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LUSZTIG INDUCTION AND ℓ-BLOCKS OF FINITE REDUCTIVE GROUPS

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To the memory of Robert Steinberg

We present a unified parametrisation of ℓ-blocks of quasisimple finite groups of Lie type in nondefining characteristic via Lusztig's induction functor in terms of e-Jordan-cuspidal pairs and e-Jordan quasicentral cuspidal pairs.

1. Introduction

The work of Fong and Srinivasan for classical matrix groups and of Schewe for certain blocks of groups of exceptional type exhibited a close relation between the ℓ-modular block structure of groups of Lie type and the decomposition of Lusztig's induction functor, defined in terms of ℓ-adic cohomology. This connection was extended to unipotent blocks of arbitrary finite reductive groups and large primes ℓ by Broué–Malle–Michel [1993], to all unipotent blocks by Cabanes–Enguehard [1994] and Enguehard [2000], to arbitrary blocks for primes $\ell \geq 7$ by Cabanes–Enguehard [1999], to nonquasi-isolated blocks by Bonnafé–Rouquier [2003] and to quasi-isolated blocks of exceptional groups at bad primes by the authors [2013].

It is the main purpose of this paper to unify and extend all of the preceding results in particular from [Cabanes and Enguehard 1999] so as to establish a statement in its largest possible generality, without restrictions on the prime ℓ, the type of group or the type of block, in terms of e-Jordan quasicentral cuspidal pairs (see Section 2 for the notation used).

Theorem A. *Let $\boldsymbol{H}$ be a simple algebraic group of simply connected type with a Frobenius endomorphism $F : \boldsymbol{H} \to \boldsymbol{H}$ endowing $\boldsymbol{H}$ with an $\mathbb{F}_q$-rational structure. Let $\boldsymbol{G}$ be an F-stable Levi subgroup of $\boldsymbol{H}$. Let ℓ be a prime not dividing q and set $e = e_\ell(q)$.*

(a) *For any e-Jordan-cuspidal pair $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ such that $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$, there exists a unique ℓ-block $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}^F$ such that all irreducible constituents of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ lie in $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$.*

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(b) *The map* $\Xi : (\boldsymbol{L}, \lambda) \mapsto b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ *is a surjection from the set of* $\boldsymbol{G}^F$*-conjugacy classes of e-Jordan-cuspidal pairs* $(\boldsymbol{L}, \lambda)$ *of* $\boldsymbol{G}$ *such that* $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ *to the set of* ℓ*-blocks of* $\boldsymbol{G}^F$.

(c) *The map* Ξ *restricts to a surjection from the set of* $\boldsymbol{G}^F$*-conjugacy classes of e-Jordan* quasicentral *cuspidal pairs* $(\boldsymbol{L}, \lambda)$ *of* $\boldsymbol{G}$ *such that* $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ *to the set of* ℓ*-blocks of* $\boldsymbol{G}^F$.

(d) *For* $\ell \geq 3$ *the map* Ξ *restricts to a bijection between the set of* $\boldsymbol{G}^F$*-conjugacy classes of e-Jordan quasicentral cuspidal pairs* $(\boldsymbol{L}, \lambda)$ *of* $\boldsymbol{G}$ *with* $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ *and the set of* ℓ*-blocks of* $\boldsymbol{G}^F$.

(e) *The map* Ξ *itself is bijective if* $\ell \geq 3$ *is good for* $\boldsymbol{G}$, *and moreover* $\ell \neq 3$ *if* $\boldsymbol{G}^F$ *has a factor* ${}^3D_4(q)$.

The restrictions in (d) and (e) are necessary (see Remark 3.15 and Example 3.16).

In fact, part (a) of the preceding result is a special case of the following characterisation of the ℓ'-characters in a given ℓ-block in terms of Lusztig induction:

Theorem B. *In the setting of Theorem A let* b *be an* ℓ*-block of* $\boldsymbol{G}^F$ *and denote by* $\mathcal{L}(b)$ *the set of e-Jordan cuspidal pairs* $(\boldsymbol{L}, \lambda)$ *of* $\boldsymbol{G}$ *such that* $\big\{\chi \in \mathrm{Irr}(b) \mid \langle \chi, R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)\rangle \neq 0\big\} \neq \varnothing$. *Then*

$$\mathrm{Irr}(b) \cap \mathcal{E}(\boldsymbol{G}^F, \ell') = \big\{\chi \in \mathcal{E}(\boldsymbol{G}^F, \ell') \mid \exists\, (\boldsymbol{L}, \lambda) \in \mathcal{L}(b) \text{ with } (\boldsymbol{L}, \lambda) \ll_e (\boldsymbol{G}, \chi)\big\}.$$

Note that at present, it is not known whether Lusztig induction $R_{\boldsymbol{L}}^{\boldsymbol{G}}$ is independent of the parabolic subgroup containing the Levi subgroup $\boldsymbol{L}$ used to define it. Our proofs will show, though, that in our case $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ is defined unambiguously.

An important motivation for this work comes from the recent reductions of most long-standing famous conjectures in modular representation theory of finite groups to questions about quasisimple groups. Among the latter, the quasisimple groups of Lie type form the by far most important part. A knowledge and suitable inductive description of the ℓ-blocks of these groups is thus of paramount importance for an eventual proof of those central conjectures. Our results are specifically tailored for use in an inductive approach by considering groups that occur as Levi subgroups inside groups of Lie type of simply connected type, that is, inside quasisimple groups.

Our paper is organised as follows; in Section 2, we set up e-Jordan (quasicentral) cuspidal pairs and discuss some of their properties. In Section 3 we prove Theorem A (see Theorem 3.14) on parametrising ℓ-blocks by e-Jordan-cuspidal and e-Jordan quasicentral cuspidal pairs and Theorem B (see Theorem 3.6) on characterising ℓ'-characters in blocks. The crucial case turns out to be when $\ell = 3$. In particular, the whole section on pages 287–289 is devoted to the situation of extra-special defect groups of order 27, excluded in [Cabanes and Enguehard 1999], which eventually turns out to behave just as the generic case. An important ingredient of Section 3 is Theorem 3.4, which shows that the distribution of ℓ'-characters in ℓ-blocks is preserved under Lusztig induction from e-split Levi subgroups. Finally, in Section 4 we collect some results relating e-Jordan-cuspidality and usual e-cuspidality.

2. Cuspidal pairs

Throughout this section, $\boldsymbol{G}$ is a connected reductive linear algebraic group over the algebraic closure of a finite field of characteristic p, and $F : \boldsymbol{G} \to \boldsymbol{G}$ is a Frobenius endomorphism endowing $\boldsymbol{G}$ with an $\mathbb{F}_q$-structure for some power q of p. By $\boldsymbol{G}^*$ we denote a group in duality with $\boldsymbol{G}$ with respect to some fixed F-stable maximal torus of $\boldsymbol{G}$, with corresponding Frobenius endomorphism also denoted by F.

e-Jordan-cuspidality. Let e be a positive integer. We will make use of the terminology of Sylow e-theory (see for instance [Broué et al. 1993]). For an F-stable maximal torus $\boldsymbol{T}$, $\boldsymbol{T}_e$ denotes its Sylow e-torus. Then a Levi subgroup $\boldsymbol{L} \leq \boldsymbol{G}$ is called *e-split* if $\boldsymbol{L} = C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L})_e)$, and $\lambda \in \mathrm{Irr}(\boldsymbol{L}^F)$ is called *e-cuspidal* if ${}^*R^{\boldsymbol{L}}_{\boldsymbol{M}\leq\boldsymbol{P}}(\lambda) = 0$ for all proper e-split Levi subgroups $\boldsymbol{M} < \boldsymbol{L}$ and any parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{L}$ containing $\boldsymbol{M}$ as Levi complement. (It is expected that Lusztig induction is in fact independent of the ambient parabolic subgroup. This would follow for example if the Mackey formula holds for $R^{\boldsymbol{G}}_{\boldsymbol{L}}$, and has been proved whenever $\boldsymbol{G}^F$ does not have any component of type ${}^2E_6(2)$, $E_7(2)$ or $E_8(2)$, see [Bonnafé and Michel 2011]. All the statements made in this section using $R^{\boldsymbol{G}}_{\boldsymbol{L}}$ are valid independent of the particular choice of parabolic subgroup — we will make clarifying remarks at points where there might be any ambiguity.)

Definition 2.1. Let $s \in \boldsymbol{G}^{*F}$ be semisimple. Following [Cabanes and Enguehard 1999, Section 1.3] we say that $\chi \in \mathcal{E}(\boldsymbol{G}^F, s)$ is *e-Jordan-cuspidal*, or *satisfies condition (J)* with respect to some $e \geq 1$ if

(J_1) $Z^\circ(C^\circ_{\boldsymbol{G}^*}(s))_e = Z^\circ(\boldsymbol{G}^*)_e$, and

(J_2) χ corresponds under Jordan decomposition (see [Digne and Michel 1991, Theorem 13.23]) to the $C_{\boldsymbol{G}^*}(s)^F$-orbit of an e-cuspidal unipotent character of $C^\circ_{\boldsymbol{G}^*}(s)^F$.

If $\boldsymbol{L} \leq \boldsymbol{G}$ is e-split and $\lambda \in \mathrm{Irr}(\boldsymbol{L}^F)$ is e-Jordan-cuspidal, then $(\boldsymbol{L}, \lambda)$ is called an *e-Jordan-cuspidal pair*.

It is shown in [Cabanes and Enguehard 1999, Proposition 1.10] that χ is e-Jordan-cuspidal if and only if it satisfies the *uniform criterion*

(U): for every F-stable maximal torus $\boldsymbol{T} \leq \boldsymbol{G}$ with $\boldsymbol{T}_e \not\leq Z(\boldsymbol{G})$ we have ${}^*R^{\boldsymbol{G}}_{\boldsymbol{T}}(\chi) = 0$.

Remark 2.2. By [Cabanes and Enguehard 1999, Proposition 1.10(ii)] it is known that e-cuspidality implies e-Jordan-cuspidality; moreover e-Jordan-cuspidality and e-cuspidality agree at least in the following situations:

(1) when $e = 1$;

(2) for unipotent characters (see [Broué et al. 1993, Corollary 3.13]);

(3) for characters lying in an ℓ'-series where ℓ is an odd prime, good for $\boldsymbol{G}$, e is the order of q modulo ℓ and either $\ell \geq 5$ or $\ell = 3 \in \Gamma(\boldsymbol{G}, F)$ as defined in [Cabanes and Enguehard 1994, Notation 1.1] (see [Cabanes and Enguehard 1999, Theorem 4.2 and Remark 5.2]); and

(4) for characters lying in a quasi-isolated ℓ'-series of an exceptional type simple group for ℓ a bad prime (this follows by inspection of the explicit results in [Kessar and Malle 2013]).

To see the first point, assume that χ is 1-Jordan-cuspidal. Suppose if possible that χ is not 1-cuspidal. Then there exists a proper 1-split Levi subgroup $\boldsymbol{L}$ of $\boldsymbol{G}$ such that ${}^*R_{\boldsymbol{L}}^{\boldsymbol{G}}(\chi)$ is nonzero. Then ${}^*R_{\boldsymbol{L}}^{\boldsymbol{G}}(\chi)(1) \neq 0$ as ${}^*R_{\boldsymbol{L}}^{\boldsymbol{G}}$ is ordinary Harish-Chandra restriction. Hence the projection of ${}^*R_{\boldsymbol{L}}^{\boldsymbol{G}}(\chi)$ to the space of uniform functions of $\boldsymbol{L}^F$ is nonzero in contradiction to the uniform criterion (U).

It seems reasonable to expect (and that is formulated as a conjecture in [Cabanes and Enguehard 1999, Section 1.11]) that e-cuspidality and e-Jordan-cuspidality agree in general. See Section 4 below for a further discussion of this.

We first establish conservation of e-Jordan-cuspidality under some natural constructions:

Lemma 2.3. *Let $\boldsymbol{L}$ be an F-stable Levi subgroup of $\boldsymbol{G}$ and $\lambda \in \mathrm{Irr}(\boldsymbol{L}^F)$. Let $\boldsymbol{L}_0 = \boldsymbol{L} \cap [\boldsymbol{G}, \boldsymbol{G}]$ and let λ_0 be an irreducible constituent of $\mathrm{Res}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda)$. Let $e \geq 1$. Then $(\boldsymbol{L}, \lambda)$ is an e-Jordan-cuspidal pair for $\boldsymbol{G}$ if and only if $(\boldsymbol{L}_0, \lambda_0)$ is an e-Jordan-cuspidal pair for $[\boldsymbol{G}, \boldsymbol{G}]$.*

Proof. Note that $\boldsymbol{L}$ is e-split in $\boldsymbol{G}$ if and only if $\boldsymbol{L}_0$ is e-split in $\boldsymbol{G}_0$. Let $\iota : \boldsymbol{G} \hookrightarrow \tilde{\boldsymbol{G}}$ be a regular embedding. It is shown in the proof of [Cabanes and Enguehard 1999, Proposition 1.10] that condition (J) with respect to $\boldsymbol{G}$ is equivalent to condition (J) with respect to $\tilde{\boldsymbol{G}}$. Since ι restricts to a regular embedding $[\boldsymbol{G}, \boldsymbol{G}] \hookrightarrow \tilde{\boldsymbol{G}}$, the same argument shows that condition (J) with respect to $\tilde{\boldsymbol{G}}$ is equivalent to that condition with respect to $[\boldsymbol{G}, \boldsymbol{G}]$. □

Proposition 2.4. *Let $s \in \boldsymbol{G}^{*F}$ be semisimple, and $\boldsymbol{G}_1 \leq \boldsymbol{G}$ an F-stable Levi subgroup with $\boldsymbol{G}_1^*$ containing $C_{\boldsymbol{G}^*}(s)$. For $(\boldsymbol{L}_1, \lambda_1)$ an e-Jordan-cuspidal pair of $\boldsymbol{G}_1$ below $\mathcal{E}(\boldsymbol{G}_1^F, s)$ define $\boldsymbol{L} := C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L}_1)_e)$ and $\lambda := \epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{L}_1} R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda_1)$. Then $Z^\circ(\boldsymbol{L}_1)_e = Z^\circ(\boldsymbol{L})_e$, and $(\boldsymbol{L}_1, \lambda_1) \mapsto (\boldsymbol{L}, \lambda)$ defines a bijection $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ between the set of e-Jordan-cuspidal pairs of $\boldsymbol{G}_1$ below $\mathcal{E}(\boldsymbol{G}_1^F, s)$ and the set of e-Jordan-cuspidal pairs of $\boldsymbol{G}$ below $\mathcal{E}(\boldsymbol{G}^F, s)$.*

We note that the character λ and hence the bijection $\Psi^{\boldsymbol{G}}_{\boldsymbol{G}_1}$ above are independent of the choice of parabolic subgroup. This is explained in the proof below.

Proof. We first show that $\Psi^{\boldsymbol{G}}_{\boldsymbol{G}_1}$ is well-defined. Let $(\boldsymbol{L}_1, \lambda_1)$ be e-Jordan-cuspidal in $\boldsymbol{G}_1$ below $\mathcal{E}(\boldsymbol{G}_1^F, s)$, so $s \in \boldsymbol{L}_1^*$. Then $\boldsymbol{L}^* := C_{\boldsymbol{G}^*}(Z^\circ(\boldsymbol{L}_1^*)_e)$ clearly is an e-split Levi subgroup of $\boldsymbol{G}^*$. Moreover we have

$$\boldsymbol{L}_1^* = C_{\boldsymbol{G}_1^*}(Z^\circ(\boldsymbol{L}_1^*)_e) = C_{\boldsymbol{G}^*}(Z^\circ(\boldsymbol{L}_1^*)_e) \cap \boldsymbol{G}_1^* = \boldsymbol{L}^* \cap \boldsymbol{G}_1^*.$$

Now $s \in \boldsymbol{L}_1^*$ by assumption, so

$$\boldsymbol{L}_1^* = \boldsymbol{L}^* \cap \boldsymbol{G}_1^* \geq \boldsymbol{L}^* \cap C_{\boldsymbol{G}^*}(s) = C_{\boldsymbol{L}^*}(s).$$

In particular, $\boldsymbol{L}_1^*$ and $\boldsymbol{L}^*$ have a maximal torus in common, so $\boldsymbol{L}_1^*$ is a Levi subgroup of $\boldsymbol{L}^*$. Thus, passing to duals, $\boldsymbol{L}_1$ is a Levi subgroup of $\boldsymbol{L} = C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L}_1)_e)$.

We clearly have $Z^\circ(\boldsymbol{L}_1)_e \leq Z^\circ(\boldsymbol{L})_e$. For the reverse inclusion, observe that $Z^\circ(\boldsymbol{L})_e \leq \boldsymbol{L}_1$, as $\boldsymbol{L}_1$ is a Levi subgroup in $\boldsymbol{L}$, so indeed $Z^\circ(\boldsymbol{L})_e \leq Z^\circ(\boldsymbol{L}_1)_e$.

Hence by [Digne and Michel 1991, Theorem 13.25], $\lambda := \epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{L}_1} R^{\boldsymbol{L}}_{\boldsymbol{L}_1}(\lambda_1)$ is irreducible since, as we saw above, $\boldsymbol{L}_1^* \geq C_{\boldsymbol{L}^*}(s)$. By [Digne and Michel 1991, Remark 13.28], λ is independent of the choice of parabolic subgroup of $\boldsymbol{L}$ containing $\boldsymbol{L}_1$ as Levi subgroup. Let's argue that λ is e-Jordan-cuspidal. Indeed, for any F-stable maximal torus $\boldsymbol{T} \leq \boldsymbol{L}$ we have by the Mackey-formula (which holds as one of the Levi subgroups is a maximal torus by a result of Deligne–Lusztig, see [Bonnafé and Michel 2011, Theorem 2]) that $\epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{L}_1} {}^*R^{\boldsymbol{L}}_{\boldsymbol{T}}(\lambda) = {}^*R^{\boldsymbol{L}}_{\boldsymbol{T}} R^{\boldsymbol{L}}_{\boldsymbol{L}_1}(\lambda_1)$ is a sum of $\boldsymbol{L}^F$-conjugates of ${}^*R^{\boldsymbol{L}_1}_{\boldsymbol{T}}(\lambda_1)$. As λ_1 is e-Jordan-cuspidal, this vanishes if $\boldsymbol{T}_e \not\leq Z^\circ(\boldsymbol{L}_1)_e = Z^\circ(\boldsymbol{L})_e$. So λ satisfies condition (U), hence is e-Jordan-cuspidal, and $\Psi^{\boldsymbol{G}}_{\boldsymbol{G}_1}$ is well-defined.

It is clearly injective, since if $(\boldsymbol{L}, \lambda) = \Psi^{\boldsymbol{G}}_{\boldsymbol{G}_1}(\boldsymbol{L}_2, \lambda_2)$ for some e-cuspidal pair $(\boldsymbol{L}_2, \lambda_2)$ of $\boldsymbol{G}_1$, then $Z^\circ(\boldsymbol{L}_1)_e = Z^\circ(\boldsymbol{L})_e = Z^\circ(\boldsymbol{L}_2)_e$, whence $\boldsymbol{L}_1 = C_{\boldsymbol{G}_1}(Z^\circ(\boldsymbol{L}_1)_e) = C_{\boldsymbol{G}_1}(Z^\circ(\boldsymbol{L}_2)_e) = \boldsymbol{L}_2$, and then the bijectivity of $R^{\boldsymbol{L}}_{\boldsymbol{L}_1}$ on $\mathcal{E}(\boldsymbol{L}_1^F, s)$ shows that $\lambda_1 = \lambda_2$ as well.

