldsymbol{z})$ in $\mathbb{P}$.

3B. *c-vectors, F-polynomials, and g-vectors.* Let us extend the notions of c-vectors, F-polynomials, and g-vectors in [Fomin and Zelevinsky 2007] to a $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients.

3B.1. *C-matrices and c-vectors.* For a $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients, each y-variable $y_i^t \in \mathrm{Trop}(\boldsymbol{y}, \boldsymbol{z})$ is, by definition, a Laurent monomial of $\boldsymbol{y}$ and $\boldsymbol{z}$ with coefficient 1. The following simple fact was observed in [Iwaki and Nakanishi 2014] in the special case.

Lemma 3.6. *Each y-variable y_i^t is actually a Laurent monomial of $\boldsymbol{y}$ with coefficient* 1.

Proof. This is equivalent to saying that the frozen coefficients $\boldsymbol{z}$ never enter in y_i^t. This is true for the initial y-variables. Then, the claim can be shown by induction on the distance between t and t_0 in $\mathbb{T}_n$, by inspecting the mutation (2-5) and the definition of the tropical sum (3-1). $\square$

Definition 3.7. Let Σ be a $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients. Let us express each y-variable y_j^t in Σ as

$$y_j^t = Y_i^t|_{\mathrm{Trop}(\boldsymbol{y},\boldsymbol{z})}(\boldsymbol{y}, \boldsymbol{z}) = \prod_{i=1}^{n} y_i^{c_{ij}^t}. \tag{3-4}$$

The resulting matrices $C^t = (c_{ij}^t)_{i,j=1}^n$ and their column vectors $c_j^t = (c_{ij}^t)_{i=1}^n$ are called the *C-matrices* and the *c-vectors* of Σ, respectively.

The following mutation/recurrence formula provides a combinatorial description of c-vectors.

Proposition 3.8. *The c-vectors of a $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$c_{ij}^{t_0} = \delta_{ij}, \tag{3-5}$$

$$c_{ij}^{t'} = \begin{cases} -c_{ik}^t & \text{if } j = k, \\ c_{ij}^t + c_{ik}^t[\varepsilon d_k b_{kj}^t]_+ + [-\varepsilon c_{ik}^t]_+ d_k b_{kj}^t & \text{if } j \neq k, \end{cases} \tag{3-6}$$

where $\varepsilon = \pm 1$ and it is independent of the choice of the sign ε.

Proof. As already remarked in the proof of Lemma 3.6, for a $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients, the mutation (2-5) is simplified as

$$y_i^{t'} = \begin{cases} y_k^{t\,-1} & \text{if } i = k, \\ y_i^t \bigl(y_k^{t\,[\varepsilon b_{ki}^t]_+}\bigr)^{d_k} \Bigl(\bigoplus_{s=0}^{d_k} y_k^{t\,\varepsilon s}\Bigr)^{-b_{ki}^t} & \text{if } i \neq k. \end{cases} \tag{3-7}$$

This is equivalent to (3-6) due to the following formula in $\mathrm{Trop}(\boldsymbol{y}, \boldsymbol{z})$:

$$\frac{1}{\bigoplus_{s=0}^{d_k} \Bigl(\prod_{j=1}^{n} y_j^{\varepsilon c_{jk}^t}\Bigr)^s} = \Bigl(\prod_{j=1}^{n} y_j^{[-\varepsilon c_{jk}^t]_+}\Bigr)^{d_k}. \tag{3-8}$$ $\square$

We observe that the above relation coincides with the one for the c-vectors of the ordinary cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, DB)$ in [Fomin and Zelevinsky 2002, Proposition 5.8]. Therefore, we have the following result.

Proposition 3.9. *The c-vectors of the $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, B)$ coincide with the c-vectors of the ordinary cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, DB)$.*

Alternatively, one can relate these c-vectors with the c-vectors of the ordinary cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, BD)$ as follows. Let us introduce

$$\tilde{c}^t_{ij} = d_i^{-1} c^t_{ij} d_j. \tag{3-9}$$

Then, $\tilde{c}^{t_0}_{ij} = \delta_{ij}$, and (3-6) is rewritten as

$$\tilde{c}^{t'}_{ij} = \begin{cases} -\tilde{c}^t_{ik} & \text{if } j = k, \\ \tilde{c}^t_{ij} + \tilde{c}^t_{ik}[\varepsilon b^t_{kj} d_j]_+ + [-\varepsilon \tilde{c}^t_{ik}]_+ b^t_{kj} d_j & \text{if } j \neq k. \end{cases} \tag{3-10}$$

Therefore, we have the following result.

Proposition 3.10. *The $\tilde{c}$-vectors, which are the column vectors in (3-9), of the $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, B)$ coincide with the c-vectors of the ordinary cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, BD)$.*

We need this alternative description for the description of the g-vectors below.

3B.2. *F-polynomials.* Thanks to Proposition 3.3, the following definition makes sense.

Definition 3.11. Let Σ be a $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients. For each $t \in \mathbb{T}_n$ and $i = 1, \dots, n$, a polynomial $F^t_i(\boldsymbol{y}, \boldsymbol{z}) \in \mathbb{Z}[\boldsymbol{y}, \boldsymbol{z}]$ is defined by the specialization of the X-function $X^t_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ of Σ with $x_1 = \cdots = x_n = 1$. They are called the *F-polynomials* of Σ.

The following mutation/recurrence formula provides a combinatorial description of F-polynomials.

Proposition 3.12 (cf. [Fomin and Zelevinsky 2007, Proposition 5.1]). *The F-polynomials for a $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$F^{t_0}_i = 1, \tag{3-11}$$

$$F^{t'}_i = \begin{cases} F^{t\,-1}_k \left(\prod_{j=1}^n y_j^{[-\varepsilon c^t_{jk}]_+} F_j^{t\,[-\varepsilon b^t_{jk}]_+} \right)^{d_k} \sum_{s=0}^{d_k} z_{k,s} \left(\prod_{j=1}^n y_j^{\varepsilon c^t_{jk}} F_j^{t\,\varepsilon b^t_{jk}} \right)^s & \textit{if } i = k, \\ F^t_i & \textit{if } i \neq k, \end{cases} \tag{3-12}$$

where $\varepsilon = \pm 1$ and it is independent of the choice of the sign ε.

Proof. By specializing the mutation (2-6) with $\mathbb{P} = \mathrm{Trop}(\boldsymbol{y}, \boldsymbol{z})$, we obtain

$$X_i^{t'} = \begin{cases} X_k^{t\,-1}\left(\prod_{j=1}^{n} X_j^{t\,[-\varepsilon b_{jk}^t]_+}\right)^{d_k} \dfrac{\sum_{s=0}^{d_k} z_{k,s}\left(\prod_{j=1}^{n} y_j^{\varepsilon c_{jk}^t} X_j^{t\,\varepsilon b_{jk}^t}\right)^s}{\bigoplus_{s=0}^{d_k}\left(\prod_{j=1}^{n} y_j^{\varepsilon c_{jk}^t}\right)^s} & \text{if } i = k, \\ X_i^t & \text{if } i \neq k. \end{cases} \tag{3-13}$$

Then, specializing it with $x_1 = \dots x_n = 1$, and using (3-8), we obtain (3-12). $\square$

3B.3. *G-matrices and g-vectors.* Let Σ be the $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, B)$. Let $\mathbb{Z}[\boldsymbol{x}^{\pm 1}, \boldsymbol{y}, \boldsymbol{z}]$ be the one in Proposition 3.3. Following [Fomin and Zelevinsky 2007], we introduce a $\mathbb{Z}^n$-grading in $\mathbb{Z}[\boldsymbol{x}^{\pm 1}, \boldsymbol{y}, \boldsymbol{z}]$ as follows:

$$\deg(x_i) = \boldsymbol{e}_i, \quad \deg(y_i) = -\boldsymbol{b}_j, \quad \deg(z_{i,r}) = 0. \tag{3-14}$$

Here, $\boldsymbol{e}_i$ is the i-th unit vector of $\mathbb{Z}^n$, and $\boldsymbol{b}_j = \sum_{i=1}^n b_{ij}\boldsymbol{e}_i$ is the j-th column of the initial matrix $B = (b_{ij})_{i,j=1}^n$. Note that $\deg(\hat{y}_i) = 0$ by (2-7).

Proposition 3.13 (cf. [Fomin and Zelevinsky 2007, Proposition 6.1]). *The X-functions are homogeneous with respect to the $\mathbb{Z}^n$-grading.*

Proof. We repeat the original argument of Fomin and Zelevinsky, by induction on the distance between t and t_0 in $\mathbb{T}_n$. Using (2-6) and Lemma 3.5 specialized to a $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients, we have

$$X_i^{t'} = \begin{cases} X_k^{t\,-1}\left(\prod_{j=1}^{n} X_j^{t\,[-\varepsilon b_{jk}^t]_+}\right)^{d_k} \dfrac{\sum_{s=0}^{d_k} z_{k,s} Y_k^{t\,\varepsilon s}|_{\mathcal{F}}(\hat{\boldsymbol{y}}, \boldsymbol{z})}{\bigoplus_{s=0}^{d_k} z_{k,s} Y_k^{t\,\varepsilon s}|_{\mathrm{Trop}(\boldsymbol{y},\boldsymbol{z})}(\boldsymbol{y}, \boldsymbol{z})} & \text{if } i = k, \\ X_i^t & \text{if } i \neq k. \end{cases} \tag{3-15}$$

Then, the right-hand side is homogeneous due to the induction hypothesis. $\square$

Definition 3.14. Let Σ be the $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients and initial matrix $(\boldsymbol{x}, \boldsymbol{y}, B)$. Thanks to Proposition 3.13, the degree vector $\deg(X_i^t)$ of each X-function X_i^t of Σ is defined. Let us express it as

$$\deg(X_j^t) = \sum_{i=1}^{n} g_{ij}^t \boldsymbol{e}_i. \tag{3-16}$$

The resulting matrices $G^t = (g_{ij}^t)_{i,j=1}^n$ and their column vectors $g_j^t = (g_{ij}^t)_{i=1}^n$ are called the *G-matrices* and the *g-vectors* of Σ, respectively.

The following mutation/recurrence formula provides a combinatorial description of g-vectors.

Proposition 3.15. *The g-vectors of the $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, B)$ satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$g_{ij}^{t_0} = \delta_{ij}, \tag{3-17}$$

$$g_{ij}^{t'} = \begin{cases} -g_{ik}^{t} + \sum_{\ell=1}^{n} g_{i\ell}^{t}[-\varepsilon b_{\ell k}^{t} d_k]_+ - \sum_{\ell=1}^{n} b_{i\ell}[-\varepsilon c_{\ell k}^{t} d_k]_+ & \text{if } j = k, \\ g_{ij}^{t} & \text{if } j \neq k, \end{cases} \tag{3-18}$$

where $\varepsilon = \pm 1$ and it is independent of the choice of the sign ε.

Proof. This is obtained by comparing the degrees of both sides of (3-13). □

By using the $\tilde{c}$-vectors in (3-9), the relation (3-18) is rewritten as follows.

$$g_{ij}^{t'} = \begin{cases} -g_{ik}^{t} + \sum_{\ell=1}^{n} g_{i\ell}^{t}[-\varepsilon b_{\ell k}^{t} d_k]_+ - \sum_{\ell=1}^{n} b_{i\ell} d_\ell [-\varepsilon \tilde{c}_{\ell k}^{t}]_+ & \text{if } j = k, \\ g_{ij}^{t} & \text{if } j \neq k. \end{cases} \tag{3-19}$$

Having Proposition 3.10 in mind, we observe that this relation coincides with the one for the g-vectors of the ordinary cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, BD)$ in [Fomin and Zelevinsky 2007, Proposition 6.6]. Therefore, we have the following result.

Proposition 3.16. *The g-vectors of the $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, B)$ coincide with the g-vectors of the ordinary cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, BD)$.*

For the sake of completeness, we also present the counterpart of Proposition 3.10. Let us introduce

$$\tilde{g}_{ij}^{t} = d_i g_{ij}^{t} d_j^{-1}. \tag{3-20}$$

Then, the relation (3-18) is also rewritten as

$$\tilde{g}_{ij}^{t'} = \begin{cases} -\tilde{g}_{ik}^{t} + \sum_{\ell=1}^{n} \tilde{g}_{i\ell}^{t}[-\varepsilon d_\ell b_{\ell k}^{t}]_+ - \sum_{\ell=1}^{n} d_i b_{i\ell}[-\varepsilon c_{\ell k}^{t}]_+ & \text{if } j = k, \\ \tilde{g}_{ij}^{t} & \text{if } j \neq k. \end{cases} \tag{3-21}$$

Having Proposition 3.9 in mind, we observe that this relation coincides with the one for the g-vectors of the ordinary cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, DB)$. Therefore, we have the following result.

Proposition 3.17. *The $\tilde{g}$-vectors, which are the column vectors in (3-20), of the $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, B)$ coincide with the g-vectors of the ordinary cluster pattern with principal coefficients and initial seed $(\boldsymbol{x}, \boldsymbol{y}, DB)$.*

We see a duality between the c-vectors and the g-vectors in Propositions 3.9, 3.10, 3.16, and 3.17. In particular, the c-vectors are associated with the matrix DB, while the g-vectors are associated with the matrix BD. This is somewhat suggested from the beginning in the monomial parts in the relations (2-5) and (2-6).

3B.4. *Sign-coherence.*

Definition 3.18. Let Σ be a $(\boldsymbol{d}, z)$-cluster pattern with principal coefficients. A c-vector c_j^t of Σ is said to be *sign-coherent* if it is nonzero and all components are either nonnegative or nonpositive.

Proposition 3.19 (cf. [Fomin and Zelevinsky 2007, Proposition 5.6]). *For any $(\boldsymbol{d}, z)$-cluster pattern with principal coefficients, the following two conditions are equivalent.*

(i) *Any F-polynomial $F_i^t(\boldsymbol{y}, z)$ has constant term* 1.

(ii) *Any c-vector c_i^t is sign-coherent.*

Proof. This is proved by an argument parallel to the one in [Fomin and Zelevinsky 2007, Proposition 5.6] by using the recursion relation (3-12) for the F-polynomials. We omit the details. □

In the ordinary case it was conjectured in [Fomin and Zelevinsky 2007, Conjecture 5.6] that the sign-coherence holds for any c-vector of any cluster pattern with principal coefficients. This was proved by Derksen et al. [2010, Theorem 1.7] when the initial exchange matrix B is skew-symmetric, and very recently it was proved in full generality by Gross et al. [2014, Corollary 5.5]. Since our c-vectors are identified with the c-vectors of some ordinary cluster pattern with principal coefficients by Proposition 3.9, we obtain the following theorem as a corollary of [Gross et al. 2014, Corollary 5.5].

Theorem 3.20. *Any c-vector of any $(\boldsymbol{d}, z)$-cluster pattern with principal coefficients is sign-coherent.*

As a consequence of the sign-coherence, we also obtain the following duality between the C- and G-matrices by applying [Nakanishi and Zelevinsky 2012, Equation (3.11)] (see also [Nakanishi 2012, Proposition 3.2]), which is valid under the sign-coherence property. Recall that for a skew-symmetrizable matrix B the matrix DB is still skew-symmetrizable.

Proposition 3.21 (cf. [Nakanishi and Zelevinsky 2012, Equation (3.11)]). *Let C^t and G^t be the C- and G-matrices at $t \in \mathbb{T}_n$ of any $(\boldsymbol{d}, z)$-cluster pattern Σ with principal coefficients. Let $R = (r_i \delta_{ij})_{i,j=1}^n$ be a diagonal matrix with positive diagonal entries such that RDB is skew-symmetric. Then*

$$R^{-1}D^{-1}(G^t)^T DRC^t = I. \tag{3-22}$$

Proof. This is obtained by combining [Nakanishi and Zelevinsky 2012, Equation (3.11)] with Propositions 3.9 and 3.17. □

3C. *Main formulas.* Finally, we present the main formulas expressing the x- and y-variables of any $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern Σ in any semifield $\mathbb{P}$ in terms of F-polynomials, c-vectors, and g-vectors defined for the same initial exchange matrix of Σ.

