Pacific Journal of Mathematics

REPRESENTATIONS OF KNOT GROUPS INTO $\mathrm{SL}_n(\mathbb{C})$ AND TWISTED ALEXANDER POLYNOMIALS

MICHAEL HEUSENER AND JOAN PORTI

Volume 277 No. 2 October 2015

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.277.313

REPRESENTATIONS OF KNOT GROUPS INTO $\mathrm{SL}_n(\mathbb{C})$ AND TWISTED ALEXANDER POLYNOMIALS

MICHAEL HEUSENER AND JOAN PORTI

Let Γ be the fundamental group of the exterior of a knot in the three-sphere. We study deformations of representations of Γ into $\mathrm{SL}_n(\mathbb{C})$ which are the sum of two irreducible representations. For such representations we give a necessary condition, in terms of the twisted Alexander polynomial, for the existence of irreducible deformations. We also give a more restrictive sufficient condition for the existence of irreducible deformations. We also prove a duality theorem for twisted Alexander polynomials and we describe the local structure of the representation and character varieties.

1. Introduction

Let $K \subset S^3$ be an oriented knot in the three-sphere. Its exterior is the compact three-manifold $X = S^3 \setminus \mathcal{N}(K)$. Set $\Gamma = \pi_1(X)$ and let $\varphi : \Gamma \twoheadrightarrow \mathbb{Z}$ denote the abelianization morphism, so that $\varphi(\gamma)$ is the linking number in S^3 between any loop realizing $\gamma \in \Gamma$ and K. Let

$$\alpha : \Gamma \to \mathrm{SL}_a(\mathbb{C}) \quad \text{and} \quad \beta : \Gamma \to \mathrm{SL}_b(\mathbb{C})$$

be *irreducible* and *infinitesimally regular* representations.

Both authors were partially supported by Mineco through grant MTM2012-34834. Heusener was also supported by the ANR projects ModGroup and SGT (*Structures Géométriques Triangulées*).
MSC2010: primary 57M25, 57M05; secondary 57M27.
Keywords: variety of representations, character variety, twisted Alexander polynomial, deformations.

Definition 1.1. A representation $\alpha : \Gamma \to \mathrm{SL}_a(\mathbb{C})$ is called *reducible* when it preserves a proper subspace of $\mathbb{C}^a$, otherwise it is called *irreducible*. The representation α is called *semisimple* or *completely reducible* if α is a direct sum of irreducible representations.

In what follows we call a representation $\alpha : \Gamma \to \mathrm{SL}_a(\mathbb{C})$ *infinitesimally regular* if $H^1(\Gamma; \mathfrak{sl}_a(\mathbb{C})_{\mathrm{Ad}\,\alpha}) \cong \mathbb{C}^{a-1}$.

As we assume that α is irreducible and infinitesimally regular, its character is a regular point of the character variety of Γ in $\mathrm{SL}_a(\mathbb{C})$ (Proposition 3.6). When $b = 1$, then β is trivial and hence it is infinitesimally regular.

For a given nonzero complex number $\lambda \in \mathbb{C}^*$ we consider the representation $\rho_\lambda = (\lambda^{b\varphi} \otimes \alpha) \oplus (\lambda^{-a\varphi} \otimes \beta)$, namely for all $\gamma \in \Gamma$

$$\rho_\lambda(\gamma) = \begin{pmatrix} \lambda^{b\varphi(\gamma)}\alpha(\gamma) & 0 \\ 0 & \lambda^{-a\varphi(\gamma)}\beta(\gamma) \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}), \tag{1}$$

where $a+b=n$. The representation $\rho_\lambda : \Gamma \to \mathrm{SL}_n(\mathbb{C})$ is reducible and the following question then arises:

Question 1.2. When can ρ_λ be deformed to irreducible representations?

We give necessary and sufficient conditions in terms of twisted Alexander polynomials. For this purpose we consider the representations

$$\alpha \otimes \beta^* : \Gamma \to \mathrm{Aut}(M_{a\times b}(\mathbb{C}))$$

defined by $(\alpha\otimes\beta^*)(\gamma)(A) = \alpha(\gamma)A\beta(\gamma^{-1})$ for $\gamma \in \Gamma$ and $A \in M_{a\times b}(\mathbb{C})$. Similarly, consider

$$\beta \otimes \alpha^* : \Gamma \to \mathrm{Aut}(M_{b\times a}(\mathbb{C})).$$

The corresponding twisted Alexander polynomials of degree i are denoted by

$$\Delta_i^+(t) = \Delta_i^{\alpha\otimes\beta^*}(t) \quad \text{and} \quad \Delta_i^-(t) = \Delta_i^{\beta\otimes\alpha^*}(t).$$

Recall that the twisted Alexander polynomial is a generator of the order ideal of the twisted Alexander module and hence it is unique up to multiplication with an invertible element of the group ring $\mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[t^{\pm 1}]$, i.e., ct^k, with $c \in \mathbb{C}^*$ and $k \in \mathbb{Z}$ (see Definition 2.1 for more details). We have $\Delta_i^\pm(t) = 1$ for $i > 2$ and $\Delta_2^\pm(t) \in \{0, 1\}$. We prove in Corollary 4.6 that $\alpha\otimes\beta^*$ is a *semisimple* representation, hence by Theorem 2.6 we obtain the duality formula (Corollary 4.7):

$$\Delta_i^+(t) \doteq \Delta_i^-(1/t).$$

Here $p \doteq q$ means that p and q are *associated elements* in $\mathbb{C}[\mathbb{Z}]$, i.e., there exists some unit $ct^k \in \mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[t^{\pm 1}]$, with $c \in \mathbb{C}^*$ and $k \in \mathbb{Z}$, such that $p = ct^k q$. This duality formula is a particular case of Theorem 2.6, where we establish a duality formula for twisted Alexander polynomials provided that the twisting representation is semisimple. This duality formula can also be deduced from results of Friedl, Kim, and Kitayama [Friedl et al. 2012].

We shall prove a necessary condition for the deformability of ρ_λ to irreducible representations:

Theorem 1.3. *If ρ_λ can be deformed to irreducible representations, then*

$$\Delta_1^+(\lambda^n) = \Delta_1^-(\lambda^{-n}) = 0.$$

The theorem also applies when α or β (or both) is trivial. When both α and β are trivial, this is a result obtained in 1967 independently by Burde [1967] and de Rham [1967]. The key idea is to look at the dimension of the fiber of the algebraic quotient $R(\Gamma, \mathrm{SL}_n(\mathbb{C})) \to X(\Gamma, \mathrm{SL}_n(\mathbb{C}))$. When ρ_λ can be deformed to irreducible representations, this dimension jumps among characters of reducible representations, and this translates to the twisted Alexander polynomial by means of the tangent space and cohomology with twisted coefficients.

The next result is a sufficient condition for the deformability of ρ_λ to irreducible representations:

Theorem 1.4. *If $\Delta_0^+(\lambda^n) \neq 0$ and λ^n is a simple root of $\Delta_1^+(t)$, then ρ_λ can be deformed to irreducible representations.*

Again this theorem and the next one apply for α and/or β trivial. Theorems 1.4 and 1.5 are due to [Heusener et al. 2001] when both α and β are trivial, and also related results were obtained in [Shors 1991; Frohman and Klassen 1991; Heusener and Klassen 1997; Heusener and Kroll 1998; Ben Abdelghani 2000; Ben Abdelghani and Lines 2002; Heusener and Porti 2005; Ben Abdelghani et al. 2010; Heusener and Medjerab 2014].

The outline of the proof of Theorem 1.4 is the following: the hypothesis implies that there exists a representation $\rho^+ \in R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ with the same character as ρ_λ but not conjugate to it (see Corollary 5.6). An analysis of the cohomology groups allows us to prove that ρ^+ is a smooth point of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$. Among other tools, this uses the vanishing of obstructions to integrability of Zariski tangent vectors, due to [Goldman 1984], a smoothness result of the variety of representations due to [Heusener and Medjerab 2014], and the nonvanishing of certain cup product (following the ideas of [Ben Abdelghani 2000]). Once this smoothness result is established, we realize that the dimension of the space of reducible representations is less than the dimension of the component of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ containing ρ^+.

Our next result concerns the local structure of the character variety. Let $\chi_\lambda \in X(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ denote the character of ρ_λ.

Theorem 1.5. *Under the hypotheses of Theorem 1.4*, χ_λ *belongs to precisely two components* Y *and* Z *of* $X(\Gamma, \mathrm{SL}_n(\mathbb{C}))$, *that have dimension* $n-1$ *and meet transversally at* χ_λ *along a subvariety of dimension* $n-2$. *The component* Y *contains characters of irreducible representations and* Z *consists only of characters of reducible ones.*

As in [Heusener et al. 2001] and [Heusener and Porti 2005] for $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$ respectively, the key idea for Theorem 1.5 is to study the quadratic cone of the representation ρ_λ, by identifying certain obstructions to integrability. Here we also use Luna's slice theorem, as in [Ben Abdelghani 2002].

We conclude the paper by an explicit description of the component of the variety of irreducible characters of the trefoil knot in $\mathrm{SL}_3(\mathbb{C})$ that illustrates our results.

The paper is organized as follows. Section 2 is devoted to twisted Alexander modules, and in particular to the duality theorem, Theorem 2.6. In Section 3 we review some preliminaries on the representation varieties and in Section 4 some further preliminaries on twisted cohomology and twisted invariants. Then in Section 5 we prove Theorem 1.3. The proof of the sufficient condition, Theorem 1.4, splits in Sections 6 and 7. Theorem 1.5 is proved in Section 8. Finally in Section 9 we compute $X(\Gamma, \mathrm{SL}_3(\mathbb{C}))$ for Γ the fundamental group of the trefoil knot exterior.

2. Twisted Alexander modules

The aim of this section is to introduce twisted Alexander modules and Alexander polynomials, together with their main properties. We also give a new result that we will require later: a duality theorem for Alexander polynomials twisted by semisimple representations. It relies on Franz–Milnor duality for Reidemeister torsion, but it is different, as the torsion is the ratio of the Alexander polynomials. For further background about twisted Alexander polynomials see [Kirk and Livingston 1999].

A representation of a group Γ in a finite-dimensional complex vector space V is a homomorphism $\rho : \Gamma \to \mathrm{GL}(V)$. We say that such a map gives V the structure of a Γ-module. If there is no ambiguity about the map ρ we call V itself a representation of Γ and we will often suppress the symbol ρ and write $\gamma \cdot v$ or γv for $\rho(\gamma)(v)$. Two representations $\rho : \Gamma \to \mathrm{GL}(V)$ and $\varrho : \Gamma \to \mathrm{GL}(W)$ are called *equivalent* if there exits an isomorphism $T : V \to W$ such that $\varrho(\gamma) \circ T = T \circ \rho(\gamma)$ for all $\gamma \in \Gamma$, i.e., if the Γ-modules V and W are isomorphic.

Our main reference for group cohomology is [Brown 1994]. Since we work with left-modules, for defining homology consider the right action of the inverse, as in [Kirk and Livingston 1999, (2.1)]. As the knot exterior X is an Eilenberg–MacLane space, (co)homology groups of Γ and X are naturally identified. In what follows, we will not distinguish between $H_i(\Gamma; V)$ and $H_i(X; V)$.

We give an interpretation of the low dimensional (co)homology groups. The cohomology group in dimension zero is the module of invariants, i.e.,

$$H^0(\Gamma; V) \cong V^\Gamma = \{v \in V \mid \gamma v = v \text{ for all } \gamma \in \Gamma\}.$$

The homology group in dimension zero is the co-invariant module:

$$H_0(\Gamma; V) \cong \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} V \cong V/IV$$

where $I \subset \mathbb{Z}[\Gamma]$ is the augmentation ideal and $IV \subset V$ is the subspace generated by $\{\gamma v - v \mid v \in V, \gamma \in \Gamma\}$.

We will make use of the interpretation of $H^1(\Gamma; V)$ by means of crossed morphisms, it is well suited for our purpose. A *crossed morphism* $d : \Gamma \to V$ is a map that satisfies $d(\gamma_1\gamma_2) = d(\gamma_1) + \gamma_1 d(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$. A crossed morphism d is called *principal* if there exists $v \in V$ satisfying $d(\gamma) = \gamma v - v$ for all $\gamma \in \Gamma$. Crossed morphisms are precisely the *cocycles* of the standard or bar resolution of the Γ-module V, and the principal ones are the *coboundaries*. Thus the set of crossed morphisms or cocycles is denoted by $Z^1(\Gamma; V)$ and the set of principal crossed morphisms or coboundaries by $B^1(\Gamma; V)$. In particular, the first cohomology group is

$$H^1(\Gamma; V) \cong Z^1(\Gamma; V)/B^1(\Gamma; V). \tag{2}$$

Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a finite dimensional representation of Γ. If $X_\infty \to X$ denotes the infinite cyclic covering, then

$$H_i(X_\infty; V)$$

is a finitely generated $\mathbb{C}[\mathbb{Z}]$-module, because X is compact and V is finite dimensional. Here $\mathbb{Z}$ is the group of deck transformations of the covering $X_\infty \to X$. We will sometimes interpret the elements of $\mathbb{C}[\mathbb{Z}]$ as Laurent polynomials, by using the isomorphism $\mathbb{C}[\mathbb{Z}] \xrightarrow{\sim} \mathbb{C}[t^{\pm 1}]$ that maps the generator 1 of $\mathbb{Z}$ to t.

Definition 2.1. The homology groups $H_i(X_\infty; V)$ are called the *twisted Alexander modules*, viewed as $\mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[t^{\pm 1}]$-modules. The corresponding orders are the *twisted Alexander polynomials*

$$\Delta_i^\rho(t) \in \mathbb{C}[t^{\pm 1}].$$

They are unique up to multiplication by a unit $ct^k \in \mathbb{C}[t^{\pm 1}]$, $k \in \mathbb{Z}$, $c \in \mathbb{C}^*$.

Recall that the *order* of a finitely generated $\mathbb{C}[t^{\pm 1}]$-module

$$M = \bigoplus_i \mathbb{C}[t^{\pm 1}] \big/ p_i(t)\mathbb{C}[t^{\pm 1}]$$

is $\prod_i p_i(t)$. In particular the order is nonzero if and only if M is a torsion module. Notice that this is not the same convention as in [Kirk and Livingston 1999].

Due to the indeterminacy in the definition of twisted Alexander polynomials, we shall write

$$p(t) \doteq q(t)$$

to denote that the polynomials $p(t), q(t) \in \mathbb{C}[\mathbb{Z}]$ are *associated*, i.e., they are equal up to multiplication with an element $ct^k \in \mathbb{C}[\mathbb{Z}]$, $k \in \mathbb{Z}$, $c \in \mathbb{C}^*$.

Remark 2.2. It follows from a result of M. Wada [1994, Theorem 2] that the twisted Alexander polynomial of a link exterior twisted by a representation in $\mathrm{SL}_n(\mathbb{C})$ is well defined up to powers of $\pm t^k$. It is also well known that for n even there is no sign ambiguity. We shall not need those facts, as we use essentially the structure of the Alexander module.

Let

$$V[\mathbb{Z}] = V \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}[\mathbb{Z}]$$

denote the Γ-module via the representation $\rho \otimes t^{\varphi}$. Then we have a natural isomorphism of $\mathbb{C}[\mathbb{Z}]$-modules

$$H_i(X; V[\mathbb{Z}]) \cong H_i(X_\infty; V) \tag{3}$$

(see [Kirk and Livingston 1999, Theorem 2.1]). Notice that equivalent representations give rise to isomorphic Γ-modules and hence to associated Alexander polynomials.

The *dual representation* $\rho^* : \Gamma \to \mathrm{GL}(V^*)$ is defined in the usual way by

$$\rho^*(\gamma)(f) = f \circ \rho(\gamma)^{-1} \quad \text{for all } \gamma \in \Gamma \text{ and } f \in V^* = \mathrm{Hom}(V, \mathbb{C}).$$

The following lemma is straightforward.

Lemma 2.3. *The representations ρ and ρ^* are equivalent if and only if there exists a nondegenerate bilinear form $V \otimes V \to \mathbb{C}$ which is Γ-invariant.*

Example 2.4. For any representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$, the module $V = \mathbb{C}^2$ has a skew-symmetric nondegenerate bilinear form defined by the determinant. Namely, the vectors (x_1, x_2) and $(y_1, y_2) \in \mathbb{C}^2$ are mapped to

$$\det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}.$$

In view of Lemma 2.3, ρ^* and ρ are equivalent and hence $\Delta_i^{\rho} \doteq \Delta_i^{\rho^*}$.

Recall from the introduction (see Definition 1.1) that a representation $\rho : \Gamma \to \mathrm{GL}(V)$ is called semisimple or completely reducible if ρ is the direct sum of irreducible representations.

Remark 2.5. A representation ρ is completely reducible if and only if each subspace of V stable under $\rho(\Gamma)$ has a $\rho(\Gamma)$-invariant complement.

Theorem 2.6. *Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a completely reducible representation. Then*

$$\Delta_i^{\rho}(t^{-1}) \doteq \Delta_i^{\rho^*}(t).$$

Example 2.9 below shows that the hypothesis of complete reducibility is necessary in Theorem 2.6. This duality formula can also be deduced from results of Friedl, Kim, and Kitayama [2012].

The first step in the proof of Theorem 2.6 is the following:

Lemma 2.7. *Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a completely reducible representation. The modules $H_0(X_\infty; V)$ and $H_0(X_\infty; V^*)$ are finitely generated torsion modules. In addition,*

$$\Delta_0^{\rho}(t^{-1}) \doteq \Delta_0^{\rho^*}(t).$$

Proof. First notice that if ρ is irreducible or completely reducible then the dual representation ρ^* is also irreducible or completely reducible respectively since each proper invariant subspace of ρ corresponds to a proper invariant subspace of ρ^* by the *orthogonality* relation.

We have that $H_0(X_\infty; V) \cong V/\tilde{I}V$, where $\tilde{I} \subset \mathbb{C}[\pi_1(X_\infty)]$ is the augmentation ideal. Hence, $H_0(X_\infty; V)$ is a finite dimensional $\mathbb{C}$-vector space and as $\mathbb{C}[t^{\pm 1}]$-module it cannot have a free summand. This proves that $H_0(X_\infty; V)$ is a finitely generated torsion module.

