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PATTERSON–SULLIVAN CURRENTS, GENERIC STRETCHING FACTORS AND THE ASYMMETRIC LIPSCHITZ METRIC FOR OUTER SPACE

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We quantitatively relate the Patterson–Sullivan currents and generic stretching factors for free group automorphisms to the asymmetric Lipschitz metric on outer space and to Guirardel's intersection number. Thus we show that, given $N \geq 2$ and $\varepsilon > 0$, there exists a constant $c = c(N, \varepsilon) > 0$ such that for any two trees $T, S \in \mathrm{cv}_N$ of covolume 1 and injectivity radius $\geq \varepsilon$, we have

$$|\log\langle S, \mu_T\rangle - d_L(T, S)| \leq c,$$

where d_L is the asymmetric Lipschitz metric on the Culler–Vogtmann outer space, and where μ_T is the (appropriately normalized) Patterson–Sullivan current corresponding to T. As a corollary, we show there exist constants $C_1 \geq 1$ and $C_2 \geq 1$ (depending on N, ε) such that for any T, S as above we have

$$\frac{1}{C_1}\log i_c(T, S) - C_2 \leq \log\langle S, \mu_T\rangle \leq C_1 \log i_c(T, S) + C_2,$$

where i_c is the combinatorial version of Guirardel's intersection number. We apply these results to the properties of generic stretching factors of free group automorphisms. In particular, we show that for any $N \geq 2$, there exists a constant $0 < \rho_N < 1$ such that for every automorphism φ of $F_N = F(A)$, we have

$$0 < \rho_N \leq \frac{\lambda_A(\varphi)}{\Lambda_A(\varphi)} \leq 1.$$

Here λ_A is the generic stretching factor of φ with respect to the free basis A of F_N and $\Lambda_A(\varphi)$ is the extremal stretching factor of φ with respect to A.

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1. Introduction

For an integer $N \geq 2$, the *unprojectivized outer space* cv_N is the set of all $\mathbb{R}$-trees equipped with a free discrete minimal isometric action of F_N, considered up to an F_N-equivariant isometry. We denote by cv_N^1 the set of all $T \in \mathrm{cv}_N$ such that the metric graph T/F_N has volume 1. The closure $\overline{\mathrm{cv}}_N$ of cv_N with respect to the equivariant Gromov–Hausdorff convergence topology (or equivalently [Paulin 1989], with respect to the hyperbolic length function topology) consists of all *very small* minimal isometric actions of F_N on $\mathbb{R}$-trees, again up to an F_N-equivariant isometry. There is a natural action of $\mathbb{R}_{>0}$ on $\overline{\mathrm{cv}}_N$ by multiplying the metric on a tree by a positive scalar. The subset cv_N of $\overline{\mathrm{cv}}_N$ is invariant under this action, and the quotient $\mathrm{CV}_N = \mathrm{cv}_N/\mathbb{R}_{>0}$ is the *projectivized outer space*, originally introduced by Culler and Vogtmann [1986]. The quotient $\overline{\mathrm{CV}}_N = \overline{\mathrm{cv}}_N/\mathbb{R}_{>0}$ is compact, and is called the *Thurston compactification* of CV_N. All of the above spaces admit natural $\mathrm{Out}(F_N)$-actions. The space CV_N is naturally $\mathrm{Out}(F_N)$-equivariantly homeomorphic to cv_N^1, but it is useful to remember that technically cv_N^1 and CV_N are distinct objects.

There are three main quantitative tools for studying points of $\overline{\mathrm{cv}}_N$. The first is the so-called "asymmetric Lipschitz distance". If $T \in \mathrm{cv}_N$ and $S \in \overline{\mathrm{cv}}_N$, the *extremal Lipschitz distortion* is given by

$$\Lambda(T,S) := \sup_{w \in F_N \setminus \{1\}} \frac{\|w\|_S}{\|w\|_T}.$$

It is known (see [Francaviglia and Martino 2011] for details) that this supremum is actually a maximum, and that $\Lambda(T,S)$ is the infimum of the Lipschitz constants of all the F_N-equivariant Lipschitz maps $T \to S$. It is also known that for all $T, S \in \mathrm{cv}_N^1$, we have $\Lambda(T,S) \geq 1$, and that the equality holds if and only if $T = S$. The *asymmetric Lipschitz distance* is defined as $d_L(T,S) := \log \Lambda(T,S)$, where $T, S \in \mathrm{cv}_N^1$. Although it is usually the case that $d_L(T,S) \neq d_L(S,T)$, the asymmetric distance d_L satisfies all the other properties of being a metric, and it is known that the topology defined by d_L on cv_N^1 coincides with the standard subspace topology for $\mathrm{cv}_N^1 \subseteq \mathrm{cv}_N$. Moreover, for any $T, S \in \mathrm{cv}_N^1$, there exists an (in general nonunique) d_L-geodesic path from T to S in cv_N^1, given by natural "folding lines" [loc. cit.]. The asymmetric distance d_L is a useful tool in the study of the geometry of $\mathrm{Out}(F_N)$ and it has found significant recent applications; see, for example, [Algom-Kfir 2011; 2013; Algom-Kfir and Bestvina 2012; Bestvina 2011; Francaviglia and Martino 2011; 2012; Ladra et al. 2015; White 1991].

Another two important quantitative tools for studying outer space are two notions of a "geometric intersection number". The first of these was introduced by Guirardel [2005] in the general setting of groups acting by isometries on $\mathbb{R}$-trees. Guirardel's intersection number $i(T,S)$ (where $T,S\in\overline{\mathrm{cv}}_N$) is defined as the covolume of the "core" for the action of F_N on $T\times S$. Guirardel's intersection number is symmetric and $\mathrm{Out}(F_N)$-invariant, and for $T,S\in\mathrm{cv}_N$, one always has $0\le i(T,S)<\infty$. However, for trees in $\partial\mathrm{cv}_N=\overline{\mathrm{cv}}_N\setminus\mathrm{cv}_N$, it is often the case that $i(T,S)=\infty$ and $i(\,\cdot\,,\cdot\,)$ is discontinuous when viewed as a function on $\overline{\mathrm{cv}}_N\times\overline{\mathrm{cv}}_N$. Still, Guirardel's intersection number is a highly useful tool when studying the asymptotic geometry of cv_N itself, particularly when looking at orbits of subgroups of $\mathrm{Out}(F_N)$ in cv_N^1 and cv_N. Examples of such applications can be found in [Behrstock et al. 2010; Clay et al. 2015; Clay and Pettet 2010; 2012b; Guirardel 2005; Horbez 2012].

The second notion of a "geometric intersection number" was introduced in [Kapovich and Lustig 2009]. There we constructed a *geometric intersection form* $\langle\,\cdot\,,\cdot\,\rangle:\overline{\mathrm{cv}}_N\times\mathrm{Curr}(F_N)\to\mathbb{R}_{\ge 0}$, where $\mathrm{Curr}(F_N)$ is the space of *geodesic currents* on F_N. See Section 2C below and [Kapovich 2005; 2006; Kapovich and Lustig 2007; 2009] for the more information and the background on geodesic currents. The geometric intersection form is continuous, $\mathrm{Out}(F_N)$-equivariant, and, importantly, it always gives a finite output; that is, for every $T\in\overline{\mathrm{cv}}_N$ and $\mu\in\mathrm{Curr}(F_N)$, one has $0\le\langle T,\mu\rangle<\infty$. If $T\in\overline{\mathrm{cv}}_N$ and $g\in F_N\setminus\{1\}$ then $\langle T,\eta_g\rangle=\|g\|_T$, where $\eta_g\in\mathrm{Curr}(F_N)$ is the "counting current" associated with g. By its very definition, $\langle\,\cdot\,,\cdot\,\rangle$ is an asymmetric gadget. However, its good properties, including finiteness and global continuity on $\overline{\mathrm{cv}}_N$, make the geometric intersection form a useful tool that has also found a number of significant applications to the study of the dynamics and geometry of $\mathrm{Out}(F_N)$. See, for example, [Bestvina and Feighn 2010; Bestvina and Reynolds 2012; Carette et al. 2012; Clay and Pettet 2012a; Coulbois and Hilion 2014; Coulbois et al. 2008b; Hamenstädt 2014a; 2014b; Kapovich and Lustig 2009; 2010a; 2010b; Mann and Reynolds 2013; Reynolds 2012].

For $\varepsilon\ge 0$, we denote by $\mathrm{cv}_{N,\varepsilon}^1$ the set of all $T\in\mathrm{cv}_N^1$ such that the length of the shortest simple closed loop in T/F_N is at least ε. The set $\mathrm{cv}_{N,\varepsilon}^1$ is called the *ε-thick* part of cv_N^1. Horbez [2012] showed that, for any fixed $\varepsilon>0$, if $T,S\in\mathrm{cv}_{N,\varepsilon}^1$, one has

$$(\ddagger)\qquad \frac{1}{K_1}\log i_c(T,S)-K_2\le d_L(T,S)\le K_1\log i_c(T,S)+K_2$$

for some constants $K_1\ge 1$, $K_2\ge 0$ depending only on N and ε. Here $i_c(T,S)$ is the combinatorial version of Guirardel's intersection number, where $i_c(T,S)$ is defined as the number of 2-cells in $\mathrm{Core}(T\times S)/F_N$, while $i(T,S)$ is defined as the sum of the areas of all the 2-cells in $\mathrm{Core}(T\times S)/F_N$. Thus if, for $S,T\in\mathrm{cv}_N^1$, the trees $T_0,S_0\in\mathrm{cv}_N$ are obtained from T and S by making all edges have length 1, then $i_c(T,S):=i(T_0,S_0)$. Also, following the usual convention, in ($\ddagger$) we interpret $\log 0$ as $\log 0=0$.

In the present paper, for $T,S\in\mathrm{cv}_{N,\varepsilon}^1$, we relate $\Lambda(T,S)$ to a natural quantity defined in terms of $\langle\,\cdot\,,\cdot\,\rangle$. Via Horbez' result, this connection also relates the geometric intersection form $\langle\cdot,\cdot\rangle$ to Guirardel's geometric intersection number $i(\cdot,\cdot)$. Following the results of Furman [2002] in the general set-up of word-hyperbolic groups, Kapovich and Nagnibeda [2007] associated to every $T \in \mathrm{cv}_N$ its *Patterson–Sullivan current*. In general, the Patterson–Sullivan current is naturally defined only up to a multiplication by a positive scalar. Normalizing by the geometric intersection number with T provides a canonical choice. Thus for a tree $T \in \mathrm{cv}_N$, we denote by $\mu_T \in \mathrm{Curr}(F_N)$ the *Patterson–Sullivan current* associated to T, normalized so that $\langle T, \mu_T\rangle = 1$. We refer the reader to Section 4 below and to [Furman 2002; Kapovich and Nagnibeda 2007; 2010] for the precise definitions and background information about the Patterson–Sullivan currents. A key result obtained by Kapovich and Nagnibeda [2007] shows that the map $J_{PS} : \mathrm{cv}_N^1 \to \mathrm{Curr}(F_N)$, $T \mapsto \mu_T$ is a continuous $\mathrm{Out}(F_N)$-equivariant embedding.

