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Dedicated to the memory of Professor Robert Steinberg.

In this paper we construct free resolutions of a class of closed subvarieties of affine spaces (the so-called "opposite big cells" of Grassmannians). Our class covers the determinantal varieties, whose resolutions were first constructed by A. Lascoux (*Adv. in Math.* 30:3 (1978), 202–237). Our approach uses the geometry of Schubert varieties. An interesting aspect of our work is its connection to the computation of the cohomology of homogeneous bundles (that are not necessarily completely reducible) on partial flag varieties.

1. Introduction

A classical problem in commutative algebra and algebraic geometry is to describe the syzygies of the defining ideals of interesting varieties. Let $k \le n \le m$ be positive integers. The space D_k of $m \times n$ matrices (over a field $\Bbbk$) of rank at most k is a closed subvariety of the mn-dimensional affine space of all $m \times n$ matrices. When $\Bbbk = \mathbb{C}$, a minimal free resolution of the coordinate ring $\Bbbk[D_k]$ as a module over the coordinate ring of the mn-dimensional affine space (i.e., the mn-dimensional polynomial ring) was constructed by A. Lascoux [1978]; see also [Weyman 2003, Chapter 6].

In this paper, we construct free resolutions for a larger class of singularities, *viz.*, Schubert singularities, i.e., the intersection of a singular Schubert variety and the "opposite big cell" inside a Grassmannian. The advantage of our method is that it is algebraic group-theoretic, and is likely to work for Schubert singularities in more general flag varieties. In this process, we have come up with a method to compute the cohomology of certain homogeneous vector bundles (which are not completely reducible) on flag varieties. We will work over $\Bbbk = \mathbb{C}$.

Let $N = m + n$. Let $\mathrm{GL}_N = \mathrm{GL}_N(\mathbb{C})$ be the group of $N \times N$ invertible matrices. Let B_N be the Borel subgroup of all upper-triangular matrices and B_N^- the *opposite*

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Borel subgroup of all lower-triangular matrices in GL_N. Let P be the maximal parabolic subgroup corresponding to omitting the simple root α_n, i.e., the subgroup of GL_N comprising the matrices in which the (i, j)-th entry (i.e., in row i and column j) is zero, if $n+1 \le i \le N$ and $1 \le j \le n$; in other words,

$$P = \left\{ \begin{bmatrix} A_{n\times n} & C_{n\times m} \\ 0_{m\times n} & E_{m\times m} \end{bmatrix} \in \mathrm{GL}_N \right\}.$$

We have a canonical identification of the Grassmannian of n-dimensional subspaces of $\Bbbk^N$ with GL_N/P. Let W and W_P be the Weyl groups of GL_N and of P, respectively; note that $W = S_N$ (the symmetric group) and $W_P = S_n \times S_m$. For $w \in W/W_P$, let $X_P(w) \subseteq \mathrm{GL}_N/P$ be the Schubert variety corresponding to w (i.e., the closure of the B_N-orbit of the coset $wP \in \mathrm{GL}_N/P$, equipped with the canonical reduced scheme structure). The B_N^--orbit of the coset $(\mathrm{id}\cdot P)$ in GL_N/P is denoted by $O^-_{\mathrm{GL}_N/P}$, and is usually called the *opposite big cell* in GL_N/P; it can be identified with the mn-dimensional affine space. (See Section 2.2.)

Write W^P for the set of minimal representatives (under the Bruhat order) in W for the elements of W/W_P. For $1 \le r \le n-1$, we consider certain subsets $\mathcal{W}_r$ of W^P (Definition 3.3); there is $w \in \mathcal{W}_{n-k}$ such that $D_k = X_P(w) \cap O^-_{\mathrm{GL}_N/P}$. Note that for any $w \in W^P$, $X_P(w) \cap O^-_{\mathrm{GL}_N/P}$ is a closed subvariety of $O^-_{\mathrm{GL}_N/P}$. Our main result is a description of the minimal free resolution of the coordinate ring of $X_P(w) \cap O^-_{\mathrm{GL}_N/P}$ as a module over the coordinate ring of $O^-_{\mathrm{GL}_N/P}$ for every $w \in \mathcal{W}_r$. This latter ring is a polynomial ring. We now outline our approach.

First we recall the Kempf–Lascoux–Weyman "geometric technique" of constructing minimal free resolutions. Suppose that we have a commutative diagram of varieties

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "q'"'] & \mathbb{A} \times V \arrow[r] \arrow[d, "q"] & V \\ Y \arrow[r, hook] & \mathbb{A} & \end{tikzcd} \tag{1-1}$$

where $\mathbb{A}$ is an affine space, Y a closed subvariety of $\mathbb{A}$ and V a projective variety and q is the projection to the first factor. Suppose further that the (necessarily proper) map q' is birational, and that the inclusion $Z \hookrightarrow \mathbb{A} \times V$ is a subbundle (over V) of the trivial bundle $\mathbb{A} \times V$. Let ξ be the dual of the quotient bundle on V corresponding to Z. Then the derived direct image $\boldsymbol{R}q'_*\mathcal{O}_Z$ is quasi-isomorphic to a minimal complex $F_\bullet$ with

$$F_i = \bigoplus_{j\ge 0} \mathrm{H}^j(V, \textstyle\bigwedge^{i+j}\xi) \otimes_{\mathbb{C}} R(-i-j).$$

Here R is the coordinate ring of $\mathbb{A}$; it is a polynomial ring and $R(k)$ refers to twisting with respect to its natural grading. If q' is such that the natural map

$\mathcal{O}_Y \to \boldsymbol{R}q'_*\mathcal{O}_Z$ is a quasi-isomorphism, (for example, if q' is a desingularization of Y and Y has rational singularities), then $F_\bullet$ is a minimal free resolution of $\mathbb{C}[Y]$ over the polynomial ring R.

The difficulty in applying this technique in any given situation is two-fold: one must find a suitable morphism $q' : Z \to Y$ such that the map $\mathcal{O}_Y \to \boldsymbol{R}q'_*\mathcal{O}_Z$ is a quasi-isomorphism and such that Z is a vector bundle over a projective variety V; and, one must be able to compute the necessary cohomology groups. We overcome this for opposite cells in a certain class (which includes the determinantal varieties) of Schubert varieties in a Grassmannian, in two steps.

As the first step, we need to establish the existence of a diagram as above. This is done using the geometry of Schubert varieties. We take $\mathbb{A} = O^-_{\mathrm{GL}_N/P}$ and

$$Y = Y_P(w) := X_P(w) \cap O^-_{\mathrm{GL}_N/P}.$$

Let $\tilde{P}$ be a parabolic subgroup with $B_N \subseteq \tilde{P} \subsetneq P$. The inverse image of $O^-_{\mathrm{GL}_N/P}$ under the natural map $\mathrm{GL}_N/\tilde{P} \to \mathrm{GL}_N/P$ is $O^-_{\mathrm{GL}_N/P} \times P/\tilde{P}$. Let $\tilde{w}$ be the representative of the coset $w\tilde{P}$ in $W^{\tilde{P}}$. Then $X_{\tilde{P}}(\tilde{w}) \subseteq \mathrm{GL}_N/\tilde{P}$ (the Schubert subvariety of $\mathrm{GL}_N/\tilde{P}$ associated to $\tilde{w}$) maps properly and birationally onto $X_P(w)$. We may choose $\tilde{P}$ to ensure that $X_{\tilde{P}}(\tilde{w})$ is smooth. Let $Z_{\tilde{P}}(\tilde{w})$ be the preimage of $Y_P(w)$ in $X_{\tilde{P}}(\tilde{w})$. We take $Z = Z_{\tilde{P}}(\tilde{w})$. Then V, which is the image of Z under the second projection, is a smooth Schubert subvariety of $P/\tilde{P}$. The vector bundle ξ on V that we obtain is the restriction of a homogeneous bundle on $P/\tilde{P}$. Thus we get:

$$\begin{tikzcd} Z_{\tilde{P}}(\tilde{w}) \arrow[r, hook] \arrow[d, "q'"] & O^-_{\mathrm{GL}_N/P} \times V \arrow[r] \arrow[d, "q"] & V \\ Y_P(w) \arrow[r, hook] & O^- & \end{tikzcd} \tag{1-2}$$

See Theorem 3.7 and Corollary 3.9. In this diagram, q' is a desingularization of $Y_P(w)$. Since it is known that Schubert varieties have rational singularities, we have that the map $\mathcal{O}_Y \to \boldsymbol{R}q'_*\mathcal{O}_Z$ is a quasi-isomorphism, so $F_\bullet$ is a minimal resolution.

As the second step, we need to determine the cohomology of the homogeneous bundles $\bigwedge^t \xi$ over V. There are two ensuing issues: computing cohomology of homogeneous vector bundles over Schubert subvarieties of flag varieties is difficult, and furthermore, these bundles are not usually completely reducible, so one cannot apply the Borel–Weil–Bott theorem directly. We address the former issue by restricting our class; if $w \in \mathcal{W}_r$ (for some r) then V will equal $P/\tilde{P}$. Regarding the latter issue, we inductively replace $\tilde{P}$ by larger parabolic subgroups (still inside P), such that at each stage, the computation reduces to that of the cohomology of completely reducible bundles on Grassmannians; using various spectral sequences, we are able to determine the cohomology groups that determine the minimal free resolution. See Proposition 5.5 for the key inductive step. In contrast, in Lascoux's construction of the resolution of determinantal ideals, one comes across only completely reducible bundles; therefore, one may use the Borel–Weil–Bott theorem to compute the cohomology of the bundles $\bigwedge^t \xi$. The idea of using $\mathbb{P}^1$-fibrations for the computation of cohomology on flag varieties and their Schubert varieties goes back to M. Demazure [1968; 1974]; see also the related "one-step construction" of Kempf [1976].

Computing cohomology of homogeneous bundles, in general, is difficult, and is of independent interest; we hope that our approach would be useful in this regard. The best results, as far as we know, are due to G. Ottaviani and E. Rubei [2006], which deal with general homogeneous bundles on Hermitian symmetric spaces. The only Hermitian symmetric spaces in Type A are the Grassmannians, so their results do not apply to our situation.

Since the opposite big cell $O^-_{\mathrm{GL}_N/P}$ intersects every B_N-orbit of GL_N/P, the variety $Y_P(w)$ captures all the singularities of $X_P(w)$ for every $w \in W$. In this paper, we describe a construction of a minimal free resolution of $\mathbb{C}[Y_P(w)]$ over $\mathbb{C}[O^-_{\mathrm{GL}_N/P}]$. We hope that our methods could shed some light on the problem of construction of a locally free resolution of $\mathcal{O}_{X_P(w)}$ as an $\mathcal{O}_{\mathrm{GL}_N/P}$-module.

The paper is organized as follows. Section 2 contains notations and conventions (Section 2.1) and the necessary background material on Schubert varieties (Section 2.2) and homogeneous bundles (Section 2.3). In Section 3, we discuss properties of Schubert desingularization, including the construction of Diagram (1-2). Section 4 is devoted to a review of the Kempf–Lascoux–Weyman technique and its application to our problem. Section 5 explains how the cohomology of the homogeneous bundles on certain partial flag varieties can be computed; Section 6 gives some examples. Finally, in Section 7, we describe Lascoux's resolution in terms of our approach and describe the multiplicity and Castelnuovo–Mumford regularity of $\mathbb{C}[Y_P(w)]$.

2. Preliminaries

In this section, we collect various results about Schubert varieties, homogeneous bundles, and the Kempf–Lascoux–Weyman geometric technique.

2.1. ***Notation and conventions.*** We collect the symbols used and the conventions adopted in the rest of the paper here. For details on algebraic groups and Schubert varieties, the reader may refer to [Borel 1991; Jantzen 2003; Billey and Lakshmibai 2000; Seshadri 2007].

Let $m \geq n$ be positive integers and $N = m + n$. We denote by GL_N (respectively, B_N, B_N^-) the group of all (respectively, upper-triangular, lower-triangular) invertible $N \times N$ matrices over $\mathbb{C}$. The Weyl group W of GL_N is isomorphic to the group

S_N of permutations of N symbols and is generated by the *simple reflections* s_i, for $1 \le i \le N-1$, which correspond to the transpositions $(i, i+1)$. For $w \in W$, its *length* is the smallest integer l such that $w = s_{i_1} \cdots s_{i_l}$ as a product of simple reflections. For every $1 \le i \le N-1$, there is a *minimal parabolic subgroup* P_i containing s_i (thought of as an element of GL_N) and a *maximal parabolic subgroup* $P_{\hat{\imath}}$ *not containing* s_i. Any parabolic subgroup can be written as $P_{\hat{A}} := \bigcap_{i \in A} P_{\hat{\imath}}$ for some $A \subset \{1, \dots, N-1\}$. On the other hand, for $A \subseteq \{1, \dots, N-1\}$ write P_A for the subgroup of GL_N generated by P_i, $i \in A$. Then P_A is a parabolic subgroup and $P_{\{1,\dots,N-1\} \setminus A} = P_{\hat{A}}$.

The following is fixed for the rest of this paper:

(a) P is the maximal parabolic subgroup $P_{\hat{n}}$ of GL_N;

(b) for $1 \le s \le n-1$, $\tilde{P}_s$ is the parabolic subgroup $P_{\{1,\dots,s-1,n+1,\dots,N-1\}} = \cap_{i=s}^{n} P_{\hat{\imath}}$ of GL_N;

(c) for $1 \le s \le n-1$, Q_s is the parabolic subgroup $P_{\{1,\dots,s-1\}} = \cap_{i=s}^{n-1} P_{\hat{\imath}}$ of GL_n.