In order to prove the symmetry relation it is sufficient to prove it for irreducible representations since for $\rho_1 : \Gamma \to \mathrm{GL}(V_1)$ and $\rho_2 : \Gamma \to \mathrm{GL}(V_2)$ we have

$$(\rho_1 \oplus \rho_2)^* = \rho_1^* \oplus \rho_2^* \quad \text{and} \quad \Delta_i^{\rho_1 \oplus \rho_2} \doteq \Delta_i^{\rho_1} \cdot \Delta_i^{\rho_2}.$$

First we will prove that for every irreducible representation $\rho : \Gamma \to \mathrm{GL}(V)$ with $\dim V > 1$ we have

$$\Delta_0^{\rho} \doteq 1 \doteq \Delta_0^{\rho^*}. \tag{4}$$

The irreducibility of ρ and $\dim V > 1$ imply that $IV \subset V$ is a nontrivial Γ-invariant subspace, and hence $IV = V$. It follows that $H_0(\Gamma; V) = 0$. Now, for any complex number $\lambda \in \mathbb{C}^*$ the vector space V becomes a Γ-module via $\rho \otimes \lambda^{\varphi}$, i.e., for $\gamma \in \Gamma$ and for $v \in V$ we have $\rho(\gamma) \otimes \lambda^{\varphi(\gamma)}(v) = \lambda^{\varphi(\gamma)}\rho(\gamma)v$. This Γ-module will be denoted by V_λ. Notice that V_λ is also an irreducible Γ-module since the map $v \mapsto \lambda^{\varphi(\gamma)}v$ is a homothety of V. Moreover, V_λ is a nontrivial Γ-module and hence $H_0(\Gamma; V_\lambda) = 0$ for all $\lambda \in \mathbb{C}^*$. Next, the short exact sequence of Γ-modules

$$0 \to V[\mathbb{Z}] \xrightarrow{(t-\lambda)\cdot} V[\mathbb{Z}] \to V_\lambda \to 0$$

induces a long exact sequence in homology [Brown 1994, III.§6]:

$$\cdots \to H_0(\Gamma; V[\mathbb{Z}]) \xrightarrow{(t-\lambda)\cdot} H_0(\Gamma; V[\mathbb{Z}]) \to H_0(\Gamma; V_\lambda) \to 0,$$

and $H_0(\Gamma;\, V_\lambda) = 0$ implies that the multiplication by $(t-\lambda)$ is surjective. Hence for all $\lambda \in \mathbb{C}^*$, the module $H_0(\Gamma;\, V[\mathbb{Z}])$ has no $(t-\lambda)$-torsion. Hence, $H_0(\Gamma;\, V[\mathbb{Z}]) = 0$ and $\Delta_0^\rho = 1$. Finally, ρ^* is also irreducible and $\dim V^* = \dim V > 1$. This implies in the same way that $\Delta_0^{\rho^*} = 1$

Now suppose that $\dim V = 1$, i.e., $\rho : \Gamma \to \mathrm{GL}(V) \cong \mathbb{C}^*$. Hence ρ is abelian and completely determined by a nonzero-complex number λ, meaning that for all $\gamma \in \Gamma$ and $v \in V$ we have $\rho(\gamma)(v) = \lambda^{\varphi(\gamma)} v$. So we write $\rho = \lambda^\varphi$. Now

$$H_0(\Gamma;\, V[\mathbb{Z}]) \cong V[\mathbb{Z}]/IV[\mathbb{Z}] \cong V[t^{\pm 1}]/(\lambda t - 1),$$

since λ^φ is an abelian representation and factors through $\mathbb{Z}$. Therefore $\Delta_0^{\lambda^\varphi}(t) \doteq t - \lambda^{-1}$. The dual representation $(\lambda^\varphi)^*$ is $\lambda^{-\varphi}$, as $(\lambda^\varphi)^*(\gamma)(f) = f \circ (\lambda^{\varphi(\gamma)})^{-1} = \lambda^{-\varphi(\gamma)} f$, where $\gamma \in \Gamma$ and $f \in V^*$. The same calculation as above shows that $H_0(\Gamma;\, V^*[\mathbb{Z}]) \cong V[t^{\pm 1}]/(\lambda^{-1} t - 1)$ and hence $\Delta_0^{(\lambda^\varphi)^*}(t) \doteq t - \lambda$. We obtain $\Delta_0^{(\lambda^\varphi)*}(t) \doteq \Delta_0^{\lambda^\varphi}(t^{-1})$, which proves the lemma. □

Proof of Theorem 2.6. The knot exterior X has the homotopy type of a 2-dimensional complex. Therefore $H_i(X_\infty;\, V) = 0$ for $i > 2$ and $H_2(X_\infty;\, V)$ is a free $\mathbb{C}[\mathbb{Z}]$-module. This implies that $\Delta_i^\rho \doteq 1$ for $i > 2$ and $\Delta_2^\rho \in \{0, 1\}$. According to the value of Δ_2^ρ there are two cases to study.

Assume first that $\Delta_2^\rho = 0$. This is equivalent to $H_2(X_\infty;\, V)$ being a nontrivial free $\mathbb{C}[\mathbb{Z}]$-module. By an Euler characteristic argument, $H_1(X_\infty;\, V)$ contains also a nontrivial free factor of the same rank. In particular $\Delta_1^\rho = 0$. Since $H_i(X_\infty;\, V) \cong H_i(X;\, V[\mathbb{Z}])$, the universal coefficient theorem yields that $H_i(X;\, V[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)) \neq 0$ for $i = 1, 2$. Notice also that the natural pairing $V \times V^* \to \mathbb{C}$ extends to a nondegenerate $\mathbb{C}(t)$-bilinear form

$$\big(V[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)\big) \times \big(V^*[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)\big) \to \mathbb{C}(t). \tag{5}$$

Using this bilinear form and Poincaré duality, $H_i(X, \partial X;\, V^*[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)) \neq 0$ for $i = 1, 2$. Since the homology of the 2-torus ∂X with coefficients $V^*[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)$ vanishes [Kirk and Livingston 1999, §3.3], $H_i(X;\, V^*[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)) \neq 0$ for $i = 1, 2$. Hence $\Delta_1^{\rho^*} = \Delta_2^{\rho^*} = 0$.

Next we deal with the case $\Delta_2^\rho \doteq 1$. Since this is equivalent to $H_2(X_\infty;\, V) = 0$, the homology argument in the previous paragraph gives $\Delta_2^{\rho^*} \doteq 1$. For the first Alexander polynomials we shall use Reidemeister torsion and Franz–Milnor duality. By Kitano's theorem [1996] the torsion of X with coefficients $V[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)$ is the ratio of Alexander polynomials:

$$\mathrm{TOR}(X;\, V[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)) \doteq \frac{\Delta_1^\rho}{\Delta_0^\rho},$$

see [Kirk and Livingston 1999, Theorem 3.4] for this precise statement (this is a version of Milnor's theorem [1962], see [Turaev 1986]).

Using the bilinear form (5), Franz–Milnor duality for Reidemeister torsion [Milnor 1962; Franz 1937] gives

$$\begin{aligned}\mathrm{TOR}\big(X;\, V[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)\big)(t) &\doteq \mathrm{TOR}\big(X, \partial X;\, V^*[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)\big)\big(\tfrac{1}{t}\big)\\ &\doteq \frac{\mathrm{TOR}\big(X;\, V^*[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)\big)\big(\tfrac{1}{t}\big)}{\mathrm{TOR}\big(\partial X;\, V^*[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)\big)\big(\tfrac{1}{t}\big)},\end{aligned}$$

see [Kirk and Livingston 1999, §5.1]. Since $\partial X \cong S^1 \times S^1$, $\mathrm{TOR}(\partial X;\, V^*[\mathbb{Z}] \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}(t)) \doteq 1$ [Kirk and Livingston 1999, §3.3]. Then the theorem follows from Lemma 2.7. □

Remark 2.8. Note that every representation $\rho : \Gamma \to \mathrm{O}(n)$ is completely reducible since for each stable subspace W the orthogonal complement $W^{\perp}$ is also stable. Moreover, we have $\rho^* = \rho$ and hence $\Delta_i^{\rho}(t^{-1}) \doteq \Delta_i^{\rho}(t)$ is symmetric (see [Kitano 1996, Theorem B]). It follows also from the proof of Lemma 2.7 that $\Delta_0^{\rho}(t) = (t-1)^{k_+}(t+1)^{k_-}$ where $k_+ = \dim\{v \in \mathbb{R}^n \mid \rho(\gamma)v = v \text{ for all } \gamma \in \Gamma\}$ and $k_- = \dim\{v \in \mathbb{R}^n \mid \rho(\gamma)v = (-1)^{\varphi(\gamma)}v \text{ for all } \gamma \in \Gamma\}$.

It was proved in Hillman, Silver, and Williams [2010] that $\Delta_i^{\rho}(t^{-1}) \doteq \Delta_i^{\rho}(t)$ holds if ρ^* and ρ are conjugates.

We finish this section with an example to show that the hypothesis of complete reducibility is needed in Theorem 2.6:

Example 2.9. We exhibit representations that are not completely reducible and such that the conclusion of Theorem 2.6 fails. In order to construct such a representation, we take $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ of the form

$$\rho = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{\varphi} & 0 \\ 0 & \lambda^{-\varphi} \end{pmatrix}$$

that is not abelian. It is a representation if $d \in Z^1(\Gamma; \mathbb{C}_{\lambda^2})$ and it is nonabelian if $\lambda \neq \pm 1$ and $d \notin B^1(\Gamma; \mathbb{C}_{\lambda^2})$, where $\mathbb{C}_{\lambda^2}$ denotes the Γ-module given by $\gamma \cdot z = \lambda^{2\varphi(\gamma)} z$ for $\gamma \in \Gamma$, $z \in \mathbb{C}$, see Lemma 5.5. Such a representation exists if and only if λ^2 is a root of the untwisted Alexander polynomial (in particular $\lambda \neq \pm 1$), see [Burde 1967; de Rham 1967; Heusener et al. 2001] for instance, or Lemma 5.5. As ρ is not abelian, its restriction to $\pi_1(X_\infty)$ is nontrivial but

$$\rho(\pi_1(X_\infty)) \subset \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \;\middle|\; c \in \mathbb{C} \right\}.$$

The cohomology module $H_0(X_\infty; \mathbb{C}^2)$ is isomorphic to $\mathbb{C}^2 / I\mathbb{C}^2$. Here the subspace $I\mathbb{C}^2 \subset \mathbb{C}^2$ is generated by elements of the form $v - \rho(\gamma)v$, with $\gamma \in \pi_1(X_\infty)$

and $v \in \mathbb{C}^2$, i.e., $I\mathbb{C}^2 = \left\{\binom{c}{0} \mid c \in \mathbb{C}\right\}$. So, the linear projection $\mathbb{C}^2 \to \mathbb{C}$ onto the second coordinate induces a linear isomorphism $\mathbb{C}^2/I\mathbb{C}^2 \xrightarrow{\sim} \mathbb{C}$. The action of a meridian $m \in \Gamma$ on $\mathbb{C}^2/I\mathbb{C}^2$ is multiplication by λ^{-1} and hence $H_0(X_\infty; \mathbb{C}^2) \cong \mathbb{C}[t^{\pm 1}]/(t-\lambda^{-1})$ as $\mathbb{C}[\mathbb{Z}]$-modules. Therefore, $\Delta_0^\rho(t) = t - \lambda^{-1}$. On the other hand, using that every representation in $\mathrm{SL}_2(\mathbb{C})$ is equivalent to its dual, see Example 2.4, $\Delta_0^{\rho^*}(t) \doteq t - \lambda^{-1}$, and

$$\Delta_0^\rho(t^{-1}) \doteq (t-\lambda) \text{ and } \Delta_0^{\rho^*}(t) \text{ are not associated.}$$

Notice that if $\Delta_2^\rho \doteq 1$, Franz–Milnor duality (used in the proof of Theorem 2.6) applies and it holds that $\Delta_1^\rho(t^{-1})/\Delta_0^\rho(t^{-1}) \doteq \Delta_1^{\rho^*}(t)/\Delta_0^{\rho^*}(t)$. In particular $\Delta_1^\rho(t^{-1})$ and $\Delta_1^{\rho^*}(t)$ are not associated either.

3. Varieties of representations

In this section we recall some preliminaries on the varieties of representations, we discuss representations of the peripheral subgroup $\pi_1(\partial X) \cong \mathbb{Z} \oplus \mathbb{Z}$, and we state a regularity result, Proposition 3.3 due to [Heusener and Medjerab 2014]. We also show that infinitesimal regularity implies regularity of the representation (Corollary 3.5) and its character (Proposition 3.6).

Recall that the set of all representations of Γ in $\mathrm{SL}_n(\mathbb{C})$ is called the *variety of representations* or the $\mathrm{SL}_n(\mathbb{C})$-*representation variety*:

$$R(\Gamma, \mathrm{SL}_n(\mathbb{C})) = \mathrm{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{C})).$$

It is an affine algebraic set (possibly with several components), as Γ is finitely generated. More precisely, $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ embeds in a Cartesian product $\mathrm{SL}_n(\mathbb{C}) \times \cdots \times \mathrm{SL}_n(\mathbb{C})$ by mapping each representation to the image of a generating set, and $\mathrm{SL}_n(\mathbb{C})$ is an algebraic group in $\mathbb{C}^{n^2}$. The group relations of a presentation of Γ induce the algebraic equations defining $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$. Different presentations give isomorphic algebraic sets (see [Lubotzky and Magid 1985], for instance).

The group $\mathrm{SL}_n(\mathbb{C})$ acts on $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ by conjugation. The algebraic quotient by this action is the *variety of characters* or $\mathrm{SL}_n(\mathbb{C})$-*character variety*

$$X(\Gamma, \mathrm{SL}_n(\mathbb{C})) = R(\Gamma, \mathrm{SL}_n(\mathbb{C})) /\!\!/ \mathrm{SL}_n(\mathbb{C}).$$

Recall that the GIT quotient exists since $\mathrm{SL}_n(\mathbb{C})$ is *reductive* and the representation variety is an affine algebraic set. (For more details see [Newstead 1978, 3.§3] or [Shafarevich 1994].)

To describe the Zariski tangent space to $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ and $X(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ we use crossed morphisms or cocycles.

An *infinitesimal deformation* of a representation is the same as a Zariski tangent vector to $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$. We use André Weil's construction that identifies

$Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho})$ with the Zariski tangent space to the scheme $\mathcal{R}(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ at ρ. Here $\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}$ is a Γ-module via the adjoint action, i.e., $\gamma \cdot x = \mathrm{Ad}_{\rho(\gamma)}(x)$ for $\gamma \in \Gamma$ and $x \in \mathfrak{sl}_n(\mathbb{C})$. Notice furthermore that the algebraic equations defining the representation variety may be nonreduced, hence there is an underlying affine scheme $\mathcal{R}(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ with a possible nonreduced coordinate ring. Weil's construction assigns to each cocycle $d \in Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C}))$ the infinitesimal deformation $\gamma \mapsto (1 + \varepsilon\, d(\gamma))\rho(\gamma)$ for all $\gamma \in \Gamma$, which satisfies the defining equations for $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ up to terms in the ideal (ε^2) of $\mathbb{C}[\varepsilon]$, i.e., a Zariski tangent vector to $\mathcal{R}(\Gamma, \mathrm{SL}_n(\mathbb{C}))$. Weil's construction identifies $B^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho})$ with the tangent space to the orbit by conjugation. See [Weil 1964; Lubotzky and Magid 1985; Ben Abdelghani 2002] for more details.

Let $\dim_\rho R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ denote the local dimension of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ at ρ (i.e., the maximal dimension of the irreducible components of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ containing ρ [Shafarevich 1977, Chapter II]). So we obtain:

$$\dim_\rho R(\Gamma, \mathrm{SL}_n(\mathbb{C})) \le \dim T_\rho(R(\Gamma, \mathrm{SL}_n(\mathbb{C}))) \le \dim Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}). \tag{6}$$

Definition 3.1. Let $\rho : \Gamma \to \mathrm{SL}_n(\mathbb{C})$ be a representation. We say that ρ is a *regular point* of the representation variety if

$$\dim_\rho R(\Gamma, \mathrm{SL}_n(\mathbb{C})) = \dim Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}).$$

We call ρ *infinitesimal regular* if $\dim H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}) = n - 1$.

It follows directly from (6) that a regular point is a smooth point of the representation variety. There are representations of discrete groups which are smooth points of the representation variety without being regular, as the scheme $\mathcal{R}(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ may be nonreduced. (See [Lubotzky and Magid 1985, Example 2.10] for more details.)

We also make use of the Poincaré–Lefschetz duality theorem with twisted coefficients: let M be a connected, orientable, compact m-dimensional manifold with boundary ∂M and let $\rho : \pi_1(M) \to \mathrm{SL}_n(\mathbb{C})$ be a representation. Then the cup product and the Killing form $b : \mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C}) \to \mathbb{C}$ induce a nondegenerate bilinear pairing

$$\begin{aligned} H^k(M; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}) \otimes H^{m-k}(M, \partial M; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}) &\xrightarrow{\smile} \\ H^m(M, \partial M; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho} \otimes \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}) &\xrightarrow{b} H^m(M, \partial M; \mathbb{C}) \cong \mathbb{C} \end{aligned} \tag{7}$$

and hence an isomorphism

$$H^k(M; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}) \cong H^{m-k}(M, \partial M; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho})^*,$$

for all $0 \le k \le m$. See [Johnson and Millson 1987; Porti 1997] for more details.

Lemma 3.2. *For any representation* $\varrho : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{SL}_n(\mathbb{C})$ *we have*:

$$\dim H^1(\mathbb{Z} \oplus \mathbb{Z}; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\varrho}) \geq 2(n-1).$$

In addition, $\dim H^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{sl}_n(\mathbb{C})) = 2(n-1)$ *if and only if* ϱ *is a regular point of* $R(\mathbb{Z} \oplus \mathbb{Z}, \mathrm{SL}_n(\mathbb{C}))$.

Recall that a function $\phi : R(\Gamma, \mathrm{SL}_n(\mathbb{C})) \to \mathbb{Z}$ is called *upper semicontinuous* if for all $k \in \mathbb{Z}$ the set $\phi^{-1}\big([k, \infty)\big)$ is closed. Moreover, it is easy to prove that for $q = 0, 1$ the function $\rho \mapsto \dim H^q(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho})$ is upper semicontinuous (see [Heusener and Porti 2011, Lemma 3.2], this is a particular case of the semicontinuity theorem [Hartshorne 1977, Chapter III, Theorem 12.8]).

Proof of Lemma 3.2. Poincaré duality and Euler characteristic give

$$\tfrac{1}{2} \dim H^1(\mathbb{Z} \oplus \mathbb{Z}; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\varrho}) = \dim H^0(\mathbb{Z} \oplus \mathbb{Z}; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\varrho}) = \dim \mathfrak{sl}_n(\mathbb{C})^{\mathbb{Z} \oplus \mathbb{Z}}.$$

By a result of Richardson [1979, Theorem C], every representation of $\mathbb{Z} \oplus \mathbb{Z}$ into $\mathrm{SL}_n(\mathbb{C})$ is a limit of diagonal representations, and for diagonal representations $\dim \mathfrak{sl}_n(\mathbb{C})^{\mathbb{Z} \oplus \mathbb{Z}} \geq n-1$. The general inequality follows from the upper semicontinuity of the function $\varrho \mapsto \dim H^0(\mathbb{Z} \oplus \mathbb{Z}; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\varrho})$.

For the second statement, Richardson proved in the same Theorem C that the representation variety $R(\mathbb{Z} \oplus \mathbb{Z}, \mathrm{SL}_n(\mathbb{C}))$ is an irreducible algebraic variety of dimension $(n+2)(n-1)$. It follows that $\varrho \in R(\mathbb{Z} \oplus \mathbb{Z}, \mathrm{SL}_n(\mathbb{C}))$ is a regular point if and only if $\dim Z^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{sl}_n(\mathbb{C})) = (n+2)(n-1)$.

On the other hand,

$$\begin{aligned}
\dim Z^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{sl}_n(\mathbb{C})) &= \dim H^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{sl}_n(\mathbb{C})) + \dim B^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{sl}_n(\mathbb{C}));\\
\dim B^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{sl}_n(\mathbb{C})) &= n^2 - 1 - \dim H^0(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{sl}_n(\mathbb{C}));\\
\dim H^0(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{sl}_n(\mathbb{C})) &= \tfrac{1}{2} \dim H^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{sl}_n(\mathbb{C})).
\end{aligned}$$

Hence

$$\dim Z^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{sl}_n(\mathbb{C})) = \tfrac{1}{2} \dim H^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{sl}_n(\mathbb{C})) + n^2 - 1.$$

Thus the lemma follows. (See also [Popov 2008].) □

We will require the following result:

Proposition 3.3 [Heusener and Medjerab 2014, Proposition 3.3]. *Let* α *be a point in the* $\mathrm{SL}_a(\mathbb{C})$*-representation variety* $R(\Gamma, \mathrm{SL}_a(\mathbb{C}))$. *If* α *is infinitesimally regular, then it is a regular point of* $R(\Gamma, \mathrm{SL}_a(\mathbb{C}))$ *and belongs to a unique component of dimension* $a^2 + a - 2 - \dim H^0(\Gamma; \mathfrak{sl}_a(\mathbb{C}))$.