Our main result (see Theorem 4.2 below) is:

Theorem 1.1. *Let $N \geq 2$ and $\varepsilon > 0$. Then there exist constants $0 < \delta_1 \leq \delta_2$ such that for every $T \in \mathrm{cv}_{N,\varepsilon}^1$ and every $S \in \overline{\mathrm{cv}}_N$ we have*

$$\delta_1 \leq \frac{\langle S, \mu_T\rangle}{\Lambda(T,S)} \leq \delta_2.$$

Therefore there exists a constant $c = c(N,\varepsilon) > 0$ such that for every $T \in \mathrm{cv}_{N,\varepsilon}^1$ and $S \in \mathrm{cv}_N^1$ we have

$$|\log\langle S, \mu_T\rangle - d_L(T,S)| \leq c.$$

Using the result of Horbez [2012] stated in (‡) above, Theorem 1.1 directly implies (using the notation introduced after (‡)):

Corollary 1.2. *Let $N \geq 2$ and $\varepsilon > 0$. There exist constants $C_1, C_2 \geq 1$ such that for any $T, S \in \mathrm{cv}_{N,\varepsilon}^1$, we have*

$$\frac{1}{C_1}\log i_c(T,S) - C_2 \leq \log\langle S, \mu_T\rangle \leq C_1 \log i_c(T,S) + C_2.$$

The proof of Theorem 1.1 relies on several results regarding geodesic currents, particularly one from [Kapovich and Lustig 2009] about the continuity of the already mentioned geometric intersection form on $\overline{\mathrm{cv}}_N \times \mathrm{Curr}(F_N)$, and a result from [Kapovich and Nagnibeda 2007] saying that the Patterson–Sullivan map $\mathrm{cv}_N^1 \to \mathrm{Curr}(F_N)$, $T \mapsto \mu_T$, is a continuous $\mathrm{Out}(F_N)$-equivariant embedding. The crucial point in the argument uses a result from [Kapovich and Lustig 2010a] characterizing the case $\langle S, \nu\rangle = 0$, where $S \in \overline{\mathrm{cv}}_N$ and $\nu \in \mathrm{Curr}(F_N)$ are arbitrary. This characterization implies that every current μ with full support (such as the Patterson–Sullivan current μ_T for $T \in \mathrm{cv}_N^1$) is *filling*, that is, satisfies $\langle S, \mu\rangle > 0$ for every $S \in \overline{\mathrm{cv}}_N$. Modulo the tools mentioned above, the proof of Theorem 1.1 is not difficult (although it does require an extra trick exploiting the $\mathrm{Out}(F_N)$-equivariant nature of certain functions and some nice properties of d_L). Still, Theorem 1.1

and its applications obtained here do provide a conceptual clarification regarding the quantitative relationships between the two notions of a geometric intersection number used in the study of $\mathrm{Out}(F_N)$, and about their relationship to the asymmetric Lipschitz distance.

One of our main motivations for this paper has been to better understand the properties of "generic stretching factors" for free group automorphisms.

Proposition-Definition 1.3 [Kaimanovich et al. 2007]. *For any free basis A of F_N and any $S \in \overline{\mathrm{cv}}_N$, there exists a number $\lambda_A(S) \geq 0$ with the following property.*

For a.e. trajectory $\xi = y_1 y_2 \cdots y_n \cdots$ of the simple nonbacktracking random walk on F_N with respect to A (that is, for a "random" geodesic ray $\xi = y_1 y_2 \cdots y_n \cdots$ over $A^{\pm 1}$ with $y_i \in A^{\pm 1}$), we have $\|y_1 y_2 \cdots y_n\|_A = n + o(n)$ and

$$\lim_{n\to\infty} \frac{\|y_1 y_2 \cdots y_n\|_S}{n} = \lim_{n\to\infty} \frac{\|y_1 y_2 \cdots y_n\|_S}{\|y_1 y_2 \cdots y_n\|_A} = \lambda_A(S).$$

The number $\lambda_A(S)$ is called [Kapovich 2006; Kaimanovich et al. 2007] the generic stretching factor *of S with respect to A.*

The term "nonbacktracking" in "nonbacktracking simple random walk" refers to the fact that for this random walk, if $x, y \in A \cup A^{-1}$, the transition probability for x to be followed by y is equal to $1/(2N-1)$ if $y \neq x^{-1}$ and is equal to 0 if $y = x^{-1}$. Thus the trajectories of this random walk are semi-infinite freely reduced words over $A^{\pm 1}$. Informally, the generic stretching factor $\lambda_A(S) \geq 0$ captures the distortion $\|y_1 y_2 \cdots y_n\|_S / n$, where $y_1 \cdots y_n$ is a "random" freely reduced word of length n over A, as n tends to infinity. The existence of $\lambda_A(S) \geq 0$ follows from general ergodic-theoretic considerations, as observed in [Kaimanovich et al. 2007]. As noted in Remark 4.6 below, one actually has $\lambda_A(S) > 0$ for every $S \in \overline{\mathrm{cv}}_N$.

Let A be a free basis of F_N and consider the Cayley tree $T_A \in \mathrm{cv}_N$, with all edges of length $1/N$, so that $T_A \in \mathrm{cv}_N^1$. Thus for every $w \in F_N$, we have $\|w\|_A = N\|w\|_{T_A}$, where $\|w\|_A$ is the cyclically reduced length of w over $A^{\pm 1}$. It is known that the Patterson–Sullivan current μ_{T_A} is equal to the "uniform current" ν_A on F_N corresponding to A. Using the interpretation of $\langle S, \nu_A \rangle$ as the "generic stretching factor" $\lambda_A(S)$ of $S \in \mathrm{cv}_N$ with respect to A [Kapovich 2006], as a consequence of Theorem 1.1 we also obtain (see Theorem 4.7 below):

Corollary 1.4. *Let $N \geq 2$. There exists a constant $\delta = \delta(N) \in (0, 1)$ with the following property*:

For any free basis A of F_N and any $S \in \overline{\mathrm{cv}}_N$, we have

$$(\dagger) \qquad 0 < \delta \leq \frac{\lambda_A(S)}{\Lambda(T_A, S)} \leq \frac{1}{N}.$$

We are particularly interested in relationships between generic stretching factors and extremal stretching factors in the context of Cayley trees of F_N and of elements of $\mathrm{Out}(F_N)$. Note that if A is a free basis of A then $NT_A \in \mathrm{cv}_N$ is the standard Cayley graph of F_N with respect to A, where all edges have length 1.

If $\varphi \in \mathrm{Out}(F_N)$ and $w \in F_N$, then, since φ is an outer automorphism, it acts on the conjugacy classes of elements of F_N (rather than on elements of F_N). By convention, for $\varphi \in \mathrm{Out}(F_N)$ and $w \in F_N$, if $\varphi(w)$ appears in an expression that depends only on the conjugacy class $\varphi([w])$, we will use $\varphi(w)$ to mean any representative of that conjugacy class.

Definition 1.5 (extremal and generic stretching factors of automorphisms). Let A be a free basis of F_N and let $\varphi \in \mathrm{Out}(F_N)$.

Define

$$\Lambda_A(\varphi) := \Lambda(T_A, T_A\varphi) = \sup_{w \neq 1} \frac{\|\varphi(w)\|_A}{\|w\|_A} = e^{d_L(T_A, T_A\varphi)},$$

and refer to $\Lambda_A(\varphi)$ as the *extremal stretching factor* for φ with respect to A.

Also, define $\lambda_A(\varphi) := \lambda_A(NT_A\varphi) = N\lambda_A(T_A\varphi)$.

Thus for a.e. trajectory $\xi = y_1 \cdots y_n \cdots$ of the simple nonbacktracking random walk on F_N with respect to A, we have

$$\lambda_A(\varphi) = \lim_{n\to\infty} \frac{\|\varphi(y_1 y_2 \cdots y_n)\|_A}{n} = \lim_{n\to\infty} \frac{\|\varphi(y_1 y_2 \cdots y_n)\|_A}{\|y_1 y_2 \cdots y_n\|_A}.$$

We call $\lambda_A(\varphi)$ the *generic stretching factor* of φ with respect to A.

Thus $\Lambda_A(\varphi)$ measures the maximal distortion $\|\varphi(w)\|_A/\|w\|_A$ as w varies over all nontrivial elements of F_N, while $\lambda_A(\varphi)$ captures the "generic distortion" $\|\varphi(w)\|_A/\|w\|_A$, where w is a "long random" freely reduced (or cyclically reduced) word over $A^{\pm 1}$. In practice, $\Lambda_A(\varphi)$ is easy to compute since it is known (see, e.g., [Francaviglia and Martino 2011]) that $\Lambda_A(\varphi) = \max_{1\le\|w\|\le 2}(\|\varphi(w)\|_A/\|w\|_A)$.

The generic stretching factors $\lambda_A(\varphi)$ were introduced in [Kaimanovich et al. 2007] and further studied in [Francaviglia 2009; Kapovich 2006; Kapovich and Lustig 2010a; Sharp 2010]. In particular, it is proved in [Kaimanovich et al. 2007] that for every $\varphi \in \mathrm{Out}(F_N)$, the number $\lambda_A(\varphi)$ is rational, and moreover, $2N\lambda_A(\varphi) \in \mathbb{Z}[1/(2N-1)]$ and there exists an algorithm that, given φ, computes $\lambda_A(\varphi)$. The definitions directly imply that $\lambda_A(\varphi) \le \Lambda_A(\varphi)$. However, other than this fact, the quantitative relationship between $\Lambda_A(\varphi)$ and $\lambda_A(\varphi)$ remained unclear.

Let $N \ge 2$ and $F_N = F(a_1, \dots, a_N)$ with $A = \{a_1, \dots, a_N\}$. Define

$$\rho_N := \inf_{\varphi \in \mathrm{Out}(F_N)} \frac{\lambda_A(\varphi)}{\Lambda_A(\varphi)}.$$

For every $\varphi \in \mathrm{Out}(F_N)$, we have $T_A, T_A\varphi \in \mathrm{cv}^1_{N,\varepsilon}$ with $\varepsilon = 1/N$, and thus Corollary 1.4 directly implies:

Theorem 1.6. *For every $N \ge 2$ we have $\rho_N > 0$.*

Therefore for every $\varphi \in \mathrm{Out}(F_N)$, we have

$$0 < \rho_N \leq \frac{\lambda_A(\varphi)}{\Lambda_A(\varphi)} \leq 1.$$

Our proof that $\rho_N > 0$ does not give any explicit quantitative information about ρ_N. It would be interesting to find some explicit bounds from above and below for ρ_N, and perhaps to even compute ρ_N, at least for small values of N. We show in Proposition 7.1 that $\lim_{N\to\infty} \rho_N = 0$ and that $\rho_N = O(1/N)$.