We write the elements of W in *one-line* notation: $(a_1, \dots, a_N)$ is the permutation $i \mapsto a_i$. For any $A \subseteq \{1, \dots, N-1\}$, define W_{P_A} to be the subgroup of W generated by $\{s_i : i \in A\}$. By W^{P_A} we mean the subset of W consisting of the minimal representatives (under the Bruhat order) in W of the elements of W/W_{P_A}. For $1 \le i \le N$, we represent the elements of $W^{P_{\hat{\imath}}}$ by sequences $(a_1, \dots, a_i)$ with $1 \le a_1 < \cdots < a_i \le N$ since under the action of the group $W_{P_{\hat{\imath}}}$, every element of W can be represented minimally by such a sequence.

For $w = (a_1, a_2, \dots, a_n) \in W^P$, let $r(w)$ be the integer r such that $a_r \le n < a_{r+1}$.

We identify $\mathrm{GL}_N = \mathrm{GL}(V)$ for some N-dimensional vector-space V. Let $A := \{i_1 < i_2 < \cdots < i_r\} \subseteq \{1, \dots, N-1\}$. Then $\mathrm{GL}_N / P_{\hat{A}}$ is the set of all flags

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_r \subsetneq V$$

of subspaces V_j of dimension i_j inside V. We call $\mathrm{GL}_N / P_{\hat{A}}$ a *flag variety*. If $A = \{1, \dots, N-1\}$ (i.e., $P_{\hat{A}} = B_N$), then we call the flag variety a *full flag variety*; otherwise, a *partial flag variety*. The *Grassmannian* $\mathrm{Gr}_{i,N}$ of i-dimensional subspaces of V is $\mathrm{GL}_N / P_{\hat{\imath}}$.

Let $\tilde{P}$ be any parabolic subgroup containing B_N and $\tau \in W$. The *Schubert variety* $X_{\tilde{P}}(\tau)$ is the closure inside $\mathrm{GL}_N / \tilde{P}$ of $B_N \cdot e_\tau$ where e_τ is the coset $\tau \tilde{P}$, endowed with the canonical reduced scheme structure. Hereafter, when we write $X_{\tilde{P}}(\tau)$, we mean that τ is the representative in $W^{\tilde{P}}$ of its coset. The *opposite big cell* $O^-_{\mathrm{GL}_N/\tilde{P}}$ in $\mathrm{GL}_N / \tilde{P}$ is the B_N^--orbit of the coset $(\mathrm{id} \cdot \tilde{P})$ in $\mathrm{GL}_N / \tilde{P}$. Let $Y_{\tilde{P}}(\tau) := X_{\tilde{P}}(\tau) \cap O^-_{\mathrm{GL}_N/\tilde{P}}$; we refer to $Y_{\tilde{P}}(\tau)$ as the *opposite cell* of $X_{\tilde{P}}(\tau)$.

We will write R^+, R^-, $R^+_{\tilde{P}}$, $R^-_{\tilde{P}}$, to denote, respectively, positive and negative roots for GL_N and for $\tilde{P}$. We denote by ϵ_i the character that sends the invertible diagonal matrix with $t_1, \dots, t_n$ on the diagonal to t_i.

2.2. ***Précis on*** GL_n ***and Schubert varieties.*** Let $\tilde{P}$ be a parabolic subgroup of GL_N with $B_N \subseteq \tilde{P} \subseteq P$. We will use the following proposition extensively in the sequel.

Proposition 2.2.1. *Write* $U_{\tilde{P}}^-$ *for the negative unipotent radical of* $\tilde{P}$.

(a) $O^-_{\mathrm{GL}_N/\tilde{P}}$ *can be naturally identified with* $U_{\tilde{P}}^- \tilde{P}/\tilde{P}$.

(b) *For*

$$z = \begin{bmatrix} A_{n\times n} & C_{n\times m} \\ D_{m\times n} & E_{m\times m} \end{bmatrix} \in \mathrm{GL}_N,$$

$zP \in O^-_{\mathrm{GL}_N/P}$ *if and only if* A *is invertible.*

(c) *For* $1 \le s \le n-1$, *the inverse image of* $O^-_{\mathrm{GL}_N/P}$ *under the natural map* $\mathrm{GL}_N/\tilde{P}_s \to \mathrm{GL}_N/P$ *is isomorphic to* $O^-_{\mathrm{GL}_N/P} \times P/\tilde{P}_s$ *as schemes. Every element of* $O^-_{\mathrm{GL}_N/P} \times P/\tilde{P}_s$ *is of the form*

$$\begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \bmod \tilde{P}_s \in \mathrm{GL}_N/\tilde{P}_s.$$

Moreover, two matrices

$$\begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A'_{n\times n} & 0_{n\times m} \\ D'_{m\times n} & I_m \end{bmatrix}$$

in GL_N *represent the same element modulo* $\tilde{P}_s$ *if and only if there exists a matrix* $q \in Q_s$ *such that* $A' = Aq$ *and* $D' = Dq$.

(d) *For* $1 \le s \le n-1$, $P/\tilde{P}_s$ *is isomorphic to* GL_n/Q_s. *In particular, the projection map* $O^-_{\mathrm{GL}_N/P} \times P/\tilde{P} \to P/\tilde{P}_s$ *is given by*

$$\begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \bmod \tilde{P}_s \longmapsto A \bmod \tilde{Q} \in \mathrm{GL}_n/Q \simeq P/\tilde{P}_s.$$

Proof. (a) Note that $U_{\tilde{P}}^-$ is the subgroup of GL_N generated by the (one-dimensional) root subgroups $U_\alpha, \alpha \in R^- \setminus R_{\tilde{P}}$ and that $U_{\tilde{P}}^- \tilde{P}/\tilde{P} = B_N^- \tilde{P}/\tilde{P}$. Hence under the canonical projection $\mathrm{GL}_N \to \mathrm{GL}_N/P$, $g \mapsto gP$, the subgroup U_P^- is mapped onto $O^-_{\mathrm{GL}_N/\tilde{P}}$. It is easy to check that this is an isomorphism.

(b) Suppose that $zP \in O^-_{\mathrm{GL}_N/P}$. By (a), we see that there exist matrices $A'_{n\times n}$, $C'_{n\times m}$, $D'_{m\times n}$, and $E'_{m\times m}$ such that

$$z_1 := \begin{bmatrix} I_n & 0_{n\times m} \\ D'_{m\times n} & I_m \end{bmatrix} \in U_P^-, \quad z_2 := \begin{bmatrix} A'_{n\times n} & C'_{n\times m} \\ 0_{m\times n} & E'_{m\times m} \end{bmatrix} \in P$$

$$\text{and } z = \begin{bmatrix} A_{n\times n} & C_{n\times m} \\ D_{m\times n} & E_{m\times m} \end{bmatrix} = z_1 z_2.$$

Hence $A = A'$ is invertible. Conversely, if A is invertible, then we may write $z = z_1 z_2$ where

$$z_1 := \begin{bmatrix} I_n & 0 \\ DA^{-1} & I_m \end{bmatrix} \in U_P^- \quad \text{and} \quad z_2 := \begin{bmatrix} A & C \\ 0 & E - DA^{-1}C \end{bmatrix}.$$

Since $z \in \mathrm{GL}_N$, $z_2 \in P$.

<u>(c)</u> Let $z \in U_P^- P \subseteq \mathrm{GL}_N$. Then we can write $z = z_1 z_2$ uniquely with $z_1 \in U_P^-$ and $z_2 \in P$. For, if

$$\begin{bmatrix} I_n & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \begin{bmatrix} A_{n\times n} & C_{n\times m} \\ 0_{m\times n} & E_{m\times m} \end{bmatrix} = \begin{bmatrix} I_n & 0_{n\times m} \\ D'_{m\times n} & I_m \end{bmatrix} \begin{bmatrix} A'_{n\times n} & C'_{n\times m} \\ 0_{m\times n} & E'_{m\times m} \end{bmatrix},$$

then $A = A'$, $C = C'$, $DA = D'A'$, and $DC + E = D'C' + E'$, which yields that $D' = D$ (since $A = A'$ is invertible, by (b)) and $E = E'$. Hence $U_P^- \times_{\mathbb{C}} P = U_P^- P$. Therefore, for any parabolic subgroup $P' \subseteq P$, $U_P^- \times_{\mathbb{C}} P/P' = U_P^- P/P'$. The asserted isomorphism now follows by taking $P' = \tilde{P}_s$.

For the next statement, let

$$\begin{bmatrix} A_{n\times n} & C_{n\times m} \\ D_{m\times n} & E_{m\times m} \end{bmatrix} \in \mathrm{GL}_N$$

with A invertible (which we may assume by (b)). Then we have a decomposition (in GL_N)

$$\begin{bmatrix} A & C \\ D & E \end{bmatrix} = \begin{bmatrix} A & 0_{n\times m} \\ D & I_m \end{bmatrix} \begin{bmatrix} I_n & A^{-1}C \\ 0_{m\times n} & E - DA^{-1}C \end{bmatrix}.$$

Hence

$$\begin{bmatrix} A & C \\ D & E \end{bmatrix} \equiv \begin{bmatrix} A & 0_{n\times m} \\ D & I_m \end{bmatrix} \mod \tilde{P}_s.$$

Finally,

$$\begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \equiv \begin{bmatrix} A'_{n\times n} & 0_{n\times m} \\ D'_{m\times n} & I_m \end{bmatrix} \mod \tilde{P}_s$$

if and only if there exist matrices $q \in Q_s$, $q'_{n\times m}$, and $\tilde{q}_{n\times n}$ in GL_m such that

$$\begin{bmatrix} A' & 0 \\ D' & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ D & I \end{bmatrix} \begin{bmatrix} q & q' \\ 0_{m\times n} & \tilde{q} \end{bmatrix},$$

which holds if and only if $q' = 0$, $\tilde{q} = I_m$, $A' = Aq$, and $D' = Dq$ (since A and A' are invertible).

<u>(d)</u> There is a surjective morphism of $\mathbb{C}$-group schemes $P \to \mathrm{GL}_n$,

$$\begin{bmatrix} A_{n\times n} & C_{n\times m} \\ 0_{m\times n} & E_{m\times m} \end{bmatrix} \longmapsto A.$$

This induces the required isomorphism. Notice that the element

$$\begin{bmatrix} A_{n\times n} & C_{n\times m} \\ D_{m\times n} & E_{m\times m} \end{bmatrix} \bmod \tilde{P}_s \in O^-_{\mathrm{GL}_N/P} \times P/\tilde{P}_s$$

decomposes (uniquely) as

$$\begin{bmatrix} I_n & 0 \\ DA^{-1} & I_m \end{bmatrix} \left(\begin{bmatrix} A & C \\ 0 & E \end{bmatrix} \bmod \tilde{P}_s \right)$$

Hence it is mapped to $A \bmod Q_s \in \mathrm{GL}_n/Q_s$. Now use (c). □

Discussion 2.2.2. Let $\tilde{P} = P_{\{\widehat{i_1,\dots,i_t}\}}$ with $1 \le i_1 < \cdots < i_t \le N-1$. Then using Proposition 2.2.1(a) and its proof, $O^-_{\mathrm{GL}_N/\tilde{P}}$ can be identified with the affine space of lower-triangular matrices with possible nonzero entries x_{ij} at row i and column j where (i, j) is such that there exists $l \in \{i_1, \dots, i_t\}$ such that $j \le l < i \le N$. To see this, note (from the proof of Proposition 2.2.1(a)) that we are interested in those (i, j) such that the root $\epsilon_i - \epsilon_j$ belongs to $R^- \setminus R^-_{\tilde{P}}$. Since $R^-_{\tilde{P}} = \bigcap_{k=1}^{t} R^-_{P_{\widehat{i_k}}}$, we see that we are looking for (i, j) such that $\epsilon_i - \epsilon_j \in R^- \setminus R^-_{P_{\hat{l}}}$ for some $l \in \{i_1, \dots, i_t\}$. For the maximal parabolic group $P_{\hat{l}}$, we have $R^- \setminus R^-_{P_{\hat{l}}} = \{\epsilon_i - \epsilon_j \mid 1 \le j \le l < i \le N\}$. Hence $\dim O^-_{\mathrm{GL}_N/\tilde{P}} = |R^- \setminus R^-_{\tilde{P}}|$.

Let $\alpha = \epsilon_i - \epsilon_j \in R^- \setminus R^-_{\tilde{P}}$ and $l \in \{i_1, \dots, i_t\}$. Then the Plücker coordinate $p^{(l)}_{s_\alpha}$ on the Grassmannian $\mathrm{GL}_N/P_{\hat{l}}$ lifts to a regular function on $\mathrm{GL}_N/\tilde{P}$, which we denote by the same symbol. Its restriction to $O^-_{G/\tilde{P}}$ is the $l \times l$-minor with column indices $\{1, 2, \dots, l\}$ and row indices $\{1, \dots, j-1, j+1, \dots, l, i\}$. In particular,

$$x_{ij} = p^{(j)}_{s_\alpha}|_{O^-_{G/\tilde{P}}} \quad \text{for every } \alpha = \epsilon_i - \epsilon_j \in R^- \setminus R^-_{\tilde{P}}. \tag{2.2.3}$$

Example 2.2.4. Figure 1 shows the shape of $O^-_{\mathrm{GL}_N/\tilde{P}_s}$ for some $1 \le s \le n-1$. The rectangular region labelled with a circled D is $O^-_{\mathrm{GL}_N/P}$. The trapezoidal region labelled with a circled A is $O^-_{P/\tilde{P}_s}$. In this case, the x_{ij} appearing in (2.2.3) are exactly those in the regions labelled A and B.

