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CORRECTION TO MODULAR L-VALUES OF CUBIC LEVEL

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In the paper in question, equation (2-23) is incorrect, and hence so are equations (3-6) and (3-9). We correct the statements in this note. All other statements in the paper, including the main theorems, are unaffected.

In [Knightly and Li 2012], equation (2-23) was quoted from an early draft of [Knightly and Li 2015], which at the time contained an error. It should read

$$f_2^\sigma(zg) = \frac{(p+1)\zeta}{2\omega_p(zd)} \sum_{w\in(\mathbb{Z}/p\mathbb{Z})^*} \overline{\omega_p(w)}\theta_p\left(\frac{-\frac{c}{a}w - \frac{tb}{d}w^{-1}}{p}\right) \tag{2-23}$$

for $g = \left(\begin{smallmatrix} c & dp^{-2} \\ ap & b \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \mathbb{Z}_p & (1/p^2)\mathbb{Z}_p^* \\ p\mathbb{Z}_p^* & \mathbb{Z}_p \end{smallmatrix}\right)$. It is a twisted Kloosterman sum.

Proposition 0.1 (Corrected Proposition 3.4). *Let* $\delta = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, *so that*

$$J_\delta(s, f^\sigma) = \int_{\mathbb{Q}_p^*}\int_{\mathbb{Q}_p} f^\sigma\left(\begin{pmatrix} 0 & -y \\ 1 & x \end{pmatrix}\right)\overline{\theta_p(rx)}\, dx\, \chi_p(y)\, |y|_p^{k/2-s}\, d^*y.$$

Then

$$J_\delta(s, f^\sigma) = J_\delta(s, f_2^\sigma) = \begin{cases} \dfrac{(p^3)^{k/2-s}p(p+1)\omega_p(-rp^2)}{2\zeta\,\chi_p(p^3)} & \text{if } p \nmid r, \\ 0 & \text{otherwise}, \end{cases} \tag{3-6}$$

and

$$I_\delta(s)_p = 0. \tag{3-7}$$

Proof. By (2-22), the matrix $\left(\begin{smallmatrix} 0 & -y \\ 1 & x \end{smallmatrix}\right)$ never belongs to $\mathrm{Supp}(f_1^\sigma)$, so $J_\delta(s, f^\sigma) = J_\delta(s, f_2^\sigma)$. Note that $\left(\begin{smallmatrix} 0 & -y \\ 1 & x \end{smallmatrix}\right) \in \mathrm{Supp}(f_2^\sigma)$ if and only if

$$\begin{pmatrix} 0 & -py \\ p & px \end{pmatrix} \in \begin{pmatrix} \mathbb{Z}_p & p^{-2}\mathbb{Z}_p^* \\ p\mathbb{Z}_p^* & \mathbb{Z}_p \end{pmatrix}.$$

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In this case, we may write $y = -p^{-3}u$ for $u \in \mathbb{Z}_p^*$, and $x' = px \in \mathbb{Z}_p$. Then $dx' = p^{-1}dx$, and dropping the $'$ from the notation, we have

$$\begin{aligned} &J_\delta(s, f_2^\sigma) \\ &= p\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p} f_2^\sigma\left(\begin{pmatrix} p^{-1} & \\ & p^{-1}\end{pmatrix}\begin{pmatrix} 0 & p^{-2}u \\ p & x\end{pmatrix}\right)\theta_p\left(\frac{-rx}{p}\right) dx\, \chi_p(p^{-3})(p^3)^{k/2-s}\, d^*u \\ &= \frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(p)}{2\chi_p(p^3)} \sum_{w\in(\mathbb{Z}/p\mathbb{Z})^*} \overline{\omega_p(w)}\int_{\mathbb{Z}_p^*}\overline{\omega_p(u)}\int_{\mathbb{Z}_p}\theta_p\left(\frac{-\frac{tx}{uw}}{p}\right)\theta_p\left(\frac{-rx}{p}\right) dx\, d^*u \end{aligned}$$

by (2-23). Replacing u by $(-uw)^{-1}$, the above is equal to

$$\frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(p)}{2\chi_p(p^3)} \sum_{w\in(\mathbb{Z}/p\mathbb{Z})^*} \overline{\omega_p(w)}\int_{\mathbb{Z}_p^*}\omega_p(-uw)\int_{\mathbb{Z}_p}\theta_p\left(\frac{(tu-r)x}{p}\right) dx\, d^*u.$$