As another application, we obtain (see Corollary 5.3 below):

Corollary 1.7. *Let $N \geq 2$ and $F_N = F(a_1, \dots, a_n)$ with $A = \{a_1, \dots, a_N\}$. There exists $D = D(N) \geq 1$ such that for every $\varphi \in \mathrm{Out}(F_N)$ we have*

$$\frac{1}{D} \log \lambda_A(\varphi) \leq \log \lambda_A(\varphi^{-1}) \leq D \log \lambda_A(\varphi).$$

Let $\varphi \in \mathrm{Out}(F_N)$. Recall that the *algebraic stretching factor* $\lambda(\varphi)$ is defined as

$$\lambda(\varphi) := \sup_{w\in F_N, w\neq 1} \lim_{n\to\infty} \sqrt[n]{\|\varphi^n(w)\|_S},$$

where $S \in \mathrm{cv}_N$ is an arbitrary base point. It is known that the limit in the last equality always exists, that this definition of $\lambda(\varphi)$ does not depend on the choice of $S \in \mathrm{cv}_N$, and that we always have $\lambda(\varphi) \geq 1$. An element $\varphi \in \mathrm{Out}(F_N)$ is called *exponentially growing* if $\lambda(\varphi) > 1$, and *polynomially growing* if $\lambda(\varphi) = 1$. Indeed, it is known (see, for example, [Levitt 2009]), that φ is polynomially growing if and only if for every $w \in F_N$ and $S \in \mathrm{cv}_N$, the sequence $\|\varphi^n(w)\|_S$ is bounded above by a polynomial in n.

The algebraic stretching factor $\lambda(\varphi)$ can be read off from any relative train-track representative $f : \Gamma \to \Gamma$ of φ as the maximum of the Perron–Frobenius eigenvalues for any of the canonical irreducible diagonal blocks of the (nonnegative) transition matrix $M(f)$.

As another application of the results of this paper, we explain how the generic stretching factor $\lambda_A(\varphi^n)$ grows in terms of n for an arbitrary $\varphi \in \mathrm{Out}(F_N)$. Thus we obtain (see Theorem 5.6 below) the following result, which answers Problem 9.2 posed in [Kaimanovich et al. 2007]:

Theorem 1.8. *Let A be a free basis of F_N, let $\varphi \in \mathrm{Out}(F_N)$ and let $\lambda(\varphi)$ be the algebraic stretching factor of φ. Then there exist constants $c_1, c_2 > 0$ and an integer $m \geq 0$ such that for every $n \geq 1$, we have*

$$c_1\, \lambda(\varphi)^n\, n^m \leq \lambda_A(\varphi^n) \leq c_2\, \lambda(\varphi)^n\, n^m.$$

Moreover, if φ admits an expanding train-track representative with an irreducible transition matrix (e.g., if φ is fully irreducible), then $m = 0$ and $\lambda(\varphi) > 1$.

The "polynomial growth degree" m in this result is bounded above by the number of strata of any relative train track representative f as above which have PF-eigenvalue equal to λ, and it has been determined precisely by Levitt [2009], see the proof of Proposition 5.4 below.

2. Preliminaries

2A. ***Basic terminology and notations related to outer space.*** We denote by cv_N the unprojectivized outer space, that is, the space of all free discrete minimal isometric actions of F_N on $\mathbb{R}$-trees, considered up to F_N-equivariant isometry. Denote by $\overline{\mathrm{cv}}_N$ the closure of cv_N in the equivariant Gromov–Hausdorff convergence topology (or, equivalently, in the hyperbolic length functions topology). It is known [Bestvina and Feighn 1993; Cohen and Lustig 1995; Guirardel 1998] that $\overline{\mathrm{cv}}_N$ consists of all the *very small* nontrivial minimal isometric actions of F_N on $\mathbb{R}$-trees, again considered up to F_N-equivariant isometry. Recall that a point $T \in \overline{\mathrm{cv}}_N$ is uniquely determined by its *translation length function* $\|\cdot\|_T : F_N \to [0, \infty)$, where for $w \in F_N$, we have $\|w\|_T = \inf_{x \in T} d(x, wx) = \min_{x \in T} d(x, wx)$.

The space $\overline{\mathrm{cv}}_N$ has a natural right $\mathrm{Out}(F_N)$-action, where for $w \in F_N$ and $T \in \overline{\mathrm{cv}}_N$, we have $\|w\|_{T\varphi} = \|\varphi(w)\|_T$. It is sometimes useful to convert this action to a left $\mathrm{Out}(F_N)$-action by setting $\varphi T := T\varphi^{-1}$. Define

$$\mathrm{cv}_N^1 := \{T \in \mathrm{cv}_N \mid \mathrm{vol}(T/F_N) = 1\},$$

and refer to cv_N^1 as the *volume-normalized outer space* or just *normalized outer space*. Then cv_N is an open dense $\mathrm{Out}(F_N)$-invariant subset of $\overline{\mathrm{cv}}_N$, and cv_N^1 is a closed $\mathrm{Out}(F_N)$-invariant subset of cv_N (but of course cv_N^1 is not closed in $\overline{\mathrm{cv}}_N$).

There is a natural action of $\mathbb{R}_{>0}$ on cv_N and $\overline{\mathrm{cv}}_N$ by scalar multiplication, which yields the corresponding *projectivizations* $\mathrm{CV}_N = \mathrm{cv}_N/\mathbb{R}_{>0}$ and $\overline{\mathrm{CV}}_N = \overline{\mathrm{cv}}_N/\mathbb{R}_{>0}$. For a tree $T \in \overline{\mathrm{cv}}_N$, we denote its projective class in $\overline{\mathrm{CV}}_N$ by $[T]$. Thus $[T] = \{cT \mid c > 0\}$. Note that CV_N is canonically $\mathrm{Out}(F_N)$ equivariantly homeomorphic to cv_N^1, but it is still important to remember that technically CV_N and cv_N^1 are distinct objects.

For $\varepsilon > 0$, we denote by $\mathrm{cv}_{N,\varepsilon}^1$ the set of all $T \in \mathrm{cv}_N^1$ such that the shortest nontrivial immersed circuit in the metric graph T/F_N has length $\geq \varepsilon$. Equivalently, $\mathrm{cv}_{N,\varepsilon}^1$ is the set of all $T \in \mathrm{cv}_N^1$ such that for every $w \in F_N \setminus \{1\}$, we have $\|w\|_T \geq \varepsilon$. For every $\varepsilon > 0$, the set $\mathrm{cv}_{N,\varepsilon}^1 \subseteq \mathrm{cv}_N^1$ is a closed $\mathrm{Out}(F_N)$-invariant subspace, and the quotient $\mathrm{cv}_{N,\varepsilon}^1/\mathrm{Out}(F_N)$ is compact.

A *chart* on F_N is an isomorphism $\alpha : F_N \to \pi_1(\Gamma, p)$, where Γ is a finite connected graph with all vertices of degree ≥ 3 and where p is a base vertex in Γ (which is usually suppressed). Every such α defines an open cone in cv_N consisting of assigning arbitrary positive lengths to edges of Γ and then lifting this assignment to the universal cover $\widetilde{\Gamma}$ to get an element $T \in \mathrm{cv}_N$. The intersection of such an open cone with cv_N^1 is an open simplex Δ in cv_N^1 of dimension $m-1$, where m is the number of nonoriented edges of Γ. Every point $T \in \mathrm{cv}_N$ belongs to a unique open cone of this form, and every point of cv_N^1 belongs to a unique such open simplex Δ.

The space $\overline{\mathrm{CV}}_N$ is known to be compact and finite-dimensional.

2B. ***Asymmetric Lipschitz distance.*** For points $T \in \mathrm{cv}_N$ and $S \in \overline{\mathrm{cv}}_N$, define

$$\Lambda(T,S) = \sup_{w \in F_N \setminus \{1\}} \frac{\|w\|_S}{\|w\|_T}.$$

If $T, S \in \mathrm{cv}_N^1$, we also define $d_L(T,S) := \log \Lambda(T,S)$. As noted in the Introduction, for $T, S \in \mathrm{cv}_N^1$, the quantity $d_L(T,S)$ is often called the *asymmetric Lipschitz distance* from T to S.

Remark 2.1. If $T \in \mathrm{cv}_N$ and $S \in \overline{\mathrm{cv}}_N$ then $0 < \Lambda(T,S) < \infty$. Moreover, it is known [Francaviglia and Martino 2011; White 1991] that for any open simplex $\Delta \subset \mathrm{cv}_N^1$ as in Section 2A, there exists a finite subset $C_\Delta \subseteq F_N \setminus \{1\}$ such that for every $T \in \Delta$ and every $S \in \overline{\mathrm{cv}}_N$, we have

$$\Lambda(T,S) = \max_{w \in C_\Delta} \frac{\|w\|_S}{\|w\|_T}.$$

The set C_Δ can be chosen to be contained in the subset of all elements which are represented by paths that cross at most twice over every nonoriented edge of $\Gamma = T/F_N$ for $T \in \Delta$.

Note also that from the definition, we see that for every $T \in \mathrm{cv}_N$, $S \in \overline{\mathrm{cv}}_N$ and $\varphi \in \mathrm{Out}(F_N)$, one has $\Lambda(T,S) = \Lambda(\varphi T, \varphi S)$.

2C. ***Geodesic currents.*** We refer the reader to [Kapovich 2006; Kapovich and Lustig 2007; 2009; 2010a] for detailed background on geodesic currents, and we only recall a few basic definitions and facts here. Let $\partial^2 F_N = \partial F_N \times \partial F_N \setminus \mathrm{diag}$, and endow $\partial^2 F_N$ with the subspace topology and with the diagonal F_N-action by translations. A *geodesic current* on F_N is a positive Borel measure μ on $\partial^2 F_N$ such that μ is finite on compact subsets, F_N-invariant and "flip"-invariant (where the "flip" map $\partial^2 F_N \to \partial^2 F_N$ interchanges the two coordinates). The space of all geodesic currents on F_N is denoted $\mathrm{Curr}(F_N)$. The space $\mathrm{Curr}(F_N)$ comes equipped with a natural weak*-topology and a natural left $\mathrm{Out}(F_N)$-action by affine homeomorphisms.

Let $\alpha : F_N \to \pi_1(\Gamma, p)$ be a chart on F_N, and consider $\widetilde{\Gamma}$ with the simplicial metric, where every edge has length 1. Then there is a natural F_N-equivariant quasi-isometry (given for any point $p \in \widetilde{\Gamma}$ by the orbit map $F_N \to \widetilde{\Gamma}$, $g \mapsto gp$) between F_N and $\widetilde{\Gamma}$, which induces a canonical F_N-equivariant homeomorphism between

∂F_N and $\partial\widetilde{\Gamma}$. We will therefore identify ∂F_N with $\partial\widetilde{\Gamma}$ using this homeomorphism without invoking it explicitly, whenever it is convenient.