Remark 2.2.5. $X_{\tilde{P}}(w)$ is an irreducible (and reduced) variety of dimension equal to the length of w. (Here we use that w is the representative in $W^{\tilde{P}}$.) It can be seen easily that under the natural projection $\mathrm{GL}_N/B_N \to \mathrm{GL}_N/\tilde{P}$, $X_{B_N}(w)$ maps birationally onto $X_{\tilde{P}}(w)$ for every $w \in W^{\tilde{P}}$. It is known that Schubert varieties are normal, Cohen–Macaulay and have rational singularities; see, e.g., [Brion and Kumar 2005, Section 3.4].

2.3. *Homogeneous bundles and representations.*

Let Q be a parabolic subgroup of GL_n. We collect here some results about homogeneous vector bundles on GL_n/Q. Most of these results are well known, but for some of them, we could not find a reference, so we give a proof here for the sake of completeness. Online notes of Ottaviani [1995] and of D. Snow [1994] discuss the details of many of these results.

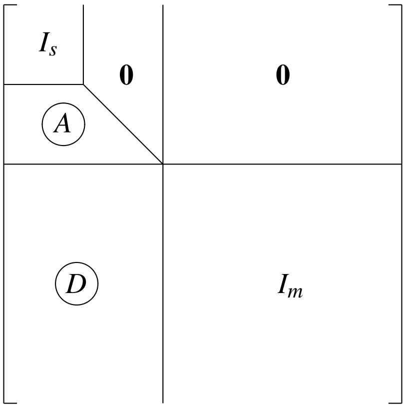

Figure 1. Shape of $O^-_{\mathrm{GL}_N/\tilde{P}_s}$.

Let L_Q and U_Q be respectively the Levi subgroup and the unipotent radical of Q. Let E be a finite-dimensional vector-space on which Q acts on the right; the vector-spaces that we will encounter have natural right action.

Definition 2.3.1. Define $\mathrm{GL}_n \times^Q E := (\mathrm{GL}_n \times E)/\sim$, where $\sim$ is the equivalence relation $(g, e) \sim (gq, eq)$ for every $g \in \mathrm{GL}_n$, $q \in Q$, and $e \in E$. Then $\pi_E : \mathrm{GL}_n \times^Q E \to \mathrm{GL}_n/Q$, $(g, e) \mapsto gQ$, is a vector bundle called the *vector bundle associated to* E (and the principal Q-bundle $\mathrm{GL}_n \to \mathrm{GL}_n/Q$). For $g \in \mathrm{GL}_n$, $e \in E$, we write $[g, e] \in \mathrm{GL}_n \times^Q E$ for the equivalence class of $(g, e) \in \mathrm{GL}_n \times E$ under $\sim$. We say that a vector bundle $\pi : \boldsymbol{E} \to \mathrm{GL}_n/Q$ is *homogeneous* if $\boldsymbol{E}$ has a GL_n-action and π is GL_n-equivariant, i.e., for every $y \in \boldsymbol{E}$, $\pi(g \cdot y) = g \cdot \pi(y)$.

In this section, we abbreviate $\mathrm{GL}_n \times^Q E$ as $\tilde{E}$. It is known that $\boldsymbol{E}$ is homogeneous if and only if $\boldsymbol{E} \simeq \tilde{E}$ for some Q-module E. (If this is the case, then E is the fibre of $\boldsymbol{E}$ over the coset Q.) A homogeneous bundle $\tilde{E}$ is said to be *irreducible* (respectively, *indecomposable*, *completely reducible*) if E is an irreducible (respectively indecomposable, completely reducible) Q-module. It is known that E is completely reducible if and only if U_Q acts trivially and that E is irreducible if and only if additionally it is irreducible as a representation of L_Q. See [Snow 1994, Section 5] or [Ottaviani 1995, Section 10] for the details.

Let $\sigma : \mathrm{GL}_n/Q \to \tilde{E}$ be a section of π_E. Let $g \in \mathrm{GL}_n$; write $[h, f] = \sigma(gQ)$. There exists a unique $q \in Q$ such that $h = gq$. Let $e = fq^{-1}$. Then $[g, e] = [h, f]$. If $[h, f'] = [h, f]$, then $f' = f$, so the assignment $g \mapsto e$ defines a function $\phi : \mathrm{GL}_n \to E$. This is Q-equivariant in the following sense:

$$\phi(gq) = \phi(g)q, \quad \text{for every } q \in Q \text{ and } g \in \mathrm{GL}_n. \tag{2.3.2}$$

Conversely, any such map defines a section of π_E. The set of sections $\mathrm{H}^0(\mathrm{GL}_n/Q, \tilde{E})$ of π_E is a finite-dimensional vector-space with $(\alpha\phi)(g) = \alpha(\phi(g))$ for every $\alpha \in \mathbb{C}$, ϕ a section of π_E, and $g \in \mathrm{GL}_n$.

Note that GL_n acts on GL_n/Q by multiplication on the left; setting $h \cdot [g, e] = [hg, e]$ for $g, h \in \mathrm{GL}_n$ and $e \in E$, we extend this to $\tilde{E}$. We can also define a natural GL_n-action on $\mathrm{H}^0(\mathrm{GL}_n/Q, \tilde{E})$ as follows. For any map $\phi : \mathrm{GL}_n \to E$, set $h \circ \phi$ to be the map $g \mapsto \phi(h^{-1}g)$. If ϕ satisfies (2.3.2), then for every $q \in Q$ and $g \in \mathrm{GL}_n$, $(h \circ \phi)(gq) = \phi(h^{-1}gq) = (\phi(h^{-1}g))q = ((h \circ \phi)(g))q$, so $h \circ \phi$ also satisfies (2.3.2). The action of GL_n on the sections is on the left:

$$(h_2h_1) \circ \phi = [g \mapsto \phi(h_1^{-1}h_2^{-1}g)] = [g \mapsto (h_1 \circ \phi)(h_2^{-1}g)] = h_2 \circ (h_1 \circ \phi).$$

In fact, $\mathrm{H}^i(\mathrm{GL}_n/Q, \tilde{E})$ is a GL_n-module for every i.

Suppose now that E is one-dimensional. Then Q acts on E by a character λ; we denote the associated line bundle on GL_n/Q by L_λ.

Discussion 2.3.3. Let $Q = P_{\widehat{i_1,\dots,i_t}}$, with $1 \le i_1 < \cdots < i_t \le n-1$. A weight λ is said to be *Q-dominant* if when we write $\lambda = \sum_{i=1}^n a_i\omega_i$ in terms of the fundamental weights ω_i, we have, $a_i \ge 0$ for all $i \notin \{i_1, \dots, i_t\}$, or equivalently, the associated line bundle (defined above) L_λ on Q/B_n has global sections. If we express λ as $\sum_{i=1}^n \lambda_i\epsilon_i$, then λ is Q-dominant if and only if for every $0 \le j \le t$, $\lambda_{i_j+1} \ge \lambda_{i_j+2} \ge \cdots \ge \lambda_{i_{j+1}}$ where we set $i_0 = 0$ and $i_{r+1} = n$. We will write $\lambda = (\lambda_1, \dots, \lambda_n)$ to mean that $\lambda = \sum_{i=1}^n \lambda_i\epsilon_i$. Every finite-dimensional irreducible Q-module is of the form $\mathrm{H}^0(Q/B_n, L_\lambda)$ for a Q-dominant weight λ. Hence the irreducible homogeneous vector bundles on GL_n/Q are in correspondence with Q-dominant weights. We describe them now. If $Q = P_{\widehat{n-i}}$, then $\mathrm{GL}_n/Q = \mathrm{Gr}_{i,n}$. (Recall that, for us, the GL_n-action on $\mathbb{C}^n$ is on the right.) On $\mathrm{Gr}_{i,n}$, we have the *tautological sequence*

$$0 \to \mathcal{R}_i \to \mathbb{C}^n \otimes \mathcal{O}_{\mathrm{Gr}_{i,n}} \to \mathcal{Q}_{n-i} \to 0 \tag{2.3.4}$$

of homogeneous vector bundles. The bundle $\mathcal{R}_i$ is called the *tautological subbundle* (of the trivial bundle $\mathbb{C}^n$) and $\mathcal{Q}_{n-i}$ is called the *tautological quotient bundle*. Every irreducible homogeneous bundle on $\mathrm{Gr}_{i,n}$ is of the form $\mathrm{S}_{(\lambda_1,\dots,\lambda_{n-i})}\,\mathcal{Q}_{n-i}^* \otimes \mathrm{S}_{(\lambda_{n-i+1},\dots,\lambda_n)}\,\mathcal{R}_i^*$ for some $P_{\widehat{n-i}}$-dominant weight λ. Here S_μ denotes the *Schur functor* associated to the partition μ. Now suppose that $Q = P_{\widehat{i_1,\dots,i_t}}$ with $1 \le i_1 < \cdots < i_t \le n-1$. Since the action is on the right, GL_n/Q projects to $\mathrm{Gr}_{n-i,n}$ precisely when $i = i_j$ for some $1 \le j \le t$. For each $1 \le j \le t$, we can take the pullback of the tautological bundles $\mathcal{R}_{n-i_j}$ and $\mathcal{Q}_{i_j}$ to GL_n/Q from $\mathrm{GL}_n/P_{\hat{i}_j}$. The irreducible homogeneous bundle corresponding to a Q-dominant weight λ is

$$\begin{aligned}&\mathrm{S}_{(\lambda_1,\dots,\lambda_{i_1})}\,\mathcal{U}_{i_1} \otimes \mathrm{S}_{(\lambda_{i_1+1},\dots,\lambda_{i_2})}(\mathcal{R}_{n-i_1}/\mathcal{R}_{n-i_2})^* \otimes \\ &\qquad \cdots \otimes \mathrm{S}_{(\lambda_{i_{t-1}+1},\dots,\lambda_{i_t})}(\mathcal{R}_{n-i_{t-1}}/\mathcal{R}_{n-i_t})^* \otimes \mathrm{S}_{(\lambda_{i_t+1},\dots,\lambda_{i_n})}(\mathcal{R}_{n-i_t})^*.\end{aligned}$$

See [Weyman 2003, Section 4.1]. Hereafter, we will write $\mathcal{U}_i = \mathcal{Q}_i^*$. Moreover, abusing notation, we will use $\mathcal{R}_i, \mathcal{Q}_i, \mathcal{U}_i$ etc. for these vector bundles on any (partial) flag variety on which they would make sense.

A Q-dominant weight is called $(i_1, \dots, i_r)$*-dominant* in [op. cit., p. 114]. Although our definition looks like Weyman's definition, we should keep in mind that our action is on the right. We only have to be careful when we apply the Borel–Weil–Bott theorem (more specifically, Bott's algorithm). In this paper, our computations are done only on Grassmannians. If μ and ν are partitions, then (μ, ν) will be Q-dominant (for a suitable Q), and will give us the vector bundle $\mathrm{S}_\mu\, \mathcal{Q}^* \otimes \mathrm{S}_\nu\, \mathcal{R}^*$ (this is where the right-action of Q becomes relevant) and to compute its cohomology, we will have to apply Bott's algorithm to the Q-dominant weight (ν, μ). (In [op. cit.], one would get $\mathrm{S}_\mu\, \mathcal{R}^* \otimes \mathrm{S}_\nu\, \mathcal{Q}^*$ and would apply Bott's algorithm to (μ, ν).) See, for example, the proof of Proposition 5.4 or the examples that follow it.

Proposition 2.3.5. *Let $Q_1 \subseteq Q_2$ be parabolic subgroups and E a Q_1-module. Let $f : \mathrm{GL}_n/Q_1 \to \mathrm{GL}_n/Q_2$ be the natural map. Then for every $i \geq 0$,*

$$R^i f_*(\mathrm{GL}_n \times^{Q_1} E) = \mathrm{GL}_n \times^{Q_2} \mathrm{H}^i(Q_2/Q_1, \mathrm{GL}_n \times^{Q_1} E).$$

Proof. For Q_2 (respectively, Q_1), the category of homogeneous vector bundles on GL_n/Q_2 (respectively, GL_n/Q_1) is equivalent to the category of finite-dimensional Q_2-modules (respectively, finite-dimensional Q_1-modules). Now, the functor f^* from the category of homogeneous vector bundles over GL_n/Q_2 to that over GL_n/Q_1 is equivalent to the restriction functor $\mathrm{Res}_{Q_1}^{Q_2}$. Hence their corresponding right-adjoint functors f_* and the induction functor $\mathrm{Ind}_{Q_1}^{Q_2}$ are equivalent; one may refer to [Hartshorne 1977, II.5, p. 110] and [Jantzen 2003, I.3.4, "Frobenius Reciprocity"] to see that these are indeed adjoint pairs. Hence, for homogeneous bundles on GL_n/Q_1, $R^i f_*$ can be computed using $R^i\mathrm{Ind}_{Q_1}^{Q_2}$. On the other hand, note that $\mathrm{Ind}_{Q_1}^{Q_2}(-)$ is the functor $\mathrm{H}^0(Q_2/Q_1, \mathrm{GL}_n \times^{Q_1} -)$ on Q_1-modules, which follows from [op. cit., I.3.3, Equation (2)]. The proposition now follows. □

3. Properties of Schubert desingularization

This section is devoted to proving some results on smooth Schubert varieties in partial flag varieties. In Theorem 3.5, we show that opposite cells of certain smooth Schubert varieties in $\mathrm{GL}_N/\tilde{P}$ are linear subvarieties of the affine variety $O^-_{\mathrm{GL}_N/\tilde{P}}$, where $\tilde{P} = \tilde{P}_s$ for some $1 \leq s \leq n-1$. Using this, we show in Theorem 3.7 that if $X_P(w) \in \mathrm{GL}_N/P$ is such that there exists a parabolic subgroup $\tilde{P} \subsetneq P$ such that the birational model $X_{\tilde{P}}(\tilde{w}) \subseteq \mathrm{GL}_N/\tilde{P}$ of $X_P(w)$ is smooth (we say that $X_P(w)$ has a *Schubert desingularization* if this happens) then the inverse image of $Y_P(w)$

inside $X_{\tilde{P}}(\tilde{w})$ is a vector bundle over a Schubert variety in $P/\tilde{P}$. This will give us a realisation of Diagram (1-2).

