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We construct a $3^{\mathbb{Z}/4}$ subfactor using an algorithm which, given generators in a spoke graph planar algebra, computes two-strand jellyfish relations. This subfactor was known to Izumi, but has not previously appeared in the literature. We systematically analyze the space of second annular consequences, adapting Jones' treatment of the space of first annular consequences in his quadratic tangles article.

This article is the natural followup to two recent articles on spoke subfactor planar algebras and the jellyfish algorithm. Work of Bigelow and Penneys explains the connection between spoke subfactor planar algebras and the jellyfish algorithm, and work of Morrison and Penneys automates the construction of subfactors where both principal graphs are spoke graphs using one-strand jellyfish. This is the published version of arXiv:1308.5197.

1. Introduction

Jones' program for constructing subfactor planar algebras starts with the observation that every subfactor planar algebra embeds in the graph planar algebra (first defined in [Jones 2000]) of its principal graph [Jones and Penneys 2011; Morrison and Walker 2010]. Following this program, one constructs a subfactor planar algebra by finding candidate generators in an appropriate graph planar algebra, and then showing they generate a subfactor planar algebra with the correct principal graph.

These methods have been used to construct a large handful of examples, some new and some well known, including the E_6 and E_8 subfactors [Jones 2001], group-subgroup subfactors [Gupta 2008], the Haagerup subfactor [Peters 2010], the extended Haagerup subfactor [BMPS 2012], the Izumi–Xu 2221 subfactor [Han 2010], certain spoke subfactors, e.g., 4442 [Morrison and Penneys 2015b], and examples related to quantum groups [LMP 2015]. These techniques have also been used to prove uniqueness results [BMPS 2012; Han 2010; Liu 2015] and obstructions to possible principal graphs [Peters 2010; Morrison 2014; Liu 2015].

Early applications of the embedding theorem to construct or obstruct subfactors were mostly ad hoc. Recent work of Bigelow and Penneys [2014], based on [Popa

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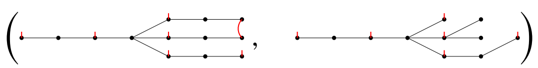
1995], has explained why some of the previous constructions work and how they fit into the same family of examples. If the principal graph of a subfactor is a spoke graph with simple arms connected to one central vertex, the planar algebra can be constructed using two-strand jellyfish relations. If both graphs are spokes, one can use one-strand relations, which are easier to compute. Recent work of Morrison and Penneys [2015b] found an algorithm to compute these one-strand relations, provided one has the generators in the graph planar algebra.

Given a set of generators in a graph planar algebra together with some local relations, we want to show evaluability, i.e., the relations can evaluate any closed diagram. The utility of the jellyfish algorithm is that for spoke graphs, it gives a systematic way to show evaluability. The key idea of the jellyfish algorithm is that given our generators, and relatively few evaluations of closed diagrams involving these generators, we can derive a collection of local relations sufficient to evaluate all closed diagrams.

This article is the natural followup to [Bigelow and Penneys 2014; Morrison and Penneys 2015b]. Our main result is an algorithm to find two-strand jellyfish relations for a subfactor planar algebra for which one of the principal graphs is a spoke graph. This algorithm requires as input the generators in a graph planar algebra. The main application of our algorithm is the construction of a subfactor known to Izumi, which has not previously appeared in the literature.

Theorem 1.1. *There exists a* $3^{\mathbb{Z}/4}$ *subfactor with principal graphs*



We describe our algorithm for the reader who is willing to take our computations on faith.

- Acquire the generators in an appropriate graph planar algebra. These generators are an assignment of numbers in a finite extension of $\mathbb{Q}$ to certain loops on a graph.
- Use a computer to evaluate certain closed diagrams with at most 4 generators. This amounts to multiplying rather large matrices, and taking the trace.
- Turn these evaluations of closed diagrams into information about inner products, and then use a computer to derive jellyfish relations for our generators. The use of the computer is limited to basic linear algebra.
- We now have an evaluable planar subalgebra of a graph planar algebra, which is necessarily a subfactor planar algebra. Compute the principal graph by a process of elimination.

By a deep theorem of Popa [1995], from a subfactor planar algebra $\mathcal{P}_\bullet$ we can always get a subfactor whose stand invariant is $\mathcal{P}_\bullet$. When $\mathcal{P}_\bullet$ is finite depth, it is a complete invariant of the associated hyperfinite subfactor [Popa 1990].

Remark 1.2. Interestingly, we construct the $3^{\mathbb{Z}/4}$ subfactor planar algebra in a graph planar algebra not associated to either of its principal graphs (see Appendix AA)! Of course, by the embedding theorem, it is also a planar subalgebra of the $3^{\mathbb{Z}/4}$ graph planar algebra, but we found the computational issues related to finding the generators easier to deal with in the other graph.

The motivation for this article is to systematically study a conjectural infinite family of 3^G spoke subfactors for certain finite abelian groups G, first studied by Izumi [2001], and later by Evans and Gannon [2011]. A 3^G subfactor has principal graph consisting of $|G|$ spokes of length 3, and the dual data is determined by the inverse law of the group G. In fact, Izumi has an unpublished construction of a $3^{\mathbb{Z}/4}$ subfactor using Cuntz algebras, analogous to his treatment for odd order G in [Izumi 2001]. Moreover, he can show such a subfactor is unique, which our approach does not attempt. In theory, all 3^G subfactors can be constructed using two-strand jellyfish [Bigelow and Penneys 2014]. The major hurdle is finding the generators in the graph planar algebra. Once given the generators, the machinery of this article produces the two-strand relations.

The foundation for this article, which underlies the previously discussed constructions and obstructions, is Jones' annular tangles point of view. Each unitary planar algebra can be orthogonally decomposed into irreducible annular Temperley–Lieb modules. In doing so, we seem to lose a lot of information, namely the action of higher genus tangles. However, we find ourselves in the simpler situation of analyzing irreducible annular Temperley–Lieb modules, which have been completely classified [Graham and Lehrer 1998; Jones 2001]. Such a module is generated by a single low-weight rotational eigenvector. This perspective is particularly useful for small index subfactors, which can only have a few small low-weight vectors.

This article is also a natural followup to Jones' exploration of quadratic tangles [2012]. There are necessarily strong quadratic relations among the few smallest low-weight generators of a subfactor planar algebra of small modulus. Jones [2012] studies the space of first annular consequences of the low-weight vectors to find explicit formulas for these relations. We provide an analogous systematic treatment of the space of second annular consequences of a set of low-weight generators of a subfactor planar algebra. Studying this space was fruitful in Peters' [2010] planar algebra construction of the Haagerup subfactor.

1A. *Outline.* In Section 2, we give the necessary background for this article, including conventions for graph planar algebras, tetrahedral structure constants, the jellyfish algorithm, and reduced trains. In Section 2D, we give a basis for the second annular consequences of a low-weight element when $\delta > 2$.

In Section 3, we analyze the space of reduced trains, in particular their projections to Temperley–Lieb and annular consequences. We then calculate many pairwise inner products of such trains and their projections. In Section 4, we provide the algorithm for computing two-strand jellyfish relations given generators in our graph planar algebras.

In Section 5, we provide the results of applying the algorithm from Section 4 to construct the $3^{\mathbb{Z}/4}$ subfactor planar algebra. We compute the principal graphs of our example in Section 6.

Finally, we have two appendices where we record the data necessary for the above computations. The generators are specified in Appendix A via their values on collapsed loops, and we give the moments and tetrahedral structure constants for our generators in Appendix B.

1B. ***The FusionAtlas*** (adapted from [Morrison and Penneys 2015b]). This article relies on some substantial calculations. In particular, our efforts to find the generators in the various graph planar algebras made use of a variety of techniques, some ad hoc, some approximate, and some computationally expensive. This article essentially does not address that work. Instead, we merely present the discovered generators and verify some relatively easy facts about them. In particular, the proofs presented in this article rely on the computer in a much weaker sense. We need to calculate certain numbers of the form $\mathrm{Tr}(PQRS)$, where P, Q, R, S are rather large martrices, and the computer does this for us. We also entered all the formulas derived in this article into Mathematica in order to evaluate the various quantities which appear in our derivation of jellyfish relations. As a reader may be interested in seeing these programs, we include a brief instruction on finding and running these programs.

The arXiv sources of this article contain a number of files in the `code` subdirectory, including:

- `Generators.nb`, which reconstructs the generators from our terse descriptions in Appendix A.
- `TwoStrandJellyfish.nb`, which calculates the requisite moments and tetrahedral structure constants of these generators, and performs the linear algebra necessary to derive the jellyfish relations.
- `GenerateLaTeX.nb`, which typesets each subsection of Section 5 for each planar algebra, and many mathematical expressions in Appendices A and B.

The Mathematica notebook `Generators.nb` can be run by itself. The final cells of that notebook write the full generators to the disk; this must be done before running `TwoStrandJellyfish.nb`. The latter notebook relies on the *FusionAtlas*, a substantial body of code the authors have developed along with Narjess Afzaly, Scott Morrison, Noah Snyder, and James Tener to perform calculations with subfactors and fusion categories. To obtain a local copy, you first need to ensure that you have *Mercurial*, the distributed version control system, installed on your machine. With that, the command

```
hg clone https://bitbucket.org/fusionatlas/fusionatlas
```

will create a local directory called `fusionatlas` containing the latest version. In the `TwoStrandJellyfish.nb` notebook, you will then need to adjust the paths appearing in the first input cell to ensure that your local copy is included. After that, running the entire notebook reproduces all the calculations described below.

We invite any interested readers to contact us with questions or queries about the use of these notebooks or the *FusionAtlas* package.

2. Background

We now give the background material for the calculations that occur in the later sections. We refer the reader to [Peters 2010; BMPS 2012; Jones 2012; 2011] for the definition of a (subfactor) planar algebra.

Notation 2.1. When we draw planar diagrams, we often suppress the external boundary disk. In this case, the external boundary is assumed to be a large rectangle whose distinguished interval contains the upper left corner. We draw one string with a number next to it instead of drawing that number of parallel strings. We shade the diagrams as much as possible, but if the parity is unknown, we often cannot know how to shade them. Finally, projections are usually drawn as rectangles with the same number of strands emanating from the top and bottom, while other elements may be drawn as circles.

Some parts of this introduction are adapted from [Morrison and Penneys 2015b; Bigelow and Penneys 2014].

2A. ***Working in graph planar algebras.*** Graph planar algebras, defined in [Jones 2000], have proven to be a fruitful place to work because of the following theorem. Strictly speaking, our constructions do not rely on this theorem. However, it motivates our search for generators in the appropriate graph planar algebra.

Theorem 2.2 [Jones and Penneys 2011; Morrison and Walker 2010]. *Every subfactor planar algebra embeds in the graph planar algebra of its principal graph.*

In [Morrison and Penneys 2015b, Section 2.2], it was observed that many of Jones' [2012] quadratic tangles formulas for subfactor planar algebras hold for certain collections of elements in unitary, spherical, shaded $*$-planar algebras which are not necessarily evaluable (see Theorem 2.8). The main example of such a planar algebra is the graph planar algebra of a finite bipartite graph. We give the necessary definitions and discuss our conventions for working in such planar algebras in this subsection.

Definition 2.3. A shaded planar $*$-algebra is *evaluable* if $\dim(\mathcal{P}_{n,\pm}) < \infty$ for all $n \geq 0$, and $\mathcal{P}_{0,\pm} \cong \mathbb{C}$ as $*$-algebras. In this case, this isomorphism must send the empty diagram to 1.

Suppose $\mathcal{P}_\bullet$ is a shaded planar $*$-algebra which is not necessarily evaluable. We call $\mathcal{P}_\bullet$ *unitary* if for all $n \geq 0$, the $\mathcal{P}_{0,\pm}$-valued sesquilinear form on $\mathcal{P}_{n,\pm}$ given by $\langle x, y\rangle = \operatorname{Tr}(y^*x)$ is positive definite (in the operator-valued sense).

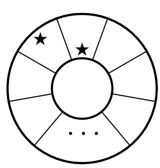
We call such a planar algebra *spherical* if, for any closed diagram in $\mathcal{P}_\bullet$ which equals a scalar multiple of the empty diagram, performing isotopy on a sphere still gives us the same scalar multiple of the appropriate empty diagram.

Remark 2.4. The above is only one possible definition of unitarity for a planar $*$-algebra. One might also want to require the existence of a faithful state on $\mathcal{P}_{0,\pm}$ which induces a C^*-algebra structure on the algebras $\mathcal{P}_{n,\pm}$ in the usual GNS way. However, the above frugal definition is sufficient for our purposes, since the following theorem holds.

Theorem 2.5. *Suppose $\mathcal{P}_\bullet$ is a spherical, unitary, shaded planar $*$-algebra which is not necessarily evaluable. If $\mathcal{Q}_\bullet \subset \mathcal{P}_\bullet$ is an evaluable planar $*$-subalgebra, then $\mathcal{Q}_\bullet$ is a subfactor planar algebra.*

Proof. Since $\mathcal{Q}_\bullet$ is evaluable, sphericality of $\mathcal{Q}_\bullet$ follows from sphericality of $\mathcal{P}_\bullet$. Now, the sesquilinear form $\langle x, y\rangle = \operatorname{Tr}(y^*x)$ on $\mathcal{Q}_{n,\pm}$ is operator-valued positive definite. Since $\mathcal{Q}_\bullet$ is evaluable, by identifying the appropriate empty diagram with $1 \in \mathbb{C}$, we get a positive definite inner product. □

Notation 2.6. Recall that the Fourier transform $\mathcal{F}$ is given by

$\mathcal{F} =$  .

For a rotational eigenvector $S \in \mathcal{P}_{n,\pm}$ corresponding to an eigenvalue $\omega_S = \sigma_S^2$, we define another rotational eigenvector $\check{S} \in \mathcal{P}_{n,\mp}$ by $\check{S} = \sigma_S^{-1}\mathcal{F}(S)$. Note that $\mathcal{F}(\check{S}) = \sigma_S S$, so $\check{\check{S}} = S$.

Definition 2.7. Suppose $\mathcal{P}_\bullet$ is a unitary, spherical, shaded planar $*$-algebra with modulus $\delta > 2$ which is not necessarily evaluable. A finite set $\mathfrak{B} \subset \mathcal{P}_{n,+}$ is called a *set of minimal generators for* $\mathcal{Q}_\bullet$ if the elements of $\mathfrak{B}$ generate the planar $*$-subalgebra

$\mathcal{Q}_\bullet \subset \mathcal{P}_\bullet$ and are linearly independent, self-adjoint, low-weight eigenvectors for the rotation, i.e, for all $S \in \mathfrak{B}$,

- $S = S^*$,
- S is uncappable, and
- $\rho(S) = \omega_S S$ for some n-th root of unity ω_S.

In the sequel, when we refer to a set of minimal generators without mentioning $\mathcal{Q}_\bullet$, assume that $\mathcal{Q}_\bullet$ is the planar $*$-subalgebra generated by $\mathfrak{B}$.

Given a set of minimal generators $\mathfrak{B}$, we get a set of dual minimal generators $\check{\mathfrak{B}} = \{\check{S} \mid S \in \mathfrak{B}\}$. We say a set of minimal generators $\mathfrak{B}$ has *scalar moments* if $\operatorname{Tr}(R)$, $\operatorname{Tr}(RS)$, $\operatorname{Tr}(RST)$ and $\operatorname{Tr}(\check{R})$, $\operatorname{Tr}(\check{R}\check{S})$, $\operatorname{Tr}(\check{R}\check{S}\check{T})$ are scalar multiples of the empty diagram in $\mathcal{P}_{0,+}$ and $\mathcal{P}_{0,-}$ respectively for each $R, S, T \in \mathfrak{B}$.

If a set of minimal generators $\mathfrak{B}$ has scalar moments, we say $\mathfrak{B}$ is

- *orthogonal* if $\langle S, T\rangle = \operatorname{Tr}(ST) = 0$ if $S \neq T$ for all $S, T \in \mathfrak{B}$, and
- *orthonormal* if $\mathfrak{B}$ is orthogonal and $\operatorname{Tr}(S^2) = \langle S, S\rangle = 1$ for all $S \in \mathfrak{B}$.

The point of working with sets of minimal generators is the following theorem.

Theorem 2.8 [Morrison and Penneys 2015b, Theorem 2.5]. *All the formulas of Section 4 of* [Jones 2012] *hold in any unitary, spherical, shaded planar $*$-algebra with modulus $\delta > 2$ for any orthonormal set of minimal generators $\mathfrak{B}$ with scalar moments.*

Assumption 2.9. For the rest of the article, unless otherwise specified, we assume $\mathcal{P}_\bullet$ is a unitary, spherical, shaded $*$-planar algebra with modulus $\delta > 2$ which is not necessarily evaluable, and $\mathfrak{B} \subset \mathcal{P}_{n,+}$ is an orthogonal set of minimal generators with scalar moments.