A nondegenerate geodesic segment γ in $\widetilde{\Gamma}$ defines a *cylinder set* $\mathrm{Cyl}_\alpha(\gamma)$ consisting of all $(X, Y) \in \partial^2 F_N$ such that the geodesic from X to Y in $\widetilde{\Gamma}$ passes through γ (in the correct direction). The sets $\mathrm{Cyl}_\alpha(\gamma)$, as γ varies among all nondegenerate geodesic edge-paths in $\widetilde{\Gamma}$, are compact and open, and form a basis for the topology on $\partial^2 F_N$. Note that for $w \in F_N$, we have $\mathrm{Cyl}_\alpha(w\gamma) = w\,\mathrm{Cyl}_\alpha(\gamma)$. If $\mu \in \mathrm{Curr}(F_N)$ and v is a nondegenerate reduced edge-path in Γ, we define the *weight* $\langle v, \mu\rangle_\alpha := \mu(\mathrm{Cyl}_\alpha(\gamma))$, where γ is any lift of v. Since the measure μ is F_N-invariant, this definition does not depend on the specific choice of the lift γ of v to $\widetilde{\Gamma}$. A current μ is uniquely determined by its collection of weights with respect to a given chart. Moreover, if $\mu_n, \mu \in \mathrm{Curr}(F_N)$ and α is a chart as above, then $\lim_{n\to\infty} \mu_n = \mu$ in $\mathrm{Curr}(F_N)$ if and only if for every nondegenerate reduced edge-path v in Γ, we have $\lim_{n\to\infty}\langle v, \mu_n\rangle_\alpha = \langle v, \mu\rangle_\alpha$.

For every $w \in F_N \setminus \{1\}$, there is an associated *counting current* $\eta_w \in \mathrm{Curr}(F_N)$, which depends only on the conjugacy class $[w]$ of w in F_N and satisfies $\eta_{w^{-1}} = \eta_w$ and $\eta_{w^n} = n\,\eta_w$ for all integers $n \geq 1$, and such that $\varphi\,\eta_w = \eta_{\varphi(w)}$ for all $\varphi \in \mathrm{Out}(F_N)$, $w \in F_N \setminus \{1\}$. The precise definition of η_w is not important at the moment, but we will recall some of its basic properties later, as necessary. The set $\{c\eta_w \mid c > 0, w \in F_N, w \neq 1\}$ of the so-called *rational currents* is dense in $\mathrm{Curr}(F_N)$.

Be aware that, in general, for a representative (even a train-track representative) $f : \Gamma \to \Gamma$ of φ, one has $\langle v, \varphi\mu\rangle_\alpha \neq \langle [f(v)], \mu\rangle_\alpha$, where $[f(v)]$ denotes the edge-path obtained from $f(v)$ by reduction (that is, the iterative contraction of any backtracking path).

2D. *Intersection form.* Kapovich and Lustig [2009] proved the existence of a continuous *geometric intersection form* between points of $\overline{\mathrm{cv}}_N$ and geodesic currents:

Proposition 2.2 [Kapovich and Lustig 2009]. *There exists a unique continuous function* $\langle\cdot,\cdot\rangle : \overline{\mathrm{cv}}_N \times \mathrm{Curr}(F_N) \to [0, \infty)$, *called the* geometric intersection form, *with the following properties*:

(1) *For any* $\mu_1, \mu_2 \in \mathrm{Curr}(F_N)$, $T \in \overline{\mathrm{cv}}_N$, $c_1, c_2 \geq 0$ *and* $r > 0$, *we have*

$$\langle rT, c_1\mu_1 + c_2\mu_2\rangle = rc_1\langle T, \mu_1\rangle + rc_2\langle T, \mu_2\rangle.$$

(2) *For any* $T \in \overline{\mathrm{cv}}_N$, $\mu \in \mathrm{Curr}(F_N)$ *and* $\varphi \in \mathrm{Out}(F_N)$, *we have*

$$\langle\varphi T, \varphi\mu\rangle = \langle T, \mu\rangle.$$

(3) *For any* $T \in \overline{\mathrm{cv}}_N$ *and* $w \in F_N \setminus \{1\}$, *we have*

$$\langle T, \eta_w\rangle = \|w\|_T.$$

(4) *For any* $T \in \mathrm{cv}_N$ (*with the associated chart* $\alpha : F_N \to \pi_1(T/F_N)$) *and any* $\mu \in \mathrm{Curr}(F_N)$, *we have*

$$\langle T, \mu\rangle = \sum_{e \in \mathrm{Edges}(T/F_N)} \tfrac{1}{2}\langle e; \mu\rangle_\alpha,$$

where the summation is taken over all oriented edges of the graph T/F_N.

3. Tree-current morphisms and extremal Lipschitz distortion

Recall that a current $\mu \in \mathrm{Curr}(F_N)$ is called *filling* if for every $S \in \overline{\mathrm{cv}}_N$, we have $\langle S, \mu\rangle > 0$.

We proved in [Kapovich and Lustig 2010a] that for a current $\mu \in \mathrm{Curr}(F_N)$ and a tree $T \in \overline{\mathrm{cv}}_N$, we have $\langle T, \mu\rangle = 0$ if and only if the support of μ is contained in the "dual algebraic lamination" of T (in the sense of [Coulbois et al. 2008a]). Using this fact, it was shown in [Kapovich and Lustig 2010a] that if μ is a current with full support, then μ is filling. We denote by $\mathrm{Curr}_{\mathrm{fill}}(F_N)$ the set of all filling $\mu \in \mathrm{Curr}(F_N)$, and endow $\mathrm{Curr}_{\mathrm{fill}}(F_N)$ with the subspace topology given by the inclusion $\mathrm{Curr}_{\mathrm{fill}}(F_N) \subseteq \mathrm{Curr}(F_N)$.

Definition 3.1 (tree-current morphism). A *tree-current morphism* is a continuous function $J : \mathrm{cv}_N^1 \to \mathrm{Curr}(F_N)$ such that for every $T \in \mathrm{cv}_N^1$ and every $\varphi \in \mathrm{Out}(F_N)$, we have $J(\varphi T) = \varphi\, J(T)$.

A *filling tree-current morphism* is a tree-current morphism $J : \mathrm{cv}_N^1 \to \mathrm{Curr}(F_N)$ such that for every $T \in \mathrm{cv}_N^1$, the current $J(T) \in \mathrm{Curr}(F_N)$ is filling.

Lemma 3.2. *The function* $\mathrm{cv}_N^1 \times \overline{\mathrm{cv}}_N \to \mathbb{R}$, $(T, S) \mapsto \Lambda(T, S)$, *is continuous.*

Proof. Let $T \in \mathrm{cv}_N^1$ be arbitrary.

Let $\Delta_1, \dots, \Delta_m$ be all the open simplices in cv_N^1 whose closures in cv_N^1 contain T.

Set $C_T = \bigcup_{i=1}^m C_{\Delta_i}$. Note that $U = \Delta_1 \cup \dots \cup \Delta_m$ is a neighborhood of T in cv_N^1. Thus for every $T' \in U$ and every $S \in \overline{\mathrm{cv}}_N$, we have

$$\Lambda(T', S) = \max_{w \in C_T} \frac{\|w\|_S}{\|w\|_{T'}}.$$

Therefore the function $\Lambda(T', S)$ is continuous on $U \times \overline{\mathrm{cv}}_N$. Since $T \in \mathrm{cv}_N^1$ was arbitrary, the conclusion of the lemma follows. □

Let J be a filling tree-current morphism. Then for any $S \in \overline{\mathrm{cv}}_N$ and $c > 0$, we have

$$\frac{\langle S, J(T)\rangle}{\Lambda(T, S)} = \frac{\langle cS, J(T)\rangle}{\Lambda(T, cS)}.$$

Also, since $J(T)$ is a filling current, for every $S \in \overline{\text{cv}}_N$, we have $\langle S, J(T)\rangle > 0$. Therefore we have a well-defined function

$$f : \text{cv}_N^1 \times \overline{\text{CV}}_N \to (0, \infty)$$

given by $f(T, [S]) = \langle S, J(T)\rangle / \Lambda(T, S)$, where $T \in \text{cv}_N^1$ and $S \in \overline{\text{cv}}_N$.

Lemma 3.3. *Let J be a filling tree-current morphism. Then the function*

$$f : \text{cv}_N^1 \times \overline{\text{CV}}_N \to (0, \infty), \quad (T, S) \mapsto \frac{\langle S, J(T)\rangle}{\Lambda(T, S)}$$

is continuous.

Proof. The conclusion of the lemma follows directly from Lemma 3.2 together with the continuity of the geometric intersection form $\langle \cdot, \cdot \rangle$. □

Corollary 3.4. *Let $K \subseteq \text{cv}_N^1$ be a compact subset, and let $J : \text{cv}_N^1 \to \text{Curr}_{\text{fill}}(F_N)$ be a filling tree-current morphism.*

Then there exist $\delta_1 = \delta_1(K, J) > 0$ and $\delta_2 = \delta_2(K, J) > 0$ such that for every $T \in K$ and every $S \in \overline{\text{cv}}_N$, we have $\delta_1 \le f(K, [S]) \le \delta_2$.

Proof. The set $K \times \overline{\text{CV}}_N$ is a compact Hausdorff space and, by Lemma 3.3, $f : K \times \overline{\text{CV}}_N \to (0, \infty)$ is a continuous function. Therefore f achieves a positive minimum δ_1 and a positive maximum δ_2 on $K \times \overline{\text{CV}}_N$, and the conclusion of the corollary follows. □

Corollary 3.5. *Let $K \subseteq \text{cv}_N^1$ be a compact subset, let $\mathcal{T}_K = \bigcup_{\varphi \in \text{Out}(F_N)} \varphi K$ and let $J : \text{cv}_N^1 \to \text{Curr}(F_N)$ be a filling tree-current morphism.*

Furthermore, let $\delta_1 = \delta_1(K, J) > 0$ and $\delta_2 = \delta_2(K, J) > 0$ be the constants provided by Corollary 3.4.

Then for every $T \in \mathcal{T}_K$ and every $[S] \in \overline{\text{CV}}_N$, we have

$$0 < \delta_1 \le \frac{\langle S, J(T)\rangle}{\Lambda(T, S)} \le \delta_2 < \infty.$$

Proof. Let $T \in \mathcal{T}_K$ and $[S] \in \overline{\text{CV}}_N$ be arbitrary.