Recall the following result about the tangent space of a Schubert variety; see [Billey and Lakshmibai 2000, Chapter 4] for details.

Proposition 3.1. *Let $\tau \in W^{\tilde{P}}$. Then the dimension of the tangent space of $X_{\tilde{P}}(\tau)$ at e_{id} is*

$$\#\{s_\alpha \mid \alpha \in R^- \setminus R^-_{\tilde{P}} \text{ and } \tau \geq s_\alpha \text{ in } W/W_{\tilde{P}}\}.$$

In particular, $X_{\tilde{P}}(\tau)$ is smooth if and only if

$$\dim X_{\tilde{P}}(\tau) = \#\{s_\alpha \mid \alpha \in R^- \setminus R^-_{\tilde{P}} \text{ and } \tau \geq s_\alpha \text{ in } W/W_{\tilde{P}}\}.$$

Notation 3.2. We adopt the following notation: Let $w = (a_1, a_2, \ldots, a_n) \in W^P$. Let $r = r(w)$, i.e., the index r such that $a_r \leq n < a_{r+1}$. Let $1 \leq s \leq r$. We write $\tilde{P} = \tilde{P}_s$. Let $\tilde{w}$ be the minimal representative of w in $W^{\tilde{P}}$. Let $c_{r+1} > \cdots > c_n$ be such that $\{c_{r+1}, \ldots, c_n\} = \{1, \ldots, n\} \setminus \{a_1, \ldots, a_r\}$; let $w' := (a_1, \ldots, a_r, c_{r+1}, \ldots, c_n) \in S_n$, the Weyl group of GL_n.

Our concrete descriptions of free resolutions will be for the following class of Schubert varieties.

Definition 3.3. Let $1 \leq r \leq n-1$. Let

$$\mathcal{W}_r = \{(n-r+1, \ldots, n, a_{r+1}, \ldots, a_{n-1}, N) \in W^P : n < a_{r+1} < \cdots < a_{n-1} < N\}.$$

The determinantal variety of $(m \times n)$ matrices of rank at most k can be realised as $Y_P(w)$, $w = (k+1, \ldots, n, N-k+1, \ldots N) \in \mathcal{W}_{n-k}$ [Seshadri 2007, Section 1.6].

Proposition 3.4. *$X_{\tilde{P}_s}(\tilde{w})$ is smooth in the following situations*:

(a) *$w \in W^P$ arbitrary and $s = 1$* [Kempf 1971].

(b) *$w \in \mathcal{W}_r$ for some $1 \leq r \leq n-1$ and $s = r$.*

Proof. For both (a) and (b): Let $w_{\max} \in W (= S_N)$ be the maximal representative of $\tilde{w}$. We claim that

$$w_{\max} = (a_s, a_{s-1}, \ldots, a_1, a_{s+1}, a_{s+2}, \ldots, a_n, b_{n+1}, \ldots, b_N) \in W.$$

Assume the claim. Then $w_{\max}$ is a 4231- and 3412-avoiding element of W; hence $X_{B_N}(w_{\max})$ is smooth; see [Lakshmibai and Sandhya 1990; Billey and Lakshmibai 2000, 8.1.1]. Since $w_{\max}$ is the maximal representative (in W) of $\tilde{w}\tilde{P}_s$, we see that $X_{B_N}(w_{\max})$ is a fibration over $X_{\tilde{P}_s}(\tilde{w})$ with smooth fibres $\tilde{P}_s/B_N$; therefore $X_{\tilde{P}_s}(\tilde{w})$ is smooth.

To prove the claim, we need to show that $X_{P_{\hat{i}}}(w_{\max}) = X_{P_{\hat{i}}}(\tilde{w})$ for every $s \leq i \leq n$ and that $w_{\max}$ is the maximal element of W with this property. This follows, since for every $\tau := (c_1, \ldots, c_N) \in W$ and for every $1 \leq i \leq N$, $X_{P_{\hat{i}}}(\tau) = X_{P_{\hat{i}}}(\tau')$ where $\tau' \in W^{P_{\hat{i}}}$ is the element with $c_1, \ldots, c_i$ written in the increasing order. □

Theorem 3.5. *Identify* $O^-_{G/\tilde{P}}$ *with* $O^-_{G/P} \times O^-_{P/\tilde{P}}$ *as in* Figure 1, *with* $O^-_{G/P}$ *thought of as* $M_{m,n}$, *the space of all* $m \times n$ *matrices. If* $w \in W^P$ *is arbitrary and* $s = 1$ (*see* Proposition 3.4(a)) *then we have an identification of* $Y_{\tilde{P}}(\tilde{w})$ *with* $\mathcal{V}_w \times \mathcal{V}'_w$, *where* $\mathcal{V}_w$ *is the linear subspace of* $O^-_{G/P}$ *given by*

$$x_{ij} = 0 \quad \textit{if} \quad \begin{cases} 1 \le j \le r(w), \ \textit{or} \\ r(w)+1 \le j \le n-1 \ \textit{and} \ a_j - n < i \le m. \end{cases}$$

and $\mathcal{V}'_w$ *is the linear subspace of* $O^-_{P/\tilde{P}}$ *given by*

$$x_{ij} = 0 \quad \textit{for every} \ \ 1 \le j \le r(w) \ \ \textit{and for every} \ \ i \ge \max\{a_j + 1, s+1\}.$$

On the other hand, if $w \in \mathcal{W}_r$ *for some* $1 \le r \le n-1$ *and* $s = r$ (*see* Proposition 3.4(b)) *then we have an identification of* $Y_{\tilde{P}}(\tilde{w})$ *with* $\mathcal{V}_w \times O^-_{P/\tilde{P}}$, *where* $\mathcal{V}_w$ *is the linear subspace of* $O^-_{G/P}$ *given by*

$$x_{ij} = 0 \quad \textit{if} \quad \begin{cases} 1 \le j \le r, \ \textit{or} \\ r+1 \le j \le n-1 \ \textit{and} \ a_j - n < i \le m. \end{cases}$$

Proof. Consider the first case: w arbitrary and $s = 1$. Since $a_1 < \cdots < a_n$, we see that for every $j \le n$ and for every $i \ge \max\{a_j + 1, s+1\}$, the reflection (i, j) equals $(1, 2, \ldots, j-1, i)$ in $W/W_{P_{\hat{j}}}$, while $\tilde{w}$ equals $(a_1, \ldots, a_j)$. Hence (i, j) is not smaller than $\tilde{w}$ in $W/W_{P_{\hat{j}}}$, so the Plücker coordinate $p^{(j)}_{(i,j)}$ vanishes on $X_{\tilde{P}}(\tilde{w})$. Therefore for such (i, j), $x_{ij} \equiv 0$ on $Y_{\tilde{P}}(\tilde{w})$, by (2.2.3).

On the other hand, note that the reflections (i, j) with $j \le n$ and $i \ge \max\{a_j + 1, s+1\}$ are precisely the reflections s_α with $\alpha \in R^- \setminus R_{\tilde{P}}$ and $\tilde{w} \not\ge s_\alpha$ in $W/W_{\tilde{P}}$. Since $X_{\tilde{P}}(\tilde{w})$ is smooth, this implies (see Proposition 3.1) that the codimension of $Y_{\tilde{P}}(\tilde{w})$ in $O^-_{\mathrm{GL}_N/\tilde{P}}$ equals

$$\#\bigl\{(i, j) \mid j \le n \ \text{ and } \ i \ge \max\{a_j + 1, s+1\}\bigr\}$$

so $Y_{\tilde{P}}(\tilde{w})$ is the linear subspace of $O^-_{\mathrm{GL}_N/\tilde{P}}$ defined by the vanishing of

$$\bigl\{x_{ij} \mid j \le n \ \text{ and } \ i \ge \max\{a_j + 1, s+1\}\bigr\}.$$

This gives the asserted identification of $Y_{\tilde{P}}(\tilde{w})$.

Now the second case: $w \in \mathcal{W}_r$ for some $1 \le r \le n-1$ and $s = r$. Note that $X_{Q_s}(w') = \mathrm{GL}_n / B_n$, because of the choice of w and s. Therefore, an argument similar to the one above, along with counting dimensions, shows that $Y_{\tilde{P}}(\tilde{w})$ is defined inside $O^-_{G/\tilde{P}}$ by

$$x_{ij} = 0 \quad \text{if} \quad \begin{cases} 1 \le j \le r, \ \text{or} \\ r+1 \le j \le n-1 \ \text{and} \ a_j - n < i \le m. \end{cases}$$

This gives the asserted identification of $Y_{\tilde{P}}(\tilde{w})$. □

Let $Z_{\tilde{P}}(\tilde{w}) := Y_P(w) \times_{X_P(w)} X_{\tilde{P}}(\tilde{w}) = (O^-_{\mathrm{GL}_N/P} \times P/\tilde{P}) \cap X_{\tilde{P}}(\tilde{w})$. Write p for the composite map

$$Z_{\tilde{P}}(\tilde{w}) \to O^-_{\mathrm{GL}_N/P} \times P/\tilde{P} \to P/\tilde{P},$$

where the first map is the inclusion (as a closed subvariety) and the second map is projection. Using Proposition 2.2.1(c) and (d), we see that

$$p\left(\begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \bmod \tilde{P}\right) = A \mod Q_s.$$

(A is invertible by Proposition 2.2.1(b).) Using the injective map

$$B_n \longrightarrow B_N, \quad A \mapsto \begin{bmatrix} A & 0_{n\times m} \\ 0_{m\times n} & I_m \end{bmatrix},$$

B_n can be thought of as a subgroup of B_N. With this identification, we have the following Proposition:

Proposition 3.6. *$Z_{\tilde{P}}(\tilde{w})$ is B_n-stable (for the action on the left by multiplication). Further, p is B_n-equivariant.*

Proof. Let

$$z := \begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \in \mathrm{GL}_N$$

be such that $z\tilde{P} \in Z_{\tilde{P}}(\tilde{w})$. Since $X_{B_N}(\tilde{w}) \to X_{\tilde{P}}(\tilde{w})$ is surjective, we may assume that $z \pmod{B_N} \in X_{B_N}(\tilde{w})$, i.e., $z \in \overline{B_N \tilde{w} B_N}$. Then for every $A' \in B_n$,

$$\begin{bmatrix} A' & 0_{n\times m} \\ 0_{m\times n} & I_m \end{bmatrix} z = \begin{bmatrix} A'A & 0 \\ D & I_m \end{bmatrix} =: z'.$$

Then $z' \in \overline{B_N \tilde{w} B_N}$, so $z' \pmod{\tilde{P}} \in X_{\tilde{P}}(\tilde{w})$. By Proposition 2.2.1(b), we have that A is invertible, and hence AA' is invertible; this implies (again by Proposition 2.2.1(b)) that $z' \pmod{\tilde{P}} \in Z_{\tilde{P}}(\tilde{w})$. Thus $Z_{\tilde{P}}(\tilde{w})$ is B_n-stable. Also, $p(A'z) = p(z') = A'A = A'p(z)$. Hence p is B_n-equivariant. □

Theorem 3.7. *With notation as above,*

(a) *The natural map $X_{\tilde{P}}(\tilde{w}) \to X_P(w)$ is proper and birational. In particular, the map $Z_{\tilde{P}}(\tilde{w}) \to Y_P(w)$ is proper and birational.*

(b) *$X_{Q_s}(w')$ is the fibre of the natural map $Z_{\tilde{P}}(\tilde{w}) \to Y_P(w)$ at $e_{\mathrm{id}} \in Y_P(w)$ (with w' as in Notation 3.2).*

(c) *Suppose that w and s satisfy the conditions of Proposition 3.4. Then $X_{Q_s}(w')$ is the image of p. Further, p is a fibration with fibre isomorphic to $\mathcal{V}_w$.*

(d) *Suppose that* w *and* s *satisfy the conditions of Proposition 3.4. Then* p *identifies* $Z_{\tilde{P}}(\tilde{w})$ *as a subbundle of the trivial bundle* $O^-_{\mathrm{GL}_N/P} \times X_{Q_s}(w')$, *which arises as the restriction of the vector bundle on* GL_n/Q_s *associated to the* Q_s*-module* $\mathcal{V}_w$ (*which, in turn, is a* Q_s*-submodule of* $O^-_{\mathrm{GL}_N/P}$).

We believe that all the assertions above hold without the hypothesis that $X_{\tilde{P}}(\tilde{w})$ is smooth.

Proof. (a) The map $X_{\tilde{P}}(\tilde{w}) \hookrightarrow \mathrm{GL}_N/\tilde{P} \to \mathrm{GL}_N/P$ is proper and its (scheme-theoretic) image is $X_P(w)$; hence $X_{\tilde{P}}(\tilde{w}) \to X_P(w)$ is proper. Birationality follows from the fact that $\tilde{w}$ is the minimal representative of the coset $w\tilde{P}$ (see Remark 2.2.5).