We now construct an inverse map. For this, let $(\boldsymbol{L}, \lambda)$ be an e-Jordan-cuspidal pair of $\boldsymbol{G}$ below $\mathcal{E}(\boldsymbol{G}^F, s)$, and $\boldsymbol{L}^* \leq \boldsymbol{G}^*$ dual to $\boldsymbol{L}$. Set

$$\boldsymbol{L}_1^* := C_{\boldsymbol{G}_1^*}(Z^\circ(\boldsymbol{L}^*)_e) = C_{\boldsymbol{G}^*}(Z^\circ(\boldsymbol{L}^*)_e) \cap \boldsymbol{G}_1^* = \boldsymbol{L}^* \cap \boldsymbol{G}_1^*,$$

an e-split Levi subgroup of $\boldsymbol{G}_1^*$. Note that $s \in \boldsymbol{L}^*$, so there exists some maximal torus $\boldsymbol{T}^*$ of $\boldsymbol{G}^*$ with $\boldsymbol{T}^* \leq C_{\boldsymbol{G}^*}(s) \leq \boldsymbol{G}_1^*$, whence $\boldsymbol{L}_1^*$ is a Levi subgroup of $\boldsymbol{L}^*$. Now again

$$\boldsymbol{L}_1^* = \boldsymbol{L}^* \cap \boldsymbol{G}_1^* \geq \boldsymbol{L}^* \cap C_{\boldsymbol{G}^*}(s) = C_{\boldsymbol{L}^*}(s).$$

So the dual $\boldsymbol{L}_1 := C_{\boldsymbol{G}_1}(Z^\circ(\boldsymbol{L})_e)$ is a Levi subgroup of $\boldsymbol{L}$ such that $\epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R^{\boldsymbol{L}}_{\boldsymbol{L}_1}$ preserves irreducibility on $\mathcal{E}(\boldsymbol{L}_1^F, s)$. We define λ_1 to be the unique constituent of ${}^*R^{\boldsymbol{L}}_{\boldsymbol{L}_1}(\lambda)$ in the series $\mathcal{E}(\boldsymbol{L}_1^F, s)$. Then λ_1 is e-Jordan-cuspidal. Indeed, for any

F-stable maximal torus $\boldsymbol{T} \le \boldsymbol{L}_1$ with $\boldsymbol{T}_e \not\le Z^\circ(\boldsymbol{L})_e = Z^\circ(\boldsymbol{L}_1)_e$ we get that ${}^*R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\lambda_1)$ is a constituent of ${}^*R_{\boldsymbol{T}}^{\boldsymbol{L}}(\lambda) = 0$ by e-Jordan-cuspidality of λ. Here note that the set of constituents of ${}^*R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\eta)$, where η is a constituent of ${}^*R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda)$ different from λ_1, is disjoint from the set of irreducible constituents of ${}^*R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\lambda_1)$.

Thus we have obtained a well-defined map ${}^*\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ from e-Jordan-cuspidal pairs in $\boldsymbol{G}$ to e-Jordan-cuspidal pairs in $\boldsymbol{G}_1$, both below the series s. As the map $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ preserves the e-part of the centre, ${}^*\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}} \circ \Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ is the identity. It remains to prove that $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ is surjective. For this, let $(\boldsymbol{M}, \mu)$ be any e-Jordan-cuspidal pair of $\boldsymbol{G}$ below $\mathcal{E}(\boldsymbol{G}^F, s)$, let $(\boldsymbol{L}_1, \lambda_1) = {}^*\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}(\boldsymbol{M}, \mu)$ and $(\boldsymbol{L}, \lambda) = \Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}(\boldsymbol{L}_1, \lambda_1)$. Then we have $Z^\circ(\boldsymbol{M})_e \le Z^\circ(\boldsymbol{L}_1)_e = Z^\circ(\boldsymbol{L})_e$, so $\boldsymbol{L} = C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L})_e) \le C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{M})_e) = \boldsymbol{M}$ is an e-split Levi subgroup of $\boldsymbol{M}$. As $\boldsymbol{L}_1 \le \boldsymbol{L} \le \boldsymbol{M}$ and $\epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{M}} R_{\boldsymbol{L}_1}^{\boldsymbol{M}}$ is a bijection from $\mathcal{E}(\boldsymbol{L}_1^F, s)$ to $\mathcal{E}(\boldsymbol{M}^F, s)$, it follows that $\epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{M}} R_{\boldsymbol{L}}^{\boldsymbol{M}}$ is a bijection between $\mathcal{E}(\boldsymbol{L}^F, s)$ and $\mathcal{E}(\boldsymbol{M}^F, s)$. As λ and μ are e-Jordan-cuspidal, (J_1) implies that $Z^\circ(\boldsymbol{M}^*)_e = Z^\circ(\boldsymbol{L}^*)_e$, so $\boldsymbol{M} = \boldsymbol{L}$, that is, $(\boldsymbol{M}, \mu)$ is in the image of $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$. The proof is complete. $\square$

The above bijection also preserves relative Weyl groups.

Lemma 2.5. *In the situation and notation of Proposition 2.4 let* $(\boldsymbol{L}, \lambda) = \Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}(\boldsymbol{L}_1, \lambda_1)$. *Then* $N_{\boldsymbol{G}_1^F}(\boldsymbol{L}_1, \lambda_1) \le N_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ *and this inclusion induces an isomorphism of relative Weyl groups* $W_{\boldsymbol{G}_1^F}(\boldsymbol{L}_1, \lambda_1) \cong W_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$.

Proof. Let $g \in N_{\boldsymbol{G}_1^F}(\boldsymbol{L}_1, \lambda_1)$. Then g normalises $Z^\circ(\boldsymbol{L}_1)_e$ and hence also $\boldsymbol{L} = C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L}_1)_e)$. Thus,

$$ {}^g\lambda = \epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R_{{}^g\boldsymbol{L}_1}^{{}^g\boldsymbol{L}}({}^g\lambda_1) = \epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda_1) = \lambda $$

and the first assertion follows.

For the second assertion, let $g \in N_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ and let $\boldsymbol{T}$ be an F-stable maximal torus of $\boldsymbol{L}_1$ and θ an irreducible character of $\boldsymbol{T}^F$ such that λ_1 is a constituent of $R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\theta)$. Since $\lambda_1 \in \mathcal{E}(\boldsymbol{L}_1^F, s)$, $(\boldsymbol{T}, \theta)$ corresponds via duality (between $\boldsymbol{L}_1$ and $\boldsymbol{L}_1^*$) to the $\boldsymbol{L}_1^{*F}$-class of s, and all constituents of $R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\theta)$ are in $\mathcal{E}(\boldsymbol{L}_1^F, s)$. Consequently, $R_{\boldsymbol{L}_1}^{\boldsymbol{L}}$ induces a bijection between the set of constituents of $R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\theta)$ and the set of constituents of $R_{\boldsymbol{T}}^{\boldsymbol{L}}(\theta)$. In particular, λ is a constituent of $R_{\boldsymbol{T}}^{\boldsymbol{L}}(\theta)$. Since g stabilises λ, λ is also a constituent of $R_{{}^g\boldsymbol{T}}^{\boldsymbol{L}}({}^g\theta)$. Hence $(\boldsymbol{T}, \theta)$ and ${}^g(\boldsymbol{T}, \theta)$ are geometrically conjugate in $\boldsymbol{L}$. Let $l \in \boldsymbol{L}$ geometrically conjugate ${}^g(\boldsymbol{T}, \theta)$ to $(\boldsymbol{T}, \theta)$. Since $C_{\boldsymbol{G}^*}(s) \le \boldsymbol{G}_1^*$, we have $lg \in \boldsymbol{G}_1$ (see for instance [Kessar and Malle 2013, Lemma 7.5]). Hence $F(l)l^{-1} = F(lg)(lg)^{-1} \in \boldsymbol{G}_1 \cap \boldsymbol{L} = \boldsymbol{L}_1$. By the Lang–Steinberg theorem applied to $\boldsymbol{L}_1$, there exists $l_1 \in \boldsymbol{L}_1$ such that $l_1 l \in \boldsymbol{L}^F$. Also, since $l_1 \in \boldsymbol{G}_1$ and $g \in \boldsymbol{G}^F$, $l_1 lg \in \boldsymbol{G}_1^F$. Thus, up to replacing g by $l_1 lg$, we may assume that $g \in \boldsymbol{G}_1^F$.

Since $\boldsymbol{L}_1 = C_{\boldsymbol{G}_1}(Z^\circ(\boldsymbol{L})_e)$, it follows that $g \in N_{\boldsymbol{G}_1^F}(\boldsymbol{L}_1)$, and thus

$$ \epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda_1) = \lambda = {}^g\lambda = \epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R_{\boldsymbol{L}_1}^{\boldsymbol{L}}({}^g\lambda_1). $$

Since $R_{L_1}^{L}$ induces a bijection between the set of characters in the geometric Lusztig series of L_1^F corresponding to s (the union of series $\mathcal{E}(L_1^F, t)$, where t runs over the semisimple elements of L_1^{*F} which are L_1-conjugate to s) and the set of characters in the geometric Lusztig series of L^F corresponding to s, it suffices to prove that ${}^g\lambda_1 \in \mathcal{E}(L_1^F, t)$ for some $t \in L_1^{*F}$ which is L_1^{*F}-conjugate to s. Let T, θ and l be as above. Since $lg \in G_1$ and $g \in G_1$, it follows that $l \in G_1 \cap L = L_1$. Hence ${}^g(T, \theta)$ and (T, θ) are geometrically conjugate in L_1. The claim follows as ${}^g\lambda_1$ is a constituent of $R_{{}^gT}^{L_1}({}^g\theta)$. □

e-Jordan-cuspidality and ℓ-blocks. We next investigate the behaviour of ℓ-blocks with respect to the map $\Psi_{G_1}^{G}$. For this, let $\ell \neq p$ be a prime. We set

$$e_\ell(q) := \text{order of } q \text{ modulo } \begin{cases} \ell & \text{if } \ell \neq 2, \\ 4 & \text{if } \ell = 2. \end{cases}$$

For a semisimple ℓ'-element s of G^{*F}, we denote by $\mathcal{E}_\ell(G^F, s)$ the union of all Lusztig series $\mathcal{E}(G^F, st)$, where $t \in G^{*F}$ is an ℓ-element commuting with s. We recall that the set $\mathcal{E}_\ell(G^F, s)$ is a union of ℓ-blocks. Further, if $G_1 \leq G$ is an F-stable Levi subgroup such that G_1^* contains $C_{G^*}(s)$, then $\epsilon_{G_1}\epsilon_G R_{G_1}^{G}$ induces a bijection, which we refer to as the *Jordan correspondence*, between the ℓ-blocks in $\mathcal{E}(G_1^F, s)$ and the ℓ-blocks in $\mathcal{E}(G^F, s)$, see [Broué 1990, §2A].

Proposition 2.6. *Let $\ell \neq p$ be a prime, $s \in G^{*F}$ a semisimple ℓ'-element and $G_1 \leq G$ an F-stable Levi subgroup with G_1^* containing $C_{G^*}(s)$. Assume that b is an ℓ-block in $\mathcal{E}_\ell(G^F, s)$, and c its Jordan corresponding block in $\mathcal{E}_\ell(G_1^F, s)$. Let $e := e_\ell(q)$.*

(a) *Let (L_1, λ_1) be e-Jordan-cuspidal in G_1 and set $(L, \lambda) = \Psi_{G_1}^{G}(L_1, \lambda_1)$. If all constituents of $R_{L_1}^{G_1}(\lambda_1)$ lie in c, then all constituents of $R_L^G(\lambda)$ lie in b.*

(b) *Let (L, λ) be e-Jordan-cuspidal in G and set $(L_1, \lambda_1) = {}^*\Psi_{G_1}^{G}(L, \lambda)$. If all constituents of $R_L^G(\lambda)$ lie in b, then all constituents of $R_{L_1}^{G_1}(\lambda_1)$ lie in c.*

Proof. Note that the hypothesis of part (a) means that for any parabolic subgroup P of G_1 containing L_1 as Levi subgroup, all constituents of $R_{L_1 \leq P}^{G_1}(\lambda_1)$ lie in c. A similar remark applies to the conclusion, as well as to part (b).

For (a), note that by the definition of $\Psi_{G_1}^{G}$ we have that all constituents of

$$\epsilon_L \epsilon_{L_1} R_L^G(\lambda) = R_{L_1}^{G}(\lambda_1) = R_{G_1}^{G} R_{L_1}^{G_1}(\lambda_1)$$

are contained in $R_{G_1}^{G}(c)$, hence in b by Jordan correspondence.

In (b), suppose that η is a constituent of $R_{L_1}^{G_1}(\lambda_1)$ not lying in c. Then by Jordan correspondence, $R_{G_1}^{G}(\eta)$ does not belong to b, whence $R_{L_1}^{G}(\lambda_1)$ has a constituent not lying in b, contradicting our assumption that all constituents of $R_{L_1}^{G}(\lambda_1) = R_L^G R_{L_1}^{L}(\lambda_1) = \epsilon_L \epsilon_{L_1} R_L^G(\lambda)$ are in b. □

e-quasicentrality. For a prime ℓ not dividing q, we denote by $\mathcal{E}(\boldsymbol{G}^F, \ell')$ the set of irreducible characters of $\boldsymbol{G}^F$ lying in a Lusztig series $\mathcal{E}(\boldsymbol{G}^F, s)$, where $s \in \boldsymbol{G}^{*F}$ is a semisimple ℓ'-element. Recall from [Kessar and Malle 2013, Definition 2.4] that a character $\chi \in \mathcal{E}(\boldsymbol{G}^F, \ell')$ is said to be of *central ℓ-defect* if the ℓ-block of $\boldsymbol{G}^F$ containing χ has a central defect group and χ is said to be of *quasicentral ℓ-defect* if some (and hence any) character of $[\boldsymbol{G}, \boldsymbol{G}]^F$ covered by χ is of central ℓ-defect.

Lemma 2.7. *Let $\boldsymbol{L}$ be an F-stable Levi subgroup of $\boldsymbol{G}$, and set $\boldsymbol{L}_0 = \boldsymbol{L} \cap [\boldsymbol{G}, \boldsymbol{G}]$. Let $\ell \neq p$ be a prime.*

(a) *If $\boldsymbol{L}_0 = C_{[\boldsymbol{G},\boldsymbol{G}]}(Z(\boldsymbol{L}_0)^F_\ell)$, then $\boldsymbol{L} = C_{\boldsymbol{G}}(Z(\boldsymbol{L})^F_\ell)$.*

(b) *Let $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ and let λ_0 be an irreducible constituent of $\mathrm{Res}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda)$. Then λ_0 is of quasicentral ℓ-defect if and only if λ is of quasicentral ℓ-defect.*

Proof. Since $\boldsymbol{G} = Z^\circ(\boldsymbol{G})[\boldsymbol{G}, \boldsymbol{G}]$ and $Z^\circ(\boldsymbol{G}) \le \boldsymbol{L}$, we have that $\boldsymbol{L} = Z^\circ(\boldsymbol{G})\boldsymbol{L}_0$. Hence if $\boldsymbol{L}_0 = C_{[\boldsymbol{G},\boldsymbol{G}]}(Z(\boldsymbol{L}_0)^F_\ell)$, then $\boldsymbol{L} = C_{\boldsymbol{G}}(Z(\boldsymbol{L}_0)^F_\ell) \supseteq C_{\boldsymbol{G}}(Z(\boldsymbol{L})^F_\ell) \supseteq \boldsymbol{L}$. This proves (a). In (b), since λ is in an ℓ'-Lusztig series, the index in $\boldsymbol{L}^F$ of the stabiliser in $\boldsymbol{L}^F$ of λ_0 is prime to ℓ and on the other hand, λ_0 extends to a character of the stabiliser in $\boldsymbol{L}^F$ of λ_0. Thus, $\lambda(1)_\ell = \lambda_0(1)_\ell$. Since $[\boldsymbol{L}_0, \boldsymbol{L}_0] = [\boldsymbol{L}, \boldsymbol{L}]$, the assertion follows by [Kessar and Malle 2013, Proposition 2.5(a)]. □

Remark 2.8. The converse of assertion (a) of Lemma 2.7 fails in general, even when we restrict to $e_\ell(q)$-split Levi subgroups: let ℓ be odd and $\boldsymbol{G} = \mathrm{GL}_\ell$ with F such that $\boldsymbol{G}^F = \mathrm{GL}_\ell(q)$ with $\ell \mid (q-1)$. Let $\boldsymbol{L}$ a 1-split Levi subgroup of type $\mathrm{GL}_{\ell-1} \times \mathrm{GL}_1$. Then $Z(\boldsymbol{L})^F_\ell \cong C_\ell \times C_\ell$ and $\boldsymbol{L} = C_{\boldsymbol{G}}(Z(\boldsymbol{L})^F_\ell)$. But $Z(\boldsymbol{L}_0)^F_\ell \cong C_\ell \cong Z([\boldsymbol{G}, \boldsymbol{G}])^F_\ell$, hence $C_{[\boldsymbol{G},\boldsymbol{G}]}(Z(\boldsymbol{L}_0)^F_\ell) = [\boldsymbol{G}, \boldsymbol{G}]$.