Then there exist $T' \in K$ and $\varphi \in \text{Out}(F_N)$ such that $T = \varphi T'$. By φ-equivariance of J, we have $J(T) = \varphi J(T')$. Define $S' = \varphi^{-1} S$, so that $\varphi S' = S$. Then

$$\frac{\langle S, J(T)\rangle}{\Lambda(T, S)} = \frac{\langle \varphi S', \varphi J(T')\rangle}{\Lambda(\varphi T', \varphi S')} = \frac{\langle S', J(T')\rangle}{\Lambda(T', S')} = f(T', [S']) \in [\delta_1 . \delta_2],$$

where the last inclusion holds by Corollary 3.4 since $T' \in K$. □

Note that Corollary 3.5 does not require the tree-current morphism $J : \text{cv}_N^1 \to \text{Curr}_{\text{fill}}(F_N)$ to be injective, although in the specific applications of interest to us J will be injective.

4. Patterson–Sullivan currents and extremal Lipschitz distortion

4A. ***Volume entropy and the Patterson–Sullivan currents.*** We only give here a brief summary of basic definitions and facts regarding Patterson–Sullivan currents for points of cv_N. We refer the reader to [Furman 2002; Coornaert 1993; Kaimanovich 1991; Kapovich and Nagnibeda 2007] for more detailed background information about Patterson–Sullivan measures and Patterson–Sullivan currents in the context of word-hyperbolic groups and Gromov-hyperbolic spaces.

Let $T \in \mathrm{cv}_N$, where $N \geq 2$. Since F_N and T are F_N-equivariantly quasi-isometric, there is a natural identification of ∂F_N and ∂T, which we will use later on.

The *volume entropy* $h(T)$ of T is defined as

$$h(T) := \lim_{R\to\infty} \frac{\log\bigl(\#\{w \in F_N \mid d_T(p, wp) \leq R\}\bigr)}{R},$$

where $p \in T$ is an arbitrary base point. It is known that the above definition does not depend on the choice of a base-point $p \in T$ and that we have $h(T) > 0$ for every $T \in \mathrm{cv}_N$. It is also known that $h(T)$ is exactly the critical exponent of the *Poincaré series*

$$\Pi_p(s) = \sum_{w\in F_N} e^{-s d_T(p,wp)}.$$

In other words, $\Pi_p(s)$ converges for all $s > h(T)$ and diverges for all $s \leq h(T)$. It is also known that as $s \to h+$, any weak limit ν of the measures

$$\frac{1}{\Pi_p(s)} \sum_{w\in F_N} e^{-s d_T(p,wp)} \,\mathrm{Dirac}(wp)$$

is a probability measure supported on $\partial T = \partial F_N$. Any such ν is called a *Patterson–Sullivan* measure on ∂F_N corresponding to T, and the measure class of ν is canonically determined by T. As follows from general results of Furman [2002], in this case there exists a unique, up to a scalar multiple, geodesic current μ in the measure class of $\nu \times \nu$ on $\partial^2 F_N$. We call the unique scalar multiple μ_T of μ such that $\langle T, \mu_T\rangle = 1$, the *Patterson–Sullivan current* for $T \in \mathrm{cv}_N$. One also has that the current μ_T has full support (this follows, for example, both from the general results of Furman [2002] and from the explicit formulas for μ_T obtained in [Kapovich and Nagnibeda 2007]).

Proposition 4.1. *The map*

$$J_{PS} : \mathrm{cv}_N^1 \to \mathrm{Curr}(F_N),\ T \mapsto \mu_T$$

is a filling tree-current morphism.

Proof. Since μ_T has full support, by a result of Kapovich and Lustig [2010a, Corollary 1.3], it follows that $\mu_T \in \mathrm{Curr}_{\mathrm{fill}}(F_N)$. The fact that J_{PS} is a continuous $\mathrm{Out}(F_N)$-equivariant map was proved by Kapovich and Nagnibeda [2007]. Thus J_{PS} is indeed a filling tree-current morphism, as claimed. □

The fact that for $T \in \mathrm{cv}_N^1$, the Patterson–Sullivan current μ_T is filling, i.e., that $\langle S, \mu_T\rangle \neq 0$ for every $S \in \overline{\mathrm{cv}}_N$, is quite nontrivial and does not follow directly from Proposition 2.2. This fact, which requires a general result from [Kapovich and Lustig 2010a] characterizing the case where $\langle S, \mu\rangle = 0$ (where $S \in \overline{\mathrm{cv}}_N$ and $\mu \in \mathrm{Curr}(F_N)$), is, in a sense, the place where the real "magic" in the proofs of the main results of the present paper happens.

We now obtain Theorem 1.1 from the Introduction:

Theorem 4.2. *Let $N \geq 2$ and $\varepsilon > 0$. Then there exist constants $\delta_2 \geq \delta_2 > 0$ such that for every $T \in \mathrm{cv}_{N,\varepsilon}^1$, $S \in \overline{\mathrm{cv}}_N$ we have*

$$\delta_1 \leq \frac{\langle S, \mu_T\rangle}{\Lambda(T, S)} \leq \delta_2.$$

Therefore there exists a constant $c > 0$ such that for every $T \in \mathrm{cv}_{N,\varepsilon}^1$ and $S \in \mathrm{cv}_N^1$, we have

$$|\log\langle S, \mu_T\rangle - d_L(T, S)| \leq c.$$

Proof. Since $\mathrm{cv}_{N,\varepsilon}^1/\mathrm{Out}(F_N)$ is compact and the action of $\mathrm{Out}(F_N)$ on $\mathrm{cv}_{N,\varepsilon}^1$ is properly discontinuous, there exists a compact subset $K \subseteq \mathrm{cv}_{N,\varepsilon}^1$ such that

$$\mathrm{cv}_{N,\varepsilon}^1 = \mathcal{T}_K = \bigcup_{\varphi \in \mathrm{Out}(F_N)} \varphi K.$$

By Proposition 4.1, the map $J_{PS} : \mathrm{cv}_N^1 \to \mathrm{Curr}(F_N)$ is a filling tree-current morphism. The conclusion of the theorem now follows from Corollary 3.5. □

4B. ***Uniform currents and generic stretching factors.*** Kapovich and Nagnibeda also provide reasonably explicit description of μ_T in terms of its weights on the "cylinder subsets" of $\partial^2 F_N$. The details of that description are not immediately relevant for the present paper. However, in the case where $T \in \mathrm{cv}_N^1$ and where T/F_N is a regular metric graph (that is, a regular graph where all edges have the same length), one can give a more precise description of μ_T as a "uniform current" corresponding to T and relate μ_T to the exit measure of the simple nonbacktracking random walk on T. We briefly recall here the description of uniform currents for the standard N-roses, that is for points of cv_N^1 corresponding to free bases of F_N.

Let $A = \{a_1, \ldots, a_N\}$ be a free basis of F_N. Let R_N be the graph given by a wedge of N loop-edges $e_1, \ldots, e_N$ at a vertex x_0. By identifying e_i with $a_i \in F_N$, we get an identification of $\alpha_A : F_N \xrightarrow{\cong} \pi_1(R_N, x_0)$, that is, a chart on F_N. We give each edge of R_N length $1/N$, so that R_N becomes a metric graph of volume 1.

Then the universal cover $T_A := \widetilde{R}_N$ is an $\mathbb{R}$-tree, which can be thought of as the Cayley graph of F_N with respect to A, but where all edges have length $1/N$. The group F_N has a natural free and discrete isometric left action on T_A by covering transformations, with $T_A/F_N = R_N$. Thus T_A is a point of cv_N^1.

The *uniform current* ν_A on F_N corresponding to A is defined explicitly by its weights. Namely, for every nontrivial freely reduced word v over $A^{\pm 1}$, we have

$$\langle v, \nu_A \rangle_{\alpha_A} = \frac{1}{N(2N-1)^{|v|-1}}.$$

One can check that this assignment of weights does define a geodesic current and that $\langle T_A, \nu_A \rangle = 1$. Moreover, in this case we also have:

Proposition 4.3. *Let $N \geq 2$ and let A be a free basis of F_N. Then $\mu_{T_A} = \nu_A$; that is, the Patterson–Sullivan current corresponding to T_A is exactly the uniform current ν_A.*

The above fact is not explicitly stated in [Kapovich and Nagnibeda 2007] but it easily follows from the explicit formulas for the weights for Patterson–Sullivan currents they obtained in the same work. Alternatively, one knows, for example, by the results of [Coornaert 1993; Lyons 1994], that for T_A the uniform visibility measure m_A on $\partial F_N = \partial T_A$ is a Patterson–Sullivan measure for T_A. Since $\nu_A \in \mathrm{Curr}(F_N)$ is in the measure class of $m_A \times m_A$ and since $\langle T_A, \nu_A \rangle = 1$, it follows from the definition of the Patterson–Sullivan current that $\mu_{T_A} = \nu_A$. Note that for any other $S \in \mathrm{cv}_N$, the intersection number $\langle S, \nu_A \rangle$ measures the distortion of a "long random geodesic" in T_A with respect to S.

Recall that in the Introduction, given a free basis A of F_N, $S \in \overline{\mathrm{cv}}_N$ and $\varphi \in \mathrm{Out}(F_N)$, we defined the generic stretching factors $\lambda_A(S)$ and $\lambda_A(\varphi)$.

Lemma 4.4. *For any free basis A of F_N and any $S \in \overline{\mathrm{cv}}_N$, we have*

$$\lambda_A(S) \leq \frac{1}{N} \Lambda(T_A, S).$$

Proof. Since all edges in T_A have length $1/N$, for every $w \in F_N$, we have $\|w\|_A = N \|w\|_{T_A}$. Then for a random trajectory $\xi = y_1 y_2 \cdots y_n \cdots$ of the simple nonbacktracking random walk on F_N with respect to A we have

$$\begin{aligned}\lambda_A(S) &= \lim_{n\to\infty} \frac{\|y_1 \cdots y_n\|_S}{\|y_1 \cdots y_n\|_A} = \lim_{n\to\infty} \frac{\|y_1 \cdots y_n\|_S}{N\|y_1 \cdots y_n\|_{T_A}} \\ &= \frac{1}{N} \lim_{n\to\infty} \frac{\|y_1 \cdots y_n\|_S}{\|y_1 \cdots y_n\|_{T_A}} \leq \frac{1}{N} \sup_{w \neq 1} \frac{\|w\|_S}{\|w\|_{T_A}} = \frac{1}{N} \Lambda(T_A, S). \qquad \square\end{aligned}$$

A key fact about generic stretching factors, originally established in [Kapovich 2006, Proposition 9.1] in slightly more limited context, is:

Proposition 4.5. *Let A be a free basis of F_N (where $N \geq 2$) and let $S \in \overline{\mathrm{cv}}_N$. Then*

$$\langle S, \nu_A \rangle = \lambda_A(S).$$

Proof. By [Kapovich 2006, Proposition 7.3], for a.e. trajectory $\xi = y_1 y_2 \cdots y_n \cdots$ of the simple nonbacktracking random walk on F_N with respect to A, we have

$$\lim_{n\to\infty} \frac{1}{n} \eta_{y_1 \cdots y_n} = \nu_A.$$

Therefore, by Proposition 2.2, for any $S \in \overline{\mathrm{cv}}_N$, we have

$$\langle S, \nu_A \rangle = \lim_{n\to\infty} \frac{1}{n} \langle S, \eta_{y_1\cdots y_n} \rangle = \lim_{n\to\infty} \frac{\|y_1 \cdots y_n\|_S}{n} = \lambda_A(S). \qquad \square$$

Remark 4.6. Since the current ν_A has full support and therefore ν_A is filling, Proposition 4.5 implies that for every $S \in \overline{\mathrm{cv}}_N$, we have $\lambda_A(S) > 0$. (From the definition of $\lambda_A(S)$, one only knows that $\lambda_A(S) \geq 0$ and it is not a priori obvious that the case $\lambda_A(S) = 0$ cannot occur.)

