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In memoriam: Robert Steinberg

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Special Issue

In memoriam

Robert Steinberg

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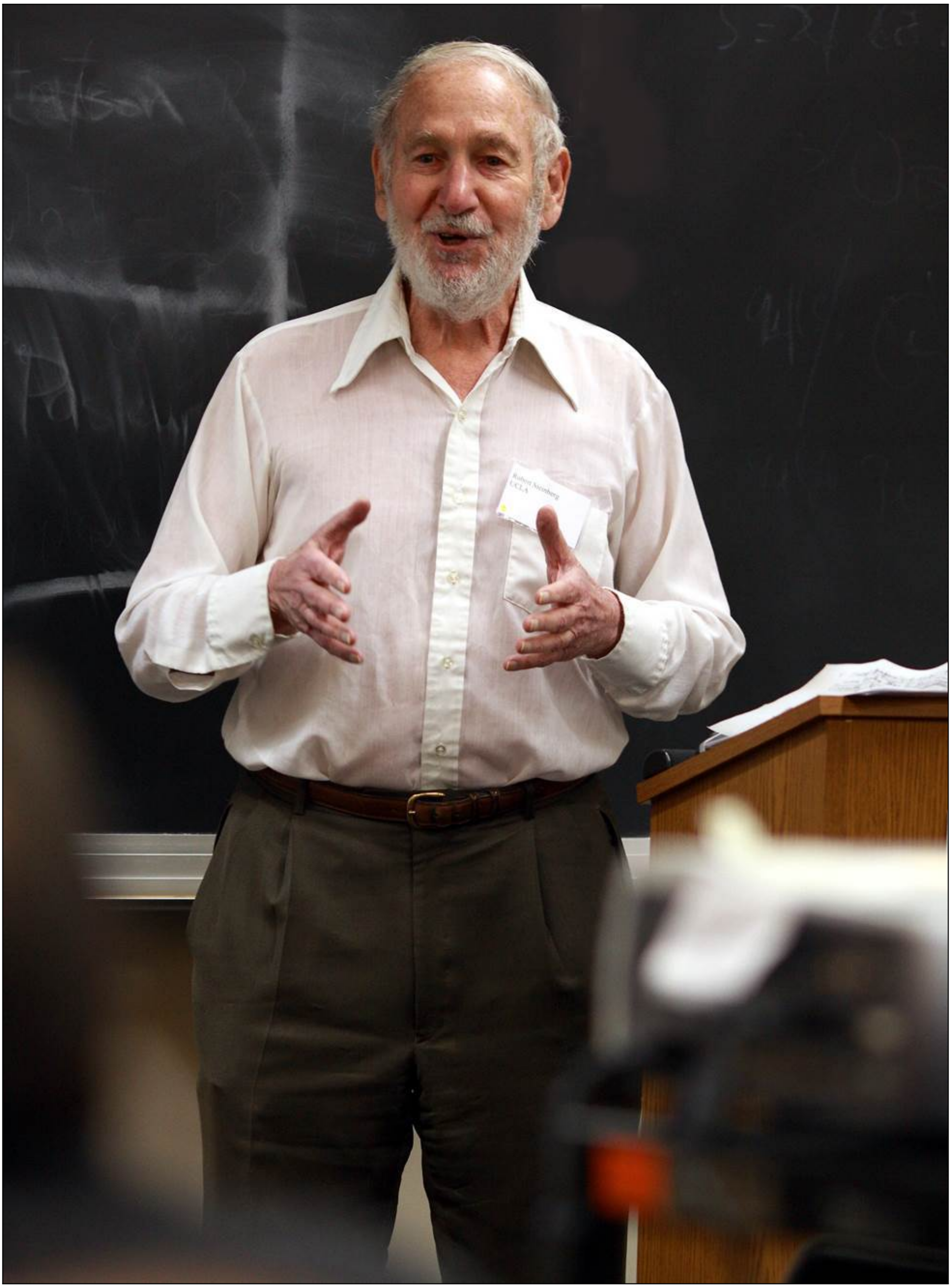

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ROBERT STEINBERG (1922–2014): IN MEMORIAM

V. S. Varadarajan

He touched nothing that he did not adorn.

The quotation above is by Samuel Johnson, writing about his friend Oliver Goldsmith. I think it is the most satisfying way to describe the work and legacy of Robert Steinberg, who passed away on May 25, 2014 on his 92nd birthday. His towering stature as one of the great masters of the theory of algebraic groups and finite groups, the vast scope, depth, and beauty of his papers (some key ones were published in the PJM), and his gigantic presence in the algebraic scene in Southern California, are the reasons that led the Board of Governors of the PJM to request that a special volume of the PJM be published in his memory. In this brief essay I shall try to sketch a portrait of a master who wore his mantle of greatness with unassuming simplicity and charm.

This is a melancholy task for me, to write about someone who was a good friend and role model for me for nearly fifty years. In these days of ever multiplying awards, million dollar grants, medals, and so on, it is refreshing, even humbling, to talk about a man who never sought the limelight, who worked quietly on the problems that appealed to him, and evolved into one of the great masters and innovators of the theory of semisimple algebraic groups. The problems he worked on and considered important became the central problems of the subject. His influence on the subject was enormous. Even after he retired he could surprise experts with new and easier proofs of some of the fundamental theorems of the subject. His monumental set of lecture notes on Chevalley groups [1968] has been studied by hundreds of mathematicians (I myself lectured twice on them) and will appear as a publication of the AMS. In spite of his greatness he was a gentle and modest man, aware of his gifts certainly, but accepting them and trying to get the job done.

His work is widely available in his *Collected papers* [1997] and its scope is extraordinary. It is a very difficult task to present his work in one short essay and I will not even attempt it, nor do I have the competence for it. But I will describe some highlights so that most of the readers will get some idea of what he achieved in his lifetime. I thank Professor Alexander Merkurjev for enlightening me on the impact of Steinberg's work on algebraic K-theory and other parts of mathematics.

Keywords: Robert Steinberg, memorial issue.

He told me that he and his collaborators have used every major theorem of Steinberg in their work.

I was spiritually close to Steinberg as a mathematician. In his words I was also a semisimple mathematician, as he said when he first introduced me to his close friend and collaborator Tonny Springer, a Dutch semisimple mathematician. However I was more interested in the transcendental aspects of real semisimple Lie groups, such as infinite dimensional representations and harmonic analysis.

After his beloved wife Maria passed away, he gradually lost the desire and will to do things, and I became closer to him in those days by visiting him as frequently as I could. His passing away was traumatic to his nephew and nieces and to all of his friends and relatives.

He was born in Romania but his parents settled in Canada very soon afterwards. I am sure he was deeply influenced by the wide open spaces of Canada and thereby acquired his lifelong love for long hikes and camping trips. He and Maria spent a part of almost every summer by hiking and camping in the high sierras. Maria's strength of mind and decisiveness blended well with his gentle personality, and they became one soul.

He studied under Richard Brauer and got his doctorate degree in 1948. He came to UCLA in 1948 and never left it. In 1985 he was given the Leroy P. Steele Prize of the AMS for lifetime achievement. He was elected to the National Academy of Sciences in the same year. He wrote a letter to me on that occasion and said that this proves he still has friends. He was awarded the Jeffery–Williams Prize of the Canadian Mathematical Society in 1990. He was an avid fan of basketball and hockey, and the Bruins and Lakers were his favorite teams, and Jerry West his all-time favorite player. He was generally taciturn but always charming, and could open up to close friends.

To understand roughly the scope of his achievement, it is essential to know what simple and semisimple groups are. In 1894, Elie Cartan classified all simple Lie algebras over $\mathbb{C}$, and found that they fall into four infinite families (the *classical algebras*), and five isolated ones (the *exceptional algebras*). This is the same as the classification of simply connected complex Lie groups which are essentially simple. The semisimple groups are, up to a cover, products of simple groups. The classical groups (so christened by Weyl) are the group of matrices of determinant 1, the orthogonal (or spin) groups, and the symplectic groups. These groups have the remarkable property that they make sense over any field or even any commutative ring with unit. Over a finite field they become finite groups which are almost simple and these were studied intensively by Dickson in the late nineteenth century. It is a natural question to ask if the exceptional groups also make sense over finite fields. In the early 1950s, Chevalley had started to study algebraic groups over fields of characteristic 0 by using the exponential map and coming down to the Lie algebras. But this method was not very successful and certainly could not touch the case when the field had positive characteristic. But Borel changed the entire landscape by studying the algebraic groups directly using algebraic geometric methods, proving the existence of what are now called Borel subgroups, and their conjugacy, over any algebraically closed field. Chevalley then used Borel's work as a starting point and completed the classification of all semisimple affine algebraic groups by methods of algebraic geometry (up to a finite cover, semisimple groups are products of simple groups, and reductive groups are products of semisimple groups and tori). He found that the simple groups are classified in the same way as Cartan's. He then discovered the further remarkable fact that any semisimple group is naturally a *group scheme* over $\mathbb{Z}$, and hence it makes sense to look at its points over any field (this is an oversimplification). In particular it makes sense to speak of the simple groups over finite fields, and this process led Chevalley to discover new simple finite groups hitherto unknown. The groups he constructed over any field became known as the *Chevalley groups*.

In my opinion, the fact that the semisimple groups are really group schemes over $\mathbb{Z}$ accounts for their great importance, depth, and vitality. Over arbitrary fields it led Borel, Chevalley, Tits, Steinberg, Lusztig, Deligne, Curtis, and others to erect a beautiful theory of their structure and representations. Over the real and p-adic fields they become Lie groups on which one could do geometry and analysis, as Weyl, Gel'fand, Mautner, Harish-Chandra, Mostow, Bruhat, Kazhdan, and others did. Over the adeles their structure and representation theory led Langlands to formulate his program linking the harmonic analysis on the adelic groups to the most fundamental aspects of algebraic number theory, the so-called Langlands program, which has inspired and animated a huge number of mathematicians of his and later generations.

Chevalley's discovery that semisimple groups are group schemes over $\mathbb{Z}$ was the mathematical context when Steinberg started his research. In his words, he wanted to become a semisimple mathematician, and soon became one. His field was the entire theory of Chevalley groups and the associated finite groups, their structure and their linear representations. He had important things to say on all aspects of these groups. But the striking fact was that he used only elementary methods, including basic algebraic geometry, and seldom ventured into the cohomological aspects. I feel he resembled Harish-Chandra in this: he got to where he wanted to go with very simple ideas and methods.

In his *Collected papers*, he discussed all his papers, elaborating some fine points and putting his work within the framework of current knowledge, occasionally adding some personal reminiscences. About one paper he wrote that it was entirely worked out in the High Sierras when he was in his sleeping bag looking at the stars! About another paper he wrote that this was his only paper for which he got money from the Russians when they translated it, and mentions that the translation of his *Lectures on Chevalley groups* [1975] fetched him no money as it was before *glasnost*!!

New finite simple groups. In what follows I shall describe some highlights of his vast opus. His first major work was a couple of papers starting with the famous *Variations on a theme of Chevalley* [1959] in the PJM, where he constructed new families of finite simple groups not covered by the Chevalley groups and obtained for them structure properties similar to those of the Chevalley groups. Suzuki and Ree and others followed him with further families of new finite simple groups, all of them collectively known as the *twisted Chevalley groups*. The Chevalley groups and their twists were called finite simple groups of Lie type, and the great classification theorem of finite simple groups is just the statement that apart from the cyclic groups of prime order p, the alternating groups A_n ($n \geq 5$), and 26 *sporadic* groups, a finite simple group is of Lie type.

Generators and relations for Chevalley groups. In the famous paper *Générateurs, relations et revêtements de groupes algébriques* [1962], Steinberg considers Chevalley groups corresponding to a root system Σ and field K. They are generated by unipotent elements $g_r(t)$ with $r \in \Sigma$, $t \in K$. Among all the relations between the generators there are (obvious) ones (R) that can be written uniformly for all Σ, K. He then considers the abstract group $\widehat{G}$ generated by symbols $x_r(t)$ ($r \in \Sigma$, $t \in K$) subject to the relations (R) and the natural surjective homomorphism

$$\pi : \widehat{G} \longrightarrow G.$$

Thus, $\mathrm{Ker}(\pi)$ describes all the relations between the generators modulo the obvious relations. Steinberg proves the remarkable result that the covering π is central, i.e., $\mathrm{Ker}(\pi)$ is contained in the center of $\widehat{G}$, and that π is a universal central extension. J. Milnor has used Steinberg's construction in the case of a general linear group over an arbitrary ring S to define the group $K_2(S)$ that describes the relations between the elementary matrices over S modulo the obvious relations. The corresponding group $\widehat{G}$ is known as the *Steinberg group* of S. Thus, this paper of Steinberg made a great impact on the development of higher algebraic K-theory. The kernel of π was studied in a profound manner by Moore and Matsumoto over a p-adic field, and their work led to deep relationships with the norm residue symbol of number theory. Among other things the work of Moore and Matsumoto highlighted the importance of the two-fold covering of the symplectic group, the so-called metaplectic group, over the local fields and the adeles. The adelic metaplectic group was the platform which Weil used in his reformulation of Siegel's work on quadratic forms.

Regular elements of semisimple algebraic groups [Steinberg 1965]***.*** This is one of his most admired and beautiful papers. Here he studies conjugacy classes of regular elements in a semisimple group G. For simplicity let us assume that the ground field is algebraically closed. An element $g \in G$ is called *regular* if the dimension of the orbit of g under the action of G by conjugacy has maximal dimension (which is $\dim(G) - \operatorname{rank}(G)$). In this paper he proves that the regular conjugacy classes have an affine space section in the algebraic geometric sense, for every simply connected semisimple group. For example, for $\mathrm{SL}(n)$ we get the space of companion matrices, a result that goes back at least to Gantmacher. Actually he does not restrict himself to the algebraically closed ground field and proves that if G is a simply connected quasisplit group over a field K (that is, it contains a Borel subgroup defined over K), then every conjugacy class defined over K contains an element defined over K. As a consequence of the main result, Steinberg proves that every principal homogeneous space of a quasisplit semisimple group admits reduction to a maximal torus. This result yields the solution of the famous Serre conjecture:

> *If K is a field of cohomological dimension 1, then all principal homogeneous spaces of a connected algebraic group over K are trivial.*

The result that the regular conjugacy classes have a section in the algebraic geometric sense led to an interesting interaction between us. I was looking at this question on a semisimple *Lie algebra* over $\mathbb{C}$. Kostant had constructed a beautiful affine cross section for the regular orbits of the adjoint representation (which reduces to the companion matrices for $\mathfrak{sl}(n)$), roughly at the same time as Steinberg's work. When I looked at the Lie algebra problem, it occurred to me that by making use of some ideas of Harish-Chandra I could obtain a proof of many of Kostant's results in a very simple way. I had this published in the American Journal of Mathematics and left a reprint in Bob's mail box. He then asked me to come to his office and explained the corresponding global result. I treasure the memory of that discussion between us which had no element of condescension in it, when I was a young researcher and he was at the peak of his powers.

The Steinberg representation. The complex representations of the finite Chevalley groups are difficult to construct, even though Green had quite early worked out the irreducible characters of $\mathrm{GL}(n)$. The final results were obtained by Deligne and Lusztig who realized the representations using certain étale cohomology spaces. But Steinberg found one of the most important and ubiquitous ones very early in his career. It is now called the *Steinberg representation*, and one can find a masterful essay on its various incarnations in his *Collected papers*. For a Chevalley group G over a finite field, if B is a Borel subgroup, and 1_B^G is the representation of G induced by the trivial representation of B, then St is the unique irreducible component of 1_B^G which does not occur in any 1_P^G where P is any parabolic subgroup containing B properly. Correspondingly, there is a formula for its character as an alternating sum of the characters of the 1_P^G. Remarkably, this character formula makes sense in a p-adic field and its properties play a fundamental role in the harmonic analysis on the p-adic semisimple groups, as developed by Harish-Chandra, Jacquet, and others. Borel and Serre proved, using the cohomology of the Bruhat–Tits buildings, that St is an irreducible square integrable (hence unitary) representation of the p-adic group.

The Steinberg representation also plays a basic role in the Langlands correspondence. For example, an elliptic curve over Q has split multiplicative reduction at a prime p if and only if the unitary automorphic representation associated to it by the Langlands correspondence has for its component at p the Steinberg representation. In general, under the correspondence, ignoring scalar twists by one dimensional representations, a Steinberg representation at p corresponds to a Galois representation for which the image of a decomposition group at p contains a regular unipotent element.[1]

For lack of time I cannot discuss some of the other major discoveries in his work. I mention the new and easier proofs of the isomorphism and isogeny theorems of algebraic semisimple groups, which say that an isomorphism (isogeny) between semisimple algebraic groups is always induced by an isomorphism (isogeny) of their corresponding root data and conversely. The other item is his new and simpler counterexample to Hilbert's 14th problem, which asks one to prove that the ring of invariant polynomials of a linear action of *any* algebraic group is finitely generated. For semisimple groups over the complex field this was proved for $\mathrm{SL}(n)$ by Hilbert, and for all semisimple groups over a field of characteristic 0 by Weyl, as a consequence of his famous result that all finite dimensional representations of a semisimple Lie algebra are direct sums of irreducible representations. In prime characteristic the Weyl reducibility fails to hold and one needs a weakening of it, called *geometric reductivity*, conjectured by Mumford and proved by Haboush. The finite generation of invariants then follows from geometric reductivity, as was shown by Nagata. So to find counterexamples to the finite generation of invariants, one has to leave the category of semisimple or even reductive groups. Nagata found a counterexample for a finite-dimensional action of a product of the additive groups. In the late 1990s, Steinberg found much simpler classes of examples in all characteristics, and made a thorough analysis of the problem, sharpening Nagata's construction and relating the examples to plane cubic curves and their geometry.

I know I have given only a brief discussion of a very minute part of Steinberg's work which is astonishing in its scope, depth, and beauty. His profound insights about semisimple groups, and the easy grace and charm of his personality, cannot ever be forgotten by people who came into contact with him. I have known very few like him.

[1] These remarks on the Steinberg representation and elliptic curves were pointed out to me by Professor Don Blasius.

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CELLULARITY OF CERTAIN QUANTUM ENDOMORPHISM ALGEBRAS

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Dedicated to the memory of Robert Steinberg.

For any ring $\widetilde{A}$ such that $\mathbb{Z}[q^{\pm 1/2}] \subseteq \widetilde{A} \subseteq \mathbb{Q}(q^{1/2})$, let $\Delta_{\widetilde{A}}(d)$ be an $\widetilde{A}$-form of the Weyl module of highest weight $d \in \mathbb{N}$ of the quantised enveloping algebra $\mathrm{U}_{\widetilde{A}}$ of $\mathfrak{sl}_2$. For suitable $\widetilde{A}$, we exhibit for all positive integers r an explicit cellular structure for $\mathrm{End}_{\mathrm{U}_{\widetilde{A}}}(\Delta_{\widetilde{A}}(d)^{\otimes r})$. This algebra and its cellular structure are described in terms of certain Temperley–Lieb-like diagrams. We also prove general results that relate endomorphism algebras of specialisations to specialisations of the endomorphism algebras. When ζ is a root of unity of order bigger than d we consider the U_ζ-module structure of the specialisation $\Delta_\zeta(d)^{\otimes r}$ at $q \mapsto \zeta$ of $\Delta_{\widetilde{A}}(d)^{\otimes r}$. As an application of these results, we prove that knowledge of the dimensions of the simple modules of the specialised cellular algebra above is equivalent to knowledge of the weight multiplicities of the tilting modules for $\mathrm{U}_\zeta(\mathfrak{sl}_2)$. As an example, in the final section we independently recover the weight multiplicities of indecomposable tilting modules for $\mathrm{U}_\zeta(\mathfrak{sl}_2)$ from the decomposition numbers of the endomorphism algebras, which are known through cellular theory.

1. Introduction

1A. ***Notation.*** Let A be the ring $\mathbb{Z}[q^{\pm 1/2}]$ where q is an indeterminate, and let U_A be the Lusztig A-form [1988; 1990; 1993] of the quantised enveloping algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ [Drinfeld 1987; Jimbo 1986; Chari and Pressley 1994], which has basis consisting of products of "divided powers" of the generators of $\mathfrak{sl}_2$ and binomials in the Cartan generators. Let $\Delta_A(d)$ be the Weyl module for U_A with highest weight $d \in \mathbb{N}$. This has dimension $d+1$ and quantum dimension equal to the quantum number $[d+1]$, where for any integer n,

$$[n] = [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

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Keywords: quantum invariants, cellular algebras, tilting modules.

For any commutative A-algebra $\tilde{A}$, we write $\mathrm{U}_{\tilde{A}} := \tilde{A} \otimes_A \mathrm{U}_A$, and similarly for $\Delta_{\tilde{A}}(d)$, etc. For any positive integer r, let $E_r(d, \tilde{A}) := \mathrm{End}_{\mathrm{U}_{\tilde{A}}}(\Delta_{\tilde{A}}(d)^{\otimes r})$.

Let $s_1, \dots, s_{N-1}$ be the standard Coxeter generators of Sym_N. For $w \in \mathrm{Sym}_N$, write $\ell(w)$ for its length as a word in the generators s_i, and define the left set $L(w) := \{i \mid \ell(s_i w) < \ell(w)\}$; the right set $R(w)$ is defined similarly.

1B. ***The main result.*** Let $K = \mathbb{Q}(q^{1/2})$ be the field of fractions of A. Writing B_r for the r-string braid group (r a positive integer), it is known that there is an action of B_r on $\Delta_A(d)^{\otimes r}$, in which the standard generators of the braid group act on successive tensor factors via the R-matrix $\check{R}$. This is evident over K, and from [Lehrer and Zhang 2006; 2010] and [Andersen et al. 2008] or [Andersen 2012] (using [Kirillov and Reshetikhin 1990]) in the above integral form. This action respects the $\mathrm{U}_{\tilde{A}}$-action on the tensor space, and so there is a homomorphism

$$\eta : \tilde{A}B_r \longrightarrow \mathrm{End}_{\mathrm{U}_{\tilde{A}}}(\Delta_{\tilde{A}}(d)^{\otimes r}) = E_r(d, \tilde{A}). \tag{1-1}$$

We define A using $q^{1/2}$ instead of q because then, with the usual definitions of U_q, the R-matrix is defined over A with respect to a basis of weight vectors.

In [Lehrer and Zhang 2006] it was shown that when $\tilde{A} = K$, η is surjective. This provides a means of studying the relevant endomorphism algebras. When $d = 2$ this surjectivity was proved in [Andersen 2012] for most $\tilde{A}$. We haven't been able to establish this result for $d > 2$. However, inspired in part by the methods used in [loc. cit.] we show in this paper that the endomorphism algebras have a nice cellular structure, even though the R-matrix generators satisfy a polynomial equation of degree $d + 1$.

We shall work with the Temperley–Lieb algebra $\mathrm{TL}_N(\tilde{A})$, which has generators f_i, $i = 1, \dots, N-1$ and relations

$$\begin{cases} f_i f_j f_i = f_i & \text{if } |i-j| = 1, \\ f_i f_j = f_j f_i & \text{if } |i-j| > 1, \\ f_i^2 = (q+q^{-1}) f_i. \end{cases}$$

This has an $\tilde{A}$-basis consisting of planar diagrams, as explained in [Graham and Lehrer 1996, §1] (see also [2003; 2004]); these are in one-to-one correspondence with the set of fully commutative elements of Sym_N; see [Fan and Green 1997].

Theorem 1.1. *Let $d \geq 1$ be an integer. For any $\tilde{A}$ such that $[d]!$ is invertible in $\tilde{A}$, the algebra $E_r(d, \tilde{A})$ is isomorphic to a cellular subalgebra of* $\mathrm{TL}_{rd}(\tilde{A})$. *In particular, it has an $\tilde{A}$-basis labelled by planar diagrams $D \in \mathrm{TL}_{rd}(\tilde{A})$ such that $L(D), R(D) \subseteq \{d, 2d, \dots, (r-1)d\}$, where the left and right sets $L(D)$ and $R(D)$ are as in Definition 3.2 below.*

We remark that the cellular subalgebra in Theorem 1.1 has an identity different from that of $\mathrm{TL}_{rd}(\tilde{A})$, and is therefore not a unital subalgebra.

Note that the planar diagrams are labelled by the set Sym_{rd}^c of fully commutative elements in Sym_{rd}; the requirement in the theorem is equivalent to taking those $w \in \mathrm{Sym}_{rd}^c$ such that $L(w), R(w) \subseteq \{d, 2d, \ldots, (r-1)d\}$ (see [Fan and Green 1997]).

We shall give further details of the cellular structure below, both in terms of diagrams, and in terms of pairs of standard tableaux.

2. The case $d = 1$

2A. ***The Temperley–Lieb action.*** It is known (see, for example, [Lehrer and Zhang 2010, §3.4]) that in this case, the R-matrix acts on $\Delta_K(1)^{\otimes 2}$ with eigenvalues $q^{1/2}$ and $-q^{3/2}$. If we adjust the map η of (1-1) by sending the generators to $T_i := q^{1/2}R_i$, where R_i is the relevant R-matrix, then η factors through the algebra $H_r(A) := AB_r/\langle (T_i + q^{-1})(T_i - q)\rangle$, which is well known to be the Hecke algebra, and has A-basis $\{T_w \mid w \in \mathrm{Sym}_r\}$. We therefore have, after tensoring with $\tilde{A}$,

$$\mu : H_r(\tilde{A}) \longrightarrow \mathrm{End}_{\mathrm{U}_{\tilde{A}}}(\Delta_{\tilde{A}}(1)^{\otimes r}) = E_r(1, \tilde{A}). \tag{2-1}$$

Moreover it is a special case of the main result of [Du et al. 1998] (see also [Andersen et al. 2008]) that μ is surjective for any choice of $\tilde{A}$, even when $\tilde{A}$ is taken to be A. Further, the arguments in [Lehrer and Zhang 2010, Theorem 3.5], generalised to the integral case, show that the kernel of μ is the ideal generated by the element $a_3 := \sum_{w \in \mathrm{Sym}_3} (-q)^{-\ell(w)} T_w$; hence, for any $\tilde{A}$, we have an isomorphism

$$\eta : H_r(\tilde{A})/\langle a_3\rangle \cong \mathrm{TL}_r(\tilde{A}) \xrightarrow{\sim} \mathrm{End}_{\mathrm{U}_{\tilde{A}}}(\Delta_{\tilde{A}}(1)^{\otimes r}) = E_r(1, \tilde{A}), \tag{2-2}$$

where $\mathrm{TL}_r(\tilde{A}) := H_r(\tilde{A})/\langle a_3\rangle$ is the r-string Temperley–Lieb algebra. The generator f_i acts as $q - T_i$ on $\Delta_{\tilde{A}}(1)^{\otimes r}$. It is easily shown that $f_i^2 = (q + q^{-1}) f_i$, and that the other Temperley–Lieb relations are satisfied.

2B. ***Projection to $\Delta_{\tilde{A}}(d)$.*** Now it is elementary that

$$\Delta_K(1)^{\otimes d} \cong \Delta_K(d) \oplus \Delta', \tag{2-3}$$

where Δ' is the direct sum of simple modules $\Delta_K(i)$ with $i < d$. We therefore have a canonical projection $p_d : \Delta_K(1)^{\otimes d} \longrightarrow \Delta_K(d)$, which may be considered an element of $E_d(1, K) = \mathrm{End}_{\mathrm{U}_K}(\Delta_K(1)^{\otimes d})$.

Lemma 2.1. *The projection p_d is the image under μ (see (2-1)) of the element $e_d := P_d(q)^{-1} \sum_{w \in \mathrm{Sym}_d} q^{\ell(w)} T_w \in H_d(\tilde{A})$, where $P_d(q) = q^{d(d-1)/2}[d]!$.*

Proof. We begin by showing that for $i = 1, \ldots, d-1$,

$$T_i p_d = p_d T_i = q p_d \tag{2-4}$$

as endomorphisms of $\Delta_K(1)^{\otimes d}$.