(b) The fibre at $e_{\mathrm{id}} \in Y_P(w)$ of the map $Y_{\tilde{P}}(\tilde{w}) \to Y_P(w)$ is $\{0\} \times \mathcal{V}'_w$ (contained in $\mathcal{V}_w \times \mathcal{V}'_w = Y_{\tilde{P}}(\tilde{w})$). Since $Z_{\tilde{P}}(\tilde{w})$ is the closure of $Y_{\tilde{P}}(\tilde{w})$ inside $O^-_{\mathrm{GL}_N/P} \times P/\tilde{P}$ and $X_{Q_s}(w')$ is the closure of $\mathcal{V}'_w$ inside $P/\tilde{P}$ (note that, as a subvariety of $O^-_{P/\tilde{P}}$, $Y_{Q_s}(w')$ is identified with $\mathcal{V}'_w$), we see that fibre of $Z_{\tilde{P}}(\tilde{w}) \to Y_P(w)$ at $e_{\mathrm{id}} \in Y_P(w)$ is $X_{Q_s}(w')$.

(c) From Theorem 3.5 it follows that

$$Y_{\tilde{P}}(\tilde{w}) = \left\{ \begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \bmod \tilde{P} \;\middle|\; A \in \mathcal{V}'_w \text{ and } D \in \mathcal{V}_w \right\}.$$

Hence $p(Y_{\tilde{P}}(\tilde{w})) = \mathcal{V}'_w \subseteq X_{Q_s}(w')$. Since $Y_{\tilde{P}}(\tilde{w})$ is dense inside $Z_{\tilde{P}}(\tilde{w})$ and $X_{Q_s}(w')$ is closed in GL_n/Q_s, we see that $p(Z_{\tilde{P}_r}(\tilde{w})) \subseteq X_{Q_s}(w')$. The other inclusion $X_{Q_s}(w') \subseteq p(Z_{\tilde{P}_r}(\tilde{w}))$ follows from (b). Hence, $p(Z_{\tilde{P}_r}(\tilde{w}))$ equals $X_{Q_s}(w')$.

Next, to prove the second assertion in (c), we shall show that for every $A \in \mathrm{GL}_n$ with $A \bmod Q_s \in X_{Q_s}(w')$,

$$p^{-1}(A \bmod Q_s) = \left\{ \begin{bmatrix} A & 0_{n\times m} \\ D & I_m \end{bmatrix} \bmod \tilde{P} \;\middle|\; D \in \mathcal{V}_w \right\}. \tag{3.8}$$

Towards proving this, we first observe that $p^{-1}(e_{\mathrm{id}})$ equals $\mathcal{V}_w$ (in view of Theorem 3.5). Next, we observe that every B_n-orbit inside $X_{Q_s}(w')$ meets $\mathcal{V}'_w$ (which equals $Y_{Q_s}(w')$); further, p is B_n-equivariant (see Proposition 3.6). The assertion (3.8) now follows.

(d) First observe that for the action of right multiplication by GL_n on $O^-_{G/P}$ (being identified with $M_{m,n}$, the space of $m \times n$ matrices), $\mathcal{V}_w$ is stable; we thus get the homogeneous bundle $\mathrm{GL}_n \times^{Q_s} \mathcal{V}_w \to \mathrm{GL}_n/Q_s$ (Definition 2.3.1). Now to prove the assertion about $Z_{\tilde{P}_s}(\tilde{w}))$ being a vector bundle over $X_{Q_s}(w')$, we will show that there is a commutative diagram given as below, with ψ an isomorphism:

$$\begin{tikzcd}
Z_{\tilde{P}_s}(\tilde{w}) \arrow[drr, "\phi"] \arrow[dr, "\psi"] \arrow[ddr, "p"'] & & \\
& (\mathrm{GL}_n \times^{Q_s} \mathcal{V}_w)|_{X_{Q_s}(w')} \arrow[r] \arrow[d] & \mathrm{GL}_n \times^{Q_s} \mathcal{V}_w \arrow[d, "\alpha"] \\
& X_{Q_s}(w') \arrow[r, "\beta"] & \mathrm{GL}_n/Q_s
\end{tikzcd}$$

The map α is the homogeneous bundle map and β is the inclusion. Define ϕ by

$$\phi : \begin{bmatrix} A & 0_{n\times m} \\ D & I_m \end{bmatrix} \bmod \tilde{P} \longmapsto (A, D)/\sim .$$

Using Proposition 2.2.1(c) and (3.8), we conclude the following: ϕ is well defined and injective; $\beta \cdot p = \alpha \cdot \phi$; hence, by the universal property of products, the map ψ exists; and, finally, the injective map ψ is in fact an isomorphism (by dimension considerations). ☐

Corollary 3.9. *If $X_{\tilde{P}}(\tilde{w})$ is smooth, then we have the following realisation of the diagram in* (1-2):

$$\begin{tikzcd}
Z_{\tilde{P}}(\tilde{w}) \arrow[r, hook] \arrow[d, "q'"] & O^-_{\mathrm{GL}_N/P} \times X_{Q_s}(w') \arrow[r] \arrow[d, "q"] & X_{Q_s}(w') \\
Y_P(w) \arrow[r, hook] & O^-_{\mathrm{GL}_N/P} &
\end{tikzcd} .$$

Example 3.10. This example shows that even with $r = s$, $X_{Q_s}(w')$ need not be smooth for arbitrary $w \in W^P$. Let $n = m = 4$ and $w = (2, 4, 7, 8)$. Then $r = 2$; take $s = 2$. Then we obtain $w_{\max} = (4, 2, 7, 8, 5, 6, 3, 1)$, which has a 4231 pattern.

4. Free resolutions

The Kempf–Lascoux–Weyman geometric technique. We now summarise the geometric technique of computing free resolutions, following [Weyman 2003, Chapter 5]. Consider (1-1). There is a natural map $f : V \to \mathrm{Gr}_{r,d}$ (where $r = \mathrm{rk}_V Z$ and $d = \dim \mathbb{A}$) such that the inclusion $Z \subseteq \mathbb{A} \times V$ is the pullback of the tautological sequence (2.3.4); here $\mathrm{rk}_V Z$ denotes the rank of Z as a vector bundle over V, i.e., $\mathrm{rk}_V Z = \dim Z - \dim V$. Let $\xi = (f^*\mathcal{Q})^*$. Write R for the polynomial ring $\mathbb{C}[\mathbb{A}]$ and $\mathfrak{m}$ for its homogeneous maximal ideal. (The grading on R arises as follows. In (1-1), $\mathbb{A}$ is thought of as the fibre of a trivial vector bundle, so it has a distinguished point, its origin. Now, being a subbundle, Z is defined by linear equations in each fibre; i.e., for each $v \in V$, there exist $s := (\dim \mathbb{A} - \mathrm{rk}_V Z)$ linearly independent linear polynomials $\ell_{v,1}, \dots, \ell_{v,s}$ that vanish along Z and define it. Now $Y = \{y \in \mathbb{A} \mid \text{there exists } v \in V \text{ such that } \ell_{v,1}(y) = \cdots = \ell_{v,s}(y) = 0\}$. Hence Y is defined by homogeneous polynomials. This explains why the resolution obtained below is graded.) Let $\mathfrak{m}$ be the homogeneous maximal ideal, i.e., the ideal defining the origin in $\mathbb{A}$.

Theorem 4.1 [Weyman 2003, basic theorem 5.1.2]. *With notation as above, there is a finite complex* $(F_\bullet, \partial_\bullet)$ *of finitely generated graded free* R*-modules that is quasi-isomorphic to* $\boldsymbol{R}q'_*\mathcal{O}_Z$, *with*

$$F_i = \bigoplus_{j \geq 0} \mathrm{H}^j(V, \textstyle\bigwedge^{i+j}\xi) \otimes_{\mathbb{C}} R(-i-j),$$

and $\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}$. *Furthermore, the following are equivalent:*

(a) *Y has rational singularities; i.e.,* $\boldsymbol{R}q'_*\mathcal{O}_Z$ *is quasi-isomorphic to* $\mathcal{O}_Y$;

(b) $F_\bullet$ *is a minimal* R*-free resolution of* $\mathbb{C}[Y]$, *i.e.,* $F_0 \simeq R$ *and* $F_{-i} = 0$ *for all* $i > 0$.

We give a sketch of the proof because one direction of the equivalence is only implicit in the proof of [op. cit., 5.1.3].

Sketch of the proof. One constructs a suitable q_*-acyclic resolution $\mathcal{I}^\bullet$ of the Koszul complex that resolves $\mathcal{O}_Z$ as an $\mathcal{O}_{\mathbb{A}\times V}$-module so that the terms in $q_*\mathcal{I}^\bullet$ are *finitely generated free* graded R-modules. One places the Koszul complex on the negative horizontal axis and thinks of $\mathcal{I}^\bullet$ as a second-quadrant double complex, thus to obtain a complex $G_\bullet$ of finitely generated free R-modules whose homology at the i-th position is $R^{-i}q_*\mathcal{O}_Z$. Then, using standard homological considerations, one constructs a subcomplex $(F_\bullet, \partial_\bullet)$ of $G_\bullet$ that is quasi-isomorphic to $G_\bullet$ with $\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ (we say that $F_\bullet$ is *minimal* if this happens), and since $\mathrm{H}_i(G_\bullet) = 0$ for every $|i| \gg 0$, $F_i = 0$ for every $|i| \gg 0$. Now using the minimality of $F_\bullet$, we see that $R^iq_*\mathcal{O}_Z = 0$ for every $i \geq 1$ if and only if $F_{-i} = 0$ for every $i \geq 1$. When one of these conditions holds, then $F_\bullet$ becomes a minimal free resolution of $q_*\mathcal{O}_Z$ which is a finitely generated $\mathcal{O}_Y$-module, and therefore $q_*\mathcal{O}_Z = \mathcal{O}_Y$ if and only if $q_*\mathcal{O}_Z$ is generated by one element as an $\mathcal{O}_Y$-module if and only if $q_*\mathcal{O}_Z$ is a generated by one element as an R-module if and only if F_0 is a free R-module of rank one if and only if $F_0 = R(0)$ since $\mathrm{H}^0(V, \bigwedge^0\xi) \otimes R$ is a summand of F_0. □

Our situation. We now apply Theorem 4.1 to our situation. We keep the notation of Theorem 3.7. Theorem 4.1 and Corollary 3.9 yield the following result:

Theorem 4.2. *Suppose that* $X_{\tilde{P}_s}(\tilde{w})$ *is smooth. Write* $\mathcal{U}_w$ *for the restriction to* $X_{Q_s}(w')$ *of the vector bundle on* GL_n/Q_s *associated to the* Q_s*-module* $(O^-_{\mathrm{GL}_N/P}/\mathcal{V}_w)^*$. *(This is the dual of the quotient of* $O^-_{\mathrm{GL}_N/P} \times X_{Q_s}(w')$ *by* $Z_{\tilde{P}_s}(\tilde{w})$.*) Then we have a* *minimal R-free resolution* $(F_\bullet, \partial_\bullet)$ *of* $\mathbb{C}[Y_P(w)]$ *with*

$$F_i = \bigoplus_{j \geq 0} \mathrm{H}^j(X_{Q_s}(w'), \textstyle\bigwedge^{i+j} \mathcal{U}_w) \otimes_{\mathbb{C}} R(-i-j).$$

In the first case, $Q_s = B_n$, so p makes $Z_{\tilde{P}_1}(\tilde{w})$ a vector bundle on a smooth Schubert subvariety $X_{B_1}(w')$ of GL_n / B_n. In the second case, w' is the maximal word in S_n, so $X_{Q_r}(w') = \mathrm{GL}_n / Q_r$; see Discussion 4.3 for further details.

Computing the cohomology groups required in Theorem 4.2 in the general situation of Kempf's desingularization (Proposition 3.4(a)) is a difficult problem, even though the relevant Schubert variety $X_{B_n}(w')$ is smooth. Hence we are forced to restrict our attention to the subset of W^P considered in Proposition 3.4(b).

The stipulation that, for $w \in \mathcal{W}_r$, w sends n to N is not very restrictive. This can be seen in two (related) ways. Suppose that w does not send n to N. Then, firstly, $X_P(w)$ can be thought of as a Schubert subvariety of a smaller Grassmannian. Or, secondly, $\mathcal{U}_w$ will contain the trivial bundle $\mathcal{U}_n$ as a summand, so $\mathrm{H}^0(\mathrm{GL}_n/Q_r, \xi) \neq 0$, i.e., $R(-1)$ is a summand of F_1. In other words, the defining ideal of $Y_P(w)$ contains a linear form.

Discussion 4.3. We give some more details of the situation in Proposition 3.4(b) that will be used in the next section. Let

$$w = (n-r+1, n-r+2, \ldots, n, a_{r+1}, \ldots, a_{n-1}, N) \in \mathcal{W}_r.$$

The space of $(m \times n)$ matrices is a GL_n-module with a right action; the subspace $\mathcal{V}_w$ is Q_r-stable under this action. Thus $\mathcal{V}_w$ is a Q_r-module, and gives an associated vector bundle $(\mathrm{GL}_n \times^{Q_r} \mathcal{V}_w)$ on GL_n/Q_r. The action on the right of GL_n on the space of $(m \times n)$ matrices breaks by rows; each row is a natural n-dimensional representation of GL_n. For each $1 \leq j \leq m$, there is a unique $r \leq i_j \leq n-1$ such that $a_{i_j} < j+n \leq a_{i_j+1}$. (Note that $a_r = n$ and $a_n = N$.) In row j, $\mathcal{V}_w$ has rank $n - i_j$, and is a subbundle of the natural representation. Hence the vector bundle associated to the j-th row of $\mathcal{V}_w$ is the pullback of the tautological subbundle (of rank $(n-i_j)$) on $\mathrm{Gr}_{n-i_j,n}$. We denote this by $\mathcal{R}_{n-i_j}$. Therefore $(\mathrm{GL}_n \times^{Q_r} \mathcal{V}_w)$ is the vector bundle $\mathcal{R}_w := \bigoplus_{j=1}^m \mathcal{R}_{n-i_j}$. Let $\mathcal{Q}_w := \bigoplus_{j=1}^m \mathcal{Q}_{i_j}$ where $\mathcal{Q}_{i_j}$ is the tautological quotient bundle corresponding to $\mathcal{R}_{n-i_j}$. Then the vector bundle $\mathcal{U}_w$ on GL_n/Q_r that was defined in Theorem 4.2 is $\mathcal{Q}_w^*$.