We can now obtain Corollary 1.4 from the Introduction:

Theorem 4.7. *Let $N \geq 2$. Then there exists a constant $\delta = \delta(N) \in (0, 1)$ with the following property*:

For any free basis A of F_N and any $S \in \overline{\mathrm{cv}}_N$, we have

$$0 < \delta \leq \frac{\lambda_A(S)}{\Lambda(T_A, S)} \leq \frac{1}{N}.$$

Proof. Let A be a free basis of F_N and let $S \in \overline{\mathrm{cv}}_N$ be arbitrary. By Lemma 4.4, we have

$$\frac{\lambda_A(S)}{\Lambda(T_A, S)} \leq \frac{1}{N}.$$

Let $\delta = \delta_1(\varepsilon, N) > 0$ be the constant provided by Theorem 4.2. By decreasing this constant if necessary, we can always assume that $0 < \delta_1 < 1$. Note that the length of the shortest essential circuit in T_A is equal to $1/N$.

Since $0 < \varepsilon \leq 1/N$, it follows that $T_A \in \mathrm{cv}^1_{N,\varepsilon}$. Since $\mu_{T_A} = \nu_A$ and $\langle S, \nu_A \rangle = \lambda_A(S)$, by Theorem 4.2 we have

$$0 < \delta_1 \leq \frac{\langle S, \mu_{T_A} \rangle}{\Lambda(T_A, S)} = \frac{\langle S, \nu_A \rangle}{\Lambda(T_A, S)} = \frac{\lambda_A(S)}{\Lambda(T_A, S)} \leq \frac{1}{N},$$

as required. $\square$

5. Extremal, generic and algebraic stretching factors for free group automorphisms

We recall the notions of extremal and generic stretching factors from Definition 1.5 in the Introduction:

Definition 5.1 (extremal and generic stretching factors of automorphisms). Let A be a free basis of F_N and let $\varphi \in \mathrm{Out}(F_N)$.

Define

$$\Lambda_A(\varphi) := \Lambda(T_A, T_A\varphi) = \sup_{w \neq 1} \frac{\|\varphi(w)\|_A}{\|w\|_A} = e^{d_L(T_A, T_A\varphi)},$$

and refer to $\Lambda_A(\varphi)$ as the *extremal stretching factor* for φ with respect to A.

Also, define $\lambda_A(\varphi) := \lambda_A(N T_A\varphi) = N\lambda_A(T_A\varphi)$.

Thus for a.e. trajectory $\xi = y_1 \cdots y_n \cdots$ of the simple nonbacktracking random walk on F_N with respect to A, we have

$$\lambda_A(\varphi) = \lim_{n\to\infty} \frac{\|\varphi(y_1 y_2 \cdots y_n)\|_A}{n} = \lim_{n\to\infty} \frac{\|\varphi(y_1 y_2 \cdots y_n)\|_A}{\|y_1 y_2 \cdots y_n\|_A}.$$

We call $\lambda_A(\varphi)$ the *generic stretching factor* of φ with respect to A.

First, we obtain, in a slightly restated form, Theorem 1.6 from the Introduction:

Theorem 5.2. *For every $N \geq 2$, there exists $0 < \tau_N \leq 1$ such that if A is a free basis of F_N and $\varphi \in \mathrm{Out}(F_N)$ then*

$$0 < \tau_N \leq \frac{\lambda_A(\varphi)}{\Lambda_A(\varphi)} \leq 1.$$

Proof. Let A be a free basis of F_N. Recall that, by definition, for $\varphi \in \mathrm{Out}(F_N)$ we have $\lambda_A(\varphi) = N\lambda_A(T_A\varphi)$ and $\Lambda_A(\varphi) = \Lambda(T_A, T_A\varphi)$. Therefore, by Lemma 4.4, we have $\lambda_A(\varphi) \leq \Lambda_A(\varphi)$, so that $\lambda_A(\varphi)/\Lambda_A(\varphi) \leq 1$. Since for any $\varphi \in \mathrm{Out}(F_N)$, we have $T_A, T_A\varphi \in \mathrm{cv}^1_{N,\varepsilon}$ with $\varepsilon = 1/N$, the statement of the theorem now follows directly from Theorem 4.7. □

For two sequences $x_n > 0$, $y_n > 0$ (where $n \geq 1$), we say that x_n *grows like* y_n, if there exist $0 < c < c' < \infty$ such that for every $n \geq 1$, we have $c \leq x_n/y_n \leq c'$.

We now obtain Corollary 1.7 from the Introduction:

Corollary 5.3. *Let $N \geq 2$ and $F_N = F(a_1, \ldots, a_n)$ with $A = \{a_1, \ldots, a_N\}$. There exists $D = D(N) \geq 1$ such that for every $\varphi \in \mathrm{Out}(F_N)$, we have*

$$\frac{1}{D} \log \lambda_A(\varphi) \leq \log \lambda_A(\varphi^{-1}) \leq D \log \lambda_A(\varphi).$$

Proof. It follows from [Algom-Kfir and Bestvina 2012, Theorem 24] that there exists $D' = D'(N) \geq 1$ such that for every $\varphi \in \mathrm{Out}(F_N)$, we have

$$\frac{1}{D'} d_L(T_A, T_A\varphi) \leq d_L(T_A\varphi, T_A) \leq D' d_L(T_A, T_A\varphi).$$

Note that $d_L(T_A, T_A\varphi) = \log \Lambda(T_A, T_A\varphi) = \log \Lambda_A(\varphi)$ and that

$$d_L(T_A\varphi, T_A) = d_L(T_A, T_A\varphi^{-1}) = \log \Lambda(T_A, T_A\varphi^{-1}) = \log \Lambda_A(\varphi^{-1}).$$

Theorem 5.2 now implies that there exists $D'' = D''(N) \geq 1$ such that for every $\varphi \in \mathrm{Out}(F_N)$, we have

$$(**) \qquad \frac{1}{D''} \log \lambda_A(\varphi) - D'' \leq \log \lambda_A(\varphi^{-1}) \leq D'' \log \lambda_A(\varphi) + D''.$$

It was proved in [Francaviglia 2009; Kapovich and Lustig 2010a] (and also follows from Theorem 5.2) that the set $\Omega_N := \{\lambda_A(\varphi) \mid \varphi \in \mathrm{Out}(F_N)\}$ is a discrete subset of $[1, \infty)$. It was established in [Kaimanovich et al. 2007] that $\lambda_A(\varphi) = 1$ if and only if φ is a *permutational automorphism* with respect to A, that is, if and only if, after a possible composition with an inner automorphism, φ is induced by a permutation of A, with possibly inverting some elements of A. Note that φ is permutational with respect to A if and only if φ^{-1} is permutational with respect to A, so that for $\varphi \in \mathrm{Out}(F_N)$, $\lambda_A(\varphi^{-1}) = 1$ if and only if $\lambda_A(\varphi) = 1$. It was also proved in [loc. cit.] that the minimum of $\lambda_A(\varphi)$, taken over all nonpermutational φ, is equal to $1 + (2N-3)/(2N^2 - N)$. Therefore $(**)$ implies that there exists $D = D(N) \geq 1$ such that for every nonpermutational $\varphi \in \mathrm{Out}(F_N)$, we have

$$(\diamond) \qquad \frac{1}{D} \log \lambda_A(\varphi) \leq \log \lambda_A(\varphi^{-1}) \leq D \log \lambda_A(\varphi).$$

If φ is permutational, then so is φ^{-1}. In this case we have $\log \lambda_A(\varphi^{-1}) = \log \lambda_A(\varphi) = 0$ and $(\diamond)$ holds as well. Thus $(\diamond)$ holds for every $\varphi \in \mathrm{Out}(F_N)$, which completes the proof. $\square$

Recall that for $\varphi \in \mathrm{Out}(F_N)$, the *algebraic stretching factor* $\lambda(\varphi)$ is defined as

$$\lambda(\varphi) = \sup_{w \in F_N, w \neq 1} \lim_{n \to \infty} \sqrt[n]{\|\varphi^n(w)\|_S},$$

where $S \in \mathrm{cv}_N$ is an arbitrary base-point. As noted earlier, this definition of $\lambda(\varphi)$ does not depend on the choice of $S \in \mathrm{cv}_N$. The algebraic stretching factor $\lambda(\varphi)$ can be read off from any relative train-track representative $f : \Gamma \to \Gamma$ of φ as the maximum of the Perron–Frobenius eigenvalues for any of the canonical irreducible diagonal blocks of the (nonnegative) transition matrix $M(f)$.

Corollary 5.5 below describes, given $\varphi \in \mathrm{Out}(F_N)$, the asymptotics of $\Lambda(S, S\varphi^n)$ as n tends to infinity (where $S \in \overline{\mathrm{cv}}_N$ is an arbitrary point, the choice of which does not affect this asymptotics). The statement of Corollary 5.5 is probably known to the experts. Since the proof is not yet available in the literature, and since we need Corollary 5.5 for the applications in this paper, we include the proof here.

Proposition 5.4. *Let* $\varphi \in \mathrm{Out}(F_N)$.

(1) *Let* $q \geq 1$ *and let* $\alpha = \varphi^q$ *admit an improved relative train-track* (in the sense of [Bestvina et al. 2000]) *representative* $f : \Gamma \to \Gamma$. *Put* $\lambda := 1$ *if* α *is polynomially growing* (*that is*, *if* f *has no exponentially growing strata*) *and otherwise let*

$\lambda > 1$ be the largest Perron–Frobenius eigenvalue of the exponentially growing strata of $f : \Gamma \to \Gamma$.