By symmetry, it suffices to prove (2-4) for $i = 1$. Now

$$\begin{aligned}\Delta_K(1)^{\otimes d} &= \Delta_K(1) \otimes \Delta_K(1) \otimes \Delta_K(1)^{\otimes(d-2)} \\ &\cong (\Delta_K(0) \oplus \Delta_K(2)) \otimes \Delta_K(1)^{\otimes(d-2)} \\ &\cong (\Delta_K(0) \otimes \Delta_K(1)^{\otimes(d-2)}) \oplus (\Delta_K(2) \otimes \Delta_K(1)^{\otimes(d-2)})\end{aligned}$$

But p_d acts as zero on the first summand (since the highest occurring weight is $d-2$) and T_1 acts as q on the second summand. This proves the relation (2-4). Now since $f_i = \mu(q - T_i)$, this shows that p_d is the "Jones idempotent" of $\mathrm{TL}_d(K)$, defined by the relations $f_i p_d = p_d f_i = 0$ for all i.

It follows that if p'_d is the unique idempotent in $H_d(K)$ corresponding to the algebra homomorphism $T_w \mapsto q^{\ell(w)}$, then $p_d = \mu(p'_d)$. But this idempotent is precisely the element e_d in the statement. □

The next statement is immediate.

Corollary 2.2. *Let $\tilde{A} = A[[d]!^{-1}]$. Then*

$$\Delta_{\tilde{A}}(1)^{\otimes rd} \cong \Delta_{\tilde{A}}(d)^{\otimes r} \oplus \Gamma, \tag{2-5}$$

where Γ is a $\mathrm{U}_{\tilde{A}}$-submodule, and the corresponding projection $p \in \mathrm{End}_{rd}(1, \tilde{A})$ such that $p(\Delta_{\tilde{A}}(1)^{\otimes rd}) = \Delta_{\tilde{A}}(d)^{\otimes r}$ is given by $p = p_d^{\otimes r}$, where we now consider p_d as an element of $E_d(1, \tilde{A}) \subset E_d(1, K)$.

3. Endomorphisms of $\Delta_{\tilde{A}}(d)^{\otimes r}$

3A. ***Identification of $E_r(d, \tilde{A})$.*** Throughout this section we take $\tilde{A}$ to be $\tilde{A} = A[[d]!^{-1}]$. Recall that $E_r(d, \tilde{A}) = \mathrm{End}_{\mathrm{U}_{\tilde{A}}}(\Delta_{\tilde{A}}(d)^{\otimes r})$. We are now in a position to identify $E_r(d, \tilde{A})$ on the nose, as a subalgebra of $\mathrm{TL}_{rd}(\tilde{A}) \cong \mathrm{End}_{\mathrm{U}_{\tilde{A}}}(\Delta_{\tilde{A}}(1)^{\otimes rd})$. This will lead to the identification of the cellular structure on $E_r(d, \tilde{A})$.

Proposition 3.1. *There is an isomorphism $E_r(d, \tilde{A}) \xrightarrow{\sim} p\mathrm{TL}_{rd}(\tilde{A})p$, where p is the idempotent $p = p_d^{\otimes r}$ of $\mathrm{TL}_{rd}(\tilde{A})$ described above.*

Proof. For any endomorphism $\alpha \in E_r(d, \tilde{A})$ we obtain an endomorphism $\tilde{\alpha}$ of $\Delta_{\tilde{A}}(1)^{\otimes rd}$ by extending α by zero, using the decomposition (2-5), that is, by defining $\tilde{\alpha}$ to be zero on Γ. The map $\alpha \mapsto \tilde{\alpha}$ is an inclusion $E_r(d, \tilde{A}) \hookrightarrow E_{rd}(1, \tilde{A})$, and its image is clearly the space of endomorphisms $\beta \in E_{rd}(1, \tilde{A})$ such that $\ker(\beta) \supseteq \Gamma$ and $\mathrm{Im}(\beta) \subset \Delta_{\tilde{A}}(d)^{\otimes r}$ (as in the decomposition (2-5)). This image is $p\mathrm{TL}_{rd}(\tilde{A})p$. □

3B. ***Temperley–Lieb diagrams.*** The key step in proving cellularity is the identification of a certain $\tilde{A}$-basis of $p\mathrm{TL}_{rd}(\tilde{A})p$. This will be done in terms of certain diagrams. The Temperley–Lieb algebra $\mathrm{TL}_{rd}(\tilde{A})$ has $\tilde{A}$-basis consisting of planar diagrams from rd to rd, in the language of [Graham and Lehrer 1998]. These

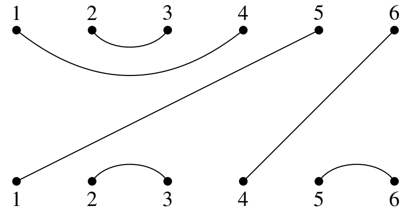


Figure 1. A planar diagram from 6 to 6.

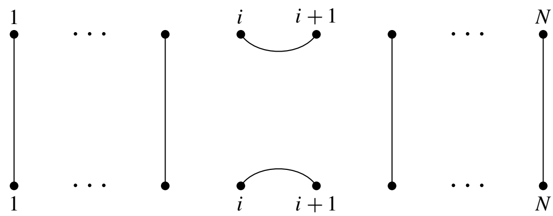


Figure 2. The generator f_i as a planar diagram from N to N.

diagrams are in bijection with the set Sym^c_{rd} of fully commutative elements [Fan and Green 1997] of Sym_{rd}, which in turn is in bijection with those elements of Sym_{rd} which correspond, under the Robinson–Schensted correspondence, to pairs of standard tableaux with two rows.

We shall describe now how to obtain a pair $(S(D), R(D))$ of standard tableaux directly from a planar diagram D. We use the planar diagram from 6 to 6 in Figure 1 to illustrate the description.

Each planar diagram from N to N consists of a set of N nonintersecting arcs. These may be through-arcs, joining an upper node to a lower node, or upper (top to top) or lower (bottom to bottom). The latter two are referred to as horizontal arcs. The diagrams are multiplied in the usual way, by concatenation, with each closed circle being replaced by $[2] = q + q^{-1}$. The generator f_i corresponds to the diagram in Figure 2. Note that if there are t through-arcs, then there are equally many top arcs and bottom arcs, and if this number is k, then $t + 2k = N$.

Now to each such planar diagram D, we associate an ordered pair $(S(D), T(D))$ of standard tableaux with two rows, as follows. Let $i_1, \dots, i_k$ be the right nodes of the upper arcs written in ascending order. Then $S(D)$ has second row $i_1, \dots, i_k$, and first row the complement of $\{i_1, \dots, i_k\}$, written in ascending order. Note that the first row has $t + k \geq k$ elements. The tableau $T(D)$ is defined similarly, using the sequence $j_1, \dots, j_k$ of right ends of the lower arcs. Note that both $S(D)$ and $T(D)$ correspond to the partition $(t + k, k)$, and hence the diagram corresponds via the Robinson–Schensted correspondence to an element $w(D) \in \mathrm{Sym}_N$, which is fully commutative (see [Fan and Green 1997, Definition 3.3.1]).

Say that a horizontal arc is *small* if its vertices are $i, i+1$ for some i.

Definition 3.2. The *left set* $L(D)$ of a planar diagram D is the set of left vertices of the small upper arcs of D. Similarly, the right set $R(D)$ is the set of left vertices of the small lower arcs of D.

It is well known, and proved in a straightforward way using the Robinson–Schensted correspondence, that in the notation from Section 1A we have $L(D) = L(w(D))$, and similarly $R(D) = R(w(D))$.

For the diagram D in Figure 1, $L(D) = \{2\}$, while $R(D) = \{2, 5\}$. The tableaux $S(D)$ and $T(D)$ are given by

$$S(D) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \text{and} \quad T(D) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2}\end{array}.$$

Note that if $\mathfrak{D}(S) := \{i \mid i+1 \text{ is in a lower row than } i\}$ is the descent set of a standard tableau S, then $L(D) = \mathfrak{D}(S(D))$ and $R(D) = \mathfrak{D}(T(D))$.

4. Proof of the main theorem

In this section we prove Theorem 1.1, and give some of its consequences. We keep the convention $\tilde{A} = A[([d]!)^{-1}]$ from Section 3.

4A. ***A key lemma.*** We begin by proving the following key result.

Lemma 4.1. *The $\tilde{A}$-algebra $p\mathrm{TL}_{dr}(\tilde{A})p$ has $\tilde{A}$-basis given by the set of elements pDp, where D is a diagram in $\mathrm{TL}_{dr}(\tilde{A})$ such that*

$$L(D) \cup R(D) \subseteq \{d, 2d, \dots, (r-1)d\}.$$

Proof. The $\tilde{A}$-algebra $E_r(d, \tilde{A}) \cong p\mathrm{TL}_{rd}(\tilde{A})p$ is evidently spanned by the elements pDp, where D ranges over planar diagrams from rd to rd. But for $i = 1, \dots, d-1$, we have seen that $p_d f_i = f_i p_d = 0$. It follows that $pDp = 0$ unless $L(D)$ and $R(D)$ are both contained in $\{d, 2d, \dots, (r-1)d\}$. Let $\mathcal{B}(d, r)$ be the set of planar diagrams satisfying these conditions. By the above remarks, it will suffice to show that

$$\{pDp \mid D \in \mathcal{B}(d, r)\} \text{ is linearly independent.} \tag{4-1}$$

To prove (4-1) it suffices to work over the field K; in particular we are reduced to showing that

$$|\mathcal{B}(d, r)| = \dim_K\big(\mathrm{End}_{\mathrm{U}_K}(\Delta_K(d)^{\otimes r}\big). \tag{4-2}$$

We shall prove (4-2) essentially by showing that both sides of (4-2) satisfy the same recurrence. Let us begin with the left side.

Observe that if a diagram $D \in \mathcal{B}(d,r)$ has t through-arcs, it may be thought of as a pair of diagrams D_1, D_2, where the D_i are monic diagrams from t to rd. Recall that a diagram from t to N ($t \le N$) is monic if it has t through-arcs. One thinks of D_1 as the top half of D, and D_2 as the * of the bottom half of D, where * is the cellular involution on the Temperley–Lieb category that reflects diagrams in a horizontal line. It follows that if we write $|\mathcal{B}(d,r)| = b(d,r)$ and $|\mathcal{B}(d,r;t)| = b(d,r;t)$, where $\mathcal{B}(d,r;t)$ is the set of monic planar diagrams $D: t \to rd$ such that $L(D) \subseteq \{d, 2d, \dots, (r-1)d\}$, then

$$b(d,r) = \sum_{0 \le t \le dr} b(d,r;t)^2. \tag{4-3}$$

Now consider the right side of (4-2). Define the positive integers $m(d,r;t)$ by

$$\Delta_K(d)^{\otimes r} \cong \bigoplus_{t=0}^{dr} m(d,r;t)\Delta_K(t). \tag{4-4}$$

Thus the $m(d,r;t)$ are multiplicities, and $m(d,r;t) = 0$ unless $t \equiv rd \pmod 2$. Moreover, we obviously have, if $m(d,r) := \dim_K\bigl(\mathrm{End}_{\mathrm{U}_K}(\Delta_K(d)^{\otimes r}\bigr)$,

$$m(d,r) = \sum_{0 \le t \le dr} m(d,r;t)^2. \tag{4-5}$$

It is clear that in view of (4-3) and (4-5), the lemma will follow if we prove that for all d, r and t,

$$m(d,r;t) = b(d,r;t). \tag{4-6}$$

We shall prove (4-6) by induction on r. If $r = 1$, then

$$m(d,1;t) = b(d,1;t) = \begin{cases} 0 & \text{if } t \ne d, \\ 1 & \text{if } t = d. \end{cases} \tag{4-7}$$

Now by the Clebsch–Gordan formula, we have, for any integer n,

$$\Delta_K(d) \otimes \Delta_K(n) \cong \Delta_K(d+n) \oplus \Delta_K(d+n-2) \oplus \cdots \oplus \Delta_K(|d-n|).$$

It follows that

$$m(d,r+1;t) = \sum_{s=t-d}^{t+d} m(d,r;s), \tag{4-8}$$

where $m(d,r;s) = 0$ if $s < 0$ or if $s > dr$.

We shall complete the proof of the lemma by showing that the numbers $b(d,r;t)$ satisfy a recurrence analogous to (4-8). For this observe that any diagram D in

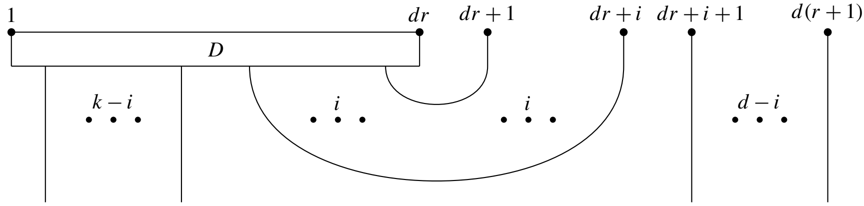


Figure 3. From diagram D to diagram D'.

$\mathcal{B}(d, r; k)$ gives rise to a unique diagram in $\mathcal{B}(d, r+1; k+d-2i)$, for $0 \le i \le \min\{d, k\}$, as depicted in Figure 3, and each diagram $D' \in \mathcal{B}(d, r+1; t)$ arises in this way from a unique diagram in $\mathcal{B}(d, r; k)$ for a uniquely determined k. In fact, $k = t - d + 2i$ where i is the number of arcs in D' whose right vertices belong to $\{dr+1, \cdots, d(r+1)\}$. It follows that

$$b(d, r+1; t) = \sum_{s=t-d}^{t+d} b(d, r; s), \tag{4-9}$$

where $b(d, r; s) = 0$ if $s < 0$ or if $s > dr$.

Comparing (4-8) with (4-9), and taking into account (4-7), it follows that $m(d, r; k) = b(d, r; k)$ for all d, r and k. This completes the proof of (4-6) above, and hence of the lemma. $\square$

4B. *Cellular structure.*

Proof of Theorem 1.1. We have seen that $E_r(d, \tilde{A}) \cong p\mathrm{TL}_{rd}(\tilde{A})p$, and that the latter algebra has the basis $\mathcal{B}(d, r)$, as stated in the theorem. It remains only to show that $p\mathrm{TL}_{rd}(\tilde{A})p$ has a cellular structure. Following [Graham and Lehrer 1996, Definition 1.1] we need to produce a cell datum $(\Lambda, M, C, {}^*)$ for $p\mathrm{TL}_{rd}(\tilde{A})p$.

Take Λ to be the poset $\{t \in \mathbb{Z} \mid 0 \le t \le dr \text{ and } dr - t \in 2\mathbb{Z}\}$, ordered as integers. For $t \in \Lambda$, let $M(t) := \mathcal{B}(d, r; t)$, the set of monic planar diagrams $D : t \to dr$ such that $L(D) \subseteq \{d, 2d, \ldots, (r-1)d\}$ (see Section 3B and the proof of Lemma 4.1). Then the map $C : \amalg_{t\in\Lambda} M(t) \times M(t) \longrightarrow p\mathrm{TL}_{rd}(\tilde{A})p$ is defined by $C(D_1, D_2) = pD_1 \circ D_2^* p$, where $\circ$ indicates concatenation of diagrams. We shall henceforth simply use juxtaposition to indicate composition in the Temperley–Lieb category. Since each diagram $D \in \mathcal{B}(r, d)$ is expressible uniquely as $D = D_1 D_2^*$ for some $t \in \Lambda$ and $D_1, D_2 \in M(t)$, it follows from Lemma 4.1 that C is a bijection from $\amalg_{t\in\Lambda} M(t) \times M(t)$ to a basis of $p\mathrm{TL}_{rd}(\tilde{A})p$. Finally, the anti-involution * is the restriction to $p\mathrm{TL}_{rd}(\tilde{A})p$ of the anti-involution on $\mathrm{TL}_{dr}(\tilde{A})$, namely, reflection in a horizontal line. Since $p^* = p$, we have $C(D_1, D_2)^* = (pD_1D_2^*p)^* = pD_2D_1^*p = C(D_2, D_1)$.

If $S, T \in M(t)$, we shall write $C(S, T) = C^t_{S,T}$, and for this proof only, write

$$\mathscr{A} = p\mathrm{TL}_{rd}(\tilde{A})p \quad \text{and} \quad \mathscr{A}(<i) = \sum_{\substack{j<i,\\ S,T\in M(j)}} \tilde{A}C^j_{S,T}.$$

It remains only to prove the axiom (C3) of [Graham and Lehrer 1996, Definition 1.1]. For this, let $S_1, S_2 \in M(s)$ and $T_1, T_2 \in M(t)$. Then

$$C^s_{S_1,S_2}C^t_{T_1,T_2} = pS_1(S_2^* pT_1)T_2^* p, \tag{4-10}$$

so that if $s < t$, the left side is in $\mathscr{A}(<t)$, and there is nothing to prove. Hence we take $s \geq t$.

Now $S_2^* pT_1$ is a morphism from t to s, and hence is an $\tilde{A}$-linear combination of planar diagrams D from t to s. Thus the left side of (4-10) is an $\tilde{A}$-linear combination of elements of the form $pS_1DT_2^* p$. If D is not monic, then $pS_1DT_2^* p \in \mathscr{A}(<t)$; if D is monic, then clearly $pS_1DT_2^* p = pS'T_2^* p$ for some monic $S' : t \to dr$.

It follows from (4-10) that modulo $\mathscr{A}(<t)$, $C^s_{S_1,S_2}C^t_{T_1,T_2} = \sum_{S\in\mathscr{B}(d,r;t)} a(S)C^t_{S,T_2}$, and $a(S)$ is independent of T_2. This proves the axiom (C3), and hence the cellularity of $\mathscr{A}$. The proof of Theorem 1.1 is now complete. □

5. Endomorphism algebras and specialisation

We shall prove in this section results showing how the multiplicities of the indecomposable summands of the specialisations of $\Delta_A(d)^{\otimes r}$ corresponding to homomorphisms $A \to k$ where k is a field, relate to the dimensions of the simple modules for the corresponding endomorphism rings. It turns out that this is a consequence of a result on tilting modules which is valid for general quantum groups. Therefore in Sections 5A and 5B we deal with this general situation. Then in Section 5C we deduce the explicit consequences in our $\mathfrak{sl}_2$ case where we take advantage of our cellularity result from Section 4 on the endomorphism rings.

5A. ***Integral endomorphism algebras and specialisation.*** We now provide some rather general base change results for Hom-spaces between certain representations of quantum groups. So in this section we shall work with a general quantum group U_q over K with integral form U_A. We denote by k an arbitrary field (in this section k may even be any commutative noetherian A-algebra) made into an A-algebra by specializing q to $\zeta \in k \setminus \{0\}$ and set $\mathrm{U}_\zeta = \mathrm{U}_A \otimes_A k$. When M is a U_A-module we write M_q and M_ζ for the corresponding U_q- and U_ζ-modules, respectively.

For each dominant weight λ we write $\Delta_q(\lambda)$, $\Delta_A(\lambda)$ and $\Delta_\zeta(\lambda)$ for the Weyl modules for U_q, U_A and U_ζ respectively. Similarly, we have the dual Weyl modules $\nabla_q(\lambda)$, $\nabla_A(\lambda)$ and $\nabla_\zeta(\lambda)$ respectively. Then it is well known that, writing w_0 for the longest element of the Weyl group,

$$\nabla_\zeta(\lambda) = \Delta_\zeta(-w_0\lambda)^*,$$

and similarly for $\nabla_A(\lambda)$ and $\nabla_q(\lambda)$.

We shall make repeated use of the following result. For any two weights $\lambda, \mu \in X$, we have

$$\operatorname{Ext}^i_{\mathrm{U}_A}(\Delta_A(\lambda), \nabla_A(\mu)) = \begin{cases} A & \text{if } \lambda = \mu \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5-1}$$

This is proved exactly as in the corresponding classical case (see, for example, [Jantzen 2003, Proposition II.B.4]) by invoking the quantised Kempf vanishing theorem proved in general in [Ryom-Hansen 2003].

Lemma 5.1. *Let M, N be U_A-modules that are finitely generated as A-modules. If M has a filtration by Weyl modules $\Delta_A(\lambda)$ and N has a filtration by dual Weyl modules $\nabla_A(\mu)$, then $\operatorname{Hom}_{\mathrm{U}_A}(M, N)$ is a free A-module of rank equal to $\dim_{\mathbb{Q}(q)} \operatorname{Hom}_{\mathrm{U}_q}(M_q, N_q)$. Further, we have*

$$\operatorname{Hom}_{\mathrm{U}_\zeta}(M_\zeta, N_\zeta) \simeq \operatorname{Hom}_{\mathrm{U}_A}(M_A, N_A) \otimes_A k.$$

Proof. We have a spectral sequence with E_2-terms

$$E_2^{-p,q} = \operatorname{Tor}^A_p(\operatorname{Ext}^q_{\mathrm{U}_A}(M, N), k)$$

converging to $\operatorname{Ext}^{q-p}_{\mathrm{U}_\zeta}(M_\zeta, N_\zeta)$. By (5-1) we have $E_2^{-p,q} = 0$ if either $q > 0$ or $q = 0 < p$. Hence the spectral sequence collapses and we can read off the result. □

Corollary 5.2. *Let V be a U_A-module which satisfies the assumption*

$$V^* \otimes_A V \text{ has a } \nabla_A\text{-filtration.} \tag{5-2}$$

Then $\operatorname{End}_{\mathrm{U}_\zeta}(V_\zeta^{\otimes r}) \simeq \operatorname{End}_{\mathrm{U}_A}(V^{\otimes r}) \otimes_A k.$

Proof. We have $\operatorname{End}_{\mathrm{U}_A}(V^{\otimes r}) \simeq \operatorname{Hom}_{\mathrm{U}_A}(\Delta_A(0), (V^* \otimes V)^{\otimes r})$ because $\Delta_A(0)$ is the trivial U_A-module A. By the assumption (5-2), we may apply Lemma 5.1 to obtain the statement. □

As usual we denote by ρ half the sum of the positive roots. Recall the concept of strongly multiplicity-free modules from [Lehrer and Zhang 2006]. A U_q-module V_q is *strongly multiplicity-free* if the weights of U_q occurring in V_q form a chain in the usual ordering on weights.

There are significant cases where the above result applies:

Proposition 5.3. *Suppose $V = \Delta_A(\lambda)$ for some dominant weight λ. Assume that V_q is strongly multiplicity-free, and that $-w_0\lambda + \mu + \rho$ is dominant for each weight μ of V. Then $V^* \otimes V$ has a ∇_A-filtration.*

Proof. Recall that U_A has a triangular decomposition $\mathrm{U}_A = \mathrm{U}_A^+ \mathrm{U}_A^0 \mathrm{U}_A^-$, and each weight μ defines a 1-dimensional representation of the subalgebra $\mathrm{U}_A^0 \mathrm{U}_A^-$, which we also denote by μ.

We have $V^* = \nabla_A(\lambda')$ where $\lambda' = -w_0\lambda$. Moreover ∇_A is realised as the induction functor $\mathrm{Ind}_{\mathrm{U}_A^0 \mathrm{U}_A^-}^{\mathrm{U}_A}$. Hence by a standard property of induction,

$$V^* \otimes V = \mathrm{Ind}_{\mathrm{U}_A^0 \mathrm{U}_A^-}^{\mathrm{U}_A}(\lambda') \otimes V = \mathrm{Ind}_{\mathrm{U}_A^0 \mathrm{U}_A^-}^{\mathrm{U}_A}(\lambda' \otimes V),$$

where in this formula the last occurrence of V is its restriction to $\mathrm{U}_A^0 \mathrm{U}_A^-$. Now the hypothesis that V_q is strongly multiplicity-free implies that the weights of V are linearly ordered. But the weights of $\lambda' \otimes V$ are $\{\lambda' + \mu\}$, where μ runs over the weights of V. This set is therefore a linearly ordered chain, and accordingly, $\lambda' \otimes V$ has a $\mathrm{U}_A^0 \mathrm{U}_A^-$-module filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_d = \lambda' \otimes V,$$

where $d = \dim V_q$, with the quotients F_i/F_{i-1} running over the $\mathrm{U}_A^0 \mathrm{U}_A^-$-modules $\lambda' + \mu$. Our hypothesis, together with (the quantised) Kempf's vanishing theorem imply that the higher (degree > 0) cohomology of the corresponding line bundles vanishes, and hence that induction is exact on this filtration. We therefore have a corresponding filtration of U_A-modules

$$0 \subset \nabla_A(F_1) \subset \cdots \subset \nabla_A(F_d) = \nabla_A(\lambda' \otimes V) = V^* \otimes V. \qquad \square$$

Corollary 5.4. *The conclusion of Proposition 5.3 holds in the following cases.*

(1) *V is a Weyl module with minuscule highest weight. This includes the natural modules in types A, C and D (but not type B).*

(2) *V is any Weyl module for $\mathrm{U}_A(\mathfrak{sl}_2)$.*

(3) *V is the Weyl module in type G_2 with highest weight $2\alpha_1 + \alpha_2$, where α_1 and α_2 denote the two simple roots, with α_2 long.*

Proof. When V is minuscule, it is well known that for any weight μ of V we have $(\mu, \alpha^\vee) = \pm 1$ or 0, and hence (1) is clear. The case of $\mathfrak{sl}_2$ is evident, while in the case of type G_2, the weights of the Weyl module in question are the short roots, together with 0. This easily gives (3). $\square$

5B. ***Multiplicities of tilting modules and dimensions of irreducibles.*** In this section we shall prove some rather general results which will allow us to relate multiplicities of indecomposable tilting summands in tensor powers of certain representations of quantum groups to the dimensions of simple modules for the corresponding endomorphism algebras.

We note that the results of this section are similar in spirit to those of [Brundan and Kleshchev 1999, §3], which in turn have their genesis in some aspects of [Mathieu and Papadopoulos 1999, §3].

Theorem 5.5. *Let k be a field, U a k-algebra, and M a finite-dimensional* (*over k*) U*-module. Let $E = \mathrm{End}_{\mathrm{U}}(M)$, and assume that for each indecomposable direct summand M' of M, we have $E'/\operatorname{Rad} E' \simeq k$ where $E' = \mathrm{End}_{\mathrm{U}}(M')$. Then*

$$\frac{E}{\operatorname{Rad} E} \simeq \bigoplus_i M_{d_i}(k),$$

where $M_d(k)$ is the algebra of $n \times n$ matrices over k, i runs over the isomorphism classes of indecomposable U*-modules* (*of course only a finite number occur*), *and the d_i are the multiplicities of the indecomposable summands of M.*

Proof. Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a decomposition of M into indecomposables. Then any endomorphism $\phi \in E$ may be written $\phi = (\phi_{ij})_{1 \le i,j \le n}$, where ϕ_{ij} is in $\mathrm{Hom}_{\mathrm{U}}(M_j, M_i)$.

Now by Fitting's lemma, any endomorphism of M_i is either an automorphism or is nilpotent. With the notation $E_i := \mathrm{End}_{\mathrm{U}}(M_i)$, it follows that for each i, the set $R_i := \{\psi \in E_i \mid \psi$ is not an automorphism$\}$ is a nilpotent ideal of E_i. In particular there is an integer N_i such that $R_i^{N_i} = 0$.

Next, suppose that we have a sequence $i = i_1, i_2, \ldots, i_{p+1} = i$, and $\phi_j := \phi_{i_j, i_{j+1}} \in \mathrm{Hom}_{\mathrm{U}}(M_{i_{j+1}}, M_{i_j})$ for $j = 1, 2, \ldots, p$. Consider $\psi_1 := \phi_1 \ldots \phi_{p-1}\phi_p$ in $\mathrm{Hom}_{\mathrm{U}}(M_i, M_i)$. We shall show that:

(5-3) $\quad \psi_1$ *is an automorphism* $\implies$
the M_{i_j} are all isomorphic, and ϕ_j is an isomorphism for each j.

