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Dedicated to the memory of Professor Robert Steinberg

In this paper we construct free resolutions of a certain class of closed subvarieties of affine space of symmetric matrices (of a given size). Our class covers the symmetric determinantal varieties (i.e., determinantal varieties in the space of symmetric matrices), whose resolutions were first constructed by Józefiak, Pragacz and Weyman (1981). Our approach follows the techniques developed by Kummini, Lakshmibai, Pramathanath and Seshadri (2015), and uses the geometry of Schubert varieties.

1. Introduction

This paper is a sequel to [Kummini et al. 2015]. Lascoux [1978] constructed a minimal free resolution of the coordinate ring of the determinantal varieties (consisting of $m \times n$ matrices (over $\mathbb{C}$) of rank at most k, considered as a closed subvariety of the mn-dimensional affine space of all $m \times n$ matrices), as a module over the mn-dimensional polynomial ring (the coordinate ring of the mn-dimensional affine space).

In [Kummini et al. 2015], the authors construct free resolutions for a larger class of singularities, *viz.*, Schubert singularities, i.e., the intersection of a singular Schubert variety and the "opposite big cell" inside a Grassmannian.

Józefiak, Pragacz and Weyman [1981] constructed a minimal free resolution of the coordinate ring of the determinantal varieties (in the space of symmetric matrices) as a module over the coordinate ring of the space of symmetric matrices. In this paper we construct free resolutions for a certain class of closed subvarieties of the affine space of symmetric matrices, which includes the symmetric determinantal varieties. The technique adopted in [Kummini et al. 2015] is algebraic group-theoretic, and we follow this approach.

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We now describe the results of this paper. Let n be a positive integer. Let $V = \mathbb{C}^{2n}$ and let $(\cdot,\cdot)$ be a nondegenerate skew-symmetric bilinear form on V. Let $H = \mathrm{SL}(V)$ and $G = \mathrm{SP}(V) = \{Z \in \mathrm{SL}(V) \mid Z \text{ leaves the form } (\cdot,\cdot) \text{ invariant}\}$. We take the matrix of the form, with respect to the standard basis of V, to be

$$F = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$$

where J is the antidiagonal $(1, \dots, 1)$, in this case of size n. To simplify our notation we will normally omit specifying the size of J as it will be obvious from the context. We may realize $\mathrm{SP}(V)$ as the fixed point set of the involution $\sigma : H \to H$ given by $\sigma(Z) = F(Z^T)^{-1}F^{-1}$ (cf. [Steinberg 1968]).

Denoting by T_H and B_H the maximal torus in H consisting of diagonal matrices and the Borel subgroup in H consisting of upper triangular matrices, respectively, we have that T_H and B_H are stable under σ and we set $T_G = T_H^\sigma$, $B_G = B_H^\sigma$. It is easily checked that T_G is a maximal torus in G and B_G is a Borel subgroup in G.

Thus we obtain

$$W_G \hookrightarrow W_H$$

where W_G, W_H denote the Weyl groups of G, H respectively (with respect to T_G, T_H respectively). Further, σ induces an involution on W_H:

$$w = (a_1, \cdots, a_{2n}) \in W_H, \quad \sigma(w) = (c_1, \cdots, c_{2n}), \quad c_i = 2n + 1 - a_{2n+1-i}$$

and

$$W_G = W_H^\sigma.$$

Thus we obtain

$$W_G = \{(a_1 \cdots a_{2n}) \in S_{2n} \mid a_i = 2n + 1 - a_{2n+1-i},\ 1 \le i \le 2n\}.$$

(here, S_{2n} is the symmetric group on $2n$ letters). Thus $w = (a_1 \cdots a_{2n}) \in W_G$ is known once $(a_1 \cdots a_n)$ is known. We shall denote an element $(a_1 \cdots a_{2n})$ in W_G by just $(a_1 \cdots a_n)$. Further, for $w \in W_G$, denoting by $X_G(w)$ (resp. $X_H(w)$), the associated Schubert variety in G/B_G (resp. H/B_H), we have that under the canonical inclusion $G/B_G \hookrightarrow H/B_H$, $X_G(w) = X_H(w) \cap G/B_G$, scheme-theoretically.

Let $P = P_{\hat{n}}$, the maximal parabolic subgroup of G corresponding to omitting the simple root α_n, the set of simple roots of G being indexed as in [Bourbaki 1968]. Let $1 \le k < r \le n$ be positive integers, and let $w \in \mathcal{W}_{k,r}$ (cf. Notation 3.2). Our main result (cf. Theorem 3.22) is a description of the minimal free resolution of the coordinate ring of $Y_P(w) := X_P(w) \cap O^-_{G/P}$, the *opposite cell* of $X_P(w)$, as a module over the coordinate ring of $O^-_{G/P}$. For this, as in [Kummini et al. 2015], we use the Kempf–Lascoux–Weyman "geometric technique" of constructing minimal free resolutions; in fact we use the same notation and description of this technique as in [Kummini et al. 2015].

Suppose that we have a commutative diagram of varieties

$$
\begin{array}{ccccc}
Z & \hookrightarrow & \mathbb{A} \times V & \longrightarrow & V \\
\downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \\
Y & \hookrightarrow & \mathbb{A} & &
\end{array}
\tag{1.1}
$$

where $\mathbb{A}$ is an affine space, Y a closed subvariety of $\mathbb{A}$ and V a projective variety. The map q is first projection, q' is proper and birational, and the inclusion $Z \hookrightarrow \mathbb{A} \times V$ is a subbundle (over V) of the trivial bundle $\mathbb{A} \times V$. Let ξ be the dual of the quotient bundle on V corresponding to Z. Then the derived direct image $\mathbf{R}q'_*\mathcal{O}_Z$ is quasi-isomorphic to a minimal complex $F_\bullet$ with

$$F_i = \bigoplus_{j \geq 0} H^j\big(V, \textstyle\bigwedge^{i+j}\xi\big) \otimes_{\mathbb{C}} R(-i-j).$$

Here R is the coordinate ring of $\mathbb{A}$; it is a polynomial ring and $R(k)$ refers to twisting with respect to its natural grading. If q' is such that the natural map $\mathcal{O}_Y \longrightarrow \mathbf{R}q'_*\mathcal{O}_Z$ is a quasi-isomorphism (for example, if q' is a desingularization of Y and Y has rational singularities) then $F_\bullet$ is a minimal free resolution of $\mathbb{C}[Y]$ over the polynomial ring R.

In applying this technique in any given situation, there are two main steps involved: one must find a suitable Z and a suitable morphism $q' : Z \longrightarrow Y$ such that the map $\mathcal{O}_Y \longrightarrow \mathbf{R}q'_*\mathcal{O}_Z$ is a quasi-isomorphism and such that Z is a vector bundle over a projective variety V; and, one must be able to compute the necessary cohomology groups. We carry this out for opposite cells $Y_P(w)$, $w \in \mathcal{W}_{k,r}$.

As the first step, we establish the existence of a diagram as above, using the geometry of Schubert varieties. We now describe this briefly.

We take $\mathbb{A} = O^-_{G/P}$ and $Y = Y_P(w)$. Let $\widetilde{P}$ be the two-step parabolic subgroup $P_{\widehat{r-k},\hat{n}}$ of G, and let $\tilde{w}$ be the minimal representative of $w\widetilde{P}$ in $W^{\widetilde{P}}$ (that is, the set of minimal coset representatives in W, under the Bruhat order, of $W/W_{\widetilde{P}}$, where $W_{\widetilde{P}}$ is the Weyl group of $\widetilde{P}$). Let $w' := (k+1, \dots, r, n, \dots, r+1, k, \dots, 1) \in S_n$, the Weyl group of GL_n. Let $Z_{\widetilde{P}}(\tilde{w}) := Y_P(w) \times_{X_P(w)} X_{\widetilde{P}}(w)$ $(= (O^-_{G/P} \times P/\widetilde{P}) \cap X_{\widetilde{P}}(w))$. Then it turns out that $Z_{\widetilde{P}}(\tilde{w})$ is smooth (cf. Definition 3.20), and is a desingularization of $Y_P(w)$. Write p for the composite map $Z_{\widetilde{P}}(\tilde{w}) \hookrightarrow O^-_{G/P} \times P/\widetilde{P} \to P/\widetilde{P}$ where the first map is the inclusion and the second map is the projection. We have (cf. Theorem 3.22) that p identifies $Z_{\widetilde{P}}(\tilde{w})$ as a subbundle of the trivial bundle $O^-_{G/P} \times X_{P'_{\widehat{r-k}}}(w')$ over $X_{P'_{\widehat{r-k}}}(w')$, which arises as the restriction (to $X_{P'_{\widehat{r-k}}}(w')$) of a certain homogeneous vector bundle on $\mathrm{GL}_n / P'_{\widehat{r-k}}$. With $V := X_{P'_{\widehat{r-k}}}(w')$, we get:

$$
\begin{array}{ccccc}
Z_{\widetilde{P}}(\tilde{w}) & \hookrightarrow & O^-_{G/P} \times V & \longrightarrow & V \\
\downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \\
Y_P(w) & \hookrightarrow & O^-_{G/P} & &
\end{array}
\tag{1.2}
$$

In this diagram, q' is a desingularization of $Y_P(w)$. Since it is known that Schubert varieties have rational singularities, we have that the map $\mathbb{O}_Y \longrightarrow \mathbf{R}q'_*\mathbb{O}_Z$ is a quasi-isomorphism, so $F_\bullet$ is a minimal resolution.

At the second step, we need to determine the cohomology of the bundles $\bigwedge^t \xi$ over V. In the above situation, $V = X_{P'_{\widehat{r-k}}}(w') \hookrightarrow \mathrm{GL}_n / P'_{\widehat{r-k}}$. As can be easily seen, $X_{P'_{\widehat{r-k}}}(w')$ is a Grassmannian, namely, $\mathrm{GL}_r / P''_{\widehat{r-k}}$; the bundles $\bigwedge^t \xi$ (on $\mathrm{GL}_r / P''_{\widehat{r-k}}$) are also homogeneous, but are not of Bott type: they are not completely reducible (so one can not apply the Bott algorithm for computing the cohomology). This can be resolved in two ways. In [Ottaviani and Rubei 2006] the authors determine the cohomology of general homogeneous bundles on Hermitian symmetric spaces, and thus their results can be used to determine $H^\bullet(V, \bigwedge^t \xi)$. Alternatively, using a technique from [Weyman 2003], we may compute the resolution of a related space (whose associated homogeneous vector bundle is of Bott type) from which we retrieve the resolution of the coordinate ring of $Y_P(w)$ as a subcomplex.

We hope to extend the results of this paper to Schubert varieties in the orthogonal Grassmannian. Details will appear in a subsequent paper.

The paper is organized as follows. Section 2 contains notations and conventions and the necessary background material on Schubert varieties in the flag variety (Section 2.1) and Schubert varieties in the symplectic flag variety (Sections 2.2 and 2.3) and homogeneous bundles (Section 2.4). In Section 3, we discuss properties of Schubert desingularization, including the construction of Diagram 1.2. Section 4 is devoted to a review of the Kempf–Lascoux–Weyman technique and completes step one of the two part process of the geometric technique. Section 5 explains how the cohomology groups of the homogeneous bundles constructed in step one may be calculated.

2. Preliminaries

In this section we collect various results about Schubert varieties in the flag variety and symplectic flag variety, homogeneous vector bundles, and the Bott algorithm.

2.1. ***Notation and conventions in type A.*** We collect the symbols used and the conventions adopted in the rest of the paper here. For details on algebraic groups and Schubert varieties, the reader may refer to [Borel 1991; Jantzen 2003; Billey and Lakshmibai 2000; Seshadri 2007].