Then there exists an integer $m \geq 0$ such that for every $S \in \mathrm{cv}_N$, there are some constants $0 < C_1 \leq C_2 < \infty$ such that for every $n \geq 1$,

$$C_1 \lambda^{n/q} n^m \leq \Lambda(S, S\varphi^n) \leq C_2 \lambda^{n/q} n^m.$$

(2) *If φ admits a train-track representative $f : \Gamma \to \Gamma$ with an irreducible transition matrix and with the Perron–Frobenius eigenvalue $\lambda > 1$, then for every $S \in \mathrm{cv}_N$, there exist $0 < C_1 \leq C_2 < \infty$ such that for every $n \geq 1$,*

$$C_1 \lambda^n \leq \Lambda(S, S\varphi^n) \leq C_2 \lambda^n.$$

Proof. (1) Let $T \in \mathrm{cv}_N^1$ be the point corresponding to the improved relative train-track $f : \Gamma \to \Gamma$, where all edges of Γ are given equal length. Put $L = \{1\}$ if f has no exponentially growing strata. Otherwise let $\lambda_1 \geq \cdots \geq \lambda_k > 1$ be all the Perron–Frobenius eigenvalues of the exponentially growing strata of f and put $L = \{\lambda_1, \dots, \lambda_k, 1\}$. Finally put $\lambda = \max L$. Thus $\lambda \geq 1$ and $\lambda = 1$ if and only if f has no exponential strata.

A result of Levitt [2009, Theorem 6.2] shows that there is a finite subset M of $\mathbb{Z}_{\geq 0}$ such that for every nontrivial $w \in F_N$, there is some $(\lambda', m') \in L \times M$ such that the sequence $\|\alpha^n(w)\|_T$ grows like $(\lambda')^n n^{m'}$. Moreover, there exists some element $1 \neq w_0 \in F_N$ such that $\|\alpha^n(w_0)\|_T$ grows as $\lambda^n n^m$ and such that if some other $w \neq 1$ has $\|\alpha^n(w)\|_T$ growing as $\lambda^n n^{m'}$ then $m' \leq m$.

Let $D = C_\Delta$ be the finite subset of F_N as in Remark 2.1, where Δ is the open simplex in cv_N^1 containing T. Therefore for every $n \geq 1$, we have $\Lambda(T, T\varphi^n) = \max_{w \in D}(\|\alpha^n(w)\|_T / \|w\|_T)$. Moreover, through replacing D by $D \cup \{w_0\}$, we can assume that $w_0 \in D$.

It follows that $\Lambda(T, T\alpha^n) = \max_{w \in D}(\|\alpha^n(w)\|_T / \|w\|_T)$ grows like $\lambda^n n^m$.

Now let $n \geq 1$ and write $n = qn_1 + r$, where $n_1 \geq 0$ and $0 \leq r \leq q-1$ are integers. As we have seen, $\Lambda(T, T\alpha^{n_1}) = \max_{w \in D}(\|\varphi^{n_1}(w)\|_T / \|w\|_T)$ grows like $\lambda^{n_1} n_1^m$. Since $0 \leq r \leq q-1$, applying φ^r distorts $\|\cdot\|_T$ by a bounded multiplicative amount. Therefore $\Lambda(T, T\varphi^n) = \max_{w \in D}(\|\varphi^n(w)\|_T / \|w\|_T)$ grows as $\lambda^{n/q}(n/q)^m$, that is, as $\lambda^{n/q} n^m$.

Since T and S are F_N-equivariantly quasi-isometric, it follows that $\Lambda(S, S\varphi^n) = \Lambda_A(\varphi^n)$ also grows like $\lambda^{n/q} n^m$, and the conclusion of part (1) of the proposition follows.

(2) The proof of part (2) is known (e.g., see Theorem 8.1 in [Francaviglia and Martino 2011]) and is simpler than the proof of part (1), and we leave the details to the reader. The key point is that in this case for every nontrivial $w \in F_N$ such that the conjugacy class of w is not φ-periodic, the sequence $\|\varphi^n(w)\|_S$ grows like λ^n. $\square$

Corollary 5.5. *Let $\varphi \in \mathrm{Out}(F_N)$, let $S \in \mathrm{cv}_N$ and let $\lambda(\varphi)$ be the algebraic stretching factor of φ.*

Then there is an integer $m \geq 0$ such that for every $S \in \mathrm{cv}_N$, there are some $C_1, C_2 > 0$ such that

$$C_1\,\lambda(\varphi)^n\, n^m \leq \Lambda(S, S\varphi^n) \leq C_2\,\lambda(\varphi)^n\, n^m$$

for all $n \geq 1$.

Proof. It is known [Bestvina et al. 2000] that some positive power $\alpha = \varphi^q$ of φ admits an improved relative train track representative.

In this case we have $\lambda(\alpha) = \lambda(\varphi^q) = \lambda(\varphi)^q$, so that $[\lambda(\alpha)]^{1/q} = \lambda(\varphi)$. The conclusion of the corollary now follows directly from part (1) of Proposition 5.4. □

Now Corollary 5.5 (applied to $S = T_A$, which gives $\Lambda(S, S\varphi^n) = \Lambda_A(\varphi^n)$) and Theorem 5.2 directly imply Theorem 1.8 from the Introduction:

Theorem 5.6. *Let A be a free basis of F_N, let $\varphi \in \mathrm{Out}(F_N)$ and let $\lambda(\varphi)$ be the algebraic stretching factor of φ. Then there exist constants $c_1, c_2 > 0$ and an integer $m \geq 0$ such that for every $n \geq 1$, we have*

$$c_1\,\lambda(\varphi)^n\, n^m \leq \lambda_A(\varphi^n) \leq c_2\,\lambda(\varphi)^n\, n^m.$$

Moreover, if φ admits an expanding train-track representative with an irreducible transition matrix (e.g., if φ is fully irreducible), then $m = 0$ and $\lambda(\varphi) > 1$. □

Example 5.7. To demonstrate that the case $\lambda > 1, m > 0$ in Theorem 5.6 can indeed occur, we consider an example explained on p. 1138 in [Levitt 2009]. Let $N = 4$ and $F_4 = F(A)$ with $A = \{a_1, b_1, a_2, b_2\}$. Let an automorphism $\varphi : F(A) \to F(A)$ be given by

$$\varphi(a_1) = a_1 b_1, \quad \varphi(b_1) = a_1, \quad \varphi(a_2) = a_2 b_1 a_1, \quad \varphi(b_2) = a_2.$$

For the A-rose R_A, the map $f : R_A \to R_A$, given by the same formula as φ, is both a global train-track and a 2-strata relative train-track representative for φ. The bottom stratum is $\{a_1, b_1\}$ and the top stratum is $\{a_2, b_2\}$. The transition matrices for both strata are the same and are equal to $B = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which has the Perron–Frobenius eigenvalue $\lambda = (1 + \sqrt{5})/2$. The transition matrix for f has the form $M = \left(\begin{smallmatrix} B & 0 \\ C & B \end{smallmatrix}\right)$, where $C = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. By iterating M one can see that $\|\varphi^n(a_2)\|_A$ grows like $n\lambda^n$. One can then show that in this case $\Lambda_A(\varphi^n)$ also grows as $n\lambda^n$. Therefore, by Theorem 5.2, $\lambda_A(\varphi^n)$ grows as $n\lambda^n$ as well.

6. Other examples of filling tree-current morphisms

The Patterson–Sullivan map $J_{PS} : \mathrm{cv}_N^1 \to \mathrm{Curr}(F_N)$, $T \mapsto \mu_T$, is just one, albeit natural and useful, example of a filling tree-current morphism. There are many other filling tree-current morphisms $J : \mathrm{cv}_N^1 \to \mathrm{Curr}(F_N)$, and Corollary 3.5 is applicable to all such J. We indicate here some sources of such J, following the approach of Reiner Martin [1995]. The main idea is that if $t \mapsto \rho(t) > 0$ is a monotone decreasing continuous function which approaches 0 as $t \to \infty$ "sufficiently quickly", then

$$J_\rho : \mathrm{cv}_N^1 \to \mathrm{Curr}(F_N), \quad T \mapsto \sum_{[w]\neq[1]} \rho(\|w\|_T)\eta_w$$

is a filling tree-current morphism.

The summation here can be taken either over all nontrivial conjugacy classes $[w]$ of elements of F_N (or over an $\mathrm{Out}(F_N)$-invariant set of such conjugacy classes, although in the latter case one has to take additional care to ensure that the current $J_\rho(T)$ is filling).

Let us first observe that such a function J_ρ is, by its construction, always $\mathrm{Out}(F_N)$-equivariant: for any $T \in \mathrm{cv}_N^1$ and $\varphi \in \mathrm{Out}(F_N)$, we have

$$\varphi(J_\rho(T)) = \sum_{[w]\neq[1]} \rho(\|w\|_T)\varphi(\eta_w) = \sum_{[w]\neq[1]} \rho(\|w\|_T)\eta_{\varphi(w)}$$

and

$$J_\rho(\varphi T) = \sum_{[w]\neq[1]} \rho(\|w\|_{\varphi T})\eta_w = \sum_{[w]\neq[1]} \rho(\|\varphi^{-1}(w)\|_T)\eta_w$$
$$\underset{\text{with } u=\varphi^{-1}(w)}{=} \sum_{[u]\neq[1]} \rho(\|u\|_T)\eta_{\varphi(u)} = \varphi(J_\rho(T)),$$

so that J_ρ is indeed $\mathrm{Out}(F_N)$-equivariant.

We provide here a representative result of the kind described above:

Proposition 6.1. *The function*

$$J : \mathrm{cv}_N^1 \to \mathrm{Curr}(F_N), \quad T \mapsto \sum_{[w]\neq[1]} e^{-e^{\|w\|_T}}\eta_w,$$

where the sum is taken over all nontrivial root-free conjugacy classes $[w]$ of elements of F_N, is an injective filling tree-current morphism.

Proof. Fix a free basis A of F_N and let $T_A \in \mathrm{cv}_N^1$ be the Cayley graph of F_N with respect to A, where all edges in T_A have length $1/N$. For $w \in F_N$ denote by $\|w\|_A$ the cyclically reduced length of w over $A^{\pm 1}$. Thus $\|w\|_A = N\|w\|_{T_A}$. We let $R_A = T_A/F_N$ be the quotient metric graph, which is a wedge of N loop-edges of length $1/N$ corresponding to elements of A. Let $\alpha_A : F_N \to \pi_1(R_A)$ be the associated chart.

Let $T \in \mathrm{cv}_N^1$ be arbitrary and let U be a compact neighborhood of T in cv_N^1. There exists a constant $C \geq 1$ such that for every $w \in F_N$ and every $T' \in U$, we have $\|w\|_{T'}/C \le \|w\|_A \le C\|w\|_{T'}$. Note that for $n \ge 1$, the number of conjugacy classes $[w]$ with $\|w\|_A \le n$ is at most $(2N)^n$.

To show that for each $T' \in U$, $J(T')$ is a geodesic current we only need to verify that $J(T')$ takes finite values on all the two-sided cylinder sets in $\partial^2 F_N$ determined by the chart α_A. Since every cylinder is contained in a cylinder determined by a single edge, it suffices to show that for every oriented edge e of R_A, we have $\langle e, J(T')\rangle_{\alpha_A} < \infty$.