Remark 3.4. For an irreducible representation $\alpha : \Gamma \to \mathrm{SL}_a(\mathbb{C})$, it holds that $H^0(\Gamma; \mathfrak{sl}_a(\mathbb{C})_{\mathrm{Ad}\,\alpha}) = 0$. Indeed, if $X \in \mathfrak{sl}_a(\mathbb{C})$ commutes with $\alpha(\gamma)$ for all $\gamma \in \Gamma$, then Schur's lemma implies that X is a scalar matrix and hence $X = 0$.

As a corollary we obtain from Proposition 3.3 and Remark 3.4:

Corollary 3.5. *If an* irreducible *representation* $\alpha : \Gamma \to \mathrm{SL}_a(\mathbb{C})$ *is infinitesimally regular then it is a regular point of* $R(\Gamma, \mathrm{SL}_a(\mathbb{C}))$ *of local dimension* $a^2 + a - 2$.

One has furthermore:

Proposition 3.6. *If an irreducible representation* $\alpha : \Gamma \to \mathrm{SL}_a(\mathbb{C})$ *is infinitesimally regular, then its character is a smooth point of* $X(\Gamma, \mathrm{SL}_a(\mathbb{C}))$ *of local dimension* $a - 1$.

Proof. By Corollary 3.5, α is a regular point of $R(\Gamma, \mathrm{SL}_a(\mathbb{C}))$ of local dimension $a^2 + a - 2$. As α is irreducible, the fiber of the projection $R(\Gamma, \mathrm{SL}_a(\mathbb{C})) \to X(\Gamma, \mathrm{SL}_a(\mathbb{C}))$ at α has dimension $a^2 - 1$. The dimension of this fiber is an upper semicontinuous function, therefore the dimension of $X(\Gamma, \mathrm{SL}_a(\mathbb{C}))$ at α is at least $a - 1$. On the other hand, the dimension of the Zariski tangent space of $X(\Gamma, \mathrm{SL}_a(\mathbb{C}))$ at α is at most $\dim H^1(\Gamma; \mathfrak{sl}_a(\mathbb{C})_{\mathrm{Ad}\,\alpha})$ (this follows from Luna's slice as α is irreducible, see [Lubotzky and Magid 1985, Theorem 2.15]). Hence we have equality of dimensions and the proposition follows. □

4. Twisted cohomology and twisted polynomials

In this section we prove that $\alpha \otimes \beta^*$ and $\beta \otimes \alpha^*$ are completely reducible representations, so that the duality theorem (Theorem 2.6) applies to them. Our assumption that $\alpha : \Gamma \to \mathrm{SL}_a(\mathbb{C})$ and $\beta : \Gamma \to \mathrm{SL}_b(\mathbb{C})$ are irreducible will be crucial for the conclusion.

Decomposition of $\mathfrak{sl}_n(\mathbb{C})$**.** Consider the action of Γ on the space of matrices with a rows and b columns $M_{a\times b}(\mathbb{C})$:

$$\Gamma \times M_{a\times b}(\mathbb{C}) \to M_{a\times b}(\mathbb{C}), \quad (\gamma, A) \mapsto \lambda^{n\varphi(\gamma)}\alpha(\gamma)A\beta(\gamma^{-1}). \tag{8}$$

The corresponding Γ-module is denoted by

$$\mathcal{M}^+_{\lambda^n} = M_{a\times b}(\mathbb{C})_{\alpha\otimes\beta^*\otimes\lambda^{n\varphi}}.$$

Similarly, we consider the module

$$\mathcal{M}^-_{\lambda^{-n}} = M_{b\times a}(\mathbb{C})_{\beta\otimes\alpha^*\otimes\lambda^{-n\varphi}}.$$

Notice that those modules occur as factors in the decomposition of $\mathfrak{sl}_n(\mathbb{C})$ as Γ-modules via the adjoint action $\mathrm{Ad}\,\rho_\lambda$:

$$\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda} = \mathfrak{sl}_a(\mathbb{C})_{\mathrm{Ad}\,\alpha} \oplus \mathfrak{sl}_b(\mathbb{C})_{\mathrm{Ad}\,\beta} \oplus \mathbb{C} \oplus \mathcal{M}^+_{\lambda^n} \oplus \mathcal{M}^-_{\lambda^{-n}}.$$

This can be visualized as

$$\mathfrak{sl}_n(\mathbb{C})_{\operatorname{Ad}\rho_\lambda} \cong \begin{pmatrix} \mathfrak{sl}_a(\mathbb{C})_{\operatorname{Ad}\alpha} & \mathcal{M}^+_{\lambda^n} \\ \mathcal{M}^-_{\lambda^{-n}} & \mathfrak{sl}_b(\mathbb{C})_{\operatorname{Ad}\beta} \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} b\,\mathrm{Id}_a & 0 \\ 0 & -a\,\mathrm{Id}_b \end{pmatrix}.$$

Duality. For every $\lambda \in \mathbb{C}^*$ we have a nondegenerate bilinear form

$$\Psi : \mathcal{M}^+_{\lambda^n} \times \mathcal{M}^-_{\lambda^{-n}} \to \mathbb{C}, \quad (A, B) \mapsto \operatorname{tr}(AB), \tag{9}$$

which is Γ-invariant: $\Psi(A, B) = \Psi(\gamma A, \gamma B)$ for all $\gamma \in \Gamma$. As an immediate consequence, we have Poincaré and Kronecker dualities:

$$H_i(X; \mathcal{M}^\pm_{\lambda^{\pm n}}) \cong H_{3-i}(X, \partial X; \mathcal{M}^\mp_{\lambda^{\mp n}})^*; \tag{10}$$

$$H^i(X; \mathcal{M}^\pm_{\lambda^{\pm n}}) \cong H^{3-i}(X, \partial X; \mathcal{M}^\mp_{\lambda^{\mp n}})^*; \tag{11}$$

$$H_i(X; \mathcal{M}^\pm_{\lambda^{\pm n}}) \cong H^i(X; \mathcal{M}^\mp_{\lambda^{\mp n}})^*. \tag{12}$$

The i-th twisted Alexander polynomials of the Γ-modules $\mathcal{M}^\mp_1$ are denoted by

$$\Delta_i^+ = \Delta_i^{\alpha\otimes\beta^*} \quad \text{and} \quad \Delta_i^- = \Delta_i^{\beta\otimes\alpha^*}.$$

Taking $\rho = \alpha \otimes \beta^*$, then $\rho^* = \beta \otimes \alpha^*$ by (9). In order to apply Theorem 2.6 to those polynomials, we need to show that $\rho = \alpha \otimes \beta^*$ is completely reducible; this motivates the next subsection.

Linear algebraic groups. We follow Humphreys' book [1975] as general reference for linear algebraic groups. A linear algebraic group G contains a unique largest normal solvable subgroup, which is automatically closed. Its identity component is then the largest connected normal solvable subgroup of G; it is called the *radical* of G, denoted by $R(G)$. The subgroup of unipotent elements in $R(G)$ is normal in both $R(G)$ and G; it is called the *unipotent radical* of G, denoted by $R_u(G)$. We have that $R(G)/R_u(G)$ is a torus. Hence $R(G)$ is a torus if and only if $R_u(G)$ is trivial.

Recall that a representation $\rho : \Gamma \to \mathrm{SL}(V)$ is called completely reducible if it is a direct sum of irreducible representations, see Definition 1.1.

Theorem 4.1 [Nagata 1961/1962, Theorem 3]. *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be an algebraic group. Then $R_u(G)$ is trivial if and only if each rational representation of G is completely reducible.* □

Here a representation $\rho : G \to \mathrm{GL}(V)$ is called *rational* if, with respect to a basis of V, the matrix entries of $\rho(g)$ are polynomial functions in the $n^2 + 1$ coordinate functions x_{ij} $(1 \le i, j \le n)$ and $1/\det$ of $\mathrm{GL}_n(\mathbb{C})$.

Remark 4.2. A nontrivial, connected algebraic group G is called *reductive* if $R_u(G)$ is trivial. Since the Zariski closure of a matrix group is in general not connected we will avoid the term reductive in what follows.

Lemma 4.3. *Let Γ be a group and let $\rho : \Gamma \to \mathrm{SL}_n(\mathbb{C})$ be an irreducible representation. Then the unipotent radical $R_u(G)$ of the Zariski closure G of $\rho(\Gamma) \subset \mathrm{SL}_n(\mathbb{C})$ is trivial.*

Proof. Suppose that $R_u(G) \subset \mathrm{SL}_n(\mathbb{C})$ is nontrivial. Every unipotent subgroup of $\mathrm{GL}_n(\mathbb{C})$ has a nonzero vector fixed by all elements of the group (see [Humphreys 1975, 17.5]). Then the subspace $W \subset \mathbb{C}^n$ of fixed vectors of $R_u(G)$ is nonzero. By normality, this subspace is preserved by G, hence by $\rho(\Gamma)$, which contradicts the irreducibility of ρ. □

Lemma 4.4. *Let $\alpha : \Gamma \to \mathrm{SL}_a(\mathbb{C})$ and $\beta : \Gamma \to \mathrm{SL}_b(\mathbb{C})$ be irreducible. Then the unipotent radical $R_u(G)$ of the Zariski closure G of $(\alpha \oplus \beta)(\Gamma) \subset \mathrm{SL}_a(\mathbb{C}) \times \mathrm{SL}_b(\mathbb{C})$ is trivial.*

Proof. Let $p_a : \mathrm{SL}_a(\mathbb{C}) \times \mathrm{SL}_b(\mathbb{C}) \to \mathrm{SL}_a(\mathbb{C})$ denote the projection. Then $p_a((\alpha \oplus \beta)(\Gamma)) = \alpha(\Gamma)$ and therefore $p_a(R_u(G))$ is contained in the unipotent radical $R_u(G_a)$ of the Zariski closure G_a of $\alpha(\Gamma)$ in $\mathrm{SL}_a(\mathbb{C})$. (The image of an unipotent element under a morphism of algebraic groups is unipotent [Humphreys 1975, 15.3].) Now, $R_u(G_a)$ is trivial by Lemma 4.3 and hence $p_a(R_u(G))$ is trivial. It follows in the same way that $p_b(R_u(G))$ is trivial and hence $R_u(G) = \{1\}$. □

Remark 4.5. The same argument of Lemma 4.4 proves that the Zariski closure of a completely reducible linear representation has trivial unipotent radical.

Corollary 4.6. *The Γ-modules $\mathcal{M}^{\pm}_{\lambda^{\pm n}}$ are completely reducible.*

Proof. By Lemma 4.4 the unipotent radical $R_u(G)$ of the Zariski closure G of $(\alpha \oplus \beta)(\Gamma) \subset \mathrm{SL}_a(\mathbb{C}) \times \mathrm{SL}_b(\mathbb{C})$ is trivial. Hence Nagata's theorem (Theorem 4.1) implies that every rational representation of G is completely reducible. In particular, the restriction to G of the rational representation $\mathrm{SL}_a(\mathbb{C}) \times \mathrm{SL}_b(\mathbb{C}) \to \mathrm{GL}(M_{a\times b}(\mathbb{C}))$, given by

$$(A, B) \cdot X = AXB^{-1},$$

for all $(A, B) \in \mathrm{SL}_a(\mathbb{C}) \times \mathrm{SL}_b(\mathbb{C})$ and for all $X \in M_{a\times b}(\mathbb{C})$, is completely reducible.

Since $(\alpha \oplus \beta)(\Gamma)$ is Zariski dense in G, we obtain that $\mathcal{M}^{+}_{1}$ is a completely reducible Γ-module. Finally, the action of $\gamma \in \Gamma$ on $X \in \mathcal{M}^{+}_{\lambda^n}$, given by Equation (8), and the action $\gamma \cdot X = \alpha(\gamma) X \beta(\gamma^{-1})$ differ only by a homothety. Therefore, $\mathcal{M}^{+}_{\lambda^n}$ is a completely reducible Γ-module. The proof for $\mathcal{M}^{-}_{\lambda^{-n}}$ is similar. □

Corollary 4.7. $$\Delta^{+}_{i}(t) \doteq \Delta^{-}_{i}(t^{-1}).$$

Proof. The corollary follows directly from Theorem 2.6 and Corollary 4.6. □

5. Necessary condition

The goal of this section is to prove Theorem 1.3. More precisely, we will prove that if the representation $\rho_\lambda = (\lambda^{b\varphi} \otimes \alpha) \oplus (\lambda^{-a\varphi} \otimes \beta)$, as defined in (1), can be deformed to irreducible representations, then $\Delta_1^+(\lambda^n) = 0$. Recall that throughout the paper we assume that α and β are irreducible and infinitesimally regular.

Lemma 5.1. *Assume that ρ_λ belongs to a component of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ that contains irreducible representations. Then*

$$\dim Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda}) \geq n^2 + n - 2.$$

Proof. It is sufficient to prove the inequality for an irreducible representation $\rho \in R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$, because the dimension of $Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho})$ is an upper semi-continuous function on ρ and because irreducibility is a Zariski-open condition. We have

$$\dim Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}) = \dim H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}) + \dim B^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}).$$

Now, $\dim B^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}) = n^2 - 1$ because ρ is irreducible.

Next we apply Poincaré duality to the long exact sequence of the pair $(X, \partial X)$:

$$H^1(X; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}) \to H^1(\partial X; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}) \to H^2(X, \partial X; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}). \tag{13}$$

Poincaré duality (7) implies isomorphy between $H^1(X; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho})$ and the dual space $H^2(X, \partial X; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho})^*$. Moreover, the maps of (13) are dual to each other. So:

$$\tfrac{1}{2} \dim H^1(\partial X; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}) \leq \dim H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho}). \tag{14}$$

The claimed inequality of the statement follows from Lemma 3.2. □

Lemma 5.2. *Under the hypothesis of Lemma 5.1 we have*

$$\dim H^1(\Gamma; \mathcal{M}^+_{\lambda^n}) > \dim H^0(\Gamma; \mathcal{M}^+_{\lambda^n})$$

or

$$\dim H^1(\Gamma; \mathcal{M}^-_{\lambda^{-n}}) > \dim H^0(\Gamma; \mathcal{M}^-_{\lambda^{-n}}).$$

We shall see in Remark 5.4 below that we get both inequalities.

Proof. Here we use the decomposition of Γ-modules (see Section 4):

$$\mathfrak{sl}_n(\mathbb{C})_{Ad\rho_\lambda} = \mathfrak{sl}_a(\mathbb{C})_{\mathrm{Ad}\,\alpha} \oplus \mathfrak{sl}_b(\mathbb{C})_{\mathrm{Ad}\,\beta} \oplus \mathbb{C} \oplus \mathcal{M}^+_{\lambda^n} \oplus \mathcal{M}^-_{\lambda^{-n}}. \tag{15}$$

We aim to apply Lemma 5.1, so we compute the dimension of the space of 1-cocycles for each Γ-module in (15). For each Γ-module $\mathfrak{m}$, we use the formula

$$\begin{aligned} \dim Z^1(\Gamma; \mathfrak{m}) &= \dim H^1(\Gamma; \mathfrak{m}) + \dim B^1(\Gamma; \mathfrak{m}) \\ &= \dim H^1(\Gamma; \mathfrak{m}) + \dim \mathfrak{m} - \dim H^0(\Gamma; \mathfrak{m}). \end{aligned} \tag{16}$$

Ordering the terms as they appear in (16):

$$\begin{aligned}
\dim Z^1(\Gamma;\mathfrak{sl}_a(\mathbb{C})_{\mathrm{Ad}\,\alpha}) &= (a-1)+(a^2-1)-0,\\
\dim Z^1(\Gamma;\mathfrak{sl}_b(\mathbb{C})_{\mathrm{Ad}\,\beta}) &= (b-1)+(b^2-1)-0,\\
\dim Z^1(\Gamma;\mathbb{C}) &= 1+1-1,\\
\dim Z^1(\Gamma;\mathcal{M}^{\pm}_{\lambda^{\pm n}}) &= \dim H^1(\Gamma;\mathcal{M}^{\pm}_{\lambda^{\pm n}})+ab-\dim H^0(\Gamma;\mathcal{M}^{\pm}_{\lambda^{\pm n}}).
\end{aligned}$$

The first two lines use that α and β are irreducible and infinitesimally regular, the last one that $\dim \mathcal{M}^{\pm}_{\lambda^{\pm n}} = ab$. Adding up the dimensions of the terms in (15) and using Lemma 5.1 and the fact that $a+b=n$, we obtain

$$\begin{aligned}
n^2+n-2 \le n^2+n-3+\dim H^1(\Gamma;\mathcal{M}^{+}_{\lambda^{n}})-\dim H^0(\Gamma;\mathcal{M}^{+}_{\lambda^{n}})\\
+\dim H^1(\Gamma;\mathcal{M}^{-}_{\lambda^{-n}})-\dim H^0(\Gamma;\mathcal{M}^{-}_{\lambda^{-n}}),
\end{aligned}$$

which proves the lemma. □

For later use we remark on the following computation, made during the last proof. Notice that it does not use that ρ_λ can be deformed to irreducible representations (but it uses that α and β are irreducible and infinitesimally regular):

$$\begin{aligned}
(17)\quad \dim Z^1(\Gamma;\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda}) = n^2+n-3+\dim H^1(\Gamma;\mathcal{M}^{+}_{\lambda^{n}})-\dim H^0(\Gamma;\mathcal{M}^{+}_{\lambda^{n}})\\
+\dim H^1(\Gamma;\mathcal{M}^{-}_{\lambda^{-n}})-\dim H^0(\Gamma;\mathcal{M}^{-}_{\lambda^{-n}}).
\end{aligned}$$

Lemma 5.3. *Let* $\rho_\lambda:\Gamma\to \mathrm{SL}_n(\mathbb{C})$ *be given by* $\rho_\lambda=(\lambda^{b\varphi}\otimes\alpha)\oplus(\lambda^{-a\varphi}\otimes\beta)$. *Then* $\dim H^1(\Gamma;\mathcal{M}^{\pm}_{\lambda^{\pm n}}) > \dim H^0(\Gamma;\mathcal{M}^{\pm}_{\lambda^{\pm n}})$ *if and only if* $\Delta_1^{\mp}(\lambda^{\mp n})=0$.

Proof. Recall that $\Delta_i^{\pm}$ is the order of $H_i(X_\infty;\mathcal{M}_1^{\pm})\cong H_i(X;\mathcal{M}_1^{\pm}[\mathbb{Z}])$. We have a short exact sequence of Γ-modules

$$0\to \mathcal{M}_1^{-}[\mathbb{Z}] \xrightarrow{(t-\lambda^{-n})\cdot} \mathcal{M}_1^{-}[\mathbb{Z}] \to \mathcal{M}^{-}_{\lambda^{-n}}\to 0$$

which gives the following long exact sequence in homology [Brown 1994, III.§6]:

$$\begin{aligned}
\ldots\to H_1(\Gamma;\mathcal{M}_1^{-}[\mathbb{Z}]) \xrightarrow{(t-\lambda^{-n})\cdot} H_1(\Gamma;\mathcal{M}_1^{-}[\mathbb{Z}]) \to H_1(\Gamma;\mathcal{M}^{-}_{\lambda^{-n}}) \xrightarrow{\partial}\\
H_0(\Gamma;\mathcal{M}_1^{-}[\mathbb{Z}]) \xrightarrow{(t-\lambda^{-n})\cdot} H_0(\Gamma;\mathcal{M}_1^{-}[\mathbb{Z}]) \to H_0(\Gamma;\mathcal{M}^{-}_{\lambda^{-n}})\to 0.
\end{aligned}$$

Thus $\Delta_1^{-}(\lambda^{-n})=0$ if and only if $\ker\partial$ is nontrivial.