To see (5-3), let $\psi_j = \phi_j \ldots \phi_p \phi_1 \ldots \phi_{j-1} \in \mathrm{Hom}(M_{i_j}, M_{i_j})$. If ψ_j is an automorphism for each j, then for each j, ϕ_{j-1} is injective and ϕ_j is surjective, whence each ϕ_j is an automorphism, and we are done. If not, then there is some j such that ψ_j is nilpotent. It follows that $\psi_1^N = 0$ for large N, which is a contradiction. This proves (5-3).

Now let J be the subspace of E consisting of the endomorphisms ϕ such that ϕ_{ij} is not invertible for each pair i, j. If

$$J_{ij} := \{\phi_{ij} \in \mathrm{Hom}_{\mathrm{U}}(M_j, M_i) \mid \phi_{ij} \text{ is not invertible}\},$$

then again by Fitting's lemma, J_{ij} is an (E_i, E_j) bimodule, and using the observation (5-3) above, it is clear that J is an ideal of E. We shall show that J is nilpotent.

Let $\phi^{(1)}, \ldots, \phi^{(\ell)}$ be a sequence of elements of J. Then

$$(\phi^{(1)} \ldots \phi^{(\ell)})_{ij} = \sum_{k_1, k_2, \ldots, k_{\ell-1}} \phi^{(1)}_{ik_1} \phi^{(2)}_{k_1 k_2} \cdots \phi^{(\ell)}_{k_{\ell-1} j},$$

where the sum is over all sequences $k_1, k_2, \ldots, k_{\ell-1}$ with $1 \le k_i \le n$ for all i.

Now we have seen that for any j, if $R_j = \operatorname{Rad} E_j$, then there is an integer N_j such that $R_j^{N_j} = 0$. If we take $\ell \ge N_1 + N_2 + \cdots + N_n + 2$, then there some index a that occurs among the k_i at least $N_a + 1$ times. Then each summand in the expression for $(\phi^{(1)} \ldots \phi^{(\ell)})_{ij}$ contains a product of N_a noninvertible elements of E_a for some a, and hence is 0. Thus $J^{N_1+\cdots+N_n+2} = 0$.

Finally, it is clear that since $E_i/R_i \simeq k$ for each i, $E/J \simeq \bigoplus_{i=1}^n M_{d_i}(k)$. $\square$

The proof above actually yields the following corollary of the Artin–Wedderburn theorem.

Corollary 5.6. *Let M be as in Theorem 5.5 but drop the assumption on the endomorphism rings of direct summands of M. Then there are division rings D_i over k such that*

$$\frac{E}{\operatorname{Rad} E} \simeq \bigoplus_i M_{d_i}(D_i).$$

Proof. In this case Fitting's lemma yields that E_i/R_i is a division algebra D_i over k, and the argument above proves the assertion. $\square$

The application to our situation arises through the following property of finite-dimensional tilting modules for quantum groups. Let k be a field considered as an A-algebra via $q \mapsto \zeta \in k \setminus \{0\}$ and let U_ζ be as in Section 5A.

Proposition 5.7. *Let M be a finite-dimensional indecomposable tilting module for U_ζ and set $E = \operatorname{End}_{\mathrm{U}_\zeta}(M)$. Then $E/\operatorname{Rad} E \simeq k$.*

Proof. By the Ringel–Donkin classification [Donkin 1993] (see [Andersen 1992] for the adaption to the quantum case) of indecomposable tilting modules we get that M has a unique highest weight $\lambda \in X^+$ and that the weight space M_λ is 1-dimensional. Therefore any $\varphi \in \operatorname{End}_{\mathrm{U}_\zeta}(M)$ is given by a scalar $a \in k$ on M_λ. But then $\varphi - a\ \mathrm{id}_M$ is not an automorphism; i.e., $\varphi - a\ \mathrm{id}_M \in \operatorname{Rad} E$. $\square$

We denote the indecomposable tilting module for U_ζ with highest weight λ by $\mathcal{T}_\zeta(\lambda)$ and for an arbitrary tilting module $\mathcal{T}$ for U_ζ we write $(\mathcal{T} : \mathcal{T}_\zeta(\lambda))$ for the multiplicity with which $\mathcal{T}_\zeta(\lambda)$ occurs as a summand of $\mathcal{T}$. Then Theorem 5.5 together with Proposition 5.7 give the following result.

Corollary 5.8. *For any tilting module $\mathcal{T}$ for U_ζ and any $\lambda \in X^+$ we have*

$$(\mathcal{T} : \mathcal{T}_\zeta(\lambda)) = \dim_k L_\zeta(\lambda),$$

where $L_\zeta(\lambda)$ is the simple module for $E = \operatorname{End}_{\mathrm{U}_\zeta}(\mathcal{T})$ corresponding to λ.

5C. ***Multiplicities for*** $\mathbf{U}_\zeta(\mathfrak{sl}_2)$**.** We now apply the above general results to $\mathfrak{sl}_2$. With k and ζ as above, the indecomposable tilting modules in this case are $\mathcal{T}_\zeta(m)$ with $m \in \mathbb{N}$. If ζ is not a root of unity in k then the category of finite-dimensional U_ζ-modules is semisimple and behaves exactly like the corresponding category for the generic quantum group U_q.

From now on we assume that ζ is a root of unity; for the specialisation U_ζ, etc., we assume that the homomorphism $A \to k$ is given by $q \mapsto \zeta$ (so $q^{1/2} \mapsto \sqrt{\zeta}$) and we set $\ell = \mathrm{ord}(\zeta^2)$. If d is a positive integer with $d < \ell$ we have $\Delta_\zeta(d) = \mathcal{T}_\zeta(d)$ and all the tensor powers $\mathcal{T}_r = \Delta_\zeta(d)^{\otimes r}$ are also tilting modules. We set $E_\zeta(d, r) = \mathrm{End}_{\mathrm{U}_\zeta}(\mathcal{T}_r)$. By Lemma 5.1 we have

$$E_\zeta(d, r) = E_r(d, \tilde{A}) \otimes_{\tilde{A}} k,$$

where as before $\tilde{A} = A[([d]!)^{-1}]$. Note that our assumption $\ell > d$ ensures that the specialization $\phi_\zeta : A \to k$ factors through $\tilde{A}$ making k into an $\tilde{A}$-algebra.

Our cellularity results from Section 3 imply that

$$E_\zeta(d, r) \cong p_\zeta \mathrm{TL}_{dr}(k) p_\zeta, \tag{5-4}$$

where p_ζ is the specialisation at $q = \zeta$ of the idempotent $p \in \mathrm{TL}_{dr}(\tilde{A})$. Note that in $\mathrm{TL}_{dr}(k) = \mathrm{TL}_{dr,\zeta}(k)$ the generators f_i satisfy $f_i^2 = (\zeta + \zeta^{-1}) f_i$.

The simple modules for the cellular algebra $p_\zeta \mathrm{TL}_{dr}(k) p_\zeta$ are parametrised by the poset $\Lambda = \{m \in \mathbb{Z} \mid 0 \le m \le dr \text{ and } dr - m \in 2\mathbb{Z}\}$; see Section 4B. We denote the simple module associated with $m \in \Lambda$ by $L_\zeta(m)$.

Theorem 5.9. *In the above notation, in particular assuming $\ell = \mathrm{ord}(\zeta^2) > d$, we have for $m \in \Lambda$,*

$$(\mathcal{T}_r : \mathcal{T}_\zeta(m)) = \dim_k L_\zeta(m).$$

This multiplicity is the rank of the matrix whose rows and columns are labelled by $\mathcal{B}(d, r; m)$ (see Section 4A) and whose (D_1, D_2)-entry is the coefficient of the identity map $m \to m$ (in the Temperley–Lieb category) in the expansion of $D_2^* p_\zeta D_1$ as a linear combination of diagrams from m to m.

Proof. The equality in the theorem is an immediate consequence of Corollary 5.8. To see the second statement note that $L_\zeta(m)$ is realised as follows: Let $W_\zeta(m)$ be the cell module corresponding to m. This has k-basis C_S, $S \in \mathcal{B}(d, r; m)$, the monic diagrams D from m to dr such that $L(D) \subseteq \{d, 2d, \dots, (r-1)d\}$. We may think of C_S as $p_\zeta S$, and then the $E_\zeta(d, r)$-action is by left composition: for $x \in E_\zeta(d, r)$, $x C_S = \sum_{T \in \mathcal{B}(d,r;m)} a(T, S) C_T$, where

$$x p_\zeta S = \sum_{T \in \mathcal{B}(d,r;m)} a(T, D) p_\zeta T + \text{lower terms},$$

where “lower” means “having fewer through-arcs”.

There is an invariant form $(\,-\,,\,-\,)$ on $W_\zeta(m)$, defined by

$$C^{m\;\,2}_{S,T} \in (C_S, C_T)C^m_{S,T} + E_\zeta(d,r)(<m) \quad \text{for } S, T \text{ in } \mathcal{B}(d,r;m). \tag{5-5}$$

The radical $\mathrm{Rad}_\zeta(m)$ of this form is a submodule of $W_\zeta(m)$, and

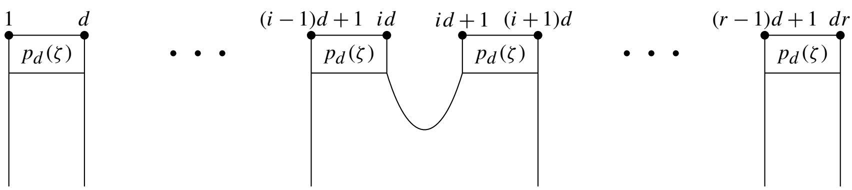
$$L_\zeta(m) = W_\zeta(m)/\,\mathrm{Rad}_\zeta(M).$$

It is therefore evident that $\dim L_\zeta(m)$ is equal to the rank of the Gram matrix $M_{m,\zeta}$, whose rows and columns are indexed by $\mathcal{B}(d,r;m)$, and whose (S,T)-entry is (C_S, C_T).

Finally, since $C^{m\;\,2}_{S,T} = p_\zeta S(T^* p_\zeta S)T^* p_\zeta$, and noting that $T^* p_\zeta S$ is a linear combination of diagrams from m to m, it follows from (5-5) that (C_S, C_T) is the coefficient of $\mathrm{id} : m \to m$. □

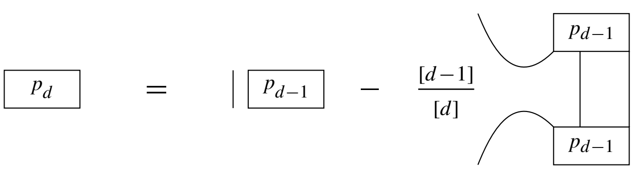
Since $\dim W_\zeta(dr) = 1$ and the coefficient of $\mathrm{id} : d \to d$ in $p_d(\zeta)$ is 1, it is immediate from the theorem that the multiplicity of $\mathcal{T}_\zeta(dr)$ is 1. We finish this section with a less trivial example.

Example 5.10. Take $k = dr - 2$ and recall that $d < \ell$. We shall compute the multiplicity of $\mathcal{T}_\zeta(k)$ in $\Delta_\zeta(d)^{\otimes r}$ for any d, r. Here $\mathcal{B}(d,r;dr-2) = \{S_1, S_2, \ldots, S_{r-1}\}$, where S_i is as shown in the figure:



Now by repeated use of the diagrammatic recursion

(*)



it is straightforward to compute the Gram matrix $M_{dr-2,\zeta}$ of the invariant form (see the proof above). One shows that

$$(S_i, S_j) = \begin{cases} 0 & \text{if } j \neq i \text{ or } i \pm 1, \\ \frac{[2]_{\zeta^d}}{[d]_\zeta} & \text{if } j = i, \\ (-1)^{d+1}[d]_\zeta^{-1} & \text{if } j = i \pm 1. \end{cases}$$

Hence the Gram matrix of the invariant form is the $(r-1)\times(r-1)$ matrix

$$M_{dr-2,\zeta} = \frac{1}{[d]_\zeta}\begin{pmatrix} \delta & (-1)^{d+1} & 0 & \cdots & \cdots & 0 \\ (-1)^{d+1} & \delta & (-1)^{d+1} & 0 & \cdots & \vdots \\ 0 & (-1)^{d+1} & \delta & (-1)^{d+1} & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & & \ddots & (-1)^{d+1} \\ 0 & \cdots & \cdots & 0 & (-1)^{d+1} & \delta \end{pmatrix},$$

where $\delta = \zeta^d + \zeta^{-d} = [2]_{\zeta^d}$.

Now it is easily shown by induction that any $n\times n$ matrix of the form

$$A = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ 1 & a_2 & b_2 & 0 & \cdots & \vdots \\ 0 & 1 & a_3 & b_3 & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & & \ddots & b_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 & a_n \end{pmatrix}$$

with entries in a principal ideal domain may be transformed by row and column operations into

$$A' = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 1 & 0 & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & D \end{pmatrix},$$

where $D = \det(A)$. It follows that the rank of the Gram matrix $M_{dr-2,\zeta}$ is $r-1$ if $\det M_{dr-2,\zeta} \neq 0$, while if $\det M_{dr-2,\zeta} = 0$, the rank is $r-2$.

Now the determinant of $[d]_\zeta M_{dr-2,\zeta}$ is easily computed (cf. [Graham and Lehrer 1996, Equation 6.18.2]), and using this, we see that

$$\det M_{dr-2,\zeta} = (-1)^{(d+1)(r+1)}([d]_\zeta)^{-(r-1)}[r]_{(-1)^{d+1}\zeta^d}.$$

It therefore follows that the multiplicity of $\mathcal{T}_\zeta(dr-2)$ in $\Delta_\zeta(d)^{\otimes r}$ is

$$\begin{cases} r-1 & \text{if } [r]_{(-1)^{d+1}\zeta^d} \neq 0, \\ r-2 & \text{otherwise.} \end{cases}$$

Finally, observe that

$$[r]_{(-1)^{d+1}\zeta^d} = 0 \iff \zeta^{2dr} = 1.$$

Hence if we write (using the convention that for any root of unity ξ, we denote by $|\xi|$ or by $\mathrm{ord}(\xi)$ the multiplicative order of ξ)

$$\ell = \begin{cases} |\zeta| & \text{if } |\zeta| \text{ is odd,} \\ \frac{1}{2}|\zeta| & \text{if } |\zeta| \text{ is even,} \end{cases} \tag{5-6}$$

then $\ell = |\zeta^2|$, whence the multiplicity of $\mathcal{T}_\zeta(dr-2)$ in $\Delta_\zeta(d)^{\otimes r}$ is given by

$$(\mathcal{T}_r : \mathcal{T}_\zeta(dr-2)) = \begin{cases} r-1 & \text{if } \ell \nmid dr, \\ r-2 & \text{if } \ell \mid dr. \end{cases} \tag{5-7}$$

This shows also by standard cellular theory that the cell module $W_\zeta(dr-2)$ of $E_\zeta(d,r))$ is simple if $\ell \nmid dr$, while if $\ell \mid dr$, then $W_\zeta(dr-2)$ has composition factors $L_\zeta(d,r;dr-2)$ and $L_\zeta(d,r;dr)$ (the latter being the trivial module), each with multiplicity one.

6. Complex roots of unity

In this section we take $k = \mathbb{C}$ and fix a root of unity $\zeta \in \mathbb{C}$. As before we set $\ell = \mathrm{ord}(\zeta^2)$. In this case the structure of the tilting modules $\mathcal{T}_\zeta(m)$ is well understood, and hence, when $\ell > d$, provides an alternative approach to the computation of the multiplicities $\mu_\zeta(d,r;m) := (\Delta_\zeta(d)^{\otimes r} : \mathcal{T}_\zeta(m))$, and thus of the dimensions of the simple modules for the cellular algebra $E_\zeta(d,r)$ (see Theorem 5.9). In this section we demonstrate how this is done. We then show how these results on tilting modules may alternatively be deduced from results on the decomposition numbers of the algebras $E_\zeta(d,r)$, which are also proved in this section.

6A. *Structure of tilting modules.*

Proposition 6.1. *The indecomposable tilting module $\mathcal{T}_\zeta(m)$ for $\mathrm{U}_\zeta = \mathrm{U}_\zeta(\mathfrak{sl}_2)$ with highest weight m has the following description.*

(1) *If either $m < \ell$ or $m \equiv -1 \pmod{\ell}$ then $\mathcal{T}_\zeta(m) \simeq \Delta_\zeta(m)$ is irreducible.*

(2) *Write $m = a\ell + b$, where $a \geq 1$ and $0 \leq b < \ell - 1$. Then $\mathcal{T}_\zeta(m)$ is the unique nontrivial extension*

$$0 \longrightarrow \Delta_\zeta(m) \longrightarrow \mathcal{T}_\zeta(m) \longrightarrow \Delta_\zeta(m-2b-2) \longrightarrow 0.$$

Proof. This result is certainly well known and follows from the results of [Soergel 1998]. As we haven't been able to find a reference where this is explicitly stated we sketch the easy proof.

Denote by $\mathscr{L}_\zeta(m)$ the simple U_ζ-module with highest weight $m \in \mathbb{N}$ (not to be confused with the simple $E_\zeta(d,r)$-module $L_\zeta(m)$). It follows from the strong linkage principle [Andersen 2003] (or by direct calculations) that $\mathscr{L}_\zeta(m) = \Delta_\zeta(m)$ if and only if m satisfies the conditions in (1); in particular, (1) holds.

So assume $m = a\ell + b$ with a and b as in (2). The module $\Delta(a\ell-1)\otimes_{\mathbb{C}}\Delta_\zeta(b+1)$ has a Weyl filtration with factors $\Delta_\zeta(m)$, $\Delta_\zeta(m-2), \cdots, \Delta_\zeta(m-2(b+1))$. Note that the first and the last factors belong to the same linkage class and that none of the other factors are in this class. Hence by the linkage principle [loc. cit.] there is a summand $\mathscr{T}$ of $\Delta_\zeta(a\ell-1)\otimes_{\mathbb{C}}\Delta_\zeta(b+1)$ which has these two Weyl factors, i.e., fits into an exact sequence

$$0 \longrightarrow \Delta_\zeta(m) \longrightarrow \mathscr{T} \longrightarrow \Delta_\zeta(m-2b-2) \longrightarrow 0.$$

By case (1) we see that $\Delta_\zeta(a\ell-1)\otimes_{\mathbb{C}}\Delta_\zeta(b+1)$ is tilting. Hence so is our summand $\mathscr{T}$. The proof of case (2) will therefore be complete if we prove that $\mathscr{T}$ is indecomposable. This in turn would follow if there were no nontrivial homomorphisms $\mathscr{T}$ of $\Delta_\zeta(a\ell-1)\otimes_{\mathbb{C}}\Delta_\zeta(b+1) \longrightarrow \mathscr{L}_\zeta(m)$, for if the last sequence splits, there would be such a homomorphism. To check the last statement, we need the quantised Steinberg tensor product theorem [Andersen and Wen 1992, Theorem 1.10] for simple modules, $\mathscr{L}_\zeta(m) \simeq \mathscr{L}_\zeta(a\ell)\otimes\mathscr{L}_\zeta(b)$ (again in the case at hand this can alternatively be checked by direct calculations).

Using this together with the self-duality of simple modules and the result in (1) we get

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{U}_\zeta}\big(\Delta_\zeta(a\ell-1)\otimes_{\mathbb{C}}\Delta_\zeta(b+1), \mathscr{L}_\zeta(m)\big)\\
&\quad\simeq \mathrm{Hom}_{\mathrm{U}_\zeta}\big(\mathscr{L}_\zeta(a\ell-1)\otimes_{\mathbb{C}}\mathscr{L}_\zeta(b+1), \mathscr{L}_\zeta(m)\big)\\
&\quad\simeq \mathrm{Hom}_{\mathrm{U}_\zeta}\big(\mathscr{L}_\zeta((a-1)\ell)\otimes_{\mathbb{C}}\mathscr{L}_\zeta(\ell-1)\otimes_{\mathbb{C}}\mathscr{L}_\zeta(b+1), \mathscr{L}_\zeta(a\ell)\otimes_{\mathbb{C}}\mathscr{L}_\zeta(b)\big)\\
&\quad\simeq \mathrm{Hom}_{\mathrm{U}_\zeta}\big(\mathscr{L}_\zeta((a-1)\ell)\otimes_{\mathbb{C}}\mathscr{L}_\zeta(b+1)\otimes_{\mathbb{C}}\mathscr{L}_\zeta(b), \mathscr{L}_\zeta(a\ell)\otimes_{\mathbb{C}}\mathscr{L}_\zeta(\ell-1)\big)\\
&\quad\simeq \mathrm{Hom}_{\mathrm{U}_\zeta}\big(\mathscr{L}_\zeta((a-1)\ell)\otimes_{\mathbb{C}}\mathscr{L}_\zeta(b+1)\otimes_{\mathbb{C}}\mathscr{L}_\zeta(b), \mathscr{L}_\zeta((a+1)\ell-1)\big).
\end{aligned}$$

The last Hom-space is 0 because, by our condition on b, the weight $(a+1)\ell-1$ is strictly larger than all weights of $\mathscr{L}_\zeta((a-1)\ell)\otimes_{\mathbb{C}}\mathscr{L}_\zeta(b+1)\otimes_{\mathbb{C}}\mathscr{L}_\zeta(b)$. □

Since the weights of $\Delta_\zeta(m)$ are $m, m-2, \cdots, -m$, each occurring with multiplicity one, we deduce the following result.

Corollary 6.2. *We have*

$$\dim \mathscr{T}_\zeta(m)_t = \begin{cases} 1 & \text{if } t = m-2i,\ 0 \le i \le m \text{ in case (1)},\\ 2 & \text{if } t = m-2j,\ b+1 \le j \le m-(b+1) \text{ in case (2)},\\ 1 & \text{if } t = m-2j,\ \text{with } 0 \le j \le b \text{ or } m \ge j \ge m-b \text{ in case (2)},\\ 0 & \text{otherwise.}\end{cases}$$

6B. ***Multiplicities and dimensions.*** Now the equation

$$\Delta_\zeta(d)^{\otimes r} \cong \bigoplus_{m=0}^{dr} \mu_\zeta(d,r;m)\mathcal{T}_\zeta(m). \tag{6-1}$$

may be used to relate the multiplicities to the dimensions of the weight spaces. For this purpose, we make the following definitions.

Definition 6.3. (1) Let $w(d,r;m) := \dim(\Delta_\zeta(d)^{\otimes r})_m$. This is independent of ζ.

(2) Let $a_m = a_m(d,r) := \big|\big\{(i_1,\ldots,i_r) \bigm| 0 \le i_j \le d \text{ for all } j \text{ and } \sum_j i_j = m\big\}\big|$.

Note that $a_m = a_{dr-m}$ for all m.

Lemma 6.4. (1) *For* $0 \le m \le dr$, $m \equiv dr \pmod 2$, $w(d,r;m) = a_{(m+dr)/2}$.

(2) *We have*

$$w(d,r;m) = \mu_\zeta(d,r;m) + \sum_{j=1}^{(dr-m)/2} \dim \mathcal{T}_\zeta(m+2j)_m \mu_\zeta(d,r;m+2j).$$

The first statement follows easily from the fact that $\Delta_\zeta(d)^{\otimes r}$ has q-character $[d+1]^r$, while the second arises from (6-1) by taking the dimension of the m-weight spaces on both sides, taking into account that $\mathcal{T}_\zeta(t)$ has only weights m that satisfy $m = t-2i$, $i \ge 0$, and $rd \ge m \ge -rd$.

Lemma 6.4(2) may be used to determine the multiplicities $\mu_\zeta(d,r;m)$ recursively. We shall do this for the case considered in Example 5.10.

Example 6.5. Let us compute $\mu_\zeta(d,r,dr-2)$. By Lemma 6.4(2),

$$w(d,r;dr-2) = \mu_\zeta(d,r;dr-2) + \dim \mathcal{T}_\zeta(dr)_{dr-2}.$$

Moreover, it follows from Corollary 6.2 that

$$\dim \mathcal{T}_\zeta(dr)_{dr-2} = \begin{cases} 2 & \text{if } b = 0, \\ 1 & \text{if } b \ne 0. \end{cases}$$

Noting that by Lemma 6.4(1) we have $w(d,r,dr-2) = a_{dr-1} = a_1 = r$, we get

$$\mu_\zeta(d,r;dr-2) = \begin{cases} r-1 & \text{if } \ell \nmid dr, \\ r-2 & \text{if } \ell \mid dr, \end{cases}$$

in accord with (5-7).

Example 6.6. In Example 6.5 we considered multiplicities $\mu_\zeta(d,r;t)$, where t was large, namely $t = dr-2$. We now consider the case where t is small.

Assume $t < \ell$. Then we may apply [Andersen and Paradowski 1995, Formula 3.20 (1)]. Using the notation from Section 4A this formula reads in our case

$$\mu_\zeta(d,r;t) = \sum_{j\ge 0} m(d,r;t+2j\ell) - \sum_{i>0} m(d,r;2i\ell-t-2).$$

Recall that the multiplicities $m(d,r;t)$ are given by the recursion relation (4-8); i.e., they may be calculated by induction on r.

In fact this formula is valid in general: maintaining the notation of Example 6.6 (except that the integer t below may now be arbitrary) we have the following result.

Proposition 6.7. *Let $t \in \mathbb{N}$.*

(1) *If $t \equiv -1 \pmod{\ell}$ then $\mu_\zeta(d,r;t) = m(d,r;t)$.*

(2) *If $t \not\equiv -1 \pmod{\ell}$ then, writing $t = a\ell + b$ with $0 \le b \le \ell - 2$, we have*

$$\begin{aligned}\mu_\zeta(d,r;t) &= \sum_{j\ge 0} m(d,r;t+2j\ell) - \sum_{i\ge 1} m(d,r;t-2b-2+2i\ell)\\ &= \sum_{j\ge 0} m(d,r;t+2j\ell) - \sum_{i\ge a+1} m(d,r;2i\ell-t-2).\end{aligned}$$

Proof. This follows easily from the description of the indecomposable tilting modules $\mathcal{T}_\zeta(m)$ in Proposition 6.1 by taking characters in the relation $\Delta_\zeta(d)^{\otimes r} \cong \bigoplus_m \mu(d,r;m)\mathcal{T}_\zeta(m)$. Let $\mathscr{C}_1$ be the set of positive integers occurring in case (1) of Proposition 6.1, and similarly let $\mathscr{C}_2$ be those occurring in case (2).

If we denote by c_t the q-character of $\Delta_q(t)$, then Proposition 6.1 shows that if $t \in \mathscr{C}_1$, then $\operatorname{char}(\mathcal{T}_\zeta(t)) = c_t$, while if $t \in \mathscr{C}_2$, then $\operatorname{char}(\mathcal{T}_\zeta(t)) = c_t + c_{t-2b-2}$. Now substitute these values and compare coefficients of c_t in the equation

$$\sum_{t\in\mathbb{N}} m(d,r;t)c_t = \sum_{t\in\mathscr{C}_1} \mu_\zeta(d,r;t)\operatorname{char}(\mathcal{T}_\zeta(t)) + \sum_{t\in\mathscr{C}_2} \mu_\zeta(d,r;t)\operatorname{char}(\mathcal{T}_\zeta(t)).$$

One obtains $\mu_\zeta(d,r;t) = m(d,r;t)$ if $t \equiv -1 \pmod{\ell}$, while if $t = a\ell + b$ with $a \ge 0$ and $0 \le b \le \ell - 2$, we have

$$m(d,r;t) = \mu_\zeta(d,r;t) + \mu_\zeta(d,r;(a+2)\ell - b - 2). \tag{6-2}$$

Now for any integer $t = a\ell + b \ge 0$ such that $t \not\equiv -1 \pmod{\ell}$, write $g(t) = (a+2)\ell - b - 2$; then $g(t) \not\equiv -1 \pmod{\ell}$, and the relation above reads $m(d,r;t) = \mu_\zeta(d,r;t) + \mu_\zeta(d,r;g(t))$. It follows that

$$\mu_\zeta(d,r;t) = \sum_{i\ge 0} m(d,r;g^{2i}(t)) - \sum_{j\ge 0} m(d,r;g^{2j+1}(t)).$$

The statements (1) and (2) are now immediate. □

As these multiplicities are also dimensions of simple modules for our cellular algebra from Section 4, we may rewrite these formulae as follows (again using notation from Section 4A).