5. Cohomology of homogeneous vector bundles

It is, in general, difficult to compute the cohomology groups $\mathrm{H}^j(\mathrm{GL}_n/Q_r, \bigwedge^t \mathcal{U}_w)$ in Theorem 4.2 for arbitrary $w \in \mathcal{W}_r$. In this section, we will discuss some approaches. We believe that this is a problem of independent interest. Our method involves replacing Q_r inductively by increasingly bigger parabolic subgroups, so we give the general setup below.

Setup 5.1. Let $1 \leq r \leq n-1$. Let $m_r, \dots, m_{n-1}$ be nonnegative integers such that $m_r + \cdots + m_{n-1} = m$. Let Q be a parabolic subgroup of GL_n such that $Q \subseteq P_{\hat{i}}$ for every $r \leq i \leq n-1$ such that $m_i > 0$. We consider the homogeneous vector bundle $\xi = \bigoplus_{i=r}^{n-1} \mathcal{U}_i^{m_i}$ on GL_n/Q, We want to compute the vector-spaces $\mathrm{H}^j(\mathrm{GL}_n/Q_r, \bigwedge^t \xi)$.

Lemma 5.2. *Let $f : X' \to X$ be a fibration with fibre some Schubert subvariety Y of some (partial) flag variety. Then $f_* \mathbb{O}_{X'} = \mathbb{O}_X$ and $R^i f_* \mathbb{O}_{X'} = 0$ for every $i \geq 1$. In particular, for every locally free coherent sheaf L on X, $\mathrm{H}^i(X', f^* L) = \mathrm{H}^i(X, L)$ for every $i \geq 0$.*

Proof. The first assertion is a consequence of Grauert's theorem [Hartshorne 1977, III.12.9] and the fact (see, for example, [Seshadri 2007, Theorem 3.2.1]) that

$$\mathrm{H}^i(Y, \mathbb{O}_Y) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The second assertion follows from the projection formula and the Leray spectral sequence. □

Proposition 5.3. *Let $m_i, r \leq i \leq n-1$ be as in Setup 5.1. Let*

$$Q' = \bigcap_{\substack{r \leq i \leq n-1 \\ m_i > 0}} P_{\hat{i}}.$$

Then $\mathrm{H}^*(\mathrm{GL}_n/Q, \bigwedge^t \xi) = \mathrm{H}^*(\mathrm{GL}_n/Q', \bigwedge^t \xi)$ for every t.

Proof. The assertion follows from Lemma 5.2, noting that $\bigwedge^t \xi$ on GL_n/Q is the pullback of $\bigwedge^t \xi$ on GL_n/Q', under the natural morphism $\mathrm{GL}_n/Q \to \mathrm{GL}_n/Q'$. □

Proposition 5.4. *For all j, $\mathrm{H}^j(\mathrm{GL}_n/Q, \xi) = 0$.*

Proof. We want to show that $\mathrm{H}^j(\mathrm{GL}_n/Q, \mathcal{U}_i) = 0$ for every $r \leq i \leq n-1$ and for every j. By Lemma 5.2 (and keeping Discussion 2.3.3 in mind), it suffices to show that $\mathrm{H}^j(\mathrm{Gr}_{n-i,n}, \mathcal{U}_i) = 0$ for every $r \leq i \leq n-1$ and for every j. To this end, we apply the Bott's algorithm [Weyman 2003, (4.1.5)] to the weight

$$\alpha := (\underbrace{0, \dots, 0}_{n-i}, 1, \underbrace{0, \dots, 0}_{i-1}).$$

Note that there is a permutation σ such that $\sigma \cdot \alpha = \alpha$, yielding the proposition. □

An inductive approach. We are looking for a way to compute $\mathrm{H}^*(\mathrm{GL}_n/Q, \bigwedge^t \xi)$ for a homogeneous bundle

$$\xi = \bigoplus_{i \in A} \mathcal{U}_i^{\oplus m_i}$$

where $A \subseteq \{r, \dots, n-1\}$ and $m_i > 0$ for every $i \in A$. Using Proposition 5.3, we assume that $Q = P_{\hat{A}}$. (Using Proposition 5.8 below, we may further assume that $m_i \geq 2$, but this is not necessary for the inductive argument to work.)

Let j be such that $Q \subseteq P_{\hat{j}}$ and $\mathcal{Q}_j$ (equivalently $\mathcal{U}_j$) be of least dimension; in other words, j is the smallest element of A. If $Q = P_{\hat{j}}$ (i.e., $|A| = 1$), then the $\bigwedge^t \xi$ is completely reducible, and we may use the Borel–Weil–Bott theorem to compute the cohomology groups. Hence suppose that $Q \neq P_{\hat{j}}$; write $Q = Q' \cap P_{\hat{j}}$ nontrivially, with Q' being a parabolic subgroup. Consider the diagram

$$\begin{tikzcd}
\mathrm{GL}_n/Q \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathrm{GL}_n/P_{\hat{j}} \\
\mathrm{GL}_n/Q' &
\end{tikzcd}$$

Note that $\bigwedge^t \xi$ decomposes as a direct sum of bundles of the form $(p_1)^*\eta \otimes (p_2)^*(\bigwedge^{t_1} \mathcal{U}_j^{\oplus m_j})$ where η is a homogeneous bundle on GL_n/Q'. We must compute

$$\mathrm{H}^*\big(\mathrm{GL}_n/Q,\ (p_1)^*\eta \otimes (p_2)^*(\textstyle\bigwedge^{t_1} \mathcal{U}_j^{\oplus m_j})\big).$$

Using the Leray spectral sequence and the projection formula, we can compute this from

$$\mathrm{H}^*\big(\mathrm{GL}_n/Q',\ \eta \otimes R^*(p_1)_*(p_2)^*(\textstyle\bigwedge^{t_1} \mathcal{U}_j^{\oplus m_j})\big).$$

Now $\bigwedge^{t_1} \mathcal{U}_j^{\oplus m_j}$, in turn, decomposes as a direct sum of $\mathrm{S}_\mu \mathcal{U}_j$, so we must compute

$$\mathrm{H}^*\big(\mathrm{GL}_n/Q',\ \eta \otimes R^*(p_1)_*(p_2)^*\, \mathrm{S}_\mu \mathcal{U}_j\big).$$

The Leray spectral sequence and the projection formula respect the various direct-sum decompositions mentioned above. It would follow from Proposition 5.5 below that for each μ, at most one of the $R^p(p_1)_*(p_2)^*\, \mathrm{S}_\mu \mathcal{U}_j$ is nonzero, so the abutment of the spectral sequence is, in fact, an equality.

Proposition 5.5. *With notation as above, let θ be a homogeneous bundle on $\mathrm{GL}_n/P_{\hat{j}}$. Then $R^i {p_1}_* {p_2}^* \theta$ is the locally free sheaf associated to the vector bundle $\mathrm{GL}_n \times^{Q'} \mathrm{H}^i(Q'/Q, {p_2}^*\theta|_{Q'/Q})$ over GL_n/Q'.*

Proof. This proposition follows from Proposition 2.3.5. □

We hence want to determine the cohomology of the restriction of $\mathrm{S}_\mu \mathcal{U}_j$ on Q'/Q. It follows from the definition of j that Q'/Q is a Grassmannian whose tautological quotient bundle and its dual are, respectively, $\mathcal{Q}_j|_{Q'/Q}$ and $\mathcal{U}_j|_{Q'/Q}$. We can therefore compute $\mathrm{H}^i(Q'/Q, \mathrm{S}_\mu \mathcal{U}_j|_{Q'/Q})$ using the Borel–Weil–Bott theorem.

Example 5.6. Suppose that $n = 6$ and that $Q = P_{\{\widehat{2,4}\}}$. Then we have the diagram

$$\begin{tikzcd}
\mathrm{GL}_6/Q \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathrm{GL}_6/P_{\hat{2}} \\
\mathrm{GL}_6/P_{\hat{4}} &
\end{tikzcd}$$

The fibre of p_1 is isomorphic to $P_{\hat{4}}/Q$ which is a Grassmannian of two-dimensional subspaces of a four-dimensional vector-space. Let $\mu = (\mu_1, \mu_2)$ be a weight. Then we can compute the cohomology groups $\mathrm{H}^*(P_{\hat{4}}/Q,\ \mathrm{S}_\mu \mathcal{U}_2|_{P_{\hat{4}}/Q})$ applying the Borel–Weil–Bott theorem [Weyman 2003, (4.1.5)] to the sequence $(0, 0, \mu_1, \mu_2)$. Note that $\mathrm{H}^*(P_{\hat{4}}/Q,\ \mathrm{S}_\mu \mathcal{U}_2|_{P_{\hat{4}}/Q})$ is, if it is nonzero, $\mathrm{S}_\lambda W$ where W is a four-dimensional vector-space that is the fibre of the dual of the tautological quotient bundle of $\mathrm{GL}_4/P_{\hat{4}}$ and λ is a partition with at most four parts. Hence, by Proposition 5.5, we see that $R^i(p_1)_*(p_2)^*\,\mathrm{S}_\mu \mathcal{U}_2$ is, if it is nonzero, $\mathrm{S}_\lambda \mathcal{U}_4$ on $\mathrm{GL}_6/P_{\hat{4}}$.

We summarise the above discussion as a theorem:

Theorem 5.7. *For $w \in \mathcal{W}_r$ the modules in the free resolution of $\mathbb{C}[Y_P(w)]$ given in Theorem 4.2 can be computed.*

We end this section with some observations.

Proposition 5.8. *Suppose that there exists i such that $r + 1 \le i \le n - 1$ and such that ξ contains exactly one copy of $\mathcal{U}_i$ as a direct summand. Let*

$$\xi' = \mathcal{U}_{i-1} \oplus \bigoplus_{\substack{j=1 \\ i_j \neq i}}^{m} \mathcal{U}_{i_j}.$$

Then $\mathrm{H}^*(\mathrm{GL}_n/Q, \bigwedge^t \xi) = \mathrm{H}^*(\mathrm{GL}_n/Q, \bigwedge^t \xi')$ for every t.

Proof. Note that ξ' is a subbundle of ξ with quotient $\mathcal{U}_i/\mathcal{U}_{i-1}$. We claim that $\mathcal{U}_i/\mathcal{U}_{i-1} \simeq L_{\omega_{i-1}-\omega_i}$, where for $1 \le j \le n$, ω_j is the j-th fundamental weight. Assume the claim. Then we have an exact sequence

$$0 \longrightarrow \textstyle\bigwedge^t \xi' \longrightarrow \bigwedge^t \xi \longrightarrow \bigwedge^{t-1} \xi' \otimes L_{\omega_{i-1}-\omega_i} \longrightarrow 0$$

Let

$$Q' = \bigcap_{\substack{r \le l \le n-1 \\ l \neq i}} P_{\hat{l}};$$

then $Q = Q' \cap P_{\hat{i}}$. Let $p : \mathrm{GL}_n/Q \to \mathrm{GL}_n/Q'$ be the natural projection; its fibres are isomorphic to $Q'/Q \simeq \mathrm{GL}_2/B_N \simeq \mathbb{P}^1$. Note that $\bigwedge^{t-1} \xi' \otimes L_{\omega_{i-1}}$ is the pullback along p of some vector bundle on GL_n/Q'; hence it is constant on the fibres of p.

On the other hand, L_{ω_i} is the ample line bundle on $\mathrm{GL}_n / P_{\hat{\imath}}$ that generates its Picard group, so $L_{-\omega_i}$ restricted to any fibre of p is $\mathcal{O}(-1)$. Hence the bundle $\bigwedge^{t-1}\xi' \otimes L_{\omega_{i-1}-\omega_i}$ on any fibre of p is a direct sum of copies of $\mathcal{O}(-1)$ and hence it has no cohomology. By Grauert's theorem [Hartshorne 1977, III.12.9], $R^i p_*(\bigwedge^{t-1}\xi' \otimes L_{\omega_{i-1}-\omega_i}) = 0$ for every i, so, using the Leray spectral sequence, we conclude that $\mathrm{H}^*(\mathrm{GL}_n/Q, \bigwedge^{t-1}\xi' \otimes L_{\omega_{i-1}-\omega_i}) = 0$. This gives the proposition.