Theorem 3.22 (cf. [Fomin and Zelevinsky 2007, Proposition 3.13]). *For any $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern in $\mathbb{P}$,*

$$y_i^t = \prod_{j=1}^{n} y_j^{c_{ji}^t} \prod_{j=1}^{n} F_j^t|_{\mathbb{P}}(\boldsymbol{y}, \boldsymbol{z})^{b_{ji}^t}. \tag{3-23}$$

Proof. We apply Lemma 3.5 to a $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients, and we obtain

$$\hat{y}_i^t = Y_i^t(\hat{\boldsymbol{y}}, \boldsymbol{z}). \tag{3-24}$$

On the other hand, specializing (2-7) to the same $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern with principal coefficients, we have

$$\hat{y}_i^t = Y_i^t|_{\mathrm{Trop}(\boldsymbol{y},\boldsymbol{z})}(\boldsymbol{y}, \boldsymbol{z}) \prod_{j=1}^{n} X_j^t(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})^{b_{ji}^t} = \prod_{j=1}^{n} y_j^{c_{ji}^t} \prod_{j=1}^{n} X_j^t(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})^{b_{ji}^t}, \tag{3-25}$$

where we used (3-4) in the second equality. Thus, we have

$$Y_i^t(\hat{\boldsymbol{y}}, \boldsymbol{z}) = \prod_{j=1}^{n} y_j^{c_{ji}^t} \prod_{j=1}^{n} X_j^t(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})^{b_{ji}^t}. \tag{3-26}$$

Now, we set $x_1 = \cdots = x_n = 1$. Then, $\hat{\boldsymbol{y}} = \boldsymbol{y}$, and we obtain

$$Y_i^t(\boldsymbol{y}, \boldsymbol{z}) = \prod_{j=1}^{n} y_j^{c_{ji}^t} \prod_{j=1}^{n} F_j^t(\boldsymbol{y}, \boldsymbol{z})^{b_{ji}^t}. \tag{3-27}$$

Finally, evaluating it in $\mathbb{P}$, we obtain (3-23). □

Theorem 3.23 (cf. [Fomin and Zelevinsky 2007, Corollary 6.3]). *For any $(\boldsymbol{d}, \boldsymbol{z})$-cluster pattern in $\mathbb{P}$,*

$$x_i^t = \left(\prod_{j=1}^{n} x_j^{g_{ji}^t}\right) \frac{F_i^t|_{\mathcal{F}}(\hat{\boldsymbol{y}}, \boldsymbol{z})}{F_i^t|_{\mathbb{P}}(\boldsymbol{y}, \boldsymbol{z})}. \tag{3-28}$$

Proof. First, we obtain the following equality in exactly the same way as [Fomin and Zelevinsky 2007, Theorem 3.7], and we skip its derivation:

(3-29)
$$x_i^t = \frac{X_i^t|_{\mathcal{F}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})}{F_i^t|_{\mathbb{P}}(\boldsymbol{y}, \boldsymbol{z})}.$$

On the other hand, by the definition of the g-vectors, we have

(3-30)
$$X_i^t\biggl(\dots, \gamma_i x_i, \dots; \dots, \prod_{j=1}^{n} \gamma_k^{-b_{ki}} y_i, \dots; \dots, z_{i,r}, \dots\biggr) = \biggl(\prod_{j=1} \gamma_j^{g_{ji}^t}\biggr) X_i^t(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}).$$

By setting $\gamma_i = x_i^{-1}$, we have

(3-31)
$$F_i^t(\hat{\boldsymbol{y}}, \boldsymbol{z}) = \biggl(\prod_{j=1} x_j^{-g_{ji}^t}\biggr) X_i^t(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}).$$

Combining it with (3-29), we obtain (3-28). □

3D. ***Example.*** Let us consider the example in Section 2C again. From the data in Table 1, one can read off the following data for the C-matrix $C(t)$, the G-matrix $G(t)$, and the F-polynomials $F_i(t)$ for the seed $\Sigma(t)$ with principal coefficients therein.

$$C(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} F_1(1) = 1, \\ F_2(1) = 1, \end{cases}$$

$$C(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} F_1(2) = 1 + z y_1 + y_1^2, \\ F_2(2) = 1, \end{cases}$$

$$C(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad G(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} F_1(3) = 1 + z y_1 + y_1^2, \\ F_2(3) = 1 + y_2 + z y_1 y_2 + y_1^2 y_2, \end{cases}$$

$$C(4) = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}, \quad G(4) = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}, \quad \begin{cases} F_1(4) = 1 + 2y_2 + y_2^2 \\ \qquad\qquad + z y_1 y_2 + z y_1 y_2^2 + y_1^2 y_2^2, \\ F_2(4) = 1 + y_2 + z y_1 y_2 + y_1^2 y_2, \end{cases}$$

$$C(5) = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}, \quad G(5) = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad \begin{cases} F_1(5) = 1 + 2y_2 + y_2^2 \\ \qquad\qquad + z y_1 y_2 + z y_1 y_2^2 + y_1^2 y_2^2, \\ F_2(5) = 1 + y_2, \end{cases}$$

$$C(6) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad G(6) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} F_1(6) = 1, \\ F_2(6) = 1 + y_2, \end{cases}$$

$$C(7) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G(7) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} F_1(7) = 1, \\ F_2(7) = 1. \end{cases}$$

Acknowledgements

We thank Anne-Sophie Gleitz, Kohei Iwaki, and Michael Shapiro for useful discussions and communications.

References

[Chekhov and Shapiro 2014] L. Chekhov and M. Shapiro, "Teichmüller spaces of Riemann surfaces with orbifold points of arbitrary order and cluster variables", *Int. Math. Res. Not.* **2014**:10 (2014), 2746–2772. MR 3214284 Zbl 1301.30042

[Derksen et al. 2010] H. Derksen, J. Weyman, and A. Zelevinsky, "Quivers with potentials and their representations, II: Applications to cluster algebras", *J. Amer. Math. Soc.* **23**:3 (2010), 749–790. MR 2012c:16044 Zbl 1208.16017

[Fomin and Zelevinsky 2002] S. Fomin and A. Zelevinsky, "Cluster algebras, I: Foundations", *J. Amer. Math. Soc.* **15**:2 (2002), 497–529. MR 2003f:16050 Zbl 1021.16017

[Fomin and Zelevinsky 2003] S. Fomin and A. Zelevinsky, "Cluster algebras, II: Finite type classification", *Invent. Math.* **154**:1 (2003), 63–121. MR 2004m:17011 Zbl 1054.17024

[Fomin and Zelevinsky 2007] S. Fomin and A. Zelevinsky, "Cluster algebras, IV: Coefficients", *Compos. Math.* **143**:1 (2007), 112–164. MR 2008d:16049 Zbl 1127.16023

[Gekhtman et al. 2005] M. Gekhtman, M. Shapiro, and A. Vainshtein, "Cluster algebras and Weil–Petersson forms", *Duke Math. J.* **127**:2 (2005), 291–311. Correction at **139**:2 (2007), 407–409. MR 2006d:53103 Zbl 1079.53124

[Gleitz 2014] A.-S. Gleitz, "Quantum affine algebras at roots of unity and generalised cluster algebras", preprint, 2014. arXiv 1410.2446

[Gross et al. 2014] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, "Canonical bases for cluster algebras", preprint, 2014. arXiv 1411.1394

[Iwaki and Nakanishi 2014] K. Iwaki and T. Nakanishi, "Exact WKB analysis and cluster algebras, II: Simple poles, orbifold points, and generalized cluster algebras", preprint, 2014. arXiv 1409.4641

[Nakanishi 2012] T. Nakanishi, "Tropicalization method in cluster algebras", pp. 95–115 in *Tropical geometry and integrable systems* (Glasgow, 2011), edited by C. Athorne et al., Contemp. Math. **580**, Amer. Math. Soc., Providence, RI, 2012. MR 2985390

[Nakanishi and Zelevinsky 2012] T. Nakanishi and A. Zelevinsky, "On tropical dualities in cluster algebras", pp. 217–226 in *Algebraic groups and quantum groups* (Nagoya, 2010), edited by S. Ariki et al., Contemp. Math. **565**, Amer. Math. Soc., Providence, RI, 2012. MR 2932428 Zbl 06296873

[Plamondon 2011] P.-G. Plamondon, "Cluster algebras via cluster categories with infinite-dimensional morphism spaces", *Compos. Math.* **147**:6 (2011), 1921–1954. MR 2862067 Zbl 1244.13017

[Rupel 2013] D. Rupel, "Greedy bases in rank 2 generalized cluster algebras", preprint, 2013. arXiv 1309.2567

Received November 27, 2014. Revised January 21, 2015.

TOMOKI NAKANISHI
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERISTY
CHIKUSA-KU
NAGOYA 464-8602
JAPAN
nakanisi@math.nagoya-u.ac.jp

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.219

INEQUALITIES OF ALEXANDROV–FENCHEL TYPE FOR CONVEX HYPERSURFACES IN HYPERBOLIC SPACE AND IN THE SPHERE

YONG WEI AND CHANGWEI XIONG

By applying the unit normal flow to well-known inequalities in hyperbolic space $\mathbb{H}^{n+1}$ and in the sphere $\mathbb{S}^{n+1}$, we derive some new inequalities of Alexandrov–Fenchel type for closed convex hypersurfaces in these spaces. We also use the inverse mean curvature flow in the sphere to prove an optimal Sobolev-type inequality for closed convex hypersurfaces in the sphere.

1. Introduction

Let $N^{n+1}(c)$ be the simply connected space form of constant sectional curvature c and $\psi : \Sigma^n \to N^{n+1}(c)$ be a closed hypersurface. Denote the k-th order mean curvature of Σ by p_k (see Section 2A). Inequalities about the integrals $\int_\Sigma p_k \, d\mu$ have attracted much attention for a long time. Among them the most famous one is the classical Minkowski inequality for closed convex surfaces $\Sigma \subset \mathbb{R}^3$, which can be written as

$$\left(\frac{1}{\omega_2}\int_\Sigma p_1 \, d\mu\right)^2 \geq \frac{|\Sigma|}{\omega_2}, \tag{1-1}$$

with equality if and only if Σ is a sphere. Here ω_n is the area of $\mathbb{S}^n(1)$ and $|\Sigma| = \int_\Sigma d\mu$ is the area of Σ with respect to the induced metric from $\mathbb{R}^3$. The general inequality is the Alexandrov–Fenchel inequality [Alexandrov 1937; 1938; Fenchel 1936] which states that for convex hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$,

$$\frac{1}{\omega_n}\int_\Sigma p_k \, d\mu \geq \left(\frac{1}{\omega_n}\int_\Sigma p_l \, d\mu\right)^{(n-k)/(n-l)} \quad \text{for } 0 \leq l < k \leq n, \tag{1-2}$$

with equality if and only if Σ is a sphere. See [Chang and Wang 2011; Guan and Li 2009; McCoy 2005; Schneider 1993] for other references on Alexandrov–Fenchel inequalities for closed hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$.

MSC2010: primary 53C44; secondary 53C42.
Keywords: isoperimetric inequality, convex hypersurface, Alexandrov–Fenchel-type inequality, k-th order mean curvature, Gauss–Bonnet curvature.

It is natural to generalize the Minkowski inequality and Alexandrov–Fenchel inequalities to the hypersurfaces in space forms. See, for example, [Borisenko and Miquel 1999; Gallego and Solanes 2005; Natário 2015]. Recently, the following optimal inequalities of Alexandrov–Fenchel type in $\mathbb{H}^{n+1}$ were obtained (see [Ge et al. 2013; 2014b; Li et al. 2014; Wang and Xia 2014]): for $1 \le k \le n$ and any closed horospherical convex hypersurface $\Sigma \subset \mathbb{H}^{n+1}$,

$$\frac{1}{\omega_n}\int_\Sigma p_k \, d\mu \ge \left(\left(\frac{|\Sigma|}{\omega_n}\right)^{2/k} + \left(\frac{|\Sigma|}{\omega_n}\right)^{2(n-k)/kn}\right)^{k/2}, \tag{1-3}$$

with equality if and only if Σ is a geodesic sphere in $\mathbb{H}^{n+1}$. In particular, when $k = 2$, Li, Wei and Xiong [Li et al. 2014] proved that (1-3) holds under the weaker condition that Σ is star-shaped and 2-convex. In the proof of (1-3), the geometric flow was used and was an important tool. However, so far there is no inequality comparing $\int_\Sigma p_k \, d\mu$ and $\int_\Sigma p_l \, d\mu$ in $\mathbb{H}^{n+1}$ like (1-2) in $\mathbb{R}^{n+1}$. And one also wants to know whether there exist other inequalities of Alexandrov–Fenchel type in $\mathbb{H}^{n+1}$ for closed hypersurfaces under a weaker condition than horospherical convex. Besides, in space forms, the integrals $\int_\Sigma p_k \, d\mu$ are essentially the so-called quermassintegrals from convex geometry and integral geometry (see, e.g., [Solanes 2006] for the transformation formula) and many attempts have been devoted to establishing the relationships for quermassintegrals. See [Santaló 1976; Solanes 2003] and the references therein. So in this paper we are interested in obtaining new inequalities between the integrals $\int_\Sigma p_k \, d\mu$.

The Minkowski inequality and the Alexandrov–Fenchel inequalities can be viewed as the generalizations of the classical isoperimetric inequality, which compares the area of the hypersurface Σ and the volume of the domain enclosed by Σ. The Minkowski inequality (1-1) was used by Minkowski himself to prove the isoperimetric inequality for closed convex surfaces (see [Minkowski 1903; Osserman 1978]). Recently, J. Natário [2015] reversed Minkowski's idea and derived a new Minkowski-type inequality for closed convex surfaces in the hyperbolic space $\mathbb{H}^3$ from the isoperimetric inequality by using the unit normal flow. In this paper, first, we deal with the higher dimensional case by adapting Natário's method [2015]. We will derive some new inequalities of Alexandrov–Fenchel type for closed convex hypersurfaces in $\mathbb{H}^{n+1}$ and in $\mathbb{S}^{n+1}$, starting from the isoperimetric inequality.

Let Σ be a closed and convex hypersurface in $\mathbb{H}^{n+1}$. We say a hypersurface Σ is convex if all the principal curvatures of Σ are nonnegative everywhere. Then by the well-known result of do Carmo and Warner [1970], Σ is embedded and bounds a convex body in $\mathbb{H}^{n+1}$. Inspired by [Natário 2015], we flow the initial hypersurface Σ by its unit outer normal ν. The resulting hypersurfaces are $\Sigma_t = \psi_t(\Sigma)$, where $\psi_t(x) = \exp_{\psi(x)}(t\nu(x))$, $x \in \Sigma$. The Σ_t are also called the parallel hypersurfaces of Σ. From Steiner's formula [Allendoerfer 1948], we can compute the area of Σ_t and the volume of the domain Ω_t enclosed by Σ_t. In Natário's paper, the area of Σ_t was obtained by using the first and second variation formulas with the help of the Gauss–Bonnet formula. Steiner's formula can also be obtained by using the precise expressions of the geodesics in space forms (see Section 2C). Since $\mathbb{H}^{n+1}$ has constant negative curvature $c = -1$ and Σ is convex, it follows that Σ_t can be well-defined for all $t \geq 0$. Define a function $r(t)$ such that $|\Sigma_t| = |S_{r(t)}|$. Then the isoperimetric inequality (see [Schmidt 1940; Ros 2005]) implies that $\mathrm{Vol}(\Omega_t) \leq \mathrm{Vol}(B_{r(t)})$, where $S_{r(t)}$ and $B_{r(t)}$ are the geodesic sphere and geodesic ball of radius $r(t)$ in $\mathbb{H}^{n+1}$, respectively. Applying the isoperimetric inequality to Σ_t for sufficiently large t, we obtain the following inequalities of Alexandrov–Fenchel type in $\mathbb{H}^{n+1}$.

Theorem 1.1. *Let Σ^n be a closed and convex hypersurface in $\mathbb{H}^{n+1}$ with $n \geq 3$. Then*

$$\sum_{k=0}^{n} \frac{2k-n}{n\omega_n} \int_\Sigma C_n^k p_k \, d\mu \geq \left(\frac{1}{\omega_n} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \right)^{(n-2)/n}. \tag{1-4}$$

A direct calculation shows that if Σ is a geodesic sphere, then the equality in (1-4) holds. However, we do not obtain the rigidity (i.e.,we don't know whether the equality in (1-4) implies that Σ is a geodesic sphere). In Remark 3.2, we note that when the hypersurface $\Sigma \subset \mathbb{H}^{n+1}$ is sufficiently small, the inequality (1-4) reduces to one of the Alexandrov–Fenchel inequalities in Euclidean space.

Besides the isoperimetric inequality, there are many other known inequalities in hyperbolic space. If we use the warped product model for the hyperbolic space $\mathbb{H}^{n+1}$, i.e., $\mathbb{H}^{n+1} = \mathbb{R}^+ \times \mathbb{S}^n$ with the metric $g = dr^2 + \sinh^2 r \, g_{\mathbb{S}^n}$, then there are two important functions on the hypersurface Σ in $\mathbb{H}^{n+1}$. One is the weight function $f = \cosh r$, and the other one is the support function $u = \langle Df, \nu \rangle$. Recently, the following inequality of Alexandrov–Fenchel type with weight f was proved by de Lima and Girão [2015]: for any mean convex and star-shaped closed hypersurface Σ in $\mathbb{H}^{n+1}$,

$$\frac{1}{\omega_n} \int_\Sigma f p_1 \, d\mu \geq \left(\frac{|\Sigma|}{\omega_n} \right)^{(n+1)/n} + \left(\frac{|\Sigma|}{\omega_n} \right)^{(n-1)/n}, \tag{1-5}$$

with equality if and only if Σ is a geodesic sphere centered at the origin in $\mathbb{H}^{n+1}$. For more weighted inequalities of Alexandrov–Fenchel type in different ambient spaces, readers can refer to the recent papers [Brendle et al. 2014; Ge et al. 2014a; 2015]. We remark that in [Ge et al. 2014a], the weighted Alexandrov–Fenchel-type inequalities were used to prove the Penrose-type inequality for the new Gauss–Bonnet–Chern mass in asymptotically hyperbolic graphs. Thus it is an interesting question to establish new inequalities with weight.