Let N be positive integer. We denote by GL_N (respectively, B_N, B_N^-) the group of all (respectively, upper triangular, lower triangular) invertible $N \times N$ matrices over $\mathbb{C}$. The Weyl group W of GL_N is isomorphic to the group S_N of permutations of N symbols and is generated by the *simple reflections* s_i, which correspond to the transpositions $(i, i+1)$, for $1 \le i \le N-1$. For $w \in W$, its *length* is the smallest integer l such that $w = s_{i_1} \cdots s_{i_l}$ as a product of simple reflections. For every

$1 \le i \le N-1$, there is a *minimal parabolic subgroup* P_i containing s_i (thought of as an element of GL_N) and a *maximal parabolic subgroup* $P_{\hat{i}}$ *not containing* s_i. Any parabolic subgroup can be written as $P_{\widehat{A}} := \bigcap_{i \in A} P_{\hat{i}}$ for some $A \subset \{1, \dots, N-1\}$. On the other hand, for $A \subseteq \{1, \dots, N-1\}$ write P_A for the subgroup of GL_N generated by P_i for $i \in A$. Then P_A is a parabolic subgroup and $P_{\{1,\dots,N-1\}\setminus A} = P_{\widehat{A}}$.

We write the elements of W in *one-line* notation: $(a_1, \dots, a_N)$ is the permutation $i \mapsto a_i$. For any $A \subseteq \{1, \dots, N-1\}$, define W_{P_A} to be the subgroup of W generated by $\{s_i : i \in A\}$. By W^{P_A} we mean the subset of W consisting of the minimal representatives (under the Bruhat order) in W of the elements of W/W_{P_A}. For $1 \le i \le N$, we represent the elements of $W^{P_{\hat{i}}}$ by sequences $(a_1, \dots, a_i)$ with $1 \le a_1 < \cdots < a_i \le N$ since under the action of the group $W_{P_{\hat{i}}}$, every element of W can be represented minimally by such a sequence.

We identify $\mathrm{GL}_N = \mathrm{GL}(V)$ for some N-dimensional vector-space V. Let $A := \{i_1 < i_2 < \cdots < i_r\} \subseteq \{1, \dots, N-1\}$. Then $\mathrm{GL}_N / P_{\widehat{A}}$ is the set of all flags $0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_r \subsetneq V$ of subspaces V_j of dimension i_j inside V. We call $\mathrm{GL}_N / P_{\widehat{A}}$ a *flag variety*. If $A = \{1, \dots, N-1\}$ (i.e., $P_{\widehat{A}} = B_N$), then we call the flag variety a *full flag variety*; otherwise, a *partial flag variety*. The *Grassmannian* $\mathrm{Grass}_{i,N}$ of i-dimensional subspaces of V is $\mathrm{GL}_N / P_{\hat{i}}$.

Let $\widetilde{P}$ be any parabolic subgroup containing B_N and $\tau \in W$. The *Schubert variety* $X_{\widetilde{P}}(\tau)$ is the closure inside $\mathrm{GL}_N / \widetilde{P}$ of $B_N \cdot e_w$ where e_w is the coset $\tau \widetilde{P}$, endowed with the canonical reduced scheme structure. Hereafter, when we write $X_{\widetilde{P}}(\tau)$, we mean that τ is the representative in $W^{\widetilde{P}}$ of its coset. The *opposite big cell* $O^-_{\mathrm{GL}_N / \widetilde{P}}$ in $\mathrm{GL}_N / \widetilde{P}$ is the B_N^--orbit of the coset $(\mathrm{id} \cdot \widetilde{P})$ in $\mathrm{GL}_N / \widetilde{P}$. Let $Y_{\widetilde{P}}(\tau) := X_{\widetilde{P}}(\tau) \cap O^-_{\mathrm{GL}_N / \widetilde{P}}$; we refer to $Y_{\widetilde{P}}(\tau)$ as the *opposite cell* of $X_{\widetilde{P}}(\tau)$.

We will write R^+, R^-, $R^+_{\widetilde{P}}$, $R^-_{\widetilde{P}}$, to denote respectively, positive and negative roots for GL_N and for $\widetilde{P}$. We denote by ϵ_i the character that sends the invertible diagonal matrix with $t_1, \dots, t_n$ on the diagonal to t_i.

2.2. ***Notation and conventions in type C.*** Below we review the properties of symplectic Schubert varieties relevant to this paper. For a more in-depth introduction the reader may refer to [Lakshmibai and Raghavan 2008, Chapter 6].

Let n be a positive integer. Let $V = \mathbb{C}^{2n}$ and let $(\cdot\,,\cdot)$ be a nondegenerate skew-symmetric bilinear form on V. Let $H = \mathrm{SL}(V)$ and $G = \mathrm{SP}(V) = \{Z \in \mathrm{SL}(V) \mid Z \text{ leaves the form } (\cdot\,,\cdot) \text{ invariant}\}$. We take the matrix of the form, with respect to the standard basis of V, to be

$$F = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$$

where J is the antidiagonal $(1, \dots, 1)$, in this case of size n. To simplify our notation we will normally omit specifying the size of J as it will be obvious from the context.

We may realize SP(V) as the fixed point set of the involution $\sigma : H \to H$ given by $\sigma(Z) = F(Z^T)^{-1}F^{-1}$ (cf. [Steinberg 1968]). That is,

$$\begin{aligned} G &= \{Z \in \mathrm{SL}(V) \mid Z^T F Z = F\} \\ &= \{Z \in \mathrm{SL}(V) \mid F^{-1}(Z^T)^{-1}F = Z\} \\ &= \{Z \in \mathrm{SL}(V) \mid F(Z^T)^{-1}F^{-1} = Z\} \\ &= H^\sigma. \end{aligned}$$

Denote by T_H and B_H the maximal torus in H consisting of diagonal matrices and the Borel subgroup in H consisting of upper triangular matrices, respectively. It is easily seen that T_H and B_H are stable under σ and we set $T_G = T_H^\sigma$, $B_G = B_H^\sigma$. It is easily checked that T_G is a maximal torus in G and B_G is a Borel subgroup in G.

Thus we obtain

$$W_G \hookrightarrow W_H$$

where W_G, W_H denote the Weyl groups of G, H, respectively (with respect to T_G, T_H, respectively). Further, σ induces an involution on W_H:

$$w = (a_1, \cdots, a_{2n}) \in W_H, \quad \sigma(w) = (c_1, \cdots, c_{2n}), \quad c_i = 2n + 1 - a_{2n+1-i}$$

and

$$W_G = W_H^\sigma.$$

Thus we obtain

$$W_G = \{(a_1 \cdots a_{2n}) \in S_{2n} \mid a_i = 2n + 1 - a_{2n+1-i},\ 1 \le i \le 2n\}.$$

(here, S_{2n} is the symmetric group on $2n$ letters). Thus $w = (a_1 \cdots a_{2n}) \in W_G$ is known once $(a_1 \cdots a_n)$ is known. We shall denote an element $(a_1 \cdots a_{2n})$ in W_G by just $(a_1 \cdots a_n)$. For example, $(4231) \in S_4$ represents $(42) \in W_G$, $G = \mathrm{SP}(4)$.

The involution σ induces an involution on $X(T_H)$, the character group of T_H:

$$\chi \in X(T_H), \quad \sigma(\chi)(D) = \chi(\sigma(D)), \quad D \in T_H.$$

Let ϵ_i, for $1 \le i \le 2n$, be the character in $X(T_H)$, $\epsilon_i(D) = d_i$, the i-th entry in $D \in T_H$. We have

$$\sigma(\epsilon_i) = -\epsilon_{2n+1-i}$$

Now it is easily seen that the under the canonical surjective map

$$\varphi : X(T_H) \to X(T_G)$$

we have

$$\varphi(\epsilon_i) = -\varphi(\epsilon_{2n+1-i}), \quad 1 \le i \le 2n.$$

Let $R_H := \{\epsilon_i - \epsilon_j, 1 \le i, j \le 2n\}$ be the root system of H (relative to T_H), and $R_H^+ := \{\epsilon_i - \epsilon_j, 1 \le i < j \le 2n\}$ the set of positive roots (relative to B_H). We have the following:

(a) σ leaves R_H (resp. R_H^+) stable.

(b) For $\alpha, \beta \in R_H, \varphi(\alpha) = \varphi(\beta) \Leftrightarrow \alpha = \sigma(\beta)$.

(c) φ is equivariant for the canonical action of W_G on $X(T_H)$, $X(T_G)$.

(d) $R_H^\sigma = \{\pm(\epsilon_i - \epsilon_{2n+1-i}), 1 \le i \le n\}$.

Let R_G (resp. R_G^+) be the set of roots of G with respect to T_G (resp. the set of positive roots with respect to B_G). Using the above facts and the explicit nature of the adjoint representation of G on Lie G, we deduce that

$$R_G = \varphi(R_H), \quad R_G^+ = \varphi(R_H^+).$$

In particular, R_G (resp. R_G^+) gets identified with the orbit space of R_H (resp. R_H^+) modulo the action of σ. Thus we obtain the following identification:

$$R_G = \{\pm(\epsilon_i \pm \epsilon_j),\ 1 \le i < j \le n\} \cup \{\pm 2\epsilon_i,\ i = 1, \dots, n\},$$
$$R_G^+ = \{(\epsilon_i \pm \epsilon_j),\ 1 \le i < j \le n\} \cup \{2\epsilon_i,\ i = 1, \dots, n\}.$$

The set S_G of simple roots in R_G^+ is given by

$$S_G := \{\alpha_i = \epsilon_i - \epsilon_{i+1},\ 1 \le i \le n-1\} \cup \{\alpha_n = 2\epsilon_n\}.$$

Let us denote the simple reflections in W_G by $\{s_i, 1 \le i \le n\}$, namely, $s_i =$ reflection with respect to $\epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n-1$, and $s_n =$ reflection with respect to $2\epsilon_n$. Then we have

$$s_i = \begin{cases} r_i r_{2n-i}, & \text{if } 1 \le i \le n-1, \\ r_n, & \text{if } i = n, \end{cases} \tag{2.2.1}$$

where r_i denotes the transposition $(i, i+1)$ in S_{2n} for $1 \le i \le 2n-1$.

For $w \in W_G$, let us denote by $l(w, W_H)$ (resp. $l(w, W_G)$) the length of w as an element of W_H (resp. W_G). For $w = (a_1, \cdots, a_{2n}) \in W_H$, denote

$$m(w) := \#\{i \le n \mid a_i > n\}. \tag{2.2.2}$$

Then for $w = (a_1, \cdots, a_{2n}) \in W_G$, we have $l(w, W_G) = \frac{1}{2}\big(l(w, W_H) + m(w)\big)$.

Proposition 2.2.3 [Lakshmibai and Raghavan 2008, Proposition 6.2.5.1]. *Let $w \in W_G$; let $X_G(w)$ (resp. $X_H(w)$) be the associated Schubert variety in G/B_G (resp. H/B_H). Under the canonical inclusion $G/B_G \hookrightarrow H/B_H$, we have $X_G(w) = X_H(w) \cap G/B_G$. Further, the intersection is scheme-theoretic.*

Notation 2.2.4. For the remainder of the paper we fix the following notation. Let $1 \le k < r \le n$ be positive integers. Let $Q = Q_{\hat{n}}$ to be the parabolic subgroup of H corresponding to omitting the root α_n and $P = P_{\hat{n}}$ to be the parabolic subgroup of G corresponding to omitting the root α_n. Let $\widetilde{P}$ be the two-step parabolic subgroup $P_{\widehat{r-k},\hat{n}}$ of G. Let $\widetilde{Q}$ be the three step parabolic subgroup $Q_{\widehat{r-k},\hat{n},\widehat{2n-(r-k)}}$ in H. Note that $P = Q^\sigma$ and $\widetilde{P} = \widetilde{Q}^\sigma$. Finally, we identify $P/\widetilde{P}$ with $\mathrm{GL}_n / P'_{\widehat{r-k}}$ where $P'_{\widehat{r-k}}$ is the parabolic subgroup of GL_n corresponding to omitting the root α_{r-k}.