Now to prove the claim, note that $\mathcal{U}_i/\mathcal{U}_{i-1} \simeq (\mathcal{R}_{n-i+1}/\mathcal{R}_{n-i})^*$. Let $e_1, \dots, e_n$ be a basis for $\mathbb{C}^n$ such that the subspace spanned by $e_i, \dots, e_n$ is B_N-stable for every $1 \le i \le n$. (Recall that we take the right action of B_N on $\mathbb{C}^n$.) Hence $\mathcal{R}_{n-i+1}/\mathcal{R}_{n-i}$ is the invertible sheaf on which B_N acts through the character $\omega_i - \omega_{i-1}$, which implies the claim. □

Remark 5.9 (determinantal case). Recall (see the paragraph after Definition 3.3) that $Y_P(w) = D_k$ if $w = (k+1, \dots, n, N-k+1, \dots N) \in \mathcal{W}_{n-k}$. In this case,

$$\mathcal{U}_w = \mathcal{U}_{n-k}^{\oplus(m-k+1)} \oplus \bigoplus_{i=n-k+1}^{n-1} \mathcal{U}_i.$$

Therefore

$$\mathrm{H}^*(\mathrm{GL}_n/Q_{n-k}, \textstyle\bigwedge^*\xi) = \mathrm{H}^*(\mathrm{GL}_n/Q_{n-k}, \bigwedge^*\mathcal{U}_{n-k}^{\oplus m}) = \mathrm{H}^*(\mathrm{GL}_n / P_{\widehat{n-k}}, \bigwedge^*\mathcal{U}_{n-k}^{\oplus m})$$

where the first equality comes from a repeated application of Proposition 5.8 and the second one follows by Lemma 5.2, applied to the natural map $f : \mathrm{GL}_n/Q \to \mathrm{GL}_n / P_{\widehat{n-k}}$. Hence our approach recovers Lascoux's resolution of the determinantal ideal [Lascoux 1978]; see also [Weyman 2003, Chapter 6].

6. Examples

We illustrate our approach with two examples. Firstly, we compute the resolution of a determinantal variety using the inductive method from the last section.

Example 6.1 ($n \times m$ matrices of rank $\le k$). If $k = 1$, then $w = (2, \dots, n, n+m)$, and, hence, $\xi = \mathcal{U}_{n-1}^{\oplus m}$. Since this would not illustrate the inductive argument, let us take $k = 2$.

Consider the ideal generated by the 3×3 minors of a 4×3 matrix of indeterminates. It is generated by four cubics, which have a linear relation. Hence minimal free resolution of the quotient ring looks like

$$0 \longrightarrow R(-4)^{\oplus 3} \longrightarrow R(-3)^{\oplus 4} \longrightarrow R \longrightarrow 0. \tag{6.2}$$

Note that $w = (3, 4, 6, 7)$ and $\xi = \mathcal{U}_2^{\oplus 2} \bigoplus \mathcal{U}_3$. Write $G = \mathrm{GL}_4$ and $Q = P_{\widehat{\{2,3\}}}$. Then $j = 2$, $Q' = P_{\hat{3}}$ and $Q'/Q \simeq \mathrm{GL}_3 / P_{\hat{2}} \simeq \mathbb{P}^2$. Now there is a decomposition

$$\textstyle\bigwedge^t \xi = \bigoplus_{|\mu| \le t} \mathrm{S}_{\mu'}\,\mathbb{C}^2 \otimes \mathrm{S}_\mu\,\mathcal{U}_2 \otimes \bigwedge^{t-|\mu|}\mathcal{U}_3$$

Hence we need to consider only $\mu = (\mu_1, \mu_2) \leq (2, 2)$. On $Q'/Q \simeq \mathrm{GL}_3 / P_{\hat{2}}$, we would apply the Borel–Weil–Bott theorem [op. cit., (4.1.5)] to the weight $(0, \mu_1, \mu_2)$ to compute the cohomology of $\mathrm{S}_\mu \mathcal{U}_j$. Thus we see that we need to consider only $\mu = (0, 0)$, $\mu = (2, 0)$ and $\mu = (2, 1)$. From this, we conclude that

$$R^i (p_1)_*(p_2)^*(\mathrm{S}_{\mu'} \mathbb{C}^2 \otimes \mathrm{S}_\mu \mathcal{U}_2) = \begin{cases} \mathcal{O}_{G/P_{\hat{3}}} & \text{if } i = 0 \text{ and } \mu = (0, 0), \\ \bigwedge^2 \mathcal{U}_3 & \text{if } i = 1 \text{ and } \mu = (2, 0), \\ (\bigwedge^3 \mathcal{U}_3)^{\oplus 2} & \text{if } i = 1 \text{ and } \mu = (2, 1), \\ 0 & \text{otherwise.} \end{cases}$$

We have to compute the cohomology groups of $(R^i (p_1)_*(p_2)^*(\mathrm{S}_{\mu'} \mathbb{C}^2 \otimes \mathrm{S}_\mu \mathcal{U}_2)) \otimes \bigwedge^{t-|\mu|} \mathcal{U}_3$ on $G/P_{\hat{3}}$. Now, $\mathrm{H}^*(G/P_{\hat{3}}, \bigwedge^i \mathcal{U}_3) = 0$ for every $i > 0$. Further

$$\begin{aligned} \textstyle\bigwedge^2 \mathcal{U}_3 \otimes \mathcal{U}_3 &\simeq \textstyle\bigwedge^3 \mathcal{U}_3 \oplus \mathrm{S}_{2,1} \mathcal{U}_3 && \text{for } \mu = (2, 0) \text{ and } t = 3, \\ \textstyle\bigwedge^2 \mathcal{U}_3 \otimes \bigwedge^2 \mathcal{U}_3 &\simeq \mathrm{S}_{2,1,1} \mathcal{U}_3 \oplus \mathrm{S}_{2,2} \mathcal{U}_3 && \text{for } \mu = (2, 0) \text{ and } t = 4, \\ \textstyle\bigwedge^2 \mathcal{U}_3 \otimes \bigwedge^3 \mathcal{U}_3 &\simeq \mathrm{S}_{2,2,1} \mathcal{U}_3 && \text{for } (\mu = (2, 0) \text{ or } \mu = (2, 1)) \text{ and } t = 5, \\ \textstyle\bigwedge^3 \mathcal{U}_3 \otimes \mathcal{U}_3 &\simeq \mathrm{S}_{2,1,1} \mathcal{U}_3 && \text{for } \mu = (2, 1) \text{ and } t = 4, \\ \textstyle\bigwedge^3 \mathcal{U}_3 \otimes \bigwedge^3 \mathcal{U}_3 &\simeq \mathrm{S}_{2,2,2} \mathcal{U}_3 && \text{for } \mu = (2, 1) \text{ and } t = 6. \end{aligned}$$

Again, by applying the Borel–Weil–Bott theorem [loc. cit.] for $G/P_{\hat{3}}$, we see that $\mathrm{S}_{2,2} \mathcal{U}_3$, $\mathrm{S}_{2,2,1} \mathcal{U}_3$ and $\mathrm{S}_{2,2,2} \mathcal{U}_3$ have no cohomology. Therefore we conclude that

$$\mathrm{H}^j (G/Q, \textstyle\bigwedge^t \xi) = \begin{cases} \bigwedge^0 \mathbb{C}^{\oplus 4} & \text{if } t = 0 \text{ and } j = 0, \\ \bigwedge^3 \mathbb{C}^{\oplus 4} & \text{if } t = 3 \text{ and } j = 2, \\ (\bigwedge^4 \mathbb{C}^{\oplus 4})^{\oplus 3} & \text{if } t = 4 \text{ and } j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

These ranks agree with the expected ranks from (6.2).

Example 6.3. Let $n = 6$, $m = 6$, $k = 4$ and $w = (5, 6, 8, 9, 11, 12)$. For this, $Q = P_{\{\widehat{2,\ldots,5}\}}$ and $\mathcal{U}_w = \mathcal{U}_2^{\oplus 2} \oplus \mathcal{U}_3 \oplus \mathcal{U}_4^{\oplus 2} \oplus \mathcal{U}_5$. After applying Propositions 5.3 and 5.8, we reduce to the situation $Q = P_{\{\widehat{2,4}\}}$ and $\xi = \mathcal{U}_2^{\oplus 3} \oplus \mathcal{U}_4^{\oplus 3}$. Write $\xi = (\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2) \oplus (\mathbb{C}^3 \oplus \mathcal{U}_4)$. Now we project away from $\mathrm{GL}_6 / P_{\hat{2}}$.

$$\begin{tikzcd} \mathrm{GL}_6 / Q \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathrm{GL}_6 / P_{\hat{2}} \\ \mathrm{GL}_6 / P_{\hat{4}} & \end{tikzcd}$$

The fibre of p_1 is isomorphic to $P_{\hat{4}}/Q$ which is a Grassmannian of two-dimensional subspaces of a four-dimensional vector-space. We use the spectral sequence

$$\mathrm{H}^j (G/P_{\hat{4}}, R^i p_{1*} \textstyle\bigwedge^t \xi) \implies \mathrm{H}^{i+j} (G/Q, \textstyle\bigwedge^t \xi). \tag{6.4}$$

Observe that $\bigwedge^t \xi = \bigoplus_{t_1} \bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2) \otimes \bigwedge^{t-t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_4)$; the above spectral sequence respects this decomposition. Further, using the projection formula, we see that we need to compute

$$\mathrm{H}^j\big(G/P_{\hat{4}},\ (R^i p_{1*} \textstyle\bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2)) \otimes \bigwedge^{t-t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_4)\big).$$

Now, $R^i p_{1*} \bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2)$ is the vector bundle associated to the $P_{\hat{4}}$-module

$$\mathrm{H}^i(P_{\hat{4}}/Q,\ \textstyle\bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2)|_{P_{\hat{4}}/Q}) = \mathrm{H}^i(P_{\hat{4}}/Q,\ \bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2|_{P_{\hat{4}}/Q})).$$

Note that $\mathcal{U}_2|_{P_{\hat{4}}/Q}$ is the dual of the tautological quotient bundle of $P_{\hat{4}}/Q \simeq \mathrm{GL}_4/P_{\hat{2}}$; we denote this also, by abuse of notation, by $\mathcal{U}_2$. Note, further, that $\bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2) = \bigoplus_{\mu \vdash t_1} \mathrm{S}_{\mu'} \mathbb{C}^3 \otimes \mathrm{S}_{\mu} \mathcal{U}_2$. We need only consider $\mu \leq (3, 3)$. From the Borel–Weil–Bott theorem [Weyman 2003, (4.1.5)], it follows that

$$\mathrm{H}^i(P_{\hat{4}}/Q,\ \mathrm{S}_\mu \mathcal{U}_2) = \begin{cases} \bigwedge^0(\mathbb{C}^{\oplus 4}) & \text{if } i = 0 \text{ and } \mu = (0, 0), \\ \bigwedge^3(\mathbb{C}^{\oplus 4}) & \text{if } i = 2 \text{ and } \mu = (3, 0), \\ \bigwedge^4(\mathbb{C}^{\oplus 4}) & \text{if } i = 2 \text{ and } \mu = (3, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Therefore we conclude that

$$R^i p_{1*} \textstyle\bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2) = \begin{cases} \mathcal{O}_{\mathrm{GL}_4/P_{\hat{2}}} & \text{if } i = 0 \text{ and } t_1 = 0, \\ \bigwedge^3 \mathcal{U}_4 & \text{if } i = 2 \text{ and } t_1 = 3, \\ (\bigwedge^4 \mathcal{U}_4)^{\oplus 3} & \text{if } i = 2 \text{ and } t_1 = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore for each pair (t, t_1) at most one column of the summand of the spectral sequence (6.4) is nonzero; hence the abutment in (6.4) is in fact an equality.

Fix a pair (t, t_1) and an integer l. Then we have

$$\begin{aligned} \mathrm{H}^l(G/Q,\ \textstyle\bigwedge^t \xi) = \mathrm{H}^l(G/P_{\hat{4}},\ \bigwedge^t(\mathbb{C}^3 \otimes \mathcal{U}_4)) \\ \oplus \mathrm{H}^{l-2}(G/P_{\hat{4}},\ \textstyle\bigwedge^3 \mathcal{U}_4 \otimes \bigwedge^{t-3}(\mathbb{C}^3 \otimes \mathcal{U}_4)) \\ \oplus \mathrm{H}^{l-2}(G/P_{\hat{4}},\ (\textstyle\bigwedge^4 \mathcal{U}_4)^{\oplus 3} \otimes \bigwedge^{t-4}(\mathbb{C}^3 \otimes \mathcal{U}_4)). \end{aligned}$$

Write $h^i(-) = \dim_{\mathbb{C}} \mathrm{H}^i(-)$. Note that $\bigwedge^t(\mathbb{C}^3 \otimes \mathcal{U}_4) \simeq \bigoplus_{\lambda \vdash t} \mathrm{S}_{\lambda'} \mathbb{C}^3 \otimes \mathrm{S}_\lambda \mathcal{U}_4$, by the Cauchy formula. Write $d_{\mu'} = \dim_{\mathbb{C}} \mathrm{S}_{\mu'} \mathbb{C}^{\oplus 3}$. Thus, from the above equation, we see, that for every l and for every t,

$$\begin{aligned} (6.5)\quad h^l(\textstyle\bigwedge^t \xi) = \displaystyle\sum_{\mu \vdash t} d_{\mu'} h^l(\mathrm{S}_\mu \mathcal{U}_4) \\ + \sum_{\mu \vdash t-3} d_{\mu'} h^{l-2}(\textstyle\bigwedge^3 \mathcal{U}_4 \otimes \mathrm{S}_\mu \mathcal{U}_4) + 3 \displaystyle\sum_{\mu \vdash t-4} d_{\mu'} h^{l-2}(\textstyle\bigwedge^4 \mathcal{U}_4 \otimes \mathrm{S}_\mu \mathcal{U}_4) \end{aligned}$$