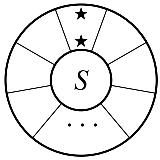
Since we do not assume our generators in $\mathfrak{B}$ are orthonormal, our formulas will differ slightly in appearance from those of [Jones 2012] and [Morrison and Penneys 2015b].

Remark 2.10. For diagram evaluation, it is useful to have our standard equations for our set of minimal generators in one place. For $S \in \mathfrak{B}$,

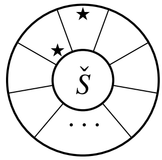
$$\begin{array}{llll} S = S^* & \mathcal{F}^2 = \rho & \rho(S) = \omega_S S & \mathcal{F}(S) = \sigma_S \check{S} \\ \check{S} = \check{S}^* & \sigma_S^2 = \omega_S & \rho(\check{S}) = \omega_S \check{S} & \mathcal{F}(\check{S}) = \sigma_S S. \end{array}$$

When moving $\star$ on the distinguished interval of a generator, the resulting diagram is multiplied by some exponent of σ_S:

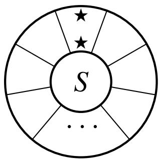
- If you shift $\star$ counterclockwise by one strand, multiply by σ_S and switch $\check{}$:



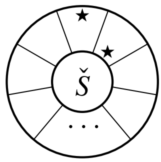
$= \sigma_S$



- If you shift $\star$ clockwise by one strand, multiply by σ_S^{-1} and switch $\check{}$:



$= \sigma_S^{-1}$



Using notation from [Jones 2012], for $P, Q, R \in \mathfrak{B}$, we set $a_R^{PQ} = \operatorname{Tr}(PQR)$ and $b_R^{PQ} = \operatorname{Tr}(\check{P}\check{Q}\check{R})$.

Remark 2.11. Once we have determined our set of minimal generators $\mathfrak{B}$ has scalar moments, the next thing to do is to verify that the complex spans of $\mathfrak{B} \cup \{f^{(n)}\}$ and $\check{\mathfrak{B}} \cup \{f^{(n)}\}$ form algebras under the usual multiplication. If this is the case, for $P, Q \in \mathfrak{B}$, we necessarily have

$$PQ = \frac{\operatorname{Tr}(PQ)}{[n+1]} f^{(n)} + \sum_{R \in \mathfrak{B}} \frac{a_R^{PQ}}{\|R\|^2} R. \tag{1}$$

Immediately, we get that all higher moments of $\mathfrak{B}$, $\check{\mathfrak{B}}$ are scalars, as are certain tetrahedral structure constants (see Remark 2.15 and Example 2.17). For example, we have that

$$\operatorname{Tr}(PQRS) = \frac{\operatorname{Tr}(PQ)\operatorname{Tr}(RS)}{[n+1]} + \sum_{T \in \mathfrak{B}} \frac{a_T^{PQ}}{\|T\|^2} a_T^{RS}. \tag{2}$$

for $P, Q, R, S \in \mathfrak{B}$.

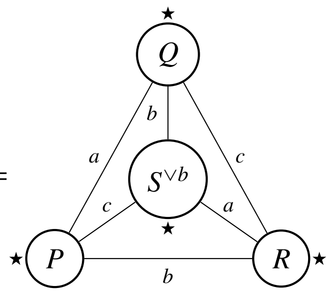
Assumption 2.12. We now assume the complex spans of $\mathfrak{B} \cup \{f^{(n)}\}$ and $\check{\mathfrak{B}} \cup \{f^{(n)}\}$ form algebras under the usual multiplication.

Remark 2.13. The assumptions of this subsection are significant. A randomly chosen subset of a graph planar algebra will not satisfy Assumption 2.9. Given an orthogonal set of minimal generators $\mathfrak{B}$ with scalar moments, it is still possible it will not satisfy Assumption 2.12. For example, if we start with a $\mathfrak{B}$ satisfying Assumptions 2.9 and 2.12 and we discard one element, the resulting set together with $f^{(n)}$ may not span an algebra.

2B. *Tetrahedral structure constants.* We will also need the tetrahedral structure constants defined in [Jones 2003, Section 3.3].

Definition 2.14. For $P, Q, R, S \in \mathfrak{B}$, we define

$$\Delta_{a,b}(P, Q, R \mid S) \quad =$$



where $c = 2n - a - b$, and

$$S^{\vee b} = \begin{cases} S & \text{if } b \text{ is even,} \\ \check{S} & \text{if } b \text{ is odd.} \end{cases}$$

Note that the $\Delta_{a,b}(P, Q, R \mid S)$ for $P, Q, R, S \in \mathfrak{B}$ determine all the tetrahedral structure constants by [Jones 2003, Section 3.3].

Remark 2.15. For this article, we only need the following tetrahedral structure constants:

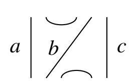
- $\Delta_{n-1,2}(P, Q, R \mid S)$
- $\Delta_{n,1}(P, Q, R \mid S)$
- $\Delta_{n-1,1}(P, Q, R \mid S) = \overline{\Delta_{n,1}(R, Q, P \mid S)}$.

By Assumption 2.12, we can express the second and third tetrahedral structure constants above in terms of the moments and chiralities of $\mathfrak{B}$ and $\check{\mathfrak{B}}$, since one of $a, b, c \geq n$. We do this computation in Example 2.17. Thus for convenience, we will just write $\Delta(P, Q, R \mid S)$ instead of $\Delta_{n-1,2}(P, Q, R \mid S)$, and we will only write subscripts a, b if $a \neq n-1$ or $b \neq 2$. For each of our planar algebras in this article, we give the tetrahedral structure constants $\Delta(P, Q, R \mid S)$ in Appendix B.

Since we will use it repeatedly, we reproduce the following well-known fact for convenience.

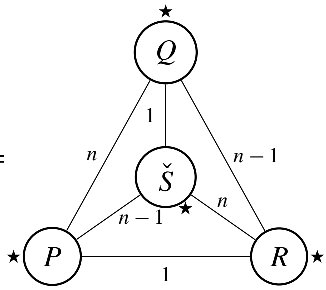
Fact 2.16 [Morrison 2015; Reznikoff 2007]. *The coefficient of the below Temperley–Lieb diagram in the Jones–Wenzl idempotent $f^{(k)}$ is given by*

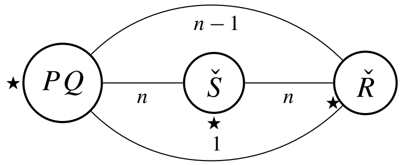
$$\operatorname*{coeff}_{\in f^{(k)}}\Big(\ \ \Big) = (-1)^{b+1}\frac{[a+1][c+1]}{[k]}.$$

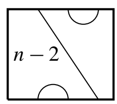
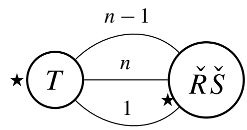


Example 2.17. In the following calculation, we use (1) for the third equality and 2.16 for the coefficient in the Jones–Wenzl idempotent appearing in the third line.

$\Delta_{n,1}(P, Q, R \mid S)$

$=$ 

$= \sigma_R^{-1}$ 

$= \sigma_R^{-1} \dfrac{\operatorname{Tr}(PQ)\operatorname{Tr}(\check{R}\check{S})}{[n+1]} \underset{\in f^{(n)}}{\operatorname{coeff}}\Bigg($  $\Bigg) + \displaystyle\sum_{T\in\mathfrak{B}} \sigma_R^{-1} \frac{a_T^{PQ}}{\|T\|^2}$ 

$$= (-1)^{n-1}\sigma_R^{-1} \frac{\operatorname{Tr}(PQ)\operatorname{Tr}(\check{R}\check{S})}{[n][n+1]} + \sum_{T\in\mathfrak{B}} \sigma_T \sigma_R^{-1} \frac{a_T^{PQ} b_T^{RS}}{\|T\|^2}.$$

By symmetry, we get

$$\begin{aligned}\Delta_{n-1,1}(P, Q, R \mid S) &= \overline{\Delta_{n,1}(R, Q, P \mid S)} \\ &= (-1)^{n-1}\sigma_P \frac{\operatorname{Tr}(QR)\operatorname{Tr}(\check{P}\check{S})}{[n][n+1]} + \sum_{T\in\mathfrak{B}} \sigma_T^{-1}\sigma_P \frac{a_T^{QR} b_T^{SP}}{\|T\|^2}.\end{aligned}$$

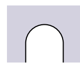
Lemma 2.18. *We have the following symmetries*:

$$\begin{aligned}\Delta(P, Q, R \mid S) &= \overline{\Delta(R, Q, P \mid S)} \\ &= \omega_P\omega_R^{-1}\Delta(R, S, P \mid Q) \\ &= \omega_P\omega_R^{-1}\overline{\Delta(P, S, R \mid Q)} \\ &= \sigma_P^{1-n}\sigma_Q^{n-1}\sigma_R^{n-1}\sigma_S^{1-n}\Delta(Q^{\vee(n-1)}, P^{\vee(n-1)}, S^{\vee(n-1)} \mid R^{\vee(n-1)}) \\ &= \sigma_P^{1-n}\sigma_Q^{n-1}\sigma_R^{n-1}\sigma_S^{1-n}\overline{\Delta(S^{\vee(n-1)}, P^{\vee(n-1)}, Q^{\vee(n-1)} \mid R^{\vee(n-1)})} \\ &= \sigma_P^{1-n}\sigma_Q^{n+1}\sigma_R^{n-1}\sigma_S^{-1-n}\Delta(S^{\vee(n-1)}, R^{\vee(n-1)}, Q^{\vee(n-1)} \mid P^{\vee(n-1)}) \\ &= \sigma_P^{1-n}\sigma_Q^{n+1}\sigma_R^{n-1}\sigma_S^{-1-n}\overline{\Delta(Q^{\vee(n-1)}, R^{\vee(n-1)}, S^{\vee(n-1)} \mid P^{\vee(n-1)})}\end{aligned}$$

Proof. Immediate from drawing diagrams using unitarity and sphericality of $\mathcal{P}_\bullet$. □

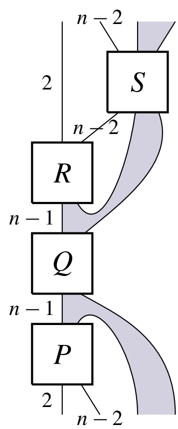
Remark 2.19. As in [Morrison and Peters 2014; Morrison and Penneys 2015b], when doing calculations in the graph planar algebra, we work with the lopsided convention rather than the spherical convention (see [Morrison and Peters 2014]). The lopsided convention treats shaded and unshaded contractible loops differently, which has the advantage that there are fewer square roots, so arithmetic is easier.

The translation map $\natural\colon \mathscr{P}_\bullet^{\text{spherical}} \to \mathscr{P}_\bullet^{\text{lopsided}}$ between the conventions from [Morrison and Peters 2014] is not a planar algebra map, but it commutes with the action of the planar operad up to a scalar. We determine the scalar by first drawing the tangle in a standard rectangular form where each box has the same number of strings attached to the top and bottom. We then get one factor of $\delta^{\pm 1}$ for each critical point which is shaded above, and the power of δ corresponds to the sign of the critical point:

 $\longleftrightarrow \delta,$  $\longleftrightarrow \delta^{-1}.$

Correction factors for the lopsided convention for the Fourier transform and the trace were worked out in [Morrison and Penneys 2015b, Examples 2.6 and 2.7], and we derive another correction factor in the next example.

Example 2.20. We find the correction factors for the lopsided convention when calculating $\Delta(P, Q, R \mid S)$. We have

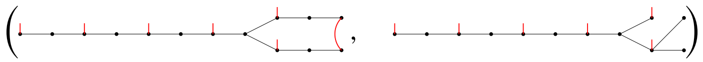
$\Delta(P, Q, R \mid S) = \operatorname{Tr}$ (),

where the shading assumes n is even. The above diagram contributes a factor of δ^{-1}, and the trace tangle contributes no factors of δ. When n is odd, the above diagram contributes a factor of δ, and the trace tangle contributes a factor of δ. (See [Morrison and Penneys 2015b, Example 2.6] as well.) Hence we have the formula

$$\Delta(P, Q, R \mid S) = \natural\Delta(P, Q, R \mid S) = \begin{cases} \delta^{-1}\Delta(\natural P, \natural Q, \natural R \mid \natural S) & \text{if } n \text{ is even,} \\ \delta^{2}\Delta(\natural P, \natural Q, \natural R \mid \natural S) & \text{if } n \text{ is odd.} \end{cases}$$

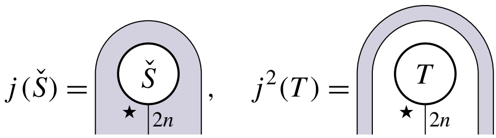
Assumption 2.21. For the rest of the article, we assume that for all $P, Q, R, S \in \mathfrak{B}$, the tetrahedral structure constants $\Delta(P, Q, R \mid S)$ are scalar multiples of the empty diagram.

2C. *The jellyfish algorithm and reduced trains.* The *jellyfish algorithm* was invented in [BMPS 2012] to construct the extended Haagerup subfactor planar algebra with principal graphs

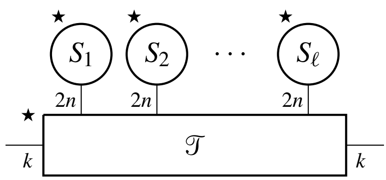


One uses the jellyfish algorithm to evaluate closed diagrams on a set of minimal generators. There are two ingredients:

(i) The generators in $\mathfrak{B} \subset \mathcal{P}_{n,+}$ satisfy *jellyfish relations*, i.e., for each generator S, T,

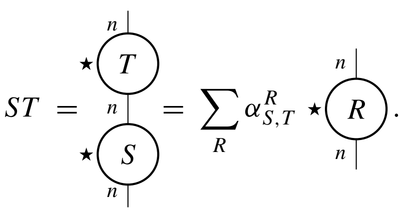


can be written as linear combinations of *trains*. Trains are diagrams where any region meeting the distinguished interval of a generator meets the distinguished interval of the external disk, i.e.,



where $S_1, \ldots, S_\ell \in \mathfrak{B}$, and $\mathcal{T}$ is a single Temperley–Lieb diagram.

(ii) The generators in $\mathfrak{B}$ are uncappable and together with the Jones–Wenzl projection $f^{(n)}$ form an algebra under the usual multiplication



(Note that the Mathematica package *FusionAtlas* also multiplies in this order; reading from left to right in products corresponds to reading from bottom to top in planar composites.)

Given these two ingredients, one can evaluate any closed diagram using the following two step process.

(i) Pull all generators S to the outside of the diagram using the jellyfish relations, possibly getting diagrams with more S's.

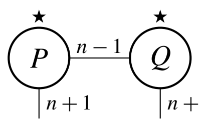
(ii) Use uncappability and the algebra property to iteratively reduce the number of generators. Any nonzero train which is a closed diagram is either a Temperley–Lieb diagram, has a capped generator, or has two generators S, T connected by at least n strings, giving ST. Each of these cases can be simplified using the relations, still giving a linear combination of trains.

Section 4 is devoted to our procedure for computing the jellyfish relations necessary for the first part of the jellyfish algorithm. The second part is rather easy, and amounts to verifying equation (1) (see the beginning of Section 5).

Definition 2.22. A $\mathfrak{B}$-train is called *reduced* if no two generators are connected by more than $n-1$ strands, and no generator is connected to itself.

Example 2.23. In $\mathcal{P}_{n+1,+}$, the set of reduced trains is given by

$$\left\{ P \underset{n-1}{\circ} Q = \right.$$

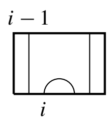


$$\left| \; P, Q \in \mathfrak{B} \right\}.$$

To describe the reduced trains in $\mathcal{P}_{n+2,+}$, we introduce the following notation.

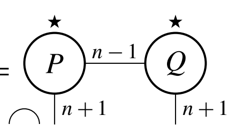
Definition 2.24. Let $C_i[P \circ_{n-1} Q] \in \mathcal{P}_{n+2,+}$ for $i = 1, \ldots, 2n+3$ be the reduced train obtained from $P \circ_{n-1} Q$ by putting C_i underneath, where C_i is the diagram given by

$$C_i =$$



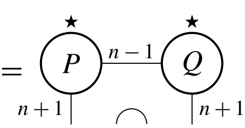
This can be thought of as multiplying C_i by $P \circ_{n-1} Q$ for a fixed arrangement of boundary strings. For example, we have, for $P, Q \in \mathfrak{B}$,

$$C_1\big[P \underset{n-1}{\circ} Q\big] =$$



and

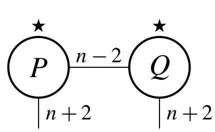
$$C_{n+2}\big[P \underset{n-1}{\circ} Q\big] =$$



Example 2.25. In $\mathcal{P}_{n+2,+}$, we have many more reduced trains. First, we have those annular consequences of the $P \circ_{n-1} Q$'s which are still trains in $\mathcal{P}_{n+2,+}$. These are exactly the $C_i[P \circ_{n-1} Q]$ for $i = 1, \ldots, 2n+3$.