One might hope for further good properties of the bijection of Proposition 2.6 with respect to (quasi-)centrality. In this direction, we observe the following:

Lemma 2.9. *In the situation of Proposition 2.4, if $(\boldsymbol{L}, \lambda)$ is of central ℓ-defect for a prime ℓ with $e_\ell(q) = e$, then so is $(\boldsymbol{L}_1, \lambda_1) = {}^*\Psi^{\boldsymbol{G}}_{\boldsymbol{G}_1}(\boldsymbol{L}, \lambda)$, and we have $Z(\boldsymbol{L})^F_\ell = Z(\boldsymbol{L}_1)^F_\ell$.*

Proof. By assumption, we have that $\lambda(1)_\ell = |\boldsymbol{L}^F : Z(\boldsymbol{L})^F|_\ell$. Now $Z(\boldsymbol{L})$ lies in every maximal torus of $\boldsymbol{L}$, hence in $\boldsymbol{L}_1$, so we have that $Z(\boldsymbol{L})^F_\ell \le Z(\boldsymbol{L}_1)^F_\ell$. As $\lambda = \epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R^{\boldsymbol{L}}_{\boldsymbol{L}_1}(\lambda_1)$, we obtain $\lambda(1)_\ell = \lambda_1(1)_\ell |\boldsymbol{L}^F : \boldsymbol{L}_1^F|_\ell$, whence

$$\lambda_1(1)_\ell = \lambda(1)_\ell |\boldsymbol{L}^F : \boldsymbol{L}_1^F|_\ell^{-1} = |\boldsymbol{L}_1^F|_\ell |Z(\boldsymbol{L})^F|_\ell^{-1} \ge |\boldsymbol{L}_1^F : Z(\boldsymbol{L}_1)^F|_\ell.$$

But clearly $\lambda_1(1)_\ell \le |\boldsymbol{L}_1^F : Z(\boldsymbol{L}_1)^F|_\ell$, so we have equality throughout, as claimed. □

Example 2.10. The converse of Lemma 2.9 does not hold in general. To see this, let $\boldsymbol{G} = \mathrm{PGL}_\ell$ with $\boldsymbol{G}^F = \mathrm{PGL}_\ell(q)$, $\boldsymbol{L} = \boldsymbol{G}$, and $\boldsymbol{G}_1 \le \boldsymbol{G}$ an F-stable maximal torus such that $\boldsymbol{G}_1^F$ is a Coxeter torus of $\boldsymbol{G}^F$, of order Φ_ℓ. Assume that $\ell \mid (q-1)$

(so $e = 1$). Then $\boldsymbol{L}_1 = \boldsymbol{G}_1$. Here, any $\lambda_1 \in \mathrm{Irr}(\boldsymbol{L}_1^F)$ is e-(Jordan-)cuspidal, and certainly of central ℓ-defect, and $|Z(\boldsymbol{L}_1)_\ell^F| = (\Phi_\ell)_\ell = \ell$ for $\ell \geq 3$, while clearly $Z(\boldsymbol{L})_\ell^F = Z(\boldsymbol{G})_\ell^F = 1$. Furthermore

$$\lambda(1)_\ell = \lambda_1(1)_\ell [\boldsymbol{L}^F : \boldsymbol{L}_1^F]_\ell = [\boldsymbol{L}^F : \boldsymbol{L}_1^F]_\ell,$$

since λ_1 is linear. Since $|Z(\boldsymbol{L}^F)|_\ell = 1$ and $|\boldsymbol{L}_1^F|_\ell > 1$, it follows that

$$\lambda(1)_\ell |Z(\boldsymbol{L}^F)|_\ell < |\boldsymbol{L}^F|_\ell,$$

hence λ is not of central ℓ-defect (and not even of quasicentral ℓ-defect).

Example 2.11. We also recall that e-(Jordan-)cuspidal characters are not always of central ℓ-defect, even when ℓ is a good prime: let $\boldsymbol{G}^F = \mathrm{SL}_{\ell^2}(q)$ with $\ell \mid (q-1)$, so $e = 1$. Then for $\boldsymbol{T}$ a Coxeter torus and $\theta \in \mathrm{Irr}(\boldsymbol{T}^F)$ in general position, $R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)$ is e-(Jordan-) cuspidal but not of quasicentral ℓ-defect.

For the next definition note that the property of being of (quasi)-central ℓ-defect is invariant under automorphisms of $\boldsymbol{G}^F$.

Definition 2.12. Let $\ell \neq p$ be a prime and $e = e_\ell(q)$. A character $\chi \in \mathcal{E}(\boldsymbol{G}^F, \ell')$ is called *e-Jordan quasicentral cuspidal* if χ is e-Jordan cuspidal and the $C_{\boldsymbol{G}^*}(s)^F$-orbit of unipotent characters of $C^\circ_{\boldsymbol{G}^*}(s)^F$ which corresponds to χ under Jordan decomposition consists of characters of quasicentral ℓ-defect, where $s \in \boldsymbol{G}^{*F}$ is a semisimple ℓ'-element such that $\chi \in \mathcal{E}(\boldsymbol{G}^F, s)$. An *e-Jordan quasicentral cuspidal pair of* $\boldsymbol{G}$ is a pair $(\boldsymbol{L}, \lambda)$ such that $\boldsymbol{L}$ is an e-split Levi subgroup of $\boldsymbol{G}$ and $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ is an e-Jordan quasicentral cuspidal character of $\boldsymbol{L}^F$.

We note that the set of e-Jordan quasicentral cuspidal pairs of $\boldsymbol{G}$ is closed under $\boldsymbol{G}^F$-conjugation. Also, note that Lemma 2.3 remains true upon replacing the e-Jordan-cuspidal property by the e-Jordan quasicentral cuspidal property. This is because, with the notation of Lemma 2.3, the orbit of unipotent characters corresponding to λ under Jordan decomposition is a subset of the orbit of unipotent characters corresponding to λ_0 under Jordan decomposition. Finally we note that the bijection $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ of Proposition 2.6 preserves e-quasicentrality since, with the notation of the proposition, λ_1 and λ correspond to the same orbit of unipotent characters under Jordan decomposition.

3. Lusztig induction and ℓ-blocks

Here we prove our main results on the parametrisation of ℓ-blocks in terms of e-Harish-Chandra series, in arbitrary Levi subgroups of simple groups of simply connected type. As in Section 2, $\ell \neq p$ will be prime numbers, q a power of p and $e = e_\ell(q)$.

Preservation of ℓ-blocks by Lusztig induction. We first extend [Cabanes and Enguehard 1999, Theorem 2.5]. The proof will require three auxiliary results:

Lemma 3.1. *Let $\boldsymbol{G}$ be connected reductive with a Frobenius endomorphism F endowing $\boldsymbol{G}$ with an $\mathbb{F}_q$-rational structure. Let $\boldsymbol{M}$ be an e-split Levi of $\boldsymbol{G}^F$ and c an ℓ-block of $\boldsymbol{M}^F$. Suppose that*

(1) *the set $\{d^{1,\boldsymbol{M}^F}(\mu) \mid \mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')\}$ is linearly independent*; *and*

(2) *there exists a subgroup $Z \le Z(\boldsymbol{M})_\ell^F$ and a block d of $C_{\boldsymbol{G}}^\circ(Z)^F$ such that all irreducible constituents of $R_{\boldsymbol{M}}^{C_{\boldsymbol{G}}^\circ(Z)}(\mu)$, where $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$, lie in the block d.*

Then there exists a block b of $\boldsymbol{G}^F$ such that all irreducible constituents of $R_{\boldsymbol{M}}^{\boldsymbol{G}}(\mu)$, where $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$, lie in the block b.

Proof. We adapt the argument of [Kessar and Malle 2013, Proposition 2.16]. Let $\chi \in \mathrm{Irr}(\boldsymbol{G}^F, \ell')$ be such that $\langle R_{\boldsymbol{M}}^{\boldsymbol{G}}(\mu), \chi\rangle \neq 0$ for some $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$. Then $\langle \mu, {}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)\rangle \neq 0$. In particular, $c.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi) \neq 0$. All constituents of ${}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)$ lie in $\mathcal{E}(\boldsymbol{M}^F, \ell')$, so by assumption (1) it follows that $d^{1,\boldsymbol{M}^F}(c.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)) \neq 0$. Since $d^{1,\boldsymbol{M}^F}(c.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi))$ vanishes on ℓ-singular elements of $\boldsymbol{M}^F$, we have that

$$\langle d^{1,\boldsymbol{M}^F}(c.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)), c.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)\rangle = \langle d^{1,\boldsymbol{M}^F}(c.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)), d^{1,\boldsymbol{M}^F}(c.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi))\rangle \neq 0.$$

If φ and φ' are irreducible ℓ-Brauer characters of $\boldsymbol{M}^F$ lying in different ℓ-blocks of $\boldsymbol{M}^F$, then $\langle \varphi, \varphi'\rangle = 0$ (see for instance [Nagao and Tsushima 1989, Chapter 3, Exercise 6.20(ii)]). Thus,

$$\langle d^{1,\boldsymbol{M}^F}(c.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)), c'.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)\rangle = \langle d^{1,\boldsymbol{M}^F}(c.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)), d^{1,\boldsymbol{M}^F}(c'.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)\rangle = 0$$

for all blocks c' of $\boldsymbol{M}^F$ different from c. So, $\langle d^{1,\boldsymbol{M}^F}(c.{}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)), {}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)\rangle \neq 0$ from which it follows that $\langle d^{1,\boldsymbol{M}^F}(\mu'), {}^*R_{\boldsymbol{M}}^{\boldsymbol{G}}(\chi)\rangle \neq 0$ for some $\mu' \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$.

Continuing as in the proof of [Kessar and Malle 2013, Proposition 2.12] gives the required result. Note that condition (1) of this proposition is not necessarily met as stated, since μ' may be different from μ. However, μ and μ' are in the same block of $\boldsymbol{M}^F$ which is sufficient to obtain the conclusion of the lemma. □

Lemma 3.2. *Let $\boldsymbol{G}$ be connected reductive with a Frobenius endomorphism F. Suppose that $\boldsymbol{G}$ has connected centre and $[\boldsymbol{G}, \boldsymbol{G}]$ is simply connected. Let $\boldsymbol{G} = \boldsymbol{X}\boldsymbol{Y}$ such that either $\boldsymbol{X}$ is an F-stable product of components of $[\boldsymbol{G}, \boldsymbol{G}]$ and $\boldsymbol{Y}$ is the product of the remaining components with $Z(\boldsymbol{G})$, or vice versa. Suppose further that $\boldsymbol{G}^F/\boldsymbol{X}^F\boldsymbol{Y}^F$ is an ℓ-group. Let $\boldsymbol{N}$ be an F-stable Levi subgroup of $\boldsymbol{Y}$ and set $\boldsymbol{M} = \boldsymbol{X}\boldsymbol{N}$. Let c be an ℓ-block of $\boldsymbol{M}^F$ and let c' be an ℓ-block of $\boldsymbol{N}^F$ covered by c. Suppose that there exists a block b' of $\boldsymbol{Y}^F$ such that every irreducible constituent of $R_{\boldsymbol{N}}^{\boldsymbol{Y}}(\tau)$ where $\tau \in \mathrm{Irr}(c') \cap \mathcal{E}(\boldsymbol{N}^F, \ell')$ lies in b'. Then there exists a block b of $\boldsymbol{G}^F$ such that every irreducible constituent of $R_{\boldsymbol{M}}^{\boldsymbol{G}}(\mu)$ where $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$ lies in b.*

Proof. We will use the extension of Lusztig induction to certain disconnected groups as in [Cabanes and Enguehard 1999, Section 1.1]. Let

$$\begin{aligned} \boldsymbol{G}_0 &= [\boldsymbol{G}, \boldsymbol{G}] = [\boldsymbol{X}, \boldsymbol{X}] \times [\boldsymbol{Y}, \boldsymbol{Y}], \\ \boldsymbol{M}_0 &= \boldsymbol{G}_0 \cap \boldsymbol{M} = [\boldsymbol{X}, \boldsymbol{X}] \times ([\boldsymbol{Y}, \boldsymbol{Y}] \cap \boldsymbol{N}). \end{aligned}$$

Then, $\boldsymbol{G}_0^F \subseteq \boldsymbol{X}^F\boldsymbol{Y}^F$ and $\boldsymbol{M}_0^F \subseteq \boldsymbol{X}^F\boldsymbol{N}^F$. Let $\boldsymbol{T}$ be an F-stable maximal torus of $\boldsymbol{M}$. Since $\boldsymbol{G}$ and hence also $\boldsymbol{M}$ has connected centre, $\boldsymbol{M} = \boldsymbol{M}_0^F\boldsymbol{T}^F$ and $\boldsymbol{G}^F = \boldsymbol{G}_0^F\boldsymbol{T}^F$. Further, $A := \boldsymbol{X}^F\boldsymbol{Y}^F \cap \boldsymbol{T}^F = \boldsymbol{X}^F\boldsymbol{N}^F \cap \boldsymbol{T}^F$ and $\boldsymbol{X}^F\boldsymbol{Y}^F = \boldsymbol{G}_0^F A = (\boldsymbol{G}_0 A)^F$, $\boldsymbol{X}^F\boldsymbol{N}^F = \boldsymbol{M}_0^F A = (\boldsymbol{M}_0 A)^F$. As in [Cabanes and Enguehard 1999, Section 1.1], we denote by $\mathcal{E}(\boldsymbol{X}^F\boldsymbol{Y}^F, \ell')$ the set of irreducible characters of $\boldsymbol{X}^F\boldsymbol{Y}^F$ that appear in the restriction of elements of $\mathcal{E}(\boldsymbol{G}^F, \ell')$ to $\boldsymbol{X}^F\boldsymbol{Y}^F$.

Let $\chi \in \mathcal{E}(\boldsymbol{G}^F, \ell')$. Since $\boldsymbol{G}^F/\boldsymbol{X}^F\boldsymbol{Y}^F$ is an ℓ-group, by [Cabanes and Enguehard 1999, Proposition 1.3(i)], $\mathrm{Res}^{\boldsymbol{G}^F}_{\boldsymbol{X}^F\boldsymbol{Y}^F}(\chi)$ is irreducible. Now if $\chi' \in \mathrm{Irr}(\boldsymbol{G}^F)$ has the same restriction to $\boldsymbol{X}^F\boldsymbol{Y}^F$ as χ, then again since $\boldsymbol{G}^F/\boldsymbol{X}^F\boldsymbol{Y}^F$ is an ℓ-group, either $\chi' = \chi$ or $\chi' \notin \mathcal{E}(\boldsymbol{G}^F, \ell')$. In other words, the restriction from $\mathbb{Z}\mathcal{E}(\boldsymbol{G}^F, \ell')$ to $\mathbb{Z}\mathcal{E}(\boldsymbol{X}^F\boldsymbol{Y}^F, \ell')$ is a bijection. Similarly, the restriction from $\mathbb{Z}\mathcal{E}(\boldsymbol{M}^F, \ell')$ to $\mathbb{Z}\mathcal{E}(\boldsymbol{X}^F\boldsymbol{N}^F, \ell')$ is a bijection.

In particular, every block of $\boldsymbol{G}^F$ covers a unique block of $\boldsymbol{X}^F\boldsymbol{Y}^F$. Since $\boldsymbol{G}^F/\boldsymbol{X}^F\boldsymbol{Y}^F$ is an ℓ-group, there is a bijection (through covering) between the set of blocks of $\boldsymbol{G}^F$ and the set of blocks of $\boldsymbol{X}^F\boldsymbol{Y}^F$. Hence, by the injectivity of restriction from $\mathbb{Z}\mathcal{E}(\boldsymbol{G}^F, \ell')$ to $\mathbb{Z}\mathcal{E}(\boldsymbol{X}^F\boldsymbol{Y}^F, \ell')$, it suffices to prove that there is a block b_0 of $\boldsymbol{X}^F\boldsymbol{Y}^F$ such that every irreducible constituent of $\mathrm{Res}^{\boldsymbol{G}^F}_{\boldsymbol{X}^F\boldsymbol{Y}^F} R^{\boldsymbol{G}}_{\boldsymbol{M}}(\mu)$ as μ ranges over $\mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$ lies in b_0.

Following [Cabanes and Enguehard 1999, Section 1.1], we have $\mathrm{Res}^{\boldsymbol{G}^F}_{\boldsymbol{X}^F\boldsymbol{Y}^F} R^{\boldsymbol{G}}_{\boldsymbol{M}} = R^{\boldsymbol{G}_0 A}_{\boldsymbol{M}_0 A} \mathrm{Res}^{\boldsymbol{M}^F}_{\boldsymbol{X}^F\boldsymbol{N}^F}$ on $\mathrm{Irr}(\boldsymbol{M}^F)$ (where here $R^{\boldsymbol{G}_0 A}_{\boldsymbol{M}_0 A}$ is Lusztig induction in the disconnected setting). Thus, it suffices to prove that there is a block b_0 of $\boldsymbol{X}^F\boldsymbol{Y}^F$ such that every irreducible constituent of $R^{\boldsymbol{G}_0 A}_{\boldsymbol{M}_0 A}\mathrm{Res}^{\boldsymbol{M}^F}_{\boldsymbol{X}^F\boldsymbol{N}^F}(\mu)$ as μ ranges over $\mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$ is contained in b_0.

By the above arguments applied to $\boldsymbol{M}^F$ and $\boldsymbol{X}^F\boldsymbol{N}^F$, there is a unique block c_0 of $\boldsymbol{X}^F\boldsymbol{N}^F$ covered by c. The surjectivity of restriction from $\mathbb{Z}\mathcal{E}(\boldsymbol{M}^F, \ell')$ to $\mathbb{Z}\mathcal{E}(\boldsymbol{X}^F\boldsymbol{N}^F, \ell')$ implies that it suffices to prove that there is a block b_0 of $\boldsymbol{X}^F\boldsymbol{Y}^F$ such that every irreducible constituent of $R^{\boldsymbol{G}_0 A}_{\boldsymbol{M}_0 A}(\mu)$ for $\mu \in \mathrm{Irr}(c_0) \cap \mathcal{E}(\boldsymbol{X}^F\boldsymbol{N}^F, \ell')$ is contained in b_0.

The group $I := \{(x, x^{-1}) \mid x \in \boldsymbol{X}^F \cap \boldsymbol{Y}^F\} \leq \boldsymbol{X} \times \boldsymbol{Y}$ is the kernel of the multiplication map $\boldsymbol{X}^F \times \boldsymbol{Y}^F \to \boldsymbol{X}^F\boldsymbol{Y}^F$. Identifying $\boldsymbol{X}^F\boldsymbol{Y}^F$ with $\boldsymbol{X}^F \times \boldsymbol{Y}^F/I$ through multiplication, $\mathrm{Irr}(\boldsymbol{X}^F\boldsymbol{Y}^F)$ is the subset of $\mathrm{Irr}(\boldsymbol{X}^F \times \boldsymbol{Y}^F)$ consisting of characters whose kernel contains I. Since $\boldsymbol{X}^F \cap \boldsymbol{Y}^F \leq \boldsymbol{X} \cap \boldsymbol{Y} \leq Z(\boldsymbol{G}) \leq \boldsymbol{M}$, I is also the kernel of the multiplication map $\boldsymbol{X}^F \times \boldsymbol{N}^F \to \boldsymbol{X}^F\boldsymbol{N}^F$ and we may identify $\mathrm{Irr}(\boldsymbol{X}^F\boldsymbol{Y}^F)$ with the subset of $\mathrm{Irr}(\boldsymbol{X}^F \times \boldsymbol{N}^F)$ consisting of characters whose kernel contains I.

Any parabolic subgroup of $\boldsymbol{G}_0$ containing $\boldsymbol{M}_0$ as Levi subgroup is of the form $[\boldsymbol{X}, \boldsymbol{X}]\boldsymbol{P}$, where $\boldsymbol{P}$ is a parabolic subgroup of $[\boldsymbol{Y}, \boldsymbol{Y}]$ containing $\boldsymbol{N}\cap[\boldsymbol{Y}, \boldsymbol{Y}]$ as Levi subgroup. Let $\boldsymbol{U} := R_u(\boldsymbol{X}\boldsymbol{P}) = R_u(\boldsymbol{P}) \le [\boldsymbol{Y}, \boldsymbol{Y}]$ and denote by $\mathcal{L}^{-1}(\boldsymbol{U})$ the inverse image of $\boldsymbol{U}$ under the Lang map $\boldsymbol{G} \to \boldsymbol{G}$ given by $g \mapsto g^{-1}F(g)$.