Applying the same method as in Theorem 1.1 to inequality (1-5), we can obtain a new inequality as follows:

Theorem 1.2. *Let Σ^n be a closed and convex hypersurface in $\mathbb{H}^{n+1}$. Then*

$$\frac{1}{\omega_n}\int_\Sigma (f+u)\sum_{k=0}^n C_n^k p_k \, d\mu \geq \left(\frac{1}{\omega_n}\int_\Sigma \sum_{k=0}^n C_n^k p_k \, d\mu\right)^{(n+1)/n}. \tag{1-6}$$

We remark that if Σ is a geodesic sphere centered at the origin, then the equality in (1-6) holds. But as before we do not obtain the rigidity.

Next we will use the same method to derive inequalities for closed convex hypersurfaces in $\mathbb{S}^{n+1}$. In this case, we can prove the rigidity result.

Theorem 1.3. *Let Σ^n be a closed and convex hypersurface in $\mathbb{S}^{n+1}$ with $n \geq 2$. Then*

$$\omega_n \leq \sum_{s=\frac{1-(-1)^n}{2},+2}^{n} \sqrt{(E(s))^2+(F(s))^2}, \tag{1-7}$$

where "+2" means that the step-length of the summation for s is 2 and

$$E(s) = \sum_{\substack{p+q=(n\pm s)/2\\ p,q\geq 0}} \sum_{\substack{q\leq k\leq n-p\\ 2\mid k}} C_n^k \frac{1}{2^n} C_{n-k}^p C_k^q (-1)^{\frac{k}{2}+k-q} \int_\Sigma p_k \, d\mu,$$

$$F(s) = \sum_{\substack{p+q=(n\pm s)/2\\ p,q\geq 0}} \sum_{\substack{q\leq k\leq n-p\\ 2\nmid k}} C_n^k \frac{1}{2^n} C_{n-k}^p C_k^q (-1)^{[\frac{k}{2}]+k-q} \times (-1)^{\chi_{\{2(p+q)-n\leq 0\}}} \int_\Sigma p_k \, d\mu,$$

Moreover, the equality holds in (1-7) if and only if Σ^n is a geodesic sphere.

When $n=2$, it is easy to check that

$$E(0)=2\pi, \qquad F(0)=0,$$
$$E(2)=|\Sigma|-2\pi, \qquad F(2)=\int_\Sigma p_1 \, d\mu,$$

using the Gauss–Bonnet theorem $|\Sigma|+\int_\Sigma p_2 \, d\mu = 4\pi$ (see Section 2B). So (1-7) implies the Minkowski-type inequality in the sphere

$$\left(\int_\Sigma p_1 \, d\mu\right)^2 \geq |\Sigma|(4\pi-|\Sigma|), \tag{1-8}$$

which is just Theorem 0.2 in [Natário 2015]. See also [Blaschke 1938; Knothe 1952; Santaló 1963]. Makowski and Scheuer [2013] proved (1-8) by using the inverse curvature flow in sphere. To get a better feeling of the inequality (1-7), we also give the precise expressions of (1-7) in the case of $n=3$ and $n=4$; see Remark 3.3.

Finally, in the last part of this paper, we use the inverse mean curvature flow in the sphere [Makowski and Scheuer 2013; Gerhardt 2015] to prove the following optimal inequalities for strictly convex hypersurfaces in sphere $\mathbb{S}^{n+1}$.

Theorem 1.4. *Let Σ^n be a closed and strictly convex hypersurface in $\mathbb{S}^{n+1}$. Then we have the optimal inequality*

$$\int_\Sigma L_k \, d\mu \geq C_n^{2k} (2k)! \omega_n^{2k/n} |\Sigma|^{(n-2k)/n} \quad \text{for } k \leq n/2. \tag{1-9}$$

Equality holds in (1-9) if and only if Σ is a geodesic sphere. Here L_k is the Gauss–Bonnet curvature of the induced metric on Σ (see Section 2B for details).

The proof of Theorem 1.4 uses a similar idea as in [Brendle et al. 2014; de Lima and Girão 2015; Guan and Li 2009; Ge et al. 2013; 2014b; Li et al. 2014]. We define a curvature quantity $Q(t)$ which is monotone nonincreasing under the inverse mean curvature flow in the sphere. Then we obtain the inequality (1-9) by comparing the initial value $Q(0)$ with the limit $\lim_{t \to T^*} Q(t)$. We remark that since Σ is a closed and strictly convex hypersurface in $\mathbb{S}^{n+1}$, a well-known result due to do Carmo and Warner [1970] implies that Σ is embedded and is contained in an open hemisphere.

When $k = 1$, the inequality (1-9) reduces to

$$\int_\Sigma p_2 \, d\mu + |\Sigma| \geq \omega_n^{2/n} |\Sigma|^{(n-2)/n}, \tag{1-10}$$

which was already proved by Makowski and Scheuer [2013]. One can compare (1-10) with the case $k = 2$ of the Alexandrov–Fenchel-type inequality (1-3) in $\mathbb{H}^{n+1}$; that is,

$$\int_\Sigma p_2 \, d\mu - |\Sigma| \geq \omega_n^{2/n} |\Sigma|^{(n-2)/n}, \tag{1-11}$$

which was proved by Li, Wei and Xiong [Li et al. 2014] for star-shaped and 2-convex hypersurfaces in $\mathbb{H}^{n+1}$. For $k > 1$, inequalities of the same type as (1-9) were proved by Ge, Wang and Wu [Ge et al. 2013; 2014b] for horospherical convex hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$.

2. Preliminaries

2A. ***k-th order mean curvature.*** Let Σ be a closed hypersurface in $N^{n+1}(c)$ with unit outward normal ν. The second fundamental form h of Σ is defined by

$$h(X, Y) = \langle \overline{\nabla}_X \nu, Y \rangle$$

for any two tangent vector fields X, Y on Σ. For an orthonormal basis $\{e_1, \ldots, e_n\}$ of Σ, the components of the second fundamental form are given by $h_{ij} = h(e_i, e_j)$ and $h_i^j = g^{jk} h_{ki}$, where g is the induced metric on Σ. The principal curvatures

$\kappa = (\kappa_1, \dots, \kappa_n)$ are the eigenvalues of h with respect to g. The k-th order mean curvature of Σ for $1 \le k \le n$ is defined as

$$p_k = \frac{1}{C_n^k}\sigma_k(\kappa) = \frac{1}{C_n^k}\sum_{i_1<i_2<\cdots<i_k} \kappa_{i_1}\cdots\kappa_{i_k}, \tag{2-1}$$

or equivalently as

$$p_k = \frac{1}{C_n^k}\sigma_k(h_i^j) = \frac{1}{C_n^k k!}\delta_{j_1\cdots j_k}^{i_1\cdots i_k} h_{i_1}^{j_1}\cdots h_{i_k}^{j_k}, \tag{2-2}$$

where $\delta_{j_1\cdots j_k}^{i_1\cdots i_k}$ is the generalized Kronecker delta given by

$$\delta_{j_1\cdots j_k}^{i_1\cdots i_k} = \det\begin{pmatrix} \delta_{j_1}^{i_1} & \delta_{j_1}^{i_2} & \cdots & \delta_{j_1}^{i_k} \\ \delta_{j_2}^{i_1} & \delta_{j_2}^{i_2} & \cdots & \delta_{j_2}^{i_k} \\ \vdots & \vdots & \vdots & \vdots \\ \delta_{j_k}^{i_1} & \delta_{j_k}^{i_2} & \cdots & \delta_{j_k}^{i_k} \end{pmatrix}.$$

We have the following Newton–MacLaurin inequalities (see, e.g., [Guan 2006]).

Lemma 2.1. *For $\kappa \in \overline{\Gamma}_k^+$, $1 \le k \le n$, where $\overline{\Gamma}_k^+$ is the closure of the Garding cone*

$$\Gamma_k^+ = \{\kappa \in \mathbb{R}^n \mid p_j(\kappa) > 0, \forall j \le k\},$$

we have the following Newton–MacLaurin inequalities

$$p_1 p_{k-1} \ge p_k,$$
$$p_1 \ge p_2^{1/2} \ge \cdots \ge p_k^{1/k}.$$

Moreover, equalities above hold for some $\kappa \in \Gamma_k^+$ if and only if $\kappa = c(1, \dots, 1)$, where c is a constant.

2B. ***Gauss–Bonnet curvature L_k.*** Given an n-dimensional Riemannian manifold (M, g), the Gauss–Bonnet curvature L_k, where $k \le n/2$, is defined by (see, e.g., [Ge et al. 2014b; 2014c])

$$L_k = \frac{1}{2^k}\delta_{j_1 j_2\cdots j_{2k-1} j_{2k}}^{i_1 i_2\cdots i_{2k-1} i_{2k}} R_{i_1 i_2}{}^{j_1 j_2}\cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}. \tag{2-3}$$

For a closed hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$, recall the Gauss equation

$$R_{ij}{}^{kl} = h_i^k h_j^l - h_i^l h_j^k.$$

Then the Gauss–Bonnet curvature of the induced metric on $\Sigma^n \subset \mathbb{R}^{n+1}$ is

$$\begin{aligned} L_k &= \delta_{j_1 j_2\cdots j_{2k-1} j_{2k}}^{i_1 i_2\cdots i_{2k-1} i_{2k}} h_{i_1}^{j_1} h_{i_2}^{j_2}\cdots h_{i_{2k-1}}^{j_{2k-1}} h_{i_{2k}}^{j_{2k}} \\ &= (2k)! C_n^{2k} p_{2k}. \end{aligned} \tag{2-4}$$

For a closed hypersurface $\Sigma^n \subset \mathbb{S}^{n+1}$, the Gauss equations are

$$R_{ij}{}^{kl} = (h_i^k h_j^l - h_i^l h_j^k) + (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k). \tag{2-5}$$

Then by a straightforward calculation, we have

$$\begin{aligned} L_k &= \delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} (h_{i_1}^{j_1} h_{i_2}^{j_2} + \delta_{i_1}^{j_1} \delta_{i_2}^{j_2}) \cdots (h_{i_{2k-1}}^{j_{2k-1}} h_{i_{2k}}^{j_{2k}} + \delta_{i_{2k-1}}^{j_{2k-1}} \delta_{i_{2k}}^{j_{2k}}) \\ &= \sum_{i=0}^{k} C_k^i (n-2k+1)(n-2k+2)\cdots(n-2k+2i)(2k-2i)!\,\sigma_{2k-2i} \\ &= \sum_{i=0}^{k} C_k^i (n-2k+1)(n-2k+2)\cdots(n-2k+2i)(2k-2i)!\,C_n^{2k-2i}\, p_{2k-2i} \\ &= \sum_{i=0}^{k} C_k^i \frac{n!}{(n-2k)!} p_{2k-2i} \\ &= C_n^{2k}(2k)! \sum_{i=0}^{k} C_k^i\, p_{2k-2i}. \end{aligned}$$

Similarly, for a closed hypersurface $\Sigma^n \subset \mathbb{H}^{n+1}$, its Gauss–Bonnet curvature is

$$L_k = C_n^{2k}(2k)! \sum_{i=0}^{k} C_k^i (-1)^i\, p_{2k-2i}. \tag{2-6}$$

Finally, note that throughout our paper, we assume that the hypersurface $\Sigma \subset N^{n+1}(c)$ is closed and convex. It follows that Σ is homeomorphic to the n-sphere (see [do Carmo and Warner 1970]). Then if the dimension of Σ is even, the Gauss–Bonnet–Chern theorem [Chern 1944] implies that

$$\int_\Sigma L_{n/2}\, d\mu = n!\, \omega_n. \tag{2-7}$$

Equation (2-7) will be used in the following sections. Also (2-7) shows that when $2k = n$, the inequality (1-9) is an equality.

2C. ***Unit normal flow and Steiner's formula.*** Let $\psi : \Sigma \to N^{n+1}(c)$ be a closed and convex hypersurface in the simply connected space form $N^{n+1}(c)$ of constant sectional curvature c. Denote by Ω the domain enclosed by Σ. The area of Σ is denoted by $|\Sigma|$ and the volume of Ω is denoted by $|V|$. As we mentioned in Section 1, we flow the initial hypersurface Σ by its unit outer normal ν. The resulting hypersurfaces are $\Sigma_t = \psi_t(\Sigma)$, where $\psi_t(x) = \exp_{\psi(x)}(t\nu(x)), x \in \Sigma$. The Σ_t are also called the parallel hypersurfaces of Σ. Denote by Ω_t the domain bounded by Σ_t. The convexity assumption of Σ and the curvature of $N^{n+1}(c)$ guarantee that the Σ_t are well-defined in the following range:

$$\begin{aligned} &t \in \left[0, \tfrac{\pi}{2}\right) && \text{for } c = 1, \\ &t \geq 0 && \text{for } c = 0 \text{ or } -1. \end{aligned}$$

Further, denote the area of Σ_t and the volume of Ω_t by $|\Sigma_t|$ and $|V_t|$, respectively. Then Steiner's formula [Allendoerfer 1948] implies that

$$|\Sigma_t| = \begin{cases} \sum_{k=0}^n \int_\Sigma C_n^k p_k \, d\mu \, t^k & \text{if } c = 0, \\ \sum_{k=0}^n \int_\Sigma C_n^k p_k \, d\mu \cosh^{n-k} t \sinh^k t & \text{if } c = -1, \\ \sum_{k=0}^n \int_\Sigma C_n^k p_k \, d\mu \cos^{n-k} t \sin^k t & \text{if } c = 1. \end{cases} \tag{2-8}$$

and

$$|V_t| = \begin{cases} |V| + \sum_{k=0}^n \int_\Sigma C_n^k p_k \, d\mu \frac{1}{k+1} t^{k+1} & \text{if } c = 0, \\ |V| + \sum_{k=0}^n \int_\Sigma C_n^k p_k \, d\mu \int_0^t \cosh^{n-k} s \sinh^k s \, ds & \text{if } c = -1, \\ |V| + \sum_{k=0}^n \int_\Sigma C_n^k p_k \, d\mu \int_0^t \cos^{n-k} s \sin^k s \, ds & \text{if } c = 1. \end{cases} \tag{2-9}$$

We give a simple proof of (2-8) and (2-9) here. First, when $c = 0$, the parallel hypersurface can be expressed as $\psi_t = \psi + t\nu$ (see [Montiel and Ros 1991]). So $(\psi_t)_*(e_i) = (1 + t\kappa_i)e_i$. Therefore the area element of Σ_t is

$$d\mu_t = (1 + t\kappa_1) \cdots (1 + t\kappa_n) \, d\mu,$$

which implies that the areas $|\Sigma_t|$ of the parallel hypersurfaces Σ_t are equal to

$$|\Sigma_t| = \int_\Sigma (1 + t\kappa_1) \cdots (1 + t\kappa_n) \, d\mu = \sum_{k=0}^n \int_\Sigma C_n^k p_k \, d\mu \, t^k.$$

Note that $\Sigma_t = \psi_t(\Sigma)$ are parallel hypersurfaces of Σ given by

$$\psi_t(x) = \exp_{\psi(x)}(t\nu(x))$$

for $x \in \Sigma$. By integrating and using the co-area formula, we obtain

$$|V_t| = |V| + \int_0^t |\Sigma_s| \, ds = |V| + \sum_{k=0}^n \int_\Sigma C_n^k p_k \, d\mu \frac{1}{k+1} t^{k+1}.$$

Similarly, when $c = -1$, $\psi_t = \cosh t \, \psi + \sinh t \, \nu$ (see [Montiel and Ros 1991]) and so $(\psi_t)_*(e_i) = (\cosh t + \sinh t \, \kappa_i)e_i$. Therefore the area element of Σ_t is

$$d\mu_t = (\cosh t + \sinh t \, \kappa_1) \cdots (\cosh t + \sinh t \, \kappa_n) \, d\mu,$$

which implies

$$\begin{aligned}|\Sigma_t| &= \int_\Sigma (\cosh t + \sinh t\,\kappa_1)\cdots(\cosh t + \sinh t\,\kappa_n)\,d\mu \\ &= \sum_{k=0}^{n} \int_\Sigma C_n^k\, p_k\, d\mu \cosh^{n-k} t \sinh^k t.\end{aligned}$$

Then by integrating, we obtain

$$|V_t| = |V| + \int_0^t |\Sigma_s|\,ds = |V| + \sum_{k=0}^{n} \int_\Sigma C_n^k\, p_k\, d\mu \int_0^t \cosh^{n-k} s \sinh^k s\, ds.$$

Finally, the case $c = 1$ can also be proved by noting that $\psi_t = \cos t\,\psi + \sin t\,\nu$, where $t \in \left[0, \frac{\pi}{2}\right)$.