Definition 2.2.5. A square $m \times m$ matrix X is *persymmetric* if $JX = X^T J$. Or, equivalently, if JX is symmetric.

Remark 2.2.6. We denote by Mat_n the space of $n \times n$ matrices. Let K be the subgroup of H consisting of matrices of the form

$$\begin{bmatrix} \mathrm{Id}_n & 0 \\ Y & \mathrm{Id}_n \end{bmatrix}, \qquad Y \in \mathrm{Mat}_n\,.$$

The canonical morphism $H \to H/Q$ induces a morphism $\psi_H : K \to H/Q$. We have that ψ_H is an open immersion, and $\psi_H(K)$ gets identified with the *opposite big cell* $O^-_{H/Q}$ in H/Q.

The cell $O^-_{H/Q}$ is σ-stable and by [Lakshmibai and Raghavan 2008, Corollary 6.2.4.3], we can identify the *opposite big cell* $O^-_{G/P}$ as

$$O^-_{G/P} = (O^-_{H/Q})^\sigma = \{z \in K \mid JY^T J = Y\}.$$

So $O^-_{G/P}$ is the subspace of K with Y persymmetric. Thus we can identify $O^-_{G/P}$ with the space of symmetric $n \times n$ matrices, Sym_n, under the map $O^-_{G/P} \longrightarrow \mathrm{Sym}_n$ given by

$$\begin{bmatrix} \mathrm{Id}_n & 0 \\ Y & \mathrm{Id}_n \end{bmatrix} \mapsto JY.$$

2.3. ***Opposite cells in Schubert varieties in the symplectic flag variety.*** A matrix $z \in \mathrm{SL}(V)$ with $n \times n$ block form

$$\begin{bmatrix} A_{n\times n} & C_{n\times n} \\ D_{n\times n} & E_{n\times n} \end{bmatrix}$$

is an element of G if and only if $z^T F z = F$, i.e., if and only if the following conditions hold on the $n \times n$ blocks:

$$A^T JD = D^T JA, \tag{2.3.1}$$

$$C^T JE = E^T JC, \tag{2.3.2}$$

$$J = (A^T JE - D^T JC) = (E^T JA - C^T JD). \tag{2.3.3}$$

The following proposition will prove useful throughout the rest of the paper.

Proposition 2.3.4. *Write U_P^- for the negative unipotent radical of P.*

(a) *$O_{G/P}^-$ can be naturally identified with $U_P^- P/P$*

(b) *For*

$$z = \begin{bmatrix} A_{n\times n} & C_{n\times n} \\ D_{n\times n} & E_{n\times n} \end{bmatrix} \in G,$$

$zP \in O_{G/P}^-$ if and only if A is invertible.

(c) *The inverse image of $O_{G/P}^-$ under the natural map $G/\widetilde{P} \to G/P$ is isomorphic to $O_{G/P}^- \times P/\widetilde{P}$ as schemes. Every element of $O_{G/P}^- \times P/\widetilde{P}$ is of the form*

$$\begin{bmatrix} A_{n\times n} & 0 \\ D_{n\times n} & J(A^T)^{-1}J \end{bmatrix} \bmod \widetilde{P} \in G/\widetilde{P}.$$

Moreover, two matrices

$$\begin{bmatrix} A_{n\times n} & 0_{n\times n} \\ D_{n\times n} & J(A^T)^{-1}J \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A'_{n\times n} & 0_{n\times n} \\ D'_{n\times n} & J(A'^T)^{-1}J \end{bmatrix}$$

in G represent the same element modulo $\widetilde{P}$ if and only if there exists a matrix $q \in P'_{\widehat{r-k}}$ (as defined in Notation 2.2.4) such that $A' = Aq$ and $D' = Dq$.

(d) *$P/\widetilde{P}$ is isomorphic to $\mathrm{GL}_n/P'_{\widehat{r-k}}$. In particular, the projection map $O_{G/P}^- \times P/\widetilde{P} \to P/\widetilde{P}$ is given by*

$$\begin{bmatrix} A_{n\times n} & 0 \\ D_{n\times n} & J(A^T)^{-1}J \end{bmatrix} \bmod \widetilde{P} \longmapsto A \bmod P'_{\widehat{r-k}} \in \mathrm{GL}_n/P'_{\widehat{r-k}} \cong P/\widetilde{P}.$$

Proof. (a): Note that U_P^- is the subgroup of G generated by the root subgroups $U_{-\alpha}$ for $\alpha \in R^+ \setminus R_P^+$. Under the canonical projection $G \to G/P$, $g \mapsto gP$, U_P^- is mapped isomorphically onto its image $O_{G/P}^-$ (cf. [Billey and Lakshmibai 2000, Section 4.4.4]). Thus we obtain the identification of $O_{G/P}^-$ with $U_P^- P/P$.

(b): Suppose that $zP \in O_{G/P}^-$. By (a) this means that $\exists$ $n \times n$ matrices A', C', D', E' such that

$$z_1 = \begin{bmatrix} \mathrm{Id}_n & 0 \\ D' & \mathrm{Id}_n \end{bmatrix} \in U_P^- \quad \text{and} \quad z_2 = \begin{bmatrix} A' & C' \\ 0 & E' \end{bmatrix} \in P \quad \text{with} \quad z = \begin{bmatrix} A & C \\ D & E \end{bmatrix} = z_1 z_2.$$

Hence $A = A'$, and A' invertible implies A invertible.

Conversely, suppose A is invertible. Let

$$z = \begin{bmatrix} A & C \\ D & E \end{bmatrix} \in G.$$

Then A, C, D, E satisfy properties (2.3.1)–(2.3.2). Since A is invertible we may write

$$z = z_1 z_2 \quad \text{where} \quad z_1 = \begin{bmatrix} \mathrm{Id}_n & 0 \\ DA^{-1} & \mathrm{Id}_n \end{bmatrix}, z_2 = \begin{bmatrix} A & C \\ 0 & E - DA^{-1}C \end{bmatrix}$$

We shall now show that $z_1, z_2 \in G$. First, we note that (2.3.1) implies that

$$J(DA^{-1}) = (DA^{-1})^T J. \tag{2.3.5}$$

Then (2.3.5) shows that $z_1 \in U_P^-$, and hence $z_1 \in G$.

Now $z_1 \in G$ implies $z_1^{-1} \in G$, and $z \in G$ by assumption. Hence $z_2 = z z_1^{-1} \in G$. Further, since A is invertible, $z_2 \in P$. Hence the coset $zP = z_1 P$, which in view of the fact that $z_1 \in U_P^-$, implies by part (a) that $zP \in O_{G/P}^-$.

(c): Let $z \in U_P^- P \subset G$. Then we can write $z = z_1 z_2$ uniquely with $z_1 \in U_P^-$, $z_2 \in P$. Suppose that

$$\begin{bmatrix} \mathrm{Id}_n & 0 \\ D_{n\times n} & \mathrm{Id}_n \end{bmatrix}\begin{bmatrix} A_{n\times n} & C_{n\times n} \\ 0_{n\times n} & E_{n\times n} \end{bmatrix} = \begin{bmatrix} \mathrm{Id}_n & 0 \\ D'_{n\times n} & \mathrm{Id}_n \end{bmatrix}\begin{bmatrix} A'_{n\times n} & C'_{n\times n} \\ 0_{n\times n} & E'_{n\times n} \end{bmatrix}$$

then $A = A'$, $C = C'$, $DA = D'A'$ and $DC + E = D'C' + E'$, which yields that $D' = D$ (since $A = A'$ is invertible), and then $E = E'$. Hence $U_P^- \times_{\mathbb{C}} P = U_P^- P$. Thus for any parabolic subgroup $P' \subseteq P$, $U_P^- \times_{\mathbb{C}} P/P' = U_P^- P/P'$. The asserted isomorphism follows by part (a) from taking $P' = \widetilde{P}$.

To see the second assertion consider

$$z = \begin{bmatrix} A_{n\times n} & C_{n\times n} \\ D_{n\times n} & E_{n\times n} \end{bmatrix} \in G$$

with $zP \in O_{G/P}^-$. Note that the $n \times n$ block matrices satisfy properties (2.3.1)–(2.3.3) and by (b), A is invertible.

We have by the first part of (c) that the coset zP is an element of $O_{G/P}^- \times P/\widetilde{P}$, since $zP \in O_{G/P}^-$.

Claim. *We have a decomposition of z in G,*

$$\begin{bmatrix} A & C \\ D & E \end{bmatrix} = y_1 y_2 \textit{ where } y_1 = \begin{bmatrix} A & 0 \\ D & J(A^T)^{-1}J \end{bmatrix} \in G,\ y_2 = \begin{bmatrix} \mathrm{Id}_n & A^{-1}C \\ 0 & \mathrm{Id}_n \end{bmatrix} \in \widetilde{P}.$$

We first check that $z = y_1 y_2$. We need the following identity

$$JA^T J(E - DA^{-1}C) = \mathrm{Id}_n, \tag{2.3.6}$$

which follows from

$$\begin{aligned} JA^T J(E - DA^{-1}C) &= J(A^T JE - A^T JDA^{-1}C) \\ &= J(A^T JE - D^T JAA^{-1}C) && (2.3.1) \\ &= JJ && (2.3.3) \\ &= \mathrm{Id}_n\,. \end{aligned}$$

So that

$$\begin{aligned} DA^{-1}C + J(A^T)^{-1}J &= DA^{-1}C + J(A^T)^{-1}JJA^TJ(E - DA^{-1}C) && (2.3.6)\\ &= DA^{-1}C + E - DA^{-1}C \\ &= E. \end{aligned}$$

With this it is easily verified that $z = y_1 y_2$.

It is clear that $y_1 \in G$. To show $y_2 \in G$ we need to check that $J(A^{-1}C)^T J = A^{-1}C$.

$$\begin{aligned} (A^{-1}C)^T J &= (A^{-1}C)^T JJA^TJ(E - DA^{-1}C) && (2.3.6)\\ &= C^T J(E - DA^{-1}C) \\ &= E^T JC - C^T JDA^{-1}C && (2.3.2)\\ &= (E - DA^{-1}C)^T JC && (2.3.5)\\ &= (E - DA^{-1}C)^T JAJJA^{-1}C \\ &= (JA^TJ(E - DA^{-1}C))^T J(A^{-1}C) \\ &= J(A^{-1}C) && (2.3.6) \end{aligned}$$

Thus $y_2 \in G$. It is clear additionally that $y_2 \in \widetilde{P}$ (in fact $y_2 \in B_G$).

Hence our claim follows and we have

$$\begin{bmatrix} A & C \\ D & E \end{bmatrix} = \begin{bmatrix} A & 0 \\ D & J(A^T)^{-1}J \end{bmatrix} \bmod \widetilde{P}.$$

Finally,

$$\begin{bmatrix} A_{n\times n} & 0_{n\times n} \\ D_{n\times n} & J(A^T)^{-1}J \end{bmatrix} = \begin{bmatrix} A'_{n\times n} & 0_{n\times n} \\ D'_{n\times n} & J(A'^T)^{-1}J \end{bmatrix} \quad \bmod \widetilde{P}$$

if and only if there exist matrices $q \in P'_{\widehat{r-k}}$, and $q' \in \mathrm{Mat}_n$ such that

$$\begin{bmatrix} A' & 0_{n\times n} \\ D' & J(A^T)^{-1}J \end{bmatrix} = \begin{bmatrix} A & 0_{n\times n} \\ D & J(A'^T)^{-1}J \end{bmatrix} \begin{bmatrix} q & q' \\ 0_{n\times n} & J(q^T)^{-1}J \end{bmatrix},$$

which holds if and only if $q' = 0$, $A' = Aq$ and $D' = Dq$ (since A and A' are invertible).