The Deligne–Lusztig variety associated to $R_{\boldsymbol{M}_0 A}^{\boldsymbol{G}_0 A}$ (with respect to $\boldsymbol{X}\boldsymbol{P}$) is

$$\mathcal{L}^{-1}(\boldsymbol{U}) \cap \boldsymbol{G}_0 A.$$

Since $\boldsymbol{T} = (\boldsymbol{T} \cap \boldsymbol{M}_0)Z(\boldsymbol{G})$, $\boldsymbol{U}$ is normalised by $\boldsymbol{T}$ and in particular by A. Hence,

$$\begin{aligned}\mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{G}_0 A = (\mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{G}_0)A &= [\boldsymbol{X}, \boldsymbol{X}]^F(\mathcal{L}^{-1}(\boldsymbol{U})\cap[\boldsymbol{Y}, \boldsymbol{Y}])A \\ &= [\boldsymbol{X}, \boldsymbol{X}]^F(A\cap\boldsymbol{X}^F)(\mathcal{L}^{-1}(\boldsymbol{U})\cap[\boldsymbol{Y}, \boldsymbol{Y}])(A\cap\boldsymbol{Y}^F).\end{aligned}$$

For the last equality, note that

$$A = \boldsymbol{X}^F\boldsymbol{Y}^F \cap \boldsymbol{T} = (\boldsymbol{X}^F \cap \boldsymbol{T})(\boldsymbol{Y}^F \cap \boldsymbol{T}) = (\boldsymbol{X}^F \cap A)(\boldsymbol{Y}^F \cap A).$$

Now, $\mathcal{L}^{-1}(\boldsymbol{U}) \cap \boldsymbol{Y} = (\mathcal{L}^{-1}(\boldsymbol{U}) \cap [\boldsymbol{Y}, \boldsymbol{Y}])\boldsymbol{S}^F$ for any F-stable maximal torus $\boldsymbol{S}$ of $\boldsymbol{Y}$. Applying this with $\boldsymbol{S} = \boldsymbol{T} \cap \boldsymbol{Y}$, we have $(\mathcal{L}^{-1}(\boldsymbol{U}) \cap [\boldsymbol{Y}, \boldsymbol{Y}])(A \cap \boldsymbol{Y}^F) = \mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{Y}$. Also, $[\boldsymbol{X}, \boldsymbol{X}]^F(A\cap\boldsymbol{X}^F) = \boldsymbol{X}^F$. Altogether this gives $\mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{G}_0 A = \boldsymbol{X}^F(\mathcal{L}^{-1}(\boldsymbol{U}) \cap \boldsymbol{Y})$. Further, $\mathcal{L}^{-1}(\boldsymbol{U}) \cap \boldsymbol{Y}$ is the variety underlying $R_{\boldsymbol{N}}^{\boldsymbol{Y}}$ (with respect to the parabolic subgroup $\boldsymbol{P}Z(\boldsymbol{G})$). Hence, for any $\tau_1 \in \mathrm{Irr}(\boldsymbol{X}^F)$, $\tau_2 \in \mathrm{Irr}(\boldsymbol{Y}^F)$ such that I is in the kernel of $\tau_1\tau_2$, we have

$$R_{\boldsymbol{M}_0 A}^{\boldsymbol{G}_0 A}(\tau_1\tau_2) = \tau_1 R_{\boldsymbol{N}}^{\boldsymbol{Y}}(\tau_2).$$

Further, $\tau_1\tau_2 \in \mathcal{E}(\boldsymbol{X}^F\boldsymbol{N}^F, \ell')$ if and only if $\tau_1 \in \mathcal{E}(\boldsymbol{X}^F, \ell')$ and $\tau_2 \in \mathcal{E}(\boldsymbol{N}^F, \ell')$.

To conclude note that c' is the unique block of $\boldsymbol{N}^F$ covered by c_0 and $c_0 = dc'$, where d is a block $\boldsymbol{X}^F$. Let b' be the block of $\boldsymbol{Y}^F$ in the hypothesis. Then, setting $b_0 = db'$ gives the desired result. □

We will also make use of the following well-known extension of [Enguehard 2008, Proposition 1.5].

Lemma 3.3. *Suppose that q is odd. Let $\boldsymbol{G}$ be connected reductive with a Frobenius endomorphism F. Suppose that all components of $\boldsymbol{G}$ are of classical type A, B, C or D and that $Z(\boldsymbol{G})/Z^\circ(\boldsymbol{G})$ is a 2-group. Let $s \in \boldsymbol{G}^{*F}$ be semisimple of odd order. Then all elements of $\mathcal{E}(\boldsymbol{G}^F, s)$ lie in the same 2-block of $\boldsymbol{G}^F$.*

Proof. Since s has odd order and $Z(\boldsymbol{G})/Z^\circ(\boldsymbol{G})$ is a 2-group, $C_{\boldsymbol{G}^*}(s)$ is connected. On the other hand, since all components of $\boldsymbol{G}^*$ are of classical type and s has odd order, $C^\circ_{\boldsymbol{G}^*}(s)$ is a Levi subgroup of $\boldsymbol{G}$. Thus, $C_{\boldsymbol{G}^*}(s)$ is a Levi subgroup of $\boldsymbol{G}^*$ and by Jordan correspondence the set of 2-blocks of $\boldsymbol{G}^F$ which contain a character of $\mathcal{E}(\boldsymbol{G}^F, s)$ is in bijection with the set of unipotent 2-blocks of $\boldsymbol{C}^F$, where $\boldsymbol{C}$ is a Levi subgroup of $\boldsymbol{G}$ in duality with $C_{\boldsymbol{G}^*}(s)$. Since all components of $\boldsymbol{C}$ are also of classical type, the claim follows by [Enguehard 2008, Proposition 1.5(a)]. □

We now have the following extension of [Cabanes and Enguehard 1999, Theorem 2.5] to all primes.

Theorem 3.4. *Let $\boldsymbol{H}$ be a simple algebraic group of simply connected type with a Frobenius endomorphism $F : \boldsymbol{H} \to \boldsymbol{H}$ endowing $\boldsymbol{H}$ with an $\mathbb{F}_q$-rational structure. Let $\boldsymbol{G}$ be an F-stable Levi subgroup of $\boldsymbol{H}$. Let ℓ be a prime not dividing q and set $e = e_\ell(q)$. Let $\boldsymbol{M}$ be an e-split Levi subgroup of $\boldsymbol{G}$ and let c be a block of $\boldsymbol{M}^F$. Then there exists a block b of $\boldsymbol{G}^F$ such that every irreducible constituent of $R_{\boldsymbol{M}}^{\boldsymbol{G}}(\mu)$ for every $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$ lies in b.*

Proof. Let $\dim(\boldsymbol{G})$ be minimal such that the claim of the theorem does not hold. Let $s \in \boldsymbol{M}^{*F}$ be a semisimple ℓ'-element with $\mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell') \subseteq \mathcal{E}(\boldsymbol{M}^F, s)$. Then all irreducible constituents of $R_{\boldsymbol{M}}^{\boldsymbol{G}}(\mu)$ where $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$ are in $\mathcal{E}(\boldsymbol{G}^F, s)$.

First suppose that s is not quasi-isolated and let $\boldsymbol{G}_1$ be a proper F-stable Levi subgroup of $\boldsymbol{G}$ whose dual contains $C_{\boldsymbol{G}^*}(s)$. Let $\boldsymbol{M}^*$ be a Levi subgroup of $\boldsymbol{G}^*$ in duality with $\boldsymbol{M}$ and set $\boldsymbol{M}_1^* = C_{\boldsymbol{G}_1^*}(Z^\circ(\boldsymbol{M}^*)_e)$. Then, as in the proof of Proposition 2.4, $\boldsymbol{M}_1^*$ is an e-split Levi subgroup of $\boldsymbol{G}_1^*$ and letting $\boldsymbol{M}_1$ be the dual of $\boldsymbol{M}_1^*$ in $\boldsymbol{G}$, $\boldsymbol{M}_1$ is an e-split Levi subgroup of $\boldsymbol{G}_1$. Further, $\boldsymbol{M}_1^* \geq C_{\boldsymbol{M}^*}(s)$. Hence there exists a unique block say c_1 of $\boldsymbol{M}_1^F$ such that $\mathrm{Irr}(c_1) \cap \mathcal{E}(\boldsymbol{M}_1^F, \ell') \subseteq \mathcal{E}(\boldsymbol{M}_1^F, s)$ and such that c_1 and c are Jordan corresponding blocks.

By induction our claim holds for $\boldsymbol{G}_1$ and the block c_1 of $\boldsymbol{M}_1$. Let b_1 be the block of $\boldsymbol{G}_1^F$ such that every irreducible constituent of $R_{\boldsymbol{M}_1}^{\boldsymbol{G}_1}(\mu)$ where $\mu \in \mathrm{Irr}(c_1) \cap \mathcal{E}(\boldsymbol{M}_1^F, \ell')$ lies in b_1 and let b be the Jordan correspondent of b_1 in $\boldsymbol{G}^F$.

Now let $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, s)$ and let χ be an irreducible constituent of $R_{\boldsymbol{M}}^{\boldsymbol{G}}(\mu)$. Let μ_1 be the unique character in $\mathrm{Irr}(\boldsymbol{M}_1^F, s)$ with $\mu = \pm R_{\boldsymbol{M}_1}^{\boldsymbol{M}}(\mu_1)$. Then, $\mu_1 \in \mathrm{Irr}(c_1)$ and

$$R_{\boldsymbol{M}}^{\boldsymbol{G}}(\mu) = R_{\boldsymbol{M}}^{\boldsymbol{G}}\big(R_{\boldsymbol{M}_1}^{\boldsymbol{M}}(\mu_1)\big) = R_{\boldsymbol{G}_1}^{\boldsymbol{G}}\big(R_{\boldsymbol{M}_1}^{\boldsymbol{G}_1}(\mu_1)\big).$$

All irreducible constituents of $R_{\boldsymbol{M}_1}^{\boldsymbol{G}_1}(\mu_1)$ lie in b_1. Hence, by the above equation and by the Jordan decomposition of blocks, χ lies in b, a contradiction.

So, we may assume from now on that s is quasi-isolated in $\boldsymbol{G}^*$. By [Cabanes and Enguehard 1999, Theorem 2.5], we may assume that ℓ is bad for $\boldsymbol{G}$ and hence for $\boldsymbol{H}$. So $\boldsymbol{H}$ is not of type A. If $\boldsymbol{H}$ is of type B, C or D, then $\ell = 2$ and we have a contradiction by Lemma 3.3.

Thus $\boldsymbol{H}$ is of exceptional type. Suppose that $s = 1$. By [Broué et al. 1993, Theorem 3.2] $\boldsymbol{G}^F$ satisfies an e-Harish-Chandra theory above each unipotent e-cuspidal pair $(\boldsymbol{L}, \lambda)$ and by [Enguehard 2000, Theorems A and A.bis], all irreducible constituents of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ lie in the same ℓ-block of $\boldsymbol{G}^F$.

So we may assume that $s \neq 1$. We consider the case that $\boldsymbol{G} = \boldsymbol{H}$. Then by [Kessar and Malle 2013, Theorem 1.4], $\boldsymbol{G}^F$ satisfies an e-Harish-Chandra theory above each e-cuspidal pair $(\boldsymbol{L}, \lambda)$ below $\mathcal{E}(\boldsymbol{G}^F, s)$ and by [Kessar and Malle 2013, Theorem 1.2], all irreducible constituents of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ lie in the same ℓ-block of $\boldsymbol{G}^F$.

So, we may assume that $\boldsymbol{G}$ is proper in $\boldsymbol{H}$. If $\boldsymbol{H}$ is of type G_2, F_4 or E_6, then $\ell = 2$, all components of $\boldsymbol{G}$ are of classical type. For G_2 and F_4 we have that $Z(\boldsymbol{H})$ and therefore $Z(\boldsymbol{G})$ is connected. If $\boldsymbol{H}$ is of type E_6, since 2 is bad for $\boldsymbol{G}$, $\boldsymbol{G}$ has a component of type D_n, $n \geq 4$. By rank considerations, $[\boldsymbol{G}, \boldsymbol{G}]$ is of type D_4 or D_5. Since $|Z(\boldsymbol{H})/Z^\circ(\boldsymbol{H})| = 3$ it follows again that $Z(\boldsymbol{G})$ is connected. In either case we get a contradiction by Lemma 3.3.

So, $\boldsymbol{H}$ is of type E_7 or E_8. Since $\boldsymbol{G}$ is proper in $\boldsymbol{H}$, 5 is good for $\boldsymbol{G}$, hence $\ell = 3$ or 2. Also, we may assume that at least one of the two assumptions of Lemma 3.1 fails to hold for $\boldsymbol{G}$, $\boldsymbol{M}$ and c.

Suppose that $\ell = 3$. Since $\boldsymbol{G}$ is proper in $\boldsymbol{H}$ and 3 is bad for $\boldsymbol{G}$, either $[\boldsymbol{G}, \boldsymbol{G}]$ is of type E_6, or $\boldsymbol{H}$ is of type E_8 and $[\boldsymbol{G}, \boldsymbol{G}]$ is of type $E_6 + A_1$ or of type E_7. In all cases, $Z(\boldsymbol{G})$ is connected (note that if $\boldsymbol{H}$ is of type E_7, then $[\boldsymbol{G}, \boldsymbol{G}]$ is of type E_6, whence the order of $Z(\boldsymbol{G})/Z^\circ(\boldsymbol{G})$ divides both 2 and 3). If $\boldsymbol{G} = \boldsymbol{M}$, there is nothing to prove, so we may assume that $\boldsymbol{M}$ is proper in $\boldsymbol{G}$. Let

$$\boldsymbol{C} := C^\circ_{\boldsymbol{G}}(Z(\boldsymbol{M})^F_3) \geq \boldsymbol{M}.$$

We claim that there is a block, say d, of $\boldsymbol{C}^F$ such that for all $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$, every irreducible constituent of $R^{\boldsymbol{C}}_{\boldsymbol{M}}(\mu)$ lies in d. Indeed, since $\boldsymbol{M}$ is proper in $\boldsymbol{G}$ and since $Z(\boldsymbol{G})$ is connected, by [Cabanes and Enguehard 1993, Proposition 2.1] $\boldsymbol{C}$ is proper in $\boldsymbol{G}$. Also, by direct calculation either $\boldsymbol{C}$ is a Levi subgroup of $\boldsymbol{G}$ or 3 is good for $\boldsymbol{C}$. In the first case, the claim follows by the inductive hypothesis since $\boldsymbol{M}$ is also e-split in $\boldsymbol{C}$. In the second case, we are done by [Cabanes and Enguehard 1999, Theorem 2.5].

Thus, we may assume that assumption (1) of Lemma 3.1 does not hold. Hence, by [Cabanes and Enguehard 1999, Theorem 1.7], 3 is bad for $\boldsymbol{M}$. Consequently, $\boldsymbol{M}$ has a component of nonclassical type. Since $\boldsymbol{M}$ is proper in $\boldsymbol{G}$, this means that $[\boldsymbol{G}, \boldsymbol{G}]$ is of type $E_6 + A_1$ or of type E_7 and $[\boldsymbol{M}, \boldsymbol{M}]$ is of type E_6. Suppose $[\boldsymbol{G}, \boldsymbol{G}]$ is of type $E_6 + A_1$. Since $[\boldsymbol{M}, \boldsymbol{M}]$ is of type E_6, and since 3 is good for groups of type A, the result follows from Lemma 3.2, applied with $\boldsymbol{X}$ being the component of $\boldsymbol{G}$ of type E_6, and [ibid., Theorem 2.5].

So we have $[\boldsymbol{G}, \boldsymbol{G}]$ of type E_7 and $[\boldsymbol{M}, \boldsymbol{M}]$ of type E_6. Suppose that s is not quasi-isolated in $\boldsymbol{M}^*$. Then c is in Jordan correspondence with a block, say c' of a proper F-stable Levi subgroup, say $\boldsymbol{M}'$ of $\boldsymbol{M}$. The prime 3 is good for any proper Levi subgroup of $\boldsymbol{M}$, hence by [ibid., Theorem 1.7] condition (1) of Lemma 3.1 holds for the group $\boldsymbol{M}'$ and the block c'. By Jordan decomposition of blocks, this condition also holds for $\boldsymbol{M}$ and c, a contradiction. So, s is quasi-isolated in $\boldsymbol{M}^*$. Since as pointed out above, $\boldsymbol{G}$ has connected centre, so does $\boldsymbol{M}$ whence s is isolated in $\boldsymbol{M}^*$. Also, note that since s is also quasi-isolated in $\boldsymbol{G}^*$, by the same reasoning s is isolated in $\boldsymbol{G}^*$. Inspection shows that the only possible case for this is when s has order three with $C_{\boldsymbol{G}^*}(s)$ of type $A_5 + A_2$, $C_{\boldsymbol{M}^*}(s)$ of type $3A_2$. Since s is supposed to be a $3'$-element, this case does not arise here.

Now suppose that $\ell = 2$. Since $Z(\boldsymbol{H})/Z^\circ(\boldsymbol{H})$ has order dividing 2, by Lemma 3.3 we may assume that $\boldsymbol{G}$ has at least one nonclassical component, that is we are in one of the cases $[\boldsymbol{G}, \boldsymbol{G}] = E_6$, or $\boldsymbol{H} = E_8$ and $[\boldsymbol{G}, \boldsymbol{G}] = E_6 + A_1$ or E_7. Again, in all cases, $Z(\boldsymbol{G})$ is connected and consequently $C_{\boldsymbol{G}^*}(s)$ is connected and s is isolated.

Suppose first that $[\boldsymbol{G}, \boldsymbol{G}] = E_7$. We claim that all elements of $\mathcal{E}(\boldsymbol{G}^F, s)$ lie in the same 2-block. Indeed, let $\bar{s}$ be the image of s under the surjective map $\boldsymbol{G}^* \to [\boldsymbol{G}, \boldsymbol{G}]^*$ induced by the regular embedding of $[\boldsymbol{G}, \boldsymbol{G}]$ in $\boldsymbol{G}$. By [Kessar and Malle 2013, Table 4], all elements of $\mathcal{E}([\boldsymbol{G}, \boldsymbol{G}]^F, \bar{s})$ lie in the same 2-block, say d of $[\boldsymbol{G}, \boldsymbol{G}]^F$. So, any block of $\boldsymbol{G}^F$ which contains a character in $\mathcal{E}(\boldsymbol{G}^F, s)$ covers d. By general block theoretical reasons, there are at most $|\boldsymbol{G}^F/[\boldsymbol{G}, \boldsymbol{G}]^F|_{2'}$ 2-blocks of $\boldsymbol{G}^F$ covering a given d. Now since s is a $2'$-element, $C_{[\boldsymbol{G},\boldsymbol{G}]^*}(\bar{s})$ is connected. Thus, if $\mu \in \mathcal{E}([\boldsymbol{G}, \boldsymbol{G}]^F, \bar{s})$, then there are $|\boldsymbol{G}^F/[\boldsymbol{G}, \boldsymbol{G}]^F|_{2'}$ different $2'$-Lusztig series of $\boldsymbol{G}^F$ containing an irreducible character covering μ. Since characters in different $2'$-Lusztig series lie in different 2-blocks, the claim follows.

By the claim above, we may assume that either $[\boldsymbol{G}, \boldsymbol{G}] = E_6$ or $[\boldsymbol{G}, \boldsymbol{G}] = E_6 + A_1$. Since s is isolated of odd order in $\boldsymbol{G}^*$, by [Kessar and Malle 2013, Table 1] all components of $C_{\boldsymbol{G}^*}(s)$ are of type A_2 or A_1. Consequently, all components of $C_{\boldsymbol{M}^*}(s)$ are of type A. Suppose first that $\boldsymbol{M}$ has a nonclassical component. Then $[\boldsymbol{M}, \boldsymbol{M}]$ is of type E_6, and $[\boldsymbol{G}, \boldsymbol{G}] = E_6 + A_1$. This may be ruled out by Lemma 3.2, applied with $\boldsymbol{X}$ equal to the product of the component of type E_6 with $Z(\boldsymbol{G})$ and $\boldsymbol{Y}$ equal to the component of type A_1.