3. The results by the method of unit normal flow

3A. ***The Euclidean case.*** To demonstrate the method which will be used to prove Theorems 1.1–1.3, in this subsection we first consider the simple case that Σ is a closed and convex hypersurface in $\mathbb{R}^{n+1}$. Let Σ_t be the parallel hypersurfaces of Σ and Ω_t be the domain enclosed by Σ_t. Then Σ_t is well-defined for all $t \geq 0$. For all $t \geq 0$, the isoperimetric inequality (see [Osserman 1978]) in Euclidean space $\mathbb{R}^{n+1}$ implies

$$\left(\frac{|\Sigma_t|}{\omega_n}\right)^{n+1} \geq \left((n+1)\frac{|V_t|}{\omega_n}\right)^{n}. \tag{3-1}$$

Substitute Steiner's formulas (2-8), (2-9) into (3-1). If n is odd, then comparing the coefficient of $t^{n(n+1)}$ in (3-1) yields

$$\int_\Sigma p_n\, d\mu \geq \omega_n, \tag{3-2}$$

which is a special Alexandrov–Fenchel inequality.

If n is even, (2-4) and the Gauss–Bonnet–Chern theorem (2-7) imply that $\int_\Sigma p_n\, d\mu = \omega_n$. Thus expanding the two sides of (3-1) and comparing the coefficients of $t^{n(n+1)}$, $t^{n(n+1)-1}$ and $t^{n(n+1)-2}$, we can get

$$\left(\frac{1}{\omega_n}\int_\Sigma p_{n-1}\, d\mu\right)^2 \geq \frac{1}{\omega_n}\int_\Sigma p_{n-2}\, d\mu, \tag{3-3}$$

which is also an Alexandrov–Fenchel inequality. In particular, when $n = 2$, (3-3) reduces to the classical Minkowski inequality (1-1).

3B. ***The hyperbolic case, I.*** In this subsection, we prove Theorem 1.1. Assume that Σ is a closed and convex hypersurface in $\mathbb{H}^{n+1}$. Then the parallel hypersurfaces Σ_t are well-defined for all $t \geq 0$. Recall that the area of a geodesic sphere S_r and the volume of a geodesic ball B_r with radius r in the hyperbolic space $\mathbb{H}^{n+1}$ are

$$S(r) := |S_r| = \omega_n \sinh^n r,$$
$$V(r) := \mathrm{Vol}(B_r) = \omega_n \int_0^r \sinh^n s \, ds.$$

Now define a function $r(t)$ such that $|\Sigma_t| = S(r(t))$. That is,

$$\sum_{k=0}^{n} \cosh^{n-k} t \sinh^k t \int_\Sigma C_n^k p_k \, d\mu = \omega_n \sinh^n r(t). \tag{3-4}$$

Then the isoperimetric inequality (see [Schmidt 1940; Ros 2005]) implies

$$|V_t| \leq V(r(t)) \quad \text{for } t \geq 0. \tag{3-5}$$

From this inequality, we can get some information for Σ.

First we get a rough estimate for $r(t)$. When $t \to +\infty$, $\cosh^{n-k} t \sinh^k t = \sinh^n t\,(1 + o(1))$. Thus from $|\Sigma_t| = S(r(t))$, we get

$$\sinh^n t\,(1 + o(1)) \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu = \omega_n \sinh^n r(t),$$

which implies

$$r(t) = t + \frac{1}{n} \ln\left(\frac{1}{\omega_n} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k \, d\mu \right) + o(1). \tag{3-6}$$

However, this estimate for $r(t)$ is not enough. For our purposes, we should make better use of $|\Sigma_t| = S(r(t))$ as follows. The case of $n = 2$ was considered by Natário [2015], so we assume that $n \geq 3$ in the following calculation. Since we will examine (3-5) for sufficiently large t, we only care about the terms involving e^{nt} and $e^{(n-2)t}$. The other terms are $o(e^{(n-2)t})$. It is straightforward to check that

$$\cosh^{n-k} t \sinh^k t = \frac{1}{2^n} e^{nt} + \frac{1}{2^n} (n - 2k) e^{(n-2)t} + \cdots .$$

Consequently (3-4) implies

$$\frac{1}{2^n} \sum_{k=0}^{n} (e^{nt} + (n-2k) e^{(n-2)t} + \cdots) \int_\Sigma C_n^k p_k \, d\mu = \omega_n \left(\frac{1}{2^n} e^{nr} - \frac{1}{2^n} n e^{(n-2)r} + \cdots \right). \tag{3-7}$$

On the other hand, from Steiner's formula (2-9), we have

$$\begin{aligned}
|V_t| &= |V| + \sum_{k=0}^{n} \int_0^t \cosh^{n-k} s \sinh^k s\, ds \int_\Sigma C_n^k p_k\, d\mu \\
&= |V| + \frac{1}{2^n} \sum_{k=0}^{n} \int_0^t (e^{ns} + (n-2k)e^{(n-2)s} + \cdots)\, ds \int_\Sigma C_n^k p_k\, d\mu \\
&= |V| + \frac{1}{2^n} \sum_{k=0}^{n} \Big(\frac{1}{n} e^{nt} + \frac{n-2k}{n-2} e^{(n-2)t} + \cdots \Big) \int_\Sigma C_n^k p_k\, d\mu \\
&= \frac{1}{2^n} \frac{1}{n} e^{nt} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k\, d\mu + \frac{1}{2^n} e^{(n-2)t} \sum_{k=0}^{n} \frac{n-2k}{n-2} \int_\Sigma C_n^k p_k\, d\mu + \cdots,
\end{aligned}$$

and

$$\begin{aligned}
V(r(t)) &= \omega_n \int_0^r \sinh^n s\, ds \\
&= \omega_n \int_0^r \Big(\frac{1}{2^n} e^{ns} - \frac{1}{2^n} n e^{(n-2)s} + \cdots \Big)\, ds \\
&= \frac{\omega_n}{2^n} \frac{1}{n} e^{nr} - \frac{\omega_n}{2^n} \frac{n}{n-2} e^{(n-2)r} + \cdots .
\end{aligned}$$

Now taking (3-7) into account yields

$$\begin{aligned}
V(r(t)) &= \frac{\omega_n}{2^n} e^{(n-2)r} + \frac{1}{2^n} \frac{1}{n} \sum_{k=0}^{n} (e^{nt} + (n-2k)e^{(n-2)t} + \cdots) \int_\Sigma C_n^k p_k\, d\mu \\
&\qquad\qquad - \frac{\omega_n}{2^n} \frac{n}{n-2} e^{(n-2)r} \\
&= \frac{\omega_n}{2^n} \Big(\frac{-2}{n-2} \Big) e^{(n-2)r} + \frac{1}{2^n} \frac{1}{n} e^{nt} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k\, d\mu \\
&\qquad\qquad + \frac{1}{2^n} \frac{1}{n} e^{(n-2)t} \sum_{k=0}^{n} (n-2k) \int_\Sigma C_n^k p_k\, d\mu + \cdots .
\end{aligned}$$

Noting (3-6), we have

$$\begin{aligned}
V(r(t)) &= \frac{\omega_n}{2^n} \Big(\frac{-2}{n-2} \Big) e^{(n-2)t} \Big(\frac{1}{\omega_n} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k\, d\mu \Big)^{(n-2)/n} \\
&\quad + \frac{1}{2^n} \frac{1}{n} e^{nt} \sum_{k=0}^{n} \int_\Sigma C_n^k p_k\, d\mu + \frac{1}{2^n} \frac{1}{n} e^{(n-2)t} \sum_{k=0}^{n} (n-2k) \int_\Sigma C_n^k p_k\, d\mu + \cdots .
\end{aligned}$$

Now $|V_t| \le V(r(t)), t \to +\infty$ gives us

$$\frac{1}{2^n}\sum_{k=0}^{n}\left(\frac{n-2k}{n-2}-\frac{n-2k}{n}\right)\int_\Sigma C_n^k\, p_k\, d\mu \le \frac{\omega_n}{2^n}\frac{-2}{n-2}\left(\frac{1}{\omega_n}\sum_{k=0}^{n}\int_\Sigma C_n^k\, p_k\, d\mu\right)^{(n-2)/n},$$

or equivalently

$$\sum_{k=0}^{n}\frac{2k-n}{n}\int_\Sigma C_n^k\, p_k\, d\mu \ge \omega_n\left(\frac{1}{\omega_n}\sum_{k=0}^{n}\int_\Sigma C_n^k\, p_k\, d\mu\right)^{(n-2)/n} \quad \text{for } n\ge 3. \tag{3-8}$$

Hence we complete the proof of Theorem 1.1.

Remark 3.1. It is easy to check that for a geodesic sphere in $\mathbb{H}^{n+1}$, the equality holds in (3-8). However this method can not yield the rigidity result; i.e., we cannot conclude that Σ is a geodesic sphere if the equality holds in (3-8).

Remark 3.2. We also remark that for a small hypersurface $\Sigma \subset \mathbb{H}^{n+1}$ (i.e., with small diameter), the inequality (3-8) can reduce to the Euclidean inequalities (3-2) and (3-3). For example, we first assume $n = 4$. For 4-dimensional hypersurface $\Sigma \subset \mathbb{H}^5$, the Gauss–Bonnet–Chern formula (2-7) implies

$$\int_\Sigma (p_4 - 2p_2 + 1)\, d\mu = \frac{1}{4!}\int_\Sigma L_2\, d\mu = \omega_4. \tag{3-9}$$

Substituting (3-9) into the inequality (3-8) gives that

$$\left(1+\frac{2}{\omega_4}\int_\Sigma (p_3+p_2-p_1-1)\, d\mu\right)^2 \ge 1+\frac{4}{\omega_4}\int_\Sigma (p_3+2p_2+p_1)\, d\mu.$$

Expanding the left-hand side of the above inequality, and comparing both sides by orders (note that Σ is a small hypersurface), we obtain that

$$\left(\frac{1}{\omega_4}\int_\Sigma p_3\, d\mu\right)^2 \ge \frac{1}{\omega_4}\int_\Sigma p_2\, d\mu. \tag{3-10}$$

This is just the inequality (3-3) for hypersurfaces in Euclidean space $\mathbb{R}^5$. For the general even-dimensional case, by using the Gauss–Bonnet–Chern formula,

$$\int_\Sigma \sum_{k=0}^{n/2} C_{n/2}^k (-1)^k\, p_{n-2k}\, d\mu = \frac{1}{n!}\int_\Sigma L_{n/2}\, d\mu = \omega_n.$$

We can also reduce the inequality (3-8) to the Euclidean version (3-3) for small hypersurfaces $\Sigma \subset \mathbb{H}^{n+1}$. For the odd-dimensional case, the argument is similar.

3C. ***The hyperbolic case, II.*** In this subsection, we will prove Theorem 1.2. Since the method is similar to that of the last subsection, we just sketch it.

Here we need the following model of the hyperbolic space. Let $\mathbb{R}^{n+2}_1$ be the Minkowski space with the Lorentzian metric

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_{n+1} y_{n+1} - x_{n+2} y_{n+2}.$$

Then the $(n+1)$-dimensional hyperbolic space can be defined by

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+2}_1 \mid \langle x, x\rangle = -1, x_{n+2} \geq 1\}$$

with the induced metric from $\mathbb{R}^{n+2}_1$.

Fix a point $a = (0, \dots, 0, -1)$. Then it is easy to check that the weight function and the support function can be written down as

$$f = \langle \psi, a\rangle,$$
$$u = \langle \nu, a\rangle.$$

Next define a family of parallel hypersurfaces $\Sigma_t = \psi_t(\Sigma)$, where $\psi_t(x) = \exp_{\psi(x)}(t\nu(x))$, $x \in \Sigma$, and $\nu(x)$ is the outward unit normal of Σ. In fact, $\psi_t = \cosh t\, \psi + \sinh t\, \nu$. And since the initial hypersurface is convex, Σ_t is well-defined for all $t \geq 0$. Then $(\psi_t)_*(e_i) = (\cosh t + \kappa_i \sinh t)e_i$ and

$$\kappa_i(t) = \frac{\tanh t + \kappa_i}{1 + \kappa_i \tanh t}.$$

For convenience, we define a function $Q(t)$ by

$$Q_n(t) = (1 + t\kappa_1)\cdots(1 + t\kappa_n) = 1 + C^1_n p_1 t + \cdots + C^n_n p_n t^n.$$

Then the mean curvature of Σ_t is

$$p_1(t) = \frac{n \cosh t \sinh t\, Q_n(\tanh t) + Q_n'(\tanh t)}{n \cosh^2 t\, Q_n(\tanh t)}.$$

Note that $p_1(t) \to 1$ as $t \to +\infty$. So for sufficiently large t, Σ_t is mean convex. And $\langle \nu_t, a\rangle = \langle \sinh t\, \psi + \cosh t\, \nu, a\rangle \geq 0$ for sufficiently large t, which implies Σ_t is star-shaped for these t. Thus, we can apply (1-5) to Σ_t:

$$\begin{aligned}\frac{1}{\omega_n}\int_\Sigma &\langle \cosh t\, \psi + \sinh t\, \nu, a\rangle\, p_1(t) \cosh^n t\, Q_n(\tanh t)\, d\mu \\ &\geq \left(\frac{1}{\omega_n}\int_\Sigma \cosh^n t\, Q_n(\tanh t)\, d\mu\right)^{\frac{n+1}{n}} + \left(\frac{1}{\omega_n}\int_\Sigma \cosh^n t\, Q_n(\tanh t)\, d\mu\right)^{\frac{n-1}{n}}.\end{aligned}$$

Let $t \to +\infty$. Taking into account that $\tanh t \to 1$, $p_1(t) \to 1$ and $\sinh t = \cosh t\,(1 + o(1))$, we obtain (1-6). So we have finished the proof.

3D. ***The spherical case.*** We now prove Theorem 1.3. Assume that Σ is a closed and convex hypersurface in $\mathbb{S}^{n+1}$. Then the parallel hypersurface Σ_t is well-defined for $t \in \big[0, \frac{\pi}{2}\big)$. Recall that the area of a geodesic sphere S_r and the volume of a geodesic ball B_r with radius r in the sphere $\mathbb{S}^{n+1}$ are

$$\begin{aligned} S(r) &= \omega_n \sin^n r, \\ V(r) &= \omega_n \int_0^r \sin^n s \, ds. \end{aligned}$$

Now since $|V_t|$ is increasing in t, when t satisfies

$$|V_t| = V\Big(\frac{\pi}{2}\Big) = \omega_n \int_0^{\pi/2} \sin^n r \, dr,$$

the isoperimetric inequality (see [Ros 2005]) implies $|\Sigma_t| \geq S(\frac{\pi}{2}) = \omega_n$ for this t. Therefore, a weaker requirement is

$$\max_{t\in[0,\pi/2)} |\Sigma_t| \geq \omega_n. \tag{3-11}$$

Then the key point is to estimate $\max_{t\in[0,\pi/2)}|\Sigma_t|$. Direct computation shows that

$$\begin{aligned} \cos^{n-k} t \sin^k t &= \Big(\frac{e^{it}+e^{-it}}{2}\Big)^{n-k}\Big(\frac{e^{it}-e^{-it}}{2i}\Big)^k \\ &= \frac{1}{2^n}\sum_{p=0}^{n-k}\sum_{q=0}^{k} C_{n-k}^p C_k^q \cos\Big((2(p+q)-n)t - \frac{k\pi}{2}\Big)(-1)^{k-q}. \end{aligned}$$

Then Steiner's formula (2-8) implies

$$\begin{aligned} |\Sigma_t| &= \sum_{k=0}^{n} C_n^k \cos^{n-k} t \sin^k t \int_\Sigma p_k \, d\mu \\ &= \sum_{k=0}^{n} C_n^k \frac{1}{2^n}\sum_{p=0}^{n-k}\sum_{q=0}^{k} C_{n-k}^p C_k^q \cos\Big((2(p+q)-n)t - \frac{k\pi}{2}\Big)(-1)^{k-q}\int_\Sigma p_k \, d\mu \\ &= \sum_{\substack{0\leq k\leq n \\ 2|k}} C_n^k \frac{1}{2^n}\sum_{p=0}^{n-k}\sum_{q=0}^{k} C_{n-k}^p C_k^q (-1)^{\frac{k}{2}} \cos((2(p+q)-n)t)(-1)^{k-q}\int_\Sigma p_k \, d\mu \\ &\quad + \sum_{\substack{0\leq k\leq n \\ 2\nmid k}} C_n^k \frac{1}{2^n}\sum_{p=0}^{n-k}\sum_{q=0}^{k} C_{n-k}^p C_k^q (-1)^{[\frac{k}{2}]} \sin((2(p+q)-n)t)(-1)^{k-q}\int_\Sigma p_k \, d\mu. \end{aligned}$$

Next let $2(p+q)-n=\pm s$ and sum up in terms of s first. We get

$$
\begin{aligned}
|\Sigma_t| = &\sum_{s=\frac{1-(-1)^n}{2},+2}^{n} \sum_{\substack{p+q=(n\pm s)/2\\ p,q\ge 0}} \sum_{\substack{q\le k\le n-p\\ 2|k}} C_n^k \frac{1}{2^n} C_{n-k}^p C_k^q \\
&\qquad\qquad \times(-1)^{\frac{k}{2}+k-q}\cos(st)\int_\Sigma p_k\,d\mu \\
&+\sum_{s=\frac{1-(-1)^n}{2},+2}^{n} \sum_{\substack{p+q=(n\pm s)/2\\ p,q\ge 0}} \sum_{\substack{q\le k\le n-p\\ 2\nmid k}} C_n^k \frac{1}{2^n} C_{n-k}^p C_k^q (-1)^{[\frac{k}{2}]+k-q} \\
&\qquad\qquad \times(-1)^{\chi\{2(p+q)-n\le 0\}}\sin(st)\int_\Sigma p_k\,d\mu \\
\le &\sum_{s=\frac{1-(-1)^n}{2},+2}^{n} \sqrt{(E(s))^2+(F(s))^2},
\end{aligned}
\tag{3-12}
$$

in the notation of Theorem 1.3.