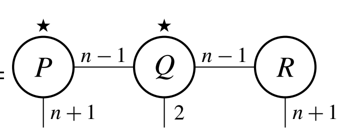
Now the only reduced trains which are nonzero when we put a copy of $f^{(2n+4)}$ underneath, for $P, Q, R \in \mathfrak{B}$, are

$$P \underset{n-2}{\circ} Q =$$



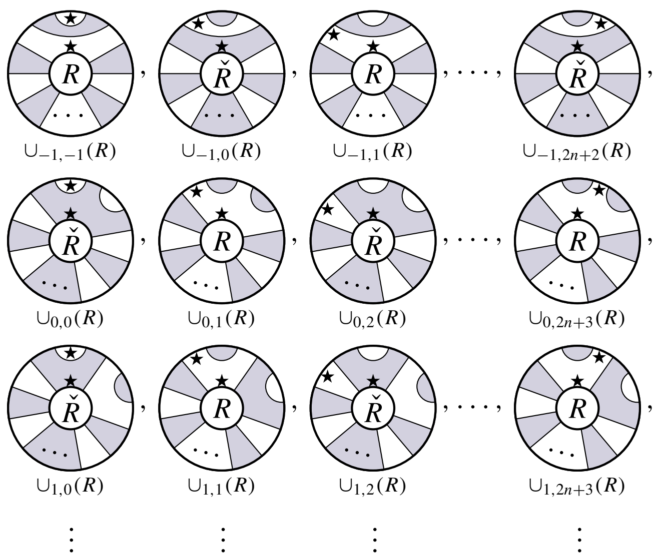
and

$$P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R =$$



2D. ***The second annular basis.*** Given a nonzero low-weight rotational eigenvector $R \in \mathcal{P}_{n,+}$, the space $\mathfrak{A}_{n+2}(R) \subset \mathcal{P}_{n+2,+}$ of second annular consequences of R is spanned by diagrams with two cups on the outer boundary. We now describe a distinguished basis of $\mathfrak{A}_{n+2}(R)$ when $\delta > 2$ along the lines of [Jones 2001; 2012].

Definition 2.26. The element $\cup_{i,j}(R) \in \mathfrak{A}_{n+2}(R)$ is the annular consequence of R given in the following diagrams, where each row consists of $2n+4$ elements.

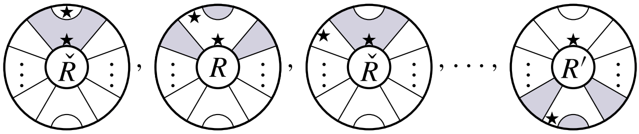


The index i specifies the number of through strings separating the two cups (counting clockwise from the cup at 12 o'clock in the above diagrams). Here $i = -1$ denotes two nested cups. The j refers to the number of strings separating the external boundary interval at 12 o'clock from the interval for the external $\star$, counting counterclockwise (and subtract 1 for nested cups). Note that $n+k$ strings separating the cups is the same as a rotation (up to switching the shading) of $n-k$ strings separating the cups.

The *second annular basis* of $\mathfrak{A}_{n+2}(R)$ the set of $\cup_{i,j}(R)$ such that $-1 \le i \le n$, and

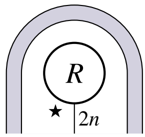
$$j \in \begin{cases} \{-1, 0, \ldots, 2n+2\} & \text{if } i = -1, \\ \{0, \ldots, 2n+3\} & \text{if } -1 < i < n, \\ \{0, \ldots, n+1\} & \text{if } i = n. \end{cases}$$

If $i = n$, the $n+2$ elements corresponding to $j = 0, \ldots, n+1$ are given below. Here the shading on the bottom in the first three pictures depends on the parity of n, while the shading on the top of the final picture depends on the parity of n and whether R' is R or $\check{R}$.



$\cup_{n,0}(R)$, $\cup_{n,1}(R)$, $\cup_{n,2}(R)$, …, $\cup_{n,n+1}(R)$

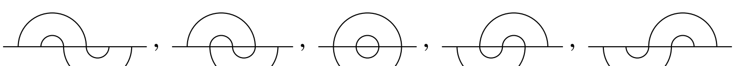
Remark 2.27. Note that

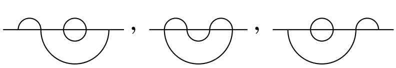
$$\cup_{-1,-1}(R) = j^2(R) =$$  .

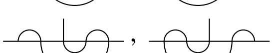
Recall that the inner product is defined by $\langle x, y\rangle = \mathrm{Tr}(x^* y)$, which is the same as connecting all strings of x^* and y. Computing inner products amongst the $\cup_{i,j}(R)$'s amounts to examining the relative positions of caps along the interface between the two diagrams. Since R is uncappable, the entire diagram is zero if a cap from one of the $\cup_{i,j}(R)$'s reaches the other copy of R.

It is easy to see that pairing $\cup_{i,j}(R)$ with $\cup_{i',k}(R)$ is nonzero only if $|i - i'| < 3$. When the scalar is nonzero differs for the cases $i = -1$ and $i \geq 0$, and there are some exceptional cases when $i = n - 1, n$.

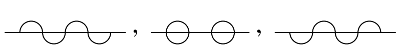
- When $i = -1$, there are exactly 5, 3, and 2 ways of getting a nonzero scalar when pairing $\cup_{-1,j}(R)$ with $\cup_{i',k}(R)$ for $i' = -1$, 0, and 1 respectively. They correspond to the following relative positions of caps along the interface.

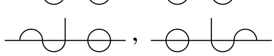
 $i' = -1$,

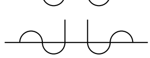
 $i' = 0$,

 $i' = 1$.

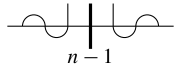
- For $0 \leq i \leq n-2$, there are exactly 3, 2, and 1 ways of getting a nonzero scalar when pairing $\cup_{i,j}(R)$ with $\cup_{i',k}(R)$ for $i' = i$, $i+1$, and $i+2$ respectively. The relative positions of caps corresponding to the case $i = 0$ are.

 $i' = 0$,

 $i' = 1$,

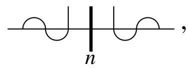
 $i' = 2$.

- For $i = n - 1$, there is an additional way of getting a nonzero scalar when pairing $\cup_{n-1,j}(R)$ with $\cup_{n-1,k}(R)$, which makes up for the fact that there is no $\cup_{n+1,k}(R)$. The relative position of caps given by



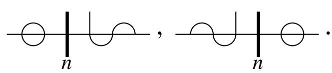
can be interpreted as $(j-k) \equiv -1$ or $n+2 \bmod (2n+4)$, depending on the location of the $\star$ above the line. In the former case, the diagram contributes σ^{-1}, and in the latter, σ^n.

- The case $i = n$ is more subtle. When $i' = n-2$, there are two ways of pairing $\cup_{n,j}(R)$ with $\cup_{n-2,k}(R)$ to get a nonzero scalar, which correspond to the $\star$ placement of

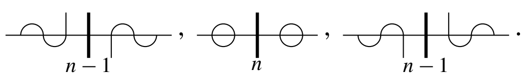
  

  i.e., $(j-k) \equiv -1$ or $n+1 \bmod (2n+4)$. In the former case, the diagram contributes a scalar of σ^{-1}, and in the latter, $\sigma^n \sigma^{-1}$.

  When $i' = n-1$, there are four ways to get a nonzero scalar, which correspond to the $\star$ placement of

  Finally, when $i' = n$, there are three ways to get a nonzero scalar, corresponding to

  (Note that the $\star$ placement is determined.)

The following proposition now follows from the above discussion.

Proposition 2.28. *Assuming $R = R^*$ and $\|R\|^2 = \mathrm{Tr}(R^2) = 1$, we have the following inner products* (*linear on the* right):

$$\langle \cup_{i',k}(R), \cup_{-1,j}(R)\rangle = $$

		$(j-k) \bmod (2n+4)$				
		-2	-1	0	1	2
i'	-1	ω_R^{-1}	σ_R^{-1}	$[2]^2$	σ_R	ω_R
	0	0	$[2]\sigma_R^{-1}$	$[2]$	$[2]\sigma_R$	0
	1	0	0	1	σ_R	0

and is zero otherwise.

For $0 \le i', i \le n-1$, *we have*

$$\langle \cup_{i',k}(R), \cup_{i,j}(R)\rangle = \begin{array}{c|c|ccc|} & & \multicolumn{3}{c}{(j-k) \mod (2n+4)} \\ \cline{3-5} & & -1 & 0 & 1 \\ \cline{2-5} & -2 & \sigma_R^{-1} & 0 & 0 \\ & -1 & [2]\sigma_R^{-1} & [2] & 0 \\ i'-i & 0 & \sigma_R^{-1} & [2]^2 & \sigma_R \\ & 1 & 0 & [2] & [2]\sigma_R \\ & 2 & 0 & 0 & \sigma_R \\ \cline{2-5} \end{array}$$

and is zero otherwise, *with the exception that*

$$\langle \cup_{n-1,k}(R), \cup_{n-1,j}\rangle = \sigma_R^n \quad \textit{if } j-k \equiv n+2 \mod 2n+4.$$

For $i = n$ *and* $i' < n$, *we have*

$$\langle \cup_{i',k}(R), \cup_{n-1,j}(R)\rangle = \begin{array}{c|c|cccc|} & & \multicolumn{4}{c}{(j-k) \mod (2n+4)} \\ \cline{3-6} & & -1 & 0 & n+1 & n+2 \\ \cline{2-6} i' & n-2 & \sigma_R^{-1} & 0 & \sigma_R^n\sigma_R^{-1} & 0 \\ & n-1 & [2]\sigma_R^{-1} & [2] & [2]\sigma_R^n\sigma_R^{-1} & [2]\sigma_R^n \\ \cline{2-6} \end{array}$$

and is zero otherwise.

Finally, *if* $i = i' = n$, *then we have*

$$\langle \cup_{n,k}(R), \cup_{n,j}(R)\rangle = \begin{cases} \sigma_R^N\sigma_R^{-1} & \textit{if } (j-k) \equiv -1 \mod (n+2) \textit{ and } j = n+1, \\ \sigma_R^{-1} & \textit{if } (j-k) \equiv -1 \mod (n+2) \textit{ and } j < n+1, \\ [2]^2 & \textit{if } (j-k) \equiv 0 \mod (n+2), \\ \sigma_R & \textit{if } (j-k) \equiv 1 \mod (n+2) \textit{ and } j > 0, \\ \sigma_R^n\sigma_R & \textit{if } (j-k) \equiv -1 \mod (n+2) \textit{ and } j = 0, \\ 0 & \textit{else.} \end{cases}$$

Remark 2.29. The concerned reader may wonder if we have missed a case or two amidst this muddle of indices. Be reassured that we have checked these inner products numerically for the generators of our example directly in the graph planar algebra. See Section 4D for more details.

Remark 2.30. In this article, we do not give a formula for the dual basis $\widehat{\cup}_{i,j}(R)$ in terms of the $\cup_{i,j}(R)$'s, i.e., the change of basis matrix from the annular basis to the dual annular basis. Instead, we find the dual annular basis for our examples by inverting the matrix of inner products given by Proposition 2.28.

As in [Morrison and Penneys 2015b, Remark 3.7], if W is the matrix of inner products of the $\cup_{i,j}(R)$'s, then the change of basis matrix from the column vectors representing the annular basis U to the column vectors representing the dual basis $\widehat{U}$ is $\overline{W^{-1}}$, i.e., $\overline{W^{-1}}U = \widehat{U}$. (The inner product is linear on the *right*.) If $\hat{c}$ is the row vector of coefficients in the dual basis for an annular consequence x, i.e., $x = \hat{c} \cdot \widehat{U}$, then the row vector of coefficients in the annular basis is given by $c = \hat{c}\,\overline{W^{-1}}$.

It would certainly be useful to have a general formula for the dual annular basis in terms of the annular basis. While such a computation is routine, it would be demanding, and we leave it for another time.

3. Projections and inner products of trains

As in the previous section, we continue to use Assumptions 2.9, 2.12, and 2.21.

To derive two-strand jellyfish relations, we need to analyze all reduced $\mathfrak{B}$-trains in $\mathcal{P}_{n+2,+}$, in particular their projections to $\mathcal{TL}_{n+2,+}$, their projections to the space of second annular consequence of $\mathfrak{B}$, and their pairwise inner products.

We express some projections to Temperley–Lieb and annular consequences in terms of dual bases. We will use the following formula repeatedly.

Remark 3.1. Suppose $\{v_1, \ldots, v_k\} \subset V$ is a basis for the finite dimensional Hilbert space V. Let $\{\hat{v}_1, \ldots, \hat{v}_k\}$ be the dual basis V, defined by $\langle \hat{v}_i, v_j\rangle = \delta_{i,j}$, where the inner product is linear on the right. If $x \in V$, we have $x = \sum_{i=1}^{k} \langle v_i, x\rangle \hat{v}_i$.

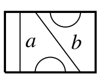
In what follows, P, Q, R, S, T are always elements of $\mathfrak{B}$. We will first need a few results about certain Temperley–Lieb dual basis elements.

3A. ***Some Temperley–Lieb dual basis elements.*** We now discuss certain elements of the basis which is dual to the usual diagrammatic basis of $\mathcal{TL}_k$.

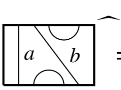
Lemma 3.2. *If* $a, b \geq 0$ *and* $a + b = n$, *then* $[a+2][b+1] - [a+1][b] = [n+2]$.

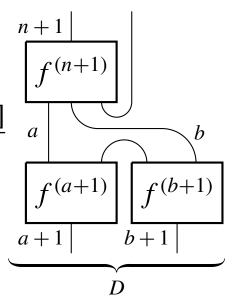
Proof. Immediate from the formula

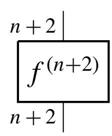
$$[k][\ell] = \sum_{\substack{|k-\ell|<j<k+\ell \\ j \equiv |k-\ell|+1 \mod 2}} [j]. \qquad \square$$

Lemma 3.3. *The element dual to* 

$\in \mathcal{TL}_{n+2,+}$ *is given by*

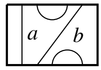


$= \frac{[a+1][b+1]}{[n+2]^2}$ 

$- \frac{(-1)^b[a+1]}{[n+2][n+3]}$ 

.

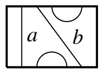
To find the element dual to



$\in \mathcal{TL}_{n+2,+}$,

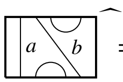
maintain the coefficients and vertically reflect the diagrams in the above formula.

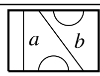
Proof. Note that the middle diagram D in the above equation has nonzero inner product only with 1_{n+2} and



.

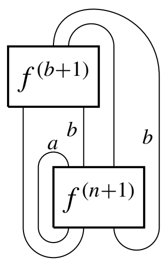
We already know that $\hat{1}_{n+2} = f^{(n+2)}/[n+3]$, so we have



$= 1 \Big/ \Big\langle D,$


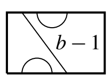
$\Big\rangle \left(D - \langle D, 1_{n+2}\rangle \dfrac{f^{(n+2)}}{[n+3]}\right).$

A routine calculation computes the necessary inner products. First,

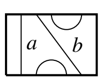
$\langle D, 1_{n+2}\rangle =$


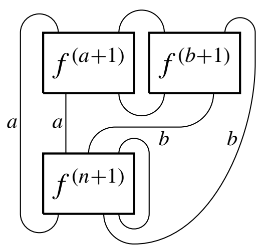
$= \dfrac{(-1)^b[n+2]}{[b+1]}$,

since the only diagram in the top $f^{(b+1)}$ which contributes to the closed diagram is



(the coefficient of this diagram in $f^{(b+1)}$ is given in Fact 2.16). Next, we calculate

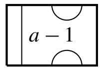
(3) $\Big\langle D,$


$\Big\rangle =$


$$= [n+2]\left(\frac{[a+2]}{[a+1]} - \frac{[b]}{[b+1]}\right) \tag{4}$$

$$= \frac{[n+2]^2}{[a+1][b+1]}, \tag{5}$$

where (4) follows since the only two terms in the top $f^{(a+1)}$ which contribute to the closed diagram are 1_{a+1} and

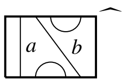
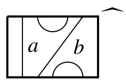


Equation (5) now follows by Lemma 3.2.