So finally suppose that all components of $\boldsymbol{M}$ are of classical type. Then, $C_{\boldsymbol{M}^*}(s) = C^\circ_{\boldsymbol{M}^*}(s)$ is a Levi subgroup of $\boldsymbol{M}$ with all components of type A. Hence, the first hypothesis of Lemma 3.1 holds by the Jordan decomposition of blocks and [Cabanes and Enguehard 1999, Theorem 1.7]. So, we may assume that the second hypothesis of Lemma 3.1 does not hold. Let

$$\boldsymbol{C} := C^\circ_{\boldsymbol{G}}(Z(\boldsymbol{M}^F)_2).$$

Since $\boldsymbol{M}$ is a proper e-split Levi subgroup of $\boldsymbol{G}$, and since $Z(\boldsymbol{G})$ is connected, by [Cabanes and Enguehard 1993, Proposition 2.1] $\boldsymbol{C}$ is proper in $\boldsymbol{G}$. By induction, we may assume that $\boldsymbol{C}$ is not a Levi subgroup of $\boldsymbol{G}$. In particular, the intersection of $\boldsymbol{C}$ with the component of type E_6 of $\boldsymbol{G}$ is proper in that component and hence all components of $\boldsymbol{C}$ are of type A or D. If all components of $\boldsymbol{C}$ are of type A, then 2 is good for $\boldsymbol{C}$ and the second hypothesis of Lemma 3.1 holds by [Cabanes and Enguehard 1999, Theorem 2.5]. Thus we may assume that $\boldsymbol{C}$ has a component of type D. Since all components of $\boldsymbol{C}$ are classical, by Lemma 3.3, we may assume that $Z(\boldsymbol{C})/Z^\circ(\boldsymbol{C})$ is not a 2-group and consequently $\boldsymbol{C}$ has a component of type A_n, with $n \equiv 2 \pmod 3$. But by the Borel–de Siebenthal algorithm, a group of type E_6 has no subsystem subgroup of type $D_m + A_n$ with $n \geq 1$ and $m \geq 4$. $\square$

Characters in ℓ-blocks. Using the results collected so far, it is now easy to characterise all characters in ℓ'-series inside a given ℓ-block in terms of Lusztig induction.

Definition 3.5. As in [Cabanes and Enguehard 1999, Section 1.11] (see also [Broué et al. 1993, Definition 3.1]) for e-split Levi subgroups $\boldsymbol{M}_1, \boldsymbol{M}_2$ of $\boldsymbol{G}$ and $\mu_i \in \operatorname{Irr}(\boldsymbol{M}_i^F)$, we write $(\boldsymbol{M}_1, \mu_1) \le_e (\boldsymbol{M}_2, \mu_2)$ if $\boldsymbol{M}_1 \le \boldsymbol{M}_2$ and μ_2 is a constituent of $R_{\boldsymbol{M}_1}^{\boldsymbol{M}_2}(\mu_1)$ (with respect to some parabolic subgroup of $\boldsymbol{M}_2$ with Levi subgroup $\boldsymbol{M}_1$). We let $\ll_e$ denote the transitive closure of the relation $\le_e$.

As pointed out in [Cabanes and Enguehard 1999, Section 1.11] it seems reasonable to expect that the relations $\le_e$ and $\ll_e$ coincide. While this is known to hold for unipotent characters (see [Broué et al. 1993, Theorem 3.11]), it is open in general.

We put ourselves in the situation and notation of Theorem A.

Theorem 3.6. *Let b be an ℓ-block of $\boldsymbol{G}^F$ and denote by $\mathcal{L}(b)$ the set of e-Jordan-cuspidal pairs $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ such that there is $\chi \in \operatorname{Irr}(b)$ with $\langle \chi, R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)\rangle \neq 0$. Then*

$$\operatorname{Irr}(b) \cap \mathcal{E}(\boldsymbol{G}^F, \ell') = \big\{\chi \in \mathcal{E}(\boldsymbol{G}^F, \ell') \mid \exists\, (\boldsymbol{L}, \lambda) \in \mathcal{L}(b) \text{ with } (\boldsymbol{L}, \lambda) \ll_e (\boldsymbol{G}, \chi)\big\}.$$

Proof. Let b be as in the statement and first assume that $\chi \in \operatorname{Irr}(b) \cap \mathcal{E}(\boldsymbol{G}^F, \ell')$. If χ is not e-Jordan-cuspidal, then it is not e-cuspidal, so there exists a proper e-split Levi subgroup $\boldsymbol{M}_1$ such that χ occurs in $R_{\boldsymbol{M}_1}^{\boldsymbol{G}}(\mu_1)$ for some $\mu_1 \in \mathcal{E}(\boldsymbol{M}_1^F, \ell')$. Thus inductively we obtain a chain of e-split Levi subgroups $\boldsymbol{M}_r \lneq \ldots \lneq \boldsymbol{M}_1 \lneq \boldsymbol{M}_0 := \boldsymbol{G}$ and characters $\mu_i \in \mathcal{E}(\boldsymbol{M}_i^F, \ell')$ (with $\mu_0 := \chi$) such that $(\boldsymbol{M}_r, \mu_r)$ is e-Jordan cuspidal and such that $(\boldsymbol{M}_i, \mu_i) \le_e (\boldsymbol{M}_{i-1}, \mu_{i-1})$ for $i = 1, \ldots, r$, whence $(\boldsymbol{M}_r, \mu_r) \ll_e (\boldsymbol{G}, \chi)$. Let b_r be the ℓ-block of $\boldsymbol{M}_r^F$ containing μ_r. Now Theorem 3.4 yields that for each i there exists a block, say b_i, of $\boldsymbol{M}_i^F$ such that all constituents of $R_{\boldsymbol{M}_i}^{\boldsymbol{M}_{i-1}}(\zeta_i)$ lie in b_{i-1} for all $\zeta_i \in \operatorname{Irr}(b_i) \cap \mathcal{E}(\boldsymbol{M}_i^F, \ell')$. In particular, χ lies in b_0, so $b_0 = b$, and thus $(\boldsymbol{M}_r, \mu_r) \in \mathcal{L}(b)$.

For the reverse inclusion, let $(\boldsymbol{L}, \lambda) \in \mathcal{L}(b)$ and $\chi \in \operatorname{Irr}(\boldsymbol{G}^F, \ell')$ such that $(\boldsymbol{L}, \lambda) \ll_e (\boldsymbol{G}, \chi)$. Thus there exists a chain of e-split Levi subgroups $\boldsymbol{L} = \boldsymbol{M}_r \lneq \ldots \lneq \boldsymbol{M}_0 = \boldsymbol{G}$ and characters $\mu_i \in \operatorname{Irr}(\boldsymbol{M}_i^F)$ with $(\boldsymbol{M}_i, \mu_i) \le_e (\boldsymbol{M}_{i-1}, \mu_{i-1})$. Again, an application of Theorem 3.4 allows us to conclude that $\chi \in \operatorname{Irr}(b)$. □

ℓ-blocks and derived subgroups. In the following two results, which will be used in showing that the map Ξ in Theorem A is surjective, $\boldsymbol{G}$ is connected reductive with Frobenius endomorphism F, and $\boldsymbol{G}_0 := [\boldsymbol{G}, \boldsymbol{G}]$. Here, in the cases that the Mackey formula is not known to hold we assume that $R_{\boldsymbol{L}_0}^{\boldsymbol{G}_0}$ and $R_{\boldsymbol{L}}^{\boldsymbol{G}}$ are with respect to a choice of parabolic subgroups $\boldsymbol{P}_0 \ge \boldsymbol{L}_0$ and $\boldsymbol{P} \ge \boldsymbol{L}$ such that $\boldsymbol{P}_0 = \boldsymbol{G}_0 \cap \boldsymbol{P}$.

Lemma 3.7. *Let b be an ℓ-block of $\boldsymbol{G}^F$ and let b_0 be an ℓ-block of $\boldsymbol{G}_0^F$ covered by b. Let $\boldsymbol{L}$ be an F-stable Levi subgroup of $\boldsymbol{G}$, $\boldsymbol{L}_0 = \boldsymbol{L} \cap \boldsymbol{G}_0$ and let $\lambda_0 \in \operatorname{Irr}(\boldsymbol{L}_0^F)$. Suppose that every irreducible constituent of $R_{\boldsymbol{L}_0}^{\boldsymbol{G}_0}(\lambda_0)$ is contained in b_0. Then* *there exists* $\lambda \in \mathrm{Irr}(\boldsymbol{L}^F)$ *and* $\chi \in \mathrm{Irr}(b)$ *such that* λ_0 *is an irreducible constituent of* $\mathrm{Res}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda)$ *and* χ *is an irreducible constituent of* $R^{\boldsymbol{G}}_{\boldsymbol{L}}(\lambda)$.

Proof. Since $\boldsymbol{G} = Z^\circ(\boldsymbol{G})\boldsymbol{G}_0$, by [Bonnafé 2006, Proposition 10.10] we have that

$$R^{\boldsymbol{G}}_{\boldsymbol{L}} \mathrm{Ind}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda_0) = \mathrm{Ind}^{\boldsymbol{G}^F}_{\boldsymbol{G}_0^F} R^{\boldsymbol{G}_0}_{\boldsymbol{L}_0}(\lambda_0).$$

Note that the result in [Bonnafé 2006] is only stated for the case that $\boldsymbol{G}$ has connected centre but the proof does not use this hypothesis. The right hand side of the above equality evaluated at 1 is nonzero. Let $\chi' \in \mathrm{Irr}(\boldsymbol{G}^F)$ be a constituent of the left hand side of the equality. There exists $\lambda \in \mathrm{Irr}(\boldsymbol{L}^F)$ and χ_0 in $\mathrm{Irr}(\boldsymbol{G}_0^F)$ such that λ is an irreducible constituent of $\mathrm{Ind}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda_0)$, χ' is an irreducible constituent of $R^{\boldsymbol{G}}_{\boldsymbol{L}}(\lambda)$, χ_0 is an irreducible constituent of $R^{\boldsymbol{G}_0}_{\boldsymbol{L}_0}(\lambda_0)$ and χ' is an irreducible constituent of $\mathrm{Ind}^{\boldsymbol{G}^F}_{\boldsymbol{G}_0^F}(\chi_0)$. Since $\chi_0 \in \mathrm{Irr}(b_0)$, χ' lies in a block, say b', of $\boldsymbol{G}^F$ which covers b_0. Since b also covers b_0 and since $\boldsymbol{G}^F/\boldsymbol{G}_0^F$ is abelian, there exists a linear character, say θ of $\boldsymbol{G}^F/\boldsymbol{G}_0^F$ such that $b = b' \otimes \theta$ (see [Kessar and Malle 2013, Lemma 2.2]). Now the result follows from [Bonnafé 2006, Proposition 10.11] with $\chi = \chi' \otimes \theta$. $\square$

Lemma 3.8. *Let* b *be an* ℓ*-block of* $\boldsymbol{G}^F$ *and let* $\boldsymbol{L}$ *be an* F*-stable Levi subgroup of* $\boldsymbol{G}$ *and* $\lambda \in \mathrm{Irr}(\boldsymbol{L}^F)$ *such that every irreducible constituent of* $R^{\boldsymbol{G}}_{\boldsymbol{L}}(\lambda)$ *is contained in* b. *Let* $\boldsymbol{L}_0 = \boldsymbol{L} \cap \boldsymbol{G}_0$ *and let* $\lambda_0 \in \mathrm{Irr}(\boldsymbol{L}_0^F)$ *be an irreducible constituent of* $\mathrm{Res}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda)$. *Then there exists an* ℓ*-block* b_0 *of* $\boldsymbol{G}_0^F$ *covered by* b *and an irreducible character* χ_0 *of* $\boldsymbol{G}_0^F$ *in the block* b_0 *such that* χ_0 *is a constituent of* $R^{\boldsymbol{G}_0}_{\boldsymbol{L}_0}(\lambda_0)$.

Proof. Arguing as in the proof of Lemma 3.7, there exist $\chi \in \mathrm{Irr}(\boldsymbol{G}^F)$, $\lambda' \in \mathrm{Irr}(\boldsymbol{L}^F)$ and χ_0 in $\mathrm{Irr}([\boldsymbol{G},\boldsymbol{G}]^F)$ such that λ' is an irreducible constituent of $\mathrm{Ind}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda_0)$, χ is an irreducible constituent of $R^{\boldsymbol{G}}_{\boldsymbol{L}}(\lambda')$, χ_0 is an irreducible constituent of $R^{[\boldsymbol{G},\boldsymbol{G}]}_{\boldsymbol{L}_0}(\lambda_0)$ and χ is an irreducible constituent of $\mathrm{Ind}^{\boldsymbol{G}^F}_{[\boldsymbol{G},\boldsymbol{G}]^F}(\chi_0)$. Now, $\lambda = \theta \otimes \lambda'$ for some linear character θ of $\boldsymbol{L}^F/\boldsymbol{L}_0^F$. By [Bonnafé 2006, Proposition 10.11], $\theta \otimes \chi$ is an irreducible constituent of $R^{\boldsymbol{G}}_{\boldsymbol{L}}(\lambda)$, and hence $\theta \otimes \chi \in \mathrm{Irr}(b)$. Further, $\theta \otimes \chi$ is also a constituent of $\mathrm{Ind}^{\boldsymbol{G}^F}_{[\boldsymbol{G},\boldsymbol{G}]^F}(\chi_0)$, hence b covers the block of $[\boldsymbol{G},\boldsymbol{G}]^F$ containing χ_0. $\square$

Unique maximal abelian normal subgroups. A crucial ingredient for proving injectivity of the map in parts (d) and (e) of Theorem A is a property related to the nonfailure of factorisation phenomenon of finite group theory, which holds for the defect groups of many blocks of finite groups of Lie type and which was highlighted by Cabanes [1994]: for a prime ℓ, an ℓ-group is said to be *Cabanes* if it has a unique maximal abelian normal subgroup.

Now first consider the following setting: let $\boldsymbol{G}$ be connected reductive. For $i = 1, 2$, let $\boldsymbol{L}_i$ be an F-stable Levi subgroup of $\boldsymbol{G}$ with $\lambda_i \in \mathcal{E}(\boldsymbol{L}_i^F, \ell')$, and let u_i denote the ℓ-block of $\boldsymbol{L}_i^F$ containing λ_i. Suppose that $C_{\boldsymbol{G}}(Z(\boldsymbol{L}_i^F)_\ell) = \boldsymbol{L}_i$ and that λ_i is of quasicentral ℓ-defect. Then by [Kessar and Malle 2013, Propositions

2.12, 2.13, 2.16] there exists a block b_i of $\boldsymbol{G}^F$ such that all irreducible characters of $R^{\boldsymbol{G}}_{\boldsymbol{L}_i}(\lambda_i)$ lie in b_i and $(Z(\boldsymbol{L}_i^F)_\ell, u_i)$ is a b_i-Brauer pair.

Lemma 3.9. *In the above situation, assume further that for $i = 1, 2$ there exists a maximal b_i-Brauer pair (P_i, c_i) such that $(Z(\boldsymbol{L}_i^F)_\ell, u_i) \trianglelefteq (P_i, c_i)$, and such that P_i is Cabanes with $Z(\boldsymbol{L}_i^F)_\ell$ as the unique maximal abelian normal subgroup of P_i. If $b_1 = b_2$ then the pairs $(\boldsymbol{L}_1, \lambda_1)$ and $(\boldsymbol{L}_2, \lambda_2)$ are $\boldsymbol{G}^F$-conjugate.*

Proof. Suppose that $b_1 = b_2$. Since maximal b_1-Brauer pairs are $\boldsymbol{G}^F$-conjugate, it follows that ${}^g(Z(\boldsymbol{L}_2^F)_\ell, u_2) \le {}^g(P_2, c_2) = (P_1, c_1)$ for some $g \in \boldsymbol{G}^F$. By transport of structure, ${}^gZ(\boldsymbol{L}_2^F)_\ell$ is a maximal normal abelian subgroup of P_1, hence ${}^gZ(\boldsymbol{L}_2^F)_\ell = Z(\boldsymbol{L}_1^F)_\ell$. By the uniqueness of inclusion of Brauer pairs it follows that ${}^g(Z(\boldsymbol{L}_2^F)_\ell, u_2) = (Z(\boldsymbol{L}_1)_\ell^F, u_1)$. Since $\boldsymbol{L}_i = C_{\boldsymbol{G}}(Z(\boldsymbol{L}_i^F)_\ell)$, this means that ${}^g\boldsymbol{L}_2 = \boldsymbol{L}_1$. Further, since λ_i is of quasicentral ℓ-defect, by [Kessar and Malle 2013, Proposition 2.5(f)], λ_i is the unique element of $\mathcal{E}(\boldsymbol{L}_i^F, \ell') \cap \mathrm{Irr}(u_i)$. Thus ${}^gu_2 = u_1$ implies that ${}^g\lambda_2 = \lambda_1$ and $(\boldsymbol{L}_1, \lambda_1)$ and $(\boldsymbol{L}_2, \lambda_2)$ are $\boldsymbol{G}^F$-conjugate as required. □

By the proof of Theorems 4.1 and 4.2 of [Cabanes and Enguehard 1999] we also have:

Proposition 3.10. *Let $\boldsymbol{G}$ be connected reductive with simply connected derived subgroup. Suppose that $\ell \ge 3$ is good for $\boldsymbol{G}$, and $\ell \ne 3$ if $\boldsymbol{G}^F$ has a factor ${}^3D_4(q)$. Let b be an ℓ-block of $\boldsymbol{G}^F$ such that the defect groups of b are Cabanes. If $(\boldsymbol{L}, \lambda)$ and $(\boldsymbol{L}', \lambda')$ are e-Jordan-cuspidal pairs of $\boldsymbol{G}$ such that $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$, $\lambda' \in \mathcal{E}(\boldsymbol{L}'^F, \ell')$ with $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda) = b = b_{\boldsymbol{G}^F}(\boldsymbol{L}', \lambda')$, then $(\boldsymbol{L}, \lambda)$ and $(\boldsymbol{L}', \lambda')$ are $\boldsymbol{G}^F$-conjugate.*