(d): There is a surjective morphism of $\mathbb{C}$-group schemes $P \to \mathrm{GL}_n$:

$$\begin{bmatrix} A & C \\ 0 & E \end{bmatrix} \to A.$$

This induces the required isomorphism. The element

$$\begin{bmatrix} A & C \\ D & E \end{bmatrix} \bmod \widetilde{P} \in O^-_{G/P} \times P/\widetilde{P}$$

decomposes uniquely as

$$\begin{bmatrix} \mathrm{Id}_n & 0 \\ DA^{-1} & \mathrm{Id}_n \end{bmatrix} \left(\begin{bmatrix} A & C \\ 0 & E - DA^{-1}C \end{bmatrix} \bmod \widetilde{P} \right)$$

and hence it is mapped to $A \bmod P'_{\widehat{r-k}}$. □

2.4. ***Homogeneous bundles and representations.*** Let Q be a parabolic subgroup of GL_n. We collect here some results about homogeneous vector bundles on GL_n/Q. Most of these results are well-known, but for some of them, we could not find a reference, so we give a proof here for the sake of completeness. Online notes of G. Ottaviani [1995] and of D. Snow [1994] discuss the details of many of these results.

Let L_Q and U_Q be respectively the Levi subgroup and the unipotent radical of Q. Let E be a finite-dimensional vector-space on which Q acts on the right.

Definition 2.4.1. Define $\mathrm{GL}_n \times^Q E := (\mathrm{GL}_n \times E)/\sim$ where $\sim$ is the equivalence relation $(g, e) \sim (gq, eq)$ for every $g \in \mathrm{GL}_n$, $q \in Q$ and $e \in E$. Then π_E : $\mathrm{GL}_n \times^Q E \longrightarrow \mathrm{GL}_n/Q$, $(g, e) \mapsto gQ$, is a vector bundle called the *vector bundle associated to* E (and the principal Q-bundle $\mathrm{GL}_n \longrightarrow \mathrm{GL}_n/Q$). For $g \in \mathrm{GL}_n$, $e \in E$, we write $[g, e] \in \mathrm{GL}_n \times^Q E$ for the equivalence class of $(g, e) \in \mathrm{GL}_n \times E$ under $\sim$. We say that a vector bundle $\pi : \mathbf{E} \longrightarrow \mathrm{GL}_n/Q$ is *homogeneous* if $\mathbf{E}$ has a GL_n-action and π is GL_n-equivariant, i.e, for every $y \in \mathbf{E}$, $\pi(g \cdot y) = g \cdot \pi(y)$.

Remark 2.4.2. There is a similar construction in the case when E is a left Q-module.

In this section, we abbreviate $\mathrm{GL}_n \times^Q E$ as $\widetilde{E}$. It is known that $\mathbf{E}$ is homogeneous if and only if $\mathbf{E} \simeq \widetilde{E}$ for some Q-module E. (If this is the case, then E is the fiber of $\mathbf{E}$ over the coset Q.) A homogeneous bundle $\widetilde{E}$ is said to be *irreducible* (respectively *indecomposable*, *completely reducible*) if E is an irreducible (respectively indecomposable, completely reducible) Q-module. It is known that E is completely reducible if and only if U_Q acts trivially and that E is irreducible if and only if additionally it is irreducible as a representation of L_Q. See [Snow 1994, Section 5] or [Ottaviani 1995, Section 10] for the details.

Discussion 2.4.3. For the cohomology group computations in this paper, we will primarily be interested in the case when GL_n/Q is a Grassmannian. Thus let $Q = P_{\hat{m}}$, with $1 \le m \le n-1$. A weight λ is said to be *Q-dominant* if and only if when we express λ as $\sum_{i=1}^n \lambda_i \epsilon_i$ (where ϵ_i, for $1 \le i \le n$, is the character that sends a diagonal matrix in T to its i-th entry), then $\lambda_1 \ge \ldots \ge \lambda_m$ and $\lambda_{m+1} \ge \ldots \ge \lambda_n$. We will write $\lambda = (\lambda_1, \ldots, \lambda_n)$ to mean that $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$. Every finite-dimensional irreducible Q-module is of the form $H^0(Q/B_n, L_\lambda)$ for a Q-dominant weight λ. Hence the irreducible homogeneous vector bundles on GL_n/Q are in correspondence with Q-dominant weights. We describe them now. If $Q = P_{\widehat{n-i}}$, then $\mathrm{GL}_n/Q = \mathrm{Grass}_{i,n}$. (Recall that, for us, the GL_n-action on $\mathbb{C}^n$ is on the right.) On $\mathrm{Grass}_{i,n}$, we have the *tautological sequence*

$$0 \longrightarrow \mathcal{R}_i \longrightarrow \mathbb{C}^n \otimes \mathcal{O}_{\mathrm{Grass}_{i,n}} \longrightarrow \mathcal{Q}_{n-i} \longrightarrow 0 \tag{2.4.4}$$

of homogeneous vector bundles. The bundle $\mathcal{R}_i$ is called the *tautological subbundle* (of the trivial bundle $\mathbb{C}^n$) and $\mathcal{Q}_{n-i}$ is called the *tautological quotient bundle*. Every irreducible homogeneous bundle on $\mathrm{Grass}_{i,n}$ is of the form $S_{(\lambda_1,\cdots,\lambda_{n-i})}\mathcal{Q}^*_{n-i} \otimes S_{(\lambda_{n-i+1},\cdots,\lambda_n)}\mathcal{R}^*_i$ for some $P_{\widehat{n-i}}$-dominant weight λ. Here S_μ denotes the *Schur functor* associated to the partition μ (cf. [Fulton and Harris 1991, §6.1]).

A Q-dominant weight is called (m)-*dominant* in [Weyman 2003, p. 114]. Although our definition looks like Weyman's definition, we should keep in mind that our action is on the right. We only have to be careful when we apply the Borel–Weil–Bott theorem (more specifically, the Bott algorithm). In this paper, our computations are done only on Grassmannians. If μ and ν are partitions, then (μ, ν) will be Q-dominant (for a suitable Q), and will give us the vector bundle $S_\mu \mathcal{Q}^* \otimes S_\nu \mathcal{R}^*$ (this is where the right-action of Q becomes relevant) and to compute its cohomology, we will have to apply the Bott algorithm to the Q-dominant weight (ν, μ). (In [Weyman 2003], one would get $S_\mu \mathcal{R}^* \otimes S_\nu \mathcal{Q}^*$ and would apply the Bott algorithm to (μ, ν).) □

We now give a brief description of the Bott algorithm for computing the cohomology of irreducible homogeneous vector bundles on GL_n / Q [Weyman 2003, Remark 4.1.5].

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a weight. As in [Weyman 2003, Remark 4.1.5] we define an action of the permutation $\nu_i = (i, i+1)$ on the set of weights in the following way:

$$\nu_i \alpha = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1} - 1, \alpha_i + 1, \alpha_{i+2}, \dots, \alpha_n). \tag{2.4.5}$$

The Bott algorithm may be applied to our case as follows. Let $Q = P_{\hat{m}}$, with $1 \le m \le n-1$ and let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a Q-dominant weight with associated homogeneous vector bundle $V(\lambda) := S_{(\lambda_1,\dots,\lambda_m)}\mathcal{Q}^* \otimes S_{(\lambda_{m+1},\dots,\lambda_n)}\mathcal{R}^*$. We will apply the Bott algorithm to $\lambda' = (\lambda_{m+1}, \dots, \lambda_n, \lambda_1, \dots, \lambda_m)$ in keeping with the last paragraph of Discussion 2.4.3.

If λ' is nonincreasing, then $H^0(\mathrm{GL}_n/Q, V(\lambda)) = S_{\lambda'}\mathbb{C}^n$ and $H^i(\mathrm{GL}_n/Q, V(\lambda)) = 0$ for $i > 0$. Otherwise we start to apply the exchanges of type (2.4.5) to λ', trying to move smaller numbers on the left to the right. Two possibilities can occur:

(1) We apply an exchange of type (2.4.5) and it leaves the sequence unchanged. In this case $H^i(\mathrm{GL}_n / Q, V(\lambda)) = 0$ for $i \ge 0$.

(2) After applying j exchanges, we transform λ' into a nonincreasing sequence β. Then we have $H^i(\mathrm{GL}_n/Q, V(\lambda)) = 0$ for $i \ne j$ and $H^j(\mathrm{GL}_n/Q, V(\lambda)) = S_\beta \mathbb{C}^n$.

3. Properties of Schubert desingularization in type C

Recall the following result about the tangent space of a Schubert variety, see [Billey and Lakshmibai 2000, Chapter 4] for details.

Proposition 3.1. *Let Q be a parabolic subgroup of* SL_{2n}. *Let $\tau \in W^Q$. Then the dimension of the tangent space of $X_Q(\tau)$ at e_{id} is*

$$\#\{s_\alpha \mid \alpha \in R^- \backslash R_Q^- \quad \textit{and} \quad \tau \geq s_\alpha \textit{ in } W/W_Q\}.$$

In particular, $X_Q(\tau)$ *is smooth if and only if*

$$\dim X_Q(\tau) = \#\{s_\alpha \mid \alpha \in R^- \backslash R_Q^- \quad \textit{and} \quad \tau \geq s_\alpha \textit{ in } W/W_Q\}.$$

Notation 3.2. For an integer i with $1 \leq i \leq n$ we define $i' = 2n + 1 - i$. Let $1 \leq k < r \leq n$. Then

$$\mathcal{W}_{k,r} = \begin{cases} (k+1, \ldots, r, n', \ldots, (r+1)', k', \ldots, 1') \in W^P, & \text{if } r < n, \\ (k+1, \ldots, r, k', \ldots, 1') \in W^P, & \text{if } r = n. \end{cases}$$

Let $1 \leq k < r \leq n$ be integers. Let $w \in \mathcal{W}_{k,r}$ with $\tilde{w}$ its minimal representative in $W^{\widetilde{P}}$.

Proposition 3.3. *The Schubert variety $X_{\widetilde{Q}}(\tilde{w})$ in $H/\widetilde{Q}$ is smooth.*

Proof. Let $w_{\max} \in W_H (= S_{2n})$ be the maximal representative of $\tilde{w}$. Then

$$w_{\max} = \begin{cases} \big([r, k+1][1', k'][(r+1)', n'][n, (r+1)][k, 1][(k+1)', r']\big), & \text{if } r < n, \\ \big([r, k+1][1', k'][k, 1][(k+1)', r']\big), & \text{if } r = n. \end{cases}$$

To see this we need to show that $X_{P_{\hat{i}}}(w_{\max}) = X_{P_{\hat{i}}}(\tilde{w})$ for $i = r-k,\ n,\ 2n-(r-k)$ and that $w_{\max}$ is the maximal element of W_H with this property. But this follows from the fact that for $\tau = (c_1, \ldots, c_{2n}) \in W_H$ and $1 \leq i \leq 2n$ we have that $X_{P_{\hat{i}}}(\tau) = X_{P_{\hat{i}}}(\tau')$ where $\tau' \in W^{P_{\hat{i}}}$ is the element with $c_1, \ldots, c_i$ written in increasing order.

Thus $X_{B_H}(w_{\max})$ is the inverse image of $X_{\widetilde{Q}}(\tilde{w})$ under the natural morphism $H/B_H \to H/\widetilde{Q}$. As $w_{\max}$ is a 4231 and 3142 avoiding element of W_H we have that $X_{B_H}(w_{\max})$ is nonsingular (see [Billey and Lakshmibai 2000, 8.1.1]). Since the morphism $H/B_H \to H/\widetilde{Q}$ has nonsingular fibers (namely $\widetilde{Q}/B_H$), $X_{\widetilde{Q}}(\tilde{w})$ must be smooth. □

Proposition 3.4. *The Schubert variety $X_{\widetilde{P}}(\tilde{w})$ in $G/\widetilde{P}$ is smooth .*

Proof. Let $w_{\max}$ be as in the proof of Proposition 3.3. Then clearly $w_{\max}$ is in W_G and $X_{B_G}(w_{\max})$ is the inverse image of $X_{\widetilde{P}}(\tilde{w})$ under the natural morphism $G/B_G \to G/\widetilde{P}$.