Let $T' \in U$ and let e be an edge of R_A. For every integer $n \ge 1$, set

$$b_n(e, T') := \sum_{0.9n \le \|[w]\|_A \le 1.1n} e^{-e^{\|w\|_{T'}}} \langle e, \eta_w\rangle_{\alpha_A}.$$

Then $\langle e, J(T')\rangle_{\alpha_A} \le \sum_{n=1}^{\infty} b_n(e, T')$. The weight $\langle e, \eta_w\rangle_{\alpha_A}$ is equal to $1/N$ times the number of occurrences of $e^{\pm 1}$ in the cyclically reduced circuit γ_w in R_A representing $[w]$. Hence $\langle e, \eta_w\rangle_{\alpha_A} \le (1/N)\|w\|_A$. Since $T' \in U$, we have $\|w\|_{T'} \ge \|w\|_A/C$. Hence for every $n \ge 1$ and $T' \in U$, we have

$$\begin{aligned} b_n(e, T') &= \sum_{\|[w]\|_A \in I} e^{-e^{\|w\|_{T'}}} \langle e, \eta_w\rangle_{\alpha_A} \le \frac{1}{N} \sum_{\|[w]\|_A \in I} e^{-e^{\|w\|_A/C}} \|w\|_A \\ &\le \frac{1}{N} \sum_{\|[w]\|_A \in I} e^{-e^{0.9n/C}} 1.1n \ \le \frac{1.1n}{N} e^{-e^{0.9n/C}} (2N)^{1.1n} \\ &= \frac{1.1n}{N} e^{-e^{0.9n/C}} e^{1.1n \log(2N)} \ = \frac{1.1n}{N} e^{1.1n\log(2N) - e^{0.9n/C}}, \end{aligned}$$

where $I = [0.9n, 1.1n]$. From here we see that

$$\langle e, J(T')\rangle_{\alpha_A} \le \sum_{n=1}^{\infty} b_n(e, T') \le C_1,$$

where $C_1 = C_1(U) < \infty$ is some constant depending only on U.

Thus for every $T' \in U$, $J(T')$ is indeed a geodesic current on F_N, and, in particular, $J(T) \in \mathrm{Curr}(F_N)$.

Note that the current $J(T)$ has full support. Indeed, for every nontrivial freely reduced word v over $A^{\pm 1}$, there exists a root-free cyclically reduced word w over $A^{\pm 1}$ containing v as a subword. Then $\langle v, \eta_w\rangle_{\alpha_A} > 0$ and hence, from the definition of $J(T)$, we see that $\langle v, J(T)\rangle_{\alpha_A} > 0$. Thus indeed $J(T)$ has full support and therefore, by a result of Kapovich and Lustig [2010a], the current $J(T)$ is filling.

Since an automorphism of F_N permutes the set of all root-free nontrivial conjugacy classes in F_N, it follows from the definition of J that for every $T \in \mathrm{cv}_N^1$ and every $\varphi \in \mathrm{Out}(F_N)$, we have $J(\varphi T) = \varphi J(T)$.

Thus we have constructed an $\mathrm{Out}(F_N)$-equivariant map $J : \mathrm{cv}_N^1 \to \mathrm{Curr}_{\mathrm{fill}}(F_N)$.

We next observe that the map J is continuous. The proof of the continuity of J is similar to the proof that $J(T)$ is a current. Let $T \in \mathrm{cv}_N^1$, let U be a compact neighborhood of T in cv_N^1 and let v be a nontrivial freely reduced word over $A^{\pm 1}$. Then for every $T' \in U$, we have

$$\langle v, T' \rangle_{\alpha_A} = \sum_{[w]} \langle v, e^{-e^{\|w\|_{T'}}} \eta_w \rangle_{\alpha_A} = \sum_{[w]} e^{-e^{\|w\|_{T'}}} \langle v, w \rangle_{\alpha_A}.$$

One can then show, by an argument similar to that used above, that there exist positive constants $M_w > 0$ (also depending on U and v but independent of $T' \in U$) such that for every $T' \in U$, we have $e^{-e^{\|w\|_{T'}}} \langle v, w \rangle_{\alpha_A} \le M_w$ and that $\sum_{[w]} M_w < \infty$. By the Weierstrass M-test, it follows that the series

$$\sum_{[w]} e^{-e^{\|w\|_{T'}}} \langle v, w \rangle_{\alpha_A},$$

viewed as the sum of a functions on U, converges uniformly on U and that its sum $\langle v, T' \rangle_{\alpha_A}$ is a continuous function on U.

Since v was arbitrary, the explicit description of the topology on $\mathrm{Curr}(F_N)$ (see [Kapovich 2006]) implies that J is a continuous function on cv_N^1, as required.

It remains to show that J is injective. Fix an enumeration, without repetitions, $w_1, w_2, \dots$ of representatives of all the nontrivial root-free conjugacy classes in F_N. Thus for every root-free nontrivial $w \in F_N$, there exist unique distinct $m, n \ge 1$ such that $[w] = [w_m]$ and $[w^{-1}] = [w_n]$.

For every $i \ge 1$, set $q_i = (w_i^{-\infty}, w_i^{\infty}) \in \partial^2 F_N$ and set $Q_i = \{q_i\}$. Note that for $i, j \ge 1$, we have $\eta_{w_j}(Q_i) = 1$ if $[w_i] = [w_j^{\pm 1}]$ and $\eta_{w_i}(Q_i) = 0$ otherwise. Then, by definition of J, for every $T \in \mathrm{cv}_N^1$ and $i \ge 1$, we have $J(T)(Q_i) = 2e^{-e^{\|w_i\|_T}}$. Since the function $t \mapsto 2e^{-e^t}$ is strictly monotone and thus injective, it follows that knowing the current $J(T)$, we can recover $\|w_i\|_T$ for all $i \ge 1$. Hence we can recover the length function $\|\cdot\|_T : F_N \to \mathbb{R}$ and so we can also recover T itself. Thus J is injective, as required. □

7. Open problems

As we have seen in Theorem 1.6, if $N \ge 2$, $A = \{a_1, \dots, a_N\}$ is a fixed free basis of $F_N = F(A)$, then for

$$\rho_N = \inf_{\varphi \in \mathrm{Out}(F_N)} \frac{\lambda_A(\varphi)}{\Lambda_A(\varphi)},$$

we have $\rho_N > 0$. In fact, one can show:

Proposition 7.1. *We have* $\lim_{N\to\infty} \rho_N = 0$, *and moreover*, $\rho_N = O(1/N)$; *that is*, $\limsup_{N\to\infty} N\rho_N < \infty$.

Proof. For $N \geq 2$ and $m \geq 1$, let $\varphi_{N,m} : F(A) \to F(A)$ be given by $\varphi_{N,m}(a_1) = a_1 a_2^m$ and $\varphi_{N,m}(a_i) = a_i$ for $2 \leq i \leq N$. It is not hard to see that

$$\Lambda_A(\varphi_{N,m}) = \sup_{w \neq 1} \frac{\|\varphi_{N,m}(w)\|_A}{\|w\|_A} = m+1.$$

For any freely reduced $w \in F(A)$, we have

$$\|\varphi_{N,m}(w)\|_A \leq (m+1)(a_1; w)_A + \sum_{i=2}^{N} (a_i; w)_A,$$

where $(a_j; w)_A$ is the number of occurrences of $a_j^{\pm 1}$ in w. On the other hand, if $w_n \in F(A)$ is a "long random" freely reduced word of length n, then asymptotically we have $(a_i; w_n)_A / n \xrightarrow{n\to\infty} 1/N$ for $i = 1, \ldots, N$. Therefore

$$\begin{aligned} \lambda_A(\varphi_{N,m}) &\leq \lim_{n\to\infty} \frac{(m+1)(a_1; w)_A + \sum_{i=2}^{N} (a_i; w)_A}{n} \\ &= (m+1)\frac{1}{N} + \frac{N-1}{N} = \frac{m}{N} + 1. \end{aligned}$$

Hence

$$\rho_N \leq \frac{\lambda_A(\varphi_{N,m})}{\Lambda_A(\varphi_{N,m})} \leq \frac{1 + \frac{m}{N}}{m+1}.$$

By taking $m = N$, we see that $\rho_N \leq 2/(N+1) \xrightarrow{n\to\infty} 0$. Thus $\lim_{N\to\infty} \rho_N = 0$ and $\limsup_{N\to\infty} N\rho_N < \infty$. □

Theorem 1.6 and Proposition 7.1 naturally raise the following:

Problem 7.2. Are the values ρ_N algorithmically computable in terms of N? What are the exact values of ρ_N for small N, say for $N = 2, 3, 4$? Is it true that $\rho_N \in \mathbb{Q}$? What can be said about the precise asymptotics of ρ_N as $N \to \infty$? (Note that Proposition 7.1 shows that ρ_N decays at least as fast as $1/N$.)

Theorem 1.1 also motivates the definition of a new notion of a continuous symmetric and $\mathrm{Out}(F_N)$-invariant intersection number $I : \mathrm{cv}_N^1 \times \mathrm{cv}_N^1 \to \mathbb{R}_{>0}$, where for $T, S \in \mathrm{cv}_N^1$, we define $I(T, S) := \langle S, \mu_T \rangle \langle T, \mu_S \rangle$. The function $I(\cdot, \cdot)$ was originally suggested to us by Arnaud Hilion, as it appears to be relevant for attempting to define an analogue of the Weil–Petersson metric on cv_N^1.

Since the Patterson–Sullivan currents are normalized so that $\langle T, \mu_T \rangle = 1$, for $T = S$, we have $I(T, T) = 1$.

Problem 7.3. (a) Is it true that for every $T, S \in \mathrm{cv}_N^1$, we have $I(T, S) \geq 1$?

(b) Is it true that for $T, S \in \mathrm{cv}_N^1$, we have $I(T, S) = 1$ if and only if $T = S$?

It was shown in [Kaimanovich et al. 2007] that if A is a free basis of F_N and $\varphi \in \mathrm{Out}(F_N)$ then $\lambda_A(\varphi) \geq 1$ and that $\lambda_A(\varphi) = 1$ if and only if $T_A\varphi = T_A$. If

B is another free basis of F_N and $\varphi \in \mathrm{Aut}(F_N)$ is such that $T_A\varphi = T_B$, then $\langle T_B, \mu_{T_A}\rangle = \lambda_A(\varphi)$ and $\langle T_A, \mu_{T_B}\rangle = \lambda_A(\varphi^{-1})$. It follows that if A, B are free bases of F_N then $I(T_A, T_B) \geq 1$ and that $I(T_A, T_B) = 1$ if and only if $T_A = T_B$. However, beyond this fact nothing appears to be known about the above question.

Recently Pollicott and Sharp [2014], using a different approach, defined and studied a Weil–Petersson type metric on cv_N^1. It would be interesting to investigate the relationship of their metric to the quantity $I(T, S)$ defined above.

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