Next we claim that $\ker\partial$ is nontrivial if and only if $H^1(\Gamma;\mathcal{M}^{+}_{\lambda^{n}})$ has higher dimension than $H^0(\Gamma;\mathcal{M}^{+}_{\lambda^{n}})$. It follows from Lemma 2.7 and Equation (3) that the $\mathbb{C}[\mathbb{Z}]$-module $H_0(\Gamma;\mathcal{M}_1^{-}[\mathbb{Z}])$ is torsion, i.e., it is a finite dimensional $\mathbb{C}$-vector space. Hence, by exactness, $\operatorname{rank}\partial=\dim H_0(\Gamma;\mathcal{M}^{-}_{\lambda^{-n}})$ and

$$\begin{aligned}\dim\ker\partial &= \dim H_1(\Gamma;\mathcal{M}^-_{\lambda^{-n}}) - \operatorname{rank}\partial\\ &= \dim H_1(\Gamma;\mathcal{M}^-_{\lambda^{-n}}) - \dim H_0(\Gamma;\mathcal{M}^-_{\lambda^{-n}})\\ &= \dim H^1(\Gamma;\mathcal{M}^+_{\lambda^n}) - \dim H^0(\Gamma;\mathcal{M}^+_{\lambda^n}),\end{aligned}$$

by Kronecker duality (12), which proves the claim. Of course the same proof applies by symmetry for the opposite signs $\pm$ and $\mp$. □

Proof of Theorem 1.3. By Lemmas 5.2 and 5.3 we get that if ρ_λ can be deformed to irreducible representations, then $\Delta_1^+(\lambda^n)=0$ or $\Delta_1^-(\lambda^{-n})=0$. Corollary 4.7 yields $\Delta_1^+(\lambda^n)=\Delta_1^-(\lambda^{-n})=0$. □

Remark 5.4. Notice that in the situation of Theorem 1.3, from Lemma 5.3 and Corollary 4.7 we get both inequalities

$$\dim H^1(\Gamma;\mathcal{M}^\pm_{\lambda^{\pm n}}) > \dim H^0(\Gamma;\mathcal{M}^\pm_{\lambda^{\pm n}}).$$

We will later need the following construction. Given a 1-cochain $c\in C^1(\Gamma;\mathcal{M}^+_{\lambda^n})$, i.e., a map $c:\Gamma\to\mathcal{M}^+_{\lambda^n}$, consider the map $\rho_\lambda^c:\Gamma\to \mathrm{SL}_n(\mathbb{C})$ given by

$$\rho_\lambda^c(\gamma)=\begin{pmatrix}\mathrm{Id}_a & c(\gamma)\\ 0 & \mathrm{Id}_b\end{pmatrix}\rho_\lambda(\gamma),\qquad \gamma\in\Gamma. \tag{18}$$

Lemma 5.5. *The map $\rho_\lambda^c:\Gamma\to\mathrm{SL}_n(\mathbb{C})$ given by* (18) *is a representation if and only if c is a cocycle*, *i.e.*, $c\in Z^1(\Gamma;\mathcal{M}^+_{\lambda^n})$. *For such c*, *there is equivalence between these conditions*:

(i) *ρ_λ^c is conjugate to ρ_λ.*

(ii) *c is a coboundary* (*i.e.*, $c\in B^1(\Gamma;\mathcal{M}^+_{\lambda^n})$).

(iii) *ρ_λ^c is completely reducible.*

Proof. The equivalence between being a representation and the cocycle condition is a straightforward computation; so is the equivalence between being conjugate to ρ_λ and the coboundary condition. The equivalence with complete reducibility comes from the fact that there is a unique orbit of completely reducible representations in the fiber of the map $R(\Gamma,\mathrm{SL}_n(\mathbb{C}))\to X(\Gamma,\mathrm{SL}_n(\mathbb{C}))$ [Lubotzky and Magid 1985]. Hence two completely reducible representations having the same character are conjugates. □

The following corollary generalizes a result of G. Burde [1967] and G. de Rham [1967]:

Corollary 5.6. *There exists a reducible*, *not completely reducible representation $\rho_\lambda^c:\Gamma\to\mathrm{SL}_n(\mathbb{C})$ such that $\chi_{\rho_\lambda^c}=\chi_{\rho_\lambda}$ if and only if λ^n is a root of the product of twisted Alexander polynomials $\Delta_1^+(t)\Delta_0^+(t)$.*

Proof. By Lemma 5.5, such a representation exists if and only if $H^1(\Gamma; \mathcal{M}^{+}_{\lambda^n})$ or $H^1(\Gamma; \mathcal{M}^{-}_{\lambda^{-n}})$ does not vanish. By Kronecker duality this is equivalent to saying that $H_1(\Gamma; \mathcal{M}^{+}_{\lambda^n})$ or $H_1(\Gamma; \mathcal{M}^{-}_{\lambda^{-n}})$ does not vanish. Then, the long exact sequence in the proof of Lemma 5.3 shows that this is equivalent to one of $H_1(\Gamma; \mathcal{M}^{\pm}_1[\mathbb{Z}])$ or $H_0(\Gamma; \mathcal{M}^{\pm}_1[\mathbb{Z}])$ to have $(t-\lambda^{\pm n})$-torsion. With the duality of polynomials, Corollary 4.7, this proves the lemma. □

This corollary also applies when $\alpha=\beta=1$ and $\lambda=\pm 1$. Since $\Delta_0(t)=(t-1)$, the vanishing $\Delta_0((\pm 1)^2)=0$ corresponds to the representations

$$\gamma \mapsto \pm \begin{pmatrix} 1 & d(\gamma) \\ 0 & 1 \end{pmatrix}, \qquad \gamma \in \Gamma,$$

where $d : \Gamma \to (\mathbb{C}, +)$ is any group morphism.

6. Infinitesimal deformations and cup products

Throughout this section and the next one we assume the hypothesis of Theorem 1.4, namely that $\Delta^{+}_0(\lambda^n) \neq 0$ and λ^n is a simple root of Δ^{+}_1. By Corollary 4.7, we also have that $\Delta^{-}_0(\lambda^{-n}) \neq 0$ and λ^{-n} is a simple root of Δ^{-}_1. Thus the $\mathbb{C}[\mathbb{Z}]$-module $H_0(\Gamma; \mathcal{M}^{\pm}_1[\mathbb{Z}])$ has no $(t-\lambda^{\pm n})$-torsion and $H_1(\Gamma; \mathcal{M}^{\pm}_1[\mathbb{Z}])$ has a single $\mathbb{C}[t^{\pm 1}]/(t-\lambda^{\pm n})$-factor. Furthermore, the following proposition gives more details on the cohomology.

Proposition 6.1. *Assume $\Delta^{+}_0(\lambda^n) \neq 0$ and λ^n is a simple root of Δ^{+}_1. Then*

(i) $\dim H^i(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}}) = \begin{cases} 1 & \text{if } i = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$

(ii) *The $(t-\lambda^{\pm n})$-torsion of $H^q(\Gamma; \mathcal{M}^{\pm}_1[\mathbb{Z}])$ is zero for $q \neq 2$ and cyclic of the form $\mathbb{C}[\mathbb{Z}]/(t-\lambda^{\pm n})$ for $q=2$.*

Proof. In order to prove the first assertion, we use the long exact sequence in the proof of Lemma 5.3. The hypothesis on the twisted Alexander polynomials gives that the $(t-\lambda^{\pm n})$-torsion of $H_i(\Gamma; \mathcal{M}^{\pm}_1[\mathbb{Z}])$ is zero for $i \neq 1$ and $t-\lambda^{\pm n}$ for $i=1$. The long exact sequence gives that $H_i(X; \mathcal{M}^{\pm}_{\lambda^{\pm n}})$ has dimension 1 if $i=1,2$ and dimension 0 otherwise. Hence the first assertion follows from Kronecker duality, (12).

For the second assertion, we use the universal coefficient theorem for cohomology: for any representation $\rho : \Gamma \to \mathrm{GL}(V)$ we have

$$\text{(19)} \quad \overline{H}^q(X; V^*[\mathbb{Z}]) \cong \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}]}\big(H_q(X; V[\mathbb{Z}]), \mathbb{C}[\mathbb{Z}]\big) \oplus \mathrm{Ext}_{\mathbb{C}[\mathbb{Z}]}\big(H_{q-1}(X; V[\mathbb{Z}]), \mathbb{C}[\mathbb{Z}]\big),$$

where $\overline{H}^q(X; V^*[\mathbb{Z}])$ denotes the group $H^q(X; V^*[\mathbb{Z}])$ with the conjugate $\mathbb{C}[\mathbb{Z}]$-module structure. For a detailed argument see pp. 638–639 in [Kirk and Livingston

1999]. We apply (19) to the representation $\alpha \otimes \beta^*$ and its dual $\beta \otimes \alpha^*$. By the hypothesis on the twisted Alexander polynomials, the $(t-\lambda^{\pm n})$-torsion of $H_i(\Gamma, \mathcal{M}_1^{\pm}[\mathbb{Z}])$ is zero for $i \neq 1$ and $t-\lambda^{\pm n}$ for $i=1$. Notice that $H_q(X; \mathcal{M}_1^{\pm}[\mathbb{Z}])$ are torsion $\mathbb{C}[\mathbb{Z}]$-modules, and the $(t-\lambda^{\pm n})$-torsion of $\overline{H}^2(X; \mathcal{M}_1^{\pm}[\mathbb{Z}])$ is $(t-\lambda^{\mp n})$. The claim follows since $-t\lambda^{\pm n}(t^{-1}-\lambda^{\mp n}) = (t-\lambda^{\pm n})$. $\square$

From now on we fix cocycles

$$d_\pm \in Z^1(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}})$$

whose cohomology classes do not vanish. Because $H^1(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}}) \cong \mathbb{C}$, the elements $d_\pm$ are unique up to adding a coboundary and up to multiplying by a nonzero scalar. Our next goal is to show that the cohomology class of the cup product $\varphi \smile d_\pm$ does not vanish in $H^2(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}})$. For that purpose we shall use the dual numbers.

Dual numbers. The *algebra of dual numbers* is defined to be

$$\mathbb{C}_\varepsilon = \mathbb{C}[\varepsilon]/\varepsilon^2.$$

Similarly define $\mathbb{C}_\varepsilon[\mathbb{Z}] = \mathbb{C}[\mathbb{Z}] \otimes_{\mathbb{C}} \mathbb{C}_\varepsilon$ and

$$\mathcal{M}^{\pm}_{\lambda^{\pm n}(1\pm\varepsilon)} = \big(\mathcal{M}_1^{\pm}[\mathbb{Z}] \otimes_{\mathbb{C}} \mathbb{C}_\varepsilon\big)/\big(t-\lambda^{\pm n}(1\pm\varepsilon)\big).$$

Lemma 6.2. *If $\lambda^n \in \mathbb{C}^*$ is a simple root of Δ_1^+ such that $\Delta_0^+(\lambda^n) \neq 0$, then* $\dim H^1(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}(1\pm\varepsilon)}) = 1$.

Proof. Notice that $\Delta_0^-(\lambda^{-n}) \neq 0$ and λ^{-n} is a simple root of Δ_1^- by Corollary 4.7. We have that $H^1(\Gamma; \mathcal{M}_1^+[\mathbb{Z}] \otimes \mathbb{C}_\varepsilon) \cong H^1(\Gamma; \mathcal{M}_1^+[\mathbb{Z}]) \otimes \mathbb{C}_\varepsilon$ since $\mathbb{C}_\varepsilon$ is a trivial Γ-module (isomorphic to $\mathbb{C}^2$). As before, the short exact sequence

$$0 \to \mathcal{M}_1^+[\mathbb{Z}] \otimes_{\mathbb{C}} \mathbb{C}_\varepsilon \xrightarrow{(t-\lambda^n(1+\varepsilon))\cdot} \mathcal{M}_1^+[\mathbb{Z}] \otimes_{\mathbb{C}} \mathbb{C}_\varepsilon \to \mathcal{M}^+_{\lambda^n(1+\varepsilon)} \to 0$$

gives a long exact sequence in cohomology (see [Brown 1994, III.§6]),

$$\begin{aligned}\cdots \to H^i(\Gamma; \mathcal{M}_1^+[\mathbb{Z}]) \otimes \mathbb{C}_\varepsilon &\xrightarrow{(t-(\lambda^n(1+\varepsilon))\cdot} H^i(\Gamma; \mathcal{M}_1^+[\mathbb{Z}]) \otimes \mathbb{C}_\varepsilon \\ &\to H^i(\Gamma; \mathcal{M}^+_{\lambda^n(1+\varepsilon)}) \to H^{i+1}(\Gamma; \mathcal{M}_1^+[\mathbb{Z}]) \otimes \mathbb{C}_\varepsilon \to \cdots .\end{aligned}$$

Note that for $\mu \neq \lambda^n$, $\mu \in \mathbb{C}$, multiplication by $t-\lambda^n(1+\varepsilon)$ induces an automorphism of $\mathbb{C}_\varepsilon[\mathbb{Z}]/(t-\mu)^k$. Therefore, we are interested in the $(t-\lambda^n)$-torsion of $H^q(\Gamma; \mathcal{M}_1^+[\mathbb{Z}])$ described by Proposition 6.1: it vanishes for $q \neq 2$ and it is $\mathbb{C}[\mathbb{Z}]/(t-\lambda^n)$ for $q=2$. Hence, multiplication by $(t-\lambda^n(1+\varepsilon))$ on $H^i(\Gamma; \mathcal{M}_1^+[\mathbb{Z}] \otimes \mathbb{C}_\varepsilon)$ is an isomorphism for $i \neq 2$. In order to understand the effect of the multiplication on $H^2(\Gamma; \mathcal{M}_1^+[\mathbb{Z}] \otimes \mathbb{C}_\varepsilon)$ it is sufficient to consider multiplication by $(t-\lambda^n(1+\varepsilon))$ on

$$\mathbb{C}[\mathbb{Z}]/(t-\lambda^n) \otimes \mathbb{C}_\varepsilon \cong \mathbb{C}[\mathbb{Z}]/(t-\lambda^n) \oplus \varepsilon\mathbb{C}[\mathbb{Z}]/(t-\lambda^n).$$

Since $t-\lambda^n$ vanishes in this ring, multiplication by $(t-\lambda^n(1+\varepsilon))$ is equivalent to multiplication by $-\varepsilon\lambda^n$ on $\mathbb{C}_\varepsilon \cong \mathbb{C}\oplus\varepsilon\mathbb{C}$. Therefore, its kernel and cokernel have $\mathbb{C}$-dimension 1, which proves $\dim_{\mathbb{C}} H^1(\Gamma;\, \mathcal{M}^+_{\lambda^n(1+\varepsilon)}) = 1$.

By symmetry the same argument yields $\dim_{\mathbb{C}} H^1(\Gamma;\, \mathcal{M}^-_{\lambda^{-n}(1-\varepsilon)}) = 1$. □

Cup product and Bockstein homomorphism. Let A_1, A_2 and A_3 be Γ-modules. The cup product of two cochains $c_i \in C^1(\Gamma;\, A_i)$, $i=1,2$ is the cochain $c_1 \smile c_2 \in C^2(\Gamma;\, A_1\otimes A_2)$ defined by

$$c_1 \smile c_2(\gamma_1,\gamma_2) := c_1(\gamma_1)\otimes \gamma_1 c_2(\gamma_2). \tag{20}$$

Here $A_1\otimes A_2$ is a Γ-module via the diagonal action.

It is possible to combine the cup product with any Γ-invariant, bilinear map $b: A_1\otimes A_2 \to A_3$. So we obtain a cup product

$$_b\!\smile\, : C^1(\Gamma;\, A_1)\otimes C^1(\Gamma;\, A_2) \xrightarrow{\smile} C^1(\Gamma;\, A_1\otimes A_2) \xrightarrow{b} C^2(\Gamma;\, A_3).$$

For details see [Brown 1994, V.3]. In what follows we are mainly interested in the case where the bilinear form is simply the matrix multiplication, i.e.,

$$\mathbb{C}\otimes \mathcal{M}^\pm_{\lambda^{\pm n}} \to \mathcal{M}^\pm_{\lambda^{\pm n}} \quad \text{or} \quad \mathfrak{sl}_a(\mathbb{C})\otimes \mathcal{M}^+_{\lambda^n} \to \mathcal{M}^+_{\lambda^n}.$$

Hence we will write simply "$\smile$" for such a cup product when no confusion can arise.

Let $b: A_1\otimes A_2 \to A_3$ be bilinear and let $\tau: A_2\otimes A_1 \to A_1\otimes A_2$ be the twist operator. Then for $c_i \in C^1(\Gamma;\, A_i)$, $i=1,2$, we define the cup product

$$_{b\circ\tau}\!\smile\, : C^1(\Gamma;\, A_2)\otimes C^1(\Gamma;\, A_1) \to C^2(\Gamma;\, A_3).$$

Again we are mainly interested in matrix multiplication and we will write simply "$_\tau\!\smile$" for such a cup product when no confusion can arise.

Example 6.3. Let $c_a \in C^1(\Gamma;\, \mathfrak{sl}_a(\mathbb{C}))$ and $d \in C^1(\Gamma;\, \mathcal{M}^+_{\lambda^n})$ be given. Then

$$c_a \smile d(\gamma_1,\gamma_2) = c_a(\gamma_1)\gamma_1\, d(\gamma_2) = \lambda^{n\varphi(\gamma_1)} c_a(\gamma_1)\alpha(\gamma_1)\, d(\gamma_2)\beta(\gamma_1)^{-1}$$

and

$$d\; _\tau\!\smile c_a(\gamma_1,\gamma_2) = \gamma_1 c_a(\gamma_2)\, d(\gamma_1) = \alpha(\gamma_1)c_a(\gamma_2)\alpha(\gamma_1)^{-1}\, d(\gamma_1).$$

Remark 6.4. If $z_a \in Z^1(\Gamma;\, \mathfrak{sl}_a(\mathbb{C}))$ and $d_+ \in Z^1(\Gamma;\, \mathcal{M}^+_{\lambda^n})$ are cocycles, then for $f: \Gamma \to \mathcal{M}^+_{\lambda^n}$ given by $f(\gamma) = z_a(\gamma)d_+(\gamma)$ we have

$$\delta f(\gamma_1,\gamma_2) + z_a \smile d_+(\gamma_1,\gamma_2) + d_+\; _\tau\!\smile z_a(\gamma_1,\gamma_2) = 0,$$

i.e., $d_+\; _\tau\!\smile z_a \sim -z_a \smile d$ in $C^2(\Gamma;\, \mathcal{M}^+_{\lambda^n})$.