Claim. $X_{B_G}(w_{\max})$ *is smooth.*

Note that the claim implies the required result (since the canonical morphism $G/B_G \to G/\widetilde{P}$ is a fibration with nonsingular fibers (namely, $\widetilde{P}/B_G$)). To prove the claim, as seen in the proof of Proposition 3.3, we have that $X_{B_H}(w_{\max})$ is smooth.

We conclude the smoothness of $X_{B_G}(w_{\max})$ using the following two formulas [Lakshmibai 1987, §3(VI), Remark 5.8]:

$$l_G(\theta) = \tfrac{1}{2}[l_H(\theta) + m(\theta)], \tag{1}$$

where we let $\theta \in W_G$, say, $\theta = (a_1, \cdots a_n)$. With $m(\theta) = \#\{i,\ 1 \le i \le m \mid a_i > m\}$ (cf. (2.2.2)), we have

$$\dim T_{\mathrm{id}}(\theta, G) = \tfrac{1}{2}[\dim T_{\mathrm{id}}(\theta, H) + c(\theta)], \tag{2}$$

where $c(\theta) = \#\{1 \le i \le m \mid \theta \ge s_{\epsilon_{2i}}\}$, and $T_{\mathrm{id}}(\theta, G)$ (resp $T_{\mathrm{id}}(\theta, H)$) denotes the Zariski tangent space of $X_{B_G}(\theta)$ (resp $X_{B_H}(\theta)$) at e_{id}. Note that $s_{\epsilon_{2i}}$ is just the transposition (i, i') (cf. (2.2.1)). Now taking $\theta = w_{\max}$, we have, $c(w_{\max}) = m(w_{\max})$. Hence we obtain from (1), (2) that $\dim T_{\mathrm{id}}(w_{\max}, G) = l_G(w_{\max})$, proving that $X_{B_G}(w_{\max})$ is smooth at e_{id}, and hence is nonsingular (note that for a Schubert variety X, the singular locus of X, $\mathrm{Sing}(X)$, is B-stable implying $e_{\mathrm{id}} \in \mathrm{Sing}(X)$ if $\mathrm{Sing}(X) \ne \varnothing$). Thus the claim (and hence the required result) follows. □

Remark 3.5. We have that $X_{\widetilde{P}}(\tilde{w})$ is the fixed point set under an automorphism of order two of the Schubert variety $X_{\widetilde{Q}}(\tilde{w})$ and thus is smooth, provided char $K \ne 2$ ([Edixhoven 1992, Proposition 3.4]).

Discussion 3.6. To give a characterization of $Y_{\widetilde{Q}}(\tilde{w})$ we first need a review of the structure of $O^-_{H/\widetilde{Q}}$ and its Plücker coordinates.

Recall that for the Plücker embedding of the Grassmannian $\mathrm{Grass}_{d,n}$, the *Plücker coordinate* $p_{\underline{i}}(U)$, $U \in \mathrm{Grass}_{d,n}$ and $\underline{i} = (i_1, \ldots, i_d)$ with $1 \le i_1 < \ldots < i_d < n$, is just the $d \times d$ minor of the matrix $A_{n\times d}$ with row indices $(i_1, \ldots, i_d)$ (here the matrix $A_{n\times d}$ represents the d-dimensional subspace U with respect to the standard basis).

The cell $O^-_{H/\widetilde{Q}}$ can be identified with the affine space of lower-triangular matrices with possible nonzero entries x_{ij} at row i and column j where (i, j) is such that there exists an $l \in \{r-k, n, 2n-(r-k)\}$ such that $j \le l < i \le N$. To see this, note that we are interested in those (i, j) such that the root $\epsilon_i - \epsilon_j$ belongs to $R^- \setminus R^-_{\widetilde{Q}}$. Since $R^-_{\widetilde{Q}} = R^-_{Q_{\widehat{r-k}}} \cap R_{Q_{\hat{n}}} \cap R_{Q_{\widehat{2n-(r-k)}}}$, we see that we are looking for (i, j) such that $\epsilon_i - \epsilon_j \in R^- \setminus R^-_{Q_{\hat{l}}}$, for some $l \in \{r-k, n, 2n-(r-k)\}$. For the maximal parabolic subgroup $P_{\hat{l}}$, we have, $R^- \setminus R^-_{Q_{\hat{l}}} = \{\epsilon_i - \epsilon_j \mid 1 \le j \le l < i \le N\}$. We have $\dim O^-_{H/\widetilde{Q}} = |R^- \setminus R^-_{\widetilde{Q}}|$.

Thus we have the following identification

$$O^-_{H/\widetilde{Q}} = \begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ \mathcal{D}_1 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ \mathcal{D}_3 & \mathcal{D}_4 & E' & \mathrm{Id}_{r-k} \end{bmatrix} \tag{3.7}$$

where the block matrices have possible nonzero entries x_{ij} given by

$$A'=\begin{bmatrix} x_{(r-k)+1\ 1} & \cdots & x_{(r-k)+1\ r-k} \\ \vdots & & \vdots \\ x_{n\ 1} & \cdots & x_{n\ r-k} \end{bmatrix}, \qquad E'=\begin{bmatrix} x_{2n-(r-k)+1\ n+1} & \cdots & x_{2n-(r-k)+1\ 2n-(r-k)} \\ \vdots & & \vdots \\ x_{2n\ n+1} & \cdots & x_{2n\ 2n-(r-k)} \end{bmatrix},$$

$$\mathcal{D}_1=\begin{bmatrix} x_{n+1\ 1} & \cdots & x_{n+1\ r-k} \\ \vdots & & \vdots \\ x_{2n-(r-k)\ 1} & \cdots & x_{2n-(r-k)\ r-k} \end{bmatrix}, \qquad \mathcal{D}_2=\begin{bmatrix} x_{n+1\ (r-k)+1} & \cdots & x_{n+1\ n} \\ \vdots & & \vdots \\ x_{2n-(r-k)\ (r-k)+1} & \cdots & x_{2n-(r-k)\ n} \end{bmatrix},$$

$$\mathcal{D}_3=\begin{bmatrix} x_{2n-(r-k)+1\ 1} & \cdots & x_{2n-(r-k)+1\ r-k} \\ \vdots & & \vdots \\ x_{2n\ 1} & \cdots & x_{2n\ r-k} \end{bmatrix}, \quad \mathcal{D}_4=\begin{bmatrix} x_{2n-(r-k)+1\ (r-k)+1} & \cdots & x_{2n-(r-k)+1\ n} \\ \vdots & & \vdots \\ x_{2n\ (r-k)+1} & \cdots & x_{2n\ n} \end{bmatrix}.$$

We may break the Plücker coordinates we want to understand into several cases.

Case 1: For $i > r$, $j \le r-k$ the Plücker coordinate $p^{(r-k)}_{(i,j)}$ on the Grassmannian $H/Q_{\widehat{r-k}}$ lifts to a regular function on $H/\widetilde{Q}$. Its restriction to $O^-_{H/\widetilde{Q}}$ is the $r-k \times r-k$ minor of (3.7) with column indices $\{1, 2, \dots, r-k\}$ and row indices $\{1, \dots, j-1, j+1, \dots, r-k, i\}$. This minor is the determinant of an $r-k \times r-k$ matrix with the top $(r-k)-1$ rows equal to Id_{r-k} omitting the j-th row, and the bottom row equal to the first $r-k$ entries of the i-th row of (3.7). The determinant of this matrix is thus $(-1)^{(r-k)-j}x_{ij}$. Thus for $i > r$, $j \le r-k$:

$$p^{(r-k)}_{(i,j)}|_{O^-_{H/\widetilde{Q}}} = (-1)^{(r-k)-j}x_{ij}. \tag{3.8}$$

Case 2: For $i > 2n-(r-k)$, $n < j \le 2n-(r-k)$ the Plücker coordinate $p^{(2n-(r-k))}_{(i,j)}$ on the Grassmannian $H/Q_{\widehat{2n-(r-k)}}$ lifts to a regular function on $H/\widetilde{Q}$. Its restriction to $O^-_{H/\widetilde{Q}}$ is the $2n-(r-k) \times 2n-(r-k)$ minor of (3.7) with column indices $\{1, 2, \dots, 2n-(r-k)\}$ and row indices $\{1, \dots, j-1, j+1, \dots, 2n-(r-k), i\}$. This minor is the determinant of

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 \\ \widehat{\mathcal{D}}_1 & \widehat{\mathcal{D}}_2 & \widehat{I_1} \\ [x_{i\ 1}\dots x_{i\ r-k}] & [x_{i\ (r-k)+1}\dots x_{i\ n}] & [x_{i\ n+1}\dots x_{i\ 2n-(r-k)}] \end{bmatrix} \tag{3.9}$$

where $\widehat{\mathcal{D}}_1$, $\widehat{\mathcal{D}}_2$, and $\widehat{I_1}$ are equal to, respectively, $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathrm{Id}_{n-(r-k)}$ with their $(j-n)$-th rows omitted. The determinant of (3.9) is equal to the determinant of

$$\begin{bmatrix} \widehat{I_1} \\ [x_{i\ n+1}\dots x_{i\ 2n-(r-k)}] \end{bmatrix}.$$

As above this is just an identity matrix with a single row replaced and so its determinant is just $(-1)^{2n-(r-k)-j}x_{ij}$. Thus for $i > 2n-(r-k)$, $n < j \le 2n-(r-k)$:

$$p^{(2n-(r-k))}_{(i,j)}|_{O^-_{H/\widetilde{Q}}} = (-1)^{2n-(r-k)-j}x_{ij}. \tag{3.10}$$

Case 3: For $i > 2n-(r-k)$, $r-k < j \le n$ the Plücker coordinate $p^{(2n-(r-k))}_{(i,j)}$ on the Grassmannian $H/Q_{\widehat{2n-(r-k)}}$ lifts to a regular function on $H/\widetilde{Q}$. Its restriction to $O^-_{H/\widetilde{Q}}$ is the $2n-(r-k) \times 2n-(r-k)$ minor of (3.7) with column indices $\{1, 2, \dots, 2n-(r-k)\}$ and row indices $\{1, \dots, j-1, j+1, \dots, 2n-(r-k), i\}$. This minor is the determinant of

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 \\ \widehat{A'} & \widehat{I_2} & 0 \\ \mathcal{D}_1 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} \\ [x_{i\,1} \dots x_{i\,r-k}] & [x_{i\,(r-k)+1} \dots x_{i\,n}] & [x_{i\,n+1} \dots x_{i\,2n-(r-k)}] \end{bmatrix} \tag{3.11}$$

where $\widehat{A'}$ and $\widehat{I_2}$ are equal to, respectively, A' and $\mathrm{Id}_{n-(r-k)}$ with their $j-(r-k)$-th rows omitted. The determinant of (3.11) is equal to the determinant of

$$\begin{bmatrix} \widehat{I_2} & 0 \\ \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} \\ [x_{i\,(r-k)+1} \dots x_{i\,n}] & [x_{i\,n+1} \dots x_{i\,2n-(r-k)}] \end{bmatrix}. \tag{3.12}$$