Corollary 6.8. *Let $t \in \mathbb{N}$.*

(1) *If $t \equiv -1 \pmod{\ell}$ then $\dim_{\mathbb{C}} L_\zeta(t) = b(d,r;t)$.*

(2) *If* $t \not\equiv -1 \pmod{\ell}$ *then, writing* $t = a\ell + b$ *with* $0 \le b \le \ell - 2$, *we have*

$$\dim_{\mathbb{C}} L_\zeta(t) = \sum_{j \ge 0} b(d, r; t + 2j\ell) - \sum_{i \ge a+1} b(d, r; 2i\ell - t - 2).$$

Note that the numbers $b(d, r; t)$ are dimensions of the cell modules of the cellular algebra $p\mathrm{TL}_{dr}(\tilde{A})p$ that do not change under specialisation.

6C. *Decomposition numbers.* In this section we shall determine the decomposition numbers of the cellular algebra $E_\zeta(d, r)$, and show how the weight multiplicities of the tilting modules are determined by these, giving an alternative proof of Corollary 6.2. The algebra has cell modules $W_\zeta(t)$ as implied in Section 4B and $\dim(W_\zeta(t)) = b(d, r; t)$. If $L_\zeta(t)$ is the corresponding simple module, we write $d_{st} = [W_\zeta(t) : L_\zeta(s)]$ for the multiplicity of $L_\zeta(s)$ in $W_\zeta(t)$. It is known by the theory of cellular algebras that the matrix (d_{st}) is lower unitriangular.

We have $\dim(L_\zeta(t)) = \mu_\zeta(d, r; t)$, and therefore we clearly have

$$b(d, r; t) = \sum_{s \ge t} d_{st}\mu_\zeta(d, r; s). \tag{6-3}$$

Theorem 6.9. *Maintain the notation above. Suppose* $\ell \in \mathbb{N}$ *is such that* $\ell = \mathrm{ord}(\zeta^2)$ *and* $\ell > d$, *and write* $\mathbb{N} = \mathcal{N}_1 \amalg \mathcal{N}_2$, *where* $\mathcal{N}_1 = \{t \in \mathbb{N} \mid t \equiv -1 \pmod{\ell}\}$ *and* $\mathcal{N}_2 = \mathbb{N} \setminus \mathcal{N}_1$. *Let* $g : \mathcal{N}_2 \longrightarrow \mathcal{N}_2$ *be the function defined in the proof of Proposition 6.7, viz. if* $t = a\ell + b$ *with* $0 \le b \le \ell - 2$, *then* $g(t) = (a+1)\ell + \ell - b - 2$. *Observe that* $g(t) = t + 2(\ell - b - 1) \ge t + 2$, *and that* $g(t) \equiv t \pmod 2$.

(1) *For each* $t \in \mathcal{N}_2$ *such that* $0 \le t < g(t) \le dr$ *and* $t \equiv dr \pmod 2$, *there is a nonzero homomorphism* $\theta_t : W_\zeta(g(t)) \longrightarrow W_\zeta(t)$ *which is uniquely determined up to scalar multiplication.*

(2) *The* θ_t *are the only nontrivial homomorphisms between the cell modules of* $E_\zeta(d, r)$.

(3) *Let* $t \in \mathbb{N}$ *be such that* $0 \le t \le dr$ *and* $t \equiv dr \pmod 2$. *If* $t \in \mathcal{N}_2$ *and* $g(t) \le dr$, *then* $W_\zeta(t)$ *has composition factors* $L_\zeta(t)$ *and* $L_\zeta(g(t))$, *each with multiplicity* 1. *All other cell modules are simple.*

(4) *The decomposition numbers of* $E_\zeta(d, r)$ *are all equal to* 0 *or* 1.

Note that (3) and (4) are formal consequences of (1) and (2).

Proof. We begin by observing that the statement is true when $d = 1$. In this case $E_\zeta(1, r) = \mathrm{TL}_{r,\zeta}(\mathbb{C})$, the structure of whose cell modules (as well as all homomorphisms between them) is treated in [Graham and Lehrer 1998]. In particular, Theorem 5.3 of that reference asserts that (in our notation above) if $s \ne t$, then $L_\zeta(s)$ is a composition factor of $W_\zeta(t)$ if and only if s satisfies both (i) $t + 2\ell > s > t$

and (ii) $s+t+2\equiv 0 \pmod{2\ell}$. It is an easy exercise to show that (i) and (ii) are equivalent to (iii) $t\not\equiv -1 \pmod{\ell}$ and (iv) $s=g(t)$. This yields all the statements of the theorem for this case.

Next recall that $E_\zeta(d,r)\cong p_d(\zeta)\mathrm{TL}_{dr,\zeta}(\mathbb{C})p_d(\zeta)$, where $p_d(\zeta)$ is the specialisation at ζ of the idempotent p_d. Thus we may define the exact functor $\mathcal{F}_d:\mathrm{Mod}(\mathrm{TL}_{dr,\zeta}(\mathbb{C}))\longrightarrow \mathrm{Mod}(E_\zeta(d,r))$ by $M\mapsto p_d(\zeta)M$, where Mod indicates the category of left modules for the relevant algebra. Now it is evident from the description in Section 4B of the cell module $W(t)$ and its basis $\mathcal{B}(d,r;t)$ that $\mathcal{F}_d(W_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t))=W_{E_\zeta(d,r)}(t)$ for all t with $0\le t\le dr$ and $t+dr\in 2\mathbb{Z}$.

Moreover by exactness, for any simple $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$-module L, $\mathcal{F}_d(L)$ is either a simple $E_\zeta(d,r)$-module or zero. Thus it follows (also from the explicit diagrammatic description) that $\mathcal{F}_d(L_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t))=L_{E_\zeta(d,r)}(t)$ whenever the latter is nonzero. Given the description in Section 4B of the cellular structure, and the fact that $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$ is quasihereditary when $\zeta\ne\zeta_4=\exp(\pi i/2)$, $\mathcal{F}_d$ does not kill any nontrivial simple $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$-module (this may be checked directly when $\zeta=\zeta_4$). The quasiheredity of $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$ when $\zeta\ne\zeta_4$ is well known, but may be seen as follows.

Since $\zeta+\zeta^{-1}\ne 0$, if $t\in\mathbb{N}$, $0\le t\le dr$, $t\equiv dr \pmod 2$, then for any monic diagram $u:t\to dr$, we have $u^*u=(\zeta+\zeta^{-1})^{(dr-t)/2}\,\mathrm{id}_t\ne 0$; hence, if u is thought of as an element of $W_\zeta(t)$, then $(u,u)\ne 0$. Thus, for any such t, $L_\zeta(t)\ne 0$. Although it is not needed for the proof of the theorem, the fact that if $L_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t)\ne 0$ then $\mathcal{F}_d(L_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t))\ne 0$ is verified in the same way, but requires a computation, using the recurrence (5-6) in Example 5.10 above, to show that for a nonzero element $u=p_dD\in W_\zeta(t)$, where $D:t\to dr$ is a monic diagram, we have $(u,u)\ne 0$. That such elements exist is easily verified.

By the case $d=1$ of Theorem 6.9 or, more precisely, [Graham and Lehrer 1998, Theorem 5.3] applied to $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$, if $t\in\mathcal{N}_2$, $0\le t<g(t)\le dr$ and $t\equiv dr \pmod 2$, then $W_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t)$ has composition factors $L_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t)$ and $L_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(g(t))$. All other cell modules for $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$ are simple. It follows from the previous paragraph that similarly, if $t\in\mathcal{N}_2$, $0\le t<g(t)\le dr$ and $t\equiv dr \pmod 2$, then $W_{E_\zeta(d,r)}(t)$ has composition factors $L_{E_\zeta(d,r)}(t)$ and $L_{E_\zeta(d,r)}(g(t))$, and that other cell modules for $E_\zeta(d,r)$ are simple. All statements in the theorem are now easy consequences of standard cellular theory. □

Remark 6.10. (1) From Theorem 6.9 it follows that (6-3) implies (6-2) and the other statements in Proposition 6.7. Thus the multiplicities $\mu_\zeta(d,r;t)$ are determined by Theorem 6.9.

(2) Since the dimensions $w(d,r;t)$ are known (Lemma 6.4(1)), it follows from Lemma 6.4(2) that the dimensions of the weight spaces $\mathcal{T}_\zeta(dr)_m$ are determined by Theorem 6.9.

(3) There are some analogies between this work and the modular theory developed by Erdmann [1995]. In the case $n = 2$, Erdmann dealt only with the 2-dimensional representation of $\mathfrak{gl}_2$. Nonetheless, there appear to be some similarities between her formulae and the Gram determinants of the cell modules in our situation.

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LOWER BOUNDS FOR ESSENTIAL DIMENSIONS IN CHARACTERISTIC 2 VIA ORTHOGONAL REPRESENTATIONS

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Dedicated to the memory of Robert Steinberg

We give a lower bound for the essential dimension of a split simple algebraic group of "adjoint" type over a field of characteristic 2. We also compute the essential dimension of orthogonal and special orthogonal groups in characteristic 2.

1. Introduction

Informally speaking, the essential dimension of an algebraic object can be thought of as the minimal number of independent parameters needed to define it. Essential dimension assigns a numerical invariant (a nonnegative integer) to each algebraic object and allows us to compare their relative complexity. Naturally, the fewer parameters needed for definition, the simpler the object is.

The notion of essential dimension first appeared in the work of J. Buhler and Z. Reichstein [1997] in the context of finite groups. Later on, A. Merkurjev generalized this notion to arbitrary functors from the category of fields to the category of sets; see [Berhuy and Favi 2003]. For the definition, properties, and results on essential dimension of algebraic groups and various functors, we refer to the recent surveys [Merkurjev 2013] and [Reichstein 2010].

In the past 15 years this numerical invariant has been extensively studied by many people. To the best of our knowledge, in all publications on this topic the only approach for computing the essential dimension $\mathrm{ed}(G)$ of an algebraic group G consisted of finding its upper and lower bounds. If, by lucky circumstance, both bounds for G are equal then of course their common value is $\mathrm{ed}(G)$. We remark that this strategy has worked in all cases where $\mathrm{ed}(G)$ is known.

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The aim of the current paper is two-fold. We recall that a general method for computing lower bounds of the essential dimensions of simple algebraic groups defined over fields of characteristic $\neq 2$ via orthogonal representations was developed in [Chernousov and Serre 2006]. Our first goal is to extend this approach to characteristic 2. In Section 12, we prove the incompressibility of the so-called canonical monomial quadratic forms and this result leads us to Theorem 2.1 below, which says that for any simple split "adjoint group" G defined over a field of characteristic 2 one has $\mathrm{ed}(G) \geq r+1$ where $r = \mathrm{rank}(G)$. Second, we show that for an adjoint split group G of type B_r one has $\mathrm{ed}(G) = r+1$. Thus, this result indicates that the lower bound $r+1$ of the essential dimension in Theorem 2.1 is optimal for groups of adjoint type in the general case and it seems inevitable that any future progress, if possible, will be based on case by case consideration.

2. The main theorems

We now pass to the precise description of the main results of the paper. In what follows, we assume that k is an algebraically closed field of characteristic 2 and all fields and rings under consideration will contain k.

Let G° be a simple algebraic group over k of adjoint type, and let T be a maximal torus of G°. Let $c \in \mathrm{Aut}(G^\circ)$ be such that $c^2 = 1$ and $c(t) = t^{-1}$ for every $t \in T$ (it is known that such an automorphism exists; see, e.g., [SGA 3_{III} 1970, exposé XXIV, proposition 3.16.2, p. 355]). This automorphism is inner (i.e., belongs to G°) if and only if -1 belongs to the Weyl group of (G, T). When this is the case, we put $G = G^\circ$. If not, we define G to be the subgroup of $\mathrm{Aut}(G^\circ)$ generated by G° and c. We have

- $G = G^\circ$ for types A_1, B_r, C_r, D_r (r even), G_2, F_4, E_7, E_8;
- $(G : G^\circ) = 2$ and $G = \mathrm{Aut}(G^\circ)$ for types A_r ($r \geq 2$), D_r (r odd), E_6.

Let $r = \dim(T)$ be the rank of G.

Theorem 2.1. *If G is as above, we have* $\mathrm{ed}(G) \geq r+1$.

Our second main theorem deals with orthogonal and special orthogonal groups.

Theorem 2.2. *Let q be a nondegenerate n-dimensional quadratic form over k.*

(a) *If $n = 2r$, then* $\mathrm{ed}(\mathrm{O}(q)) = r+1$.

(b) *If $n = 2r$ and r is even, then* $\mathrm{ed}(\mathrm{SO}(q)) = r+1$.

(c) *If $n = 2r$ and r is odd, then* $r \leq \mathrm{ed}(\mathrm{SO}(q)) \leq r+1$.

(d) *If $n = 2r+1$, then* $\mathrm{ed}(\mathrm{O}(q)) = \mathrm{ed}(\mathrm{SO}(q)) = r+1$.

3. Strategy of the proof of main theorems

For groups of type G_2 and F_4 in Theorem 2.1 there is an easy reduction to orthogonal groups (see Section 14 below). For all other adjoint types, orthogonal and special orthogonal groups, we follow the same approach as in [Chernousov and Serre 2006]. Namely:

(a) We construct a G-torsor θ_G over a suitable extension K/k with $\operatorname{tr.deg}_k(K) = r+1$ (see below).

(b) We show that there exists a suitable representation $\rho : G \to \mathrm{O}_N$ such that the image of θ_G in $H^1(K, \mathrm{O}_N)$ is incompressible; this implies that θ_G itself is incompressible, and Theorems 2.1 and 2.2 follow.

For the readers' convenience, we recall that a class $[\theta] \in H^1(K, G)$ is called *incompressible* if it doesn't descend to a subfield $F \subset K$ of smaller transcendence degree.

Let us start with part (a) for an adjoint group G. Let R be the root system of G with respect to T, and let R_{sh} be the (sub-) root system formed by the short roots of R. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be a basis of R_{sh}. The root lattices of R and R_{sh} are the same; hence Δ is a basis of the character group $X(T)$. This allows us to identify T with $\mathrm{G}_m \times \cdots \times \mathrm{G}_m$ using the basis Δ.

Call A_0 the kernel of "multiplication by 2" on T. Let

$$A = A_0 \times \{1, c\}$$

be the subgroup of G generated by A_0 and by the element c defined above. The group A is isomorphic to $\mu_2 \times \cdots \times \mu_2 \times \mathbb{Z}/2$.

Take $K = k(t_1, \ldots, t_r, x)$ where $t_1, \ldots, t_r$ and x are independent indeterminates. We have

$$H^1(K, A) = H^1(K, \mu_2) \times \ldots \times H^1(K, \mu_2) \times H^1(K, \mathbb{Z}/2).$$

We make the identifications

$$H^1(K, \mu_2) \simeq K^\times/(K^\times)^2 \quad \text{and} \quad H^1(K, \mathbb{Z}/2) \simeq K/\wp(K)$$

as usual. Here $\wp : K \to K$ is the Artin–Schreier map given by $\wp(a) = a^2 + a$. Then x and the t_i define elements (x) and (t_i) of $H^1(K, \mathbb{Z}/2)$ and $H^1(K, \mu_2)$, respectively. Let θ_A be the element of $H^1(K, A)$ with components $((t_1), \ldots, (t_r), (x))$. Let θ_G be the image of θ_A in $H^1(K, G)$. We will prove in Section 14:

Theorem 3.1. *(K, θ_G) is incompressible.*

Note that Theorem 3.1 implies Theorem 2.1 since $\operatorname{tr.deg} K = r+1$. Its proof relies on studying properties of the so-called monomial quadratic forms (see Section 10 below) which are also crucial for the proof of Theorem 2.2.

4. Review: quadratic spaces in characteristic 2

The purpose of this section is to review some properties of quadratic forms in characteristic 2 needed for construction of a representation of our group G with the required property explained above. To this end we will introduce the notion of a "normalization" (or "smoothing") of a quadratic form which may not be standard.

Let K be an arbitrary field of characteristic 2. Recall that a quadratic space over K is a pair (V, q) where V is a vector space over K and q is a quadratic form on V. As usual, for any $a, b \in K$ we will denote by $[a, b]$ a 2-dimensional quadratic form given by $[a, b] = ax^2 + xy + by^2$. The form $[0, 0]$ is called the hyperbolic plane and is denoted by $\mathbb{H}$. Similarly, for $a \in K$ we denote by $\langle a \rangle$ the quadratic form ax^2.

There is a special class of quadratic forms called n-fold Pfister forms; see [Elman et al. 2008]. Recall that, by definition, a quadratic form $[1, a]$ where $a \in K$ is called a 1-fold Pfister form and denoted $\langle\langle a]]$. A quadratic form isometric to

$$\langle\langle a_1, \dots, a_n]] := \langle\langle a_1, \dots, a_{n-1}\rangle\rangle_b \otimes \langle\langle a_n]]$$

for some $a_1, \dots, a_n \in K$ is called a quadratic n-fold Pfister form. In this expression, $\langle\langle a_1, \dots, a_{n-1}\rangle\rangle_b$ is a symmetric bilinear form given by

$$\langle\langle a_1, \dots, a_{n-1}\rangle\rangle_b = \langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_{n-1}\rangle_b.$$

Let K/k be a finitely generated field extension of our base field k and q a quadratic form over K. Then, if there exists another quadratic form g defined over a field L/k satisfying

- $k \subset L \subset K$,
- $\operatorname{tr.deg}_k L < \operatorname{tr.deg}_k K$, and
- $g \otimes_L K \simeq q$,

we say that q is *compressible*. Otherwise, it is *incompressible*.

The bilinear form $b_q : V \times V \to K$ (called the polar form) associated to a quadratic form q is given by

$$b_q(u, v) = q(v + u) - q(u) - q(v).$$

Its *radical* is

$$\operatorname{rad}(b_q) = \{v \in V \mid b_q(v, w) = 0 \text{ for all } w \in V\}$$

and the *quadratic radical* of q is defined as

$$\operatorname{rad}(q) = \{v \in \operatorname{rad}(b_q) \mid q(v) = 0\}.$$

Obviously, both $\operatorname{rad}(b_q)$ and $\operatorname{rad}(q)$ are vector subspaces in V.

One says that q is *regular* if $\operatorname{rad}(q) = 0$ and q is *nondegenerate* if it is regular over any field extension L/K. Note that nondegeneracy is equivalent to the property $\dim(\operatorname{rad}(b_q)) \leq 1$.

It is well-known (see [Elman et al. 2008]) that any nondegenerate quadratic form q of even dimension $n = 2m$ is isometric to $\bigoplus_{i=1}^{m} [a_i, b_i]$ where $a_i, b_i \in K$. In this case the element $c = \sum a_i b_i$ modulo $\wp(K)$ is called the *Arf invariant* of q. If q is nondegenerate and has odd dimension $n = 2m + 1$, then

$$q \simeq \bigoplus_{i=1}^{m} [a_i, b_i] + \langle c \rangle,$$

where $c \in K^\times$ is unique up to squares. This element c (modulo $(K^\times)^2$) is called the *determinant* (or *discriminant*) of q.

Let $q : V \to K$ be a quadratic form. We denote $\overline{V} := V/\operatorname{rad}(q)$ and let $\pi : V \to \overline{V}$ be the canonical map. It is straightforward to check that the mapping $\bar{q} : \overline{V} \to K$ given by $\bar{q}(\bar{v}) = q(v)$ is well defined. Thus, a quadratic space (V, q) gives rise to a quadratic space $(\overline{V}, \bar{q})$. We will see in the example below that $\bar{q}$ is nondegenerate, but first we state the following definition.

Definition 4.1. We will say that $\bar{q}$ is the (nondegenerate) *normalization* of q.

Example. Let q be a quadratic form over k. Since k is algebraically closed, it is isometric to a quadratic form

$$\langle 0 \rangle \oplus \cdots \oplus \langle 0 \rangle \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H} \quad \text{or} \quad \langle 0 \rangle \oplus \cdots \oplus \langle 0 \rangle \oplus \langle 1 \rangle \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H}.$$

It easily follows from the definition that its normalization is the quadratic form

$$\mathbb{H} \oplus \cdots \oplus \mathbb{H} \quad \text{or} \quad \langle 1 \rangle \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H}.$$

In particular, $\bar{q}$ is nondegenerate.

Lastly, we want to relate the orthogonal group of a quadratic form q to that of its normalization. Recall that given a quadratic space (V, q) the *orthogonal group* of (V, q) is

$$\mathrm{O}(V, q) = \{x \in \mathrm{GL}(V) \mid q(x(v)) = q(v) \text{ for all } v \in V\}.$$

We define a map

$$\lambda : \mathrm{O}(V, q) \longrightarrow \mathrm{O}(\overline{V}, \bar{q}).$$

by $x \mapsto \bar{x}$, where $\bar{x}(\bar{v}) = \overline{x(v)}$ for all $\bar{v} \in \overline{V}$.

Let us first show that $\bar{x}$ is well defined, i.e., $x(\operatorname{rad}(q)) \subset \operatorname{rad}(q)$ or, equivalently, $x(v) \in \operatorname{rad}(b_q)$ for $v \in \operatorname{rad}(q)$ (because x preserves length of vectors). Let $w_0 \in V$.

Since x is invertible, we have $x(w) = w_0$ for some $w \in V$. Then

$$\begin{aligned} b_q(x(v), w_0) &= q(x(v) + w_0) + q(x(v)) + q(w_0) \\ &= q(x(v) + x(w)) + q(x(v)) + q(x(w)) \\ &= q(x(v + w)) + q(x(v)) + q(x(w)) \\ &= q(v + w) + q(v) + q(w) = b_q(v, w) = 0, \end{aligned}$$

because $v \in \operatorname{rad}(q) \subset \operatorname{rad}(b_q)$. Thus, $x(v) \in \operatorname{rad}(q)$, as required.

It remains to see that $\bar{x} \in \mathrm{O}(\overline{V}, \bar{q})$. Indeed,

$$\bar{q}(\bar{x}(\bar{v})) = \bar{q}\big(\overline{x(v)}\big) = q(x(v)) = q(v) = \bar{q}(\bar{v}).$$

Thus, we have the following result:

Lemma 4.2. *The canonical map $V \to \overline{V}$ induces a natural morphism*

$$\lambda : \mathrm{O}(V, q) \longrightarrow \mathrm{O}(\overline{V}, \bar{q}).$$

5. Killing forms of simple Lie algebras over $\mathbb{Z}$

Let G be as in Theorem 2.1 and let $\tilde{G}$ be a universal simply connected covering of its connected component G°. To construct the required orthogonal representation ρ of G (see part (b) of our strategy described in Section 3) we need to know what the "normalized" Killing symmetric bilinear form $\mathcal{K}_b$ (and quadratic form $\mathcal{K}_q$) of the Lie algebra $\mathrm{Lie}(\tilde{G})$ look like.

Since our base field has characteristic 2, we begin by computing $\mathcal{K}_q$ in a Chevalley basis of the Lie algebra $\mathcal{L}$ of a split simple simply connected algebraic group defined over $\mathbb{Z}$. We then pass to k by first normalizing $\mathcal{K}_b$, i.e., by dividing all its coefficients by their gcd, and then applying the base change $\mathbb{Z} \to \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} \hookrightarrow k$.

Recall that a Chevalley basis is a canonical basis of $\mathcal{L}$ which arises from a decomposition of

$$\mathcal{L} = \mathcal{L}_0 \oplus \bigoplus_{\alpha \neq 0} \mathcal{L}_\alpha$$

into a direct sum of the weight subspaces $\mathcal{L}_\alpha$ with respect to a split maximal toral subalgebra $\mathcal{H} = \mathcal{L}_0 \subset \mathcal{L}$. Note that the set of all nontrivial weights in the above decomposition forms a simple root system and that for every root α we have $\dim(\mathcal{L}_\alpha) = 1$.

In what follows Φ will denote the set of all roots of $\mathcal{L}$ with respect to $\mathcal{H}$, $\Delta \subset \Phi$ its basis, and Φ^+ and Φ^- its positive and negative roots, respectively. It is known (see [Steinberg 1968]) that there exist elements $\{H_{\alpha_i} \mid \alpha_i \in \Delta\}$ in $\mathcal{H}$ and $X_\alpha \in \mathcal{L}_\alpha$, $\alpha \in \Phi$, such that the set

$$\{H_{\alpha_i} \mid \alpha_i \in \Delta\} \cup \{X_\alpha \mid \alpha \in \Phi^+\} \cup \{X_{-\alpha} \mid \alpha \in \Phi^+\} \tag{5.0.1}$$

forms a basis for $\mathcal{L}$, known as a *Chevalley basis*, and these generators are subject to the following relations:

- $[H_{\alpha_i}, H_{\alpha_j}] = 0$;
- $[H_{\alpha_i}, X_\alpha] = \langle \alpha, \alpha_i \rangle X_\alpha$;
- $H_\alpha := [X_\alpha, X_{-\alpha}] = \sum_{\alpha_i \in \Delta} n_i H_{\alpha_i}$, where $n_i \in \mathbb{Z}$;
- $[X_\alpha, X_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin \Phi, \\ \pm(p+1) X_{\alpha+\beta} & \text{otherwise,} \end{cases}$

where p is the greatest positive integer such that $\alpha - p\beta \in \Phi$. Here, for two roots $\alpha, \beta \in \Phi$, the scalar $\langle \alpha, \beta \rangle$ is given by

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)},$$

where $(\cdot\,, \cdot)$ denotes the standard inner product on the root lattice. It is in this Chevalley basis (5.0.1) that we will compute the Killing form $\mathcal{K}_q$ of $\mathcal{L}$.

Many people have addressed the computation of Killing forms (see, for example, [Gross and Nebe 2004; Malagon 2009; Seligman 1957; Springer and Steinberg 1970]), but we could not find in the literature explicit formulas valid in characteristic 2. Below we produce such formulas for the normalized Killing forms for each type with the use of the following known facts.

Recall that for any $X, Y \in \mathcal{L}$ one has

$$\mathcal{K}_b(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) \quad \text{and} \quad \mathcal{K}_q(X) = \mathcal{K}_b(X, X)$$

where $\mathrm{ad} : \mathcal{L} \to \mathrm{End}(\mathcal{L})$ is the adjoint representation of $\mathcal{L}$. It is straightforward to check that

$$\mathcal{K}_b(H_{\alpha_i}, X_\alpha) = 0 \quad \text{and} \quad \mathcal{K}_b(X_\alpha, X_\beta) = 0$$

for all i and for all roots $\alpha, \beta \in \Phi$ such that $\alpha + \beta \neq 0$; in particular, $\mathcal{K}_q(X_\alpha) = \mathcal{K}_b(X_\alpha, X_\alpha) = 0$. Thus, as a vector space $\mathcal{L}$ is decomposed into an orthogonal sum of its subspaces $\mathcal{H}$ and $\langle X_\alpha, X_{-\alpha} \rangle$, for $\alpha \in \Phi^+$.