Proof. This is essentially contained in Section 4 of [Cabanes and Enguehard 1999]; all references in this proof are to this paper. Indeed, let $(\boldsymbol{L}, \lambda)$ be an e-Jordan-cuspidal pair of $\boldsymbol{G}$ such that $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$. Let $\boldsymbol{T}^*$, $\boldsymbol{T}$, $\boldsymbol{K} = C^\circ_{\boldsymbol{G}}(Z(\boldsymbol{L})_\ell^F)$, $\boldsymbol{K}^*$, $\boldsymbol{M}$ and $\boldsymbol{M}^*$ be as in the notation before Lemma 4.4. Let $Z = Z(\boldsymbol{M})_\ell^F$ and let $\lambda_{\boldsymbol{K}}$ and $\lambda_{\boldsymbol{M}}$ be as in Definition 4.6, with λ replacing ζ. Then $Z \le \boldsymbol{T}$ and by Lemma 4.8, $\boldsymbol{M} = C^\circ_{\boldsymbol{G}}(Z)$. The simply connected hypothesis and the restrictions on ℓ imply that $C_{\boldsymbol{G}}(Z) = C^\circ_{\boldsymbol{G}}(Z) = \boldsymbol{M}$. Let $b_Z = \hat{b}_Z$ be the ℓ-block of $\boldsymbol{M}^F$ containing $\lambda_{\boldsymbol{M}}$. Then by Lemma 4.13, (Z, b_Z) is a self centralising Brauer pair and $(1, b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)) \le (Z, b_Z)$. Further, by Lemma 4.16 there exists a maximal b-Brauer pair (D, b_D) such that $(Z, b_Z) \le (D, b_D)$, Z is normal in D and $C_D(Z) = Z$. Note that the first three conclusions of Lemma 4.16 hold under the conditions we have on ℓ (it is only the fourth conclusion which requires $\ell \in \Gamma(\boldsymbol{G}, F)$). By Lemma 4.10 and its proof, we also have

$$(1, b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)) \le (Z(\boldsymbol{L})_\ell^F, b_{\boldsymbol{K}^F}(\boldsymbol{L}, \lambda)) \le (Z, b_Z).$$

Suppose that $\boldsymbol{N}$ is a proper e-split Levi subgroup of $\boldsymbol{G}$ containing $C^\circ_{\boldsymbol{G}}(z) = C_{\boldsymbol{G}}(z)$ for some $1 \ne z \in Z(D)\boldsymbol{G_a} \cap \boldsymbol{G_b}$. Then $\boldsymbol{N}$ contains $\boldsymbol{L}$, $\boldsymbol{M}$ and Z by Lemma 4.15(b). Since $\boldsymbol{L} \cap \boldsymbol{G_b} = \boldsymbol{K} \cap \boldsymbol{G_b}$ by Lemma 4.4(iii), it follows that $\boldsymbol{N}$ also contains $\boldsymbol{K}$ and

$\boldsymbol{K} = C_{\boldsymbol{N}}(Z(\boldsymbol{L}^F))$. Thus, replacing $\boldsymbol{G}$ with $\boldsymbol{N}$ in Lemma 4.13 we get that

$$(1, b_{\boldsymbol{N}^F}(\boldsymbol{L}, \lambda)) \leq (Z(\boldsymbol{L})_\ell^F, b_{\boldsymbol{K}^F}(\boldsymbol{L}, \lambda)) \leq (D, b_D).$$

Let $(\boldsymbol{L}', \lambda')$ be another e-Jordan-cuspidal pair of $\boldsymbol{G}$ with $\lambda' \in \mathcal{E}(\boldsymbol{L}'^F, \ell')$ such that $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda) = b = b_{\boldsymbol{G}^F}(\boldsymbol{L}', \lambda')$. Denote by $\boldsymbol{K}'$, $\boldsymbol{M}'$, D' etc. the corresponding groups and characters for $(\boldsymbol{L}', \lambda')$. Up to replacing by a $\boldsymbol{G}^F$-conjugate, we may assume that $(D', b_{D'}) = (D, b_D)$.

Suppose first that there is a $1 \neq z \in Z(D)\boldsymbol{G_a} \cap \boldsymbol{G_b}$. By Lemma 4.15(b), there is a proper e-split Levi subgroup $\boldsymbol{N}$ containing $C_{\boldsymbol{G}}(z)$. Moreover, $\boldsymbol{N}$ contains D, $\boldsymbol{L}'$, $\boldsymbol{M}'$, $\boldsymbol{K}'$ and $\boldsymbol{G_a}$ and we also have

$$(1, b_{\boldsymbol{N}^F}(\boldsymbol{L}', \lambda')) \leq (Z(\boldsymbol{L}')_\ell^F, b_{\boldsymbol{K}'^F}(\boldsymbol{L}', \lambda')) \leq (D, b_D).$$

By the uniqueness of inclusion of Brauer pairs it follows that $b_{\boldsymbol{N}^F}(\boldsymbol{L}, \lambda) = b_{\boldsymbol{N}^F}(\boldsymbol{L}', \lambda')$. Also D is a defect group of $b_{\boldsymbol{N}^F}(\boldsymbol{L}, \lambda)$. Thus, in this case we are done by induction.

So, we may assume that $Z(D) \leq \boldsymbol{G_a}$ hence $D \leq \boldsymbol{G_a}$. From here on, the proof of Lemma 4.17 goes through without change, the only property that is used being that Z is the unique maximal abelian normal subgroup of D. □

We will also need the following observation:

Lemma 3.11. *Let $P = P_1 \times P_2$ where P_1 and P_2 are Cabanes. Suppose that P_0 is a normal subgroup of P such that $\pi_i(P_0) = P_i$, $i = 1, 2$, where $\pi_i : P_1 \times P_2 \to P_i$ denote the projection maps. Then P_0 is Cabanes with maximal normal abelian subgroup $(A_1 \times A_2) \cap P_0$, where A_i is the unique maximal normal abelian subgroup of P_i, $i = 1, 2$.*

Proof. Let $A = A_1 \times A_2$. The group $A \cap P_0$ is abelian and normal in P_0. Let S be a normal abelian subgroup of P_0. Since $\pi_i(P_0) = P_i$, $\pi_i(S)$ is normal in P_i and since S is abelian, so is $\pi_i(S)$. Thus, $\pi_i(S)$ is a normal abelian subgroup of P_i and is therefore contained in A_i. So, $S \leq (\pi_1(S) \times \pi_2(S)) \cap P_0 \leq (A_1 \times A_2) \cap P_0 = A \cap P_0$ and the result is proved. □

Linear and unitary groups at $\boldsymbol{\ell = 3}$. The following will be instrumental in the proof of statement (e) of Theorem A.

Lemma 3.12. *Let q be a prime power such that $3 \mid (q-1)$ (respectively $3 \mid (q+1)$). Let $G = \mathrm{SL}_n(q)$ (respectively $\mathrm{SU}_n(q)$) and let P be a Sylow 3-subgroup of G. Then P is Cabanes unless $n = 3$ and $3 \parallel (q-1)$ (respectively $3 \parallel (q+1)$). In particular, if P is not Cabanes, then P is extra-special of order 27 and exponent 3. In this case $N_G(P)$ acts transitively on the set of subgroups of order 9 of P.*

Proof. Embed $P \leq \mathrm{SL}_n(q) \leq \mathrm{GL}_n(q)$. A Sylow 3-subgroup of $\mathrm{GL}_n(q)$ is contained in the normaliser $C_{q-1} \wr \mathfrak{S}_n$ of a maximally split torus. According to [Cabanes 1994, Lemme 4.1], the only case in which $\mathfrak{S}_n$ has a quadratic element on $(C_{q-1}^n)_3 \cap \mathrm{SL}_n(q)$ is when $n = 3$ and $3 \parallel (q-1)$. If there is no quadratic element in this action, then

P is Cabanes by [Cabanes 1994, Proposition 2.3]. In the case of $\mathrm{SU}_n(q)$, the same argument applies with the normaliser $C_{q+1} \wr \mathfrak{S}_n$ of a Sylow 2-torus inside $\mathrm{GU}_n(q)$.

Now assume we are in the exceptional case. Clearly $|P| = 27$. Let $P_1, P_2 \le P$ be subgroups of order 9, and let $u_i \in P_i$ be noncentral. Then u_i is G-conjugate to $\mathrm{diag}(1, \zeta, \zeta^2)$, where ζ is a primitive 3rd-root of unity in $\mathbb{F}_q$ (respectively $\mathbb{F}_{q^2}$). In particular, there exists $g \in G$ such that ${}^g u_1 = u_2$. Let $\bar{\ }: G \to G/Z(G)$ denote the canonical map. Then ${}^{\bar{g}}(\bar{u}_1) = \bar{u}_2$. Since the Sylow 3-subgroup $\bar{P}$ of $\bar{G}$ is abelian, there exists $\bar{h} \in N_{\bar{G}}(\bar{P})$ with ${}^{\bar{h}}(\bar{u}_1) = \bar{u}_2$. Then $h \in N_G(P)$ and ${}^h P_1 = P_2$ as $P_i = \langle Z(G), u_i\rangle$. □

Lemma 3.13. *Suppose that* $3 \parallel n$ *and* $3 \parallel (q-1)$ (*respectively* $3 \parallel (q+1)$). *Let* $\tilde{\boldsymbol{G}} = \mathrm{GL}_n$, $\boldsymbol{G} = \mathrm{SL}_n$ *and suppose that* $\tilde{\boldsymbol{G}}^F = \mathrm{GL}_n(q)$ (*respectively* $\mathrm{GU}_n(q)$). *Let s be a semisimple $3'$-element of $\tilde{\boldsymbol{G}}^F$ such that a Sylow 3-subgroup D of $C_{\boldsymbol{G}^F}(s)$ is extra-special of order 27 and let $P_1, P_2 \le D$ have order 9. There exists $g \in N_{\boldsymbol{G}^F}(D) \cap C_{\boldsymbol{G}^F}(C_{\boldsymbol{G}^F}(D))$ such that ${}^g P_1 = P_2$.*

Proof. Set $d = \frac{n}{3}$. Identify $\tilde{\boldsymbol{G}}$ with the group of linear transformations of an n-dimensional $\mathbb{F}_q$-vector space V with chosen basis $\{e_{i,r} \mid 1 \le i \le d, 1 \le r \le 3\}$. For $g \in \tilde{\boldsymbol{G}}$, write $a(g)_{i,r,j,s}$ for the coefficient of $e_{i,r}$ in $g(e_{j,s})$. Let $w \in \tilde{\boldsymbol{G}}$ be defined by $w(e_{i,r}) = e_{i+1,r}$, $1 \le i \le d$, $1 \le r \le 3$. For $1 \le i \le d$ let V_i be the span of $\{e_{i,1}, e_{i,2}, e_{i,3}\}$ and $\tilde{\boldsymbol{G}}_i = \mathrm{GL}(V_i)$ considered as a subgroup of $\tilde{\boldsymbol{G}}$ through the direct sum decomposition $V = \bigoplus_{1\le i\le d} V_i$.

Up to conjugation in $\tilde{\boldsymbol{G}}$ we may assume $F = \mathrm{ad}_w \circ F_0$, where F_0 is the standard Frobenius morphism which raises every matrix entry to its q-th power in the linear case, respectively the composition of the latter by the transpose inverse map in the unitary case. Note that then each $\tilde{\boldsymbol{G}}_i$ is F_0-stable.

Thus, given the hypothesis on the structure of D, we may assume the following up to conjugation: s has d distinct eigenvalues $\delta_1, \dots, \delta_d$ with $\delta_{i+1} = \delta_i^q$ (respectively δ_i^{-q}); V_i is the δ_i-eigenspace of s, and $C_{\tilde{\boldsymbol{G}}}(s) = \prod_{i=1}^d \tilde{\boldsymbol{G}}_i$. Further, $F(\tilde{\boldsymbol{G}}_i) = \tilde{\boldsymbol{G}}_{i+1}$ and denoting by $\Delta : \tilde{\boldsymbol{G}}_1 \to \prod_{i=1}^d \tilde{\boldsymbol{G}}_i$, $x \mapsto xF(x)\cdots F^{d-1}(x)$, the twisted diagonal map we have $C_{\tilde{\boldsymbol{G}}^F}(s) = \Delta(\tilde{\boldsymbol{G}}_1^{F^d})$. Here, $\tilde{\boldsymbol{G}}_1^{F^d} = \tilde{\boldsymbol{G}}_1^{F_0^d}$ is isomorphic to either $\mathrm{GL}_3(q^d)$ or $\mathrm{GU}_3(q^d)$. Note that $\mathrm{GU}_3(q^d)$ occurs only if d is odd.

Consider $\tilde{\boldsymbol{G}}_1^{F_0} \le \tilde{\boldsymbol{G}}_1^{F_0^d}$. Let U_1 be the Sylow 3-subgroup of the diagonal matrices in $\tilde{\boldsymbol{G}}_1^{F_0}$ of determinant 1 and let $\sigma_1 \in \tilde{\boldsymbol{G}}_1^{F_0}$ be defined by $\sigma_1(e_{1,r}) = e_{1,r+1}$ for $1 \le r \le 3$. Then $D_1 := \langle U_1, \sigma_1\rangle$ is a Sylow 3-subgroup of $\tilde{\boldsymbol{G}}_1^{F_0}$. Since by hypothesis the Sylow 3-subgroups of $C_{\boldsymbol{G}^F}(s)$ have order 27, $D := \Delta(D_1)$ is a Sylow 3-subgroup of $C_{\boldsymbol{G}^F}(s)$ with $\Delta(U_1) \cong U_1$ elementary abelian of order 9. Note that $\Delta(\sigma_1)(e_{i,r}) = e_{i,r+1}$ for $1 \le i \le d$ and $1 \le r \le 3$.

Let $\zeta \in \overline{\mathbb{F}}_q$ be a primitive third root of unity. Let $u_1 \in U_1$ be such that $u_1(e_{1,r}) = \zeta^r e_{1,r}$ for $1 \le r \le 3$. For $1 \le r \le 3$, let W_r be the span of $\{e_{1,r}, \dots, e_{d,r}\}$. Then W_r is the ζ^r-eigenspace of $\Delta(u_1)$, whence

$$C_{\tilde{G}}(D) \le C_{\tilde{G}}(\Delta(U_1)) = C_{\tilde{G}}(\Delta(u_1)) = \prod_{1\le r\le 3} \mathrm{GL}(W_r).$$

Since $\Delta(\sigma_1)(W_r) = W_{r+1}$, and $\Delta(\sigma_1)$ acts on $C_{\tilde{G}}(\Delta(U_1))$, it follows that $C_{\tilde{G}}(D) = \Delta'(\mathrm{GL}(W_1))$, where $\Delta' : \mathrm{GL}(W_1) \to \prod_{1\le r\le 3} \mathrm{GL}(W_r)$, $x \mapsto x\,{}^{\sigma}x\,{}^{\sigma^2}x$, is the twisted diagonal.

We claim that $\Delta(\tilde{\boldsymbol{G}}_1^{F_0})$ centralises $C_{\tilde{G}}(D)$. Indeed, note that $g \in \Delta(\tilde{\boldsymbol{G}}_1^{F_0})$ if and only if $a(g)_{i,r,j,s} = 0$ if $i \ne j$ and $a(g)_{i,r,i,s} = a(F_0^{i-1}(g))_{1,r,1,s} = a(g)_{1,r,1,s}$ for all i and all r, s. Also, $h \in C_{\tilde{G}}(D)$ if and only if $a(h)_{i,r,j,s} = 0$ if $r \ne s$ and $a(h)_{i,r,j,r} = a(h)_{i,1,j,1}$ for all i, j and all r. The claim follows from an easy matrix multiplication.

Let $H = [\tilde{\boldsymbol{G}}_1^{F_0}, \tilde{\boldsymbol{G}}_1^{F_0}]$ and note that $D_1 \le H$. By Lemma 3.12 applied to H any two subgroups of D_1 of order 9 are conjugate by an element of $N_H(D_1)$. The lemma follows from the claim above. □

Parametrising ℓ-blocks. We can now prove our main theorem, Theorem A, which we restate. Recall Definition 2.1 of e-Jordan (quasicentral) cuspidal pairs.

Theorem 3.14. *Let $\boldsymbol{H}$ be a simple algebraic group of simply connected type with a Frobenius endomorphism $F : \boldsymbol{H} \to \boldsymbol{H}$ endowing $\boldsymbol{H}$ with an $\mathbb{F}_q$-rational structure. Let $\boldsymbol{G}$ be an F-stable Levi subgroup of $\boldsymbol{H}$. Let ℓ be a prime not dividing q and set $e = e_\ell(q)$.*

(a) *For any e-Jordan-cuspidal pair $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ such that $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$, there exists a unique ℓ-block $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}^F$ such that all irreducible constituents of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ lie in $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$.*

(b) *The map $\Xi : (\boldsymbol{L}, \lambda) \mapsto b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ is a surjection from the set of $\boldsymbol{G}^F$-conjugacy classes of e-Jordan-cuspidal pairs $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ with $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ to the set of ℓ-blocks of $\boldsymbol{G}^F$.*

(c) *The map Ξ restricts to a surjection from the set of $\boldsymbol{G}^F$-conjugacy classes of e-Jordan quasicentral cuspidal pairs $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ with $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ to the set of ℓ-blocks of $\boldsymbol{G}^F$.*

(d) *For $\ell \ge 3$ the map Ξ restricts to a bijection between the set of $\boldsymbol{G}^F$-conjugacy classes of e-Jordan quasicentral cuspidal pairs $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ with $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ and the set of ℓ-blocks of $\boldsymbol{G}^F$.*

(e) *The map Ξ itself is bijective if $\ell \ge 3$ is good for $\boldsymbol{G}$, and moreover $\ell \ne 3$ if $\boldsymbol{G}^F$ has a factor ${}^3D_4(q)$.*

Remark 3.15. Note that (e) is best possible. See [Enguehard 2000; Kessar and Malle 2013] for counterexamples to the conclusion for bad primes, and [Enguehard 2000, p. 348] for a counterexample in the case $\ell = 3$ and $\boldsymbol{G}^F = {}^3D_4(q)$. Counterexamples in the case $\ell = 2$ and $\boldsymbol{G}$ of type A_n occur in the following situation. Let

$\boldsymbol{G}^F = \mathrm{SL}_n(q)$ with $4 \mid (q+1)$. Then $e = 2$ and the unipotent 2-(Jordan-)cuspidal pairs of $\boldsymbol{G}^F$ correspond to 2-cores of partitions of $n-1$ (see [Broué et al. 1993, §3A]). On the other hand, by [Cabanes and Enguehard 1993, Theorem 13], $\boldsymbol{G}^F$ has a unique unipotent 2-block.

Also, part (d) is best possible as the next example shows.

Example 3.16. Consider $\boldsymbol{G} = \mathrm{SL}_n$ with $n > 1$ odd, $\tilde{\boldsymbol{G}} = \mathrm{GL}_n$, and let $\boldsymbol{G}^F = \mathrm{SL}_n(q)$ be such that $q \equiv 1 \pmod{n}$ and $4 \mid (q+1)$. Then for $\ell = 2$ we have $e = e_2(q) = 2$, and $\mathbb{F}_q$ contains a primitive n-th root of unity, say ζ. Let $\tilde{s} = \mathrm{diag}(1, \zeta, \dots, \zeta^{n-1}) \in \tilde{\boldsymbol{G}}^{*F}$ and let s be its image in $\boldsymbol{G}^* = \mathrm{PGL}_n$. Then $C^\circ_{\boldsymbol{G}^*}(s)$ is the maximal 1-torus consisting of the image of the diagonal torus of $\tilde{\boldsymbol{G}}^*$. Thus, $(C^\circ_{\boldsymbol{G}^*}(s))_2 = 1 = Z^\circ(\boldsymbol{G}^*)_2$.

As $|C_{\boldsymbol{G}^*}(s)^F : C^\circ_{\boldsymbol{G}^*}(s)^F| = n$ we have $|\mathcal{E}(\boldsymbol{G}^F, s)| = n$, and all of these characters are 2-Jordan quasicentral cuspidal. We claim that all elements of $\mathcal{E}(\boldsymbol{G}^F, s)$ lie in the same 2-block of $\boldsymbol{G}^F$, so do not satisfy the conclusion of Theorem 3.14(d).