(Note that the value of the diagram that appears in (3) must be symmetric in a and b, but the quantity in (4) does not appear symmetric in a and b. This gave a hint that some quantum number identity should hold, which motivated Lemma 3.2.)

The last claim is now immediate. $\square$

Lemma 3.4. *Suppose $a, b \geq 0$ with $a+b=n$. Let D_a, D_a^*, $\hat{1}_{n+2}$ be the Temperley–Lieb dual basis elements*

$D_a =$  , $D_a^* =$  , *and* $\hat{1}_{n+2} = \dfrac{f^{(n+2)}}{[n+3]}$.

(i) $$\langle C_i[P \circ Q], \hat{1}_{n+2}\rangle = \begin{cases} \mathrm{Tr}(PQ)[n+2]^{-1} & \text{if } i = n+2, \\ 0 & \text{else.} \end{cases}$$

(ii) $$\langle C_i[P \circ Q], D_a\rangle = \begin{cases} \mathrm{Tr}(PQ)[n+2]^{-1} & \text{if } i-1 = a, \\ 0 & \text{if } i = n+2, \\ \dfrac{(-1)^b[a+1]}{[n+1][n+2]}\mathrm{Tr}(PQ) & \text{if } i = n+3, \\ 0 & \text{else.} \end{cases}$$

(iii) $\langle C_i[P \circ Q], D_a^*\rangle = \langle D_a, C_{2n+4-i}[Q \circ P]\rangle = \langle C_{2n+4-i}[P \circ Q], D_a\rangle$.

Proof. (i) We have

$$\langle C_i[P \circ Q], \hat{1}_{n+2}\rangle = \frac{1}{[n+3]}\langle C_i[P \circ Q], f^{(n+2)}\rangle,$$

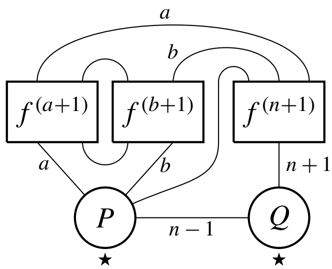
which is clearly zero if $i \neq n+2$. When $i = n+2$, it is easy to see we get $\dfrac{\mathrm{Tr}(PQ)}{[n+2]}$.

(ii) First, suppose $1 \leq i \leq n+1$. Then the inner product in question is given by

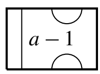
$$\frac{[a+1][b+1]}{[n+2]^2}\langle C_i[P \circ Q], D\rangle,$$

where D is the diagram in Lemma 3.3. If $i-1 \neq a$, then the resulting closed diagram is clearly zero. If $i-1 = a$, then we have

$$\frac{[a+1][b+1]}{[n+2]^2}\langle C_i[P\circ Q],\, D\rangle =$$



and the only terms in the $f^{(a+1)}$ which contribute to the value are 1_{a+1} and



.

This yields, using Lemma 3.2 and Fact 2.16,

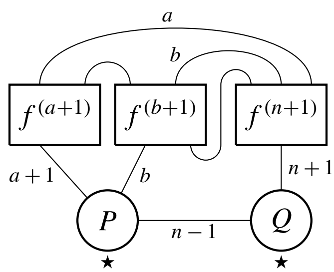
$$\frac{[a+1][b+1]}{[n+2]^2}\left(\frac{[b+2]}{[b+1]}-\frac{[a]}{[a+1]}\right)\mathrm{Tr}(PQ)=\frac{\mathrm{Tr}(PQ)}{[n+2]}.$$

Second, if $i = n+2$, then both diagrams in the formula for D_a from Lemma 3.3 contribute to the inner product, and we have

$$\begin{aligned}\langle C_{n+2}[P\circ Q],\, D_a\rangle\\ &= \frac{[a+1][b+1]}{[n+2]^2}\langle C_{n+2}[P\circ Q],\, D\rangle - \frac{(-1)^b[a+1]}{[n+2][n+3]}\langle C_{n+2}[P\circ Q],\, f^{(n+2)}\rangle\\ &= \frac{[a+1][b+1]}{[n+2]^2}\langle C_{n+2}[P\circ Q],\, D\rangle - \frac{(-1)^b[a+1]}{[n+2]^2}\,\mathrm{Tr}(PQ)\end{aligned}$$

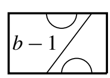
by part (i) of this lemma. Now by drawing diagrams, we get

$$\langle C_{n+2}[P\circ Q],\, D\rangle =$$



.

The only diagram in $f^{(b+1)}$ which contributes is



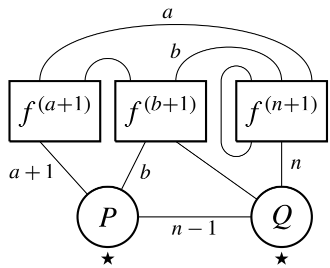
, which yields

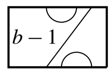
$$\frac{(-1)^b}{[b+1]}\,\mathrm{Tr}(PQ).$$

The inner product in question is thus zero.

Third, if $i = n+3$, then as in the case $1 \le i \le n+1$, we have

$$\frac{[a+1][b+1]}{[n+2]^2}\langle C_{n+3}[P \circ Q], D\rangle =$$



Again, the only diagram in $f^{(b+1)}$ which contributes is 

, which yields

$$\frac{[a+1][b+1]}{[n+1][n+2]}\left(\frac{(-1)^b}{[b+1]}\right) = \frac{(-1)^b[a+1]}{[n+1][n+2]}.$$

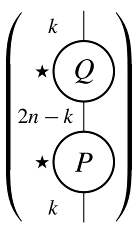
Finally, if $i > n+3$, the result is once again zero, since both diagrams in the formula for D_a from Lemma 3.3 have zero inner product with $C_i[P \circ Q]$.

(iii) The first equality follows since both sides give the same closed diagram. Note that the quantity in the middle is equal to its conjugate by part (ii) of this lemma. The second equality now follows since $\mathrm{Tr}(QP) = \mathrm{Tr}(PQ)$. □

3B. ***Projections to Temperley–Lieb.*** The first lemma below is similar to [Jones 2012, Proposition 4.5.2].

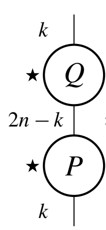
Lemma 3.5. (i) *If* $k = 0, \ldots, 2n$, *then*

$$P_{\mathcal{TL}_{k,+}}\Bigg($$



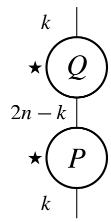
$$\Bigg) = \frac{\mathrm{Tr}(PQ)}{[k+1]} f^{(k)}.$$

(ii) *If* $k = 0, \ldots, n-1$, *then*



$$= \frac{\mathrm{Tr}(PQ)}{[k+1]} f^{(k)}.$$

Proof. For (i), notice that adding a cap to the top or bottom of

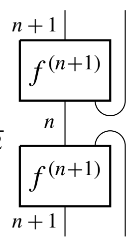
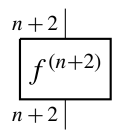


gives zero, so its projection to $\mathcal{TL}_{k,+}$ must be a constant times $f^{(k)}$. Taking traces gives the constant.

For (ii), notice that the diagram is already in Temperley–Lieb since $\mathfrak{B} \cup \{f^{(n)}\}$ spans an algebra. $\square$

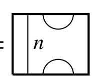
Proposition 3.6. (i) $P_{\mathcal{TL}_{n+2,+}}\big(P \underset{n-2}{\circ} Q\big) = \dfrac{\operatorname{Tr}(PQ)}{[n+3]} f^{(n+2)}$.

(ii) $P_{\mathcal{TL}_{n+2,+}}\big(P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R\big)$

$= a_R^{PQ} \Bigg(\dfrac{[n+1]}{[n+2]^2}$  $- \dfrac{[n+1]}{[n+2][n+3]}$  $\Bigg)$.

Proof. Part (i) is immediate from Lemma 3.5. For (ii), for T a diagrammatic basis element of $\mathcal{TL}_{n+2,+}$, it is clear that

$$\big\langle T, P_{\mathcal{TL}_{n+2,+}}\big(P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R\big)\big\rangle = \begin{cases} a_R^{PQ} & \text{if } T = E_{n+1} = \text{(diagram)}, \\ 0 & \text{else.} \end{cases}$$



Hence $P_{\mathcal{TL}_{n+2,+}}(P \circ_{n-1} Q \circ_{n-1} R) = a_R^{PQ} \widehat{E}_{n+1}$, where $\widehat{E}_{n+1}$ is the dual basis element of E_{n+1} in $\mathcal{TL}_{n+2,+}$. The result now follows by Lemma 3.3, using $b = 0$, $a = n$. (In particular $[b] = 0$ and $[b+1] = 1$.) $\square$

Proposition 3.7. $P_{\mathcal{TL}_{n+2,+}}(C_i[P \circ_{n-1} Q]) = \operatorname{Tr}(PQ)X$ *where X is a linear combination of Temperley–Lieb dual basis elements D_a, D_a^*, $\hat{1}_{n+2}$ (as in Lemma 3.4). The exact linear combination is given in the table below.*

i	X
1	$[2]D_0 + D_1$
$1 < i < n+1$	$D_{i-2} + [2]D_{i-1} + D_i$
$n+1$	$D_{n-1} + [2]D_n + \hat{1}_{n+2}$
$n+2$	$D_n + [2]\hat{1}_{n+2}$
$n+3$	$\hat{1}_{n+2} + [2]D_n^* + D_{n-1}^*$
$n+3 < i < 2n+3$	$D_{2n+2-i}^* + [2]D_{2n+3-i}^* + D_{2n+4-i}^*$
$2n+3$	$[2]D_0^* + D_1^*$

Proof. The only diagrammatic basis elements T in Temperley–Lieb which pair nontrivially with $C_i[P \circ_{n-1} Q]$ are those whose dual basis elements $\widehat{T}$ appear in the linear combination. The coefficients are given by $\langle T, C_i[P \circ_{n-1} Q]\rangle$. □

3C. *Projections to annular consequences.*

Definition 3.8. Let $\mathfrak{A}_{n+2}$ denote the space of second annular consequences of $\mathfrak{B}$ in $\mathcal{P}_{n+2,+}$.

The proofs of the following propositions are parallel to the proof of [Jones 2012, Proposition 4.4.1]. The inner products are only nonzero for the given annular consequences, and they are easily worked out by drawing pictures and using Lemma 3.5.

Proposition 3.9.

(i) $$P_{\mathfrak{A}_{n+2}}\big(P \underset{n-2}{\circ} Q\big) = \sum_{R\in\mathfrak{B}} a_R^{PQ}\omega_P\omega_Q^{-1}\widehat{\cup}_{-1,-1}(R) + a_R^{PQ}\sigma_R^n\widehat{\cup}_{-1,n+1}(R) + b_R^{PQ}\sigma_P\sigma_Q^{-1}\widehat{\cup}_{n,0}(R)$$
where the coefficients of the $\widehat{\cup}_{i,j}(R)$ are given by $\langle \cup_{i,j}(R), P \circ_{n-2} Q\rangle$.

(ii) $$\begin{aligned} P_{\mathfrak{A}_{n+2}}\big(P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R\big) &= \sum_{S\in\mathfrak{B}} \Delta_{n-1,2}(P, Q, R \mid S)\widehat{\cup}_{-1,-1}(S) + \frac{\sigma_S^{n+1}}{[n]}\operatorname{Tr}(SP)\operatorname{Tr}(QR)\widehat{\cup}_{-1,n}(S) \\ &\quad + \frac{\sigma_S^{n-1}}{[n]}\operatorname{Tr}(PQ)\operatorname{Tr}(RS)\widehat{\cup}_{-1,n+2}(S) + \sigma_S^n\operatorname{Tr}(PQRS)\widehat{\cup}_{0,n+1}(S) \\ &\quad + \Delta_{n-1,1}(P, Q, R \mid S)\widehat{\cup}_{n-1,0}(S) + \sigma_S^{n-1}\Delta_{n,1}(P, Q, R \mid S)\widehat{\cup}_{n-1,n+3}(S), \end{aligned}$$
where the coefficients of the $\widehat{\cup}_{i,j}(S)$ are given by $\langle \cup_{i,j}(S), P \circ_{n-1} Q \circ_{n-1} R\rangle$.

Note that in the above formula, the quartic moment and two of the three tetrahedral constants were computed in terms of the moments and chiralities of $\mathfrak{B}$ in Remark 2.11 and Example 2.17.

Proposition 3.10. *$P_{\mathfrak{A}_{n+2}}(C_i[P \circ_{n-1} Q]) = \sum_{R\in\mathfrak{B}} X_R$ where X_R is given in the table below. Here we denote $\alpha = \sigma_R^n a_R^{PQ}$, $\beta = \sigma_Q^{-1}\sigma_P b_R^{PQ}$, and $\widehat{\cup}_{i,j} = \widehat{\cup}_{i,j}(R)$.*

i	$X_R \in \mathfrak{A}_{n+2}$
1	$\beta\widehat{\cup}_{0,2n+2} + \alpha\widehat{\cup}_{n-1,n+1} + [2]\beta\widehat{\cup}_{-1,2n+2} + [2]\alpha\widehat{\cup}_{n,n+1} + \sigma_R^{-1} a_R^{PQ}\widehat{\cup}_{n-1,0}$
2	$\sigma_R\beta\widehat{\cup}_{1,2n+1} + \alpha\widehat{\cup}_{n-2,n+1} + [2]\beta\widehat{\cup}_{0,2n+2} + [2]\alpha\widehat{\cup}_{n-1,n+1} + \beta\widehat{\cup}_{-1,2n+2} + \alpha\widehat{\cup}_{n,n+1} + \sigma_R^{-1}\beta\widehat{\cup}_{-1,-1}$
$2 < i < n+1$	$\sigma_R^{i-1}\beta\widehat{\cup}_{i-1,2n-i+3} + \alpha\widehat{\cup}_{n-i,n+1} + [2]\sigma_R^{i-2}\beta\widehat{\cup}_{i-2,2n-i+4} + [2]\alpha\widehat{\cup}_{n-i+1,n+1} + \sigma_R^{i-3}\beta\widehat{\cup}_{i-3,2n-i+5} + \alpha\widehat{\cup}_{n-i+2,n+1}$
$n+1$	$\beta\widehat{\cup}_{n,0} + \alpha\widehat{\cup}_{-1,n+1} + [2]\sigma_R^{n-1}\beta\widehat{\cup}_{n-1,n+3} + [2]\alpha\widehat{\cup}_{0,n+1} + \sigma_R^{n-2}\beta\widehat{\cup}_{n-2,n+4} + \alpha\widehat{\cup}_{1,n+1} + \sigma_R^{n+1} a_R^{PQ}\widehat{\cup}_{-1,n}$
$n+2$	$\beta\widehat{\cup}_{n-1,0} + \alpha\widehat{\cup}_{0,n+1} + [2]\beta\widehat{\cup}_{n,0} + [2]\alpha\widehat{\cup}_{-1,n+1} + \sigma_R^{n-1}\beta\widehat{\cup}_{n-1,n+3}$
$n+3$	$\beta\widehat{\cup}_{n-2,0} + \sigma_R^{n+1} a_R^{PQ}\widehat{\cup}_{1,n} + [2]\beta\widehat{\cup}_{n-1,0} + [2]\alpha\widehat{\cup}_{0,n+1} + \beta\widehat{\cup}_{n,0} + \alpha\widehat{\cup}_{-1,n+1} + \sigma_R^{n-1} a_R^{PQ}\widehat{\cup}_{-1,n+2}$
$n+3 < i < 2n+2$	$\beta\widehat{\cup}_{2n+1-i,0} + \sigma_R^{i-2} a_R^{PQ}\widehat{\cup}_{i-n-2,2n+3-i} + [2]\beta\widehat{\cup}_{2n+2-i,0} + [2]\sigma_R^{i-3} a_R^{PQ}\widehat{\cup}_{i-n-3,2n+4-i} + \beta\widehat{\cup}_{2n+3-i,0} + \sigma_R^{i-4} a_R^{PQ}\widehat{\cup}_{i-n-4,2n+5-i}$
$2n+2$	$\beta\widehat{\cup}_{-1,0} + a_R^{PQ}\widehat{\cup}_{n,1} + [2]\beta\widehat{\cup}_{0,0} + [2]\sigma_R^{-1} a_R^{PQ}\widehat{\cup}_{n-1,2} + \beta\widehat{\cup}_{1,0} + \sigma_R^{-2} a_R^{PQ}\widehat{\cup}_{n-2,3} + \sigma_R\beta\widehat{\cup}_{-1,-1}$
$2n+3$	$\alpha\widehat{\cup}_{n-1,n+3} + [2]\beta\widehat{\cup}_{-1,0} + [2]a_R^{PQ}\widehat{\cup}_{n,1} + \beta\widehat{\cup}_{0,0} + \sigma_R^{-1} a_R^{PQ}\widehat{\cup}_{n-1,2}$

Remark 3.11. We check the formulas given in Propositions 3.9 and 3.10 by taking inner products directly in the graph planar algebra. See Section 4D for more details.