Next we show that for the geodesic sphere with radius $r\in\left[0,\frac{\pi}{2}\right)$, the equality holds. For this special hypersurface, $\int_\Sigma p_k\,d\mu=\omega_n\sin^n r\cot^k r=\omega_n\sin^{n-k}r\cos^k r$. Thus

$$
\begin{aligned}
|\Sigma_t| &= \omega_n\sum_{k=0}^{n} C_n^k(\cos t\sin r)^{n-k}(\sin t\cos r)^k=\omega_n\sin^n(r+t)\\
&=\omega_n\frac{1}{2^n}\sum_{q=0}^{n} C_n^q\cos\left((2q-n)(r+t)-\frac{n\pi}{2}\right)(-1)^{n-q}.
\end{aligned}
$$

For simplicity, we assume n is even. Then

$$
\begin{aligned}
|\Sigma_t| &= \omega_n\frac{1}{2^n}\sum_{q=0}^{n} C_n^q\cos((2q-n)(r+t))(-1)^{n/2+n-q}\\
&=\omega_n\sum_{s=0,2,\dots,n}\ \sum_{2q-n=\pm s}\frac{1}{2^n}C_n^q\cos(s(r+t))(-1)^{3n/2-q}\\
&=\omega_n\sum_{s=0,2,\dots,n}\ \sum_{2q-n=s}2\frac{1}{2^n}C_n^q\cos(s(r+t))(-1)^{3n/2-q},
\end{aligned}
$$

where we note that the coefficients of $\cos(s(r+t))$ for the two choices of q are the same.

Now expand $\cos(s(r+t))=\cos sr\cos st-\sin sr\sin st$. We find that all the inequalities in (3-12) become equalities for $t=\frac{\pi}{2}-r$ and $|\Sigma_t|=\omega_n\sin^n\frac{\pi}{2}=\omega_n$. Thus for the geodesic sphere, the equality in (3-12) holds.

On the other hand, assume the equality holds. Then when some t satisfies $|V_t|=V(\frac{\pi}{2})=\omega_n\int_0^{\pi/2}\sin^n r\,dr$, we must have $|\Sigma_t|=S(\frac{\pi}{2})=\omega_n$ for this t. So the isoperimetric inequality implies that $\Sigma_t = \mathbb{S}^n(1)$. Then the initial hypersurface must be a geodesic sphere.

Thus Theorem 1.3 is proved.

Remark 3.3. In Section 1, we discussed the special case $n = 2$ of (1-7), which is just the Minkowski-type inequality for convex surfaces in $\mathbb{S}^3$. Here, to get a better feeling of the inequality (1-7), we give the precise expressions for $n = 3$ and $n = 4$. For $n = 3$, we have

$$\omega_3 \le \sqrt{\Big(\frac{1}{4}\Big(|\Sigma| - 3\int_\Sigma p_2\,d\mu\Big)\Big)^2 + \Big(\frac{1}{4}\Big(3\int_\Sigma p_1\,d\mu - \int_\Sigma p_3\,d\mu\Big)\Big)^2} + \sqrt{\Big(\frac{3}{4}\Big(|\Sigma| + \int_\Sigma p_2\,d\mu\Big)\Big)^2 + \Big(\frac{3}{4}\Big(\int_\Sigma p_1\,d\mu + \int_\Sigma p_3\,d\mu\Big)\Big)^2}.$$

And for $n = 4$, we have

$$\begin{aligned}\omega_4 \le{}& \sqrt{\Big(\frac{1}{8}\Big(|\Sigma| - 6\int_\Sigma p_2\,d\mu + \int_\Sigma p_4\,d\mu\Big)\Big)^2 + \Big(\frac{1}{2}(\int_\Sigma p_1\,d\mu - \int_\Sigma p_3\,d\mu)\Big)^2} \\ &+ \sqrt{\Big(\frac{1}{2}\Big(|\Sigma| - \int_\Sigma p_4\,d\mu\Big)\Big)^2 + \Big(\int_\Sigma p_1\,d\mu + \int_\Sigma p_3\,d\mu\Big)^2} \\ &+ \frac{3}{8}\Big(|\Sigma| + 2\int_\Sigma p_2\,d\mu + \int_\Sigma p_4\,d\mu\Big).\end{aligned} \tag{3-13}$$

For a 4-dimensional hypersurface Σ in $\mathbb{S}^5$, we have the Gauss–Bonnet–Chern formula

$$\int_\Sigma (p_4 + 2p_2 + 1)\,d\mu = \frac{1}{4!}\int_\Sigma L_2\,d\mu = \omega_4. \tag{3-14}$$

Therefore, the inequality (3-13) can be further simplified by using the formula (3-14).

Remark 3.4. As in the hyperbolic case, when the hypersurface $\Sigma \subset \mathbb{S}^5$ is small, the inequality (3-13) reduces to the Euclidean version (3-3). This can be seen using a similar argument to that in Remark 3.2.

4. The results by the method of inverse mean curvature flow

In this section we give the proof of Theorem 1.4 using a different method from the one in the previous section.

4A. ***Evolution equations.*** Considering Σ as the initial hypersurface, we flow Σ in $\mathbb{S}^{n+1}$ under the flow equation $X : \Sigma \times [0, T^*) \to \mathbb{S}^{n+1}$,

$$\partial_t X = F\nu,$$

where F is a curvature function and ν is the unit normal to the flow hypersurfaces Σ_t. First we recall the following evolution equations.

Lemma 4.1 [Makowski and Scheuer 2013]. *Under the curvature flow $\partial_t X = F\nu$ in $\mathbb{S}^{n+1}$, we have*

$$\frac{d}{dt}|\Sigma_t| = n\int_{\Sigma_t} Fp_1\, d\mu_t, \tag{4-1}$$

$$\frac{d}{dt}\int_{\Sigma_t} p_m\, d\mu_t = (n-m)\int_{\Sigma_t} Fp_{m+1}\, d\mu_t - m\int_{\Sigma_t} Fp_{m-1}\, d\mu_t. \tag{4-2}$$

To simplify the notation, in the following we define

$$\widetilde{L}_k = \frac{1}{C_n^{2k}(2k)!}L_k = \sum_{i=0}^{k} C_k^i p_{2k-2i}. \tag{4-3}$$

Using Lemma 4.1, we obtain the following.

Lemma 4.2. *Under the curvature flow $\partial_t X = F\nu$ in $\mathbb{S}^{n+1}$, we have*

$$\frac{d}{dt}\int_{\Sigma_t} \widetilde{L}_k\, d\mu_t = (n-2k)\sum_{i=0}^{k} C_k^i \int_{\Sigma_t} F p_{2k-2i+1}\, d\mu_t.$$

Proof. The proof is by a direct calculation:

$$\begin{aligned}
\frac{d}{dt}\int_{\Sigma_t} \widetilde{L}_k\, d\mu_t &= \sum_{i=0}^{k} C_k^i \frac{d}{dt}\int_{\Sigma_t} p_{2k-2i}\, d\mu_t \\
&= \sum_{i=0}^{k} C_k^i \int_{\Sigma_t} \big((n-2k+2i)F p_{2k-2i+1} - 2(k-i)F p_{2k-2i-1}\big)\, d\mu_t \\
&= \sum_{i=0}^{k} C_k^i \int_{\Sigma_t} (n-2k+2i)F p_{2k-2i+1}\, d\mu_t \\
&\qquad\qquad - \sum_{i=1}^{k} C_k^{i-1} \int_{\Sigma_t} 2(k-i+1)F p_{2k-2i+1}\, d\mu_t \\
&= (n-2k)\sum_{i=0}^{k} C_k^i \int_{\Sigma_t} F p_{2k-2i+1}\, d\mu_t.
\end{aligned}$$

□

4B. ***Proof of Theorem 1.4.*** Recently, Makowski and Scheuer [2013] and Gerhardt [2015] studied the curvature flows in the sphere. If the initial hypersurface $\Sigma \subset \mathbb{S}^{n+1}$ is closed and strictly convex, then under the inverse mean curvature flow

$$\partial_t X = \frac{1}{p_1}\nu,$$

there exists a finite time $T^* < \infty$ such that the flow hypersurface Σ_t converges to an equator in $\mathbb{S}^{n+1}$ and the mean curvature of Σ_t converges to zero almost everywhere in the sense of (see Theorem 1.4 in [Makowski and Scheuer 2013])

$$\lim_{t\to T^*}\int_{\Sigma_t} p_1^{\alpha}\, d\mu_t = 0 \quad \text{for all } 1 \le \alpha < \infty. \tag{4-4}$$

For each $t \in [0, T^*)$, define the quantity $Q(t)$ by

$$Q(t) = |\Sigma_t|^{-(n-2k)/n}\int_{\Sigma_t} \widetilde{L}_k\, d\mu_t. \tag{4-5}$$

On the one hand, by Lemmas 4.2 and 2.1 (note that strictly convex implies all principal curvatures of Σ_t are positive, and certainly belong to Γ_k^+), we have

$$\begin{aligned}\frac{d}{dt}\int_{\Sigma_t}\widetilde{L}_k d\mu_t &= (n-2k)\sum_{i=0}^{k} C_k^i \int_{\Sigma_t}\frac{p_{2k-2i+1}}{p_1}\, d\mu_t\\ &\le (n-2k)\sum_{i=0}^{k} C_k^i\int_{\Sigma_t} p_{2k-2i}\, d\mu_t\\ &= (n-2k)\int_{\Sigma_t}\widetilde{L}_k\, d\mu_t.\end{aligned}$$

Equality holds if and only if Σ_t is totally umbilical. On the other hand, the area of the flow hypersurface evolves as

$$\frac{d}{dt}|\Sigma_t| = n|\Sigma_t|.$$

Therefore we obtain that the quantity $Q(t)$ is monotone nonincreasing in t; i.e.,

$$\frac{d}{dt}Q(t) \le 0. \tag{4-6}$$

Since under the inverse mean curvature flow, the flow hypersurfaces converge to an equator in $\mathbb{S}^{n+1}$ and the mean curvature of Σ_t converges to zero almost everywhere in the sense of (4-4), we have

$$\lim_{t\to T^*} Q(t) = \omega_n^{2k/n}. \tag{4-7}$$

Combining (4-6) and (4-7), we have

$$Q(0) = |\Sigma|^{-(n-2k)/n}\int_{\Sigma}\widetilde{L}_k\, d\mu \ge \lim_{t\to T^*} Q(t) = \omega_n^{2k/n}.$$

Hence noting (4-3), we obtain that

$$\int_{\Sigma} L_k\, d\mu \ge C_n^{2k}(2k)!\,\omega_n^{2k/n}|\Sigma|^{(n-2k)/n}. \tag{4-8}$$

Equality holds in (4-8) if and only if $Q(t)$ is constant in t. Then Σ_t is totally umbilical for each $t \in [0, T^*)$, and, in particular, Σ is totally umbilical and hence a geodesic sphere. The inequality (4-8) says that the induced metric of convex hypersurfaces in $\mathbb{S}^{n+1}$ satisfies the optimal Sobolev inequalities. See [Ge et al. 2014b] for further information about the Sobolev inequalities of the same type.

Acknowledgements

The authors would like to thank Professor Haizhong Li for constant encouragement and help. They also thank the referee for some helpful comments which made this paper more readable. The research of the authors was supported by NSFC No. 11271214.

References

[Alexandrov 1937] A. D. Alexandrov, "Zur Theorie der gemischten Volumina von konvexen Körpern, II: Neue Ungleichungen zwischen den gemischten Volumina und ihre Anwendungen", *Mat. Sb.* (*N.S.*) **2(44)**:6 (1937), 1205–1238. In Russian. Zbl 0018.27601

[Alexandrov 1938] A. D. Alexandrov, "Zur Theorie der gemischten Volumina von konvexen Körpern, III: Die Erweiterung zweeier Lehrsatze Minkowskis über die konvexen Polyeder auf beliebige konvexe Flachen", *Mat. Sb.* (*N.S.*) **3(45)**:1 (1938), 27–46. In Russian. Zbl 0018.42402

[Allendoerfer 1948] C. B. Allendoerfer, "Steiner's formulae on a general S^{n+1}", *Bull. Amer. Math. Soc.* **54** (1948), 128–135. MR 9,607j Zbl 0033.39503

[Blaschke 1938] W. Blaschke, *Über eine geometrische Frage von Euklid bis heute*, Hamburger Mathematiches Einzelschriften **23**, B. G. Teubner, Berlin, 1938. Zbl 0018.23404

[Borisenko and Miquel 1999] A. A. Borisenko and V. Miquel, "Total curvatures of convex hypersurfaces in hyperbolic space", *Illinois J. Math.* **43**:1 (1999), 61–78. MR 2000h:53100 Zbl 0981.53046

[Brendle et al. 2014] S. Brendle, P. K. Hung, and M. T. Wang, "Minkowski-type inequality for hypersurfaces in the Anti-de Sitter–Schwarzschild manifold", *Comm. Pure Appl. Math.* (online publication December 2014).

[do Carmo and Warner 1970] M. P. do Carmo and F. W. Warner, "Rigidity and convexity of hypersurfaces in spheres", *J. Differential Geometry* **4** (1970), 133–144. MR 42 #1014 Zbl 0201.23702

[Chang and Wang 2011] S.-Y. A. Chang and Y. Wang, "On Aleksandrov–Fenchel inequalities for k-convex domains", *Milan J. Math.* **79**:1 (2011), 13–38. MR 2012h:52005 Zbl 1229.52002

[Chern 1944] S.-s. Chern, "A simple intrinsic proof of the Gauss–Bonnet formula for closed Riemannian manifolds", *Ann. of Math.* (2) **45**:4 (1944), 747–752. MR 6,106a Zbl 0060.38103

[Fenchel 1936] W. Fenchel, "Inégalités quadratiques entre les volumes mixtes des corps convexes", *C. R. Acad. Sci., Paris* **203** (1936), 647–650. Zbl 0015.12006

[Gallego and Solanes 2005] E. Gallego and G. Solanes, "Integral geometry and geometric inequalities in hyperbolic space", *Differential Geom. Appl.* **22**:3 (2005), 315–325. MR 2006g:28011 Zbl 1077.52010

[Ge et al. 2013] Y. Ge, G. Wang, and J. Wu, "Hyperbolic Alexandrov–Fenchel quermassintegral inequalities, I", preprint, 2013. arXiv 1303.1714

[Ge et al. 2014a] Y. Ge, G. Wang, and J. Wu, "The GBC mass for asymptotically hyperbolic manifolds", *C. R. Math. Acad. Sci. Paris* **352**:2 (2014), 147–151. MR 3151884 Zbl 1285.53026

[Ge et al. 2014b] Y. Ge, G. Wang, and J. Wu, "Hyperbolic Alexandrov–Fenchel quermassintegral inequalities II", *J. Differential Geom.* **98**:2 (2014), 237–260. MR 3263518 Zbl 1301.53077

[Ge et al. 2014c] Y. Ge, G. Wang, and J. Wu, "A new mass for asymptotically flat manifolds", *Adv. Math.* **266** (2014), 84–119. MR 3262355 Zbl 1301.53032

[Ge et al. 2015] Y. Ge, G. Wang, J. Wu, and C. Xia, "A Penrose inequality for graphs over Kottler space", *Calc. Var. Partial Differential Equations* **52**:3-4 (2015), 755–782. MR 3311913 Zbl 1308.83093

[Gerhardt 2015] C. Gerhardt, "Curvature flows in the sphere", *J. Differential Geom.* **100**:2 (2015), 301–347. MR 3343834 Zbl 06451271

[Guan 2006] P. Guan, "Topics in geometric fully nonlinear equations", Lecture notes, 2006, Available at http://www.math.mcgill.ca/guan/notes.html.

[Guan and Li 2009] P. Guan and J. Li, "The quermassintegral inequalities for k-convex starshaped domains", *Adv. Math.* **221**:5 (2009), 1725–1732. MR 2010i:52021 Zbl 1170.53058

[Knothe 1952] H. Knothe, "Zur Theorie der konvexen Körper im Raum konstanter positiver Krümmung", *Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat.* (2) **2** (1952), 336–348. MR 15,819c Zbl 0049.12301

[Li et al. 2014] H. Li, Y. Wei, and C. Xiong, "A geometric inequality on hypersurface in hyperbolic space", *Adv. Math.* **253** (2014), 152–162. MR 3148549 Zbl 06285046

[de Lima and Girão 2015] L. L. de Lima and F. Girão, "An Alexandrov–Fenchel-type inequality in hyperbolic space with an application to a Penrose inequality", *Ann. Henri Poincaré* (online publication May 2015).