It is shown in [Springer and Steinberg 1970] that, for any long root $\alpha \in \Phi$,

$$\mathcal{K}_b(H_\alpha, H_\alpha) = \mathrm{Tr}(\mathrm{ad}(H_\alpha) \circ \mathrm{ad}(H_\alpha)) = 4\check{h}, \tag{5.0.2}$$

where $\check{h}$ is the dual Coxeter number of the given Lie algebra. Also, for any root $\alpha \in \Phi$, we have

$$\mathcal{K}_b(X_\alpha, X_{-\alpha}) = \tfrac{1}{2}\,\mathrm{Tr}(\mathrm{ad}(H_\alpha) \circ \mathrm{ad}(H_\alpha)). \tag{5.0.3}$$

Lastly, we need one more result from [Malagon 2009]:

$$\mathcal{K}_b(H_{\alpha_i}, H_{\alpha_j}) = 2\check{h}(\check{\alpha}_i, \check{\alpha}_j), \quad \text{where } \check{\alpha}_i = \frac{2\alpha_i}{(\alpha_i, \alpha_i)} \tag{5.0.4}$$

and $(\check{\alpha}, \check{\beta})$ is the Weyl-invariant inner product such that $(\check{\alpha}, \check{\alpha}) = 2$ for a long root α. Note that the above formula requires $(\check{\alpha}, \check{\alpha}) = 2$ for a long root α, so that for groups of type C_n and G_2 we will have to multiply the standard inner product by an appropriate scalar to match this condition.

Combining the above mentioned results, we see that for computation of $\mathcal{K}_b$ we need to know only how $\mathcal{K}_b$ looks on the Cartan subalgebra $\mathcal{H}$. Indeed, (5.0.3) allows us to compute the restriction of $\mathcal{K}_b$ to each 2-dimensional subspace $\langle X_\alpha, X_{-\alpha}\rangle$. Furthermore, for each long root α we know by (5.0.2) that

$$\mathcal{K}_b(H_\alpha, H_\alpha) = 4\check{h}.$$

Similarly, by using (5.0.4) and the fact that the Killing form is W-invariant, where W is the corresponding Weyl group, we see that $\mathcal{K}_b(H_\beta, H_\beta)$ is a constant value for all short roots β, but this value will depend on the type of Φ. Finally, we remark that if $\alpha_i, \alpha_j \in \Delta \subset \Phi$ are nonadjacent roots, then

$$\mathcal{K}_b(H_{\alpha_i}, H_{\alpha_j}) = \mathrm{Tr}\big(\mathrm{ad}(H_{\alpha_i}) \circ \mathrm{ad}(H_{\alpha_j})\big) = 0.$$

Indeed, this is equivalent to saying that $(\alpha_i, \alpha_j) = 0$, which is true for nonadjacent roots.

Below we skip straightforward computations of $\mathcal{K}_b(H_{\alpha_i}, H_{\alpha_i})$ and $\mathcal{K}_b(H_{\alpha_i}, H_{\alpha_{i+1}})$ for each type and present the final result only.

5.1. *Type A_n.* We have:

$$\mathrm{Tr}(\mathrm{ad}(H_{\alpha_i}) \circ \mathrm{ad}(H_{\alpha_i})) = 4\check{h} \quad \text{and} \quad \mathrm{Tr}(\mathrm{ad}(H_{\alpha_i}) \circ \mathrm{ad}(H_{\alpha_{i+1}})) = -2\check{h}.$$

Thus, the Killing quadratic form $\mathcal{K}_q$ restricted to the Cartan subalgebra $\mathcal{H}$ of the Lie algebra $\mathcal{L}$ of type A_n is of the form

$$\mathcal{K}_q|_{\mathcal{H}} = 4\check{h}\Big(\sum_{i=1}^{n} x_i^2\Big) - 4\check{h}\Big(\sum_{i=1}^{n-1} x_i x_{i+1}\Big).$$

and the Killing form on all of $\mathcal{L}$ is

$$\mathcal{K}_q = \mathcal{K}_q|_{\mathcal{H}} + 4\check{h}\Big(\sum_{|\Phi^+|} y_i y_{i+1}\Big).$$

To pass to the main field k we first modify (normalize) $\mathcal{K}_q$ by dividing all coefficients of $\mathcal{K}_q$ by $4\check{h}$. After doing so, our modified Killing form (still denoted by $\mathcal{K}_q$) becomes

$$\mathcal{K}_q = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} + \sum_{|\Phi^+|} y_i y_{i+1}.$$

Passing from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$, which is a field of characteristic 2, we finally would like to "diagonalize" our form. Simple computations show that a diagonalization of $\mathcal{K}_q$ looks like

$$\mathcal{K}_q \simeq \begin{cases} \bigoplus_{i=1}^{(n-1)/2} [0,0] \oplus \langle 1\rangle \oplus \bigoplus_{|\Phi^+|} [0,0], & \text{if } n \equiv 1 \pmod 4; \\ \bigoplus_{i=1}^{(n-1)/2} [0,0] \oplus \langle 0\rangle \oplus \bigoplus_{|\Phi^+|} [0,0], & \text{if } n \equiv 3 \pmod 4; \\ \bigoplus_{i=1}^{(n-1)/2} [0,0] \oplus \bigoplus_{|\Phi^+|} [0,0], & \text{if } n \text{ is even.} \end{cases}$$

Similar arguments work for each type. Below we present the final result only.

5.2. ***Type B_n.***

$$\mathcal{K}_q \simeq \begin{cases} \bigoplus_{i=1}^{(n-2)/2} [0,0] \oplus \bigoplus_{|\Phi^+_{\text{long}}|} [0,0] \oplus \langle c\rangle \oplus m\langle 0\rangle, & \text{if } n \text{ is even, with } c \in \{0,1\} \\ \bigoplus_{i=1}^{(n-1)/2} [0,0] \oplus \bigoplus_{|\Phi^+_{\text{long}}|} [0,0] \oplus m\langle 0\rangle, & \text{if } n \text{ is odd,} \end{cases}$$

where $m = 2|\Phi^+_{\text{short}}| + 1$.

5.3. ***Type C_n.***

$$\mathcal{K}_q \simeq \langle 1\rangle \oplus \bigoplus_{|\Phi^+_{long}|} [0,0] \oplus m\langle 0\rangle$$

where $m = (n-1) + 2|\Phi^+_{\text{short}}|$.

5.4. ***Type D_n.***

$$\mathcal{K}_q \simeq \begin{cases} \bigoplus_{i=1}^{(n-1)/2} [0,0] \oplus \langle 0\rangle \oplus \bigoplus_{|\Phi^+|} [0,0], & \text{if } n \text{ is odd,} \\ \bigoplus_{i=1}^{(n-2)/2} [0,0] \oplus \langle c_1\rangle \oplus \langle c_2\rangle \oplus \bigoplus_{|\Phi^+|} [0,0], & \text{if } n \text{ is even, with } c_1, c_2 \in \{0,1\}, \end{cases}$$

where one of c_1 or c_2 equals 0.

5.5. ***Type E_6.***

$$\mathcal{K}_q \simeq [0,0] \oplus [0,0] \oplus [0,0] \oplus \bigoplus_{|\Phi^+|} [0,0].$$

5.6. ***Type E_7.***

$$\mathcal{K}_q \simeq [0,0] \oplus [0,0] \oplus [0,0] \oplus \langle 1\rangle \oplus \bigoplus_{|\Phi^+|} [0,0].$$

5.7. ***Type E_8.***

$$\mathcal{K}_q \simeq [0,0] \oplus [0,0] \oplus [0,0] \oplus [0,0] \oplus \bigoplus_{|\Phi^+|} [0,0].$$

5.8. ***Type F_4.***

$$\mathcal{K}_q \simeq [0,0] \oplus \bigoplus_{|\Phi^+_{\text{long}}|} [0,0] \oplus m\langle 0\rangle,$$

where $m = 2 + |\Phi^+_{\text{short}}|$.

5.9. ***Type G_2.***

$$\mathcal{K}_q \simeq [0,0] \oplus \bigoplus_{|\Phi^+|} [0,0].$$

6. An orthogonal representation

Proposition 6.1. *Let G° be a split simple adjoint algebraic group over k of one of the following types: A_r, B_r, C_r, D_r, E_6, E_7, E_8. Then, there exists a quadratic space (V,q) over k, and an orthogonal linear representation*

$$\rho : G^\circ \longrightarrow \mathrm{O}(V,q)$$

with the following property:

$$(*) \qquad \begin{cases} q \text{ is nondegenerate;} \\ \text{the nonzero weights of } T \text{ on } V \text{ are the short roots;} \\ \text{each nonzero weight occurs with multiplicity } 1. \end{cases}$$

Proof. We treat each root system individually.

Types A_n, D_n, E_6, E_7, E_8. Let $W = \mathrm{Lie}(\tilde{G})$. Then, the adjoint representation $\tilde{G} \to \mathrm{O}(W, \mathcal{K}_q)$ factors through $\tilde{G} \to G^\circ$, so it induces the representation $\mu : G^\circ \to \mathrm{O}(W, \mathcal{K}_q)$. Let ρ be the composition of μ and the map $\lambda : \mathrm{O}(W, \mathcal{K}_q) \to \mathrm{O}(\overline{W}, \overline{\mathcal{K}_q})$ constructed in Lemma 4.2, and let $V = \overline{W}$. Inspection of the normalized Killing form $\mathcal{K}_q$ presented in Section 5 shows that ρ has the required property.

Type B_r. We take V to be the standard representation of SO_{2r+1} of dimension $2r+1$.

Type C_r. The formula for $\mathcal{K}_q$ presented in Section 5.3 shows that the adjoint representation doesn't work. So, instead of the adjoint representation of $G = \mathrm{PSp}_{2r}$, we consider its representation on the exterior square.

More precisely, let V_1 be the standard representation of $\tilde{G} = \mathrm{Sp}_{2r}$ over $\mathbb{Z}$ equipped with a standard skew-symmetric bilinear form ω. Choose a standard basis $\{e_1, \ldots, e_r, e_{-r}, \ldots, e_{-1}\}$ of V_1. There is a natural embedding $\bigwedge^2(V_1) \to V_1 \otimes V_1$ given by $v \wedge w \to v \otimes w - w \otimes v$. We extend ω to a symmetric bilinear form on $V_1 \otimes V_1$ by

$$\omega(v_1 \otimes v_2, w_1 \otimes w_2) = \omega(v_1, w_1)\omega(v_2, w_2)$$

and take its restriction (still denoted by ω) to $V_2 = \bigwedge^2(V_1)$.

Consider the natural action of G on V_2. This action preserves ω, and thus we have a natural representation $G \to \mathrm{O}(V_2, \omega)$. Let $q_2(x) = \omega(x, x)$ be the quadratic form on V_2 corresponding to ω. Denote $v_i = e_i \wedge e_{-i}$. Also if $i < j$ let $v_{ij} = e_i \wedge e_j$, $w_{ij} = e_{-i} \wedge e_{-j}$ and $u_{ij} = e_i \wedge e_{-j}$ for all $i \neq j$. It is straightforward to check that the subspaces $\langle v_i \rangle$, $\langle v_{ij}, w_{ij} \rangle$, $\langle u_{ij}, u_{ji} \rangle$ of V_2 are orthogonal to each other and that q_2 written in the bases $v_i, v_{ij}, u_{ij}, w_{ij}$ of V_2 is of the form

$$q_2 = 2\left(\sum x_i^2\right) \oplus 4\left(\sum y_{ij} z_{ij}\right).$$

Note that by dividing all coefficients of q_2 by 2 and passing from $\mathbb{Z}$ to $\mathbb{Z}/2$ we don't achieve our goal since the resulting quadratic form is "highly degenerate". So instead of considering the representation of G on V_2 we do the following. One can easily check that any (hyperplane) reflection $\tau : V_1 \to V_1$ acts trivially on a 1-dimensional subspace of V_2 spanned by $v = v_1 + \cdots + v_r$. It follows that Sp_{2r} acts trivially on $\langle v \rangle$ and hence so does G. This implies that G acts on the orthogonal complement $V = \langle v \rangle^{\perp}$ (with respect to ω). This subspace is spanned by linearly independent vectors $v_1 - v_2, v_2 - v_3, \ldots, v_{r-1} - v_r, v_{ij}, u_{ij}, w_{ij}$. In this basis of V, the restriction q of q_2 to V is of the form

$$q = 4\left(\sum x_i^2 - \sum x_i x_{i+1}\right) \oplus 4\left(\sum y_{ij} z_{ij}\right).$$

By dividing all coefficients of q by 4 and applying the base change $\mathbb{Z} \to \mathbb{Z}/2 \subset k$, we obtain an orthogonal representation of G over k with the required property. $\square$

7. The Witt group in characteristic 2

In this section we summarize Arason's results [2006b; 2006a] on the structure of the Witt group of quadratic forms over complete fields of characteristic 2 used in our present work.

Let K be a field of characteristic 2, π an indeterminate over K, and let $K((\pi))$ be the field of formal Laurent series with coefficients in K. If f is a nondegenerate quadratic form over $K((\pi))$ of even dimension, we will denote its image in the Witt group $W_q(K((\pi)))$ by f_W.

Theorem 7.1. *The Witt group $W_q(K((\pi)))$ is the additive group generated by the elements $[\alpha, \beta\pi^{-m}]_W$ and $[\alpha\pi^{-1}, \beta\pi^{-m+1}]_W$, where $m \in \mathbb{Z}$, $m \geq 0$, and $\alpha, \beta \in K$, with the condition that $[\alpha, \beta\pi^{-m}]_W$ and $[\alpha\pi^{-1}, \beta\pi^{-m+1}]_W$ are biadditive as functions of α, β and satisfy the following sets of relations*:

(7.1.1a) $$[\alpha, \beta\rho^2\pi^{-m}]_W + [\beta, \alpha\rho^2\pi^{-m}]_W = 0 \qquad \textit{if } m \textit{ is even},$$

(7.1.1b) $$[\alpha\pi^{-1}, \beta\rho^2\pi^{-m+1}]_W + [\beta\pi^{-1}, \alpha\rho^2\pi^{-m+1}]_W = 0 \qquad \textit{if } m \textit{ is even},$$

(7.1.1c) $$[\alpha, \beta\rho^2\pi^{-m}]_W + [\beta\pi^{-1}, \alpha\rho^2\pi^{-m+1}]_W = 0 \qquad \textit{if } m \textit{ is odd},$$

and

$$[\alpha, \alpha\rho^2\pi^{-2m}]_W + [\alpha, \rho\pi^{-m}]_W = 0, \tag{7.1.2a}$$

$$[\alpha\pi^{-1}, \alpha\rho^2\pi^{-2m+1}]_W + [\alpha\pi^{-1}, \rho\pi^{-m+1}]_W = 0. \tag{7.1.2b}$$

Here m runs through the nonnegative integers and α, β, and ρ run through K.

Theorem 7.2. *Let $m \geq 0$ and let $W_q(K((\pi)))_m$ be the subgroup of $W_q(K((\pi)))$ generated by $[\alpha, \beta\pi^{-i}]_W$ and $[\alpha\pi^{-1}, \beta\pi^{-i+1}]_W$, where $i \in \mathbb{Z}, 0 \leq i \leq m$ and $\alpha, \beta \in K$. Then*:

(a) *$W_q(K((\pi)))_0$ is isomorphic to $W_q(K) \oplus W_q(K)$. A generator $[\alpha, \beta]_W$ of $W_q(K((\pi)))_0$ is sent to $[\alpha, \beta]_W$ in the first summand $W_q(K)$, but a generator $[\alpha\pi^{-1}, \beta s]_W$ corresponds to $[\alpha, \beta]_W$ in the second summand.*

(b) *If $n > 0$, then $W_q(K((\pi)))_{2n}/W_q(K((\pi)))_{2n-1}$ is isomorphic to $K \wedge_{K^2} K \oplus K \wedge_{K^2} K$. The class of a generator $[\alpha, \beta\pi^{-2n}]_W$ corresponds to $\alpha \wedge \beta$ in the first summand, but the class of a generator $[\alpha\pi^{-1}, \beta\pi^{-2n+1}]_W$ corresponds to $\alpha \wedge \beta$ in the second summand.*

(c) *If $n \geq 0$, then $W_q(K((\pi)))_{2n+1}/W_q(K((\pi)))_{2n}$ is isomorphic to $K \otimes_{K^2} K$. The class of a generator $[\alpha, \beta\pi^{-2n+1}]_W$ corresponds to $\alpha \otimes \beta$, but the class of a generator $[\alpha\pi^{-1}, \beta\pi^{-2n}]_W$ corresponds to $\beta \otimes \alpha$.*

The filtration

$$W_q(K((\pi)))_0 \subset W_q(K((\pi)))_1 \subset W_q(K((\pi)))_2 \subset \cdots$$

of the group $W_q(K((\pi)))$ will be called Arason's filtration. Note that by the above theorem,

$$W_q(K((\pi)))_0 \simeq W_q(K) \oplus W_q(K),$$

so that we have two natural projections:

$$\partial_1 : W_q(K((\pi)))_0 \to W_q(K) \quad \text{and} \quad \partial_2 : W_q(K((\pi)))_0 \to W_q(K),$$

which we will call the *first and second residues* (of the zero term of Arason's filtration).

Using the fact that $[f, g] \simeq \mathbb{H}$ for all $f, g \in K((\pi))$ such that $fg \in \pi K[[\pi]]$, it is straightforward to show that the zero term of the Witt group of Arason's filtration and the first residue don't depend on the presentation $L = K((\pi))$. In other words, they don't depend on a choice of a coefficient field $\tilde{K} \subset L$ (for the notion of coefficient fields we refer to Section 9 below) nor of a choice of a uniformizer of L, and the second residue is defined up to similarity only. We leave the details of the verification to the reader.

8. Presentation of quadratic forms in the Witt group

In this section we will work with the Witt group of quadratic forms over a field of Laurent series $K((\pi))$, where the coefficient field K is of characteristic 2 and is finitely generated over k. By Theorems 7.1 and 7.2, given a nondegenerate quadratic form f defined over $K((\pi))$, we may decompose its image f_W in the Witt group as

$$f_W = f'_{m,W} + f'_{m-1,W} + \cdots + f'_{0,W} \tag{8.0.1}$$

where $f'_{i,W} \in W_q(K((\pi)))_i$ is homogeneous of degree i, i.e., a sum of elements of the form $[\alpha, \beta\pi^{-i}]$ and $[\alpha\pi^{-1}, \beta\pi^{-i+1}]$ with $\alpha, \beta \in K$. Such a decomposition is not unique. The following lemma allows us to choose the homogeneous components of f_W in a canonical way.

Lemma 8.1. *Let $\{\alpha_i\}_{i=1}^N$ be a basis for K as a K^2-vector space and let f be a nondegenerate quadratic form over $K((\pi))$. Then, f_W admits a decomposition $f_W = f_{m,W} + f_{m-1,W} + \cdots + f_{0,W}$ satisfying these conditions*:

- *If n is even, then*
$$f_{n,W} = \sum_{i<j} [\alpha_i, u_j^2\alpha_j\pi^{-n}]_W + \sum_{i<j} [\alpha_i\pi^{-1}, v_j^2\alpha_j\pi^{-n+1}]_W,$$
where $u_j, v_j \in K$.
- *If n is odd, then*
$$f_{n,W} = \sum_{i,j=1}^N [\alpha_i, u_j^2\alpha_j\pi^{-n}]_W,$$
where $u_j \in K$.

Proof. Take decomposition (8.0.1). Suppose first that $n = 2s$ is even. Write $f'_{2s,W}$ in the form

$$f'_{2s,W} = \sum [p_i, q_i\pi^{-2s}]_W + \sum [p'_i\pi^{-1}, q'_i\pi^{-2s+1}]_W,$$

where $p_i, q_i, p'_i, q'_i \in K$. Since $\{\alpha_i\}_{i=1}^N$ is a basis for K/K^2, one has

$$p_i = \sum_{i,j=1}^N e_{ij}^2\alpha_j,$$

where $e_{i,j} \in K$ and similarly for the q_i, p'_i, q'_i. Replacing the p_i, q_i, p'_i, q'_i with these expressions and using the biadditivity of $[\,\cdot\,,\cdot\,]_W$ and the fact that $[uv^2, w] = [u, v^2 w]$ for all $u, v, w \in K((\pi))$, we can write $f'_{2s,W}$ in the form

$$\begin{aligned} f'_{2s,W} &= \sum_{i,j=1}^{N} [u_i^2\alpha_i, v_j^2\alpha_j\pi^{-2s}]_W + \sum_{i,j=1}^{N} [u_i'^2\alpha_i\pi^{-1}, v_j'^2\alpha_j\pi^{-2s+1}]_W \\ &= \sum_{i,j=1}^{N} [\alpha_i, w_{ij}^2\alpha_j\pi^{-2s}]_W + \sum_{i,j=1}^{N} [\alpha_i\pi^{-1}, w_{ij}'^2\alpha_j\pi^{-2s+1}]_W, \end{aligned}$$

where $u_i, v_j, u_i', v_j' \in K$ and $w_{ij} = u_i v_j$, $w_{ij}' = u_i' v_j'$. If $i = j$, we have

$$[\alpha_i, w_{ii}^2\alpha_i\pi^{-2s}]_W \overset{(7.1.2a)}{=} [\alpha_i, w_{ii}\pi^{-s}]_W$$

and

$$[\alpha_i\pi^{-1}, w_{ii}'^2\alpha_i\pi^{-2s+1}]_W \overset{(7.1.2b)}{=} [\alpha_i\pi^{-1}, w_{ii}'\pi^{-s+1}]_W.$$

If $i > j$ we get

$$[\alpha_i, w_{ij}^2\alpha_j\pi^{-2s}]_W \overset{(7.1.1a)}{=} [\alpha_j, w_{ij}^2\alpha_i\pi^{-2s}]_W$$

and

$$[\alpha_i\pi^{-1}, w_{ij}'^2\alpha_j\pi^{-2s+1}]_W \overset{(7.1.1b)}{=} [\alpha_j\pi^{-1}, w_{ij}'^2\alpha_i\pi^{-2s+1}]_W.$$

If $n = 2s - 1$ is odd, similar arguments show that $f'_{2s-1,W}$ can be written as a sum of symbols of the form $[\alpha_i, u^2\alpha_j\pi^{-2s+1}]_W$ where $u \in K$. Collecting all summands in the above decompositions of all $f'_{2s,W}$ and $f'_{2s-1,W}$ of the same degree together, we obtain the required decomposition of f_W. □

The following proposition shows the decomposition above is unique.

Proposition 8.2. *Given a quadratic form f, its image in the Witt group can be decomposed uniquely as $f_W = f_{m,W} + f_{m-1,W} + \cdots + f_{0,W}$, where $f_{m,W}, \ldots, f_{0,W}$ are as in Lemma 8.1.*

Proof. We already know that a decomposition exists, so we only need to prove uniqueness. Suppose

$$f_W = f_{m,W} + f_{m-1,W} + \cdots + f_{0,W} = g_{n,W} + g_{n-1,W} + \cdots + g_{0,W}$$

are two different decompositions of f_W. We first claim that $n = m$. Suppose not, say $m > n$. Let us compare the images of these decompositions in the quotient group $W_q(K((s)))_m / W_q(K((s)))_{m-1}$. Since $n < m$, the image of $g_{n,W} + g_{n-1,W} + \cdots + g_{0,W}$ equals 0 whereas the other decomposition has image the class of $f_{m,W}$. We consider separately the cases when m is even and odd.

When m is even: By Lemma 8.1, write

$$f_{m,W} = \sum_{i<j} [\alpha_i, u_j^2\alpha_j s^{-m}]_W + \sum_{i<j} [\alpha_i s^{-1}, v_j^2\alpha_j s^{-m+1}]_W,$$

and by Theorem 7.2,

$$\phi : W_q(K((s)))_m / W_q(K((s)))_{m-1} \simeq K \wedge_{K^2} K \oplus K \wedge_{K^2} K$$

be the canonical isomorphism. Then,

$$\phi(f_{m,W}) = \Big(\sum_{i<j} u_j^2 (\alpha_i \wedge \alpha_j),\ \sum_{i<j} v_j^2 (\alpha_i \wedge \alpha_j) \Big).$$

Since $\{\alpha_i \wedge \alpha_j\}_{i<j}$ is a basis for $K \wedge_{K^2} K$,

$$\phi(f_{m,W}) = 0 \iff u_j^2 = v_j^2 = 0 \quad \text{for all } j.$$

This would imply that $f_{m,W} = 0$, a contradiction.

<u>When m is odd</u>: By Lemma 8.1, write

$$f_{m,W} = \sum_{i,j=1}^{N} [\alpha_i, u_j^2 \alpha_j s^{-m}]_W$$

and by Theorem 7.2,

$$\phi : W_q(K((s)))_n / W_q(K((s)))_{n-1} \simeq K \otimes_{K^2} K.$$

Then,

$$\phi(f_{m,W}) = \sum_{i,j=1}^{N} u_j^2 (\alpha_i \otimes \alpha_j).$$

Since $\{\alpha_i \otimes \alpha_j\}_{i,j=1}^{N}$ is a basis for $K \otimes_{K^2} K$,

$$\phi(f_{m,W}) = 0 \iff u_j^2 = 0 \quad \text{for all } j,$$

a contradiction.

Thus $m = n$. If m is even, we conclude from $\phi(f_{m,W}) = \phi(g_{m,W})$ that

$$\sum u_j^2 (\alpha_i \wedge \alpha_j) = \sum u_j'^2 (\alpha_i \wedge \alpha_j),$$

where $u_j'^2$ are the corresponding coefficients of $g_{m,W}$, and similarly,

$$\sum v_j^2 (\alpha_i \wedge \alpha_j) = \sum v_j'^2 (\alpha_i \wedge \alpha_j).$$

This implies that $u_j^2 = u_j'^2$ and $v_j^2 = v_j'^2$, hence $f_{m,W} = g_{m,W}$. Similarly, we can easily see that $f_{m,W} = g_{m,W}$ if m is odd. Then, from the equality

$$(f_{0,W} + \cdots + f_{m-1,W}) + f_{m,W} = (g_{0,W} + \cdots + g_{m-1,W}) + g_{m,W},$$

it follows that

$$f_{0,W} + \cdots + f_{m-1,W} = f'_{0,W} + \cdots + f'_{m-1,W}.$$

By induction, the proof is completed. □

The same arguments as in the proofs of Lemma 8.1 and Proposition 8.2 lead us to the following result needed later to establish incompressibility of the so-called canonical monomial quadratic forms.

Corollary 8.3. *Let f_W be as in (8.0.1), and assume that $f_W \in W_q(K((\pi)))_0$. Then, $f'_{m,W} + f'_{m-1,W} + \cdots + f'_{1,W} = 0$.*

9. Differential bases, 2-bases, the Cohen structure theorem, and coefficient fields

Let K/k be a finitely generated field extension. Recall that $\Omega_{K/k}$ denotes the K-vector space of Kähler differentials. A differential basis for K/k is a set of elements $\{\alpha_i\}_{i\in I}$ of K such that $\{d\alpha_i\} \subset \Omega_{K/k}$ is a vector space basis. Recall also that a set of elements $\{x_\lambda\}_{\lambda\in\Lambda}$ of K is a 2-basis for K over k if the set W of monomials in the x_λ having degree < 2 in each x_λ separately forms a vector space basis for K over its subfield $k \cdot K^2 = K^2 \subset K$. The following facts are well known.

Theorem 9.1. *Let $B = \{x_1, \dots, x_n\} \subset K$ be a subset. The following are equivalent*:

(a) *B is a separating transcendence basis for K over k.*

(b) *B is a 2-basis for K over k.*

(c) *B is a differential basis for K/k.*

Proof. See [Eisenbud 1995, Theorem 16.14]. □

Assume now that K is equipped with a discrete valuation, trivial on k. We denote its valuation ring by R and the residue field by $\overline{K}$. Since our valuation is trivial on k the residue field $\overline{K}$ contains a copy of k. Throughout we assume that v is geometric of rank 1 (for the definition of geometric valuations, see [Merkurjev 2008]), i.e., $\operatorname{tr.deg}_k K = \operatorname{tr.deg}_k \overline{K} + 1$.