Observe that w is eliminated, and the sum over w contributes $p-1$. Furthermore,

$$\int_{\mathbb{Z}_p}\theta_p\left(\frac{(tu-r)x}{p}\right) dx = \begin{cases} 1 & \text{if } u \in t^{-1}r + p\mathbb{Z}_p, \\ 0 & \text{otherwise.}\end{cases}$$

In particular, it vanishes if $p \mid r$. Assuming $p \nmid r$,

$$\begin{aligned} J_\delta(s, f^\sigma) &= \frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(p)}{2\chi_p(p^3)}(p-1)\int_{t^{-1}r+p\mathbb{Z}_p}\omega_p(-u)\, d^*u \\ &= \frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(p)}{2\chi_p(p^3)}\omega_p(-t^{-1}r) \end{aligned}$$

since the coset has multiplicative measure $1/(p-1)$. Equality (3-6) now follows, using the fact that

$$\frac{\zeta\omega_p(p)}{\omega_p(t)} = \frac{\zeta\omega_p(p)^2}{\omega_p(pt)} = \frac{\omega_p(p)^2}{\zeta}.$$

For fixed t, summing (3-6) over $\pm\zeta$ gives 0. Thus $I_\delta(s)_p = \sum_\sigma J_\delta(s, f^\sigma) = 0$. $\square$

Proposition 0.2 (Corrected Proposition 3.5). *For $a \in \mathbb{Q}^*$, let $\delta_a = \left(\begin{smallmatrix} a & -1 \\ 1 & 0\end{smallmatrix}\right)$, so that*

$$J_{\delta_a}(s, f^\sigma) = \int_{\mathbb{Q}_p^*}\int_{\mathbb{Q}_p} f^\sigma\left(\begin{pmatrix} ya & y(xa-1) \\ 1 & x\end{pmatrix}\right)\overline{\theta_p(rx)}\, dx\, \chi_p(y)\,|y|_p^{k/2-s}\, d^*y.$$

Then $J_{\delta_a}(s, f_1^\sigma)$ vanishes unless $a \in p^2\mathbb{Z}_p$ and $p \nmid r$. In this case, writing $a = p^{a_p}a_0$ for $a_0 \in \mathbb{Z}_p^* \cap \mathbb{Q}^*$, we have

$$J_{\delta_a}(s, f_1^\sigma) = \begin{cases} \dfrac{|a|_p^{2s-k} p(p+1)\,\omega_p(p^{a_p})}{2\chi_p(a^2)} \theta_p\left(\dfrac{ta}{rp^3} - \dfrac{r}{a}\right) & \text{if } a_0 \equiv 1 \bmod p, \\ 0 & \text{otherwise.} \end{cases} \tag{3-8}$$

The integral $J_{\delta_a}(s, f_2^\sigma)$ vanishes unless $a \in p^2\mathbb{Z}_p$. For such a,

$$J_{\delta_a}(s, f_2^\sigma) = \begin{cases} \dfrac{(p^3)^{k/2-s} p(p+1)\,\omega_p(-p^2 r)}{2\chi_p(p^3)\zeta} \theta_p\left(-\dfrac{ta}{rp^3}\right) & \text{if } p \nmid r, \\ 0 & \text{otherwise.} \end{cases} \tag{3-9}$$

Finally, $I_{\delta_a}(s)_p$ vanishes unless $p \nmid r$ and $a = p^{a_p} a_0$ for $a_p \geq 2$ and $a_0 \equiv 1 \bmod p\mathbb{Z}_p$. If these conditions are satisfied, then

(3-10)
$$I_{\delta_a}(s)_p = \frac{|a|_p^{2s-k} p(p+1)\,\omega_p(p^{a_p})\theta_p\left(-\frac{r}{a}\right)}{\chi_p(a^2)} \Delta_p(a) \quad \text{for } \Delta_p(a) = \begin{cases} p-1 & \text{if } a_p > 2, \\ -1 & \text{if } a_p = 2. \end{cases}$$

Remark. Only (3-9) differs from the original statement.