[Makowski and Scheuer 2013] M. Makowski and J. Scheuer, "Rigidity results, inverse curvature flows and Alexandrov–Fenchel type inequalities in the sphere", 2013. To appear in *Asian J. Math.* arXiv 1307.5764

[McCoy 2005] J. A. McCoy, "Mixed volume preserving curvature flows", *Calc. Var. Partial Differential Equations* **24**:2 (2005), 131–154. MR 2006g:53098 Zbl 1079.53099

[Minkowski 1903] H. Minkowski, "Volumen und Oberfläche", *Math. Ann.* **57**:4 (1903), 447–495. MR 1511220 JFM 34.0649.01

[Montiel and Ros 1991] S. Montiel and A. Ros, "Compact hypersurfaces: The Alexandrov theorem for higher order mean curvatures", pp. 279–296 in *Differential geometry*, edited by B. Lawson and K. Tenenblat, Pitman Monogr. Surveys Pure Appl. Math. **52**, Longman Sci. Tech., Harlow, 1991. MR 93h:53062 Zbl 0723.53032

[Natário 2015] J. Natário, "A Minkowski-type inequality for convex surfaces in the hyperbolic 3-space", *Differential Geom. Appl.* **41** (2015), 102–109. MR 3353742 Zbl 06448541

[Osserman 1978] R. Osserman, "The isoperimetric inequality", *Bull. Amer. Math. Soc.* **84**:6 (1978), 1182–1238. MR 58 #18161 Zbl 0411.52006

[Ros 2005] A. Ros, "The isoperimetric problem", pp. 175–209 in *Global theory of minimal surfaces* (Berkeley, CA, 2001), edited by D. Hoffman, Clay Math. Proc. **2**, Amer. Math. Soc., Providence, RI, 2005. MR 2006e:53023 Zbl 1125.49034

[Santaló 1963] L. A. Santaló, "A relation between the mean curvatures of parallel convex bodies in spaces of constant curvature", *Rev. Un. Mat. Argentina* **21** (1963), 131–137. In Spanish. MR 29 #6423 Zbl 0131.37801

[Santaló 1976] L. A. Santaló, *Integral geometry and geometric probability*, Encyclopedia of Mathematics and its Applications **1**, Addison-Wesley, Reading, MA, 1976. MR 55 #6340 Zbl 0342.53049

[Schmidt 1940] E. Schmidt, "Die isoperimetrischen Ungleichungen auf der gewöhnlichen Kugel und für Rotationskörper im n-dimensionalen sphärischen Raum", *Math. Z.* **46** (1940), 743–794. MR 2,262e Zbl 0023.38101

[Schneider 1993] R. Schneider, *Convex bodies: the Brunn–Minkowski theory*, Encyclopedia of Mathematics and its Applications **44**, Cambridge Univ. Press, 1993. MR 94d:52007 Zbl 0798.52001

[Solanes 2003] G. Solanes, *Integrals de curvatura i geometria integral a l'espai hiperbolic*, Ph.D. thesis, Univ. Aut. Barcelona, 2003, Available at http://mat.uab.cat/∼egallego/gil/tesi.pdf.

[Solanes 2006] G. Solanes, "Integral geometry and the Gauss–Bonnet theorem in constant curvature spaces", *Trans. Amer. Math. Soc.* **358**:3 (2006), 1105–1115. MR 2006h:53075 Zbl 1082.53075

[Wang and Xia 2014] G. Wang and C. Xia, "Isoperimetric type problems and Alexandrov–Fenchel type inequalities in the hyperbolic space", *Adv. Math.* **259** (2014), 532–556. MR 3197666 Zbl 1292.52008

Received October 6, 2013. Revised May 8, 2014.

YONG WEI
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
100084, BEIJING
CHINA

and

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE LONDON
LONDON WC1E 6BT
UNITED KINGDOM

wei-y09@mails.tsinghua.edu.cn

yong.wei@ucl.ac.uk

CHANGWEI XIONG
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
100084, BEIJING
CHINA

xiongcw10@mails.tsinghua.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.241

UPPER BOUNDS OF ROOT DISCRIMINANT LOWER BOUNDS

SIMAN WONG

For any rational number $t \in [0, 1]$, define the logarithmic Martinet function $\beta(t)$ to be the liminf of the logarithm of the root discriminant of number fields K with $r_1(K)/[K : \mathbb{Q}] = t$ as $[K : \mathbb{Q}]$ goes to infinity. Under the generalized Riemann hypothesis for Dedekind zeta functions of number fields, we show that $\beta(t) < 14.55$ for a dense subset of rational numbers $t \in [0, 1]$. We also study unconditional estimates of the growth of root discriminants by studying how the polynomial discriminant behaves under perturbation of coefficients, and by using Pisot numbers.

1. Introduction

Let K be a number field of degree n_K and absolute discriminant d_K. Denote by $r_1(K)$ and $r_2(K)$ the number of real and complex conjugate pairs of embeddings of K, and by $rd_K := |d_K|^{1/n_K}$ the *root discriminant* of K. By analyzing the explicit formula for the Dedekind zeta function $\zeta_K(s)$ of K, Stark [1974] shows that[1] as $n_K \to \infty$,

$$\log(rd_K) \geq \frac{r_1(K)}{n_K} \log(4\pi e^C) + \frac{2r_2(K)}{n_K} \log(2\pi e^C) + o(1), \tag{1}$$

where C is the Euler constant. Note that $rd_L = rd_K$ if L/K is a finite extension unramified at all finite places. This suggests that root discriminant lower bounds can be used to study ideal class groups and, more generally, numbers fields and Galois representations with restricted ramifications; see [Fontaine 1985; Masley 1978; Tate 1994] for a sample of the wide range of applications of root discriminant lower bounds.

In view of such applications, there are extensive works on sharpening root discriminant lower bounds. Let $I_{\mathbb{Q}} = \mathbb{Q} \cap [0, 1]$. Inspired by [Hajir and Maire 2001] and [Martinet 1978], to help us focus on the asymptotic nature of (1) we define the *logarithmic Martinet function* $\beta : I_{\mathbb{Q}} \to \mathbb{R}_{>0} \cup \{\infty\}$ as follows. For $t \in I_{\mathbb{Q}}$, let $R_{n,t}$

Siman Wong's work is supported in part by NSF grant DMS-0901506.
MSC2010: primary 11R29; secondary 11R37, 11R21.
Keywords: Chebotarev density theorem, class field towers, Pisot numbers, root discriminants.

[1]The asymptotic constants in this paper depend only on those quantities (if any) adorning the corresponding $\ll$ sign.

be the minimal root discriminant for number fields of degree n and with r_1 real embeddings such that $r_1/n = t$. Then

$$\beta(t) := \liminf_{n\to\infty} R_{n,t}.$$

Note that $\beta(t)$ is finite[2] for any $t \in I_{\mathbb{Q}}$: first, find a number field $K_t/\mathbb{Q}$ with $r_1(K)/n_K = t$ (see for example the proof of Theorem 1.2 below for an explicit construction). Next, let $L_1 \subset L_2 \subset \cdots$ be a totally real class field tower. Then the compositums $L_i K_t$ have bounded root discriminants and satisfy $r_1(L_i K_t)/n_{L_i K_t} = t$. We also know that $\beta(t) > 0$ for all $t \in I_{\mathbb{Q}}$; this follows from

$$\beta(t) \geq t \log(4\pi e^C) + (1-t)\log(2\pi e^C) = t \log 2 + \log(2\pi e^C),$$

which is a restatement of (1). By using a smooth form of the explicit formula and with a careful choice of kernel, this lower bound has since been improved to

$$\beta(t) \geq t \log(4\pi e^{1+C}) + (1-t)\log(4\pi e^C) = t + \log(4\pi e^C),$$

and the two constants are optimal within the framework of the explicit formula and without additional inputs about the zeros of $\zeta_K(s)$ and prime ideals of the number fields. Assuming the generalized Riemann hypothesis (GRH) for $\zeta_K(s)$, the optimal conditional lower bound from the explicit formula approach is

$$\beta(t) \geq t \log(8\pi e^{C+\pi/2}) + (1-t)\log(8\pi e^C) = \tfrac{\pi}{2} t + \log(8\pi e^C). \tag{2}$$

See [Odlyzko 1990] for a survey of the literature. Aside from this finiteness result and the aforementioned lower bounds, little is known about this function β. For example, it is not known if β is bounded on $I_{\mathbb{Q}}$ (the finiteness result for $\beta(t)$ sketched earlier depends on K_t). Hajir and Maire [2001] raise a number of interesting (and, as these authors put it, probably very difficult) questions:

- Does β extend to a continuous function on [0, 1] (which would imply that β is bounded on $I_{\mathbb{Q}}$)?
- Is β monotonically increasing?
- Is there a root discriminant lower bound of the form
$$\log(rd_K) \geq \frac{r_1(K)}{n_K}\beta(1) + \frac{2r_2(K)}{n_K}\beta(0) + o(1)?$$
- Very optimistically, is it true that $\beta(t)$ is a linear function in t and, even more boldly, do we have $\beta(t) = t\beta(1) + (1-t)\beta(0)$?

By constructing explicit Hilbert class field towers, Martinet [1978] shows that $\beta(0) < 4.53$ and $\beta(1) < 6.97$, and Hajir and Maire [2002] refine this method to

[2] We thank Professor Hajir for showing us this argument.

give $\beta(0) < 4.41$ and $\beta(1) < 6.87$; Martin [2006] has made further improvement on $\beta(t)$ for $t \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{5}{7}, 1\}$. As a comparison, note that, by (2), under GRH we have $\beta(0) \geq 3.80$ and $\beta(1) \geq 5.37$. In this paper we give a conditional proof that $\beta(t)$ is bounded by an explicit universal constant for a dense subset of $t \in I_{\mathbb{Q}}$.

Theorem 1.1. *Assume the generalized Riemann hypothesis for the Dedekind zeta functions of number fields. Fix a fraction $a/(3^b m) \in I_{\mathbb{Q}}$ with $a, b, m > 0$ and $3 \nmid m$ (we allow $3 \mid a$). Then there exist an infinite sequence of Galois extensions $K_1 \subsetneq K_2 \subsetneq \cdots$ such that $r_1(K_i)/n_{K_i} = a/(3^b m)$ for all i, and such that $\log(rd_{K_i})$ is at most*

$$19.59316 + \tfrac{m-1}{m}(2\log m + 2\log\log m + 6.813445) + O\Big(\tfrac{\log n_{K_i} + \log m}{m \cdot n_{K_i}}\Big).$$

Corollary. Assume the generalized Riemann hypothesis for the Dedekind zeta functions of number fields. Then for any fraction $a/(3^b m) \in I_{\mathbb{Q}}$ with $a, b, m > 0$ and $3 \nmid m$ (we allow $3 \mid a$), we have

$$\beta\big(\tfrac{a}{3^b m}\big) \leq 19.59316 + \tfrac{m-1}{m}(2\log m + 2\log\log m + 6.813445). \qquad \square$$

A natural way to construct number fields with a prescribed ratio $r_1(K)/n_K$ is to take the square root of a totally real algebraic integer with the appropriate number of positive embeddings. To bound the root discriminant of the field generated by such a square root, we need to keep the absolute norm of this element small. We achieve that by applying the GRH form of the effective Chebotarev density theorem to the narrow class field of an explicit infinite 3-class field tower of a real quadratic field. This produces infinitely many fields for which $r_1(K)/n_K$ take on a fixed rational value with 3-power denominator; to handle ratios with general denominators m we compose the extensions constructed above with a totally real Galois extension of degree m. Because of this last step[3] we are not able to show that $\beta(t)$ is uniformly bounded on $I_{\mathbb{Q}}$ (which would have to be the case *if* β does extend to a continuous function on [0, 1]). Since fractions with 3-power denominators are dense in $I_{\mathbb{Q}}$, Theorem 1.1 does show that $\beta(t)$ is informally bounded on a dense subset of $I_{\mathbb{Q}}$.

Remark. Our proof of Theorem 1.1 readily generalizes to function fields (for which the GRH is true unconditionally).

We do not know how to prove unconditionally that $\beta(t)$ is bounded by a universal constant for all $t \in I_{\mathbb{Q}}$. If we replace in the proof of Theorem 1.1 the conditional

[3]We thank Professor Hajir for suggesting this compositum construction. We can also directly construct totally real infinite m-class field tower using the Golod–Shafarevich construction [Roquette 1967]. This results in an upper bound $\beta(a/m) \leq c_1 \log m + c_2$ for some absolute constants c_i, just like Theorem 1.1, but these constants would be weaker than those in Theorem 1.1.

effective Chebotarev density theorem with the unconditional one, our argument only gives

$$\log(rd_{K_i}) \ll (c^{n_{K_i}})/n_{K_i} \tag{3}$$

for some absolute constant $c > 0$. We have the following unconditional improvement.

Theorem 1.2. *There exists an absolute constant $c > 0$ such that for any $t \in I_{\mathbb{Q}}$, there exist infinitely many number fields K_i (depending on t) of unbounded degree such that $r_1(K_i)/n_{K_i} = t$ and $\log(rd_{K_i}) \le cn_{K_i} \log(n_{K_i})$.*

To prove this unconditional result, we start with a polynomial $f(x)$ that splits completely over $\mathbb{Z}$. We can easily estimate the discriminant of f, and by prescribing the signs of the roots of f appropriately we can guarantee that the ratio of the number of real roots of $f(x^2)$ to the degree of $f(x^2)$ takes on any given value in $I_{\mathbb{Q}}$. To achieve irreducibility we perturb the constant term and study its effect on the discriminant and signature.

Remark. The proof of Theorems 1.1 and 1.2 come down to finding in a totally real number field algebraic integers of small absolute norm and with a prescribed number of positive embeddings. If we try to tackle this problem using Minkowski's convex body theorem, the obvious construction leads to an estimate comparable to the unconditional Chebotarev estimate (3). It would be interesting to find a geometry of numbers proof of the two theorems here.

Remark. The constants in Theorem 1.1 can be improved, but not anywhere near the records of Martinet and Hajir–Maire; to streamline the exposition we forgo such refinements. In a similar vein we leave out explicit value for the constant in Theorem 1.2.

In connection with their study on arithmetic lattices in simple Lie groups of bounded covolume, Belolipetsky and Lubotzky [2012] use Pisot numbers to construct an infinite sequence of number fields of unbounded degree with a fixed number of complex places and bounded root discriminant. On the other hand, computational data suggest that number fields with a large number of complex places tend to have large class numbers, and hence (at least heuristically) large root discriminant. The following result is the first step towards affirming this circle of ideas (and the only result we know of in this direction).

Theorem 1.3. *There exists an infinite sequence of number fields T_ℓ with $n_{T_\ell} = \ell + 1$ and $r_1(T_\ell) \in \{1, 2\}$, such that $\log(rd_{T_\ell}) \le \log(\ell + 1) + \log 3/(\ell + 1)$.*

2. Conditional estimate

For any number field $L \neq \mathbb{Q}$, denote by h_L, R_L, w_L and $\mathcal{O}_L$ its class number, regulator, number of roots of unity in K, and the ring of integers of K.

Lemma 2.1. *For any number field L with $n_L \geq 36$, we have the estimate*

$$h_L \leq 4|d_L|^{\frac{1}{2}\left(1.710172+\frac{1.292958}{\log(|d_L|^{1/2})}\right)}.$$

Proof. We prove this by finding explicit numerical values for the constants in the argument in [Lang 1986, p. 322], which is a preliminary step in the proof of the Brauer–Siegel theorem. Before we proceed with the elementary but somewhat tedious computation, we will briefly explain the idea behind the proof of the lemma.

The Brauer–Siegel theorem gives an asymptotic estimate for

$$\frac{\log(h_L R_L)}{\log(|d_L|^{1/2})}$$

as we run through an infinite sequence of number fields L with $n_L/\log|d_L| \to 0$. More precisely, the crucial exponent $\frac{1}{2}$ shows up in the main term of the asymptotic estimate, and $n_L/\log|d_L|$ appears in the error term. But if we are willing to weaken the main term of Brauer–Siegel, we can actually make this $n_L/\log|d_L|$ term go away (there are additional error terms).