(Here the cohomology is calculated over GL_6/Q on the left-hand-side and over $\mathrm{GL}_6/P_{\hat{4}}$ on the right-hand-side.) For any μ, if $d_{\mu'} \neq 0$, then $\mu_1 \leq 3$. Any μ that contributes a nonzero integer to the right-hand-side of (6.5) has at most four parts and $m_1 \leq 3$. Further, if $\mathrm{S}_\lambda \mathcal{U}_4$ is an irreducible summand of a representation on the right-hand-side of (6.5) with nonzero cohomology, then λ has at most four parts and is such that $\lambda_1 \leq 4$. Therefore for $\lambda \leq (4, 4, 4, 4)$, we compute the cohomology using the Borel–Weil–Borel theorem:

$$\mathrm{H}^i(G/P_{\hat{4}}, \mathrm{S}_\lambda \mathcal{U}_4) = \begin{cases} \bigwedge^0(\mathbb{C}^{\oplus 6}) & \text{if } i = 0 \text{ and } \lambda = 0, \\ \mathrm{S}_{(\lambda_1-2,1,1,\lambda_2,\lambda_3,\lambda_4)}(\mathbb{C}^{\oplus 6}) & \text{if } i = 2,\ \lambda_1 \in \{3, 4\}, \\ & \quad \text{and } (\lambda_2, \lambda_3, \lambda_4) \leq (1, 1, 1), \\ \mathrm{S}_{(2,2,2,2,\lambda_3,\lambda_4)}(\mathbb{C}^{\oplus 6}) & \text{if } i = 4,\ \lambda_1 = \lambda_2 = 4, \\ & \quad \text{and } (\lambda_3, \lambda_4) \leq (2, 2), \\ 0 & \text{otherwise.} \end{cases}$$

We put these together to compute $h^l(\bigwedge^t \xi)$; the result is listed in Table 1. From this we get the following resolution:

$$0 \to R(-12)^{26} \to R(-11)^{108} \to \begin{array}{c} R(-6)^{10} \\ \oplus \\ R(-10)^{153} \end{array} \to \begin{array}{c} R(-5)^{36} \\ \oplus \\ R(-7)^{36} \\ \oplus \\ R(-9)^{70} \end{array} \to \begin{array}{c} R(-3)^{45} \\ \oplus \\ R(-5)^{53} \end{array} \to \begin{array}{c} R(-2)^{20} \\ \oplus \\ R(-4)^{18} \end{array} \to R \to 0.$$

Note, indeed, that $\dim Y_Q(w) = \dim X_Q(w) = 4+4+5+5+6+6 = 30$ and that $\dim O^-_{\mathrm{GL}_N/P} = 6 \cdot 6 = 36$, so the codimension is 6. Since the variety is Cohen–Macaulay, the length of a minimal free resolution is 6.

7. Further remarks

A realisation of Lascoux's resolution for determinantal varieties. We already saw in Remark 5.9 that when $Y_P(w) = D_k$, computing $\mathrm{H}^*(\mathrm{GL}_n/Q_{n-k}, \bigwedge^* \xi)$ is reduced, by a repeated application of Proposition 5.8 to computing the cohomology groups of (completely reducible) vector bundles on the Grassmannian $\mathrm{GL}_n/P_{\widehat{n-k}}$. We thus realise Lascoux's resolution of the determinantal variety using our approach.

In this section, we give yet another desingularization of D_k (for a suitable choice of the parabolic subgroup) so that the variety V of (1-2) is in fact a Grassmannian. Recall (the paragraph after Definition 3.3 or Remark 5.9) that $Y_P(w) = D_k$ if $w = (k+1, \dots, n, N-k+1, \dots N) \in \mathcal{W}_{n-k}$. Let $\tilde{P} = P_{\widehat{\{n-k,n\}}} \subseteq \mathrm{GL}_N$. Let $\tilde{w}$ be the representative of the coset $w\tilde{P}$ in $W^{\tilde{P}}$.

Proposition 7.1. *$X_{\tilde{P}}(\tilde{w})$ is smooth and the natural map $X_{\tilde{P}}(\tilde{w}) \to X_P(w)$ is proper and birational, i.e., $X_{\tilde{P}}(\tilde{w})$ is a desingularization of $X_P(w)$.*

t	$h^0(\bigwedge^t\xi)$	$h^1(\bigwedge^t\xi)$	$h^2(\bigwedge^t\xi)$	$h^3(\bigwedge^t\xi)$	$h^4(\bigwedge^t\xi)$	$h^5(\bigwedge^t\xi)$	$h^6(\bigwedge^t\xi)$
0	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	20	0	0	0	0
4	0	0	45	0	0	0	0
5	0	0	36	0	18	0	0
6	0	0	10	0	53	0	0
7	0	0	0	0	36	0	0
8	0	0	0	0	0	0	0
9	0	0	0	0	0	0	70
10	0	0	0	0	0	0	153
11	0	0	0	0	0	0	90
12	0	0	0	0	0	0	26

Table 1. Ranks of the relevant cohomology groups.

Proof. The proof is similar to that of Proposition 3.4. Let

$$w_{\max} = (k+1, \dots, n, N-k+1, \dots N, N-k, \dots, n+1, k, \dots, 1) \in W.$$

Then $X_{B_N}(w_{\max})$ is the inverse image of $X_{\tilde{P}}(\tilde{w})$ under the natural morphism $\mathrm{GL}_N / B_N \to \mathrm{GL}_N / \tilde{P}$, and that $w_{\max}$ is a 4231 and 3412-avoiding element of $W = S_N$. □

We have $P/\tilde{P} \cong \mathrm{GL}_n / P_{\widehat{n-k}}$. As in Section 3, we have the following. Denoting by Z the preimage inside $X_{\tilde{P}}(\tilde{w})$ of $Y_P(w)$ (under the restriction to $X_{\tilde{P}}(\tilde{w})$ of the natural projection $G/\tilde{P} \to G/P$), we have $Z \subset O^- \times P/\tilde{P}$, and the image of Z under the second projection is $V := P/\tilde{P}$ ($\cong \mathrm{GL}_n / P_{\widehat{n-k}}$). The inclusion $Z \hookrightarrow O^- \times V$ is a subbundle (over V) of the trivial bundle $O^- \times V$. Denoting by ξ the dual of the quotient bundle on V corresponding to Z, we have that the homogeneous bundles $\bigwedge^{i+j}\xi$ on $\mathrm{GL}_n / P_{\widehat{n-k}}$ are completely reducible, and hence may be computed using Bott's algorithm.

Multiplicity. We describe how the free resolution obtained in Theorem 4.2 can be used to get an expression for the multiplicity $\mathrm{mult}_{\mathrm{id}}(w)$ of the local ring of the Schubert variety $X_P(w) \subseteq \mathrm{GL}_N / P$ at the point e_{id}. Notice that $Y_P(w)$ is an affine neighbourhood of e_{id}. We noticed in Section 4 that $Y_P(w)$ is a closed subvariety of $O^-_{\mathrm{GL}_N/P}$ defined by homogeneous equations. In $O^-_{\mathrm{GL}_N/P}$, e_{id} is the origin; hence in $Y_P(w)$ it is defined by the unique homogeneous maximal ideal of $\mathbb{C}[Y_P(w)]$. Therefore $\mathbb{C}[Y_P(w)]$ is the associated graded ring of the local ring of $\mathbb{C}[Y_P(w)]$ at

e_{id} (which is also the local ring of $X_P(w)$ at e_{id}). Hence $\mathrm{mult}_{\mathrm{id}}(w)$ is the normalised leading coefficient of the Hilbert series of $\mathbb{C}[Y_P(w)]$.

Observe that the Hilbert series of $\mathbb{C}[Y_P(w)]$ can be obtained as an alternating sum of the Hilbert series of the modules F_i in Theorem 4.2. Write $h^j(-) = \dim_{\mathbb{C}} \mathrm{H}^j(X_{Q_s}(w'), -)$ for coherent sheaves on $X_{Q_s}(w')$. Then the Hilbert series of $\mathbb{C}[Y_P(w)]$ is

$$\frac{1}{(1-t)^{mn}} \sum_{i=0}^{mn} \sum_{j=0}^{\dim X_{Q_s}(w')} (-1)^i h^j(\textstyle\bigwedge^{i+j} \mathcal{U}_w) t^{i+j}. \tag{7.2}$$

We may harmlessly change the range of summation in (7.2) to $-\infty < i, j < \infty$; this is immediate for j, while for i, we note that the proof of Theorem 4.1 implies that $h^j(\bigwedge^{i+j}\mathcal{U}_w) = 0$ for every $i < 0$ and for every j. Hence we may write the summation in (7.2) as (with $k = i + j$)

$$\sum_{k=0}^{\infty} (-1)^k t^k \sum_{j=0}^{\infty} (-1)^j h^j(\textstyle\bigwedge^k \mathcal{U}_w) = \sum_{k=0}^{\mathrm{rk}\,\mathcal{U}_w} (-1)^k \chi(\textstyle\bigwedge^k \mathcal{U}_w) t^k. \tag{7.3}$$

Since $\bigwedge^k \mathcal{U}_w$ is also a T_n-module, where T_n is the subgroup of diagonal matrices in GL_n, one may decompose $\bigwedge^k \mathcal{U}_w$ as a sum of rank-one T_n-modules and use the Demazure character formula to compute the Euler characteristics above.

It follows from generalities on Hilbert series (see, e.g., [Bruns and Herzog 1993, Section 4.1]) that the polynomial in (7.3) is divisible by $(1-t)^c$ where c is the codimension of $Y_P(w)$ in $O^-_{\mathrm{GL}_N/P}$, and that after we divide it and substitute $t = 1$ in the quotient, we get $\mathrm{mult}_{\mathrm{id}}(w)$. This gives an expression for $e_{\mathrm{id}}(w)$ apart from those of [Lakshmibai and Weyman 1990; Kreiman and Lakshmibai 2004].

Castelnuovo–Mumford regularity. Since $\mathbb{C}[Y_P(w)]$ is a graded quotient ring of $\mathbb{C}[O^-_{\mathrm{GL}_N/P}]$, it defines a coherent sheaf over the corresponding projective space $\mathbb{P}^{mn-1}$.

Let F be a coherent sheaf on $\mathbb{P}^n$. The *Castelnuovo–Mumford regularity* of F (with respect to $\mathcal{O}_{\mathbb{P}^n}(1)$) is the smallest integer r such that $\mathrm{H}^i(\mathbb{P}^n, F \otimes \mathcal{O}_{\mathbb{P}^n}(r-i)) = 0$ for every $1 \le i \le n$; we denote it by $\mathrm{reg}\, F$. Similarly, if $R = \Bbbk[x_0, \dots, x_n]$ is a polynomial ring over a field $\Bbbk$ with $\deg x_i = 1$ for every i and M is a finitely generated graded R-module, the *Castelnuovo–Mumford regularity* of M is the smallest integer r such that $(\mathrm{H}^i_{(x_0,\dots,x_n)}(M))_{r+1-i} = 0$ for every $0 \le i \le n+1$; we denote it by $\mathrm{reg}\, M$. (Here $\mathrm{H}^i_{(x_0,\dots,x_n)}(M)$ is the i-th *local cohomology* module of M, and is a graded R-module.) It is known that

$$\mathrm{reg}\, F = \mathrm{reg}\Bigl(\bigoplus_{i \in \mathbb{Z}} \mathrm{H}^0(\mathbb{P}^n, F \otimes \mathcal{O}_{\mathbb{P}^n}(i))\Bigr)$$

for every coherent sheaf F and that if $\operatorname{depth} M \geq 2$, then $\operatorname{reg} M = \operatorname{reg} \widetilde{M}$. See [Eisenbud 2005, Chapter 4] for details.

Proposition 7.4. *In the notation of* (1-1), $\operatorname{reg} \mathbb{C}[Y] = \max\{j : \mathrm{H}^j(V, \bigwedge^* \xi) \neq 0\}$.

Proof. Let $R = \mathbb{C}[\mathbb{A}]$. It is known that

$$\operatorname{reg} M = \max\{j : \operatorname{Tor}_i^R(\Bbbk, M)_{i+j} \neq 0 \text{ for some } i\};$$

see [loc. cit.] for a proof. The proposition now follows from noting that

$$\operatorname{Tor}_i^R(\mathbb{C}, \mathbb{C}[Y])_{i+j} \simeq \mathrm{H}^j(V, \textstyle\bigwedge^{i+j} \xi)$$

by Theorem 4.2. □

Now let $w = (n-r+1, n-r+2, \ldots, n, a_{r+1}, \ldots, a_{n-1}, N) \in \mathcal{W}_r$. We would like to determine $\operatorname{reg} \mathbb{C}[Y_P(w)] = \max\{j : \mathrm{H}^j(\mathrm{GL}_n/Q_r, \bigwedge^* \mathcal{U}_w) \neq 0\}$. Let $a_r = n$ and $a_n = N$. For $r \leq i \leq n-1$, define $m_i = a_{i+1} - a_i$. Note that $\mathcal{U}_i$ appears in $\mathcal{U}_w$ with multiplicity m_i and that $m_i > 0$. Based on the examples that we have calculated, we have the following conjecture.

Conjecture 7.5. With notation as above,

$$\operatorname{reg} \mathbb{C}[Y_P(w)] = \sum_{i=r}^{n-1} (m_i - 1)i.$$

(Note that since $Y_P(w)$ is Cohen–Macaulay, $\operatorname{reg} \mathbb{C}[Y_P(w)] = \operatorname{reg} \mathbb{O}_{Y_P(w)}$.) Consider the examples in Section 6. In Example 6.1, $m_2 = 2$, $m_3 = 1$, and $\operatorname{reg} \mathbb{C}[Y_P(w)] = (2-1)2+0 = 2$. In Example 6.3, $m_2 = m_4 = 2$ and $m_3 = m_5 = 1$, so $\operatorname{reg} \mathbb{C}[Y_P(w)] = (2-1)2+0+(2-1)4+0 = 6$, which in deed is the case, as we see from Table 1.

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