Lemma 6.5. *Consider the nonsplit exact sequence of* Γ*-modules*

$$0 \to \mathcal{M}^{\pm}_{\lambda^{\pm n}} \xrightarrow{\varepsilon\cdot} \mathcal{M}^{\pm}_{\lambda^{\pm n}(1\pm\varepsilon)} \longrightarrow \mathcal{M}^{\pm}_{\lambda^{\pm n}} \to 0.$$

Then the image of the cohomology class represented by $d_\pm$ *(in* $H^1(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}})$*) under the Bockstein homomorphism is represented by the cup product* $d_\pm \smile \varphi$ *(in* $H^2(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}})$*).*

Proof. In order to calculate the Bockstein homomorphism $\boldsymbol{b} : H^1(\Gamma; \mathcal{M}^{+}_{\lambda^n}) \to H^2(\Gamma; \mathcal{M}^{+}_{\lambda^n})$ we proceed as follows (according to the snake lemma): given a cocycle $d_+ \in Z^1(\Gamma; \mathcal{M}^{+}_{\lambda^n})$ we choose a cochain $\tilde{d}_+ \in C^1(\Gamma; \mathcal{M}^{+}_{\lambda^n(1+\varepsilon)})$ which projects onto d_+ and then we calculate $\delta_\varepsilon \tilde{d}_+ \in C^2(\Gamma; \mathcal{M}^{+}_{\lambda^n(1+\varepsilon)})$ where δ_ε denotes the coboundary operator of $C^*(\Gamma; \mathcal{M}^{+}_{\lambda^n(1+\varepsilon)})$. Since d_+ is a cocycle we obtain $\delta_\varepsilon \tilde{d}_+ = \varepsilon \cdot z$ for a 2-cocycle $z \in Z^2(\Gamma; \mathcal{M}^{+}_{\lambda^n})$ which represents the image of the Bockstein map. By abusing notation, we also denote the map constructed in this way by $\boldsymbol{b} : Z^1(\Gamma; \mathcal{M}^{+}_{\lambda^n}) \to Z^2(\Gamma; \mathcal{M}^{+}_{\lambda^n})$, even if it is only well defined in cohomology. In particular $\boldsymbol{b}(d_+) \sim z$. In order to calculate $z \in Z^2(\Gamma; \mathcal{M}^{+}_{\lambda^n})$ we choose $\tilde{d}_+ = d_+ + \varepsilon \cdot 0$:

$$\begin{aligned}
\delta_\varepsilon \tilde{d}_+(\gamma_1, \gamma_2) &= \gamma_1 \tilde{d}_+(\gamma_2) - \tilde{d}_+(\gamma_1\gamma_2) + \tilde{d}_+(\gamma_1) \\
&= \lambda^{n\varphi(\gamma_1)}(1 + \varepsilon\varphi(\gamma_1))\alpha(\gamma_1) d_+(\gamma_2)\beta(\gamma_1)^{-1} - d_+(\gamma_1\gamma_2) + d_+(\gamma_1) \\
&= \varepsilon\varphi(\gamma_1)\gamma_1 d_+(\gamma_2) = \varepsilon \cdot \varphi \smile d_+(\gamma_1, \gamma_2).
\end{aligned}$$

Therefore, $\boldsymbol{b}(d_+) \sim \varphi \smile d_+$. The calculation for $\boldsymbol{b}(d_-) \sim \varphi \smile d_-$ is similar. □

Corollary 6.6. *Assume* $\dim_{\mathbb{C}} H^1(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}}) = 1$, $H^0(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}}) = 0$ *and let* $d_\pm \in Z^1(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}})$ *be not cohomologous to zero. Then* $\dim H^1(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}(1\pm\varepsilon)}) = 1$ *if and only if the cup product* $\varphi \smile d_\pm$ *does not vanish in* $H^2(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}})$.

Proof. Consider the nonsplit exact sequence of Γ-modules

$$0 \to \mathcal{M}^{\pm}_{\lambda^{\pm n}} \xrightarrow{\varepsilon\cdot} \mathcal{M}^{\pm}_{\lambda^{\pm n}(1\pm\varepsilon)} \to \mathcal{M}^{\pm}_{\lambda^{\pm n}} \to 0$$

and the corresponding long exact sequence in cohomology:

$$0 \to H^1(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}}) \xrightarrow{\varepsilon\cdot} H^1(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}(1\pm\varepsilon)}) \to H^1(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}}) \xrightarrow{\boldsymbol{b}} H^2(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}}). \tag{21}$$

The lemma then follows from the sequence (21), as by Lemma 6.5 $\boldsymbol{b}(d_\pm) \sim \varphi \smile d_\pm$ and $\dim H^1(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}}) = 1$. □

Combining Proposition 6.1, Lemma 6.2, and Corollary 6.6, we deduce:

Corollary 6.7. *Under the hypothesis of Theorem 1.4, the cup product* $\varphi \smile d_\pm$ *does not vanish in* $H^2(\Gamma; \mathcal{M}^{\pm}_{\lambda^{\pm n}})$.

7. A not completely reducible representation ρ^+

In this section we construct a $\rho^+ \in R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ that has the same character as ρ_λ but is not completely reducible. We show that ρ^+ is a smooth point of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ and that it can be deformed to irreducible representations. This proves Theorem 1.4, because the orbit by conjugation of ρ^+ accumulates to ρ_λ.

Assume throughout this section that the hypotheses of Theorem 1.4 hold true. Namely (using Corollary 4.7), $\Delta_0^{\pm n}(\lambda^{\pm n}) \neq 0$ and $\lambda^{\pm n}$ is a simple root of $\Delta_1^{\pm}$. Recall we have fixed $d_+ \in Z^1(\Gamma; \mathcal{M}^+_{\lambda^n})$, a cocycle not homologous to zero. Let

$$\rho^+ = \begin{pmatrix} \mathrm{Id}_a & d_+ \\ 0 & \mathrm{Id}_b \end{pmatrix} \rho_\lambda.$$

By Lemma 5.5, ρ^+ is not completely reducible, hence it is not conjugate to ρ_λ, even if it has the same character. We shall prove that ρ^+ is a regular point of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ and that the local dimension is $\dim \mathrm{SL}_n(\mathbb{C}) + n - 1 = n^2 + n - 2$. Then we will argue that the reducible representations around ρ_λ form a Zariski closed algebraic set of dimension $n^2 + n - 3$, which will prove Theorem 1.4.

Let $P^+ = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \subset \mathrm{SL}_n(\mathbb{C})$ be the maximal parabolic subgroup that preserves $\mathbb{C}^a \oplus 0$. Its Lie algebra is denoted by $\mathfrak{p}^+ \subset \mathfrak{sl}_n(\mathbb{C})$. We have two short exact sequences of Γ-modules via the action of $\mathrm{Ad}\,\rho^+$:

$$0 \to \mathcal{M}^+_{\lambda^n} \to \mathfrak{p}^+ \to \mathcal{D} \to 0, \tag{22}$$

where

$$\mathcal{D} = \mathfrak{sl}_a(\mathbb{C}) \oplus \mathfrak{sl}_b(\mathbb{C}) \oplus \mathbb{C}, \tag{23}$$

and

$$0 \to \mathfrak{p}^+ \to \mathfrak{sl}_n(\mathbb{C}) \to \mathcal{M}^-_{\lambda^{-n}} \to 0. \tag{24}$$

We will use the corresponding long exact sequences in cohomology to compute $H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho^+})$. The first step is the following lemma.

Lemma 7.1. $H^0(\Gamma; \mathfrak{p}^+) = 0$.

Proof. The long exact sequence associated to (22) starts with

$$0 = H^0(\Gamma; \mathcal{M}^+_{\lambda^n}) \to H^0(\Gamma; \mathfrak{p}^+) \to H^0(\Gamma; \mathcal{D}) \xrightarrow{\boldsymbol{b}} H^1(\Gamma; \mathcal{M}^+_{\lambda^n}).$$

The group $H^0(\Gamma; \mathcal{D}) \cong \mathbb{C}$ is generated by the invariant element $\left(\begin{smallmatrix} -b\,\mathrm{Id}_a & 0 \\ 0 & a\,\mathrm{Id}_b \end{smallmatrix}\right) \in \mathcal{D}^\Gamma$. A similar calculation as in the proof of Lemma 6.5 using the snake lemma gives $\boldsymbol{b}\left(\begin{smallmatrix} -b\,\mathrm{Id}_a & 0 \\ 0 & a\,\mathrm{Id}_b \end{smallmatrix}\right) \sim n d_+$. Therefore $\boldsymbol{b} : H^0(\Gamma; \mathcal{D}) \to H^1(\Gamma; \mathcal{M}^+_{\lambda^n})$ is injective and hence $H^0(\Gamma; \mathfrak{p}^+) = 0$. □

We continue the long exact sequence in cohomology associated to (22):

$$0 \to \mathbb{C} \to H^1(\Gamma; \mathcal{M}^+_{\lambda^n}) \to H^1(\Gamma; \mathfrak{p}^+) \to H^1(\Gamma; \mathcal{D}) \xrightarrow{\boldsymbol{b}} H^2(\Gamma; \mathcal{M}^+_{\lambda^n}).$$

Since $H^i(\Gamma; \mathcal{M}^+_{\lambda^n}) \cong \mathbb{C}$ for $i = 1, 2$ by Proposition 6.1, it shortens to

$$0 \to H^1(\Gamma; \mathfrak{p}^+) \to H^1(\Gamma; \mathcal{D}) \xrightarrow{\boldsymbol{b}} H^2(\Gamma; \mathcal{M}^+_{\lambda^n}).$$

Next we aim to compute $\boldsymbol{b} : H^1(\Gamma; \mathcal{D}) \to H^2(\Gamma; \mathcal{M}^+_{\lambda^n})$. For this we use the decomposition (23). Every element in $H^1(\Gamma; \mathcal{D})$ is represented by a cocycle

$$\vartheta = \begin{pmatrix} z_a & 0 \\ 0 & z_b \end{pmatrix} + z\varphi \begin{pmatrix} -b\,\mathrm{Id}_a & 0 \\ 0 & a\,\mathrm{Id}_b \end{pmatrix} \tag{25}$$

where $z_a \in Z^1(\Gamma; \mathfrak{sl}_a(\mathbb{C}))$, $z_b \in Z^1(\Gamma; \mathfrak{sl}_b(\mathbb{C}))$, and $z \in \mathbb{C}$.

Lemma 7.2. *For a cocycle* $\vartheta \in Z^1(\Gamma; \mathcal{D})$ *as in* (25),

$$\boldsymbol{b}(\vartheta) \sim z_a \smile d_+ + d_+ \smile z_b + znd_+ \smile \varphi.$$

Proof. As in Lemma 6.5 we compute $\boldsymbol{b}(\vartheta)$ by using the snake lemma. Namely, let δ^+ be the coboundary operator of $C^*(\Gamma; \mathfrak{p}^+)$, and let $\tilde{\vartheta} \in C^1(\Gamma; \mathfrak{p}^+)$ be the composition of ϑ with the inclusion $\mathcal{D} \hookrightarrow \mathfrak{p}^+$. Then

$$\delta^+\tilde{\vartheta}(\gamma_1, \gamma_2) = \begin{pmatrix} 0 & -\gamma_1 z_a(\gamma_2) d_+(\gamma_1) + d_+(\gamma_1)\gamma_1 z_b(\gamma_2) + znd_+(\gamma_1)\varphi(\gamma_2) \\ 0 & 0 \end{pmatrix}$$

and hence $\boldsymbol{b}(\vartheta) \sim -d_+ {}_\tau\!\smile z_a + d_+ \smile z_b + znd_+ \smile \varphi$.

Finally, Remark 6.4 proves the lemma. □

Since $\varphi \smile d_\pm$ is not cohomologous to zero (Corollary 6.7) and $H^2(\Gamma; \mathcal{M}^+_{\lambda^n}) \cong \mathbb{C}$ (Proposition 6.1), we deduce:

Corollary 7.3. *The cohomology group* $H^1(\Gamma; \mathfrak{p}^+) \cong \mathbb{C}^{n-2}$ *is naturally identified to the kernel of the rank one map*:

$$H^1(\Gamma; \mathcal{D}) \cong H^1(\Gamma; \mathfrak{sl}_a(\mathbb{C})) \oplus H^1(\Gamma; \mathfrak{sl}_b(\mathbb{C})) \oplus \mathbb{C} \xrightarrow{\boldsymbol{b}} H^2(\Gamma; \mathcal{M}^+_{\lambda^n}) \cong \mathbb{C}.$$

Next we consider the long exact sequence corresponding to (24):

$$0 \to H^1(\Gamma; \mathfrak{p}^+) \to H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho^+}) \to H^1(\Gamma; \mathcal{M}^-_{\lambda^{-n}}). \tag{26}$$

Hence

$$\begin{aligned} \dim H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho^+}) &\le \dim H^1(\Gamma; \mathfrak{p}^+) + \dim H^1(\Gamma; \mathcal{M}^-_{\lambda^{-n}}) \\ &= n - 2 + 1 = n - 1. \end{aligned} \tag{27}$$

On the other hand we apply Poincaré duality to the long exact sequence of the pair $(X, \partial X)$ (see (13)) and we obtain as in Equation (14):

$$\dim H^1(\partial X; \mathfrak{sl}_n(\mathbb{C})_{\operatorname{Ad}\rho^+}) \leq 2 \dim H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\operatorname{Ad}\rho^+}) \leq 2(n-1). \tag{28}$$

Proposition 7.4. *ρ^+ is a regular point of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ of dimension n^2+n-2.*

Proof. The dimension inequality of Lemma 3.2 and the inequality (28) yield $\dim H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\operatorname{Ad}\rho^+}) = n-1$, and we apply Proposition 3.3. □

Before proving that the irreducible component of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ containing ρ^+ also contains irreducible representations, we need a remark and two lemmas.

Remark 7.5. It follows from the proof of Proposition 7.4 that inequalities (27) and (28) are equalities, therefore (26) becomes a short exact sequence:

$$0 \to H^1(\Gamma; \mathfrak{p}^+) \to H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\operatorname{Ad}\rho^+}) \to H^1(\Gamma; \mathcal{M}^-_{\lambda^{-n}}) \to 0.$$

Lemma 7.6. *The representation ρ^+ is a smooth point of $R(\Gamma, P^+)$.*

Proof. The key tool here is the vanishing of Goldman's obstructions [1984] to integrability, which relies on the naturality of these obstructions and the vanishing for $\mathfrak{sl}_n(\mathbb{C})$. (In our proof of Proposition 7.4 this vanishing is also used implicitly, since our Proposition 3.3 is taken from [Heusener and Medjerab 2014], where the vanishing is invoked.)

By Remark 7.5, the long exact sequence in cohomology associated to (24) yields an injection

$$0 \to H^2(\Gamma; \mathfrak{p}^+) \to H^2(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\operatorname{Ad}\rho^+}).$$

Now Goldman's obstructions to integrability are natural for the inclusion $\mathfrak{p}^+ \to \mathfrak{sl}_n(\mathbb{C})$. In addition, the obstructions of a cocycle in $\mathfrak{p}^+$ remain in $\mathfrak{p}^+$, because $\mathfrak{p}^+ \to \mathfrak{sl}_n(\mathbb{C})$ is a subalgebra (closed under the Lie bracket) and a Γ-submodule of $\mathfrak{sl}_n(\mathbb{C})$. Since ρ^+ is a smooth point of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$, for any cocycle in $Z^1(\Gamma; \mathfrak{p}^+)$ the infinite sequence of obstructions to integrability in $H^2(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\operatorname{Ad}\rho^+})$ vanish, so the infinite sequence of obstructions to integrability in $H^2(\Gamma; \mathfrak{p}^+)$ also vanish. This establishes that any infinitesimal deformation is formally integrable and it follows from Artin's theorem [1968] that it is actually integrable, which proves the lemma. □

Let $1 \leq k \leq n$ and let $R_k \subset R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ denote the subset of representations ρ such that $\rho(\Gamma)$ preserves a k-dimensional subspace of $\mathbb{C}^n$.

Lemma 7.7. *For all $1 \leq k \leq n$, the subset $R_k \subset R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ is Zariski-closed.*

Proof. The assertion is clear for $k = n$ since $R_n = R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$. Hence suppose that $1 \leq k < n$ and let $P(k) \subset \mathrm{SL}_n(\mathbb{C})$ denote the parabolic subgroup which preserves

$\mathbb{C}^k \times \{0\} \subset \mathbb{C}^n$. The set $R(\Gamma, P(k)) \subset R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ is Zariski-closed since it is given by a finite number of equations. Moreover, we have

$$R_k = \mathrm{SL}_n(\mathbb{C}) \cdot R(\Gamma, P(k)) = \mathrm{SL}_n(\mathbb{C})/P(k) \cdot R(\Gamma, P(k))$$

since $P(k)$ preserves $R(\Gamma, P(k))$. Finally, R_k is Zariski-closed since the quotient $\mathrm{SL}_n(\mathbb{C})/P(k)$ is complete (see [Humphreys 1995, §0.15]). □

Lemma 7.8. *The unique proper invariant subspace of* $\rho^+(\Gamma)$ *is* $\mathbb{C}^a \times \{0\}$.

Proof. We compute the possible nonzero invariant subspaces of $\rho^+(\Gamma)$ by taking a nonzero vector $v \in \mathbb{C}^n$ and considering the linear span of its orbit $\langle \rho^+(\Gamma)v \rangle$. When $v \in \mathbb{C}^a \times \{0\}$, then $\langle \rho^+(\Gamma)v \rangle = \mathbb{C}^a \times \{0\}$ because α is irreducible. So we assume that the projection of v to the quotient $\mathbb{C}^n/\mathbb{C}^a \times \{0\}$ does not vanish, and since β is irreducible, the projection of the linear span $\langle \rho^+(\Gamma)v \rangle$ is the whole $\mathbb{C}^n/\mathbb{C}^a \times \{0\}$. In particular the dimension of $\langle \rho^+(\Gamma)v \rangle$ is at least b. Notice that $\dim_{\mathbb{C}} \langle \rho^+(\Gamma)v \rangle = b$ cannot occur, because this would yield a direct sum $\mathbb{C}^n = \mathbb{C}^a \times \{0\} \oplus \langle \rho^+(\Gamma)v \rangle$; by Lemma 5.5 this would contradict the choice of ρ^+ and the nontriviality of the cohomology class of d_+. Therefore $\dim_{\mathbb{C}} \langle \rho^+(\Gamma)v \rangle > b$, so that $\langle \rho^+(\Gamma)v \rangle$ contains at least a nontrivial vector in $\mathbb{C}^a \times \{0\}$ (the kernel of the projection). Irreducibility of α gives now $\langle \rho^+(\Gamma)v \rangle = \mathbb{C}^n$. □

Let S be the component of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ that contains ρ^+. In particular, $\dim S = n^2 + n - 2$.

Proposition 7.9. *S contains irreducible representations.*

Proof. We prove the proposition by contradiction, hence assume that there is a Zariski neighborhood $U \subset S \subset R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ of ρ^+ so that all representations in U are reducible. By Lemmas 7.7 and 7.8, the choice of the U can be made so that the representations in U have only an a-dimensional invariant subspace.

In particular every representation in U is conjugate to a representation in $P^+ = P(a)$. Therefore given any Zariski neighborhood $U^+ \subset R(\Gamma, P^+)$ of ρ^+, U can be chosen so that every representation in U is conjugate to a representation in U^+. As ρ^+ is a smooth point of $R(\Gamma, P^+)$ by Lemma 7.6, ρ^+ is contained in a single irreducible component S^+ of $R(\Gamma, P^+)$, and we may chose $U^+ \subset S^+$. This yields the inclusion

$$U \subset \mathrm{SL}_n(\mathbb{C}) \cdot U^+ \subset \mathrm{SL}_n(\mathbb{C}) \cdot S^+.$$

Now we reach the contradiction by computing dimensions. Using that P^+ stabilizes S^+ we get

$$\dim U \leq \dim(\mathrm{SL}_n(\mathbb{C}) \cdot S^+) \leq \dim(\mathrm{SL}_n(\mathbb{C})/P^+) + \dim S^+,$$

where $\dim(\mathrm{SL}_n(\mathbb{C})/P^+) = n^2 - 1 - \dim \mathfrak{p}^+$, and

$$\dim S^+ = \dim H^1(\Gamma; \mathfrak{p}^+) + \dim \mathfrak{p}^+ - \dim H^0(\Gamma; \mathfrak{p}^+) = n - 2 + \dim \mathfrak{p}^+ - 0.$$

This yields $\dim U \leq n^2+n-3$, contradicting Proposition 7.4, which asserts that $\dim U = \dim S = n^2+n-2$. □

8. The neighborhood of χ_λ

The aim of this section is to prove Theorem 1.5, i.e., we determine the local structure of the character variety $X(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ at χ_λ, the character of the representation ρ_λ given by (1). For this purpose we will identify the quadratic cone of $X(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ at χ_λ by means of algebraic obstructions to integrability. Moreover, we will describe these obstructions geometrically.