Let π be a uniformizer and $I = (\pi) \subset R$ be the corresponding maximal ideal in R. Choose $a_1, \dots, a_n \in R$ such that their images $\bar{a}_1, \dots, \bar{a}_n$ under the canonical map $R \to \overline{K}$ form a differential basis for $\overline{K}/k$. Note that, by Theorem 9.1, we have $\operatorname{tr.deg}_k(\overline{K}) = n$, hence $\operatorname{tr.deg}_k(K) = n + 1$. We now claim that

$$B = \{a_1, \dots, a_n, \pi\} \tag{9.1.1}$$

is a differential basis for K/k.

Indeed, it suffices to see that $da_1, \dots, da_n, d\pi$ is a system of generators of $\Omega_{K/k}$ (because the K-vector space $\Omega_{K/k}$ has dimension $n+1$). For that in turn, it suffices to show that this is a system of generators for $\Omega_{R/k}$, for formation of differentials commutes with localization. But this easily follows from the conormal sequence

$$I/I^2 \xrightarrow{d} \overline{K} \otimes_R \Omega_{R/k} \xrightarrow{D\phi} \Omega_{\overline{K}/k} \longrightarrow 0$$

(see [Eisenbud 1995, p. 389]) and Nakayama's lemma. Thus, B is a differential basis for K/k.

We will say that a differential basis $\{a_1, a_2, \dots, a_{n+1}\}$ for K/k *comes from* $\overline{K}$ if there exists a subscript $i \in \{1, 2, \dots, n+1\}$ such that a_i is a uniformizer in K, $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1}$ are units in R, and the images of these elements in $\overline{K}$ form a differential basis for $\overline{K}/k$.

Now, let R be a complete discrete valuation ring containing a field k. Denote its quotient field by L and residue field by $\bar{L}$. We will assume throughout that the field extension $\bar{L}/k$ is finitely generated. It follows from the Cohen structure theorem [Eisenbud 1995, Theorem 7.7] that $R \simeq \bar{L}[\![\pi]\!]$ and $L \simeq \bar{L}((\pi))$, where π is a uniformizer. Such decompositions are not unique. They depend on a choice of π and a choice of a coefficient field in L, i.e., a subfield of L contained in R that maps isomorphically onto $\bar{L}$ under the canonical map $R \to \bar{L}$. Such coefficient fields do exist because the field extension $\bar{L}/k$ is separable. The following theorem describe all coefficient fields.

Theorem 9.2. *Let R be as above. If B is a differential basis for $\bar{L}/k$ then there is one-to-one correspondence between coefficient fields $\tilde{E} \subset R$ containing k and the set $\tilde{B} \subset R$ of representatives for B obtained by associating to each $\tilde{E}$ the set $\tilde{B}$ of representatives for B that it contains.*

Proof. See [Eisenbud 1995, Theorem 7.8]. □

10. Monomial quadratic forms

Let $K = k(t_1, t_2, \dots, t_n, x)$ be a pure transcendental extension of k of transcendence degree $n+1$. We say that a nondegenerate quadratic form f over K is *monomial* if it is of the form

$$f = \bigoplus_{\mu \in \mathbb{F}_2^n} m_f(\mu) t^{\mu}[1, x] \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H},$$

where $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{F}_2^n$, the $t^{\mu} = t_1^{\mu_1} t_2^{\mu_2} \cdots t_n^{\mu_n}$ are monomials in $t_1, \dots, t_n$, and $m_f(\mu)$ is the number of times a given summand appears. Note that the multiplicity $m_f(\mu)$ may be 0. Since

$$t^{\mu}[1, x] \oplus t^{\mu}[1, x] \simeq \mathbb{H} \oplus \mathbb{H},$$

we may assume without loss of generality that $m_f(\mu) = 0$ or $m_f(\mu) = 1$.

Let V be the vector subspace of $\mathbb{F}_2^n$ generated by all μ such that $m_f(\mu) = 1$. Choose a basis of V, say $\mu_1, \mu_2, \dots, \mu_s$. Then, define $u_i = t^{\mu_i}$ for $i = 1, \dots, s$. It is easy to see that $u_1, \dots, u_s$ are algebraically independent over k. Furthermore, any $\mu \in V$ can be written as $\mu = \sum_{i=1}^{s} \alpha_i u_i$, where $\alpha_i = 0$ or $\alpha_i = 1$, so that $t^{\mu} = u_1^{\alpha_1} \cdots u_s^{\alpha_s}$.

Thus, f has descent to the subfield $K' = k(u_1, \dots, u_s, x) \subset K$, and viewed over K', it is of the form

$$f = u_1[1, x] \oplus u_2[1, x] \oplus \cdots \oplus u_s[1, x] \oplus \left(\bigoplus_{\mu \in V} m_f(\mu) u^\mu [1, x] \right) \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H}$$

where u^μ are monomials in $u_1, \dots, u_s$ of length at least 2. When a monomial quadratic form f is written in such a way and is viewed over K', we say that it is a *canonical monomial form*. We also say that f has *rank* s.

For later use we need the following easy observation.

Proposition 10.1. *If f is a canonical monomial form without summands isometric to the hyperbolic plane $\mathbb{H}$, then f is anisotropic.*

Proof. The argument is similar to that in [Chernousov and Serre 2006, Proposition 5], and we leave the details to the reader. □

Our main result related to canonical monomial quadratic forms is the following.

Theorem 10.2. *If f is a canonical monomial form over K, then f is incompressible.*

The proof of this theorem will be given in Section 12.

11. Incompressibility of canonical monomial forms in "codimension 2"

In this section we establish an auxiliary result, Theorem 11.3 below, needed later on to prove Theorem 10.2. Let $K = k(x, t_1, \dots, t_n)$ be a pure transcendental extension of k of degree $n + 1$ and v the discrete valuation on K associated to t_1. It is characterized by:

$$v(t_1) = 1 \quad \text{and} \quad v(h) = 0 \text{ for all } h \in k(x, t_2, \dots, t_n)^\times.$$

Let $R \subset K$ be the corresponding valuation ring. Note that $K^2 \subset K$ is a finite field extension of degree 2^{n+1}. As usual, $K^2(a_{i_1}, \dots, a_{i_l}) \subset K$ denotes the subfield generated by K^2 and elements $a_{i_1}, \dots, a_{i_l} \in K$.

Proposition 11.1. *Let $F \subset K$ be a subfield containing k such that* $\operatorname{tr.deg}_k(F) < n + 1$. *Then there exists a differential basis $\{a_1, \dots, a_{n+1}\}$ for K/k coming from $\overline{K}$ such that $F \subset K^2(a_1, \dots, a_l)$ with $l \le$* $\operatorname{tr.deg}_k(F) < n + 1$.

Proof. Choose any 2-basis $\{b_1, \dots, b_s\}$ for F/k. By hypothesis, $s = \operatorname{tr.deg}_k(F) < n+1$. Let $L = K^2(b_1, \dots, b_s)$. Clearly, L contains F (because $F = F^2(b_1, \dots, b_s)$). Without loss of generality, we may assume that $L = K^2(b_1, \dots, b_l)$ where $l \le s$ and the set of all monomials $b_1^{\epsilon_1} \cdots b_l^{\epsilon_l}$ with $\epsilon_i = 0$ or $\epsilon_i = 1$ is linearly independent over K^2.

Let us first assume that L contains a uniformizer of v. Modifying the set $b_1, \dots, b_l$ of generators of L, if necessary, without loss of generality, we may assume that b_1 is a uniformizer of v. Let $E_0 = K^2 \subset E_1 \subset \cdots \subset E_{n+1} = K$ be a chain of quadratic extensions such that

$$E_1 = K^2(b_1),\, E_2 = K^2(b_1, b_2),\, \ldots,\, E_l = E_{l-1}(b_l) = K^2(b_1, \ldots, b_l).$$

Passing to the residues we have the chain

$$\overline{K}^2 = \bar{E}_1 \subset \bar{E}_2 \subset \cdots \subset \bar{E}_{n+1} = \overline{K}.$$

Since $[E_i : E_{i-1}] = 2$, we have $[\bar{E}_i : \bar{E}_{i-1}] \leq 2$. On the other hand, since v is a geometric valuation of rank 1 we have $[\overline{K} : \overline{K}^2] = 2^n$. It follows that $[\bar{E}_i : \bar{E}_{i-1}] = 2$ for every $2 \leq i \leq n$.

Now choose elements $\bar{a}_i \in \bar{E}_i \setminus \bar{E}_{i-1}$, $i \geq 2$. They form a 2-basis of $\overline{K}$ over k. Take any lifting a_i of $\bar{a}_i$ in E_i. Then the set $\{a_1, a_2, \ldots, a_{n+1}\}$, where $a_1 = b_1$, is a 2-basis of K over k, and hence a differential basis for K/k coming from $\overline{K}$, and it has the required property.

The case when L doesn't contain a uniformizer of v can be treated similarly. □

Let f be a canonical monomial quadratic form over K given by

$$f = \bigoplus_{\mu \in \mathbb{F}_2^n} m_f(\mu) t^{\mu}[1, x] \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H}, \tag{11.1.1}$$

where all multiplicities $m_f(\mu)$ are 1 or 0. Since f is canonical, it contains summands $t_i[1, x]$ for $i = 1, \ldots, n$.

Below we will be considering two Witt groups: $W_q(K)$ and $W_q(\hat{K})$. Here $\hat{K} \simeq k(x, t_2, \ldots, t_n)((t_1))$. There exists a natural map $W_q(K) \to W_q(\hat{K})$, and if there is no risk of confusion we will denote the image of f in both groups by f_W.

Lemma 11.2. *The form $(f_{\hat{K}})_W$ lives in $W_q(\hat{K})_0$. Its first residue is a canonical monomial form of rank $n-1$, and its second residue up to similarity is a nontrivial monomial form of rank $\leq n-1$.*

Proof. This follows from the definitions of monomial forms and the first and second residues. □

Theorem 11.3. *There exist no differential basis $B = \{a_1, \ldots, a_{n+1}\}$ for K/k and a nondegenerate quadratic form g defined over $L = K^2(a_1, \ldots, a_{n-1})$ such that $g_{K,W} = f_W$.*

Proof. Assume the contrary. Let B and g be the corresponding differential basis and quadratic form. Arguing as in Proposition 11.1, we may additionally assume that B comes from $\overline{K}$. Then, it gives rise to the coefficient field $E \subset \hat{K}$ containing all units from the set B and a presentation $\hat{K} \simeq E((t_1))$. We argue by induction on n.

Case 1: Let $a_1, \ldots, a_{n-1}$ be units in R. Since $L = K^2(a_1, \ldots, a_{n-1})$, our quadratic form g can be written in $W_q(K)$ as a sum of 2-dimensional quadratic forms of the shape

$$[a_1^{\nu_1}\cdots a_{n-1}^{\nu_{n-1}},\, u_\epsilon^2 a_1^{\epsilon_1}\cdots a_{n-1}^{\epsilon_{n-1}}] \tag{11.3.1}$$

where $\nu_i, \epsilon_i \in \{0, 1\}$, $u_\epsilon \in K$, and we use multi-index notation $\epsilon = (\epsilon_1, \ldots, \epsilon_{n-1})$. We now pass to $\hat{K} = E((t_1))$ and view g over $\hat{K}$. Writing u_ϵ in the form

$$u_\epsilon = t_1^{-s_i}\sum_{j\geq 0} e_{\epsilon j} t_1^j$$

with $e_{\epsilon j} \in E$ and using the fact that $[\alpha, \beta]_W = 0$ if $\alpha, \beta \in \hat{K}$ with $v(\alpha\beta) > 0$, we conclude that (11.3.1) can be written as a sum of symbols of the form

$$[a_1^{\nu_1}\cdots a_{n-1}^{\nu_{n-1}},\, e_{\epsilon j}^2 t_1^{-2j} a_1^{\epsilon_1}\cdots a_{n-1}^{\epsilon_{n-1}}]$$

with $e_{\epsilon j} \in E$.

Thus, $g_{\hat{K},W}$ can be written as $g_{\hat{K},W} = g_{n,W} + \cdots + g_{0,W}$ where $g_{j,W}$ are homogeneous of the form

$$g_{j,W} = \sum_{\nu,\epsilon} [a_1^{\nu_1}\cdots a_{n-1}^{\nu_{n-1}},\, e_{\epsilon j}^2 t_1^{-2j} a_1^{\epsilon_1}\cdots a_{n-1}^{\epsilon_{n-1}}]_W$$

with $e_{\epsilon j} \in E$. Since $g_{\hat{K},W} = f_W$, it lives in the zero term of Arason's filtration of $W_q(\hat{K})$, where the filtration is viewed in the presentation $\hat{K} \simeq E((t_1))$. Since $a_1, \ldots, a_{n-1} \in E$, by Corollary 8.3 we conclude that $g_n + \cdots + g_1 = 0$. Therefore,

$$f_W = g_{\hat{K},W} = g_{0,W} = \sum_{\nu,\epsilon} [a_1^{\nu_1}\cdots a_{n-1}^{\nu_{n-1}},\, e_{\epsilon 0}^2 a_1^{\epsilon_1}\cdots a_{n-1}^{\epsilon_{n-1}}]_W.$$

But this implies that the second residue of f_W is 0, which contradicts the second assertion in Lemma 11.2.

<u>Case 2</u>: Assume that $a_1, \ldots, a_{n-2}$ are units in R and a_{n-1} is a uniformizer for v. Our arguments below don't depend on the choice of a uniformizer, so by abusing notation we will assume that $a_{n-1} = t_1$. Arguing as above we find that g_W can be written as a sum of symbols of the form

$$[a_1^{\nu_1}\ldots a_{n-2}^{\nu_{n-2}} t_1^{\nu_{n-1}},\, u_\epsilon^2 a_1^{\epsilon_1}\ldots a_{n-2}^{\epsilon_{n-2}} t_1^{\epsilon_{n-1}}],$$

where $\nu_i, \epsilon_i \in \{0, 1\}$ and $u_\epsilon \in K$. Furthermore, by passing to $\hat{K} = E((t_1))$ and taking expansions of u_ϵ, we can write $g_{\hat{K},W}$ as a sum

$$g_{\hat{K},W} = g_{n,W} + \cdots + g_{1,W} + g_{0,W},$$

where each homogeneous component $g_{i,W}$ is of the form

$$g_{i,W} = \sum_{j,\nu,\epsilon} [a_1^{\nu_1} \cdots a_{n-2}^{\nu_{n-2}},\ \alpha_{\epsilon j}^2 a_1^{\epsilon_1} \cdots a_{n-2}^{\epsilon_{n-2}} t_1^{-i}]_W \\ + \sum_{j,\nu',\epsilon'} ([a_1^{\nu'_1} \cdots a_{n-2}^{\nu'_{n-2}} t_1^{-1},\ \beta_{\epsilon' j}^2 a_1^{\epsilon'_1} \cdots a_{n-2}^{\epsilon'_{n-2}} t_1^{-i+1}]_W,$$

with $\alpha_{\epsilon j}, \beta_{\epsilon' j} \in E$.

Since $g_{\hat{K},W}$ lives in the zero term of Arason's filtration, as above, application of Corollary 8.3 yields $g_n + \cdots + g_1 = 0$. Thus, $g_{\hat{K},W} = g_{0,W}$ is homogeneous of degree 0, where the component $g_{0,W}$ is a sum of symbols of the form

$$[a_1^{\nu'_1} \cdots a_{n-2}^{\nu'_{n-2}} t_1,\ \alpha_{\epsilon' 0}^2 a_1^{\epsilon'_1} \cdots a_{n-2}^{\epsilon'_{n-2}} t_1^{-1}]_W \quad \text{and} \quad [a_1^{\nu_1} \cdots a_{n-2}^{\nu_{n-2}},\ \beta_{\epsilon 0}^2 a_1^{\epsilon_1} \cdots a_{n-2}^{\epsilon_{n-2}}]_W,$$

with $\alpha_{\epsilon' 0}, \beta_{\epsilon 0} \in E$. Then, the first residue of $g_{\hat{K},W}$ (and hence of f_W) is a sum of symbols

$$[a_1^{\nu_1} \cdots a_{n-2}^{\nu_{n-2}},\ \beta_{\epsilon 0}^2 a_1^{\epsilon_1} \cdots a_{n-2}^{\epsilon'_{n-2}}]_W,$$

where $\beta_{\epsilon 0} \in E \simeq \overline{K}$.

Now recall that by construction, $B = \{a_1, \ldots, a_{n-2}, a_n, a_{n+1}\}$ is a differential basis for $E \simeq \overline{K}$ over k and that the first residue of $f_{\hat{K},W}$ is a canonical monomial form of rank $n-1$. On the other side, as we have seen above, it comes from the subfield $E^2(a_1, \ldots, a_{n-2}) \subset E$. This contradicts the induction assumption. $\square$

Corollary 11.4. *The quadratic form f does not descend to a subfield $k \subset F \subset K$ of transcendence degree $\leq n-1$.*

Proof. This follows from Proposition 11.1 and Theorem 11.3. $\square$

12. Incompressibility of canonical monomial quadratic forms

Proof of Theorem 10.2. We continue to keep the above notation. In particular, $K = k(x, t_1, t_2, \ldots, t_n)$ is a pure transcendental extension of k of transcendence degree $n+1$, equipped with the discrete valuation v associated to t_1 and R the corresponding discrete valuation ring. As a matter of notation we denote $\pi = t_1$ and $K_1 = k(t_2, \ldots, t_n, x)$. Thus, $\hat{K} \simeq K_1((\pi))$ and $\overline{K} \simeq K_1$.

Consider a canonical monomial quadratic form f over K given by (11.1.1). The proof of incompressibility of f will be carried out by induction on rank n. More precisely, we will prove by induction on n that the image f_W of f in $W_q(K)$ is incompressible. Of course, this would imply incompressibility of f itself. The base of induction $n = 0$ is obvious.

Lemma 12.1. *If $K = k(x)$ and $f = [1, x] \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H}$, then f_W is incompressible.*

Proof. Any subfield of K of transcendence degree 0 over k coincides with k. Hence, if f_W were compressible then it would be represented by a nondegenerate quadratic form defined over k, which is automatically hyperbolic. On the other hand, by Proposition 10.1, f_W is represented by an anisotropic form $[1, x]$, a contradiction. $\square$

Now let $n > 0$ and suppose that for all canonical monomial quadratic forms of rank $< n$ their classes in the Witt group are incompressible. Suppose that f_W is compressible. Then, there exists a subfield $F \subset K$ containing k (which may be assumed to have transcendence degree n over k by Proposition 11.1 and Theorem 11.3) and a nondegenerate quadratic form g over F such that $(g_K)_W = f_W$.

For the restriction $w = v|_F$ of v to F there are three possibilities.

Case 1: w is trivial. Write g as a direct sum of 2-dimensional forms $[b_i, c_i]$ with $b_i, c_i \in F \subset R$. Consider Arason's filtration of $W_q(\hat{K})$ with respect to the presentation $\hat{K} = K_1((\pi))$. Since b_i, c_i are in R, g_W lives in the zero term of Arason's filtration; moreover, its second residue is trivial. On the other hand, since $g_{\hat{K},W} = f_{\hat{K},W}$ it has nontrivial second residue by Lemma 11.2, a contradiction.

Case 2: w is nontrivial and the ramification index $e(v/w)$ is even. Then, the same arguments as in Theorem 11.3 show that the second residue of g is trivial, which is impossible since $f_{\hat{K},W} = g_{\hat{K},W}$. Indeed, arguing as in Proposition 11.1, we can choose a differential basis $B = \{a_1, \dots, a_n, \pi\}$ for K/k coming from $\bar{K}$ such that

$$F \subset K^2(a_1, \dots, a_n). \tag{12.1.1}$$

By Theorem 9.2, B gives rise to the coefficient field $E \subset \hat{K}$ containing $a_1, \dots, a_n$ and presentation $\hat{K} \simeq E((\pi))$. We then fix this presentation and below we consider the corresponding Arason's filtration.

We now pass to computing the residues of $g_{\hat{K}}$ using our presentation $\hat{K} = E((\pi))$ and inclusion (12.1.1). Since g is nondegenerate, it can be written as a direct sum of 2-dimensional forms $[b_i, c_i]$ with $b_i, c_i \in F$. In turn, in view of (12.1.1), b_i and c_i can be written as sums of elements of the form $\alpha^2_{i_1\dots i_s} a_{i_1} a_{i_2} \cdots a_{i_s}$ with $\alpha_{i_1\dots i_s} \in K$. Then arguing as in Theorem 11.3 we conclude that the image of $g_{\hat{K}}$ in $W_q(\hat{K})$ can be written as a sum of symbols

$$\left[a_{i_1} a_{i_2} \cdots a_{i_s}, \frac{\alpha^2_{j_1\dots j_p}}{\pi^{2l}} a_{j_1} a_{j_2} \cdots a_{j_p}\right]_W,$$

where $\alpha_{j_1\dots j_p} \in E$. Thus, we can write $g_{\hat{K},W}$ as the sum

$$g_W = g_{2m} + g_{2(m-1)} + \cdots + g_0,$$

where all homogeneous components g_{2i} have even degree and are sums of symbols

$$\left[a_{i_1} a_{i_2} \cdots a_{i_s}, \frac{\alpha^2_{j_1\dots j_p}}{\pi^{2i}} a_{j_1} a_{j_2} \cdots a_{j_p}\right]_W,$$

with $\alpha_{j_1 \dots j_p} \in E$. By Corollary 8.3 we obtain $g_{2m} + \dots + g_2 = 0$. Hence, the second residue of g_W is trivial.

Case 3: $e = e(v/w)$ is odd. Let $\pi' \in F$ be a uniformizer for w. Write $\pi' = u\pi^e$ where $u \in R^\times$. Our argument below doesn't depend on a choice of a uniformizer for v. So, after replacing π with $u\pi$ if necessary, we may assume without loss of generality that $u = v^2$ for some $v \in R^\times$.

Note that we may assume additionally that the extension $F \cdot K^2/K^2$ has degree 2^n, since otherwise we would be in the "codimension" 2 case and so we could apply Theorem 11.3. Then, $[\overline{F} \cdot \overline{K}^2 : \overline{K}^2] = 2^{n-1}$ (because the extension K/K^2 is unramified and $[\overline{K} : \overline{K}^2] = 2^n$). This implies that there exists a differential basis

$$B' = \{a_1, \dots, a_{n-1}, \pi'\}$$

for F/k coming from residue field $\overline{F}$ and a unit $a_n \in R^\times$ such that $B = B' \cup \{a_n\}$ is a differential basis for K/k coming from $\overline{K}$.

We now pass to the completions $\hat{F} \subset \hat{K}$ with respect to w and v, respectively. The differential bases B' and B for F/k and K/k give rise to the coefficients field E' and E and representations

$$\hat{F} = E'((\pi')) \subset \hat{K} = E((\pi))$$

such that $E' \subset E$. According to Proposition 8.2 our quadratic form g viewed over $\hat{F}$ admits a unique decomposition $g_{\hat{F},W} = g_m + \dots + g_0$. Taking into account the facts that $E_1 \subset E$, that $\pi' = v^2\pi^e$ with odd e, and that g viewed over $\hat{K}$ lives in the zero term of Arason's filtration of the field $E((\pi))$, one can easily see that the homogeneous components $g_m, \dots, g_1$ are trivial, so that $g_{\hat{F},W}$ can be written as a sum of symbols

$$[a_1^{\nu_1} \cdots a_{n-1}^{\nu_{n-1}}, u_\epsilon^2 a_1^{\epsilon_1} \cdots a_{n-1}^{\epsilon_{n-1}}] \quad \text{and} \quad [a_1^{\nu_1} \cdots a_{n-1}^{\nu_{n-1}} \pi', v_\epsilon^2 a_1^{\epsilon_1} \cdots a_{n-1}^{\epsilon_{n-1}} (\pi')^{-1}]$$

where $u_\epsilon, v_\epsilon \in E_1$. It follows that the first residue of $g_{\hat{K},W} = f_{\hat{K},W}$ lives in the subfield E_1 of $\overline{K} = E$ of transcendence degree $n-1$ over k. On the other side, this residue is a canonical monomial form of rank $n-1$, which is impossible by the induction assumption. This completes the proof of incompressibility of f. □

13. Orthogonal and special orthogonal groups

Let g be a nondegenerate n-dimensional quadratic form on a vector space V over k, and let F be any extension of k.

Orthogonal groups. It is well known (see [Knus et al. 1998, §29.E]) that if $n = 2r$ is even, then there exists a natural bijection between $H^1(F, \mathrm{O}(V, g))$ and the set of isometry classes of n-dimensional nondegenerate quadratic spaces (V', g').

Similarly, if $n = 2r+1$ is odd, then $H^1(F, \mathrm{O}(V, g))$ is in one-to-one correspondence with the set of isometry classes of $(2r+1)$-dimensional nondegenerate quadratic spaces (V', q') over F such that $\mathrm{disc}(q') = 1$. Note that any such q' is isometric to a quadratic form of the shape $([a_1, b_1] \oplus \cdots \oplus [a_r, b_r]) \oplus \langle 1 \rangle$. Then, in both cases the incompressibility of canonical monomial quadratic forms provides us with the required lower bound $\mathrm{ed}(\mathrm{O}(V, g)) \geq r+1$. What is left to finish the proof of Theorem 2.2 for orthogonal groups is to find a "good" upper bound.

Proposition 13.1. *In the above notation*, $\mathrm{ed}(\mathrm{O}(V, g)) \leq r+1$.

Proof. It suffices to show that any $2r$-dimensional nondegenerate quadratic form depends on at most $2r$ parameters. Let h be such form over F, and write $h = a_1[1, b_1] \oplus \cdots \oplus a_r[1, b_r]$. Each summand $[1, b_i]$ corresponds to a unique element $\xi_i \in H^1(F, \mathbb{Z}/2)$. Let $H = \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ be the direct sum of r copies of the constant group scheme $\mathbb{Z}/2$ and let $\xi = (\xi_1, \ldots, \xi_r)$. Choose any embedding $H \hookrightarrow \mathrm{G}_{a,k}$, which exists because k is infinite. The exact sequence

$$0 \longrightarrow H \longrightarrow \mathrm{G}_{a,k} \xrightarrow{\phi} \mathrm{G}_{a,k} \longrightarrow 0$$

gives rise to

$$F \xrightarrow{\phi} F \xrightarrow{\psi} H^1(F, H) \longrightarrow 1.$$

Let $a \in F$ be such that $\psi(a) = \xi$. It follows that ξ has descent to the subfield $k(a)$ of F. This amounts to the fact that there exist $b'_1, \ldots, b'_r \in k(a)$ such that the quadratic form $[1, b'_i]$ viewed over F is isometric to $[1, b_i]$. Therefore, h is isometric to the quadratic form $h' = a_1[1, b'_1] \oplus \cdots \oplus a_r[1, b'_r]$ defined over the subfield $k(a, a_1, \ldots, a_r)$ of F of transcendence degree (over k) at most $r+1$. □

Remark 13.2. If h has trivial Arf invariant then taking a suitable quadratic extension of $k(a, a_1, \ldots, a_r)$ contained in F, if necessary, we may also assume that h' is defined over a subfield of F of transcendence degree $\leq r+1$ and has trivial Arf invariant.

Special orthogonal groups. We first find upper bounds.

By [Knus et al. 1998, §29.E], if $n = 2r$ is even, then there exists a natural bijection between $H^1(F, \mathrm{SO}(V, g))$ and the set of isometry classes of $(2r)$-dimensional nondegenerate quadratic spaces (V', g') over F such that the Arf invariant of g' is trivial. Therefore, $\mathrm{ed}(\mathrm{SO}(V, g)) \leq r+1$, by Remark 13.2.