Proof of (3-9). Consider

$$J_{\delta_a}(s, f_2^\sigma) = \int_{\mathbb{Q}_p^*} \int_{\mathbb{Q}_p} f_2^\sigma\left(\begin{pmatrix} ya & y(xa-1) \\ 1 & x \end{pmatrix}\right) \overline{\theta_p(rx)}\, dx\, \chi_p(y)\, |y|_p^{k/2-s}\, d^*y.$$

By (2-23), the integrand is nonzero precisely when

$$\begin{pmatrix} p & \\ & p \end{pmatrix} \begin{pmatrix} ya & y(xa-1) \\ 1 & x \end{pmatrix} = \begin{pmatrix} pya & py(xa-1) \\ p & px \end{pmatrix} \in \begin{pmatrix} \mathbb{Z}_p & p^{-2}\mathbb{Z}_p^* \\ p\mathbb{Z}_p^* & \mathbb{Z}_p \end{pmatrix}.$$

Taking the determinant, this says, in particular, that $p^2 y \in p^{-1}\mathbb{Z}_p^*$, so we may write $y = u/p^3$ for $u \in \mathbb{Z}_p^*$. Replacing px by x, we have

$$J_{\delta_a}(s, f_2^\sigma) = \frac{(p^3)^{k/2-s} p\,\omega_p(p)}{\chi_p(p^3)} \int_{\mathbb{Z}_p^*} \int_{\mathbb{Z}_p} f_2^\sigma\left(\begin{pmatrix} \frac{ua}{p^2} & \frac{u}{p^2}\left(\frac{xa}{p} - 1\right) \\ p & x \end{pmatrix}\right) \theta_p\left(\frac{-rx}{p}\right) dx\, d^*u.$$

From the upper left entry, the integrand is nonzero only if $a_p \geq 2$. Assuming the latter, we also have $xa/p - 1 \in \mathbb{Z}_p^*$, so the upper right entry belongs to $p^{-2}\mathbb{Z}_p^*$ as required. Hence by (2-23), the above is equal to

$$\frac{(p^3)^{k/2-s} p(p+1)\zeta\,\omega_p(p)}{2\chi_p(p^3)} \sum_{w \in (\mathbb{Z}/p\mathbb{Z})^*} \overline{\omega_p(w)} \int_{\mathbb{Z}_p^*} \overline{\omega_p(u)} \theta_p\left(-\frac{uaw}{p^3}\right)$$
$$\int_{\mathbb{Z}_p} \overline{\omega_p\left(\frac{xa}{p} - 1\right)} \theta_p\left(\frac{-txu^{-1}\left(\frac{xa}{p} - 1\right)^{-1} w^{-1}}{p}\right) \theta_p\left(\frac{-rx}{p}\right) dx\, d^*u.$$

Note that $\omega_p(xa/p-1)=\omega_p(-1)$ since $p^2\mid a$. For the same reason,

$$\theta_p\left(\frac{-txu^{-1}\left(\frac{xa}{p}-1\right)^{-1}w^{-1}}{p}\right)=\theta_p\left(\frac{tu^{-1}w^{-1}x}{p}\right).$$

Therefore the above integral over $\mathbb{Z}_p$ equals

$$\omega_p(-1)\int_{\mathbb{Z}_p}\theta_p\left(\frac{(-r+tu^{-1}w^{-1})x}{p}\right)dx=\begin{cases}\omega_p(-1) & \text{if } u\in tr^{-1}w^{-1}+p\mathbb{Z}_p,\\ 0 & \text{otherwise.}\end{cases}$$

In particular, $J_{\delta_a}(s,f_2^\sigma)=0$ if $p\mid r$. Assuming $p\nmid r$, $J_{\delta_a}(s,f_2^\sigma)$ equals

$$\frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(-p)}{2\chi_p(p^3)}\sum_{w\in(\mathbb{Z}/p\mathbb{Z})^*}\overline{\omega_p(w)}\int_{tr^{-1}w^{-1}+p\mathbb{Z}_p}\overline{\omega_p(u)}\theta_p\left(-\frac{uaw}{p^3}\right)d^*u$$

$$=\frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(-p)}{2\chi_p(p^3)}\sum_{w\in(\mathbb{Z}/p\mathbb{Z})^*}\overline{\omega_p(w)}\,\overline{\omega_p(tr^{-1}w^{-1})}\theta_p\left(-\frac{tr^{-1}a}{p^3}\right)\frac{1}{p-1}$$

$$=\frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(-p)}{2\chi_p(p^3)}\overline{\omega_p(tr^{-1})}\theta_p\left(-\frac{tr^{-1}a}{p^3}\right).$$

Equation (3-9) follows upon using $\zeta\omega_p(p)/\omega_p(t)=\omega_p(p)^2/\zeta$. □

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