Before discussing the components of the variety of characters, we need to discuss the components of the variety of representations. In Section 7 we have constructed S a component of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ of dimension n^2+n-2 that contains ρ^+ and irreducible representations (Propositions 7.4 and 7.9).

Next we discuss a component of reducible representations. The representation variety $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ contains

$$R(\Gamma, \mathrm{SL}_a(\mathbb{C})) \times R(\Gamma, \mathrm{SL}_b(\mathbb{C})) \times R(\Gamma, \mathbb{C}^*)$$

where the inclusion is given by

$$(\alpha', \beta', \lambda') \mapsto ((\lambda')^{b\varphi} \otimes \alpha') \oplus ((\lambda')^{-a\varphi}) \otimes \beta').$$

Our hypothesis on infinitesimal regularity implies that $\alpha \in R(\Gamma, \mathrm{SL}_a(\mathbb{C}))$ and $\beta \in R(\Gamma, \mathrm{SL}_b(\mathbb{C}))$ are smooth points which are contained in unique components $V_\alpha \subset R(\Gamma, \mathrm{SL}_a(\mathbb{C}))$ and $V_\beta \subset R(\Gamma, \mathrm{SL}_b(\mathbb{C}))$ respectively. Hence we obtain an embedding

$$V_\alpha \times V_\beta \times R(\Gamma, \mathbb{C}^*) \hookrightarrow R(\Gamma, \mathrm{SL}_n(\mathbb{C})).$$

Lemma 8.1. *There exists a unique component T of $R(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ that contains*

$$V_\alpha \times V_\beta \times R(\Gamma, \mathbb{C}^*).$$

Moreover, we have $\dim T = n^2+n-3$.

Proof. By the hypothesis of Theorem 1.5 we have $\Delta_0^{\alpha\otimes\beta^*}(\lambda^n) \neq 0$ and λ^n is a simple root of $\Delta_1^{\alpha\otimes\beta^*}(t)$. Hence for all $\lambda' \neq \lambda$ which are sufficiently close to λ we have $\Delta_q^{\alpha\otimes\beta^*}((\lambda')^n) \neq 0$ for $q = 0, 1$. Hence, by the argument in the proof of Proposition 6.1 we obtain $H^q(\Gamma; \mathcal{M}^{\pm}_{(\lambda')^{\pm n}}) = 0$ for $q = 0, 1$.

Now consider the representation

$$\rho_{\lambda'} = ((\lambda')^{b\varphi} \otimes \alpha) \oplus ((\lambda')^{-a\varphi} \otimes \beta) \in V_\alpha \times V_\beta \times R(\Gamma, \mathbb{C}^*)$$

and the corresponding decomposition of $\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_{\lambda'}}$ as Γ-module:

$$\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_{\lambda'}} = \mathfrak{sl}_a(\mathbb{C})_{\mathrm{Ad}\,\alpha} \oplus \mathfrak{sl}_b(\mathbb{C})_{\mathrm{Ad}\,\beta} \oplus \mathbb{C} \oplus \mathcal{M}^{+}_{(\lambda')^{n}} \oplus \mathcal{M}^{-}_{(\lambda')^{-n}}.$$

Hence

$$\begin{aligned}\dim Z^1(\Gamma;\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_{\lambda'}}) &= \dim H^1(\Gamma;\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_{\lambda'}}) + \dim B^1(\Gamma;\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_{\lambda'}})\\ &= a-1+b-1+1+n^2-1-1 = n^2+n-3.\end{aligned}$$

On the other hand, for the $\mathrm{SL}_n(\mathbb{C})$-orbit of $V_\alpha \times V_\beta \times R(\Gamma,\mathbb{C}^*)$ we have:

$$\mathrm{SL}_n(\mathbb{C})\cdot(V_\alpha \times V_\beta \times R(\Gamma,\mathbb{C}^*)) = \mathrm{SL}_n(\mathbb{C})/P^+ \cdot (U^+\cdot V_\alpha \times V_\beta \times R(\Gamma,\mathbb{C}^*))$$

where $U^+ = \left\{\left(\begin{smallmatrix}\mathrm{id}_a & X\\ 0 & \mathrm{id}_b\end{smallmatrix}\right) \,\middle|\, X \in \mathcal{M}^+_{(\lambda')^n}\right\}$. Now the action of U^+ on $V_\alpha \times V_\beta \times R(\Gamma,\mathbb{C}^*)$ is generically free since $H^0(\Gamma;\mathcal{M}^+_{(\lambda')^n}) = 0$ and hence

$$\begin{aligned}\dim \mathrm{SL}_n(\mathbb{C})\cdot(V_\alpha \times V_\beta \times R(\Gamma,\mathbb{C}^*)) &\geq ab+ab+a^2+a-2+b^2+b-2+1\\ &= n^2+n-3.\end{aligned}$$

Therefore, $\rho_{\lambda'}$ is a smooth point of $R(\Gamma,\mathrm{SL}_n(\mathbb{C}))$ which is contained in a unique n^2+n-3-dimensional component T. Note that T is the Zariski closure of the orbit $\mathrm{SL}_n(\mathbb{C})\cdot(V_\alpha \times V_\beta \times R(\Gamma,\mathbb{C}^*))$. □

Let Y and Z denote the components of the character variety that contain the characters of S and T respectively. We have $\dim Y = \dim S - \dim \mathrm{SL}_n(\mathbb{C}) = n-1$. In addition $\dim Z \geq a-1+b-1+1 = n-1$ since T contains $V_\alpha \times V_\beta \times R(\Gamma,\mathbb{C}^*)$. Notice that the generic dimension of the orbit of $(\alpha',\beta',\lambda') \in V_\alpha \times V_\beta \times R(\Gamma,\mathbb{C}^*)$ is n^2-2. Hence, $\dim Z \leq \dim T - (n^2-2) = n-1$. Hence $\dim Z = n-1$ and $\dim T = n^2+n-3$.

Let $Z_\alpha \subset X(\Gamma,\mathrm{SL}_a(\mathbb{C}))$ and $Z_\beta \subset X(\Gamma,\mathrm{SL}_b(\mathbb{C}))$ denote the irreducible components that contain the respective projections of V_α and V_β. We have a commutative diagram

$$\begin{array}{ccc} V_\alpha \times V_\beta \times R(\Gamma,\mathbb{C}^*) & \longrightarrow & T \subset R(\Gamma,\mathrm{SL}_n(\mathbb{C}))\\ \downarrow & & \downarrow\\ Z_\alpha \times Z_\beta \times \mathbb{C}^* & \longrightarrow & Z \subset X(\Gamma,\mathrm{SL}_n(\mathbb{C}))\end{array}$$

The top row is injective but not the bottom one, as conjugation can realize permutations of rows and columns. In general those permutations are difficult to describe, but if we restrict to *irreducible* characters, this is simpler.

Lemma 8.2. *There exists a Zariski dense subset $\mathring{Z} \subset Z$ such that*:

– *If $Z_\alpha \neq Z_\beta$ (in particular if $a \neq b$), then $\mathring{Z} \cong Z_\alpha^{irr} \times Z_\beta^{irr} \times \mathbb{C}^*$.*

– *If $Z_\alpha = Z_\beta$, then $\mathring{Z} \cong Z_\alpha^{irr} \times Z_\beta^{irr} \times \mathbb{C}^*/\sim$, where the relation is defined by $(\chi_a,\chi_b,\lambda) \sim (\chi_b,\chi_a,\lambda^{-1})$, for $(\chi_a,\chi_b,\lambda) \in Z_\alpha^{irr} \times Z_\beta^{irr} \times \mathbb{C}^*$.*

Here Z_α^{irr} denotes the set of irreducible characters in Z_α. We use similar notation for other components of characters and representations.

Proof. Recall from the proof of Lemma 8.1 that T is the Zariski closure of the orbit $\mathrm{SL}_n(\mathbb{C}) \cdot (V_\alpha \times V_\beta \times R(\Gamma, \mathbb{C}^*))$. As V_α^{irr} and V_β^{irr} are dense in V_α and V_β, $\mathrm{SL}_n(\mathbb{C}) \cdot (V_\alpha^{irr} \times V_\beta^{irr} \times R(\Gamma, \mathbb{C}^*))$ is dense in T. Its projection $\mathring{Z}$ to Z is the image of $Z_\alpha^{irr} \times Z_\beta^{irr} \times \mathbb{C}^*$, which is Zariski dense. To determine this image, we use that each point in $X(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ is the character of a semisimple representation, unique up to conjugation [Lubotzky and Magid 1985]. This uniqueness implies that for $Z_\alpha \neq Z_\beta$ this is an injective map, and for $Z_\alpha = Z_\beta$ we quotient by the permutation of components, with the corresponding transformation for λ. □

Remark 8.3. When $a = b = 1$, then $Z_\alpha = Z_\beta$ consists of a single point and Z is the quotient of $\mathbb{C}^*$ by the involution $\lambda \mapsto 1/\lambda$. Hence $Z \cong \mathbb{C}$ and it is the variety of abelian characters in $\mathrm{SL}_2(\mathbb{C})$. The ring of functions invariant by this involution is generated by $\lambda + 1/\lambda$, i.e., the trace of a diagonal matrix with eigenvalues λ and $1/\lambda$ (corresponding to the character evaluated at a meridian).

We aim to show that S and T are the only components that contain ρ_λ. For this purpose we consider the quadratic cone $Q(\rho_\lambda)$ which is defined by the vanishing of an obstruction to integrability of 1-cocycles. Let

$$[.\smile.] : H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda}) \otimes H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda}) \to H^2(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda})$$

denote the *cup bracket*, which is the combination of the cup product with the Lie bracket $\mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C}) \xrightarrow{[\,.\,,\,.\,]} \mathfrak{sl}_n(\mathbb{C})$. The quadratic cone $Q(\rho_\lambda) \subset Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda})$ is defined by

$$Q(\rho_\lambda) = \{\vartheta \in Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda}) \mid [\vartheta \smile \vartheta] \sim 0\}.$$

Goldman [1984] showed that if $\vartheta \in Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda})$ is integrable then the cup bracket $[\vartheta \smile \vartheta]$ is a coboundary. In what follows we will compute the projections of this obstruction, for the projections

$$\mathrm{pr}_\pm : H^2(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda}) \to H^2(\Gamma; \mathcal{M}^\pm_{\lambda^{\pm n}}).$$

Here we use the decomposition of Γ-modules:

$$\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda} = \mathcal{D} \oplus \mathcal{M}^+_{\lambda^n} \oplus \mathcal{M}^-_{\lambda^{-n}} = \mathfrak{sl}_a(\mathbb{C}) \oplus \mathfrak{sl}_b(\mathbb{C}) \oplus \mathbb{C} \oplus \mathcal{M}^+_{\lambda^n} \oplus \mathcal{M}^-_{\lambda^{-n}}.$$

Recall that Γ acts of $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{sl}_a(\mathbb{C})$ and $\mathfrak{sl}_b(\mathbb{C})$ via the adjoint representation $\mathrm{Ad}\,\rho_\lambda$, $\mathrm{Ad}\,\alpha$ and $\mathrm{Ad}\,\beta$ respectively. For the rest of this section we will understand these modules with this action. Recall also that, by the hypotheses of Theorem 1.5, $\Delta_0^+(\lambda^n) \neq 0$ and λ^n is a simple root of Δ_1^+. By Proposition 6.1 we have $\dim H^1(\Gamma; \mathcal{M}^\pm_{\lambda^{\pm n}}) = 1$ and we fix $d_\pm \in Z^1(\Gamma; \mathcal{M}^\pm_{\lambda^{\pm n}})$ which represent nontrivial cohomology classes.

Every element in $H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C}))$ is represented by a cocycle

$$\vartheta = \begin{pmatrix} z_a & u_+ d_+ \\ u_- d_- & z_b \end{pmatrix} + z\varphi \begin{pmatrix} -b\,\mathrm{Id}_a & 0 \\ 0 & a\,\mathrm{Id}_b \end{pmatrix}, \tag{29}$$

where $z_a \in Z^1(\Gamma; \mathfrak{sl}_a(\mathbb{C}))$, $z_b \in Z^1(\Gamma; \mathfrak{sl}_b(\mathbb{C}))$ and $u_\pm, z \in \mathbb{C}$.

Lemma 8.4. *For $\vartheta \in Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C}))$ as in* (29) *we have*

$$\begin{aligned} \mathrm{pr}_+[\vartheta \smile \vartheta] &\sim 2u_+(z_a \smile d_+ + d_+ \smile z_b + znd_+ \smile \varphi), \\ \mathrm{pr}_-[\vartheta \smile \vartheta] &\sim 2u_-(d_- \smile z_a + z_b \smile d_- - znd_- \smile \varphi), \end{aligned}$$

where $\sim$ denotes being cohomologous.

Proof. The lemma follows from Remark 6.4 and a direct calculation of

$$[\vartheta \smile \vartheta](\gamma_1, \gamma_2) = [\vartheta(\gamma_1), \gamma_1 \vartheta(\gamma_2)]. \qquad \square$$

In order to understand the cup products appearing in Lemma 8.4 we introduce the complex number $l_\pm(z_a, z_b) \in \mathbb{C}$. Consider a one-parameter analytical deformation $s \mapsto \alpha_s \oplus \beta_s$ of $\alpha \oplus \beta$ in $V_\alpha \times V_\beta$ tangent to (z_a, z_b). Notice that the coefficients of the twisted Alexander polynomial $\Delta_1^{\alpha_s \otimes \beta_s^*}$ depend analytically on s. By the implicit function theorem and since λ^n is a simple root of $\Delta_1^{\alpha \otimes \beta^*}$, there is an analytical path $s \mapsto r_s^+$ of roots of $\Delta_1^{\alpha_s \otimes \beta_s^*}$ with $r_0^+ = \lambda^n$. Similarly there is a path $s \mapsto r_s^-$ of roots of $\Delta_1^{\beta_s \otimes \alpha_s^*}$ with $r_0^- = \lambda^{-n}$. We define

$$l_\pm(z_a, z_b) = \frac{d}{ds}\bigg|_{s=0} \log r_s^\pm.$$

Lemma 8.5. *The following relations hold in $Z^1(\Gamma; \mathcal{M}^\pm_{\lambda^{\pm n}})$:*

$$\begin{aligned} z_a \smile d_+ + d_+ \smile z_b &\sim -l_+(z_a, z_b) d_+ \smile \varphi, \\ d_- \smile z_a + z_b \smile d_- &\sim -l_-(z_a, z_b) d_- \smile \varphi. \end{aligned}$$

Proof. We know that $z_a \smile d_+ + d_+ \smile z_b$ is cohomologous to $x d_+ \smile \varphi$ for some $x \in \mathbb{C}$, as $H^2(\Gamma; \mathcal{M}^+_\lambda) \cong \mathbb{C}$ and $d_+ \smile \varphi \not\sim 0$ (see Proposition 6.1 and Corollary 6.7). Hence by Lemma 7.2 the cocycle

$$\zeta = \begin{pmatrix} z_a & 0 \\ 0 & z_b \end{pmatrix} + \frac{x}{n} \varphi \begin{pmatrix} -b \,\mathrm{Id}_a & 0 \\ 0 & a \,\mathrm{Id}_b \end{pmatrix} \in Z^1(\Gamma; \mathcal{D})$$

satisfies $\boldsymbol{b}(\zeta) \sim 0$ where $\boldsymbol{b} : H^1(\Gamma; \mathcal{D}) \xrightarrow{\boldsymbol{b}} H^2(\Gamma; \mathcal{M}^+_{\lambda^n})$ is the Bockstein operator of the exact cohomology sequence associated to (22).

Furthermore, by Corollary 7.3 ζ is cohomologous to the restriction of a cocycle $\zeta^+ \in Z^1(\Gamma; \mathfrak{p}^+)$. As ρ^+ is a smooth point of $R(\Gamma, P^+)$ (Lemma 7.6), we may consider a path $s \mapsto \rho_s$ in $R(\Gamma, P^+)$ tangent to ζ^+ at ρ^+, which we write as

$$\rho_s = \begin{pmatrix} 1 & d_s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_s \lambda_s^{b\varphi} & 0 \\ 0 & \beta_s \lambda_s^{-a\varphi} \end{pmatrix}.$$

In particular, by the definition of ζ we have that $s \mapsto \alpha_s$ is a deformation of α tangent to z_a, $s \mapsto \beta_s$ is a deformation of β tangent to z_b, and $\lambda_s = \lambda(1 - \frac{x}{n}s + o(s^2))$. By semicontinuity d_s is a cocycle not cohomologous to zero because $d_0 = d_+$, hence by Lemma 5.3 we obtain $\Delta_1^{\alpha_s \otimes \beta_s^*}(\lambda_s^n) = 0$. Therefore, as

$$-\frac{x}{n} = \frac{\lambda_0'}{\lambda_0} = \frac{d}{ds}\bigg|_{s=0} \log \lambda_s,$$

x equals minus the derivative of the logarithm of the root of $\Delta_1^{\alpha_s \otimes \beta_s^*}$. □

Lemmas 8.4 and 8.5 give:

Corollary 8.6. *For $\vartheta \in Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C}))$ as in* (29):

$$\mathrm{pr}_\pm[\vartheta \smile \vartheta] \sim 2u_\pm(-l_\pm(z_a, z_b) \pm zn)d_\pm \smile \varphi.$$

Since $\Delta_1^{\alpha_s \otimes \beta_s^*}(t) = \Delta_1^{\beta_s \otimes \alpha_s^*}(1/t)$ by Corollary 4.7,

$$l_+(z_a, z_b) = -l_-(z_a, z_b).$$

Hence the vanishing of the obstructions to integrability of Corollary 8.6 is equivalent to

$$u_+(-l_+(z_a, z_b) + zn) = 0 \quad \text{and} \quad u_-(-l_+(z_a, z_b) + zn) = 0. \tag{30}$$

Since As z can be interpreted as the derivative of the logarithm of λ, we view

$$-l_+(z_a, z_b) + zn$$

as the derivative of the difference between the logarithm of the root of the Alexander polynomial and the logarithm of λ^n.

Recall that by (29) every cocycle $\vartheta \in Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C}))$ is of the form

$$\vartheta = \begin{pmatrix} z_a & u_+d_+ + b_+ \\ u_-d_- + b_- & z_b \end{pmatrix} + z\varphi \begin{pmatrix} -b\,\mathrm{Id}_a & 0 \\ 0 & a\,\mathrm{Id}_b \end{pmatrix}, \tag{31}$$

where $z_a \in Z^1(\Gamma; \mathfrak{sl}_a(\mathbb{C}))$ and $z_b \in Z^1(\Gamma; \mathfrak{sl}_b(\mathbb{C}))$ are cocycles, $u_\pm, z \in \mathbb{C}$, and $b_\pm \in B^1(\Gamma; \mathcal{M}^\pm_{\lambda^{\pm n}})$ are coboundaries. Notice that this formula differs from (29) because here the coboundaries are also considered.