If $n = 2r+1$ is odd, then there exists a natural bijection between $H^1(F, \mathrm{SO}(V, g))$ and the set of isometry classes of $(2r+1)$-dimensional nondegenerate quadratic spaces (V', g') over F such that $\mathrm{disc}(g') = 1$. As we mentioned above, any such g' is isometric to a quadratic form of the shape $([a_1, b_1] \oplus \cdots \oplus [a_r, b_r]) \oplus \langle 1 \rangle$ for some $a_i, b_i \in F$. It follows that $\mathrm{ed}(\mathrm{SO}(V, g)) \leq r+1$, by Proposition 13.1.

To find a "good" lower bound, we recall that $\mathrm{SO}_{2r+1}(g) = \mathrm{O}_{2r+1}(g)_{\mathrm{red}}$, the reduced subscheme of $\mathrm{O}_{2r+1}(g)$. Thus, we have a natural closed embedding $\mathrm{SO}_{2r+1}(g) \hookrightarrow \mathrm{O}_{2r+1}(g)$. Fix a decomposition $g \simeq h \oplus \langle 1 \rangle$ where $h = \mathbb{H} \oplus \cdots \oplus \mathbb{H}$. It induces a natural closed embedding $\phi_1 : \mathrm{O}_{2r}(h) \hookrightarrow \mathrm{SO}_{2r+1}(g)$ (because $\mathrm{O}_{2r}(h)$ is smooth). Furthermore, we can view $\langle 1 \rangle$ as a subform of $[1, 0] \simeq \mathbb{H}$. This allows us to view g as a subform of a $(2r+2)$-dimensional split quadratic form $q = \mathbb{H} \oplus \cdots \oplus \mathbb{H}$ and this induces a natural map

$$\phi_2 : \mathrm{SO}_{2r+1}(g) \hookrightarrow \mathrm{O}_{2r+1}(g) \hookrightarrow \mathrm{O}_{2r+2}(q).$$

The maps ϕ_1 and ϕ_2, in turn, induce the natural maps

$$\psi_1 : H^1(F, \mathrm{O}_{2r}(h)) \to H^1(F, \mathrm{SO}_{2r+1}(g))$$

and

$$\psi_2 : H^1(F, \mathrm{SO}_{2r+1}(g)) \to H^1(F, \mathrm{O}_{2r+2}(q)).$$

It easily follows from the above discussions that ψ_1 is surjective. Also, identifying elements in $H^1(F, \mathrm{O}_{2r}(h))$ and $H^1(F, \mathrm{O}_{2r+2}(q))$ with the isometry classes of the corresponding quadratic spaces, the isometry class of a quadratic form $\bigoplus_{i=1}^{r} [a_i, b_i]$ goes to the isometry class of $\bigoplus_{i=1}^{r} [a_i, b_i] \oplus \mathbb{H}$ under the composition $\psi_2 \circ \psi_1$.

Theorem 13.3. *If g is a nondegenerate quadratic form of dimension $2r+1$ over k, then* $\mathrm{ed}\,(\mathrm{SO}_{2r+1}(g)) \geq r+1$.

Proof. Take a pure transcendental extension $K = k(x, t_1, \ldots, t_r)$ of k of degree $r+1$ and a canonical monomial form $f = t_1[1, x] \oplus \cdots \oplus t_r[1, x]$ of dimension $2r$. We will show that its image ξ under ψ_1 is incompressible. Indeed, if ξ is compressible, so is $\psi_2(\xi)$. However, $\psi_2(\xi)$ is represented by a canonical monomial form $t_1[1, x] \oplus \cdots \oplus t_r[1, x] \oplus \mathbb{H}$, which is incompressible by Theorem 10.2, a contradiction. Thus, ξ is incompressible itself, implying $\mathrm{ed}(\mathrm{SO}_{2r+1}(g)) \geq r+1$. □

14. Proof of Theorem 3.1

Types A_r, B_r, C_r, D_r, E_6, E_7, E_8. Let $\rho : G^\circ \to \mathrm{O}(V, q)$ be as in Proposition 6.1. As in [Chernousov and Serre 2006], we can extend it to $\rho_G : G \to \mathrm{O}(V, q)$. Let $\theta_O = \rho_G(\theta_G)$ be the image of θ_G in $H^1(K, \mathrm{O}(V, q))$. Consider the quadratic form q_O on V corresponding to θ_O. If $\dim(q)$ is even, then arguing as in [loc. cit.] we conclude that q_O is a canonical monomial form of rank r. By Theorem 10.2, q_O is incompressible and hence so is θ_G.

If $\dim(q)$ is odd, then we can write it as $q = \langle 1 \rangle \oplus q'$, where q' is a nondegenerate quadratic form of even dimension. The twist q_O of q by θ_O is then of the form $q_O = \langle 1 \rangle \oplus g$, where g is a canonical monomial form of rank r. Finally, the proof of Theorem 13.3 shows that q_O is incompressible as well.

Type G_2. Let F be a field of arbitrary characteristic. By [Serre 1995, Théorème 11], there is a canonical one-to-one correspondence between $H^1(F, G_2)$ and the set of isometry classes of 3-fold Pfister forms defined over F, where G_2 denotes a split group of type G_2 over F. Clearly, any 3-fold Pfister form depends on at most 3 parameters implying $\mathrm{ed}(G_2) \leq 3$. Conversely, a generic 3-fold Pfister form is a canonical monomial form of rank 2, hence incompressible. It follows $\mathrm{ed}(G_2) \geq 3$.

Type F_4. Let F be a field of arbitrary characteristic. It is known that there is a canonical one-to-one correspondence between $H^1(F, F_4)$ and the set of isomorphism classes of 27-dimensional exceptional Jordan algebras over F, where F_4 denotes a split group of type F_4 over F. To each such reduced Jordan algebra J one associates a unique (up to isometry) 5-fold Pfister form $f_5(J)$ [Petersson 2004, §4.1]. Moreover, it is known that any 5-fold Pfister form over F corresponds to some Jordan algebra J over F. Since a generic 5-Pfister form is incompressible, we conclude that $\mathrm{ed}(F_4) \geq 5$.

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COCHARACTER-CLOSURE AND SPHERICAL BUILDINGS

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In memory of Robert Steinberg

Let k be a field, let G be a reductive k-group and V an affine k-variety on which G acts. In this note we continue our study of the notion of cocharacter-closed $G(k)$-orbits in V. In earlier work we used a rationality condition on the point stabilizer of a G-orbit to prove Galois ascent/descent and Levi ascent/descent results concerning cocharacter-closure for the corresponding $G(k)$-orbit in V. In the present paper we employ building-theoretic techniques to derive analogous results.

1. Introduction

Let k be a field and let G be a reductive linear algebraic group acting on an affine variety V, with G, V and the action all defined over k. Let Δ_k be the (simplicial) spherical building of G over k, and let $\Delta_k(\mathbb{R})$ be its geometric realisation (for precise definitions, see below). In this paper we continue the study, initiated in [Bate et al. 2013; 2012; 2015], of the notion of *cocharacter-closed orbits* in V for the group $G(k)$ of k-rational points of G, and of interactions with the geometry of $\Delta_k(\mathbb{R})$. The philosophy of this paper is as follows (cf. [Bate et al. 2015]): for a point v in V, we are interested in *Galois ascent/descent* questions — given a separable algebraic extension k'/k of fields, how is the $G(k')$-orbit of v related to the $G(k)$-orbit of v? — and *Levi ascent/descent* questions — given a k-defined torus S of the stabilizer G_v, how is the $C_G(S)(k)$-orbit of v related to the $G(k)$-orbit of v? (See [Bate et al. 2015, Section 5, Paragraph 1] for an explanation of the terms Galois/Levi ascent/descent in this context.) These questions are related, and have natural interpretations in $\Delta(\mathbb{K})$.

Our results complement those of [Bate et al. 2015]: they give similar conclusions but under different assumptions. It was shown in [loc. cit.] (see Proposition 2.6 below) that Galois descent — passing from $G(k')$-orbits to $G(k)$-orbits — is always

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well-behaved. Certain results on Galois ascent were also proved [loc. cit., Theorem 5.7] under hypotheses on the stabilizer G_v. The mantra in this paper is that when the centre conjecture (see Theorem 1.2 below) is known to hold, one can use it to prove Galois ascent results, and hence deduce Levi ascent/descent results. The idea is that when the extension k'/k is separable and normal, questions of Galois ascent can be interpreted in terms of the action of the Galois group of k'/k on the building; moreover, if one has such Galois ascent questions under control, then it is easier to handle Levi ascent/descent because one may assume that the torus S is split (cf. [loc. cit., Theorem 5.4(ii)]).

When k is algebraically closed (or more generally when k is perfect), our setup is also intimately related to the *optimality formalism* of [Kempf 1978; Rousseau 1978; Hesselink 1978]. Indeed, one may interpret this formalism in the language of the centre conjecture (see [Bate et al. 2012, Section 1]). The idea is to study the G-orbits in V via limits along cocharacters of G: limits are formally defined below, but given v in V, if we take the set of cocharacters λ of G for which the limit $\lim_{a\to 0}\lambda(a)\cdot v$ exists, and interpret this set in terms of the set of $\mathbb{Q}$-points $\Delta(\mathbb{Q})$ of the building of G, then we obtain a convex subset Σ_v of $\Delta(\mathbb{Q})$. In case $G\cdot v$ is not Zariski-closed, one can find a fixed point in the set Σ_v and an associated *optimal parabolic subgroup* P of G with many nice properties: in particular, the stabilizer G_v normalises P. It is not currently known in general how to produce analogues of these optimality results over arbitrary fields (or even whether such results exist); see [Bate et al. 2013, Section 1] for further discussion. Our first main theorem gives a rational analogue of the Kempf–Rousseau–Hesselink ideas when Σ_{v,k_s} (the points of Σ_v coming from k_s-defined cocharacters of G) happens to be a subcomplex of $\Delta(\mathbb{Q})$, and also answers in this case the ascent/descent questions posed earlier.

Theorem 1.1. *Let $v\in V$. Suppose Σ_{v,k_s} is a subcomplex of $\Delta_{k_s}(\mathbb{Q})$. Then the following hold*:

(i) *Suppose $v\in V(k)$ and $G(k_s)\cdot v$ is not cocharacter-closed over k_s. Let S be any k-defined torus of G_v and set $L=C_G(S)$. Then there exists $\sigma\in Y_k(L)$ such that* $\lim_{a\to 0}\sigma(a)\cdot v$ *exists and lies outside $G(k_s)\cdot v$.*

(ii) *Suppose $v\in V(k)$. For any separable algebraic extension k'/k, $G(k')\cdot v$ is cocharacter-closed over k' if and only if $G(k)\cdot v$ is cocharacter-closed over k.*

(iii) *Let S be any k-defined torus of G_v and set $L=C_G(S)$. Then $G(k)\cdot v$ is cocharacter-closed over k if and only if $L(k)\cdot v$ is cocharacter-closed over k.*

The hypothesis that Σ_{v,k_s} is a subcomplex allows us to apply the following result — Tits' centre conjecture — in the proof of Theorem 1.1:

Theorem 1.2. *Let Θ be a thick spherical building and let Σ be a convex subcomplex of Θ such that Σ is not completely reducible. Then there is a simplex of Σ that is fixed by every building automorphism of Θ that stabilizes Σ.* (*We call such a simplex a* centre *of* Σ.)

For definitions and further details, see [Ramos-Cuevas 2013]; in particular, note that the spherical building of a reductive algebraic group is thick. The conjecture was proved by Mühlherr and Tits [2006], Leeb and Ramos-Cuevas [2011] and Ramos-Cuevas [2013], and a uniform proof for chamber subcomplexes has also now been given by Mühlherr and Weiss [2013]. The condition that Σ_{v,k_s} is a subcomplex is satisfied in the theory of complete reducibility for subgroups of G and Lie subalgebras of $\mathrm{Lie}(G)$, and our results yield applications to complete reducibility (see Theorem 1.4 below).

By [Bate et al. 2015, Theorem 5.7], the conclusions of Theorem 1.1(ii) and (iii) hold if G_v has a maximal torus that is k-defined. Our second main result gives alternative hypotheses on G_v, this time of a group-theoretic nature, for the conclusions of Theorem 1.1 to hold, without the assumption that Σ_{v,k_s} is a subcomplex. The proof of this result relies in an essential way on known cases of a strengthened version of the centre conjecture (this time from [Bate et al. 2012]).

Theorem 1.3. *Let $v \in V(k)$. Suppose that* (*a*) *G_v^0 is nilpotent*, *or* (*b*) *every simple component of G^0 has rank* 1. *Then the following hold*:

(i) *Suppose $G(k_s) \cdot v$ is not cocharacter-closed over k_s. Let S be any k-defined torus of G_v and set $L = C_G(S)$. Then there exists $\sigma \in Y_k(L)$ such that $G_v(k_s)$ normalises $P_\sigma(G^0)$ and $\lim_{a\to 0} \sigma(a) \cdot v$ exists and lies outside $G(k_s) \cdot v$.*

(ii) *For any separable algebraic extension k'/k, $G(k') \cdot v$ is cocharacter-closed over k' if and only if $G(k) \cdot v$ is cocharacter-closed over k.*

(iii) *Let S be any k-defined torus of G_v and set $L = C_G(S)$. Then $G(k) \cdot v$ is cocharacter-closed over k if and only if $L(k) \cdot v$ is cocharacter-closed over k.*

The hypothesis in Theorem 1.1(i) that v is a k-point ensures that the subset Σ_v is Galois-stable, and it is also needed in our proof of Theorem 1.3 (but see Remark 4.6). Sometimes, however, one can get away with a weaker hypothesis. This happens for G-complete reducibility in the final section of the paper, where we prove the following ascent/descent result:

Theorem 1.4. *Suppose that G is connected. Let H be a subgroup of G. Let S be a k-defined torus of $C_G(H)$ and set $L = C_G(S)$. Then H is G-completely reducible over k if and only if H is L-completely reducible over k.*

Remark 1.5. (i) Theorem 1.4 gives an alternative proof and also slightly generalises Serre's Levi ascent/descent result [Serre 1997, Proposition 3.2]; cf. [Bate et al. 2005, Corollary 3.21, Corollary 3.22] — for in the statement of Theorem 1.4, we do not require H to be a subgroup of $G(k)$.

(ii) The counterpart of Theorem 1.1(ii) (Galois ascent/descent for G-complete reducibility) was proved in [Bate et al. 2009].

We spend much of the paper recalling relevant results from geometric invariant theory and the theory of buildings. Although the basic ideas are familiar, we need to extend many of them: for instance, the material on quasi-states in Section 3D was covered in [Bate et al. 2012] for algebraically closed fields, but we need it for arbitrary fields. We work with the geometric realisations of buildings rather than with buildings as abstract simplicial complexes; some care is needed when the reductive group G has positive-dimensional centre.

The paper is laid out as follows. In Section 2, we set up notation and collect terminology and results relating to cocharacter-closedness. In Section 3, we translate our setup into the language of spherical buildings; we use notation and results from [Bate et al. 2012] on buildings, some of which we extend slightly. In Section 4, we combine the technology from both of the preceding sections to give proofs of our main results. In the final section we give our applications to the theory of complete reducibility.

2. Notation and preliminaries

Let k denote a field with separable closure k_s and algebraic closure $\bar{k}$. Let $\Gamma := \mathrm{Gal}(k_s/k) = \mathrm{Gal}(\bar{k}/k)$ denote the Galois group of k_s/k. Throughout, G denotes a (possibly nonconnected) reductive linear algebraic group defined over k, and V denotes a k-defined affine variety upon which G acts k-morphically. Let $G(k)$, $G(k_s)$, $V(k)$, $V(k_s)$ denote the k- and k_s-points of G and V; we usually identify G with $G(\bar{k})$ and V with $V(\bar{k})$. If X is a variety then we denote its Zariski closure by $\overline{X}$.

More generally, we need to consider k-points and k_s-points in subgroups that are not necessarily k-defined or k_s-defined; note that if k is not perfect then even when v is a k-point, the stabilizer G_v need not be k-defined. If k'/k is an algebraic field extension and H is a closed subgroup of G then we set $H(k') = H(\bar{k}) \cap G(k')$, and we say that a torus S of H is k'-defined if it is k'-defined as a torus of the k-defined group G. Note that a k_s-defined torus of H is a torus of $\overline{H(k_s)}$.

2A. ***Cocharacters and G-actions.*** Given a k-defined algebraic group H, we let $Y(H)$ denote the set of cocharacters of H, with $Y_k(H)$ and $Y_{k_s}(H)$ denoting the sets of k-defined and k_s-defined cocharacters, respectively. The group H acts on $Y(H)$ via the conjugation action of H on itself. This gives actions of the group of k-points $H(k)$ on $Y_k(H)$ and the group of k_s-points $H(k_s)$ on $Y_{k_s}(H)$. There is also an action of the Galois group Γ on $Y(H)$ which stabilizes $Y_{k_s}(H)$, and the Γ-fixed elements of $Y_{k_s}(H)$ are precisely the elements of $Y_k(H)$. We write $Y = Y(G)$, $Y_k = Y_k(G)$ and $Y_{k_s} = Y_{k_s}(G)$.

Definition 2.1. A function $\| \, \| : Y \to \mathbb{R}_{\geq 0}$ is called a *Γ-invariant, G-invariant norm* if:

(i) $\|g \cdot \lambda\| = \|\lambda\| = \|\gamma \cdot \lambda\|$ for all $\lambda \in Y$, $g \in G$ and $\gamma \in \Gamma$;

(ii) for any maximal torus T of G, there is a positive definite integer-valued form $(\, , \,)$ on $Y(T)$ such that $(\lambda, \lambda) = \|\lambda\|^2$ for any $\lambda \in Y(T)$.

Such a norm always exists: To see this, take a k-defined maximal torus T and any positive definite integer-valued form on $Y(T)$. Since T splits over a finite extension of k, we can average the form over the Weyl group W and over the finite Galois group of the extension to obtain a W-invariant Γ-invariant form on $Y(T)$, which defines a norm satisfying (ii). One can extend this norm to all of Y because any cocharacter is G-conjugate to one in $Y(T)$; this procedure is well-defined since the norm on $Y(T)$ is W-invariant. See [Kempf 1978] for more details. If G is simple then $\| \, \|$ is unique up to nonzero scalar multiples. We fix such a norm once and for all.

For each cocharacter $\lambda \in Y$ and each $v \in V$, we define a morphism of varieties $\phi_{v,\lambda} : \bar{k}^* \to V$ via the formula $\phi_{v,\lambda}(a) = \lambda(a) \cdot v$. If this morphism extends to a morphism $\widehat{\phi}_{v,\lambda} : \bar{k} \to V$, then we say that $\lim_{a \to 0} \lambda(a) \cdot v$ exists, and set this limit equal to $\widehat{\phi}_{v,\lambda}(0)$; note that such an extension, if it exists, is necessarily unique.

Definition 2.2. For $\lambda \in Y$ and $v \in V$, we say that λ *destabilizes* v provided $\lim_{a\to 0} \lambda(a) \cdot v$ exists, and if $\lim_{a\to 0} \lambda(a) \cdot v$ exists and does not belong to $G \cdot v$, then we say λ *properly destabilizes* v. We have an analogous notion over k: if $\lambda \in Y_k$ then we say that λ *properly destabilizes* v *over* k if $\lim_{a\to 0} \lambda(a) \cdot v$ exists and does not belong to $G(k) \cdot v$. Finally, if k'/k is an algebraic extension, and $\lambda \in Y_k$, then we say that λ *properly destabilizes* v *over* k' if $\lim_{a\to 0} \lambda(a) \cdot v$ exists and does not belong to $G(k') \cdot v$; that is, if λ — regarded as an element of $Y_{k'}(G)$ — properly destabilizes v over k'.

2B. ***R-parabolic subgroups.*** When $V = G$ and G is acting by conjugation, for each $\lambda \in Y$ we get a set $P_\lambda := \{g \in G \mid \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1} \text{ exists}\}$; this is a parabolic subgroup of G. We distinguish these parabolic subgroups by calling them *Richardson-parabolic* or *R-parabolic* subgroups. For basic properties of these subgroups, see [Bate et al. 2005, Section 6]. We recall here that $L_\lambda = C_G(\mathrm{Im}(\lambda))$ is called an *R-Levi subgroup* of P_λ, $R_u(P_\lambda)$ is the set of elements sent to $1 \in G$ in the limit, and $P_\lambda = R_u(P_\lambda) \rtimes L_\lambda$. Further, $R_u(P_\lambda)$ acts simply transitively on the set of all L_μ such that $P_\mu = P_\lambda$ (that is, on the set of all R-Levi subgroups of P_λ): note that this is a transitive action of $R_u(P_\lambda)$ on the set of *subgroups* of the form L_μ, not on the set of cocharacters for which $P_\mu = P_\lambda$. Most of these things work equally well over the field k: for example, if λ is k-defined then P_λ, L_λ and $R_u(P_\lambda)$ are; moreover, given any k-defined R-parabolic subgroup P, $R_u(P)(k)$ acts simply transitively on the set of k-defined R-Levi subgroups of P [Bate et al. 2013, Lemma 2.5]. Note that if P is k-defined and G is connected then $P = P_\lambda$ for some k-defined λ, but this can fail if G is not connected [Bate et al. 2013, Section 2].

When H is a reductive subgroup of G the inclusion $Y(H) \subseteq Y(G)$ means that we get an R-parabolic subgroup of H and of G attached to any $\lambda \in Y(H)$. When we use the notation P_λ, L_λ, etc., we are always thinking of λ as a cocharacter of G. If we need to restrict attention to the subgroup H for some reason, we write $P_\lambda(H)$, $L_\lambda(H)$, etc.

2C. ***The sets $Y(\mathbb{Q})$ and $Y(\mathbb{R})$.*** Form the set $Y(\mathbb{Q})$ by taking the quotient of $Y \times \mathbb{N}_0$ by the relation $\lambda \sim \mu$ if and only if $n\lambda = m\mu$ for some $m, n \in \mathbb{N}$, and extend the norm function to $Y(\mathbb{Q})$ in the obvious way. For any torus T in G, $Y(T, \mathbb{Q}) := Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a vector space over $\mathbb{Q}$. Now one can form real spaces $Y(T, \mathbb{R}) := Y(T, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$ for each maximal torus T of G and a set $Y(\mathbb{R})$ by glueing the $Y(T, \mathbb{R})$ together according to the way the spaces $Y(T, \mathbb{Q})$ fit together [Bate et al. 2012, Section 2.2]. The norm extends to these sets. One can define sets $Y_k(\mathbb{Q})$, $Y_k(\mathbb{R})$, $Y_k(T, \mathbb{Q})$, $Y_k(T, \mathbb{R})$, $Y_{k_s}(\mathbb{Q})$, $Y_{k_s}(\mathbb{R})$, $Y_{k_s}(T, \mathbb{Q})$ and $Y_{k_s}(T, \mathbb{R})$ analogously by restricting attention to k-defined cocharacters and maximal tori, or k_s-defined cocharacters and maximal tori, as appropriate. For the rest of the paper, $\mathbb{K}$ denotes either of $\mathbb{Q}$ or $\mathbb{R}$ when the distinction is not important. The sets $Y(\mathbb{K})$, $Y_k(\mathbb{K})$ and $Y_{k_s}(\mathbb{K})$ inherit G-, $G(k)$-, $G(k_s)$- and Γ-actions from those on Y, Y_k and Y_{k_s}, as appropriate, and each element $\lambda \in Y(\mathbb{K})$ still corresponds to an R-parabolic subgroup P_λ and an R-Levi subgroup L_λ of G (see [Bate et al. 2012, Section 2.2] for the case $\mathbb{K} = \mathbb{R}$). If H is a reductive subgroup of G then we write $Y(H, \mathbb{Q})$, etc., to denote the above constructions for H instead of G.

2D. ***G-varieties and cocharacter-closure.*** We recall the following fundamental definition from [Bate et al. 2015, Definition 1.2], which extends the one given in [Bate et al. 2013, Definition 3.8].

Definition 2.3. A subset S of V is said to be *cocharacter-closed over k (for G)* if for every $v \in S$ and $\lambda \in Y_k$ such that $v' := \lim_{a\to 0} \lambda(a) \cdot v$ exists, we have $v' \in S$.

This notion is explored in detail in [Bate et al. 2015]. In this section, we content ourselves with collecting some results from that paper, together with the earlier paper [Bate et al. 2013]. These results, most of which are also needed in the sequel, give a flavour of what is known about the notion of cocharacter-closure in the case that the subset involved is a single $G(k)$-orbit.

Remark 2.4. The geometric orbit $G \cdot v$ is Zariski-closed if and only if it is cocharacter-closed over $\bar{k}$, by the Hilbert–Mumford Theorem [Kempf 1978, Theorem 1.4].

Theorem 2.5 [Bate et al. 2015, Corollary 5.1]. *Suppose $v \in V$ is such that $G(k) \cdot v$ is cocharacter-closed over k. Then whenever $v' = \lim_{a\to 0} \lambda(a) \cdot v$ exists for some $\lambda \in Y_k$, there exists $u \in R_u(P_\lambda)(k)$ such that $v' = u \cdot v$.*

Proposition 2.6 [Bate et al. 2015, Proposition 5.5]. *Let $v \in V$ such that $G_v(k_s)$ is Γ-stable and let k'/k be a separable algebraic extension. If $G(k') \cdot v$ is cocharacter-closed over k', then $G(k) \cdot v$ is cocharacter-closed over k.*

Theorem 2.7 [Bate et al. 2015, Theorem 5.4]. *Suppose S is a k-defined torus of G_v and set $L = C_G(S)$.*

(i) *If $G(k) \cdot v$ is cocharacter-closed over k, then $L(k) \cdot v$ is cocharacter-closed over k.*

(ii) *If S is k-split, then $G(k) \cdot v$ is cocharacter-closed over k if and only if $L(k) \cdot v$ is cocharacter-closed over k.*

We note that, as described in the introduction, one of the main points of this paper is to show that the converse of Proposition 2.6 holds under certain extra hypotheses, and that the hypothesis of splitness can be removed in Theorem 2.7(ii) under the same hypotheses; see also [Bate et al. 2015, Theorem 1.5].

Our final result, a strengthening of Lemma 5.6 of [Bate et al. 2015], follows from the arguments given in the proof of that lemma.

Lemma 2.8. *Let V be an affine G-variety over k and let $v \in V(k)$. Suppose there exists $\lambda \in Y_{k_s}$ such that λ properly destabilizes v. Then there exists $\mu \in Y_k$ such that $v' = \lim_{a\to 0} \mu(a) \cdot v$ exists, v' is not $G(k_s)$-conjugate to v and $G_v(k_s)$ normalises P_μ. In particular, $G(k) \cdot v$ is not cocharacter-closed over k.*

Remark 2.9. The hypotheses of Lemma 2.8 are satisfied if $\lambda \in Y_{k_s}(Z(G^0))$ destabilizes v but does not fix v. For if $v' := \lim_{a\to 0} \lambda(a) \cdot v$ is G-conjugate to v then v' is $R_u(P_\lambda)$-conjugate to v [Bate et al. 2013, Theorem 3.3]; but $R_u(P_\lambda) = 1$, so this cannot happen.

3. Spherical buildings and Tits' centre conjecture

The simplicial building Δ_k of a semisimple algebraic group G over k is a simplicial complex, the simplices of which correspond to the k-defined parabolic subgroups of G ordered by reverse inclusion. See [Tits 1974, §5] for a detailed description. Our aim in this section is to construct for an arbitrary reductive group G over k, objects $\Delta_k(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}$ that correspond to the geometric realisation of the spherical building of G^0 over k (or the set of $\mathbb{Q}$-points thereof) when G^0 is semisimple. These are slightly more general objects (possibly with a contribution from $Z(G^0)$) when G^0 is reductive. Recall that Γ denotes the Galois group of k_s/k. Most of the notation and terminology below is developed in full detail in the paper [Bate et al. 2012] — we point the reader in particular to the constructions in [Bate et al. 2012, Sections 2, 6.3, 6.4]. For the purposes of this paper, we need to extend some of the results in [loc. cit.] (for example by incorporating the effect of the Galois group Γ), but rather than reiterating all the details, we just gather enough material to make our exposition here coherent.