To calculate this, shift the bottom row so that it becomes the $j-(r-k)$-th row of $\widehat{I_2}$. Let $M = 2n-(r-k)-j$. Then the determinant of (3.12) will be $(-1)^M$ times the determinant of

$$\begin{bmatrix} I_3 & Z \\ \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} \end{bmatrix}, \tag{3.13}$$

where I_3 is $\mathrm{Id}_{n-(r-k)}$ with the $j-(r-k)$-th row replaced by $[x_{i\,(r-k)+1} \dots x_{i\,n}]$ and Z is the zero matrix with the $j-(r-k)$-th row replaced by $[x_{i\,n+1} \dots x_{i\,2n-(r-k)}]$. Since the lower right block matrix of (3.13) commutes with its lower left block matrix we have that the determinant of (3.13) is equal to the determinant of $I_3 - Z\mathcal{D}_2$. We have that $Z\mathcal{D}_2$ is equal to the zero matrix with its $j-(r-k)$-th row replaced by

$$[x_{i\,(r-k)+1} \dots x_{i\,n}]\mathcal{D}_2.$$

And thus $I_3 - Z\mathcal{D}_2$ is equal to $\mathrm{Id}_{n-(r-k)}$ with the $j-(r-k)$-th row replaced by

$$[x_{i\,(r-k)+1} \dots x_{i\,n}] - [x_{i\,(r-k)+1} \dots x_{i\,n}]\mathcal{D}_2.$$

And so the determinant of $I_3 - Z\mathcal{D}_2$ is merely equal to the $j-(r-k)$-th entry of $I_3 - Z\mathcal{D}_2$ which is

$$x_{ij} - [x_{i\,(r-k)+1} \dots x_{i\,n}][x_{n+1\,j} \dots x_{2n-(r-k)\,j}]^T.$$

Combining all our steps, we finally have that for $i > 2n-(r-k)$, $r-k < j \le n$:

$$p^{(2n-(r-k))}_{(i,j)}\big|_{O^-_{H/\widetilde{Q}}} = (-1)^M \big(x_{ij} - [x_{i\,(r-k)+1} \dots x_{i\,n}][x_{n+1\,j} \dots x_{2n-(r-k)\,j}]^T\big). \quad \square \tag{3.14}$$

Theorem 3.15. *The opposite cell $Y_{\widetilde{Q}}(\tilde{w})$ can be identified with the subspace of $O^-_{H/\widetilde{Q}}$ given by matrices of the form*

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ 0 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ 0 & E'\mathcal{D}_2 & E' & \mathrm{Id}_{r-k} \end{bmatrix}$$

with $\mathcal{D}_2 \in \mathrm{Mat}_{n-(r-k)}$, $A' \in \mathrm{Mat}_{n-(r-k)\times r-k}$ with the bottom $n-r$ rows of A' all zero, and $E' \in \mathrm{Mat}_{r-k\times n-(r-k)}$ with the left $n-r$ columns of E' all zero.

Proof. For $j \le r-k < i$ the reflection (i, j) equals $(1, 2, \ldots, j-1, j+1, \ldots, r-k, i)$ and $\tilde{w}$ equals $(k+1, \ldots, r)$ in $W/W_{Q_{\widehat{r-k}}}$. Thus for $i > r$ and $j \le r-k$, the reflection (i, j) is not smaller than $\tilde{w}$ in $W/W_{Q_{\widehat{r-k}}}$ so the Plücker coordinate $p^{(r-k)}_{(i,j)}$ vanishes on $X_{\widetilde{Q}}(\tilde{w})$. We saw in (3.8) that for such (i, j) we have $p^{(r-k)}_{(i,j)} = (-1)^{(r-k)-j}x_{ij}$ and thus $x_{ij} \equiv 0$ on $Y_{\widetilde{Q}}(\tilde{w})$.

For $j \le n < i$ the reflection (i, j) equals $(1, 2, \ldots, j-1, j+1, \ldots, n, i)$ and $\tilde{w}$ is equal to $(k+1, \ldots, r, n', \ldots, (r+1)', k', \ldots, 1')$ in $W/W_{Q_{\hat{n}}}$. Thus there is no choice of (i, j) such that (i, j) is not smaller than $\tilde{w}$ in $W/W_{Q_{\hat{n}}}$.

For $j \le 2n-(r-k) < i$ the reflection (i, j) equals $(1, 2, \ldots, j-1, j+1, \ldots, 2n-(r-k), i)$ and $\tilde{w}$ equals $(1, \ldots, n, n', \ldots, (r+1)', k', \ldots, 1')$ in $W/W_{Q_{\widehat{2n-(r-k)}}}$. Thus for $i > 2n-(r-k)$, and $j \le 2n-r$ the reflection (i, j) is not smaller than $\tilde{w}$ in $W/W_{Q_{\widehat{2n-(r-k)}}}$. We break these into two cases, ignoring those $j \le r-k$ as we have already shown above that for $j \le r-k$ and $i > 2n-(r-k)$ we have $x_{ij} \equiv 0$ on $Y_{\widetilde{Q}}(\tilde{w})$.

The first case is for (i, j) with $i > 2n-(r-k)$, and $n < j \le 2n-r$. The fact that (i, j) is not smaller than $\tilde{w}$ in $W/W_{Q_{\widehat{2n-(r-k)}}}$ implies that the Plücker coordinate $p^{(2n-(r-k))}_{(i,j)}$ vanishes on $X_{\widetilde{Q}}(\tilde{w})$. We saw in (3.10) that for such (i, j) we have $p^{(2n-(r-k))}_{(i,j)} = (-1)^{2n-(r-k)-j}x_{ij}$ and thus $x_{ij} \equiv 0$ on $Y_{\widetilde{Q}}(\tilde{w})$.

The second case is for (i, j) with $i > 2n-(r-k)$ and $r-k < j \le n$. The reflection (i, j) is not smaller than $\tilde{w}$ in $W/W_{Q_{\widehat{2n-(r-k)}}}$ implies that the Plücker coordinate $p^{(2n-(r-k))}_{(i,j)}$ vanishes on $X_{\widetilde{Q}}(\tilde{w})$. We saw in (3.14) that for such (i, j) we have $p^{(2n-(r-k))}_{(i,j)} = (-1)^M\big(x_{ij} - [x_{i\,(r-k)+1} \ldots x_{i\,n}][x_{n+1\,j} \ldots x_{2n-(r-k)\,j}]^T\big)$. Combining these two facts we get $x_{ij} = [x_{i\,(r-k)+1} \ldots x_{i\,n}][x_{n+1\,j} \ldots x_{2n-(r-k)\,j}]^T$.

As $[x_{i\,(r-k)+1} \ldots x_{i\,n}]$ is the $(2n-(r-k)-i)$-th row of E' and $[x_{n+1\,j} \ldots x_{2n-(r-k)\,j}]^T$ is the $(2n-(r-k)-j)$-th column of $\mathcal{D}_2$ it is clear that on $Y_{\widetilde{Q}}(\tilde{w})$ we have $x_{ij} = (E'X)_{(2n-(r-k)-i)\,(2n-(r-k)-j)}$.

On the other hand note that the reflections (i, j) with $i > r$ and $j \le r-k$, and $i > 2n-(r-k)$ and $r-k < j \le 2n-r$ are precisely the reflections s_α with $\alpha \in R^- \backslash R^-_{\widetilde{Q}}$ and $\tilde{w} \not\ge s_\alpha$ in $W/W_{\widetilde{Q}}$. Since $X_{\widetilde{Q}}(\tilde{w})$ is smooth this implies by Proposition 3.1 that the codimension of $Y_{\widetilde{Q}}(\tilde{w})$ in $O^-_{H/\widetilde{Q}}$ is equal to

$\#\{(i,j) \mid i > r \text{ and } j \le r-k, \text{ or } i > 2n-(r-k) \text{ and } r-k < j \le 2n-r\}$. Above we have shown that for each such (i,j), x_{ij} either vanishes, or is completely dependent on the entries of $E'X$. Thus $Y_{\widetilde{Q}}(\tilde{w})$ is the subspace of $O^-_{H/\widetilde{Q}}$ defined by the vanishing of $\{x_{ij} \mid i > r \text{ and } j \le r-k, \text{ or } i > 2n-(r-k) \text{ and } n < j \le 2n-r\}$ and $x_{ij} = (E'X)_{(2n-(r-k)-i)\,(2n-(r-k)-j)}$ for $i > 2n-(r-k)$ and $r-k < j \le n$. □

Example 3.16. Let $k=2$, $r=4$, and $n=5$. Then $\widetilde{Q} = Q_{\hat{2},\hat{5},\hat{8}}$, $w = (3,4,6,9,10)$, and $\tilde{w} = (3,4,6,9,10,1,2,5)$. Then

$$O^-_{H/\widetilde{Q}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{41} & x_{42} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{51} & x_{52} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & 1 & 0 & 0 & 0 & 0 \\ x_{71} & x_{72} & x_{73} & x_{74} & x_{75} & 0 & 1 & 0 & 0 & 0 \\ x_{81} & x_{82} & x_{83} & x_{84} & x_{85} & 0 & 0 & 1 & 0 & 0 \\ x_{91} & x_{92} & x_{93} & x_{94} & x_{95} & x_{96} & x_{97} & x_{98} & 1 & 0 \\ x_{101} & x_{102} & x_{103} & x_{104} & x_{105} & x_{106} & x_{107} & x_{108} & 0 & 1 \end{bmatrix}.$$

And $Y_{\widetilde{P}}(\tilde{w})$ will be the subspace of $O^-_{H/\widetilde{Q}}$ given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{41} & x_{42} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{63} & x_{64} & x_{65} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{73} & x_{74} & x_{75} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & x_{83} & x_{84} & x_{85} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x_{97}x_{73}+x_{98}x_{83} & x_{97}x_{74}+x_{98}x_{84} & x_{97}x_{75}+x_{98}x_{85} & 0 & x_{97} & x_{98} & 1 & 0 \\ 0 & 0 & x_{107}x_{73}+x_{108}x_{83} & x_{107}x_{74}+x_{108}x_{84} & x_{107}x_{75}+x_{108} & 0 & x_{107} & x_{108} & 0 & 1 \end{bmatrix}.$$

Corollary 3.17. *The opposite cell $Y_{\widetilde{P}}(\tilde{w})$ can be identified with the subspace of $O^-_{G/\widetilde{P}}$ given by matrices of the form*

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ 0 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ 0 & -J(A')^T J\mathcal{D}_2 & -J(A')^T J & \mathrm{Id}_{r-k} \end{bmatrix}$$

with $J\mathcal{D}_2 \in \mathrm{Sym}_{n-(r-k)}$ and $A' \in \mathrm{Mat}_{n-(r-k)\times r-k}$ with the bottom $n-r$ rows of A' all zero.

Proof. Let $y \in Y_{\widetilde{P}}(\tilde{w}) = (Y_{\widetilde{Q}}(\tilde{w}))^\sigma \subset Y_{\widetilde{Q}}(\tilde{w})$. So y is just an element of $Y_{\widetilde{Q}}(\tilde{w})$ that is fixed under the involution σ. That is, an element which satisfies (2.3.1)–(2.3.3). Theorem 3.15 gives us that y is of the form

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ 0 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ 0 & E'\mathcal{D}_2 & E' & \mathrm{Id}_{r-k} \end{bmatrix}$$

with $\mathcal{D}_2 \in \mathrm{Mat}_{n-(r-k)}$, $A' \in \mathrm{Mat}_{n-(r-k)\times r-k}$ with the bottom $n-r$ rows of A' all zero, and $E' \in \mathrm{Mat}_{r-k\times n-(r-k)}$ with the left $n-r$ columns of E' all zero. We must now check what restrictions on y are required for it to satisfy (2.3.1)–(2.3.3). For y to satisfy (2.3.3) we know that

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ A' & \mathrm{Id}_{n-(r-k)} \end{bmatrix}^T \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix} \begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ E' & \mathrm{Id}_{n-(r-k)} \end{bmatrix} \left(= \begin{bmatrix} (A')^T J + JE' & J \\ J & 0 \end{bmatrix} \right)$$

must equal

$$\begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}$$

which implies that $E' = -J(A')^T J$.