Proposition 8.7. *The Zariski tangent spaces at ρ_λ are*

$$T_{\rho_\lambda} S = \{\vartheta \in Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})) \mid -l_+(z_a, z_b) + zn = 0\},$$
$$T_{\rho_\lambda} T = \{\vartheta \in Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})) \mid u_+ = u_- = 0\},$$

using the notation of (31) *for a cocycle $\vartheta \in Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C}))$. In particular S and T are smooth and transverse at ρ_λ.*

Proof. First at all, notice that u_+ is not identically zero on $T_{\rho_\lambda} S$, by considering the tangent vector to the path

$$s \mapsto \begin{pmatrix} 1 & sd_+ \\ 0 & 1 \end{pmatrix} \rho_\lambda.$$

Then (30) implies $-l_+(z_a, z_b)+zn=0$ on $T_{\rho_\lambda} S$. Furthermore, we know that $\dim S = n^2+n-2$ and, by (17) and Proposition 6.1, the dimension of $Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}\,\rho_\lambda})$ is n^2+n-1. This shows the equality for $T_{\rho_\lambda} S$ and proves that ρ_λ is a smooth point of S.

We follow the same lines to prove the equality for $T_{\rho_\lambda} T$. Notice that $-l_+(z_a, z_b)+zn$ is not identically zero on $T_{\rho_\lambda} T$, by considering deformations of λ that keep α and β constant. Hence $u_+ = u_- = 0$ on $T_{\rho_\lambda} T$. Moreover, $\dim T = n^2+n-3$. $\square$

We next compute the tangent space to character varieties at χ_λ. Since the representation ρ_λ is completely reducible, its orbit by conjugation is closed, hence we can apply Luna's slice theorem as in [Ben Abdelghani 2002] or [Heusener and Porti 2005, Section 9]. As a consequence of the slice theorem, since the centralizer of ρ_λ is $\mathbb{C}^*$:

$$T_{\chi_\lambda} X(\Gamma, \mathrm{SL}_n(\mathbb{C})) \cong T_0\big(H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})) /\!\!/ \mathbb{C}^*\big).$$

The action of $\mathbb{C}^*$ can be seen on the coordinates $u_\pm$: an element $\varsigma \in \mathbb{C}^*$ maps $u_\pm$ to $\varsigma^{\pm n} u_\pm$. Hence we define a new coordinate

$$u = u_+ u_-$$

and the obstructions (30) become

$$u(-l_+(z_a, z_b) + zn) = 0. \tag{32}$$

Notice that even if z_a and z_b are cocycles, the logarithmic derivative $-l_+(z_a, z_b)$ only depends on the cohomology class of (z_a, z_b) in $H^1(\Gamma; \mathfrak{sl}_a(\mathbb{C}) \oplus \mathfrak{sl}_b(\mathbb{C}))$. Also, z is the scalar that describes a cohomology class $z\varphi \in H^1(\Gamma; \mathbb{C}) = Z^1(\Gamma; \mathbb{C}) \cong \mathbb{C}$. Similarly for $u_\pm \in \mathbb{C}$ and the cohomology class $u_\pm[d_\pm] \in H^1(\Gamma; \mathcal{M}^\pm_{\lambda^{\pm n}}) \cong \mathbb{C}$. Thus we have the following:

Remark 8.8. The obstruction in (32) is well defined in $H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})) /\!\!/ \mathbb{C}^*$.

Corollary 8.9. *The Zariski tangent spaces to Y and Z are*:

$$\begin{aligned} T_{\chi_\lambda} Y &= \{[\vartheta] \in T_{\rho_\lambda} X(\Gamma, \mathrm{SL}_n(\mathbb{C})) \mid -l_+(z_a, z_b) + zn = 0\}, \\ T_{\chi_\lambda} Z &= \{[\vartheta] \in T_{\rho_\lambda} X(\Gamma, \mathrm{SL}_n(\mathbb{C})) \mid u = 0\}. \end{aligned}$$

In particular Y and Z are smooth and transverse at χ_λ.

Proof. The proof is similar to that of Proposition 8.7: we need to show that u does not vanish on the Zariski tangent space to Y and $-l_+(z_a, z_b) + zn$ does not vanish on the Zariski tangent space to Z. For the first assertion, we start with the cocycle

$$\vartheta = \begin{pmatrix} 0 & d_+ \\ d_- & 0 \end{pmatrix} \in Z^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})).$$

Following the notation of (31), since z_a, z_b and z vanish for ϑ, Proposition 8.7 implies that $\vartheta \in T_{\rho_\lambda} S$. In particular the projection of its cohomology class ϑ in $H^1(\Gamma; \mathfrak{sl}_n(\mathbb{C})) /\!\!/ \mathbb{C}^*$ is a vector tangent to Y for which $u \neq 0$. The proof that $-l_+(z_a, z_b) + zn$ is not identically zero on $T_{\chi_\lambda} Z$ is the same as in Proposition 8.7. Then one concludes by using the dimension estimates. $\square$

Notice that Corollary 8.9 and the computations of dimensions yield that χ_λ is a smooth point of both Y and Z, and that Y and Z intersect transversally at χ_λ. In particular their intersection is a variety of dimension $n-2$. Since characters in this intersection must satisfy the condition on Alexander polynomials, we have:

Corollary 8.10. *There is a neighborhood $\chi_\lambda \in U \subset X(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ such that*

$$(Y \pitchfork Z) \cap U = \left\{ (\chi_{\alpha'}, \chi_{\beta'}, \lambda') \in Z \cap U \;\middle|\; \Delta_1^{\alpha' \otimes (\beta')^*}\big((\lambda')^n\big) = 0 \right\}.$$

9. An example

Let $K \subset S^3$ be the trefoil knot and $\Gamma = \pi_1(S^3 \setminus \mathcal{N}(K))$. We use the presentation

$$\Gamma \cong \langle x, y \mid x^2 = y^3 \rangle,$$

in particular the center is the cyclic group generated by $z = x^2 = y^3$. The abelianization map $\varphi : \Gamma \to \mathbb{Z}$ satisfies $\varphi(x) = 3$, $\varphi(y) = 2$ and a meridian of the trefoil is given by $m = xy^{-1}$.

Lemma 9.1. *Every irreducible representation in $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is conjugate to α_s, where*

$$\alpha_s(x) = \begin{pmatrix} i & 0 \\ s & -i \end{pmatrix} \quad \textit{and} \quad \alpha_s(y) = \begin{pmatrix} \eta & \bar\eta - \eta \\ 0 & \bar\eta \end{pmatrix}, \tag{33}$$

for a unique $s \in \mathbb{C}$ and for $\eta \in \mathbb{C}$ a primitive sixth root of unity. Moreover, α_s is irreducible if and only if $s \neq 0, 2i$.

Proof. Let $\alpha : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation. Then by Schur's lemma $\alpha(x^2) = \alpha(y^3)$ lies in the center $\{\pm \mathrm{id}_2\}$ of $\mathrm{SL}_2(\mathbb{C})$. If we had $\alpha(x)^2 = \mathrm{id}_2$, then we would get $\alpha(x) = \pm \mathrm{id}_2$ and α would be reducible, hence $\alpha(x)^2 = \alpha(y^3) = -\mathrm{id}_2$. Furthermore, as $\alpha(y) \neq -\mathrm{id}_2$, the eigenvalues of $\alpha(y)$ are primitive sixth roots of unity. The eigenspaces of $\alpha(x)$ and $\alpha(y)$ determine four points in $\mathbb{P}^1$. These four points are distinct since α is irreducible and by conjugation we can assume that $E_{\alpha(x)}(-i) = [0:1]$ is the point at infinity, $E_{\alpha(y)}(\eta) = [1:0]$ and $E_{\alpha(y)}(\bar\eta) = [1:1]$.

The last eigenspace $E_{\alpha(x)}(-i) = [2i : s] = [1 : -is/2]$ determines the representation α up to conjugation. Hence there exists $s \in \mathbb{C}$ such that α is conjugate to α_s. Moreover, the eigenspace $E_{\alpha_s(x)}(-i)$ coincides with an eigenspace of $\alpha_s(y)$ if and only if $s \in \{0, 2i\}$. □

For any representation $\alpha \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ we consider the induced action on $\mathbb{C}^2$, as well as the action $\alpha \otimes t^\varphi$ on $\mathbb{C}^2[t^{\pm 1}]$. We aim to compute the twisted Alexander polynomials Δ_0^α and Δ_1^α, the orders for the homology of $\alpha \otimes t^\varphi$. The quotient $\Delta_1^{\alpha_s}/\Delta_0^{\alpha_s}$ has been calculated in a different way in [Kitano and Morifuji 2012, Example 4.3].

Lemma 9.2. *For any irreducible* $\alpha \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, *we have*

$$\Delta_0^\alpha \doteq 1 \ \text{ and } \ \Delta_1^\alpha \doteq t^2 + 1.$$

Proof. First, we have $\Delta_0^\alpha \doteq 1$ since α is irreducible and $\dim \mathbb{C}^2 > 1$ (see (4) in the proof of Lemma 2.7).

In order to calculate Δ_1^α we will use the amalgamated product structure of Γ

$$\Gamma \cong \langle x \rangle *_{\langle z \rangle} \langle y \rangle$$

and the corresponding Mayer–Vietoris exact sequence in group homology [Brown 1994, VII.9]. We start computing some of the terms. Since $\langle z \rangle \cong \mathbb{Z}$, the groups $H_q(\langle z \rangle, \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi})$ are the homology groups of the complex

$$0 \to \mathbb{C}^2[t^{\pm 1}] \xrightarrow{z-1} \mathbb{C}^2[t^{\pm 1}] \to 0.$$

Hence a presentation matrix of $H_0(\langle z \rangle, \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi})$ is

$$\begin{pmatrix} -t^6 - 1 & 0 \\ 0 & -t^6 - 1 \end{pmatrix}.$$

The presentation matrices for $H_0(\langle x \rangle, \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi})$ and $H_0(\langle y \rangle, \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi})$ are similarly given by (respectively)

$$\begin{pmatrix} it^3 - 1 & 0 \\ 0 & -it^3 - 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{\frac{\pi i}{3}} t^2 - 1 & 0 \\ 0 & e^{\frac{-\pi i}{3}} t^2 - 1 \end{pmatrix}.$$

Since $H_0(\langle x \rangle, \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi})$ and $H_0(\langle y \rangle, \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi})$ are torsion modules it follows that $H_1(\langle x \rangle, \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi}) \cong H_1(\langle y \rangle, \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi}) = 0$. Hence Mayer–Vietoris gives a short exact sequence

$$0 \to H_1(\Gamma; \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi}) \to H_0(\langle z \rangle, \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi}) \to \\ H_0(\langle x \rangle, \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi}) \oplus H_0(\langle y \rangle, \mathbb{C}^2[t^{\pm 1}]_{\alpha \otimes t^\varphi}) \to 0.$$

Using this sequence and the presentation matrices we obtain

$$\Delta_1^\alpha = \frac{(t^6+1)^2}{(t^3+i)(t^3-i)\big(t^2-e^{\frac{\pi i}{3}}\big)\big(t^2-e^{\frac{-\pi i}{3}}\big)} = t^2+1. \qquad \square$$

It follows that Theorems 1.4 and 1.5 apply for α irreducible and $\lambda \in \mathbb{C}$ satisfying $\lambda^6 = -1$. Namely Theorem 1.4 yields:

Corollary 9.3. *When $\alpha \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is irreducible and $\lambda^6 = -1$,*

$$(\lambda^\varphi \otimes \alpha) \oplus (\lambda^{-2\varphi} \otimes \mathbf{1}) : \Gamma \to \mathrm{SL}_3(\mathbb{C})$$

can be deformed to irreducible representations.

To illustrate Theorem 1.5, we discuss next the variety of characters.

Varieties of characters. The variety of characters $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ has two components, the abelian one and the one that contains irreducible representations, denoted by $X_0(\Gamma, \mathrm{SL}_2(\mathbb{C}))$. Let $\chi_s \in X_0(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ denote the character of α_s defined in (33). The following is well known but we provide a proof for completeness and because it is quite straightforward from Lemma 9.1.

Lemma 9.4. *The map $s \mapsto \chi_s$ defines an isomorphism $\mathbb{C} \cong X_0(\Gamma, \mathrm{SL}_2(\mathbb{C}))$.*

Proof. Using Lemma 9.1, the regular map $f : \mathbb{C} \to X_0(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ given by $f(s) = \chi_s$ restricts to a bijection between $\{s \in \mathbb{C} \mid s \neq 0, 2i\}$ and the set of characters of irreducible representations $X^{irr} \subset X_0(\Gamma, \mathrm{SL}_2(\mathbb{C}))$. A direct calculation gives for the meridian $m = xy^{-1}$ that $\chi_s(m) = i\bar\eta + s(\bar\eta - \eta)$ is a linear function in s. Hence there exists a regular map $g : X_0(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}$ such that $g \circ f = \mathrm{id}_{\mathbb{C}}$. Since the image of f contains X^{irr}, $f \circ g \circ f = f$ implies

$$f \circ g|_{X^{irr}} = \mathrm{id}_{X^{irr}}.$$

Both f and g are regular morphisms (defined on the whole variety, not only on an open subset), hence density yields:

$$f \circ g = \mathrm{id}_{X_0(\Gamma, \mathrm{SL}_2(\mathbb{C}))}$$

establishing the isomorphism. $\square$

For any $\lambda \in \mathbb{C}^*$ the map $\alpha \mapsto (\lambda^\varphi \otimes \alpha) \oplus (\lambda^{-2\varphi} \otimes \mathbf{1})$ induces an embedding

$$i_\lambda : X_0(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to X(\Gamma, \mathrm{SL}_3(\mathbb{C})).$$

Let $X_\lambda = i_\lambda(X_0(\Gamma, \mathrm{SL}_2(\mathbb{C})))$ denote its image, that consists of characters of reducible representations. We know that when $\lambda^6 = -1$, X_λ is contained in a two dimensional component that contains irreducible characters. Before describing the global structure of $X(\Gamma, \mathrm{SL}_3(\mathbb{C}))$, we discuss the incidence between the X_λ when $\lambda^6 = -1$.

Let $\tilde\sigma : R(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ be the involution such that

$$\tilde\sigma(\alpha)(x) = -\alpha(x) \quad \text{and} \quad \tilde\sigma(\alpha)(y) = \alpha(y),$$

for every $\alpha \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, namely $\tilde\sigma(\alpha) = (-1)^\varphi \otimes \alpha$. Denote by σ the induced involution on $X_0(\Gamma, \mathrm{SL}_2(\mathbb{C}))$. A straightforward computation gives

$$\tilde\sigma(\alpha) \mapsto (\lambda^\varphi \otimes \tilde\sigma(\alpha)) \oplus (\lambda^{-2\varphi} \otimes \mathbf{1}) = ((-\lambda)^\varphi \otimes \alpha) \oplus (\lambda^{-2\varphi} \otimes \mathbf{1})$$

and hence $i_\lambda \circ \sigma = i_{-\lambda}$. It follows that $X_\lambda = X_{-\lambda}$. Notice also that $\tilde\sigma(\alpha_s)$ is conjugate to α_{2i-s}.

Lemma 9.5. *For $\lambda \neq \pm\lambda'$ satisfying $\lambda^6 = (\lambda')^6 = -1$, X_λ and $X_{\lambda'}$ intersect at a single point $i_\lambda(\chi_s)$, with $s \in \{0, 2i\}$. In particular $X_\lambda \cap X_{\lambda'}$ is the character of a diagonal representation.*

This gives a configuration of three lines $X_{e^{\pi i/6}}$, X_i, $X_{e^{5\pi i/6}}$, that intersect pairwise at one point. We shall prove that there is a single component of $X(\Gamma, \mathrm{SL}_3(\mathbb{C}))$ that contains irreducible representations, and we shall describe how the three lines meet in this component.

Irreducible characters in $X(\Gamma, \mathrm{SL}_3(\mathbb{C}))$. Let $\rho \in R(\Gamma, \mathrm{SL}_3(\mathbb{C}))$ be an irreducible representation. We denote $\rho(x) = A$ and $\rho(y) = B$. The matrix $A^2 = B^3$ is a central element of $\mathrm{SL}_3(\mathbb{C})$ because ρ is irreducible. The center of $\mathrm{SL}_3(\mathbb{C})$ consists of three diagonal matrices $\{\mathrm{id}_3, \omega\, \mathrm{id}_3, \omega^2\, \mathrm{id}_3\}$, where $\omega^2 + \omega + 1 = 0$.

Lemma 9.6. $$A^2 = B^3 = \mathrm{id}_3.$$

Proof. We need to exclude the cases $A^2 = B^3 = \omega\, \mathrm{id}_3$ or equal to $\omega^2\, \mathrm{id}_3$. Seeking a contradiction, assume $A^2 = B^3 = \omega\, \mathrm{id}_3$. The equality $A^2 = \omega\, \mathrm{id}_3$ implies that one eigenvalue of A has multiplicity at least two. Of course multiplicity three is not compatible with irreducibility, thus A has a two-dimensional eigenspace. On the other hand, $B^3 - \omega\, \mathrm{id}_3 = 0$ combined with $\det(B) = 1$ yields that the minimal polynomial of B has also degree two. Hence B has also a two dimensional eigenspace. The intersection of the two dimensional eigenspaces of A and B is a proper invariant subspace, contradicting irreducibility. The same argument applies to $A^2 = B^3 = \omega^2\, \mathrm{id}_3$. □

By the discussion in the proof of the previous lemma, the minimal polynomial of A is $A^2 - \mathrm{id}_3 = 0$ and the minimal polynomial of B is $B^3 - \mathrm{id}_3 = 0$. Therefore, the matrices A and B are conjugate to

$$A \sim \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \text{and} \quad B \sim \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix},$$

where $\omega^2+\omega+1=0$. The corresponding eigenspaces are the plane $E_A(-1)$ and the lines $E_A(1)$, $E_B(1)$, $E_B(\omega)$ and $E_B(\omega^2)$. The eigenspaces determine the representation, as they determine the matrices A and B, that have fixed eigenvalues. Of course $E_A(1)\cap E_A(-1)=0$ and $E_B(1)$, $E_B(\omega)$ and $E_B(\omega^2)$ are also in general position. Since ρ is irreducible, the five eigenspaces are in general position. For instance $E_A(1)\cap(E_B(1)\oplus E_B(\omega))=0$, because otherwise $E_B(1)\oplus E_B(\omega)=E_A(1)\oplus(E_A(-1)\cap(E_B(1)\oplus E_B(\omega)))$ would be a proper invariant subspace.

In order to parametrize the conjugacy classes of the irreducible representations, we fix some normalizations of those eigenspaces. The invariant lines correspond to fixed points in the projective plane $\mathbb{P}^2$. The first normalization is that $E_A(-1)$ corresponds to the line at infinity, so that the 4 invariant lines are points in the affine plane $\mathbb{C}^2$ in general position. We further fix the three fixed points of B, corresponding to an affine frame. Then the fourth point (the line $E_A(1)$) is a point in $\mathbb{C}^2$ that does not lie in the affine lines spanned by any two of the fixed points of B. This gives rise to the subvariety $\{\rho_{s,t}\in R(\Gamma,\mathrm{SL}_3(\mathbb{C}))\mid (s,t)\in\mathbb{C}^2\}$, where the representation $\rho_{s,t}$ is given by

$$\rho_{s,t}(x)=\begin{pmatrix}1&0&0\\ s&-1&0\\ t&0&-1\end{pmatrix}\quad\text{and}\quad\rho_{s,t}(y)=\begin{pmatrix}1&\omega-1&\omega^2-1\\ 0&\omega&0\\ 0&0&\omega^2\end{pmatrix}.\tag{34}$$

Here ω is a primitive third root of unity, i.e., $\omega^2+\omega+1=0$. The eigenspaces of B determine points of $\mathbb{P}^2$:

$$E_B(1)=[1:0:0],\ \ E_B(\omega)=[1:1:0]\text{ and }E_B(\omega^2)=[1:0:1].$$

The eigenspaces of A determine a projective line (at infinity) and a point:

$$E_A(-1)=\langle[0:1:0],[0:0:1]\rangle\text{ and }E_A(1)=[2:s:t].$$

Lemma 9.7. *For $(s,t)\in\mathbb{C}^2$, the representation $\rho_{s,t}$ is reducible if and only if $s=0$, $t=0$, or $s+t=2$.*

Proof. The representation $\rho_{s,t}$ is constructed so that the points in $\mathbb{P}^2$ fixed by $B=\rho_{s,t}(x)$ and the line $E_A(-1)\subset\mathbb{P}^2$ are fixed. So $\rho_{s,t}$ is reducible if and only if the projective point $E_A(1)$ belongs to one of the lines spanned by two of the fixed points of B. This condition is equivalent to one of the three equations $s=0$, $t=0$ or $s+t=2$, one for each line. □

It follows from the proof that, when $E_A(1)$ equals one of the fixed projective points of B, then A preserves also the two lines through that point that are B-invariant. More precisely, we have:

Remark 9.8. If two of the equations $\{s=0\}$, $\{t=0\}$ and $\{s+t=2\}$ hold true, then $\rho_{s,t}$ preserves a complete flag in $\mathbb{C}^3$ and therefore it is conjugate to an upper triangular representation. Notice that it has the same character as a diagonal representation.