We now resume the proof of the lemma. The residue at $s=1$ of $\zeta_L(s)$ is equal to

$$\kappa(L) = 2^{r_1(L)}(2\pi)^{r_2(L)} h_L R_L/(w_L|d_L|^{1/2}).$$

Take the logarithm of both sides, recall that $|d_L| > 1$ if $L \neq \mathbb{Q}$ and we get

$$\frac{\log(h_L R_L)}{\log(|d_L|^{1/2})} = \frac{\log(\kappa(L)) - r_1(L)\log 2 - r_2(L)\log(2\pi) + \log(w_L)}{\log(|d_L|^{1/2})} + 1. \tag{4}$$

Next, combining the functional equation of $\zeta_L(s)$ with the positivity of the integral representation of $\zeta_L(s)$ for real $s > 1$, we find that (see [Lang 1986, Lemma XVI.1])

$$(2^{-2r_2(L)}\pi^{-n_L}\cdot|d_L|)^{s/2}\Gamma\left(\tfrac{s}{2}\right)^{r_1(L)}\Gamma(s)^{r_2(L)}\cdot\zeta_L(s)\cdot s(s-1) \geq \kappa(L)|d_L|^{1/2}(2\pi)^{-r_2(L)},$$

so

$$\begin{aligned}\kappa(L) &\leq 2^{-r_2(L)s}\pi^{-n_L s/2}(2\pi)^{r_2(L)}|d_L|^{(s-1)/2}\Gamma\left(\tfrac{s}{2}\right)^{r_1(L)}\Gamma(s)^{r_2(L)}\zeta_L(s)\cdot s(s-1)\\ &\leq 2^{r_2(L)(1-s)}\pi^{r_2(L)-n_L s/2}|d_L|^{(s-1)/2}\Gamma\left(\tfrac{s}{2}\right)^{r_1(L)}\Gamma(s)^{r_2(L)}\zeta_{\mathbb{Q}}(s)^{n_L}\cdot s(s-1).\end{aligned}$$

Set $s = 1 + 1/\alpha$ with $\alpha > 0$. Then

$$\zeta_L\left(1+\tfrac{1}{\alpha}\right) \leq \zeta_{\mathbb{Q}}\left(1+\tfrac{1}{\alpha}\right)^{n_L} = \left(1+\sum_{m=2}^{\infty}\frac{1}{m^{1+\frac{1}{\alpha}}}\right)^{n_L} \leq \left(1+\int_1^{\infty}\frac{dt}{t^{1+\frac{1}{\alpha}}}\right)^{n_L} = (1+\alpha)^{n_L}.$$

Thus

$$\begin{aligned}
&\log(\kappa(L))\\
&\quad\le -\frac{r_2(L)}{\alpha}\log 2+\big(r_2(L)-\tfrac{1}{2}n_L\big(1+\tfrac{1}{\alpha}\big)\big)\log\pi+r_1(L)\log\Gamma\big(\tfrac{1}{2}+\tfrac{1}{2\alpha}\big)\\
&\qquad\quad +r_2(L)\log\Gamma\big(1+\tfrac{1}{\alpha}\big)+\tfrac{1}{\alpha}\log|d_L^{1/2}|+n_L\log(1+\alpha)+\log\big(1+\tfrac{1}{\alpha}\big)-\log\alpha\\
&\quad=r_2(L)\big(\log\Gamma\big(1+\tfrac{1}{\alpha}\big)+\log\pi-\tfrac{\log 2}{\alpha}\big)+n_L\big(\log(1+\alpha)-\tfrac{\log\pi}{2}\big(1+\tfrac{1}{\alpha}\big)\big)\\
&\qquad\quad +r_1(L)\log\Gamma\big(\tfrac{1}{2}+\tfrac{1}{2\alpha}\big)+\tfrac{1}{\alpha}\log|d_L^{1/2}|+\log\big(1+\tfrac{1}{\alpha}\big)-\log\alpha.
\end{aligned}$$

Substitute this into the right side of (4) and we get that

$$\begin{aligned}
\frac{\log(h_LR_L)}{\log(|d_L|^{1/2})}\le 1+\tfrac{1}{\alpha}+\frac{1}{\log(|d_L|^{1/2})}&\Big(r_2(L)\big(\log\Gamma\big(1+\tfrac{1}{\alpha}\big)-\big(1+\tfrac{1}{\alpha}\big)\log 2\big)\\
&+n_L\big(\log(1+\alpha)-\tfrac{\log\pi}{2}\big(1+\tfrac{1}{\alpha}\big)\big)+r_1(L)\big(\log\Gamma\big(\tfrac{1}{2}+\tfrac{1}{2\alpha}\big)-\log 2\big)\\
&+\log\big(1+\tfrac{1}{\alpha}\big)-\log\alpha+\log w_L\Big).
\end{aligned}$$

We check that if $\alpha>\alpha_0:=0.23048745595$ then the coefficients of the $r_1(L)$ term and the $r_2(L)$ term above are both negative. Thus for $\alpha>\alpha_0$,

$$\frac{\log(h_LR_L)}{\log(|d_L|^{1/2})}\le 1+\tfrac{1}{\alpha}+\frac{n_L\big(\log(1+\alpha)-\tfrac{\log\pi}{2}\big(1+\tfrac{1}{\alpha}\big)\big)+\log\big(1+\tfrac{1}{\alpha}\big)-\log\alpha+\log w_L}{\log(|d_L|^{1/2})}.$$

The roots of unity in K form a cyclic group, so w_L is the largest positive integer w for which K contains a primitive w-root of unity. Thus n_L is divisible by

$$w_L\prod_{p\mid w_L}\frac{p-1}{p}\ge\frac{w_L}{2}\prod_{\substack{p\mid w_L\\ p>2}}\frac{2}{3}\ge\frac{w_L}{2}\Big(\frac{2}{3}\Big)^{\frac{\log w_L}{\log 3}}=\frac{1}{2}w_L^{\frac{\log 2}{\log 3}}.$$

Thus $w_L\le(2n_L)^{\log 3/\log 2}\le 3{n_L}^{1.6}$, whence $\log w_L\le 1.6\log n_L+\log 3$. We check that $0.1x>\log x$ for $x\ge 36$, so for $n_L\ge 36$ and $\alpha>\alpha_0$,

$$\frac{\log(h_LR_L)}{\log(|d_L|^{1/2})}\le 1+\tfrac{1}{\alpha}+\frac{n_L\big(\log(1+\alpha)+0.1-\tfrac{\log\pi}{2}\big(1+\tfrac{1}{\alpha}\big)\big)+\log\big(1+\tfrac{1}{\alpha}\big)-\log\alpha+\log 3}{\log(|d_L|^{1/2})}.$$

We check that $\log(1+\alpha)+0.1-(\log\pi/2)\left(1+\frac{1}{\alpha}\right)$ vanishes at $\alpha_1 := 1.408110244096$. Set $\alpha=\alpha_1$ and we get

(5)
$$\frac{\log(h_L R_L)}{\log(|d_L|^{1/2})} \le 1+\tfrac{1}{\alpha_1}+\frac{\log\left(1+\frac{1}{\alpha_1}\right)-\log\alpha_1+\log 3}{\log(|d_L|^{1/2})} = 1.710172+\frac{1.292958}{\log(|d_L|^{1/2})}.$$

Friedman [1989, Theorem B] shows that $R_L > \frac{1}{4}$ for all $L\neq\mathbb{Q}$ except for the following three totally complex sextic fields:

L	d_L	R_L	h_L	w_L
$x^6-x^5+2x^4-2x^3+2x^2-2x+1$	-10051	0.20521	1	2
$x^6-x^5-x^4+2x^3-x+1$	-10571	0.21320	1	2
$x^6-3x^5+5x^4-5x^3+5x^2-3x+1$	-12671	0.23722	1	2

Set $R_L > \frac{1}{4}$ and we get, except possibly for these three fields,

$$\log(\tfrac{1}{4}h_L) < \log(|d_L|^{1/2})\left(1.710172+\frac{1.292958}{\log(|d_L|^{1/2})}\right). \tag{6}$$

Exponentiate both sides and we get

$$h_L \le 4|d_L|^{\frac{1}{2}\left(1.710172+\frac{1.292958}{\log(|d_L|^{1/2})}\right)},$$

which is the estimate in the lemma. And since $h_L=1$ for these three fields, this estimate is applicable as well. □

Lemma 2.2. *Assume the generalized Riemann hypothesis for the Dedekind zeta functions of number fields. Then for any totally real number field L of degree $m\ge 18$ and for any integer $0\le m'\le m$, there exists a quadratic extension $L_{m'}/L$ with signature $(r_1,r_2)=(2m-2m',m')$ and*

$$\log(rd_{L_{m'}}) \le 1.855086\log(rd_L)+3.372400+\frac{\log\log|d_L|+\log 280}{n_L}.$$

Proof. Denote by $C_{L,n}$ the narrow ray class group of L (of modulus $\mathcal{O}_L$), and by $H_{L,n}$ the corresponding narrow ray class field of L. Denote by $\mathcal{O}_L^\times$ the group of units of $\mathcal{O}_L$ and by $\mathcal{O}_{L,+}^\times$ the subgroup of totally positive units. Then

$$\begin{aligned}\#C_{L,n} &= h_L\cdot 2^{[L:\mathbb{Q}]}/[\mathcal{O}_L^\times:\mathcal{O}_{L,+}^\times] \qquad \text{by [Lang 1986, Theorem VI.2]}\\ &\le h_L\cdot 2^{[L:\mathbb{Q}]}.\end{aligned}$$

Since $H_{L,n}/L$ is unramified at all finite places,

$$|d_{H_{L,n}}| = |d_L|^{[H_{L,n}:L]} \le |d_L|^{h_L\cdot 2^{[L:\mathbb{Q}]}}. \tag{7}$$

Denote by $\phi_1, \dots, \phi_m$ the distinct real embeddings of L. Apply the GRH form of the effective Chebotarev density theorem ([Lagarias and Odlyzko 1977, Corollary 1.2]; see [Oesterlé 1979, Theorem 4] for a version with explicit constants) to the Galois extension $H_{L,n}/L$ and we see that for any integer $0 \le m' \le m$, there exists a prime ideal $\mathfrak{p}_{m'} \subset \mathcal{O}_L$ such that

(i) $\mathrm{Norm}_{L/\mathbb{Q}}(\mathfrak{p}_{m'}) \le 70(\log|d_{H_{L,n}}|)^2$, and

(ii) $\mathfrak{p}_{m'}$ is principal and is generated by an element $\pi_{m'} \in \mathcal{O}_L$ with $\phi_i(\pi_{m'}) > 0$ if and only if $i \le m'$.

The sign conditions mean that $L_{m'} := L(\sqrt{\pi_{m'}})$ has exactly $2m'$ real embeddings. Since $\pi_{m'}$ is a uniformizer, $L_{m'}/L$ is a quadratic extension unramified outside $\mathfrak{p}_{m'}$ and 2. Let $\mathfrak{Q} \subset \mathcal{O}_{L_{m'}}$ be a prime lying above 2 that *ramifies* in $L_{m'}/L$. By [Serre 1979, Remark 1 on p. 58], the exponent of $\mathfrak{Q}$ in the different ideal of $L_{m'}/L$ is at most $1 + \mathrm{ord}_{\mathfrak{Q}}(2)$. Consequently, $\mathrm{Disc}(L_{m'}/L)$ divides $\mathfrak{p}_{m'} \prod_{\mathfrak{q}|2} \mathfrak{q}^{1+\mathrm{ord}_{\mathfrak{q}}(2)} = \mathfrak{p}_{m'} \prod_{\mathfrak{q}|2} \mathfrak{q} \cdot 2\mathcal{O}_L$, so in particular

$$\mathrm{Disc}(L_{m'}/L) \quad \text{divides} \quad \mathfrak{p}_{m'} \cdot 2^2\mathcal{O}_L. \tag{8}$$

Thus

$$\begin{aligned} |d_{L_{m'}}| &= \mathrm{Norm}_{L/\mathbb{Q}}(\mathrm{Disc}(L_{m'}/L)) \cdot |d_L|^{[L_{m'}:L]} \\ &\le \mathrm{Norm}_{L/\mathbb{Q}}(\mathfrak{p}_{m'} \cdot 2^2\mathcal{O}_L) \cdot d_L^2 && \text{by (8)} \\ &\le \left[70 \cdot h_L \cdot 2^{n_L} \log|d_L| \cdot 2^{2n_L}\right]^2 \cdot d_L^2 && \text{by (7).} \end{aligned}$$

Since $n_{L_{m'}} = 2n_L$, the logarithm of the root discriminant of $L_{m'}$ is bounded by

$$\log(rd_{L_{m'}}) \le \frac{\log 70}{n_L} + \frac{\log h_L}{n_L} + \log 2 + \frac{\log\log|d_L|}{n_L} + \log 4 + \log(rd_L).$$

Since $n_{L_{m'}} \ge 36$, apply Lemma 2.1 and we get

$$\begin{aligned} \log(rd_{L_{m'}}) \le \frac{\log|d_L|}{2n_L}\left(1.710172 + \frac{1.292958}{\log(|d_L|^{1/2})}\right) + \frac{\log 4}{n_L} \\ + \frac{\log 70}{n_L} + \log 2 + \frac{\log\log|d_L|}{n_L} + \log 4 + \log(rd_L) \\ \le 1.855086\log(rd_L) + 3.372400 + \frac{\log\log|d_L| + \log 280}{n_L}. \end{aligned}$$

□

Remark. The proof of the lemma (and its subsequent application) does not require that the element $\pi_{m'}$ be a generator of a prime ideal; it is enough that it is not a square, has small norm, and has the prescribed number of positive embeddings. Thus the use of the conditional effective Chebotarev density theorem is an overkill; instead we could apply the GRH form of the Perron formula to the Hecke L-series of the narrow class group $C_{L,n}$ and sieve out the desired positivity conditions using orthogonality relations. But this alternative argument still requires the GRH and would lengthen the proof, so we opt for a streamlined approach via the conditional effective Chebotarev density theorem.

Proof of Theorem 1.1. Schmithals [1980] shows that the elementary 3-class group of the real quadratic field $k = \mathbb{Q}(\sqrt{3321607})$ has rank 3. Combining this with refinement of earlier work of Koch and Venkov [1975] and Schoof [1986] shows that k has an infinite 3-class field tower. Set $K_0 := k$ and denote by K_{i+1} the 3-Hilbert class field of K_i, all viewed as subfields of a fixed algebraic closure of $\mathbb{Q}$. Since K_0 is totally real and every $[K_{i+1} : K_i]$ is odd, that means every K_i is totally real.

Since K_i/k is unramified for all $i \geq 1$, we have

(9)
$$rd_{K_i} = rd_k = \sqrt{39345017}, \qquad \frac{\log\log|d_{K_i}|}{n_{K_i}} = \frac{\log(n_{K_i}/2)}{n_{K_i}} + \frac{\log\log\sqrt{39345017}}{n_{K_i}}.$$

Fix $i \geq 18$; then for any integer $0 \leq m' \leq n_{K_i}$, Lemma 2.2 furnishes an extension $K_{i,m'}/\mathbb{Q}$ of degree $2n_{K_i}$ with signature $(2n_{K_i} - 2m', m')$ and

$$\begin{aligned} \log(rd_{K_{i,m'}}) &\leq 1.855086\log(rd_k) + 3.372400 + O\Big(\frac{\log n_{K_i}}{n_{K_i}}\Big) \\ &= 19.593159 + O\Big(\frac{\log n_{K_i}}{n_{K_i}}\Big). \end{aligned} \tag{10}$$

We now consider the $m = 1$ case of the theorem, so fix $t = a/3^b \in I_{\mathbb{Q}}$ with $b > 0$ and $0 \leq a \leq 3^b$ (we allow $3 \mid a$). Since the K_i are 3-class field towers of k, for i sufficiently large we have $3^b \mid n_{K_i}$, so for such i we can choose $0 \leq m' \leq [K_i : \mathbb{Q}]$ so that $2m'/n_{K_{i,m'}} = m'/n_{K_i} = t$. Apply (10) and we are done.