Any y clearly satisfies (2.3.2). And finally for y to satisfy (2.3.1),

$$\begin{bmatrix} 0 & \mathcal{D}_2 \\ 0 & -J(A')^T J\mathcal{D}_2 \end{bmatrix}^T \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix} \begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ A' & \mathrm{Id}_{n-(r-k)} \end{bmatrix} \left(= \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{D}_2^T J \end{bmatrix} \right)$$

must equal

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ A' & \mathrm{Id}_{n-(r-k)} \end{bmatrix}^T \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathcal{D}_2 \\ 0 & -J(A')^T J\mathcal{D}_2 \end{bmatrix} \left(= \begin{bmatrix} 0 & 0 \\ 0 & J\mathcal{D}_2 \end{bmatrix} \right)$$

which implies that $J\mathcal{D}_2 = \mathcal{D}_2^T J$, or equivalently $J\mathcal{D}_2 \in \mathrm{Sym}_{n-(r-k)}$. □

Remark 3.18. We may identify $O^-_{P/\widetilde{P}}$ with $O^-_{\mathrm{GL}_n/P'_{\widehat{r-k}}}$ under the map

$$\begin{bmatrix} \mathcal{A} & 0 \\ 0 & J(\mathcal{A}^T)^{-1}J \end{bmatrix} \mapsto \mathcal{A}.$$

Remark 3.19. Let V_w be the linear subspace of Sym_n given by $x_{ij} = 0$ if $j \le r-k$ or $i < n-(r-k)$. And let V'_w be the linear subspace of $O^-_{\mathrm{GL}_n/P'_{\widehat{r-k}}}$ given by $x_{ij} = 0$ if $i > r$ and $j \le r-k$.

Consider the map $\delta : Y_{\widetilde{P}}(\tilde{w}) \hookrightarrow O^-_{G/\widetilde{P}} = O^-_{G/P} \times O^-_{P/\widetilde{P}} \cong O^-_{G/P} \times O^-_{\mathrm{GL}_n/P'_{\widehat{r-k}}}$, where the first map is inclusion, the second is simply the product decomposition, and the final map is from Remark 3.18. This map is given explicitly by

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ 0 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ 0 & -J(A')^T J\mathcal{D}_2 & -J(A')^T J & \mathrm{Id}_{r-k} \end{bmatrix} \mapsto$$

$$\left(\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ 0 & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ -\mathcal{D}_2 A' & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ J(A')^T J\mathcal{D}_2 A' & -J(A')^T J\mathcal{D}_2 & 0 & \mathrm{Id}_{r-k} \end{bmatrix}, \begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ A' & \mathrm{Id}_{n-(r-k)} \end{bmatrix} \right).$$

Consider the isomorphism $\gamma : O^-_{G/P} \times O^-_{\mathrm{GL}_n/P'_{\widehat{r-k}}} \to \mathrm{Sym}_n \times O^-_{\mathrm{GL}_n/P'_{\widehat{r-k}}}$ (cf. Remark 2.2.6) given by

$$\left(\begin{bmatrix} \mathrm{Id}_n & 0 \\ L & \mathrm{Id}_n \end{bmatrix}, \begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ N & \mathrm{Id}_{n-(r-k)} \end{bmatrix} \right) \mapsto \left((LN)^T JN, \begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ N & \mathrm{Id}_{n-(r-k)} \end{bmatrix} \right).$$

We have that under the map $\gamma \circ \delta$, $Y_{\widetilde{P}}(\tilde{w})$ gets identified with $V_w \times V'_w$. This follows by a simple computation and Corollary 3.17.

Definition 3.20. Now let $Z_{\widetilde{P}}(\tilde{w}) := Y_P(w) \times_{X_P(w)} X_{\widetilde{P}}(\tilde{w})$. Then $Z_{\widetilde{P}}(\tilde{w}) = (O^-_{G/P} \times P/\widetilde{P}) \cap X_{\widetilde{P}}(\tilde{w})$. Hence $Z_{\widetilde{P}}(\tilde{w})$ is smooth, being open in the smooth $X_{\widetilde{P}}(\tilde{w})$ (cf. Proposition 3.3).

Write p for the composite map $Z_{\widetilde{P}}(\tilde{w}) \to O^-_{G/P} \times P/\widetilde{P} \to P/\widetilde{P}$ ($\cong \mathrm{GL}_n / P'_{\widehat{r-k}}$) where the first map is the inclusion and the second map is the projection. Using Proposition 2.3.4(c) and (d) we see that

$$p\left(\begin{bmatrix} A & 0 \\ D & J(A^T)^{-1} J \end{bmatrix} \pmod{\widetilde{P}} \right) = A \pmod{P'_{\widehat{r-k}}}.$$

Note that A is invertible by 2.3.4(b).

Using the injective map

$$A \in B_n \longmapsto \begin{bmatrix} A & 0_{n\times n} \\ 0_{n\times n} & J(A^T)^{-1} J \end{bmatrix} \in B_G,$$

B_n can be thought of as a subgroup of B_G. With this identification we have the following proposition.

Proposition 3.21. *$Z_{\widetilde{P}}(\tilde{w})$ is B_n-stable for the action on the left by multiplication. Further p is B_n equivariant.*

Proof. Let $z \in \mathrm{SP}_{2n}$ such that $z\widetilde{P} \in Z_{\widetilde{P}}(\tilde{w})$. Then by Proposition 2.3.4(c) we may write

$$z = \begin{bmatrix} A & 0 \\ D & J(A^T)^{-1} J \end{bmatrix} \bmod \widetilde{P},$$

such that $z\widetilde{P} \in Z_{\widetilde{P}}(\tilde{w})$. Since $X_{B_G}(\tilde{w}) \to X_{\widetilde{P}}(\tilde{w})$ is surjective, we may assume that $z \pmod{B_G} \in X_{B_G}(\tilde{w})$, i.e., $z \in \overline{B_G \tilde{w} B_G}$. Then for every $A' \in B_n$:

$$\begin{bmatrix} A' & 0_{n\times n} \\ 0_{n\times n} & J(A'^T)^{-1}J \end{bmatrix} z = \begin{bmatrix} A'A & 0 \\ J(A'^T)^{-1}JD & J(A'^T)^{-1}(A'^T)^{-1}J \end{bmatrix} =: z'.$$

Then $z' \in \overline{B_G \tilde{w} B_G}$, so $z' \pmod{\widetilde{P}} \in X_{\widetilde{P}}(\tilde{w})$. By Proposition 2.3.4(b), we have that A is invertible, and hence AA'. This implies again by Proposition 2.3.4(b) that $z' \pmod{\widetilde{P}} \in Z_{\widetilde{P}}(\tilde{w})$. Thus $Z_{\widetilde{P}}(\tilde{w})$ is B_n stable. Also $p(A'z) = p(z') = A'A = A'p(z)$. Hence p is B_n-equivariant. □

Theorem 3.22. *With notation as above, let* $w' := (k+1, \dots, r, n, \dots, r+1, k, \dots, 1)$ *be an element of* S_n*, the Weyl group of* GL_n*. Then*:

(a) *The natural map* $X_{\widetilde{P}}(\tilde{w}) \longrightarrow X_P(w)$ *is proper and birational. In particular, the map* $Z_{\widetilde{P}}(\tilde{w}) \longrightarrow Y_P(w)$ *is proper and birational. And therefore,* $Z_{\widetilde{P}}(\tilde{w})$ *is a desingularization of* $Y_P(w)$.

(b) $X_{P'_{\widehat{r-k}}}(w')$ *is the fiber of the natural map* $Z_{\widetilde{P}}(\tilde{w}) \longrightarrow Y_P(w)$ *at* $e_{\mathrm{id}} \in Y_P(w)$.

(c) $X_{P'_{\widehat{r-k}}}(w')$ *is the image of* p*. Further,* p *is a fibration with fiber isomorphic to* V_w.

(d) p *identifies* $Z_{\widetilde{P}}(\tilde{w})$ *as a subbundle of the trivial bundle* $O^-_{G/P} \times X_{P'_{\widehat{r-k}}}(w')$*, which arises as the restriction of the vector bundle on* $\mathrm{GL}_n / P'_{\widehat{r-k}}$ *associated to the* $P'_{\widehat{r-k}}$*-module* V_w *(which, in turn, is a* $P'_{\widehat{r-k}}$*-submodule of* $O^-_{G/P}$*).*

Proof. (a): The map $X_{\widetilde{P}}(\tilde{w}) \hookrightarrow G/\widetilde{P} \to G/P$ is proper and its (scheme-theoretic) image is $X_P(w)$, hence $X_{\widetilde{P}}(\tilde{w}) \to X_P(w)$ is proper. Birationality follows from the fact that $\tilde{w}$ is the minimal representative of the coset $w\widetilde{P}$.

(b): The fiber at $e_{\mathrm{id}} \in Y_P(w)$ of the map $Y_{\widetilde{P}}(\tilde{w}) \longrightarrow Y_P(w)$ is $0 \times V'_w$, inside $V_w \times V'_w = Y_{\widetilde{P}}(\tilde{w})$. Since $Z_{\widetilde{P}}(\tilde{w})$ is the closure of $Y_{\widetilde{P}}(\tilde{w})$ inside $O^-_{G/P} \times P/\widetilde{P}$ and $X_{P'_{\widehat{r-k}}}(w')$ is the closure of V'_w inside $P/\widetilde{P}$ (note that as a subvariety of $O^-_{P/\widetilde{P}}$, $Y_{P'_{\widehat{r-k}}}(w')$ is identified with V'_w), we see that the fiber at e_{id} (belonging to $Y_P(w)$) of $Z_{\widetilde{P}}(\tilde{w}) \longrightarrow Y_P(w)$ is $X_{P'_{\widehat{r-k}}}(w')$.

(c): From Remark 3.19 we have $p(Y_{\widetilde{P}}(\tilde{w})) = V'_w \subseteq X_{P'_{\widehat{r-k}}}(w')$. Since $Y_{\widetilde{P}}(\tilde{w})$ is dense inside $Z_{\widetilde{P}}(\tilde{w})$ and $X_{P'_{\widehat{r-k}}}(w')$ is closed in $\mathrm{GL}_n / P'_{\widehat{r-k}}$ we see that $p(Z_{\widetilde{P}}(\tilde{w})) \subseteq X_{P'_{\widehat{r-k}}}(w')$. The other inclusion $X_{P'_{\widehat{r-k}}}(w') \subseteq p(Z_{\widetilde{P}}(\tilde{w}))$ follows from (b). Hence, $p(Z_{\widetilde{P}}(\tilde{w})) = X_{P'_{\widehat{r-k}}}(w')$. To prove the second assertion of (c) we shall show that for every $A \in \mathrm{GL}_n$ with $A \bmod P'_{\widehat{r-k}} \in X_{P'_{\widehat{r-k}}}(w')$, we have that $p^{-1}(A \bmod P'_{\widehat{r-k}})$ is isomorphic to V_w.

To prove this we first observe that $p^{-1}(e_{\mathrm{id}})$ is isomorphic to V_w in view of Remark 3.19. Next observe that every B_n-orbit inside $X_{P'_{\widehat{r-k}}}(w')$ meets V'_w (which equals $Y_{P'_{\widehat{r-k}}}(w')$); further p is B_n-equivariant by Proposition 3.21 and hence every fiber is isomorphic to the fiber at e_{id}, i.e., isomorphic to V_w.