Let $\tilde{\boldsymbol{T}}$ be a maximal torus of $\tilde{\boldsymbol{G}}$ in duality with $C_{\tilde{\boldsymbol{G}}^*}(s)$ and let $\tilde{\theta} \in \mathrm{Irr}(\tilde{\boldsymbol{T}}^F)$ in duality with $\tilde{s}$. Let $\boldsymbol{T} = \tilde{\boldsymbol{T}} \cap \boldsymbol{G}$, and let $\theta = \tilde{\theta}|_{\boldsymbol{T}^F}$. Since $\tilde{s}$ is regular, $\tilde{\lambda} := R^{\tilde{\boldsymbol{G}}}_{\tilde{\boldsymbol{T}}}(\theta) \in \mathrm{Irr}(\tilde{\boldsymbol{G}}^F)$, and $\mathcal{E}(\tilde{\boldsymbol{G}}^F, \tilde{s}) = \{\tilde{\lambda}\}$. Further, $\tilde{\lambda}$ covers every element of $\mathcal{E}(\boldsymbol{G}^F, s)$. By [Bonnafé 2005, Proposition 10.10(b*)],

$$R^{\boldsymbol{G}}_{\boldsymbol{T}}(\theta) = \mathrm{Res}^{\tilde{\boldsymbol{G}}^F}_{\boldsymbol{G}^F} R^{\tilde{\boldsymbol{G}}}_{\tilde{\boldsymbol{T}}}(\tilde{\theta}) = \mathrm{Res}^{\tilde{\boldsymbol{G}}^F}_{\boldsymbol{G}^F}(\tilde{\lambda}).$$

Thus, every element of $\mathcal{E}(\boldsymbol{G}^F, s)$ is a constituent of $R^{\boldsymbol{G}}_{\boldsymbol{T}}(\theta)$. On the other hand, since $\tilde{\boldsymbol{T}}$ is the torus of diagonal matrices, we have $\boldsymbol{T} = C_{\boldsymbol{G}}(\boldsymbol{T}_2^F)$ by explicit computation. Hence by [Kessar and Malle 2013, Propositions 2.12, 2.13(1), 2.16(1)], all constituents of $R^{\boldsymbol{G}}_{\boldsymbol{T}}(\theta)$ lie in a single 2-block of $\boldsymbol{G}^F$.

Proof of Theorem 3.14. Parts (a) and (b) are immediate from Theorem 3.4 and the proof of Theorem 3.6. We next consider part (e), where it remains to show injectivity under the given assumptions. By [Cabanes and Enguehard 1999, Theorem 4.1 and Remark 5.2] only $\ell = 3$ and $\boldsymbol{G}$ of (possibly twisted) type A_n remains to be considered. Note that the claim holds if $3 \in \Gamma(\boldsymbol{G}, F)$ by [Cabanes and Enguehard 1999, Section 5.2]. Thus we may assume that the ambient simple algebraic group $\boldsymbol{H}$ of simply connected type is either SL_m or E_6, and $3 \notin \Gamma(\boldsymbol{G}, F)$. By Proposition 3.10 the claim holds for all blocks whose defect groups are Cabanes.

Let first $\boldsymbol{H} = \mathrm{SL}_m$ and $\boldsymbol{G} \le \boldsymbol{H}$ be an F-stable Levi subgroup. As $3 \notin \Gamma(\boldsymbol{G}, F)$ we have $3 \mid (q-1)$ when F is untwisted. We postpone the twisted case for a moment. Embed $\boldsymbol{H} \hookrightarrow \tilde{\boldsymbol{H}} = \mathrm{GL}_m$. Then $\tilde{\boldsymbol{G}} = \boldsymbol{G}Z(\boldsymbol{H})$ is an F-stable Levi subgroup of $\tilde{\boldsymbol{H}}$, so has connected centre. Moreover, as $\tilde{\boldsymbol{H}}$ is self-dual, so is its Levi subgroup $\tilde{\boldsymbol{G}}$. In particular, $3 \in \Gamma(\tilde{\boldsymbol{G}}, F)$. Now let b be a 3-block of $\boldsymbol{G}^F$ in $\mathcal{E}_3(\boldsymbol{G}^F, s)$, with $s \in \boldsymbol{G}^{*F}$ a semisimple $3'$-element. Let $\tilde{b}$ be a block of $\tilde{\boldsymbol{G}}$ covering b, contained in $\mathcal{E}_3(\tilde{\boldsymbol{G}}^F, \tilde{s})$, where $\tilde{s}$ is a preimage of s under the induced map $\tilde{\boldsymbol{G}}^* \to \boldsymbol{G}^*$. Since $3 \mid (q-1)$, $C_{\tilde{\boldsymbol{G}}}(\tilde{s})^F$ has a single unipotent 3-block, and so by [Cabanes and Enguehard 1999, Proposition 5.1] a Sylow 3-subgroup $\tilde{D}$ of $C_{\tilde{G}}(\tilde{s})^F$ is a defect group of $\tilde{b}$. Thus, $D := \tilde{D} \cap \boldsymbol{G} = \tilde{D} \cap \boldsymbol{H}$ is a defect group of b.

Now $C_{\tilde{G}}(\tilde{s})$ is an F-stable Levi subgroup of $\tilde{\boldsymbol{G}}$, so also an F-stable Levi subgroup of $\tilde{\boldsymbol{H}} = \mathrm{GL}_m$. As such, it is a direct product of factors GL_{m_i} with $\sum_i m_i = m$. Assume that there is more than one F-orbit on the set of factors. Then by Lemma 3.11 the Sylow 3-subgroup $\tilde{D}$ of $C_{\tilde{G}}(\tilde{s})^F$ has the property that $D = \tilde{D} \cap \boldsymbol{H}$ is Cabanes and we are done. Hence, we may assume that F has just one orbit on the set of factors of $C_{\tilde{G}}(\tilde{s})$. But this is only possible if F has only one orbit on the set of factors of $\tilde{\boldsymbol{G}}$. This implies that $\tilde{\boldsymbol{G}}^F \cong \mathrm{GL}_n(q^{m/n})$ and $\boldsymbol{G}^F \cong \mathrm{SL}_n(q^{m/n})$ for some $n \mid m$.

Exactly the same arguments apply when F is twisted, except that now $3 \mid (q+1)$. So replacing q by $q^{m/n}$ we may now suppose that $\boldsymbol{G} = \mathrm{SL}_n$ with $3 \notin \Gamma(\boldsymbol{G}, F)$. Assume that the defect groups of b are not Cabanes. Let $(\boldsymbol{L}, \lambda)$ be an e-Jordan-cuspidal pair for b with $\lambda \in \mathcal{E}(\boldsymbol{L}^F, s)$ and let $\tilde{\boldsymbol{L}} = Z^\circ(\tilde{\boldsymbol{G}})\boldsymbol{L}$. There exists an irreducible character $\tilde{\lambda}$ of $\tilde{\boldsymbol{L}}^F$ covering λ, an irreducible constituent $\tilde{\chi}$ of $R_{\tilde{\boldsymbol{L}}}^{\tilde{\boldsymbol{G}}}(\tilde{\lambda})$ and an irreducible constituent, say χ of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ such that $\tilde{\chi}$ covers χ. By Lemma 2.3, $(\tilde{\boldsymbol{L}}, \tilde{\lambda})$ is e-Jordan-cuspidal. Let $\tilde{b}$ be the block of $\tilde{\boldsymbol{G}}^F$ associated to $(\tilde{\boldsymbol{L}}, \tilde{\lambda})$, contained in $\mathcal{E}_3(\tilde{\boldsymbol{G}}^F, \tilde{s})$. So, $\tilde{b}$ covers b.

As seen above $C_{\tilde{G}}(\tilde{s})^F$ has a single unipotent 3-block and a Sylow 3-subgroup $\tilde{D}$ of $C_{\tilde{G}}(\tilde{s})^F$ is a defect group of $\tilde{b}$ and $D := \tilde{D} \cap \boldsymbol{G}$ is a defect group of b. Moreover F has a single orbit on the set of factors of $C_{\tilde{G}}(\tilde{s})$. By Lemma 3.12, $C_{\tilde{G}}(\tilde{s})^F = \mathrm{GL}_3(q^{\frac{n}{3}})$ or $\mathrm{GU}_3(q^{\frac{n}{3}})$, 3 does not divide $\frac{n}{3}$ and D is extra-special of order 27 and exponent 3. Also, $\tilde{\boldsymbol{L}}$ is an e-split Levi subgroup isomorphic to a direct product of 3 copies of $\mathrm{GL}_{\frac{n}{3}}$.

Let $U = Z(\boldsymbol{L})_3^F$ and let c be the 3-block of $\boldsymbol{L}^F$ containing λ. From the structure of $\tilde{\boldsymbol{L}}$ given above, $|U| = 9$ and $\boldsymbol{L} = C_{\boldsymbol{G}}(U)$. Thus, by [Cabanes and Enguehard 1999, Theorem 2.5], (U, c) is a b-Brauer pair. Let (D, f) be a maximal b-Brauer pair such that $(U, c) \le (D, f)$.

Let $(\boldsymbol{L}', \lambda')$ be another e-Jordan-cuspidal pair for b with $\lambda' \in \mathcal{E}({\boldsymbol{L}'}^F, s)$. Let $U' = Z(\boldsymbol{L}')_3^F$ and let c' be the 3-block of ${\boldsymbol{L}'}^F$ containing λ', so $|U'| = 9$ and (U', c') is also a b-Brauer pair. Since all maximal b-Brauer pairs are $\boldsymbol{G}^F$-conjugate, there exists $h \in \boldsymbol{G}^F$ such that ${}^h(U', c') \le (D, f)$. Thus, U and ${}^hU'$ are subgroups of order 9 of D. By Lemma 3.13, there exists $g \in N_{\boldsymbol{G}^F}(D) \cap C_{\boldsymbol{G}^F}(C_{\boldsymbol{G}^F}(D))$ such that ${}^{gh}U' = U$. Since g centralises $C_{\boldsymbol{G}^F}(D)$, ${}^g f = f$ and since g normalises D, ${}^g D = D$. Hence

$$(U, {}^{gh}c') = {}^{gh}(U', c') \le {}^g(D, f) = (D, f).$$

By the uniqueness of inclusion of Brauer pairs we get that ${}^{gh}(U', c') = (U, c)$. Thus ${}^{gh}\boldsymbol{L}' = \boldsymbol{L}$ and ${}^{gh}c' = c$. Since U is abelian of maximal order in D, (U, c) is a self-centralising Brauer pair. In particular, there is a unique irreducible character in c with U in its kernel. Since $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$, U is contained in the kernel of λ. Hence ${}^{gh}\lambda' = \lambda$ and injectivity is proved for type A.

Finally suppose that $\boldsymbol{H}$ is of type E_6. By our preliminary reductions we may assume that $\boldsymbol{G}$ has only factors of type A and $3 \notin \Gamma(\boldsymbol{G}, F)$. Thus $\boldsymbol{G}$ must have at least one factor of type A_2 or A_5. The remaining possibilities hence are: $\boldsymbol{G}$ is of type A_5, $2A_2 + A_1$, or $2A_2$. Note that for $\boldsymbol{G}$ of type $2A_2 + A_1$, the A_1-factor of the derived subgroup $[\boldsymbol{G}, \boldsymbol{G}]$ splits off, and that $2A_2$ is a Levi subgroup of A_5. So it suffices to show the claim for Levi subgroups of this particular Levi subgroup $\boldsymbol{G}$ of type A_5. Since $\boldsymbol{H}$ is simply connected, $[\boldsymbol{G}, \boldsymbol{G}] \cong \mathrm{SL}_6$ and thus virtually the same arguments as for the case of $\boldsymbol{G} = \mathrm{SL}_n$ apply. This completes the proof of (e).

Part (d) follows whenever $\ell \geq 3$ is good for $\boldsymbol{G}$, and $\ell \neq 3$ if $\boldsymbol{G}^F$ has a factor ${}^3D_4(q)$, since then by (e) there is a unique e-Jordan-cuspidal pair for any ℓ-block, and its (unipotent) Jordan correspondent has quasicentral ℓ-defect by [Cabanes and Enguehard 1994, Proposition 4.3] and Remark 2.2. So now assume that either $\ell \geq 3$ is bad for $\boldsymbol{G}$, or that $\ell = 3$ and $\boldsymbol{G}^F$ has a factor ${}^3D_4(q)$.

Note that it suffices to prove the statement for quasi-isolated blocks, since then it follows tautologically for all others using the Jordan correspondence, Proposition 2.4 and the remarks after Definition 2.12. Here note that by Lemma 2.5 the bijections of Proposition 2.4 extend to conjugacy classes of pairs. We first prove surjectivity. For this, by Lemma 3.7, Lemma 2.7 and by parts (a) and (b), we may assume that $\boldsymbol{G} = [\boldsymbol{G}, \boldsymbol{G}]$. Further, since $[\boldsymbol{G}, \boldsymbol{G}]$ is simply connected, hence a direct product of its components, we may assume that $\boldsymbol{G}$ is simple. Then surjectivity for unipotent blocks follows from [Enguehard 2000, Theorems A and A.bis], while for all other quasi-isolated blocks it is shown in [Kessar and Malle 2013, Theorem 1.2] (these also include the case that $\boldsymbol{G}^F = {}^3D_4(q)$).

Now we prove injectivity. If $\boldsymbol{G} = \boldsymbol{H}$, then the claim for unipotent blocks follows from [Enguehard 2000, Theorems A and A.bis], while for all other quasi-isolated blocks it is shown in [Kessar and Malle 2013, Theorem 1.2] (these also include the case that $\boldsymbol{G}^F = {}^3D_4(q)$). Note that in Table 4 of [Kessar and Malle 2013], each of the lines 6, 7, 10, 11, 14 and 20 give rise to two e-cuspidal pairs and so to two e-Harish-Chandra series, but each e-Jordan cuspidal pair $(\boldsymbol{L}, \lambda)$ which corresponds to these lines has the Cabanes property of Lemma 3.9, so they give rise to different blocks.

So, we may assume that $\boldsymbol{G} \neq \boldsymbol{H}$, and thus $\ell = 3$. Suppose first that $\boldsymbol{G}^F$ has a factor ${}^3D_4(q)$. Then $\boldsymbol{H}$ is of type E_6, E_7 or E_8, there is one component of $[\boldsymbol{G}, \boldsymbol{G}]$ of type D_4 and all other components are of type A. Denote by $\boldsymbol{G}_2$ the component of type D_4, and by $\boldsymbol{G}_1$ the product of the remaining components with $Z^\circ(\boldsymbol{G})$. We note that $Z(\boldsymbol{G}_1)/Z^\circ(\boldsymbol{G}_1)$ is a $3'$-group. Indeed, if $\boldsymbol{H}$ is of type E_7 or E_8, then $Z(\boldsymbol{G})/Z^\circ(\boldsymbol{G})$ is of order prime to 3, hence the same is true of $Z(\boldsymbol{G}_1)/Z^\circ(\boldsymbol{G}_1)$ and if $\boldsymbol{H}$ is of type E_6, then $\boldsymbol{G}_1 = Z^\circ(\boldsymbol{G})$.

Now, $\boldsymbol{G}^F = \boldsymbol{G}_1^F \times \boldsymbol{G}_2^F$. So, the map $((\boldsymbol{L}_1, \lambda_1), (\boldsymbol{L}_2, \lambda_2)) \to (\boldsymbol{L}_1\boldsymbol{L}_2, \lambda_1\lambda_2)$ is a bijection between pairs of e-Jordan cuspidal pairs for $\boldsymbol{G}_1^F$ and $\boldsymbol{G}_2^F$ and e-Jordan cuspidal pairs for $\boldsymbol{G}^F$. The bijection preserves conjugacy and quasicentrality. All components of $\boldsymbol{G}_1$ are of type A and as noted above 3 does not divide the order of $Z(\boldsymbol{G}_1)/Z^\circ(\boldsymbol{G}_1)$, hence by [Cabanes and Enguehard 1999, Section 5.2] we may assume that $\boldsymbol{G} = \boldsymbol{G}_2$, in which case we are done by [Enguehard 2000, Theorem A] and [Kessar and Malle 2013, Lemma 6.13].

Thus, $\boldsymbol{G}^F$ has no factor ${}^3D_4(q)$. Set $\boldsymbol{G}_0 := [\boldsymbol{G}, \boldsymbol{G}]$. Since 3 is bad for $\boldsymbol{G}$, and $\boldsymbol{G}$ is proper in $\boldsymbol{H}$, we are in one of the following cases: $\boldsymbol{H}$ is of type E_7 and $\boldsymbol{G}_0$ is simple of type E_6, or $\boldsymbol{G}$ is of type E_8 and $\boldsymbol{G}_0$ is of type E_6, $E_6 + A_1$ or E_7. In all cases, note that $Z(\boldsymbol{G})$ is connected,

Let $s \in \boldsymbol{G}^{*F}$ be a quasi-isolated semisimple $3'$-element. Let $\bar{s}$ be the image of s under the surjection $\boldsymbol{G}^* \to \boldsymbol{G}_0^*$. Since $Z(\boldsymbol{G})$ is connected, s is isolated in $\boldsymbol{G}^*$ and consequently $\bar{s}$ is isolated in $\boldsymbol{G}_0^*$. In particular, if $\boldsymbol{G}_0$ has a component of type A_1, then the projection of $\bar{s}$ into that factor is the identity. Since s has order prime to 3, this means that if $\boldsymbol{G}_0$ has a component of type E_6, then $C_{\boldsymbol{G}_0^*}(\bar{s})$ is connected. We will use this fact later. Also, we note here that $\bar{s} \neq 1$ as otherwise the result would follow from [Enguehard 2000] and the standard correspondence between unipotent blocks and blocks lying in central Lusztig series. Finally, we note that by [Kessar and Malle 2013, Theorem 1.2] the conclusion of parts (a) and (d) of the theorem holds for $\boldsymbol{G}_0^F$ as all components of $\boldsymbol{G}_0$ are of different type (so e is the same for the factors of $\boldsymbol{G}_0^F$ as for $\boldsymbol{G}^F$).

Let b be a 3-block of $\boldsymbol{G}^F$ in the series s and $(\boldsymbol{L}, \lambda)$ be an e-Jordan quasicentral cuspidal pair for b such that $s \in \boldsymbol{L}^{*F}$ and $\lambda \in \mathcal{E}(\boldsymbol{L}^F, s)$. Let $\boldsymbol{L}_0 = \boldsymbol{L} \cap \boldsymbol{G}_0$ and let λ_0 be an irreducible constituent of the restriction of λ to $\boldsymbol{L}_0^F$. By Lemma 3.8 there exists a block b_0 of $\boldsymbol{G}_0^F$ covered by b, and such that all irreducible constituents of $R_{\boldsymbol{L}_0}^{\boldsymbol{G}_0}(\lambda_0)$ belong to b. By Lemma 2.3 and the remarks following Definition 2.12, $(\boldsymbol{L}_0, \lambda_0)$ is an e-Jordan quasicentral cuspidal pair of $\boldsymbol{G}_0^F$ for b_0.