Now, let $m > 1$ be coprime to 3. Then $\phi(6m) = 2\phi(2m) < 2m$, so by [Washington 1982, Proposition 2.7],

$$|d_{\mathbb{Q}(\zeta_{6m})}| \leq \frac{(6m)^{\phi(6m)}}{2^{\phi(6m)}3^{\phi(6m)/2}} = m^{\phi(6m)}3^{\phi(6m)/2} < m^{2m}3^m = (\sqrt{3}m)^{2m}.$$

The GRH form of the effective Chebotarev density theorem then furnishes a prime $p \equiv 1 \pmod{6m}$ with

$$\begin{aligned} p &\leq 70(\log|d_{\mathbb{Q}(\zeta_{6m})}|)^2 \\ &< 70\big(\log(\sqrt{3}m)^{2m}\big)^2 \\ &\leq 70\cdot 4m^2(\log m + \log\sqrt{3})^2 \end{aligned}$$

which is to say (since $m \geq 2$)

$$p < 70\cdot 13m^2\log^2 m. \tag{11}$$

Denote by M_m the unique degree m subfield of the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$. The conductor-discriminant formula gives $|d_{M_m}| \leq p^{m-1}$, so by (11),

$$\log|d_{M_m}| \leq (m-1)(2\log m + 2\log\log m + \log(70\cdot 13)). \tag{12}$$

The only finite prime that ramifies in $K_i/\mathbb{Q}$ (resp. $M_m/\mathbb{Q}$) is $39345017 \equiv 2 \pmod 3$ (resp. $p \equiv 1 \pmod 3$), so K_i and M_m are linearly disjoint over $\mathbb{Q}$. It follows that

$$[K_i M_m : \mathbb{Q}] = m\cdot n_{K_i} \quad \text{and} \quad |d_{K_i M_m}| = |d_{K_i}|^m |d_{M_m}|^{n_{K_i}}.$$

Thus

$$\log|d_{K_i M_m}| = m\log|d_{K_i}| + n_{K_i}\log|d_{M_m}|, \tag{13}$$

whence, by (9) and (12),

$$\begin{aligned}\log(rd_{K_i M_m}) &= \log(rd_{K_i}) + \log(rd_{M_m})\\ &\leq 8.743940 + \tfrac{m-1}{m}(2\log m + 2\log\log m + 6.813445).\end{aligned}$$

Both terms on the right side of (13) are greater than 1. Since $x+y \leq xy$ if both $x, y \geq 1$, it follows from (13) that

$$\begin{aligned}\frac{\log\log|d_{K_i M_m}|}{n_{K_i M_m}} &= \frac{\log m + \log\log|d_{K_i}| + \log n_{K_i} + \log\log|d_{M_m}|}{m\cdot n_{K_i}}\\ &\leq 2\frac{\log n_{K_i}}{m\cdot n_{K_i}} + O\left(\frac{\log m}{m\cdot n_{K_i}}\right), \qquad \text{by (9), (12).}\end{aligned}$$

Since $[\mathbb{Q}(\zeta_p) : M_m]$ is even, M_m is fixed by the unique order-2 element of the cyclic group $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. That means M_m, and hence $K_i M_m$, is totally real. Apply Lemma 2.2 and we see that for any $0 \leq m \leq m\cdot n_{K_i}$ there exists an extension $K_{i,m'}$ with signature $(2m\cdot n_{K_i} - 2m', m')$ and

$$\begin{aligned}\log(rd_{K_{i,m'}}) \leq 1.855096\big(8.743940 + \tfrac{m-1}{m}(2\log m + 2\log\log m + 6.813445)\big)&\\ +3.372400 + O\left(\frac{\log n_{K_i} + \log m}{m\cdot n_{K_i}}\right),&\end{aligned}$$

and Theorem 1.1 follows for general $m > 1$. □

3. Unconditional estimate

Fix an integer $n \geq 1$. For each $0 \leq j \leq n$, pick $\sigma_j \in \{\pm 1\}$ and define

$$f_n(x) := \prod_{i=1}^{n}(x - (2i)\sigma_i), \quad g_n(x) := f_n(x) + 2.$$

Lemma 3.1. *For $n \geq 6$, the roots γ_i of $g_n(x)$ are all real and pairwise distinct, and up to relabeling we have $|\gamma_j - (2j)\sigma_j| < 1$ for all i. In particular, $g_n(x)$ has as many positive roots as $f_n(x)$.*

Proof. For any $1 \leq j \leq n$ we can write

$$f_n(x) = (x - (2j)\sigma_j) \prod_{i \neq j} (x - (2i)\sigma_i). \tag{14}$$

Since $|(2i)\sigma_i - (2j)\sigma_j| \geq 2|i-j|$ for all $i \neq j$, if $|x - (2j)\sigma_j| \leq 1$ then the product on the right side of (14) does not change sign and has absolute value at least $\prod_{i \neq j}(2|i-j|-1)$. This latter product is taken over $n-1$ odd integers between 1 and $2n-3$, with each odd integer appearing at most twice. So if $|x - (2j)\sigma_j| \leq 1$ and $n \geq 3$, then

$$\Big|\prod_{j \neq i} (x - (2i)\sigma_i)\Big| \geq \prod_{\ell=1}^{[\frac{n-1}{2}]} (2\ell - 1)^2 \geq \Big(2\Big[\frac{n-1}{2}\Big] - 1\Big)^2 \geq \Big(\frac{n-3}{2}\Big)^2.$$

To recapitulate, for $|x - (2j)\sigma_j| \leq 1$ and $n \geq 3$, the polynomial $f_n(x)$ is equal to $x - (2j)\sigma_j$ times a product that, within this closed interval, takes on a constant sign and has absolute value at least $((n-3)/2)^2$. Note that $x - (2j)\sigma_j$ takes values ∓ 1 at $(2j)\sigma_j \pm 1$. So for $n \geq 6$, one of $f_n((2j)\sigma_j \pm 1)$ is $\leq -\frac{9}{4}$ and the other is $\geq \frac{9}{4}$. Thus $g_n(x) := f_n(x) + 2$ takes a negative value at exactly one of the two end points of the *closed* interval

$$[(2j)\sigma_j - 1, (2j)\sigma_j + 1]$$

and it takes positive value in the middle. By continuity, $g_n(x)$ must have a root in one of the *open* intervals

$$((2j)\sigma_j - 1, (2j)\sigma_j) \quad \text{or} \quad ((2j)\sigma_j, (2j)\sigma_j + 1). \tag{15}$$

As we run through all $1 \leq j \leq n$, these $2n$ *open* intervals are pairwise disjoint, and the two open intervals in (15) are both contained in the positive x-axis if and only if $\sigma_j > 0$. That means if $n \geq 6$, then the degree-n polynomial $g_n(x)$ has exactly n distinct real roots, and its unique root in the union of the intervals in (15) has the same sign as σ_j. This completes the proof of the lemma. □

Lemma 3.2. *As $n \to \infty$ we have the estimate* $\log|\mathrm{disc}(g_n(x^2))| \ll n^2 \log n$.

Proof. For any polynomial $G(x)$, from the definition of polynomial discriminant we see that

$$|\mathrm{disc}(G(x^2))| = |\mathrm{disc}(G(x))|^2 \cdot 2^{\deg G} \cdot |\text{constant term of } G(x)|.$$

Consequently,

$$\log|\mathrm{disc}(g_n(x^2))| \le 2\log|\mathrm{disc}(g_n(x))| + 2n\log 2 + \sum_{i=1}^{n}\log(2i)$$
$$\ll \log|\mathrm{disc}(g_n(x))| + n\log n.$$

By Lemma 3.1, if $n \ge 6$ then the roots of $g_n(x)$ are pairwise distinct and each one is of distance less than 1 from exactly one of the $(2j)\sigma_j$. Thus

$$\log|\mathrm{disc}(g_n(x))| \le \sum_{1\le i\ne j\le n} 2\log|2i+2j+2| \ll n^2\log(2n+2) \ll n^2\log n.$$

Combine this with (16) and the lemma follows. □

Proof of Theorem 1.2. Given $0 \le n' \le n$, choose $\sigma_j \in \{\pm 1\}$ $(0 \le j \le n)$ so that exactly n' of them are positive. With respect to these σ_j, the corresponding polynomial $g_n(x^2)$ is Eisenstein at 2, and so it is irreducible over $\mathbb{Q}$. By construction it has exactly $2n'$ real embedding. Denote by $N_n/\mathbb{Q}$ the degree $2n$ extension defined by a root of $g_n(x^2)$. It is totally real if $n \ge 6$, by Lemma 3.1. By Lemma 3.2, we have $\log(rd_{N_n}) \ll n_{N_n}\log(n_{N_n})$, and the theorem follows. □

4. Small root discriminants via Pisot numbers

A real algebraic integer θ is called a Pisot number if every conjugate of θ other than θ itself has absolute value less than 1 (these other conjugates need not be real). A celebrated theorem of Salem [1944] says that the set of Pisot numbers is a closed subset of the real line.

Lemma 4.1. *Any integer $a \ge 2$ is a nonisolated limit point of the set of Pisot numbers.*

Proof. This is a standard fact about Pisot numbers; we give the details following the hint in [Salem 1963, p. 21] since we need the explicit polynomials later on. Consider the polynomial

$$f_{n,a}(x) = x^n(x-a) - 1.$$

Clearly $f_{n,a}(0) \ne 0$, and

$$f_{n,a}\left(\frac{an}{n+1}\right) = \left(\frac{an}{n+1}\right)^n\left(\frac{an}{n+1} - a\right) - 1 = \left(\frac{n}{n+1}\right)^n\left(\frac{-a^{n+1}}{n+1}\right) - 1 < 0.$$

Thus the roots of the derivative $f'_{n,a}(x) = (n+1)x^{n-1}(x - an/(n+1))$ are not roots of $f_{n,a}$, whence $f_{n,a}$ is separable. Since $f_{n,a}(a) = -1$ and

$$f_{n,a}(a+1/n) = \frac{(a+1/n)^n}{n} - 1 > \frac{(1+1)^n - n}{n} \ge \frac{(n\cdot 1^{n-1}\cdot 1) - n}{n} \ge 0 \quad \text{for } n \ge 2,$$

it follows that $f_{n,a}$ has a real root in the interval $(a, a+1/n)$ for $n \geq 2$. And since $f'_{n,a}$ has no root in $(a, a+1/n)$, the mean value theorem implies that $f_{n,a}$ has a *unique* real root $\theta_{n,a}$ in this interval. Our next step is to show that the remaining roots of $f_{n,a}$ all have absolute value less than 1.

First, suppose $a > 2$. By Rouché's theorem, the number of roots of $f_{n,a}$ inside the unit circle is equal to that of az^n, counted with multiplicity. For future reference, note that up until this point our argument does not require that a be an integer.

Take $a > 2$ to be an integer. Since $f_{n,a}$ has degree $n+1$, combine the conclusion of the two paragraphs above and it follows that $\theta_{n,a}$ is a Pisot number for all $n \geq 2$. And since $\lim_{n\to\infty} \theta_{n,a} = a$, we see that a is a nonisolated limit point of the set of Pisot numbers.

Now, fix $n \geq 2$, and let $a \to 2$ from the right side. By the conclusion of the second paragraph (which is valid for $a > 2$), it follows that $f_{n,2}$ has n roots with absolute value at most 1. Suppose it does have a root ζ with absolute value 1. Then $\zeta - \zeta^{-n} = 2$, which is impossible. Thus for any fixed $n \geq 2$, all roots of $f_{n,2}$ except for $\theta_{n,2}$ have absolute value less than 1. We can now continue as in the case of integer $a > 2$ above, and the lemma follows. $\square$

Proof of Theorem 1.3. First, note that $f_{n,a}$ is irreducible over $\mathbb{Q}$; otherwise by Gauss's lemma, it has a nontrivial monic irreducible factor over $\mathbb{Z}$ with all roots having absolute value less than 1, which is impossible. Thus $T_n := \mathbb{Q}(\theta_{n,2})$ is an extension of $\mathbb{Q}$ of degree $n+1$.

Since $f_{n,2}(0) = -1$ and since $f'_{n,2}$ is negative on the interval $(0, 1)$, that means $f_{n,2}$ has no real root on the interval $[0, 1]$. Thus $\theta_{n,2}$ is the only real root of $f_{n,2}$ on the positive real axis. Since $f'_{n,2}$ has no root on the negative real axis, the mean value theorem implies that $f_{n,2}$ has at most one negative real root. Consequently, $f_{n,2}$ has at most two real roots. Since $f_{n,2}$ does have at least one real root and since $\deg(f_{n,2}) = n+1$, it follows that $r_1(T_n) = 1$ or 2 depending on whether n is even or odd. It remains to bound the root discriminant of T_n.

As α runs through the roots of $f_{n,2}$, we see that the absolute value of the polynomial discriminant of $f_{n,2}$ is

$$\begin{aligned}
\prod_{\alpha} |f'_{n,2}(\alpha)| &= \Big|\prod_{\alpha} \alpha\Big|^{n-1} \cdot (n+1)^{n+1} \cdot \prod_{\alpha} \big|\alpha - \tfrac{2n}{n+1}\big| \\
&= (n+1)^{n+1} \cdot \big|f\big(1 - \tfrac{2}{n+1}\big)\big| \\
&= (n+1)^{n+1} \cdot \big|\big(1 - \tfrac{2}{n+1}\big)^n \big(1 - \tfrac{2}{n+1} - 2\big) - 1\big| \\
&\leq 3(n+1)^{n+1} \qquad\qquad \text{for } n \geq 2,
\end{aligned}$$

and the theorem follows. $\square$

Acknowledgements

We thank Professors Hajir and Maire for bringing this question to our attention, Professors Gunnells and Kusner for useful discussions, and the referee for detailed and constructive comments.

References

[Belolipetsky and Lubotzky 2012] M. Belolipetsky and A. Lubotzky, "Manifolds counting and class field towers", *Adv. Math.* **229**:6 (2012), 3123–3146. MR 2900437 Zbl 1241.22014

[Fontaine 1985] J.-M. Fontaine, "Il n'y a pas de variété abélienne sur $\mathbb{Z}$", *Invent. Math.* **81**:3 (1985), 515–538. MR 87g:11073 Zbl 0612.14043

[Friedman 1989] E. Friedman, "Analytic formulas for the regulator of a number field", *Invent. Math.* **98**:3 (1989), 599–622. MR 91c:11061 Zbl 0694.12006

[Hajir and Maire 2001] F. Hajir and C. Maire, "Tamely ramified towers and discriminant bounds for number fields", *Compositio Math.* **128**:1 (2001), 35–53. MR 2002g:11149 Zbl 1042.11072

[Hajir and Maire 2002] F. Hajir and C. Maire, "Tamely ramified towers and discriminant bounds for number fields, II", *J. Symbolic Comput.* **33**:4 (2002), 415–423. MR 2003h:11137 Zbl 1086.11051

[Koch and Venkov 1975] H. Koch and B. B. Venkov, "Über den p-Klassenkörperturm eines imaginär-quadratischen Zahlkörpers", pp. 57–67 in *Journées arithmétiques de Bordeaux* (Bordeaux, 1974), Astérisque **24–25**, Soc. Math. France, Paris, 1975. MR 52 #13741 Zbl 0335.12021

[Lagarias and Odlyzko 1977] J. C. Lagarias and A. M. Odlyzko, "Effective versions of the Chebotarev density theorem", pp. 409–464 in *Algebraic number fields: L-functions and Galois properties* (Durham, 1975), edited by A. Fröhlich, Academic Press, London, 1977. MR 56 #5506 Zbl 0362.12011

[Lang 1986] S. Lang, *Algebraic number theory*, Graduate Texts in Mathematics **110**, Springer, New York, 1986. MR 44 #181 Zbl 0601.12001

[Martin 2006] J. W. Martin, "Improved bounds for discriminants of number fields: improved discriminant bounds", preprint, 2006.

[Martinet 1978] J. Martinet, "Tours de corps de classes et estimations de discriminants", *Invent. Math.* **44**:1 (1978), 65–73. MR 57 #275 Zbl 0369.12007

[Masley 1978] J. M. Masley, "Class numbers of real cyclic number fields with small conductor", *Compositio Math.* **37**:3 (1978), 297–319. MR 80e:12005 Zbl 0428.12003

[Odlyzko 1990] A. M. Odlyzko, "Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: a survey of recent results", *Sém. Théor. Nombres Bordeaux* (2) **2**:1 (1990), 119–141. MR 91i:11154 Zbl 0722.11054

[Oesterlé 1979] J. Oesterlé, "Versions effectives du théorème de Chebotarev sous l'hypothése de Riemann généralisée", pp. 165–167 in *Journées arithmétiques de Luminy* (Luminy, 1978), Astérisque **61**, Soc. Math. France, Paris, 1979. Zbl 0418.12005

[Roquette 1967] P. Roquette, "On class field towers", pp. 231–249 in *Algebraic number theory* (Brighton, 1965), edited by J. W. S. Cassels and A. Fröhlich, Thompson, Washington, DC, 1967. MR 36 #1418

[Salem 1944] R. Salem, "A remarkable class of algebraic integers: proof of a conjecture of Vijayaraghavan", *Duke Math. J.* **11** (1944), 103–108. MR 5,254a Zbl 0063.06657

[Salem 1963] R. Salem, *Algebraic numbers and Fourier analysis*, D. C. Heath, Boston, 1963. MR 28 #1169 Zbl 0126.07802

[Schmithals 1980] B. Schmithals, "Konstruktion imaginärquadratischer Körper mit unendlichem Klassenkörperturm", *Arch. Math.* (*Basel*) **34**:4 (1980), 307–312. MR 82f:12017 Zbl 0448.12008

[Schoof 1986] R. Schoof, "Infinite class field towers of quadratic fields", *J. Reine Angew. Math.* **372** (1986), 209–220. MR 88a:11121 Zbl 0589.12011

[Serre 1979] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics **67**, Springer, New York, 1979. MR 82e:12016 Zbl 0423.12016

[Stark 1974] H. M. Stark, "Some effective cases of the Brauer–Siegel theorem", *Invent. Math.* **23** (1974), 135–152. MR 49 #7218 Zbl 0278.12005

[Tate 1994] J. Tate, "The non-existence of certain Galois extensions of $\mathbb{Q}$ unramified outside 2", pp. 153–156 in *Arithmetic geometry* (Tempe, AZ, 1993), edited by N. Childress and J. W. Jones, Contemp. Math. **174**, Amer. Math. Soc., Providence, RI, 1994. MR 95i:11132 Zbl 0814.11057

[Washington 1982] L. C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Mathematics **83**, Springer, New York, 1982. MR 85g:11001 Zbl 0484.12001

Received January 23, 2014. Revised February 13, 2015.

SIMAN WONG
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
LEDERLE GRADUATE RESEARCH TOWER
BOX 34515
AMHERST, MA 01003-9305
UNITED STATES

siman@math.umass.edu

Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 277 No. 1 September 2015