However, the best evidence that these formulas are correct is the fact that we can actually compute the two-strand jellyfish relations for the $3^{\mathbb{Z}/4\mathbb{Z}}$ subfactor planar algebra!

3D. *Inner products amongst trains and their projections.*

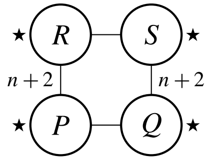
Proposition 3.12.

(i) $\left\langle P \underset{n-2}{\circ} Q, R \underset{n-2}{\circ} S \right\rangle = \dfrac{\operatorname{Tr}(PR)\operatorname{Tr}(SQ)}{[n-1]}$.

(ii) $\left\langle P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R, P' \underset{n-1}{\circ} Q' \underset{n-1}{\circ} R' \right\rangle = \dfrac{\operatorname{Tr}(PP')\operatorname{Tr}(QQ')\operatorname{Tr}(RR')}{[n]^2}$.

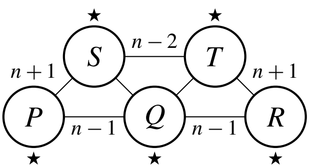
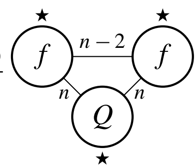
(iii) $\left\langle P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R, S \underset{n-2}{\circ} T \right\rangle = 0$.

Proof. For (i), the left-hand side equals

 .

The result now follows by Lemma 3.5 (ii).

We omit the proof of (ii), which is similar to the proof of (i). For (iii), again using Lemma 3.5 (ii), we see that the left-hand side is equal to

 $= \dfrac{\mathrm{Tr}(PS)\,\mathrm{Tr}(RT)}{[n]^2}$ 

where $f = f^{(n-1)}$. The right-hand side of the above equation is zero, since it is a linear combination of closed diagrams containing only one generator. □

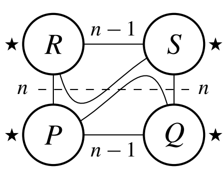
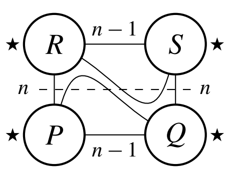
Proposition 3.13.

(i) $\left\langle C_i\left[P \underset{n-1}{\circ} Q\right], C_j\left[R \underset{n-1}{\circ} S\right]\right\rangle$
$$= \begin{cases} \mathrm{Tr}(PR)\,\mathrm{Tr}(SQ)[2][n]^{-1} & \textit{if } i = j, \\ \mathrm{Tr}(PR)\,\mathrm{Tr}(SQ)[n]^{-1} & \textit{if } |i-j| = 1, \\ \mathrm{Tr}(PRSQ) & \textit{if } (i, j) \in \{(n{+}1, n{+}3), (n{+}3, n{+}1)\}, \\ 0 & \textit{else}. \end{cases}$$

(ii) $\left\langle C_i\left[P \underset{n-1}{\circ} Q\right], R \underset{n-2}{\circ} S\right\rangle = \begin{cases} \mathrm{Tr}(PR)\,\mathrm{Tr}(SQ)[n]^{-1} & \textit{if } i = n+2, \\ 0 & \textit{else}. \end{cases}$

(iii) $\left\langle C_i\left[P \underset{n-1}{\circ} Q\right], R \underset{n-1}{\circ} S \underset{n-1}{\circ} T\right\rangle = \begin{cases} a_S^{PR}\,\mathrm{Tr}(QT)[n]^{-1} & \textit{if } i = n+1, \\ a_Q^{ST}\,\mathrm{Tr}(RP)[n]^{-1} & \textit{if } i = n+3, \\ 0 & \textit{else}. \end{cases}$

Proof. The proofs are all relatively straightforward drawing the necessary diagrams. The case in (i) which is easiest to miss is when $(i, j) \in \{(n{+}1, n{+}3), (n{+}3, n{+}1)\}$. In this case we get the following diagrams:

 $=$  $= \mathrm{Tr}(PRSQ).$ □

Proposition 3.14. (i) $\left\langle P \underset{n-2}{\circ} Q, P_{\mathcal{TL}_{n+2,+}}\left(R \underset{n-2}{\circ} S\right)\right\rangle = \dfrac{\mathrm{Tr}(PQ)\,\mathrm{Tr}(RS)}{[n{+}3]}.$

(ii) $\left\langle P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R,\, P_{\mathcal{TL}_{n+2,+}}\left(P' \underset{n-1}{\circ} Q' \underset{n-1}{\circ} R'\right)\right\rangle = a_R^{QP} a_{R'}^{P'Q'} \dfrac{[2][n+1]}{[n+2][n+3]}$.

(iii) $\left\langle P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R,\, P_{\mathcal{TL}_{n+2,+}}\left(S \underset{n-2}{\circ} T\right)\right\rangle = -\dfrac{\operatorname{Tr}(ST) a_R^{QP}[n+1]}{[n+2][n+3]}$.

Proof. This follows quickly from Proposition 3.6. For part (ii), using Proposition 3.6, the inner product in question is equal to

$$a_R^{QP} a_{R'}^{P'Q'}\Big(\frac{[n+1]^2}{[n+2]^2[n+3]} + \frac{[n+1]}{[n+2]^2}\Big) = a_R^{QP} a_{R'}^{P'Q'} \frac{[n+1]}{[n+2]^2}\Big(\frac{[n+1]+[n+3]}{[n+3]}\Big)$$
$$= a_R^{QP} a_{R'}^{P'Q'} \frac{[2][n+1]}{[n+2][n+3]}. \qquad \square$$

Proposition 3.15.

(i) $\left\langle C_i\left[P \underset{n-1}{\circ} Q\right],\, P_{\mathcal{TL}_{n+2,+}}\left(C_j\left[R \underset{n-1}{\circ} S\right]\right)\right\rangle$
$$= \begin{cases} \operatorname{Tr}(PQ)\operatorname{Tr}(RS)[2][n+2]^{-1} & \text{if } i = j, \\ \operatorname{Tr}(PQ)\operatorname{Tr}(RS)[n+2]^{-1} & \text{if } |i-j| = 1, \\ \operatorname{Tr}(PQ)\operatorname{Tr}(RS)[n+1]^{-1} & \text{if } (i,j) \in \{(n+1, n+3), (n+3, n+1)\}, \\ 0 & \text{else.} \end{cases}$$

(ii) $\left\langle C_i\left[P \underset{n-2}{\circ} Q\right],\, P_{\mathcal{TL}_{n+2,+}}\left(R \underset{n-2}{\circ} S\right)\right\rangle = \begin{cases} \operatorname{Tr}(PQ)\operatorname{Tr}(RS)[n+2]^{-1} & \text{if } i = n+2, \\ 0 & \text{else.} \end{cases}$

(iii) $\left\langle C_i\left[P \underset{n-2}{\circ} Q\right],\, P_{\mathcal{TL}_{n+2,+}}\left(R \underset{n-1}{\circ} S \underset{n-1}{\circ} T\right)\right\rangle$
$$= \begin{cases} \operatorname{Tr}(PQ) a_T^{RS}[n+2]^{-1} & \text{if } i = n+1, n+3, \\ 0 & \text{else.} \end{cases}$$

Proof. (i) The formulas can be obtained easily from Lemma 3.4 and Proposition 3.7. We work out a few interesting cases.

If $i = n+1$ and $j = n+3$, then

$$\left\langle C_i\left[P \underset{n-1}{\circ} Q\right],\, P_{\mathcal{TL}_{n+2,+}}\left(C_j\left[R \underset{n-1}{\circ} S\right]\right)\right\rangle$$
$$= \left\langle C_{n+1}\left[P \underset{n-1}{\circ} Q\right],\, \hat{1}_{n+2} + [2]D_n^* + D_{n-1}^*\right\rangle \operatorname{Tr}(RS)$$
$$= \left\langle C_{n+3}\left[P \underset{n-1}{\circ} Q\right],\, [2]D_n + D_{n-1}\right\rangle \operatorname{Tr}(RS)$$
$$= \Big(\frac{[2]}{[n+2]} - \frac{[n]}{[n+1][n+2]}\Big)\operatorname{Tr}(PQ)\operatorname{Tr}(RS)$$
$$= \frac{\operatorname{Tr}(PQ)\operatorname{Tr}(RS)}{[n+1]}.$$

If $i = n+1$ and $n+3 < j < 2n+3$, then

$$\begin{aligned}
\big\langle C_i\big[P \underset{n-1}{\circ} Q\big], P_{\mathcal{TL}_{n+2,+}}\big(C_j\big[R \underset{n-1}{\circ} S\big]\big)\big\rangle
&= \big\langle C_{n+1}\big[P \underset{n-1}{\circ} Q\big], D^*_{2n+2-j} + [2]D^*_{2n+3-j} + D^*_{2n+4-j}\big\rangle \operatorname{Tr}(RS)\\
&= \big\langle C_{n+3}\big[P \underset{n-1}{\circ} Q\big], D_j + [2]D_{j+1} + D_{j+2}\big\rangle \operatorname{Tr}(RS)\\
&= \frac{(-1)^{n-j}}{[n+1][n+2]}\big([j+1] - [2][j+2] + [j+3]\big) \operatorname{Tr}(PQ)\operatorname{Tr}(RS)\\
&= 0.
\end{aligned}$$

(ii) By Proposition 3.6, we have

$$\big\langle C_i\big[P \underset{n-1}{\circ} Q\big], P_{\mathcal{TL}_{n+2,+}}\big(R \underset{n-2}{\circ} S\big)\big\rangle = \frac{\operatorname{Tr}(RS)}{[n+3]}\big\langle C_i\big[P \underset{n-1}{\circ} Q\big], f^{(n+2)}\big\rangle,$$

which is zero unless $i = n+2$. Now by Lemma 3.3 and Proposition 3.7, the right-hand side is equal to

$$\begin{aligned}
\frac{\operatorname{Tr}(RS)\operatorname{Tr}(PQ)}{[n+3]}\langle D_n + [2]\hat{1}_{n+2}, f^{(n+2)}\rangle &= \frac{\operatorname{Tr}(PQ)\operatorname{Tr}(RS)}{[n+3]}\Big([2] - \frac{[n+1]}{[n+2]}\Big)\\
&= \frac{\operatorname{Tr}(PQ)\operatorname{Tr}(RS)}{[n+2]}.
\end{aligned}$$

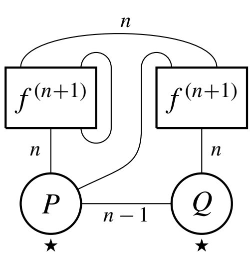
(iii) By Proposition 3.6, we have

$$\big\langle C_i\big[P \underset{n-1}{\circ} Q\big], P_{\mathcal{TL}_{n+2,+}}\big(R \underset{n-2}{\circ} S \underset{n-1}{\circ} T\big)\big\rangle = a_T^{RS}\big\langle C_i\big[P \underset{n-1}{\circ} Q\big], \widehat{E}_{n+1}\big\rangle,$$

which is clearly zero unless $n+1 \le i \le n+3$ (use the formula for $\widehat{E}_{n+1}$).

If $i = n+1$ (and similarly for $i = n+3$), then only the first diagram in Proposition 3.6 (ii) contributes to the inner product, and the value is given by

$$a_T^{RS}\frac{[n+1]}{[n+2]^2}$$



$$= \frac{\operatorname{Tr}(PQ)a_T^{RS}}{[n+2]}.$$

If $i = n+2$, by drawing similar diagrams, we see the inner product in question is equal to

$$a_T^{RS}\Big(\frac{[n+1]}{[n+2]^2} - \frac{[n+3][n+1]}{[n+2][n+2][n+3]}\Big)\operatorname{Tr}(PQ) = 0. \qquad \square$$

Remark 3.16. We now explain how to obtain the inner products

- $\langle P \underset{n-2}{\circ} Q, P_{\mathfrak{A}_{n+2}}(R \underset{n-2}{\circ} S)\rangle$,
- $\langle P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R, P_{\mathfrak{A}_{n+2}}(P' \underset{n-1}{\circ} Q' \underset{n-1}{\circ} R')\rangle$,
- $\langle P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R, P_{\mathfrak{A}_{n+2}}(S \underset{n-2}{\circ} T)\rangle$,
- $\langle C_i[P \underset{n-2}{\circ} Q], P_{\mathfrak{A}_{n+2}}(C_j[R \underset{n-2}{\circ} S])\rangle$,
- $\langle C_i[P \underset{n-2}{\circ} Q], P_{\mathfrak{A}_{n+2}}(R \underset{n-1}{\circ} S \underset{n-1}{\circ} T)\rangle$,
- $\langle C_i[P \underset{n-2}{\circ} Q], P_{\mathfrak{A}_{n+2}}(R \underset{n-2}{\circ} S)\rangle$.

First, we use the formulas for

$$P_{\mathfrak{A}_{n+2}}(P \underset{n-2}{\circ} Q), \quad P_{\mathfrak{A}_{n+2}}(P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R), \quad \text{and} \quad P_{\mathfrak{A}_{n+2}}(C_i[P \underset{n-2}{\circ} Q])$$

obtained in Propositions 3.9 and 3.10 to express each side as a linear combination of the $\widehat{\cup}_{i,j}(S)$'s. Next, we use the change of basis matrix discussed in Remark 2.30 to write the $\widehat{\cup}_{i,j}(S)$ on the right-hand side in terms of the $\cup_{i,j}(S)$. Finally, we expand the inner product in the usual way to obtain the answer.

4. Deriving formulas for two-strand box jellyfish relations

As in the previous sections, we continue Assumptions 2.9, 2.12, and 2.21.

We now go through our algorithm for determining two-strand jellyfish relations. We follow the method of [Morrison and Penneys 2015b, Section 3], which consists of three parts:

(i) Find the quadratic tangles in annular consequences.

(ii) Find the jellyfish matrix.

(iii) Invert the jellyfish matrix.

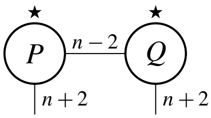
The steps in our algorithm will be clearly marked in the following three subsections.

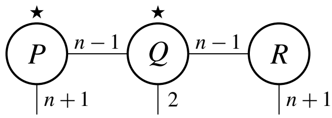
4A. ***Reduced trains in annular consequences.*** In [Morrison and Penneys 2015b], the first step was to obtain a basis for the quadratic tangles in annular consequences. Since we have quadratic and cubic trains, we call this step obtaining a basis for the reduced trains in annular consequences.

Definition 4.1. Recall from Definition 2.22 that a "reduced train" is one where no generator connects to itself, and no pair are connected by more than $n-1$ strands. Starting with our set of minimal generators $\mathfrak{B}$ satisfying Assumptions 2.9, 2.12, and 2.21, we have the reduced trains

$$\left\{C_i[P \underset{n-1}{\circ} Q] \,\middle|\, P, Q \in \mathfrak{B} \text{ and } i = 1, \ldots, 2n+3\right\} \subset \mathcal{P}_{n+2,+}$$

which are annular consequences of trains in $\mathcal{P}_{n+1,+}$, and we have the reduced trains

$\Bigg\{$ 

, 

$\Bigg|\; P, Q, R \in \mathfrak{B} \Bigg\} \subset \mathcal{P}_{n+2,+}$

which are nonzero when placing a Jones–Wenzl underneath. We let $\mathcal{RT}$ be the union of the above two sets.

Since we hope that our generators generate a subfactor planar algebra with the desired principal graph, we want some linear combination of these reduced trains to lie in annular consequences.

Definition 4.2. We set

$$\mathcal{RTAC} = (\mathcal{TL}_{n+2,+} \oplus \mathfrak{A}_{n+2}) \cap \operatorname{span}(\mathcal{RT}),$$

where $\mathcal{RTAC}$ stands for *reduced trains in annular consequences*.

Step 1 of our algorithm finds a basis for $\mathcal{RTAC}$. Since we are trying to derive box jellyfish relations, we are only interested in basis elements which are not sent to zero when we put a $f^{(2n+4)}$ underneath. Thus we make the following definition.