Lemma 9.9. *Let* $R^{irr} \subset R(\Gamma, \mathrm{SL}_3(\mathbb{C}))$ *denote the subset of irreducible representations. Then the Zariski closure* $\overline{R^{irr}} \subset R(\Gamma, \mathrm{SL}_3(\mathbb{C}))$ *is an irreducible affine variety.*

Proof. The variety $\mathbb{C}^2 \times \mathrm{SL}_3(\mathbb{C})$ is irreducible and the map $\kappa : \mathbb{C}^2 \times \mathrm{SL}_3(\mathbb{C}) \to R(\Gamma, \mathrm{SL}_3(\mathbb{C}))$ given by $\kappa(s,t,D) = D\rho_{s,t}D^{-1}$ is a regular map. The image of κ contains the irreducible representations and every representation in the image of κ is the limit of irreducible representations. Hence

$$R^{irr} \subset \kappa\big(\mathbb{C}^2 \times \mathrm{SL}_3(\mathbb{C})\big) \subset \overline{R^{irr}}$$

and $\overline{\kappa\big(\mathbb{C}^2 \times \mathrm{SL}_3(\mathbb{C})\big)} = \overline{R^{irr}}$ follows. Now the assertion of the lemma follows since the closure of the image of an irreducible variety under a regular map is irreducible. □

Theorem 9.10. *The GIT quotient* $X = \overline{R^{irr}} /\!\!/ \mathrm{SL}(3,\mathbb{C})$ *is isomorphic to* $\mathbb{C}^2$. *Moreover, the Zariski-open subset* R^{irr} *is* $\mathrm{SL}(3,\mathbb{C})$*-invariant and its GIT quotient is isomorphic to the complement of three affine lines in general position in* $\mathbb{C}^2$.

Proof. By Lemma 9.9 the affine algebraic set $\overline{R^{irr}}$ is irreducible. Since it is $\mathrm{SL}(3,\mathbb{C})$-invariant, the GIT quotient $t : \overline{R^{irr}} \to X$ exists and X is also an irreducible affine algebraic variety. Let $X^{irr} \subset X$ denote the projection of R^{irr}, which is Zariski-open and hence dense.

Consider the regular morphism $f : \mathbb{C}^2 \to X$ that maps $(s,t) \in \mathbb{C}^2$ to the character $\chi_{\rho_{s,t}}$. By construction, the image of f contains X^{irr}, because $\rho_{s,t}$ realizes every irreducible representation up to conjugacy.

There is also a regular morphism $R(\Gamma, \mathrm{SL}_3(\mathbb{C})) \to \mathbb{C}^2$ given by

$$\rho \mapsto \big(\operatorname{tr}\rho(m), \operatorname{tr}\rho(m^{-1})\big)$$

where $m = xy^{-1}$ is a meridian of the trefoil knot, which induces (after restriction) a regular map $X \to \mathbb{C}^2$, by invariance. A direct computation gives:

$$\begin{pmatrix} \operatorname{tr}\rho_{s,t}(m) \\ \operatorname{tr}\rho_{s,t}(m^{-1}) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} \omega^2 - 1 & \omega - 1 \\ \omega - 1 & \omega^2 - 1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}. \tag{35}$$

Thus, after composing with a linear map, we have a regular morphism $g : X \to \mathbb{C}^2$ that satisfies

$$g \circ f = \mathrm{id}_{\mathbb{C}^2}.$$

Since the image of f contains X^{irr}, $f \circ g \circ f = f$ implies

$$f \circ g|_{X^{irr}} = \mathrm{id}_{X^{irr}}.$$

Both f and g are regular morphisms (defined on the whole variety, not only on an open subset), hence density yields

$$f \circ g = \mathrm{id}_X,$$

establishing the isomorphism. □

Remark 9.11. It follows from Theorem 9.10 that the set of reducible characters in $X \cong \mathbb{C}^2$ consists of three lines that intersect pairwise. Those are characters of representations $(\lambda^{-\varphi} \otimes \alpha) \oplus (\lambda^{2\varphi} \otimes \mathbf{1})$, with $\alpha \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ irreducible except at the intersection points, that correspond to diagonal representations.

Notice also that there is a symmetry of order three, as the center of $\mathrm{SL}_3(\mathbb{C})$ has order three. The symmetry group is generated by

$$R(\Gamma, \mathrm{SL}_3(\mathbb{C})) \to R(\Gamma, \mathrm{SL}_3(\mathbb{C})), \quad \alpha \mapsto \omega^{\varphi} \otimes \alpha,$$

where ω is a primitive third root of unity. This symmetry maps the character with coordinates (s, t) to $(2-s-t, s)$, i.e., $\mathrm{tr}(\rho(m^{\pm 1}))$ to $\omega^{\pm 1}\,\mathrm{tr}(\rho(m^{\pm 1}))$. Its fixed point has coordinates $s = t = 2/3$ (i.e., $\mathrm{tr}(\rho(m^{\pm 1})) = 0$) and corresponds to the character of an irreducible metabelian representation. This irreducible metabelian representation is obtained by composing the surjection $\Gamma \twoheadrightarrow A_4$ with the 3-dimensional irreducible representation of A_4 (see [Serre 1978]). Note that irreducible, metabelian representations of knot groups into $\mathrm{SL}_n(\mathbb{C})$ were studied by H. Boden and S. Friedl in a series of papers [2008; 2011; 2014a; 2014b].

Remark 9.12. It is possible to combine any representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ with the irreducible 3-dimensional rational representation of $r_3 : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_3(\mathbb{C})$ of $\mathrm{SL}_2(\mathbb{C})$ (for more details see [Springer 1977] and [Heusener and Medjerab 2014]). This induces a regular map

$$(r_3)_* : X_0(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to X(\Gamma, \mathrm{SL}_3(\mathbb{C})).$$

It follows from [Heusener and Medjerab 2014, Proposition 3.1] that the image of $(r_3)_*$ is contained in the component $X \subset X(\Gamma, \mathrm{SL}_3(\mathbb{C}))$. Notice that for every matrix $A \in \mathrm{SL}_2(\mathbb{C})$ the equality $\mathrm{tr}(r_3(A)) = \mathrm{tr}(r_3(A)^{-1})$ holds. Then Equation (35) implies that the image of $(r_3)_*$ is contained in the diagonal $\{s = t\} \subset \mathbb{C}^2 \cong X$. Moreover, the map $(r_3)_*$ factors through $X_0(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ since $\mathrm{Ker}(r_3) = \{\pm \mathrm{id}\}$ is the center of $\mathrm{SL}_2(\mathbb{C})$. Hence $(r_3)_*$ is a two-fold branched covering onto its image. The branching set is the character of the binary dihedral representation $d_6 : \Gamma \to D_6 \subset \mathrm{SL}_2(\mathbb{C})$. Notice also that the restriction of r_3 onto D_6 becomes reducible, $r_3 \circ d_6 \sim \rho_{1,1}$, since dihedral groups have only one and two-dimensional irreducible representations (see [Serre 1978]).

Remark 9.13. The same argument as in Theorem 9.10 applies to torus knots $T(p, 2)$, p odd, to prove that the variety of irreducible $\mathrm{SL}_3(\mathbb{C})$-characters consist of $(p-1)(p-2)/2$ disjoint components isomorphic to $\mathbb{C}^2$ and the components of reducible characters.

Acknowledgements

We are indebted to Julien Bichon for helpful discussions and we like to thank Simon Riche for pointing out Nagata's result [1961/1962] to us. We also like to thank Stefan Friedl for providing us with the references [Friedl et al. 2012; Hillman et al. 2010]. We are particularly thankful to the anonymous referee for a thorough review and for pointing out an inaccuracy in the statement of a preliminary version of Theorem 1.5. The referee's remarks led to Lemma 8.2.

References

[Artin 1968] M. Artin, "On the solutions of analytic equations", *Invent. Math.* **5** (1968), 277–291. MR 38 #344 Zbl 0172.05301

[Ben Abdelghani 2000] L. Ben Abdelghani, "Espace des représentations du groupe d'un nœud classique dans un groupe de Lie", *Ann. Inst. Fourier* (*Grenoble*) **50**:4 (2000), 1297–1321. MR 2002d:57006a Zbl 0956.57006

[Ben Abdelghani 2002] L. Ben Abdelghani, "Variété des caractères et slice étale de l'espace des représentations d'un groupe", *Ann. Fac. Sci. Toulouse Math.* (6) **11**:1 (2002), 19–32. MR 2004h:14067 Zbl 1056.20032

[Ben Abdelghani and Lines 2002] L. Ben Abdelghani and D. Lines, "Involutions on knot groups and varieties of representations in a Lie group", *J. Knot Theory Ramifications* **11**:1 (2002), 81–104. MR 2002m:57013 Zbl 0997.57036

[Ben Abdelghani et al. 2010] L. Ben Abdelghani, M. Heusener, and H. Jebali, "Deformations of metabelian representations of knot groups into SL(3, $\mathbb{C}$)", *J. Knot Theory Ramifications* **19**:3 (2010), 385–404. MR 2011d:57027 Zbl 1195.57023

[Boden and Friedl 2008] H. U. Boden and S. Friedl, "Metabelian SL(n, $\mathbb{C}$) representations of knot groups", *Pacific J. Math.* **238**:1 (2008), 7–25. MR 2010c:57007 Zbl 1154.57004

[Boden and Friedl 2011] H. U. Boden and S. Friedl, "Metabelian SL(n, $\mathbb{C}$) representations of knot groups, II: fixed points", *Pacific J. Math.* **249**:1 (2011), 1–10. MR 2012a:57007 Zbl 1218.57002

[Boden and Friedl 2014a] H. U. Boden and S. Friedl, "Metabelian SL(n, $\mathbb{C}$) representations of knot groups III: deformations", *Q. J. Math.* **3** (2014), 817–840. Zbl 1309.57010

[Boden and Friedl 2014b] H. U. Boden and S. Friedl, "Metabelian SL(n, $\mathbb{C}$) representations of knot groups IV: twisted Alexander polynomials", *Math. Proc. Cambridge Philos. Soc.* **156**:1 (2014), 81–97. MR 3144211 Zbl 1303.57005

[Brown 1994] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics **87**, Springer, New York, 1994. MR 96a:20072 Zbl 0823.20049

[Burde 1967] G. Burde, "Darstellungen von Knotengruppen", *Math. Ann.* **173** (1967), 24–33. MR 35 #3652 Zbl 0146.45602

[Franz 1937] W. Franz, "Torsionsideale, Torsionsklassen und Torsion", *J. Reine Angew. Math.* **176** (1937), 113–125.

[Friedl et al. 2012] S. Friedl, T. Kim, and T. Kitayama, "Poincaré duality and degrees of twisted Alexander polynomials", *Indiana Univ. Math. J.* **61**:1 (2012), 147–192. MR 3029395 Zbl 1273.57009

[Frohman and Klassen 1991] C. D. Frohman and E. P. Klassen, “Deforming representations of knot groups in SU(2)”, *Comment. Math. Helv.* **66**:3 (1991), 340–361. MR 93a:57001 Zbl 0738.57001

[Goldman 1984] W. M. Goldman, “The symplectic nature of fundamental groups of surfaces”, *Adv. in Math.* **54**:2 (1984), 200–225. MR 86i:32042 Zbl 0574.32032

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977. MR 57 #3116 Zbl 0367.14001

[Heusener and Klassen 1997] M. Heusener and E. Klassen, “Deformations of dihedral representations”, *Proc. Amer. Math. Soc.* **125**:10 (1997), 3039–3047. MR 98b:57011 Zbl 1154.57004

[Heusener and Kroll 1998] M. Heusener and J. Kroll, “Deforming abelian SU(2)-representations of knot groups”, *Comment. Math. Helv.* **73**:3 (1998), 480–498. MR 99g:57008 Zbl 0910.57004

[Heusener and Medjerab 2014] M. Heusener and O. Medjerab, “Deformations of reducible representations of knot groups into $\mathrm{SL}(n, \mathbb{C})$”, arXiv:1402.4294, 2014. To appear in *Math. Slovaca*.

[Heusener and Porti 2005] M. Heusener and J. Porti, “Deformations of reducible representations of 3-manifold groups into $\mathrm{PSL}_2(\mathbb{C})$”, *Algebr. Geom. Topol.* **5** (2005), 965–997. MR 2006e:57016 Zbl 1082.57007

[Heusener and Porti 2011] M. Heusener and J. Porti, “Infinitesimal projective rigidity under Dehn filling”, *Geom. Topol.* **15**:4 (2011), 2017–2071. MR 2012j:57039 Zbl 1237.57016

[Heusener et al. 2001] M. Heusener, J. Porti, and E. Suárez Peiró, “Deformations of reducible representations of 3-manifold groups into $\mathrm{SL}_2(\mathbb{C})$”, *J. Reine Angew. Math.* **530** (2001), 191–227. MR 2002a:57002 Zbl 0964.57006

[Hillman et al. 2010] J. A. Hillman, D. S. Silver, and S. G. Williams, “On reciprocality of twisted Alexander invariants”, *Algebr. Geom. Topol.* **10**:2 (2010), 1017–1026. MR 2011j:57021 Zbl 1200.57005

[Humphreys 1975] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics **21**, Springer, New York, 1975. MR 53 #633 Zbl 0325.20039

[Humphreys 1995] J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs **43**, American Mathematical Society, Providence, RI, 1995. MR 97i:20057 Zbl 0834.20048

[Johnson and Millson 1987] D. Johnson and J. J. Millson, “Deformation spaces associated to compact hyperbolic manifolds”, pp. 48–106 in *Discrete groups in geometry and analysis*, edited by R. Howe, Progr. Math. **67**, Birkhäuser, Boston, MA, 1987. MR 88j:22010 Zbl 0664.53023

[Kirk and Livingston 1999] P. Kirk and C. Livingston, “Twisted Alexander invariants, Reidemeister torsion, and Casson–Gordon invariants”, *Topology* **38**:3 (1999), 635–661. MR 2000c:57010 Zbl 0883.57001

[Kitano 1996] T. Kitano, “Twisted Alexander polynomial and Reidemeister torsion”, *Pacific J. Math.* **174**:2 (1996), 431–442. MR 97g:57007 Zbl 0863.57001

[Kitano and Morifuji 2012] T. Kitano and T. Morifuji, “Twisted Alexander polynomials for irreducible $SL(2, \mathbb{C})$-representations of torus knots”, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **11**:2 (2012), 395–406. MR 3011996 Zbl 1255.57014

[Lubotzky and Magid 1985] A. Lubotzky and A. R. Magid, “Varieties of representations of finitely generated groups”, *Mem. Amer. Math. Soc.* **58**:336 (1985), xi+117. MR 87c:20021 Zbl 0598.14042

[Milnor 1962] J. Milnor, “A duality theorem for Reidemeister torsion”, *Ann. of Math.* (2) **76** (1962), 137–147. MR 25 #4526 Zbl 0108.36502

[Nagata 1961/1962] M. Nagata, “Complete reducibility of rational representations of a matric group”, *J. Math. Kyoto Univ.* **1** (1961/1962), 87–99. MR 26 #236 Zbl 0106.25201

[Newstead 1978] P. E. Newstead, *Introduction to moduli problems and orbit spaces*, Tata Institute Lectures on Mathematics and Physics **51**, Narosa Publishing House, New Delhi, 1978. MR 81k:14002 Zbl 0411.14003

[Popov 2008] V. L. Popov, "Irregular and singular loci of commuting varieties", *Transform. Groups* **13**:3-4 (2008), 819–837. MR 2009h:14095 Zbl 1189.17017

[Porti 1997] J. Porti, "Torsion de Reidemeister pour les variétés hyperboliques", *Mem. Amer. Math. Soc.* **128**:612 (1997), x+139. MR 98g:57034 Zbl 0928.57005

[de Rham 1967] G. de Rham, "Introduction aux polynômes d'un nœud", *Enseignement Math.* (2) **13** (1967), 187–194. MR 39 #2149 Zbl 0157.54803

[Richardson 1979] R. Richardson, "Commuting varieties of semisimple Lie algebras and algebraic groups", *Compos. Math.* **38** (1979), 311–327. MR 0535074 Zbl 0409.17006

[Serre 1978] J.-P. Serre, *Représentations linéaires des groupes finis*, revised ed., Hermann, Paris, 1978. MR 80f:20001 Zbl 0407.20003

[Shafarevich 1977] I. R. Shafarevich, *Basic algebraic geometry*, Springer, Berlin, 1977. MR 56 #5538 Zbl 0362.14001

[Shafarevich 1994] I. R. Shafarevich (editor), *Algebraic geometry. IV*, Encyclopaedia of Mathematical Sciences **55**, Springer, Berlin, 1994. MR 95g:14002 Zbl 0796.14002

[Shors 1991] D. J. Shors, *Deforming reducible representations of knot groups in SL(2)(C)*, Ph.D. thesis, University of California, Los Angeles, 1991, available at http://search.proquest.com/docview/303918617.

[Springer 1977] T. A. Springer, *Invariant theory*, Lecture Notes in Mathematics **585**, Springer, Berlin, 1977. MR 56 #5740 Zbl 0346.20020

[Turaev 1986] V. G. Turaev, "Reidemeister torsion in knot theory", *Uspekhi Mat. Nauk* **41**:1(247) (1986), 97–147, 240. In Russian; translated in *Russian Math. Surveys* **41**:1 (1986), 119–182. MR 87i:57009 Zbl 0602.57005

[Wada 1994] M. Wada, "Twisted Alexander polynomial for finitely presentable groups", *Topology* **33**:2 (1994), 241–256. MR 95g:57021 Zbl 0822.57006

[Weil 1964] A. Weil, "Remarks on the cohomology of groups", *Ann. of Math.* (2) **80** (1964), 149–157. MR 30 #199 Zbl 0192.12802

Received July 14, 2014. Revised March 11, 2015.

MICHAEL HEUSENER
LABORATOIRE DE MATHÉMATIQUES
CLERMONT UNIVERSITÉ AUVERGNE, UNIVERSITÉ BLAISE PASCAL
BP 10448, F-63000 CLERMONT-FERRAND
CNRS, UMR 6620, LM, F-63171 AUBIÈRE
FRANCE
heusener@math.univ-bpclermont.fr

JOAN PORTI
DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
CERDANYOLA DEL VALLES
08193 BARCELONA
SPAIN
porti@mat.uab.cat

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2015 is US $420/year for the electronic version, and $570/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 277 No. 2 October 2015

0030-8730(201510)277:2;1-#