(d): Define a right action of GL_n on $O^-_{G/P}$(identified with Sym_n as in Remark 2.2.6) as $g \circ v = g^T vg$ for $g \in \mathrm{GL}_n$, $v \in \mathrm{Sym}_n$. This induces an action of $P'_{\widehat{r-k}}$ on $O^-_{G/P}$ under which V_w is stable. Thus we get the homogeneous bundle

$$\mathrm{GL}_n \times^{P'_{\widehat{r-k}}} V_w \longrightarrow \mathrm{GL}_n / P'_{\widehat{r-k}}.$$

Now to prove the assertion about $Z_{\widetilde{P}}(\tilde{w}))$ being a vector bundle over $X_{P'_{\widehat{r-k}}}(w')$, we will show that there is a commutative diagram given as below, with ψ an isomorphism:

$$\begin{array}{ccccc}
Z_{\widetilde{P}}(\tilde{w}) & & & & \\
 & \searrow^{\psi} & & \overset{\phi}{\searrow} & \\
 & & (\mathrm{GL}_n \times^{P'_{\widehat{r-k}}} V_w)|_{X_{P'_{\widehat{r-k}}}(w')} & \longrightarrow & \mathrm{GL}_n \times^{P'_{\widehat{r-k}}} V_w \\
 & \searrow_{p} & \downarrow & & \downarrow \alpha \\
 & & X_{P'_{\widehat{r-k}}}(w') & \xrightarrow{\ \beta\ } & \mathrm{GL}_n / P'_{\widehat{r-k}}
\end{array}$$

The map α is the homogeneous bundle map and β is the inclusion map. Define ϕ by

$$\phi : \begin{bmatrix} A & 0_{n\times n} \\ D & J(A^T)^{-1}J \end{bmatrix} \bmod \widetilde{P} \longmapsto (A, D^T JA)/\sim .$$

Using Proposition 2.3.4(c) and Remark 3.19 we conclude the following: ϕ is well-defined and injective; $\beta \cdot p = \alpha \cdot \phi$; hence, by the universal property of products, the map ψ exists; and, finally, the injective map ψ is in fact an isomorphism (by dimension considerations). □

As an immediate consequence of Theorem 3.22 we have

Corollary 3.23. *We have the following realization of Diagram 1.2*:

$$\begin{array}{ccccc}
Z_{\widetilde{P}}(\tilde{w}) & \hookrightarrow & O^-_{G/P} \times X_{P'_{\widehat{r-k}}}(w') & \longrightarrow & X_{P'_{\widehat{r-k}}}(w') \\
\downarrow q' & & \downarrow q & & \\
Y_P(w) & \hookrightarrow & O^-_{G/P} & &
\end{array}$$

Proposition 3.24. (1) *The Schubert variety $X_{P'_{\widehat{r-k}}}(w')$ is isomorphic to the Grassmannian $\mathrm{GL}_r / P''_{\widehat{r-k}}$, where $P''_{\widehat{r-k}}$ is the parabolic subgroup in GL_r obtained by omitting α_{r-k}.*

(2) $(\mathrm{GL}_n \times^{P'_{\widehat{r-k}}} V_w)|_{X_{P'_{\widehat{r-k}}}(w')} \cong (\mathrm{GL}_n \times^{P'_{\widehat{r-k}}} V_w)|_{\mathrm{GL}_r / P''_{\widehat{r-k}}} \cong \mathrm{GL}_r \times^{P''_{\widehat{r-k}}} V_w$ *as homogeneous vector bundles.*

Proof. (1): This is clear.

(2): Consider the embedding $i : \mathrm{GL}_r \hookrightarrow \mathrm{GL}_n$ given by

$$R \mapsto \begin{bmatrix} R & 0 \\ 0 & \mathrm{Id}_{n-r} \end{bmatrix}.$$

Define the action of GL_r on Sym_n as the action induced by this embedding. This induces an action of $P''_{\widehat{r-k}}$ on Sym_n. As $i(P''_{\widehat{r-k}}) \subset P'_{\widehat{r-k}}$, the $P'_{\widehat{r-k}}$ stability of V_w implies the $P''_{\widehat{r-k}}$ stability of V_w. Hence our result follows. □

Corollary 3.25. *We have the following realization of Diagram 1.2*:

$$\begin{array}{ccccc} Z_{\widetilde{P}}(\tilde{w}) & \hookrightarrow & O^-_{G/P} \times \mathrm{GL}_r / P''_{\widehat{r-k}} & \longrightarrow & \mathrm{GL}_r / P''_{\widehat{r-k}} \\ \downarrow q' & & \downarrow q & & \\ Y_P(w) & \hookrightarrow & O^-_{G/P} & & \end{array}$$

4. Free resolutions

Kempf–Lascoux–Weyman geometric technique. We summarize the geometric technique of computing free resolutions, following [Weyman 2003, Chapter 5].

Consider Diagram 1.1. There is a natural map $f : V \longrightarrow \mathrm{Grass}_{r,d}$ (where $r = \mathrm{rk}_V Z$ and $d = \dim \mathbb{A}$) such that the inclusion $Z \subseteq \mathbb{A} \times V$ is the pull-back of the tautological sequence (2.4.4); here $\mathrm{rk}_V\, Z$ denotes the rank of Z as a vector bundle over V, i.e., $\mathrm{rk}_V\, Z = \dim Z - \dim V$. Let $\xi = (f^*\mathcal{Q})^*$. Write R for the polynomial ring $\mathbb{C}[\mathbb{A}]$ and $\mathfrak{m}$ for its homogeneous maximal ideal. (The grading on R arises as follows. In Diagram 1.1, $\mathbb{A}$ is thought of as the fiber of a trivial vector bundle, so it has a distinguished point, its origin. Now, being a subbundle, Z is defined by linear equations in each fiber; i.e., for each $v \in V$, there exist $s := (\dim \mathbb{A} - \mathrm{rk}_V\, Z)$ linearly independent linear polynomials $\ell_{v,1}, \dots, \ell_{v,s}$ that vanish along Z and define it. Now $Y = \{y \in \mathbb{A} : \text{there exists } v \in V \text{ such that } \ell_{v,1}(y) = \cdots = \ell_{v,s}(y) = 0\}$. Hence Y is defined by homogeneous polynomials. This explains why the resolution obtained below is graded.) Let $\mathfrak{m}$ be the homogeneous maximal ideal, i.e., the ideal defining the origin in $\mathbb{A}$. Then:

Theorem 4.1 [Weyman 2003, Basic Theorem 5.1.2]. *With notation as above*, *there is a finite complex* $(F_\bullet, \partial_\bullet)$ *of finitely generated graded free R-modules that is quasi-isomorphic to* $\mathbf{R}q'_*\mathcal{O}_Z$, *with*

$$F_i = \bigoplus_{j \ge 0} H^j\big(V, \textstyle\bigwedge^{i+j} \xi\big) \otimes_{\mathbb{C}} R(-i-j),$$

and $\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}$. *Furthermore*, *the following are equivalent*:

(a) *Y has rational singularities i.e.*, $\mathbf{R}q'_*\mathcal{O}_Z$ *is quasi-isomorphic to* $\mathcal{O}_Y$;

(b) *$F_\bullet$ is a minimal R-free resolution of* $\mathbb{C}[Y]$, *i.e.*, $F_0 \simeq R$ *and* $F_{-i} = 0$ *for every* $i > 0$.

A sketch of the proof is given in [Kummini et al. 2015, Section 4], and [Weyman 2003, 5.1.3] may be consulted for a more comprehensive account.

Our situation. We now apply Theorem 4.1 to our situation. We keep the notation of Theorem 3.22. Theorem 4.1 and Corollary 3.25 yield the following result:

Theorem 4.2. *Write ξ for the homogeneous vector bundle on $\mathrm{GL}_r / P''_{\widehat{r-k}}$ associated to the $P''_{\widehat{r-k}}$-module $(O^-_{G/P}/V_w)^*$ (this is the dual of the quotient of $O^-_{G/P} \times \mathrm{GL}_r / P''_{\widehat{r-k}}$ by $Z_{\widetilde{P}}(\tilde{w})$). Then we have a minimal R-free resolution $(F_\bullet, \partial_\bullet)$ of $\mathbb{C}[Y_P(w)]$ with*

$$F_i = \bigoplus_{j \geq 0} H^j(\mathrm{GL}_r / P''_{\widehat{r-k}}, \bigwedge^{i+j} \xi) \otimes_{\mathbb{C}} R(-i-j).$$

Computing the cohomology groups required in Theorem 4.2 in the general situation is a difficult problem. Techniques for computing them in our specific case are discussed in the following section.

5. Cohomology of homogeneous vector bundles

We have shown in Theorem 4.2 that the calculation of a minimal R-free resolution of $\mathbb{C}[Y_P(w)]$ comes down to the computation of the cohomology of certain homogeneous bundles over $\mathrm{GL}_r / P''_{\widehat{r-k}}$. In particular we need to calculate

$$H^\bullet\big(\mathrm{GL}_r / P''_{\widehat{r-k}}, \textstyle\bigwedge^t \xi\big) \tag{5.1}$$

for arbitrary t.

The $P''_{\widehat{r-k}}$-module $(O^-_{G/P}/V_w)^*$ is not completely reducible (the unipotent radical of $P''_{\widehat{r-k}}$ does not act trivially), and thus we can not use the Bott algorithm to compute its cohomology. In [Ottaviani and Rubei 2006] the authors determine the cohomology of general homogeneous bundles on Hermitian symmetric spaces. As $\mathrm{GL}_r / P''_{\widehat{r-k}}$ is such a space their results could be used to determine (5.1). In practice, proceeding along these lines is possible though extremely complicated.

Another approach to the calculation of these cohomologies comes from using a technique employed in [Weyman 2003, Chapter 6.3]. There the minimal R-free resolution of a related space is computed and the minimal R-free resolution of $\mathbb{C}[Y_P(w)]$ can be seen as a subresolution. In [Weyman 2003] this method is used for the case when $n = r$. That is, the case where $Y_P(w)$ is the symmetric determinental variety. In this case the authors assume that $k = 2u$ (the odd case can be reduced to this even case). They look at the subspace T_w of Sym_n given by symmetric matrices of block form

$$\begin{bmatrix} 0_{n-u \times n-u} & R \\ R^T & S_{u \times u} \end{bmatrix}.$$

Let $P'_{\widehat{n-u}}$ be the parabolic subgroup of GL_n omitting the root α_{n-u}, then T_w is a $P'_{\widehat{n-u}}$-module under the same action. If Z_w is the homogeneous vector bundle associated with T_w we have the following diagram

$$\begin{array}{ccccc} Z_w & \hookrightarrow & \mathrm{Sym}_n \times \mathrm{GL}_n / P'_{\widehat{n-u}} & \longrightarrow & \mathrm{GL}_n / P'_{\widehat{n-u}} \\ \big\downarrow{\scriptstyle q'} & & \big\downarrow{\scriptstyle q} & & \\ Y & \hookrightarrow & \mathrm{Sym}_n & & \end{array}$$

They show that the resolution of $\mathbb{C}[Y_P(w)]$ can be realized as a subresolution of the resolution of $\mathbb{C}[Y]$. In this case, the $P'_{\widehat{n-u}}$-module $(\mathrm{Sym}_n / T_w)^*$ (this is the dual of the quotient of $\mathrm{Sym}_n \times \mathrm{GL}_n / P'_{\widehat{n-u}}$ by Z_w) is completely reducible and thus the cohomology of the corresponding homogeneous vector bundles $\bigwedge^t \xi$ may be computed using the Bott algorithm, leading to this:

Theorem 5.2 [Weyman 2003, Theorem 6.3.1(c)]. *The i-th term G_i of the minimal free resolution of $\mathbb{C}[Y_P(w)]$ as an R module is given by the formula*

$$G_i = \bigoplus_{\substack{\lambda \in Q_{k-1}(2t) \\ \mathrm{rank}\,\lambda \text{ even} \\ i = t - k\frac{1}{2}\mathrm{rank}\,\lambda}} S_{\lambda^\vee}\mathbb{C}^n \otimes_{\mathbb{C}} R.$$

Here $Q_{k-1}(2t)$ is the set of partitions λ of $2t$ which in hook notation can be written as $\lambda = (a_1, \dots, a_s | b_1, \dots, b_s)$, where s is a positive integer, and for each j we have $a_j = b_j + (k-1)$. And $\lambda^\vee$ is the conjugate (or dual) partition of λ. And finally, $\mathrm{rank}\,\lambda$ is defined as being equal to l, where the largest square fitting inside λ is of size $l \times l$.

Similar methods may be used to compute a closed form formula for the minimal free resolution of $\mathbb{C}[Y_P(w)]$ as an R module in the case $r \neq n$.

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