Definition 4.3. An element of $\mathcal{RTAC}$ is called *essential* if at least one of the coefficients of the $P \circ_{n-2} Q$'s or the $P \circ_{n-1} Q \circ_{n-1} R$'s does not vanish.

Remark 4.4. If we've chosen k generators in a graph planar algebra and are hoping that they give us a subfactor planar algebra with one spoke principal graph, we expect to have at least k essential basis elements of $\mathcal{RTAC}$, i.e., one two-strand jellyfish relation for each generator.

Step 1 (a basis for $\mathcal{RTAC}$). Consider the matrix

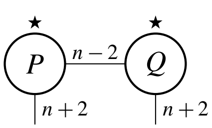
$$\big(\langle \mathcal{X} - P_{\mathcal{TL}_{n+2,+}}(\mathcal{X}) - P_{\mathfrak{A}_{n+2}}(\mathcal{X}), \mathcal{Y}\rangle\big)_{\mathcal{X},\mathcal{Y}\in\mathcal{RT}},$$

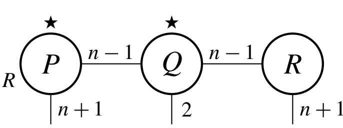
of inner products modulo Temperley–Lieb and annular consequences. (Note that the necessary inner products were derived in Propositions 3.12 and 3.14 and Remark 3.16.)

(i) Taking a basis for the null space of this matrix gives us a basis for $\mathcal{RTAC}$.

(ii) From this basis, we keep only the essential elements, which we call $X_1, \ldots, X_k$.

4B. ***Compute the jellyfish matrix.*** From Step 1, we have an expression for each essential basis element of $\mathcal{RTAC}$. Namely, the basis elements X_i can be written in the form

$\displaystyle\sum_{P,Q\in\mathfrak{B}} \alpha^i_{P,Q}$ 

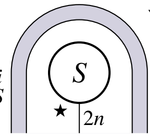
$+ \displaystyle\sum_{P,Q,R\in\mathfrak{B}} \beta^i_{P,Q,R}$ 

$+\, W_i,$

where $W_i \in \operatorname{span}\{C_i[P \circ Q] \mid P, Q \in \mathfrak{B}$ and $i = 1, \dots, 2n+3\}$. We also have an expression for X_i as an element of $\mathcal{TL}_{n+2,+} \oplus \mathfrak{A}_{n+2}$.

Step 2 (expression in the annular basis). Using Proposition 3.9, we express the X_i in terms of the dual annular basis $\widehat{\cup}_{r,s}(S)$ for $S \in \mathfrak{B}$. We then use the change of basis matrix discussed in Remark 2.30 to write the $\widehat{\cup}_{r,s}(S)$ in terms of the $\cup_{j,\ell}(S)$. Hence we may write each X_i as

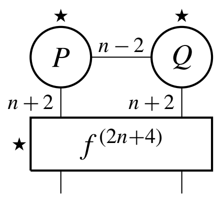
$$X_i = \left(\sum_{S\in\mathfrak{B}} \gamma_S^i \cup_{-1,-1}(S)\right) + Y_i + Z_i = \left(\sum_{S\in\mathfrak{B}} \gamma_S^i \; \text{[diagram]} \right) + Y_i + Z_i,$$



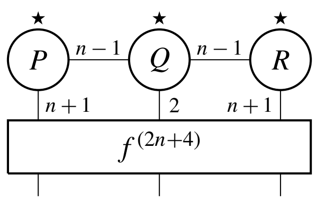
where Y_i is a linear combination of the $\cup_{j,\ell}(S)$ for $S \in \mathfrak{B}$ and $(j, \ell) \neq (-1, -1)$, and $Z_i \in \mathcal{TL}_{n+2,+}$.

Notation 4.5. For $P, Q, R, S \in \mathfrak{B}$, we use the notation

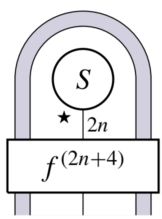
$$f\big(P \underset{n-2}{\circ} Q\big) =$$



$$f\big(P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R\big) =$$



$$f \cdot j^2(S) =$$



We also write $f \cdot X$ to denote $X \in \mathcal{P}_{n+2,+}$ in jellyfish form with a $f^{(2n+4)}$ underneath.

Step 3 (box jellyfish equations). Put an $f^{(2n+4)}$ underneath the two formulas for X_i obtained in Steps 1 and 2 to get the following equations for $i = 1, \dots, k$:

$$f \cdot X_i = \sum_{P,Q\in\mathfrak{B}} \alpha_{P,Q}^i f\big(P \underset{n-2}{\circ} Q\big) + \sum_{P,Q,R\in\mathfrak{B}} \beta_{P,Q,R}^i f\big(P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R\big) = \sum_{S\in\mathfrak{B}} \gamma_S^i f \cdot j^2(S).$$

Remark 4.6. In [Morrison and Penneys 2015b, Section 3.2], similar formulas to those obtained in Step 3 were checked by wrapping a Jones–Wenzl around the top of $P \circ_{n-1} Q$. In our case, we cannot use this check, since wrapping a Jones–Wenzl around the top of a 3-train does not give another box-train.

We now define the jellyfish matrix and the reduced trains matrix from the equations from Step 3.

Definition 4.7. The *two-strand jellyfish matrix* is the matrix J_2 whose i-th row is $(\gamma_S^i)_{S\in\mathfrak{B}}$. The *reduced trains matrix* is the matrix K_2 whose i-th row is given by concatenating the lists $(\alpha^i_{P,Q})_{P,Q\in\mathfrak{B}}$ and $(\beta^i_{R,S,T})_{R,S,T\in\mathfrak{B}}$.

Remark 4.8. Note that

$$K_2 \begin{pmatrix} f\big(P \underset{n-2}{\circ} Q\big) \\ \vdots \\ f\big(R \underset{n-1}{\circ} S \underset{n-1}{\circ} T\big) \\ \vdots \end{pmatrix}_{P,Q,R,S,T\in\mathfrak{B}} = J_2 \begin{pmatrix} f\cdot j^2(S) \\ \vdots \end{pmatrix}_{S\in\mathfrak{B}}.$$

4C. ***Invert the jellyfish matrix.*** At this point, we have accomplished most of the difficult work. Two easy steps remain.

Step 4 (invert J_2). Given the matrix J_2 from Definition 4.7 obtained via Step 3, we check if it has rank $|\mathfrak{B}|$. If it does (and we know that it should by [Bigelow and Penneys 2014]), we find a left inverse for J_2 by the formula

$$J_2^L = (J_2^* J_2)^{-1} J_2^*$$

since J_2 and $J_2^* J_2$ have the same rank.

Step 5 (box jellyfish relations). Finally, we get the *box jellyfish relations* by multiplying by J_2^L from Step 4:

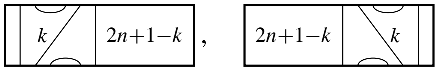
$$\begin{pmatrix} f\cdot j^2(S) \\ \vdots \end{pmatrix}_{S\in\mathfrak{B}} = J_2^L K_2 \begin{pmatrix} f\big(P \underset{n-2}{\circ} Q\big) \\ \vdots \\ f\big(P \underset{n-1}{\circ} Q \underset{n-1}{\circ} R\big) \\ \vdots \end{pmatrix}_{P,Q,R\in\mathfrak{B}}$$

which express the $f\cdot j^2(S)$ as linear combinations of reduced trains.

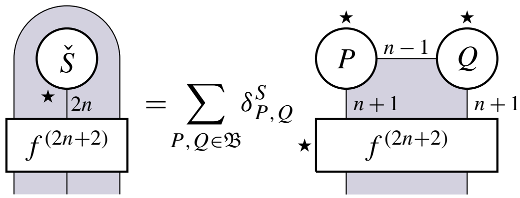
Remark 4.9. Recall that our goal was to derive two-strand jellyfish relations for our generators. These relations would be sufficient to evaluate all closed diagrams. Note that two-strand box jellyfish relations by themselves are *not* sufficient to evaluate closed diagrams!

In order to recover jellyfish relations from box jellyfish relations, we need to expand the Jones–Wenzl idempotents as in [Morrison and Penneys 2015b, Section 2.5].

When expanding $f^{(2n+4)}$ for the two-strand box jellyfish relations, terms of the form



in $f^{(2n+4)}$ yield diagrams not in jellyfish form, as they have a strand separating the generator from the outer region. Hence we also need one-strand jellyfish relations, which are obtained from one-strand box jellyfish relations of the form

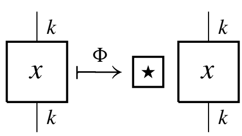


by the argument in [Morrison and Penneys 2015b, Section 2.5]. We compute the necessary one-strand box jellyfish relations using the algorithm provided there.

4D. *Checking our calculations.* Since the computer is doing all the arithmetic, it is good to check that our formulas are consistent with other methods of calculation. The computations in this section are redundant, hence we freely take shortcuts and perform spot checks when more thorough checks would be too time consuming.

The checks we perform in this subsection are done directly in the graph planar algebra. As such computations are computationally expensive, we use the following shortcut, which is known to experts. We do not prove it here as it would take us too far afield.

Proposition 4.10. *Suppose $\mathcal{P}_\bullet$ is a subfactor planar algebra. Choose an embedding of $\mathcal{P}_\bullet$ into $\mathcal{GPA}(\Gamma_+)_\bullet$, the graph planar algebra of its principal graph, and identify $\mathcal{P}_\bullet$ with its image. Define the map $\Phi : \mathcal{P}_{k,\pm} \to \mathcal{GPA}(\Gamma_+)_{k,\pm}$ by cutting down at the zero box $\star$ (the distinguished vertex of Γ_+), i.e., forgetting all loops of length $2k$ which do not start at $\star$.*



Then Φ is a $*$-algebra isomorphism under the usual multiplication, and Φ commutes with taking (partial) traces.

We remark that $\dim(\mathcal{P}_{k,\pm})$ is equal to the number of loops of length $2k$ starting at $\star$ on the principal graph, so one only needs to prove this map is injective.

To simplify calculations in the graph planar algebra, we can compute the inner product by first cutting down at $\star$ and then taking the inner product of the cut down elements in the graph planar algebra. Note that this simplification assumes we are working in the image of a subfactor planar algebra, so it cannot be used to prove that formulas hold. However, it can be used as a check for our calculations.

Using this shortcut, we check the propositions listed in the following table. The calculations are performed in the notebook `TwoStrandJellyfish.nb` in subsections called "Checking directly in the GPA" for each of our examples. Many of the computations are exact, but two are numerical. For the checks for Propositions 3.9 and 3.10, we don't check all the coefficients in the graph planar algebra; rather we only check the coefficients that our formulas tell us are nonzero.

Proposition	Checking functions	Numerical?
2.28	`CheckPairwiseInnerProductsOfSecondAC`	Yes
3.9	`CheckCoefficientsOf2TrainsInSecondAC` `CheckCoefficientsOf3TrainsInSecondAC`	No
3.10	`CheckCoefficientsOfCiQTCircsInSecondAC`	No
3.12	`CheckInnerProductBetweenTrains`	No
3.13	`CheckInnerProductWithCiQTCircs`	Yes

As a verification of the correctness of our algorithm, we also reproved the existence of the Haagerup $3^{\mathbb{Z}/3\mathbb{Z}}$ subfactor and the $3^{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ subfactor. We did not include these calculations since there are already several proofs for existence of these subfactors. We note that the formula we obtain for the two-strand jellyfish relation for $3^{\mathbb{Z}/3}$ (Haagerup) agrees with that obtained in [BMPS 2012]. We have not checked that our two-strand relations for $3^{\mathbb{Z}/2\times\mathbb{Z}/2}$ are consistent with the one-strand relations found in [Morrison and Penneys 2015b], since we use different generators.

In [Morrison and Penneys 2015b], the authors were able to check the one-strand jellyfish relations for 2221 directly in the graph planar algebra using a clever trick due to Bigelow. We cannot do these computations for our graphs. Not only are our graphs 3-supertransitive, but we also use two-strand relations, making the preparation of the two-cup Jones–Wenzl too computationally expensive.

5. Relations for $3^{\mathbb{Z}/4}$

We now record the two- and one-strand jellyfish relations for a planar algebra which we will show, in Section 6, is the $3^{\mathbb{Z}/4}$ planar algebra. The three lemmas below consist of performing the calculations described in Section 4. The proofs are simply substituting in the appropriate quantities (moments, tetrahedral structure constants) where applicable, and executing the functions in the Mathematica notebooks included with the arXiv sources of this article.

The set $\mathfrak{B} = \{A, B\}$ is an orthogonal set of minimal generators which lives in the graph planar algebra. Formulas for these generators are given in Appendix A. We first check that Assumptions 2.9, 2.12, and 2.21 hold for these generators, i.e.:

- The elements $R \in \mathfrak{B}$ are self-adjoint low-weight rotational eigenvectors with corresponding chiralities σ_R given in Appendix A. Moreover, $\mathfrak{B}$ is linearly independent and orthogonal and has scalar moments. The moments are given in Appendix B.
- The sets $\mathfrak{B} \cup \{f^{(n)}\}$ and $\check{\mathfrak{B}} \cup \{f^{(n)}\}$ span complex algebras under the usual multiplication. The program *VerifyClosedUnderMultiplication* in the notebook `TwoStrandJellyfish.nb` is used to check this.
- The tetrahedral structure constants $\Delta(P, Q, R \mid S)$ are scalars for all elements $P, Q, R, S \in \mathfrak{B}$. The tetrahedral constants are given in Appendix B.

Throughout, the notation $\lambda^{(z)}_{a_n,\dots,a_0}$ denotes the root of the polynomial $\sum_i a_i x^i$ which is closest to the approximate real number z. (The digits of precision of z are in each case chosen so that this unambiguously identifies the root.) For example, $\lambda^{(0.3278)}_{1024,0,-864,0,81}$ denotes the root of $1024x^4 - 864x^2 + 81$ which is closest to 0.3278.

5A. *Two-strand relations.*

Lemma 5.1. *The following linear combinations X_i of reduced trains lie in annular consequences. The column marked X_i gives the coefficients of the reduced trains for X_i.*

	X_1	X_2
$A \underset{n-2}{\circ} A$	1	0
$A \underset{n-2}{\circ} B$	0	1
$B \underset{n-2}{\circ} A$	$\lambda^{(0.1449i)}_{2025,0,-720,0,-16}$	1
$B \underset{n-2}{\circ} B$	$\frac{1}{45}\left(-10-3\sqrt{5}\right)$	0
$A \underset{n-1}{\circ} A \underset{n-1}{\circ} A$	$\lambda^{(0.1761i)}_{100,0,-2610,0,-81}$	$\lambda^{(2.43)}_{4,0,-1134,0,6561}$
$A \underset{n-1}{\circ} A \underset{n-1}{\circ} B$	$\lambda^{(0.3447)}_{100,0,-1030,0,121}$	$\lambda^{(0.637i)}_{4,0,-198,0,-81}$
$A \underset{n-1}{\circ} B \underset{n-1}{\circ} A$	$\lambda^{(0.1493)}_{25,0,-180,0,4}$	0
$A \underset{n-1}{\circ} B \underset{n-1}{\circ} B$	$\lambda^{(-0.3436i)}_{4,0,-8,0,-1}$	$\lambda^{(0.3733)}_{4,0,-180,0,25}$
$B \underset{n-1}{\circ} A \underset{n-1}{\circ} A$	$\lambda^{(0.4370)}_{4,0,-6,0,1}$	$\lambda^{(-0.637i)}_{4,0,-198,0,-81}$
$B \underset{n-1}{\circ} A \underset{n-1}{\circ} B$	$\lambda^{(0.05869i)}_{100,0,-290,0,-1}$	$\lambda^{(0.810)}_{4,0,-126,0,81}$
$B \underset{n-1}{\circ} B \underset{n-1}{\circ} A$	$\lambda^{(0.3976i)}_{324,0,-2232,0,-361}$	$\lambda^{(0.3733)}_{4,0,-180,0,25}$
$B \underset{n-1}{\circ} B \underset{n-1}{\circ} B$	$\lambda^{(0.4382)}_{164025,0,-34020,0,484}$	0