First suppose that $C_{\boldsymbol{G}_0}(\bar{s})$ is connected. Then all elements of $\mathcal{E}(\boldsymbol{G}_0^F, \bar{s})$ are $\boldsymbol{G}^F$-stable and in particular, b_0 is $\boldsymbol{G}^F$-stable. Now let $(\boldsymbol{L}', \lambda')$ be another e-Jordan quasicentral cuspidal pair for b. Let $\boldsymbol{L}_0' = \boldsymbol{L}' \cap \boldsymbol{G}_0$ and λ_0' be an irreducible constituent of the restriction of λ' to $\boldsymbol{L}_0'^F$. Then, as above $(\boldsymbol{L}_0', \lambda_0')$ is an e-Jordan quasicentral cuspidal pair for b_0. But there is a unique e-Jordan quasicentral cuspidal pair for b_0 up to $\boldsymbol{G}_0^F$-conjugacy. So, up to replacing by a suitable $\boldsymbol{G}_0^F$-conjugate we may assume that $(\boldsymbol{L}_0, \lambda_0) = (\boldsymbol{L}_0', \lambda_0')$, hence $\boldsymbol{L} = \boldsymbol{L}'$, and λ and λ' cover the same character $\lambda_0 = \lambda_0'$ of $\boldsymbol{L}_0^F = \boldsymbol{L}_0'^F$.

If $\mu \in \mathcal{E}(\boldsymbol{G}_0^F, \bar{s})$, then there are $|\boldsymbol{G}^F/\boldsymbol{G}_0^F|_{3'}$ different $3'$-Lusztig series of $\boldsymbol{G}^F$ containing an irreducible character covering μ. Since characters in different $3'$-Lusztig series lie in different 3-blocks, there are at least $|\boldsymbol{G}^F/\boldsymbol{G}_0^F|_{3'}$ different blocks of $\boldsymbol{G}^F$ covering b_0. Moreover, if b' is a block of $\boldsymbol{G}^F$ covering b_0, then there exists a linear character, say θ of $\boldsymbol{G}^F/\boldsymbol{G}_0^F \cong \boldsymbol{L}^F/\boldsymbol{L}_0^F$ of $3'$-degree such that $(\boldsymbol{L}, \theta \otimes \lambda)$ is an e-Jordan quasicentral cuspidal pair for b' and λ_0 appears in the restriction of $\theta \otimes \lambda$ to $\boldsymbol{L}_0^F$. Since there are at most $|\boldsymbol{L}^F/\boldsymbol{L}_0^F|_{3'} = |\boldsymbol{G}^F/\boldsymbol{G}_0^F|_{3'}$ irreducible characters of $\boldsymbol{L}^F$ in $3'$-series covering λ_0, it follows that $\lambda = \lambda'$.

Thus, we may assume that $C_{G_0}(\bar{s})$ is not connected. Hence, by the remarks above G_0 is simple of type E_7. Further $\bar{s}$ corresponds to one of the lines 5, 6, 7, 12, 13, or 14 of Table 4 of [Kessar and Malle 2013] (note that $\bar{s}$ is isolated and that e-Jordan (quasi-)central cuspidality in this case is the same as e-(quasi-)central cuspidality).

By [Kessar and Malle 2013, Lemma 5.2], $L_0 = C_{G_0}(Z(L_0^F)_3)$. In other words, (L_0, λ_0) is a good pair for b_0 in the sense of [Kessar and Malle 2013, Definition 7.10]. In particular, there is a maximal b_0-Brauer pair (P_0, c_0) such that $(Z(L_0^F)_3, b_{L_0^F}(\lambda_0)) \trianglelefteq (P_0, c_0)$. Here for a finite group X and an irreducible character η of X, we denote by $b_X(\eta)$ the ℓ-block of X containing η. By inspection of the relevant lines of Table 4 of [Kessar and Malle 2013] (and the proof of [Kessar and Malle 2013, Theorem 1.2]), one sees that the maximal Brauer pair (P_0, c_0) can be chosen so that $Z(L_0^F)_3$ is the unique maximal abelian normal subgroup of P_0.

By [Kessar and Malle 2013, Theorem 7.11] there exists a maximal b-Brauer pair (P, c) and $\nu \in \mathcal{E}(L^F, \ell')$ such that ν covers λ_0, $P_0 \le P$ and we have an inclusion of b-Brauer pairs $(Z(L^F)_3, b_{L^F}(\nu)) \trianglelefteq (P, c)$. Since λ also covers λ_0, $\lambda = \tau \otimes \nu$ for some linear character τ of $L^F/L_0^F \cong G^F/G_0^F$. Since tensoring with linear characters preserves block distribution and commutes with Brauer pair inclusion, replacing c with the block of $C_{G^F}(P_0)$ whose irreducible characters are of the form $\tau \otimes \varphi$, $\varphi \in \mathrm{Irr}(c)$, we get that there exists a maximal b-Brauer pair (P, c) such that $P_0 \le P$ and $(Z(L^F)_3, b_{L^F}(\lambda)) \trianglelefteq (P, c)$.

Being normal in G^F, $Z(G^F)_3$ is contained in the defect groups of every block of G^F, and in particular $Z(G^F)_3 \le P$. On the other hand, since G_0 has centre of order 2, $P_0 Z(G^F)_3$ is a defect group of b whence P is a direct product of P_0 and $Z(G^F)_3$. Now, $Z(L_0^F)_3$ is the unique maximal abelian normal subgroup of P_0. Hence, $Z(L^F)_3 = Z(G^F)_3 \times Z(L_0^F)_3$ is the unique maximal normal abelian subgroup of P (see Lemma 3.11). Finally note that by Lemma 2.7, λ is also of quasicentral ℓ-defect. By Lemma 3.9 it follows that up to conjugacy (L, λ) is the unique e-Jordan quasicentral cuspidal pair of G^F for b.

Finally, we show (c). In view of the part (d) just proved above, it remains to consider the prime $\ell = 2$ only. Suppose first that all components of G are of classical type. Let $s \in G^{*F}$ be semisimple of odd order and let b be a 2-block of G^F in series s. By Lemma 3.17 below there is an e-torus, say S of $C^\circ_{G^*}(s)$ such that $T^* := C_{C^\circ_{G^*}(s)}(S)$ is a maximal torus of $C^\circ_{G^*}(s)$. Let $L^* = C_{G^*}(S)$ and let L be a Levi subgroup of G in duality with L^*. Then L is an e-split subgroup of G and $T^* = C^\circ_{L^*}(s)$. Let $\lambda \in \mathrm{Irr}(L^F, s)$ correspond via Jordan decomposition to the trivial character of T^{*F}. Then (L, λ) is an e-Jordan quasicentral cuspidal pair of G.

Let $G \hookrightarrow \tilde{G}$ be a regular embedding. By part (a), Lemmas 3.3 and 3.8, there exists $g \in \tilde{G}^F$ such that $b = b_{G^F}({}^g L, {}^g \lambda)$. Now since (L, λ) is e-Jordan quasicentral cuspidal, so is $({}^g L, {}^g \lambda)$. In order to see this, first note that, up to multiplication by a suitable element of G^F and by an application of the Lang–Steinberg theorem, we may assume that g is in some F-stable maximal torus of $Z^\circ(\tilde{\boldsymbol{G}})\boldsymbol{L}$. Thus ${}^g\boldsymbol{L}=\boldsymbol{L}$, and λ and ${}^g\lambda$ correspond to the same $C_{\boldsymbol{L}^*}(s)^F$ orbit of unipotent characters of $C^\circ_{\boldsymbol{L}^*}(s)^F$.

Now suppose that $\boldsymbol{G}$ has a component of exceptional type. Then we can argue just as in the proof of surjectivity for bad ℓ in part (d). $\square$

Lemma 3.17. *Let $\boldsymbol{G}$ be connected reductive with a Frobenius morphism $F:\boldsymbol{G}\to\boldsymbol{G}$. Let $e\in\{1,2\}$ and let $\boldsymbol{S}$ be a Sylow e-torus of $\boldsymbol{G}$. Then $C_{\boldsymbol{G}}(\boldsymbol{S})$ is a torus.*

Proof. Let $\boldsymbol{C}:=[C_{\boldsymbol{G}}(\boldsymbol{S}),C_{\boldsymbol{G}}(\boldsymbol{S})]$ and assume that $\boldsymbol{C}$ has semisimple rank at least one. Let $\boldsymbol{T}$ be a maximally split torus of $\boldsymbol{C}$. Then the Sylow 1-torus of $\boldsymbol{T}$, hence of $\boldsymbol{C}$ is nontrivial. Similarly, the reductive group $\boldsymbol{C}'$ with complete root datum obtained from that of $\boldsymbol{C}$ by replacing the automorphism on the Weyl group by its negative, again has a nontrivial Sylow 1-torus. But then $\boldsymbol{C}$ also has a nontrivial Sylow 2-torus. Thus in any case $\boldsymbol{C}$ has a noncentral e-torus, which is a contradiction to its definition. $\square$

4. Jordan decomposition of blocks

Lusztig induction induces Morita equivalences between Jordan corresponding blocks. We show that this also behaves nicely with respect to e-cuspidal pairs and their corresponding e-Harish-Chandra series.

Jordan decomposition and e-cuspidal pairs. Throughout this subsection, $\boldsymbol{G}$ is a connected reductive algebraic group with a Frobenius endomorphism $F:\boldsymbol{G}\to\boldsymbol{G}$ endowing $\boldsymbol{G}$ with an $\mathbb{F}_q$-structure for some power q of p. Our results here are valid for all groups $\boldsymbol{G}^F$ satisfying the Mackey-formula for Lusztig induction. At present this is known to hold unless $\boldsymbol{G}$ has a component $\boldsymbol{H}$ of type E_6, E_7 or E_8 with $\boldsymbol{H}^F\in\{{}^2E_6(2),E_7(2),E_8(2)\}$, see Bonnafé–Michel [2011]. The following is in complete analogy with Proposition 2.4:

Proposition 4.1. *Assume that $\boldsymbol{G}^F$ has no factor ${}^2E_6(2)$, $E_7(2)$ or $E_8(2)$. Let $s\in\boldsymbol{G}^{*F}$, and $\boldsymbol{G}_1\le\boldsymbol{G}$ an F-stable Levi subgroup with $\boldsymbol{G}_1^*$ containing $C_{\boldsymbol{G}^*}(s)$. For $(\boldsymbol{L}_1,\lambda_1)$ an e-cuspidal pair of $\boldsymbol{G}_1$ below $\mathcal{E}(\boldsymbol{G}_1^F,s)$ define $\boldsymbol{L}:=C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L}_1)_e)$ and $\lambda:=\epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{L}_1}R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda_1)$. Then $(\boldsymbol{L}_1,\lambda_1)\mapsto(\boldsymbol{L},\lambda)$ defines a bijection $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ between the set of e-cuspidal pairs of $\boldsymbol{G}_1$ below $\mathcal{E}(\boldsymbol{G}_1^F,s)$ and the set of e-cuspidal pairs of $\boldsymbol{G}$ below $\mathcal{E}(\boldsymbol{G}^F,s)$.*

Proof. We had already seen in the proof of Proposition 2.4 that $\boldsymbol{L}$ is e-split and $Z^\circ(\boldsymbol{L}_1)_e=Z^\circ(\boldsymbol{L})_e$. For the well-definedness of $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ it remains to show that λ is e-cuspidal. For any e-split Levi subgroup $\boldsymbol{X}\le\boldsymbol{L}$ the Mackey formula [Bonnafé and Michel 2011, Theorem] gives

$$\epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{L}_1}{}^*R_{\boldsymbol{X}}^{\boldsymbol{L}}(\lambda)={}^*R_{\boldsymbol{X}}^{\boldsymbol{L}}R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda_1)=\sum_g R^{\boldsymbol{X}}_{\boldsymbol{X}\cap{}^g\boldsymbol{L}_1}{}^*R^{{}^g\boldsymbol{L}_1}_{\boldsymbol{X}\cap{}^g\boldsymbol{L}_1}(\lambda_1^g)$$

where the sum runs over a suitable set of double coset representatives $g \in \boldsymbol{L}^F$. Here, $\boldsymbol{X} \cap {}^g\boldsymbol{L}_1$ is e-split in $\boldsymbol{L}_1$ since $\boldsymbol{L}_1 \cap \boldsymbol{X}^g = \boldsymbol{L}_1 \cap C_{\boldsymbol{L}}(Z^\circ(\boldsymbol{X}^g)_e) = C_{\boldsymbol{L}_1}(Z^\circ(\boldsymbol{X}^g)_e)$. The e-cuspidality of λ_1 thus shows that the only nonzero terms in the above sum are those for which $\boldsymbol{L}_1 \cap \boldsymbol{X}^g = \boldsymbol{L}_1$, i.e., those with $\boldsymbol{L}_1 \le \boldsymbol{X}^g$. But then $Z^\circ(\boldsymbol{L})_e = Z^\circ(\boldsymbol{L}_1)_e = Z^\circ(\boldsymbol{X}^g)_e$, and as $\boldsymbol{X}$ is e-split in $\boldsymbol{L}$ we deduce that necessarily $\boldsymbol{X} = \boldsymbol{L}$ if ${}^*R_{\boldsymbol{X}}^{\boldsymbol{L}}(\lambda) \ne 0$. So λ is indeed e-cuspidal, and $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ is well-defined.

Injectivity was shown in the proof of Proposition 2.4, where we had constructed an inverse map with $\boldsymbol{L}_1^* := \boldsymbol{L}^* \cap \boldsymbol{G}_1^*$ and λ_1 the unique constituent of ${}^*R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda)$ in $\mathcal{E}(\boldsymbol{L}_1^F, s)$. We claim that λ_1 is e-cuspidal. Indeed, for any e-split Levi subgroup $\boldsymbol{X} \le \boldsymbol{L}_1$ let $\boldsymbol{Y} := C_{\boldsymbol{L}}(Z^\circ(\boldsymbol{X})_e)$, an e-split Levi subgroup of $\boldsymbol{L}$. Then ${}^*R_{\boldsymbol{X}}^{\boldsymbol{L}_1}(\lambda_1)$ is a constituent of

$$ {}^*R_{\boldsymbol{X}}^{\boldsymbol{L}}(\lambda) = {}^*R_{\boldsymbol{X}}^{\boldsymbol{Y}}\,{}^*R_{\boldsymbol{Y}}^{\boldsymbol{L}}(\lambda) = 0 $$

by e-cuspidality of λ, unless $\boldsymbol{Y} = \boldsymbol{L}$, whence $\boldsymbol{X} = \boldsymbol{Y} \cap \boldsymbol{L}_1 = \boldsymbol{L} \cap \boldsymbol{L}_1 = \boldsymbol{L}_1$.

Thus we have obtained a well-defined map ${}^*\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ from e-cuspidal pairs in $\boldsymbol{G}$ to e-cuspidal pairs in $\boldsymbol{G}_1$, both below the series s. The rest of the proof is again as for Proposition 2.4. □

Jordan decomposition, e-cuspidal pairs and ℓ-blocks. We next remove two of the three possible exceptions in Proposition 4.1 for characters in ℓ'-series:

Lemma 4.2. *The assertions of Proposition 4.1 remain true for $\boldsymbol{G}^F$ having no factor $E_8(2)$ whenever $s \in \boldsymbol{G}^{*F}$ is a semisimple ℓ'-element, where $e = e_\ell(q)$. In particular, $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ exists.*

Proof. Let s be a semisimple ℓ'-element. Then by [Cabanes and Enguehard 1999, Theorem 4.2] we may assume that $\ell \le 3$, so in fact $\ell = 3$. The character table of $\boldsymbol{G}^{*F} = {}^2E_6(2).3$ is known; there are 12 classes of nontrivial elements $s \in \boldsymbol{G}^{*F}$ of order prime to 6. Their centralisers $C_{\boldsymbol{G}^*}(s)$ only have factors of type A, and are connected. Thus all characters in those series $\mathcal{E}(\boldsymbol{G}^F, s)$ are uniform, so the Mackey-formula is known for them with respect to any Levi subgroup. Thus, the argument in Proposition 4.1 is applicable to those series. For $\boldsymbol{G}^F = E_7(2)$, the conjugacy classes of semisimple elements can be found in [Lübeck]. From this one verifies that again all nontrivial semisimple $3'$-elements have centraliser either of type A, or of type ${}^2D_4(q)A_1(q)\Phi_4$, or ${}^3D_4(q)\Phi_1\Phi_3$. In the latter two cases, proper Levi subgroups are either direct factors, or again of type A, and so once more the Mackey-formula is known to hold with respect to any Levi subgroup. □

Remark 4.3. The assertion of Lemma 4.2 can be extended to most ℓ'-series of $\boldsymbol{G}^F = E_8(2)$. Indeed, again by [Cabanes and Enguehard 1999, Theorem 4.2] we only need to consider $\ell \in \{3, 5\}$. For $\ell = 3$ there are just two types of Lusztig series for $3'$-elements which can not be treated by the arguments above, with corresponding centraliser $E_6(2)\Phi_3$ respectively ${}^2D_6(2)\Phi_4$. For $\ell = 5$, there are five types of Lusztig series, with centraliser ${}^2E_6(2){}^2A_2(2)$, $E_7(2)\Phi_2$, ${}^2D_7(2)\Phi_2$, $E_6(2)\Phi_3$ and ${}^2D_5(2)\Phi_2\Phi_6$ respectively. Note that the first one is isolated, so the assertion can be checked using [Kessar and Malle 2013].

Proposition 4.4. *Assume that $\boldsymbol{G}^F$ has no factor $E_8(2)$. Let $s \in \boldsymbol{G}^{*F}$ be a semisimple ℓ'-element, and $\boldsymbol{G}_1 \leq \boldsymbol{G}$ an F-stable Levi subgroup with $\boldsymbol{G}_1^*$ containing $C_{\boldsymbol{G}^*}(s)$. Assume that b is an ℓ-block in $\mathcal{E}_\ell(\boldsymbol{G}^F, s)$, and c is its Jordan correspondent in $\mathcal{E}_\ell(\boldsymbol{G}_1^F, s)$. Let $e = e_\ell(q)$.*

(a) *Let $(\boldsymbol{L}_1, \lambda_1)$ be e-cuspidal in $\boldsymbol{G}_1$, where $(\boldsymbol{L}, \lambda) = \Psi^{\boldsymbol{G}}_{\boldsymbol{G}_1}(\boldsymbol{L}_1, \lambda_1)$. If all constituents of $R^{\boldsymbol{G}_1}_{\boldsymbol{L}_1}(\lambda_1)$ lie in c, then all constituents of $R^{\boldsymbol{G}}_{\boldsymbol{L}}(\lambda)$ lie in b.*

(b) *Let $(\boldsymbol{L}, \lambda)$ be e-cuspidal in $\boldsymbol{G}$, where $(\boldsymbol{L}_1, \lambda_1) = {}^*\Psi^{\boldsymbol{G}}_{\boldsymbol{G}_1}(\boldsymbol{L}, \lambda)$. If all constituents of $R^{\boldsymbol{G}}_{\boldsymbol{L}}(\lambda)$ lie in b, then all constituents of $R^{\boldsymbol{G}_1}_{\boldsymbol{L}_1}(\lambda_1)$ lie in c.*

The proof is identical to the one of Proposition 2.6, using Proposition 4.1 and Lemma 4.2 in place of Proposition 2.4.

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