$C_1[B \underset{n-1}{\circ} A]$	$\lambda^{(-0.017098i)}_{2025,0,-3420,0,-1}$	$\sqrt{5}-2$
$C_1[B \underset{n-1}{\circ} B]$	$\frac{1}{45}(5-3\sqrt{5})$	$\lambda^{(-0.1237i)}_{81,0,-1044,0,-16}$
$C_2[B \underset{n-1}{\circ} A]$	$\lambda^{(0.039125i)}_{2025,0,-2610,0,-4}$	$\lambda^{(-0.540)}_{1,0,-14,0,4}$
$C_2[B \underset{n-1}{\circ} B]$	$\lambda^{(0.08686)}_{164025,0,-9720,0,64}$	$\lambda^{(0.2831i)}_{81,0,-792,0,-64}$
$C_3[B \underset{n-1}{\circ} A]$	$\lambda^{(-0.07243i)}_{2025,0,-180,0,-1}$	1
$C_3[B \underset{n-1}{\circ} B]$	$\frac{1}{45}(-5-\sqrt{5})$	$\lambda^{(-0.5241i)}_{81,0,-36,0,-16}$
$C_4[B \underset{n-1}{\circ} A]$	$\lambda^{(0.1266i)}_{2025,0,-3960,0,-64}$	$\lambda^{(-1.75)}_{1,0,-24,0,64}$
$C_4[B \underset{n-1}{\circ} B]$	$\frac{4\sqrt{2/5}}{9}$	$\lambda^{(0.916i)}_{81,0,-1152,0,-1024}$
$C_5[B \underset{n-1}{\circ} A]$	$\lambda^{(-0.2173i)}_{25,0,-20,0,-1}$	3
$C_5[B \underset{n-1}{\circ} B]$	$\frac{1}{15}(-5-\sqrt{5})$	$\lambda^{(-1.57i)}_{1,0,-4,0,-16}$
$C_6[B \underset{n-1}{\circ} A]$	$\lambda^{(-0.2532i)}_{2025,0,-15840,0,-1024}$	$\lambda^{(-3.50)}_{1,0,-96,0,1024}$
$C_6[B \underset{n-1}{\circ} B]$	$\lambda^{(0.3348)}_{164025,0,-87480,0,7744}$	$\lambda^{(1.66i)}_{81,0,-360,0,-1600}$
$C_7[B \underset{n-1}{\circ} A]$	$\lambda^{(0.652i)}_{25,0,-180,0,-81}$	3
$C_7[B \underset{n-1}{\circ} B]$	$\frac{1}{30}(\sqrt{5}-5)$	$\lambda^{(-1.27i)}_{1,0,1,0,-1}$
$C_8[B \underset{n-1}{\circ} A]$	$\lambda^{(-0.3798i)}_{25,0,-440,0,-64}$	$\lambda^{(-1.75)}_{1,0,-24,0,64}$
$C_8[B \underset{n-1}{\circ} B]$	$\lambda^{(0.05368)}_{164025,0,-22680,0,64}$	$\lambda^{(0.741i)}_{81,0,-72,0,-64}$
$C_9[B \underset{n-1}{\circ} A]$	$\lambda^{(0.2173i)}_{25,0,-20,0,-1}$	1
$C_9[B \underset{n-1}{\circ} B]$	$\frac{1}{90}(\sqrt{5}-5)$	$\lambda^{(-0.4240i)}_{81,0,9,0,-1}$
$C_{10}[B \underset{n-1}{\circ} A]$	$\lambda^{(-0.1174i)}_{25,0,-290,0,-4}$	$\lambda^{(-0.540)}_{1,0,-14,0,4}$
$C_{10}[B \underset{n-1}{\circ} B]$	$\lambda^{(0.016589)}_{164025,0,-14580,0,4}$	$\lambda^{(0.2290i)}_{81,0,-72,0,-4}$
$C_{11}[B \underset{n-1}{\circ} A]$	$\lambda^{(0.05129i)}_{25,0,-380,0,-1}$	$\sqrt{5}-2$
$C_{11}[B \underset{n-1}{\circ} B]$	$\frac{1}{90}(15-7\sqrt{5})$	$\lambda^{(-0.1001i)}_{81,0,-99,0,-1}$

In the next two lemmas, we use

$$J_2 = \begin{pmatrix} \lambda^{(0.1245i)}_{400,0,-5220,0,-81} & \frac{1}{10}(-5-\sqrt{5}) \\ \frac{1}{4}(27-9\sqrt{5}) & 0 \end{pmatrix}.$$

We let K_2 be the transpose of the 12×2 matrix whose entries are given by the first

12 rows and the 2 columns of the table in Lemma 5.1, and we define

$$Y = \begin{pmatrix} f\big(A \underset{n-2}{\circ} A\big) \\ f\big(A \underset{n-2}{\circ} B\big) \\ f\big(B \underset{n-2}{\circ} A\big) \\ f\big(B \underset{n-2}{\circ} B\big) \\ f\big(A \underset{n-1}{\circ} A \underset{n-1}{\circ} A\big) \\ f\big(A \underset{n-1}{\circ} A \underset{n-1}{\circ} B\big) \\ f\big(A \underset{n-1}{\circ} B \underset{n-1}{\circ} A\big) \\ f\big(A \underset{n-1}{\circ} B \underset{n-1}{\circ} B\big) \\ f\big(B \underset{n-1}{\circ} A \underset{n-1}{\circ} A\big) \\ f\big(B \underset{n-1}{\circ} A \underset{n-1}{\circ} B\big) \\ f\big(B \underset{n-1}{\circ} B \underset{n-1}{\circ} A\big) \\ f\big(B \underset{n-1}{\circ} B \underset{n-1}{\circ} B\big) \end{pmatrix}.$$

Lemma 5.2. *We have* $K_2 Y = J_2 \begin{pmatrix} f \cdot j^2(A) \\ f \cdot j^2(B) \end{pmatrix}$.

Lemma 5.3. *The elements A, B satisfy the two-strand box jellyfish relations*

$$\begin{pmatrix} f \cdot j^2(A) \\ f \cdot j^2(B) \end{pmatrix} = J_2^L K_2 Y$$

where

$$(J_2^L K_2)^T = \begin{pmatrix} 0 & \frac{1}{2}\big(\sqrt{5}-5\big) \\ \frac{1}{9}\big(3+\sqrt{5}\big) & \lambda^{(0.1001i)}_{81,0,-99,0,-1} \\ \frac{1}{9}\big(3+\sqrt{5}\big) & \lambda^{(-0.1001i)}_{81,0,-99,0,-1} \\ 0 & \frac{1}{18}\big(7+\sqrt{5}\big) \\ \sqrt{2} & 0 \\ \lambda^{(0.3706i)}_{81,0,-18,0,-4} & \lambda^{(-0.540)}_{1,0,-14,0,4} \\ 0 & \lambda^{(-0.2063)}_{1,0,-94,0,4} \\ \lambda^{(0.2172)}_{6561,0,-2430,0,100} & \lambda^{(0.512i)}_{81,0,-360,0,-100} \\ \lambda^{(-0.3706i)}_{81,0,-18,0,-4} & \lambda^{(-0.540)}_{1,0,-14,0,4} \\ \frac{\sqrt{2}}{3} & 0 \\ \lambda^{(0.2172)}_{6561,0,-2430,0,100} & \lambda^{(-0.512i)}_{81,0,-360,0,-100} \\ 0 & \lambda^{(-0.6056)}_{6561,0,-3726,0,484} \end{pmatrix}.$$

5B. ***One-strand relations.***

Lemma 5.4. *The linear combinations*

$$K_1\begin{pmatrix}A\circ A\\A\circ B\\B\circ A\\B\circ B\end{pmatrix}\quad\text{and}\quad \check{K}_1\begin{pmatrix}\check{A}\circ\check{A}\\\check{A}\circ\check{B}\\\check{B}\circ\check{A}\\\check{B}\circ\check{B}\end{pmatrix}$$

lie in annular consequences, where

$$K_1=\begin{pmatrix}1&0&0&\frac{1}{18}\left(-1-\sqrt{5}\right)\\0&1&-1&\lambda^{(0.948i)}_{81,0,45,0,-25}\end{pmatrix}\quad\text{and}\quad \check{K}_1=\begin{pmatrix}1&0&0&\frac{1}{18}\left(-1-\sqrt{5}\right)\\0&1&1&\lambda^{(0.948)}_{81,0,-45,0,-25}\end{pmatrix}.$$

Lemma 5.5. *In particular, we have*

$$K_1\begin{pmatrix}f(A\circ A)\\f(A\circ B)\\f(B\circ A)\\f(B\circ B)\end{pmatrix}=J_1\begin{pmatrix}f\cdot j(\check{A})\\f\cdot j(\check{B})\end{pmatrix}\quad\text{and}\quad \check{K}_1\begin{pmatrix}f(\check{A}\circ\check{A})\\f(\check{A}\circ\check{B})\\f(\check{B}\circ\check{A})\\f(\check{B}\circ\check{B})\end{pmatrix}=\check{J}_1\begin{pmatrix}f\cdot j(A)\\f\cdot j(B)\end{pmatrix},$$

where

$$J_1=\begin{pmatrix}\lambda^{(-0.590)}_{256,0,144,0,-81}&\frac{1}{24}\left(5+\sqrt{5}\right)\\\lambda^{(2.71i)}_{256,0,2160,0,2025}&\lambda^{(0.8325i)}_{256,0,176,0,-1}\end{pmatrix}\quad\text{and}\quad \check{J}_1=\begin{pmatrix}0&\lambda^{(-0.49923)}_{5184,0,-1296,0,1}\\0&\lambda^{(0.7277i)}_{64,0,32,0,-1}\end{pmatrix}.$$

Lemma 5.6. *The elements A and B satisfy the one-strand box jellyfish relations*

$$\begin{pmatrix}f\cdot j(\check{A})\\f\cdot j(\check{B})\end{pmatrix}=J_1^LK_1\begin{pmatrix}f(A\circ A)\\f(A\circ B)\\f(B\circ A)\\f(B\circ B)\end{pmatrix},$$

where

$$J_1^LK_1=\begin{pmatrix}\lambda^{(-0.6360)}_{16,0,-4,0,-1}&\lambda^{(-0.2303i)}_{1296,0,540,0,25}&\lambda^{(0.2303i)}_{1296,0,540,0,25}&\lambda^{(0.3327)}_{104976,0,3564,0,-1681}\\\frac{1}{4}\left(15-3\sqrt{5}\right)&\lambda^{(-0.4504i)}_{16,0,-396,0,-81}&\lambda^{(0.4504i)}_{16,0,-396,0,-81}&\frac{1}{12}\left(7\sqrt{5}-15\right)\end{pmatrix}.$$

6. Calculating principal graphs

We now know that the set of minimal generators given in Appendix A generates an evaluable subfactor planar algebra $\mathcal{P}_\bullet^{\mathbb{Z}/4}$. We must now determine the principal graphs of the $\mathcal{P}_\bullet^{\mathbb{Z}/4}$. By the next lemma, we know that the principal graphs have the desired supertransitivity since we have two-strand jellyfish relations.

Lemma 6.1. *Suppose a planar algebra $\mathcal{P}_\bullet$ is generated by uncappable elements $A_1,\dots,A_k\in\mathcal{P}_{n,+}$ such that*

(i) *the A_j's satisfy two-strand jellyfish relations*, *and*

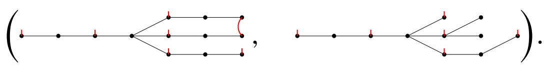
(ii) *the complex span of* $\{A_1, \ldots, A_k, f^{(n)}\}$ *forms an algebra under the usual multiplication.*

Then $\mathcal{P}_\bullet$ *is* $(n-1)$ *supertransitive.*

Proof. Similar to [Morrison and Penneys 2015b, Lemma 5.1]. □

We now determine the principal graphs of the $\mathcal{P}_\bullet^{\mathbb{Z}/4}$. These arguments are similar to those in [Morrison and Penneys 2015b, Section 5].

Theorem 6.2. *The principal graphs of* $\mathcal{P}_\bullet^{\mathbb{Z}/4}$ *are*



Proof. The modulus is $\sqrt{3+\sqrt{5}} \simeq 2.28825$, and we find that the minimal projections one past the branch from bottom to top are given by $aA + bB + cf^{(4)}$, where

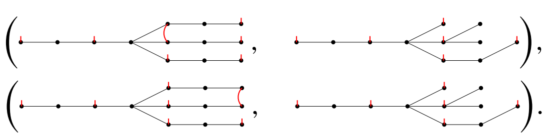
$$(a, b, c) = \begin{cases} \left(0, \frac{1}{3}, \frac{1}{3}\right), \\ \left(\frac{1}{2}, -\frac{1}{6}, \frac{1}{3}\right), \\ \left(-\frac{1}{2}, -\frac{1}{6}, \frac{1}{3}\right). \end{cases}$$

Since $\mathrm{Tr}(f^{(4)}) = 6 + 3\sqrt{5}$, all the minimal projections have trace $2+\sqrt{5}$, and the proof of [Morrison and Penneys 2015b, Theorem 5.9] shows the principal graph is correct.

To see that the dual graph is correct, we first find that the minimal projections one past the branch from bottom to top are given by $a\check{A} + b\check{B} + cf^{(4)}$, where

$$(a, b, c) = \begin{cases} \left(\lambda^{(-0.556)}_{4,0,2,0,-1}, \lambda^{(0.09003)}_{324,0,-126,0,1}, \frac{1}{3}\right), \\ \left(\lambda^{(0.2123)}_{4,0,22,0,-1}, \lambda^{(-0.3257)}_{324,0,-270,0,25}, \frac{1}{3}\left(\sqrt{5}-1\right)\right), \\ \left(\lambda^{(0.3436)}_{4,0,8,0,-1}, \frac{1}{3\sqrt{2}}, \frac{1}{3}\left(3-\sqrt{5}\right)\right) \end{cases}$$

which have traces $2+\sqrt{5}$, $3+\sqrt{5}$, $1+\sqrt{5}$ respectively. Hence there is a univalent vertex at depth 4 on the dual graph. We now run the *FusionAtlas* program *FindGraphPartners* on the 3333 graph and we see there are only two possibilities where the dual graph has a univalent vertex at depth 4:



Now the projections at depth 4 on the principal graph are self-dual since $\rho^2 = \mathrm{id}$ on $\mathrm{span}\{A, B, f^{(4)}\}$, so the only possibility is the one claimed. □

Appendix A. Generators

Suppose Γ is a simply laced graph with a distinguished subgraph $\Lambda \subset \Gamma$ such that Γ is obtained from Λ by adding A_{finite} tails to Λ. For example, when Γ is a spoke graph, we can choose Λ to be the central vertex. When $\Gamma = 2D2$ (see Section AA), we can choose Λ to be the central diamond.

By the proof of [Morrison and Penneys 2015b, Lemma A.1], a low-weight generator A is completely determined by its values on loops which stay within distance 1 of Λ. Furthermore, if Γ is obtained from Λ by adding A_{finite} tails to *distinct* vertices of Λ, then A is completely determined by its values on loops which stay inside Λ. So when Γ is a spoke graph with n spokes, we can choose Λ to be an $(n-1)$-star.

Moreover, as A is a rotational eigenvector, A is completely determined by its values on a set of rotation orbit representatives which stay in Λ.

We now describe an algorithm to recover our low-weight generator A from its values on such loops.

Remark A.1. It should seem plausible, but not at all obvious, that the recovered generator is in fact a low-weight rotational eigenvector. Proposition A.11 gives a well-defined element of the graph planar algebra. For our examples, the programs *CheckLowestWeightCondition* and *CheckRotationalEigenvector* in the notebook `Generators.nb` check that the low-weight and rotational eigenvector conditions hold respectively.

Definition A.2. For a vertex $v \in \Gamma$, we define $d(v, \Lambda)$ to be the minimal distance of v to Λ. For a loop γ whose i-th vertex is denoted $\gamma(i)$, we define $d(\gamma, \Lambda) = \max_i d(\gamma(i), \Lambda)$.

2-valent folding relation. Suppose A is an n-box. We start with a loop γ on Γ of length $2n$. If $d(\gamma, \Lambda) > 1$, we can use the 2-valent relation first considered in [Peters 2010; BMPS 2012] to fold γ inward by analyzing the capping action on 2-valent vertices as follows. We use the notation of [Morrison and Penneys 2015b].

Notation A.3. Suppose $s = \gamma(i)$ is a vertex on γ whose distance from Λ is at least 2. Let t be the vertex on the same tail 2 closer to Λ than s (possibly t is in Λ itself). Let γ' be the loop modified from γ by replacing s at position i with t. Let π be the "snipped" loop of length $2n-2$ obtained from γ or γ' by removing the i-th and $i+1$-st positions. For convenience, we let $r = \gamma(i \pm 1) = \gamma'(i \pm 1)$. For an example, see Figure 1.

Definition A.4. Applying a cap at position i to A, we have $\cap_i(A) = 0$. Evaluating this at π gives the *2-valent folding relation*

$$0 = \sqrt{\dim(r)}^{k_i} \cap_i(A)(\pi) = \sqrt{\dim(s)}^{k_i} A(\gamma) + \sqrt{\dim(t)}^{k_i} A(\gamma').$$

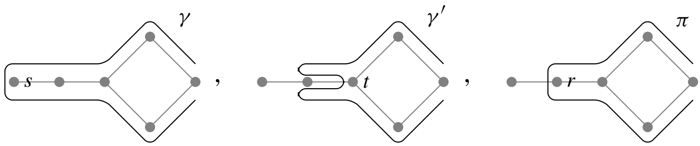
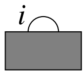
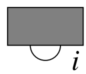
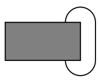
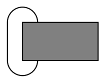


Figure 1. Example of loops and vertices appearing in the 2-valent folding relation.

Here k_i is the number of critical points in the cap strand, either 1 or 2 depending on the position of the point i around the boundary of the rectangular box:

$k_i =$

1 when we have  or ,

2 when we have  i or i .

Lemma A.5. *If $\hat{\gamma}$ is the loop of length $2n$ with $d(\hat{\gamma}, \Lambda) = 1$ obtained from γ by the 2-valent folding relation described above*, *we have*

$$A(\gamma) = (-1)^{(\|\gamma\| - \|\hat{\gamma}\|)/2} \left(\prod_i \sqrt{\frac{\dim(\hat{\gamma}(i))}{\dim(\gamma(i))}}^{\,k_i} \right) A(\hat{\gamma}), \tag{6}$$

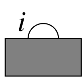
where $\|\gamma\| = \sum_i d(\gamma(i), \Lambda)$.

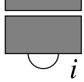
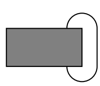
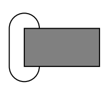
Remark A.6. In the lopsided convention, this formula is given by

$$A(\gamma) = (-1)^{(\|\gamma - \|\hat{\gamma}\|)/2} \left(\prod_i \left(\frac{\dim(\hat{\gamma}(i))}{\dim(\gamma(i))} \right)^{\ell_i} \right) A(\hat{\gamma}), \tag{7}$$

where ℓ_i is the number of minima on the cap:

$\ell_i =$

0 when we have ,

1 when we have ,  i, or i .

Tail avoiding relation. Now suppose Γ is obtained from Λ by adding A_{finite} tails to distinct vertices of Λ. Further suppose γ is a loop of length $2n$ with $d(\gamma, \Lambda) = 1$.

Notation A.7. Suppose $s = \gamma(i)$ is a vertex on γ which is distance 1 from Λ, and let $r = \gamma(i+1)$ which is necessarily in Λ. Let $\{t\}$ be the set of vertices in Λ incident to r. Let $\gamma_{i,t}$ be the loop modified from γ by replacing s at position i with t. Let π be the "snipped" loop of length $2n - 2$ obtained from γ or $\gamma_{i,t}$ by removing the i-th and $i+1$-st positions.

Definition A.8. The *tail avoiding relation* is given by

$$0 = \sqrt{\dim(r)}^{k_i} \cap_i(A)(\pi) = \sqrt{\dim(s)}^{k_i} A(\gamma) + \sum_t \sqrt{\dim(t)}^{k_i} A(\gamma_{i,t}).$$

Lemma A.9. *If γ has $d(\gamma, \Lambda) = 1$, and $\hat{\gamma}$ has $d(\hat{\gamma}, \Lambda) = 0$ and is obtained from γ by the tail avoiding relation described above, then*

$$A(\gamma) = (-1)^{\|\gamma\|} \sum_{\{i \,|\, \gamma(i) \notin \Lambda\}} \sum_{\left\{t_i \,\middle|\, \substack{t_i \sim \gamma(i\pm 1) \\ t_i \in \Lambda}\right\}} \sqrt{\frac{\dim(t_i)}{\dim(\gamma(i))}}^{k_i} A(\gamma_{i,t_i}), \tag{8}$$

where $v \sim w$ means v is incident to w (note $\gamma(i+1) = \gamma(i-1)$ if $\gamma(i) \notin \Lambda$), and k_i is as in Lemma A.5.

Remark A.10. In the lopsided convention, this formula is given by

$$A(\gamma) = (-1)^{\|\gamma\|} \sum_{\{i \,|\, \gamma(i) \notin \Lambda\}} \sum_{\left\{t_i \,\middle|\, \substack{t_i \sim \gamma(i\pm 1) \\ t_i \in \Lambda}\right\}} \left(\frac{\dim(t_i)}{\dim(\gamma(i))}\right)^{\ell_i} A(\gamma_{i,t_i}) \tag{9}$$

using similar notation from Remark A.6 and Lemma A.9.

Rotation. We still assume Γ is obtained from Λ by adding A_{finite} tails to distinct vertices of Λ.

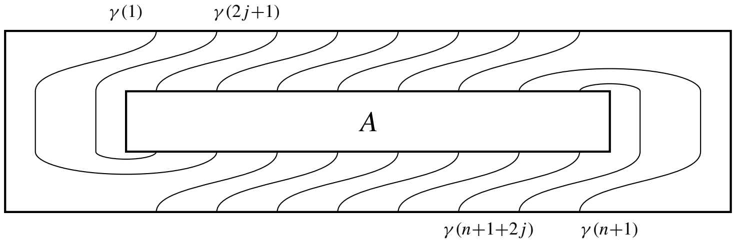
Rotation acts on the set of loops which stay in Λ, so if we are trying to specify a lowest weight vector A which is also a rotational eigenvector corresponding to eigenvalue ω, then it suffices to specify A only on a representative of each such orbit.

Proposition A.11. *Let S be a set of representatives of each rotation orbit of loops of length $2n$ in Λ. Let $A_0 : S \to \mathbb{C}$. For a loop γ of length $2n$ in Λ, let $[\gamma]$ be its representative in S. Suppose that whenever $\gamma' \in S$ is fixed by the k-fold rotation, and $\omega^k \neq 1$, then $A_0(\gamma') = 0$. Then there is a well-defined function A_1 on the loops of length $2n$ in Λ such that $A_1|_S = A_0$.*

Moreover, there is a well-defined element $A \in \mathcal{PA}(\Gamma)_n$ such that the values of A on the loops of length $2n$ on Λ is equal to A_1.

Proof. Suppose γ is a loop of length $2n$ which stays in Λ, and $\rho^{-j}(\gamma) = [\gamma]$ for some $j = 0, \ldots, n-1$. If $j \leq n/2$,

$\rho^j(A)(\gamma) = A(\rho^{-j}(\gamma)) =$



Hence for all $j = 0, \ldots, n-1$, we define

$$A_1(\gamma) = \omega^{-j}\sqrt{\frac{\dim(\gamma(2j+1))\dim(\gamma(n+2j+1))}{\dim(\gamma(1))\dim(\gamma(n+1))}}A_0([\gamma]), \tag{10}$$

modulo some modular arithmetic, namely $\gamma(b) = \gamma(b \bmod 2n)$.

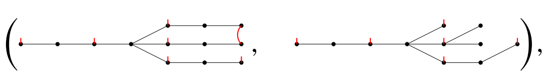
In the lopsided convention, the above equation is given by

$$A_1(\gamma) = \omega^{-j}\left(\prod_{k=1}^{2j}\frac{\dim(\gamma(1+k))}{\dim(\gamma(n+k))}\right)A_0([\gamma]). \tag{11}$$

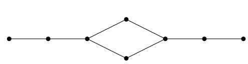
We now define $A \in \mathcal{PA}(\Gamma)_n$ as follows. First, for loops γ of length $2n$ which stay in Λ, define $A(\gamma) = A_1(\gamma)$. Next, we define A on loops γ of length $2n$ for which $d(\gamma, \Lambda) = 1$ by Lemma A.9. Finally, we define A on loops γ of length $2n$ for which $d(\gamma, \Lambda) > 1$ by Lemma A.5. $\square$

We now apply the above discussion to specify our generators by their values on a certain collection of loops. A little unusually, we find our generators in the graph planar algebra of a different graph: $\Gamma = 2\mathrm{D}2$, which has a central diamond. We label the vertices on the diamond by $\mathtt{W}$, $\mathtt{S}$, $\mathtt{E}$, $\mathtt{N}$, which stand for "west," "south," "east," "north" respectively. We denote the value of A on the collapsed loop which stays inside the central diamond by $A(w)$, where w is a word on $\{\mathtt{W}, \mathtt{S}, \mathtt{E}, \mathtt{N}\}$.

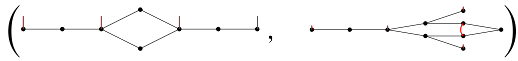
AA. ***Generators for*** $\mathbf{3^{\mathbb{Z}/4}}$**.** In an unpublished manuscript, Izumi constructs a $3^{\mathbb{Z}/4}$ subfactor with principal graphs



and he claims there is a de-equivariantization, giving a subfactor with principal graph "2-diamond-2":

2D2 = .

In an independent calculation, Morrison and Penneys [2015a] verify the existence and prove uniqueness for the 2D2 subfactor with principal graphs

2D2 = .

They solve the equation $T^2 = f^{(3)}$ in the graph planar algebra of 2D2 to get a low-weight rotational eigenvector at depth 3. Then they verify, using a universal variant of the jellyfish algorithm for finite depth subfactor planar algebras, that the planar subalgebra generated by T is evaluable and has principal graphs 2D2. They obtain a $3^{\mathbb{Z}/4}$ subfactor planar algebra as an equivariantization of the 2D2 subfactor planar algebra. Note that 2D2 has annular multiplicities $*12$, so the $3^{\mathbb{Z}/4}$ generators must be the new low-weight vectors at depth 4. See [Morrison and Penneys 2015a] for more details.

For our purposes in this article, we do *not* rely on the fact that our generators were obtained via equivariantization. Rather, we present candidate generators for $3^{\mathbb{Z}/4}$ in 2D2, show they satisfy Assumptions 2.9, 2.12, and 2.21, and use our formulas to show they generate an evaluable planar subalgebra of the graph planar algebra of 2D2, i.e., a subfactor planar algebra.

Hence we work in the graph planar algebra of 2D2 where Λ is the central diamond. The self-adjoint generators A, B for $\mathcal{P}_\bullet^{\mathbb{Z}/4}$ have chiralities $\omega_A = -1$ and $\sigma_A = i$ and $\omega_B = \sigma_B = 1$.

A assigns the below values to the indicated rotation orbit representatives of loops which remain in Λ:

0	WSWSWSWS, WSWSWSES, WSWSWNWN, WSWSWNEN, WSWSESES, WSWSENEN, WSWNWSWN, WSWNWNES, WSWNENES, WSESWSES, WSESWNWN, WSESWNEN, WSESESES, WSESENWN, WSESENEN, WSENWSEN, WSENWNES, WSENENES, WNWNWNWN, WNWNWNEN, WNWNESES, WNWNENEN, WNESWNES, WNESESEN, WNENWNEN, WNENESES, WNENENEN, ESESESES, ESESENEN, ESENESEN, ENENENEN
$\frac{1}{4}(3-\sqrt{5})$	WSWSWSEN, WSWSWNES, WSWNESES, WSESESEN, WSENWNWN, WSENENEN, WNWNWNES, WNESESES, WNENENES
$\frac{1}{4}(\sqrt{5}-3)$	WSWSESEN, WSESWSEN, WSESWNES, WSENWNEN, WSENENWN, WNWNENES, WNESWNEN
$2-\sqrt{5}$	WSWNWSEN, WSENESEN, WNESENES
$\sqrt{5}-2$	WSWNESWN

$\lambda^{(0.09279i)}_{16,0,-116,0,-1}$	WSWSESWN, WSWSENES, WSWNWSES, WSWNWNWN, WSWNENEN, WSENESES, WNWNESEN, WNESENEN, ESENENEN
$\lambda^{(-0.09279i)}_{16,0,-116,0,-1}$	WSWSWSWN, WSWNWNEN, WSWNENWN, WSESESWN, WSESENES, WNENESEN, ESESESEN
$\lambda^{(-0.3003i)}_{1,0,-11,0,-1}$	WSWNESEN
$\lambda^{(0.3003i)}_{1,0,-11,0,-1}$	WSENESWN

B assigns the below values to the indicated rotation orbit representatives of loops which remain in Λ:

0	WSWSENWN, WSWSENEN, WSWNWSEN, WSWNWNES, WSWNESWN, WSWNENES, WSESENWN, WSESENEN, WSENESEN, WNWNESES, WNESENES, WNENESES
$\lambda^{(0.2784i)}_{16,0,-1044,0,-81}$	WSWSWSEN, WSWNESES, WSESWNES, WSESESEN, WSENWNEN, WSENENWN, WNWNWNES, WNENENES
$\lambda^{(-0.2784i)}_{16,0,-1044,0,-81}$	WSWSWNES, WSWSESEN, WSESWSEN, WSENWNWN, WSENENEN, WNWNENES, WNESWNEN, WNESESES
$\frac{1}{2}\left(11-5\sqrt{5}\right)$	WSWSWSES, WSESESES, WNWNWNEN, WNENENEN
$\frac{1}{2}\left(5\sqrt{5}-11\right)$	WSWSWSWS, WSWSESES, WSESWSES, WNWNWNWN, WNWNENEN, WNENWNEN, ESESESES, ENENENEN
$\frac{1}{4}\left(\sqrt{5}-3\right)$	WSWSWSWN, WSWSENES, WSWNWNWN, WSWNENEN, WSESESWN, WNWNESEN, ESESESEN, ESENENEN
$\frac{1}{4}\left(3-\sqrt{5}\right)$	WSWSESWN, WSWNWSES, WSWNWNEN, WSWNENWN, WSESENES, WSENESES, WNESENEN, WNENESEN
$\frac{1}{2}\left(3-\sqrt{5}\right)$	WSWNWSWN, ESENESEN
$\frac{1}{2}\left(7-3\sqrt{5}\right)$	WSWSWNWN, WSESWNEN, WSENWNES, ESESENEN
$\frac{1}{2}\left(3\sqrt{5}-7\right)$	WSWSWNEN, WSESWNWN, WSENENES, WNESESEN
$\frac{1}{2}\left(3\sqrt{5}-9\right)$	WSENWSEN, WNESWNES
$2-\sqrt{5}$	WSWNESEN, WSENESWN

These entries lie in $\mathbb{Q}(\mu_{\mathbb{Z}/4})$, where $\mu_{\mathbb{Z}/4}$ is the root of

$$x^8-38x^6+100x^5+343x^4-2300x^3+5102x^2-5500x+2581$$

which is approximately $2.236+0.700i$.

Appendix B. Moments and tetrahedral constants of $3^{\mathbb{Z}/4}$

For all of our planar algebras, our generators are self-adjoint. This is a list of the moments and tetrahedral structure constants needed for our calculations.

$$\mathrm{Tr}(AA) = 4+2\sqrt{5} \qquad \mathrm{Tr}(\check{A}\check{A}) = 4+2\sqrt{5}$$
$$\mathrm{Tr}(AB) = 0 \qquad \mathrm{Tr}(\check{A}\check{B}) = 0$$
$$\mathrm{Tr}(BB) = 12+6\sqrt{5} \qquad \mathrm{Tr}(\check{B}\check{B}) = 12+6\sqrt{5}$$
$$\mathrm{Tr}(AAA) = 0 \qquad \mathrm{Tr}(\check{A}\check{A}\check{A}) = \lambda^{(-3.25)}_{4,0,-40,0,-25}$$
$$\mathrm{Tr}(AAB) = -4-2\sqrt{5} \qquad \mathrm{Tr}(\check{A}\check{A}\check{B}) = \lambda^{(6.698)}_{4,0,-180,0,25}$$
$$\mathrm{Tr}(ABB) = 0 \qquad \mathrm{Tr}(\check{A}\check{B}\check{B}) = \lambda^{(13.1)}_{4,0,-648,0,-6561}$$
$$\mathrm{Tr}(BBB) = 12+6\sqrt{5} \qquad \mathrm{Tr}(\check{B}\check{B}\check{B}) = \lambda^{(0.501)}_{4,0,-324,0,81}$$

$$\Delta(A,A,A \mid A) = -\sqrt{3+\sqrt{5}} \qquad \Delta(A,A,A \mid B) = 0$$
$$\Delta(A,A,B \mid A) = -i\sqrt{11+5\sqrt{5}} \qquad \Delta(A,A,B \mid B) = -\sqrt{2}$$
$$\Delta(A,B,A \mid B) = \sqrt{107+39\sqrt{5}} \qquad \Delta(A,B,B \mid B) = -9i\sqrt{1+\sqrt{5}}$$
$$\Delta(B,A,B \mid A) = \sqrt{47+21\sqrt{5}} \qquad \Delta(B,A,B \mid B) = 0$$
$$\Delta(B,B,B \mid B) = 9\sqrt{3-\sqrt{5}}$$

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