3A. *Definition of* $\Delta_k(\mathbb{K})$. We first form the *vector building* $V_k(\mathbb{K})$ by identifying λ in $Y_k(\mathbb{K})$ with $u \cdot \lambda$ for every $u \in R_u(P_\lambda)(k)$. The norm function on $Y_k(\mathbb{K})$ descends to $V_k(\mathbb{K})$, because it is G-invariant. This gives a well-defined function on $V_k(\mathbb{K})$, which we also call a norm, and makes $V_k(\mathbb{K})$ into a metric space.

Definition 3.1. (i) Define $\Delta_k(\mathbb{R})$ to be the unit sphere in $V_k(\mathbb{R})$ and $\Delta_k(\mathbb{Q})$ to be the projection of $V_k(\mathbb{Q}) \setminus \{0\}$ onto $\Delta_k(\mathbb{R})$.

(ii) Two points of $\Delta_k(\mathbb{K})$ are called *opposite* if they are antipodal on the sphere $\Delta_k(\mathbb{R})$.

(iii) It is clear that the conjugation action of $G(k)$ on Y_k gives rise to an action of $G(k)$ on $\Delta_k(\mathbb{K})$ by isometries, and there is a natural $G(k)$-equivariant, surjective map $\zeta : Y_k(\mathbb{K}) \setminus \{0\} \to \Delta_k(\mathbb{K})$.

(iv) The *apartments* of $\Delta_k(\mathbb{K})$ are the sets $\Delta_k(T, \mathbb{K}) := \zeta(Y_k(T, \mathbb{K}))$ where T runs over the maximal k-split tori of G.

(v) The metric space $\Delta_k(\mathbb{K})$ and its apartments have a simplicial structure, because any point $x = \zeta(\lambda)$ of $\Delta_k(\mathbb{K})$ gives rise to a k-defined parabolic subgroup P_λ of G^0 (see Section 2C); the simplicial complex consists of the proper k-defined parabolic subgroups of G^0, ordered by reverse inclusion. We write Δ_k for the spherical building of G over k regarded purely as a simplicial complex. The simplicial spherical buildings of G^0 and of $[G^0, G^0]$ are the same. Our notion of opposite is compatible with the usual one for parabolic subgroups: if $\lambda \in Y(G)$ then $P_{-\lambda}$ is an opposite parabolic to P_λ.

To avoid tying ourselves in knots, when the distinction is not important to the discussion at hand, we loosely refer to either of the objects $\Delta_k(\mathbb{Q})$ and $\Delta_k(\mathbb{R})$ as *the building of G over k*.

One can make analogous definitions of objects $\Delta_{k_s}(\mathbb{K})$ and $\Delta(\mathbb{K}) = \Delta_{\bar{k}}(\mathbb{K})$ over k_s and $\bar{k}$, respectively, with corresponding systems of apartments and maps ζ. We write Δ_{k_s} and Δ for these spherical buildings regarded as simplicial complexes.

Because we are interested in rationality results, we need to know the relationship between $\Delta_k(\mathbb{K})$ and $\Delta_{k_s}(\mathbb{K})$. Given a k-defined reductive subgroup H of G, we also want to relate $\Delta_k(H, \mathbb{K})$ to $\Delta_k(\mathbb{K})$, where $\Delta_k(H, \mathbb{K})$ denotes the building of

H over k. It is easy to see that the Γ-action on cocharacters descends (via ζ) to Γ-actions by isometries on $\Delta_{k_s}(\mathbb{K})$ and $\Delta(\mathbb{K})$.

Lemma 3.2. (i) *There are naturally occurring copies of $\Delta_k(\mathbb{K})$ inside $\Delta_{k_s}(\mathbb{K})$ and $\Delta(\mathbb{K})$. We can in fact identify $\Delta_k(\mathbb{K})$ with the set of Γ-fixed points of $\Delta_{k_s}(\mathbb{K})$.*

(ii) *Let H be a k-defined reductive subgroup of G. Then there is a naturally occurring copy of $\Delta_k(H, \mathbb{K})$ inside $\Delta_k(\mathbb{K})$.*

Proof. (i) It is clear that $Y_k(\mathbb{K}) \subseteq Y_{k_s}(\mathbb{K}) \subseteq Y(\mathbb{K})$, and $Y_k(\mathbb{K})$ is precisely the set of Γ-fixed points in $Y_{k_s}(\mathbb{K})$. Since $R_u(P_\lambda)(k)$ acts simply transitively on the set of k-defined R-Levi subgroups of P_λ, two k-defined cocharacters λ and μ are $R_u(P_\lambda)$-conjugate if and only if they are $R_u(P_\lambda)(k_s)$-conjugate if and only if they are $R_u(P_\lambda)(k)$-conjugate. Following this observation through the definition of $\Delta_k(\mathbb{K})$, $\Delta_{k_s}(\mathbb{K})$ and $\Delta(\mathbb{K})$ is enough to prove the first assertion of (i). It is clear that $\Delta_k(\mathbb{K})$ is fixed by Γ. Conversely, let $x \in \Delta_{k_s}(\mathbb{K})$ be fixed by Γ. Let P be the parabolic subgroup associated to x. Then P is k_s-defined and Γ-stable, so P is k-defined. Pick a k-defined maximal torus T of P. There exists $\lambda \in Y_{k_s}(T, \mathbb{K})$ such that $P = P_\lambda$ [Springer 1998, 8.4.4, 8.4.5]. Each $\gamma \in \Gamma$ maps λ to a $R_u(P)(k_s)$-conjugate of λ. Now $R_u(P)$ acts simply transitively on the set of Levi subgroups of P, and each maximal torus of P is contained in a unique Levi subgroup [Springer 1998, 8.4.4], so $R_u(P)$ acts freely on the set of maximal tori of P. But T is Γ-stable, so we must have that Γ fixes λ. Hence $x \in \Delta_k(\mathbb{K})$, as required.

(ii) In analogy with the first assertion of (i) (although it is slightly more subtle), the key observation is that if $\lambda, \mu \in Y_k(H)$ are $R_u(P_\lambda(G))(k)$-conjugate, then they are in fact $R_u(P_\lambda(H))(k)$-conjugate (see [Bate et al. 2011, Lemma 3.3(i)]). Observe also that the restriction of a Γ- and G-invariant norm on Y to $Y(H)$ gives a Γ- and H-invariant norm on $Y(H)$. □

Henceforth, we write $\Delta_k(\mathbb{K}) \subseteq \Delta_{k_s}(\mathbb{K}) \subseteq \Delta(\mathbb{K})$ and $\Delta_k(H, \mathbb{K}) \subseteq \Delta_k(\mathbb{K})$ without any further comment. One note of caution: the inclusion $\Delta_k(H, \mathbb{K}) \subseteq \Delta_k(\mathbb{K})$ does not in general respect the simplicial structures on these objects.

3B. ***Convex subsets.*** Because any two parabolic subgroups of G contain a common maximal torus, any two points $x, y \in \Delta(\mathbb{K})$ are contained in a common apartment and, as long as these points are not opposite each other, there is a unique geodesic $[x, y]$ joining them. This geodesic does not depend on the apartment we find containing x and y; in particular, this can be done inside $\Delta_k(\mathbb{K})$ if $x, y \in \Delta_k(\mathbb{K})$ and inside $\Delta_k(H)$ if $x, y \in \Delta_k(H)$ for some reductive subgroup H of G. This leads to the following key definitions:

Definition 3.3. (i) A subset $\Sigma \subseteq \Delta(\mathbb{K})$ is called *convex* if whenever $x, y \in \Sigma$ are not opposite then $[x, y] \subseteq \Sigma$. It follows from the discussion above that $\Delta_k(\mathbb{K})$ is a convex subset of $\Delta(\mathbb{K})$.

(ii) Given a convex subset Σ of $\Delta(\mathbb{K})$, its preimage $C := \zeta^{-1}(\Sigma) \cup \{0\}$ in $Y(\mathbb{K})$ is a union of cones $C_T := C \cap Y(T, \mathbb{K})$, where T runs over the maximal tori of G. The subset Σ is called *polyhedral* if each C_T is a polyhedral cone and Σ is said to have *finite type* if the set of cones $\{g \cdot C_{g^{-1}Tg} \mid g \in G\}$ is finite for all T.

(iii) A convex subset Σ of $\Delta(\mathbb{K})$ is called a *subcomplex* if it is a union of simplices (that is, if $\lambda, \mu \in Y(\mathbb{K})$ are such that $P_\lambda = P_\mu$, then $\zeta(\lambda) \in \Sigma$ if and only if $\zeta(\mu) \in \Sigma$) and if that union of simplices forms a subcomplex in the simplicial building Δ. In such a circumstance, we denote the subcomplex of Δ arising in this way by Σ also; note that Σ is convex in the sense of part (i) above if and only if Σ — regarded as a subcomplex of the simplicial building — is convex in the sense of simplicial buildings.

The definitions above have obvious analogues for the buildings $\Delta_k(\mathbb{K})$ and $\Delta_{k_s}(\mathbb{K})$.

There is an addition operation on the set $V(\mathbb{K})$, given as follows. Let $\varphi\colon Y(\mathbb{K}) \to V(\mathbb{K})$ be the canonical projection. Choose a maximal torus T of G and $\lambda, \mu \in Y(T, \mathbb{K})$ such that $\varphi(\lambda) = x$ and $\varphi(\mu) = y$; we define $x + y \in V(\mathbb{K})$ by $x + y = \varphi(\lambda + \mu)$. It can be shown that this does not depend on the choice of T; moreover, for any $g \in G$, $g \cdot (x + y) = g \cdot x + g \cdot y$.

3C. *The destabilizing locus and complete reducibility.* For this paper, a particularly important class of convex subsets arises from G-actions on affine varieties. Given an affine G-variety V and a point $v \in V$, set

$$\Sigma_v := \{\zeta(\lambda) \mid \lambda \in Y \text{ and } \lim_{a \to 0} \lambda(a) \cdot v \text{ exists}\} \subseteq \Delta(\mathbb{Q}).$$

We call this subset the *destabilizing locus* for v; it is a convex subset of $\Delta(\mathbb{Q})$ by [Bate et al. 2012, Lemma 5.5] (note that Σ_v coincides with $E_{V,\{v\}}(\mathbb{Q})$ in the language of [Bate et al. 2012]). Similarly we write $\Sigma_{v,k}$ (resp. Σ_{v,k_s}) for the image in $\Delta_k(\mathbb{Q})$ (resp. $\Delta_{k_s}(\mathbb{Q})$) of the k-defined (resp. k_s-defined) characters destabilizing v. If H is a reductive subgroup of G, then we write $\Sigma_{v,k}(H)$ for the destabilizing locus for v with respect to H.

Definition 3.4. A subset Σ of $\Delta(\mathbb{K})$ is called *completely reducible* if every point of Σ has an opposite in Σ.

Lemma 3.5. *Let $v \in V$. Then*:

(i) *Given $\lambda \in Y_k$ such that $\zeta(\lambda) \in \Sigma_{v,k}$, λ has an opposite in $\Sigma_{v,k}$ if and only if there exists $u \in R_u(P_\lambda)(k)$ such that $u \cdot \lambda$ fixes v, if and only if there exists $u \in R_u(P_\lambda)(k)$ such that* $\lim_{a \to 0} \lambda(a) \cdot v = u^{-1} \cdot v$.

(ii) *The subset $\Sigma_{v,k}$ is completely reducible if and only if $G(k)\cdot v$ is cocharacter-closed over k.*

(iii) *The subset Σ_v is completely reducible if and only if the orbit $G\cdot v$ is closed in V.*

Proof. We have that Σ_v (resp. $\Sigma_{v,k}$) is completely reducible if and only if for every $\lambda \in Y$ (resp. $\lambda \in Y_k$) such that $\lim_{a\to 0}\lambda(a)\cdot v$ exists, there is some $u \in R_u(P_\lambda)$ (resp. $u \in R_u(P_\lambda)(k)$) such that both $u\cdot\lambda$ and $-(u\cdot\lambda)$ destabilize v. But this is true if and only if $u\cdot\lambda$ fixes v, which is equivalent to the fact that $\lim_{a\to 0}\lambda(a)\cdot v = u^{-1}\cdot v$, by [Bate et al. 2013, Lemma 2.12]. This gives part (i). Part (ii) now follows from Theorem 2.5, and part (iii) from Remark 2.4. □

3D. *The strong centre conjecture and quasi-states.* The aim of the paper [Bate et al. 2012] is to study a strengthened version of Tits' centre conjecture for $\Delta_{k_s}(\mathbb{K})$. Let $\mathcal{G}$ denote the group of transformations of $\Delta_{k_s}(\mathbb{K})$ generated by the isometries arising from the action of $G(k_s)$ and the action of Γ. Note that elements of $\mathcal{G}$ map k_s-defined parabolic subgroups of G to k_s-defined parabolic subgroups of G, so they give rise to automorphisms of the simplicial building Δ_{k_s}. Given a convex subset Σ of $\Delta_{k_s}(\mathbb{K})$, we call a point $x \in \Sigma$ a *$\mathcal{G}$-centre* if it is fixed by all the elements of $\mathcal{G}$ that stabilize Σ setwise. We can now formulate the original centre conjecture in our setting.

Theorem 3.6. *Suppose $\Sigma \subseteq \Delta_{k_s}(\mathbb{K})$ is a convex non-completely reducible subcomplex. Then Σ has a $\mathcal{G}$-centre.*

Proof. Theorem 1.2 asserts the existence of a stable simplex in the subcomplex (note that the simplicial structure on $\Delta(\mathbb{K})$ does not "see" the difference between G and G^0, or between G^0 and its semisimple part, so the proof of the centre conjecture for subcomplexes of spherical buildings still works for the more general class of objects we have described). Now any element of $\mathcal{G}$ that fixes a simplex also fixes its barycentre (because the action is via isometries), and we are done. □

In the strong centre conjecture [Bate et al. 2012, Conjecture 2.10], one replaces convex non-completely reducible subcomplexes with convex non-completely reducible subsets. Most of [loc. cit.] deals with the special case when $k = \bar{k}$ and considers only the isometries of $\Delta_k(\mathbb{K})$ coming from the action of G. We need to take the action of Γ into account, so we briefly indicate some of the key changes that must be made to the constructions in [loc. cit.] in order to make the results go through; see also the comments in [loc. cit., Section 6.3].

Definition 3.7. We recall the notion of a $\mathbb{K}$-*quasi-state* Ξ from [Bate et al. 2012, Definition 3.1]: this is an assignment of a finite set of characters $\Xi(T)$ to each maximal torus T of G satisfying certain conditions (see [loc. cit.] for a precise statement).

The groups G and Γ act on quasi-states: given a $\mathbb{K}$-quasi-state Ξ and $g \in G$ and $\gamma \in \Gamma$ we define new quasi-states $g_* \Xi$ and $\gamma_* \Xi$ by

$$g_* \Xi(T) := g_! \Xi(g^{-1} T g), \qquad \gamma_* \Xi(T) := \gamma_!(\gamma^{-1} \cdot T),$$

where for a character χ of a torus T, $g_! \chi$ is a character of the torus gTg^{-1} given by $(g_! \chi)(gtg^{-1}) := \chi(t)$ for all $t \in T$, and similarly $\gamma_! \chi$ is a character of $\gamma \cdot T$ given by $(\gamma_! \chi)(\gamma \cdot t) := \chi(t)$ for all $t \in T$.

We say a quasi-state is *defined over a field k'* if it assigns k'-defined characters to k'-defined maximal tori. Recall also the notions of *boundedness*, *admissibility* and *quasi-admissibility* for $\mathbb{K}$-quasi-states, [Bate et al. 2012, Definitions 3.1 and 3.2].

With these definitions in hand, we can extend [loc. cit., Lemma 3.8] as follows:

Lemma 3.8. *Let Υ be a $\mathbb{K}$-quasi-state which is defined over k_s and define $\Xi := \bigcup_{\gamma \in \Gamma} \gamma_* \Upsilon$ by $\Xi(T) := \bigcup_{\gamma \in \Gamma} (\gamma_* \Upsilon)(T)$ for each maximal torus T of G. Then Ξ is a $\mathbb{K}$-quasi-state which is defined over k_s, and it is bounded* (*resp. quasi-admissible*, *admissible at λ*) *if Υ is. Moreover*, *by construction*, *Ξ is Γ-stable.*

Proof. There are two points which need to be made in order for the arguments already in the proof of [Bate et al. 2012, Lemma 3.8] to go through. First note that given a k_s-defined maximal torus T of G the set of Galois conjugates of T is finite (because T is defined over some finite extension of k). This means that, because Υ is k_s-defined, $\Xi(T)$ is still finite, so Ξ is a $\mathbb{K}$-quasi-state. Now, for boundedness we need to check that if Υ is bounded then the set $\bigcup_{\gamma \in \Gamma} \big(\bigcup_{g \in G} g_*(\gamma_* \Upsilon)(T) \big)$ is finite for some (and hence all) k_s-defined maximal tori T of G. Since we can choose any k_s-defined maximal torus T, we choose one that is actually k-defined, and then

$$\begin{aligned} g_*(\gamma_* \Upsilon)(T) &= g_!(\gamma_* \Upsilon)(g^{-1} T g) = g_! \big(\gamma_! \Upsilon \big(\gamma^{-1} \cdot (g^{-1} T g) \big) \big) \\ &= g_! \big(\gamma_! \Upsilon \big((\gamma^{-1} \cdot g)^{-1} T (\gamma^{-1} \cdot g) \big) \big) \\ &= \gamma_! (\gamma^{-1} \cdot g)_! \Upsilon \big((\gamma^{-1} \cdot g)^{-1} T (\gamma^{-1} \cdot g) \big) = \gamma_! ((\gamma^{-1} \cdot g)_* \Upsilon)(T). \end{aligned}$$

Therefore, we can write

$$\bigcup_{\gamma \in \Gamma} \bigg(\bigcup_{g \in G} (g_*(\gamma_* \Upsilon))(T) \bigg) = \bigcup_{\gamma \in \Gamma} \gamma_! \bigg(\bigcup_{g \in G} (\gamma^{-1} \cdot g)_* \Upsilon(T) \bigg).$$

Now, since Υ is bounded, the second union on the RHS is finite for every γ, and because Υ is k_s-defined and the set of Galois conjugates of a k_s-defined character is finite, the whole RHS is finite. This proves the boundedness assertion. The other assertions follow as in [loc. cit.] $\square$

Using Lemma 3.8 we can ensure that, when appropriate, the states and quasi-states constructed during the course of the paper [Bate et al. 2012] are Galois-stable. In particular, we get the following variant of [loc. cit., Theorem 5.5]:

Theorem 3.9. *Suppose $\Sigma \subseteq \Delta(\mathbb{Q})$ is a convex polyhedral non-completely reducible subset of finite type contained in a single apartment of $\Delta(\mathbb{Q})$. Then Σ has a $\mathcal{G}$-centre. In particular, if Σ is stabilized by all of Γ, then there exists a Γ-fixed point in Σ.*

Proof. Using Lemma 3.8, one can ensure that the quasi-state constructed in [Bate et al. 2012, Lemma 5.2] which is used in the proof of [loc. cit., Theorem 5.5] is also stable under the relevant elements of $\mathcal{G}$. The proofs in [loc. cit.] now go through. $\square$

Remark 3.10. In our application of Theorem 3.9 to the proof of Theorem 1.3 below, Σ is a Γ-stable subset of $\Delta_{k_s}(\mathbb{Q})$ and we want to show that Σ has a Γ-fixed point. A striking feature of Theorem 3.9 is that we do **not** require the apartment containing Σ to be k_s-defined: it can be any apartment of the building $\Delta(\mathbb{K}) = \Delta_{\bar{k}}(\mathbb{K})$.

We note here that, unfortunately, we do not know *a priori* that any cocharacter corresponding to the fixed point given by Theorem 3.9 properly destabilizes v. Moreover, we may need to consider cocharacters of $Z(G^0)$, which do not correspond to simplices of the spherical building at all. These technical issues are at the heart of many of the complications in the proofs in Section 4 below.

4. Proofs of the main results

Having put in place the building-theoretic technology needed for our proofs, we start this section with a few more technical results to be used for the main theorems. As always, V denotes a k-defined affine G-variety, and $v \in V$. One obstacle to proving Theorems 1.1 and 1.3 is that we need to deal with cocharacters that live in $Z(G^0)$, which are not detected by the simplicial building (cf. the proof of Theorem 3.6). An extra problem for Theorem 1.3 is that we need to factor out some simple components of G^0. The following results let us deal with these difficulties.

Let N be a product of certain simple factors of G^0, and let S be a torus of $Z(G^0)$. Let M be the product of the remaining simple factors of G^0 together with $Z(G^0)$. Suppose that N and S are normal in G (this implies that M is normal in G as well), and that N and S both fix v. Set $G_1 = G/NS$ and let $\pi : G \to G_1$ be the canonical projection. Since $Z(G^0)$ is k_s-defined and k_s-split, S is k_s-defined, and it is clear that N is k_s-defined. So G_1 and π are k_s-defined. We have a k_s-defined action of G_1 on the fixed point set V^{NS} (note that V^{NS} is G-stable).

Lemma 4.1. (i) *For any $\mu_1 \in Y_{k_s}(G_1)$, there exist $n \in \mathbb{N}$ and $\mu \in Y_{k_s}(M)$ such that $\pi \circ \mu = n\mu_1$.*

(ii) *Let $\lambda \in Y_{k_s}$. Then λ destabilizes v over k_s in V if and only if $\pi \circ \lambda$ destabilizes v over k_s in V^{NS}. Moreover, if this is the case then* $\lim_{a\to 0} \lambda(a) \cdot v =$

$\lim_{a\to 0}(\pi\circ\lambda)(a)\cdot v$ belongs to $R_u(P_\lambda(G))(k_s)\cdot v$ if and only if it belongs to $R_u(P_{\pi\circ\lambda}(G_1))(k_s)\cdot v$.

(iii) *$G_1(k_s)\cdot v$ is cocharacter-closed over k_s if and only if $G(k_s)\cdot v$ is cocharacter-closed over k_s.*

Proof. (i) Let $\mu_1 \in Y_{k_s}(G_1)$. Since μ_1 is k_s-defined, we can choose a k_s-defined maximal torus $T_1 \subseteq G_1$ with $\mu_1 \in Y_{k_s}(T_1)$. Since π is separable and k_s-defined, $\pi^{-1}(T_1) \subseteq G$ is k_s-defined [Springer 1998, Corollary 11.2.14]. Hence $\pi^{-1}(T_1)$ has a k_s-defined maximal torus T. Let $T' = T\cap M$, a $\bar{k}$-defined torus of M. Now $\pi(T') = \pi(T)$ is a maximal torus of G_1 by [Borel 1991, Proposition 11.14(1)]; but $\pi(T')$ is contained in T_1, so we must have $\pi(T') = T_1$. The surjection $T' \to T_1$ induces a surjection $\mathbb{Q}\otimes_{\mathbb{Z}} Y_{\bar{k}}(T') \to \mathbb{Q}\otimes_{\mathbb{Z}} Y_{\bar{k}}(T_1)$ (the map before tensoring maps onto a finite-index subgroup: e.g., by transposing the injective map on character groups [Waterhouse 1979, Theorem 7.3]), hence there exist $n\in\mathbb{N}$ and $\mu \in Y_{\bar{k}}(T')$ such that $\pi\circ\mu = n\mu_1$. As $\mu \in Y_{\bar{k}}(T) = Y_{k_s}(T)$, μ is k_s-defined as required.

(ii) The first assertion is immediate, as is the assertion that the limits coincide. Since π is an epimorphism, we have $\pi(R_u(P_\lambda(G))) = R_u(P_{\pi\circ\lambda}(G_1))$ (see [Conrad et al. 2010, Corollary 2.1.9]). Moreover, since λ normalises NS, the restriction of π to $R_u(P_\lambda(G))$ is separable (see [Conrad et al. 2010, Proposition 2.1.8(3) and Remark 2.1.11]) and k_s-defined, and hence is surjective on k_s-points (cf. [Waterhouse 1979, Corollary 18.5]). This implies that if the common limit v' is in $R_u(P_{\pi\circ\lambda}(G_1))(k_s)\cdot v$, it is contained in $R_u(P_\lambda(G))(k_s)\cdot v$. The reverse implication is clear, since π is k_s-defined.

(iii) Suppose $G(k_s)\cdot v$ is not cocharacter-closed over k_s. Then there exists $\lambda \in Y_{k_s}$ such that $\lim_{a\to 0}\lambda(a)\cdot v$ exists but does not belong to $R_u(P_\lambda(G))(k_s)\cdot v$. Then $\lim_{a\to 0}(\pi\circ\lambda)(a)\cdot v$ exists but does not belong to $R_u(P_\lambda(G_1))(k_s)\cdot v$, by part (ii). Hence $G_1(k_s)\cdot v$ is not cocharacter-closed over k_s, by Theorem 2.5.

Now suppose $G_1(k_s)\cdot v$ is not cocharacter-closed over k_s. Then there exists $\mu_1 \in Y_{k_s}(G_1)$ such that $v' := \lim_{a\to 0}\mu_1(a)\cdot v$ exists and does not belong to $R_u(P_\mu(G_1))(k_s)\cdot v$. Replacing μ_1 with a positive multiple $n\mu_1$ of μ_1 if necessary, it follows from part (i) that there exists $\mu \in Y_{k_s}$ such that $\pi\circ\mu = \mu_1$. Then $\lim_{a\to 0}\mu(a)\cdot v$ is equal to v' and v' does not belong to $R_u(P_\lambda(G))(k_s)\cdot v$, by part (ii). Hence $G(k_s)\cdot v$ is not cocharacter-closed over k_s, by Theorem 2.5. □

Remark 4.2. We insist in Lemma 4.1(i) that λ be a cocharacter of M because we need this in the proof of Theorem 1.3.

Lemma 4.3. *Let G be connected, let S be a k_s-torus of G_v and set $L = C_G(S)$. Suppose that for every $\lambda, \mu \in Y_{k_s}$ such that $P_\lambda = P_\mu$, either both of λ and μ destabilize v or neither does. Then for every $\lambda, \mu \in Y_{k_s}(L)$ such that $P_\lambda(L) = P_\mu(L)$, either both of λ and μ destabilize v or neither does.*

Proof. Let $\lambda, \mu \in Y_k(L)$ such that $P_\lambda(L) = P_\mu(L)$. We can choose $\sigma \in Y_k(S)$ such that $L_\sigma = L$ [Bate et al. 2015, Lemma 2.5]. Then $P_\lambda(L_\sigma) = P_\mu(L_\sigma)$ and there exists $n \in \mathbb{N}$ such that $P_{n\sigma+\lambda} = P_\lambda(L_\sigma)R_u(P_\sigma)$ and $P_{n\sigma+\mu} = P_\mu(L_\sigma)R_u(P_\sigma)$ [Bate et al. 2005, Lemma 6.2(i)]. By hypothesis, either both of $n\sigma + \lambda$ and $n\sigma + \mu$ destabilize v, or neither one does. In the first case, since σ fixes v, both λ and μ must destabilize v. Conversely, in the second case neither λ nor μ can destabilize v. □

Lemma 4.4. *Let T be a maximal torus of G, let $\mu_1, \dots, \mu_r \in Y(T)\setminus\{0\}$, let $\mu = \sum_{i=1}^r \mu_i$ and assume $\mu \neq 0$. Suppose $g \in G$ and $g \cdot \zeta(\mu_i) = \zeta(\mu_i)$ for all $1 \leq i \leq r$. Then $g \cdot \zeta(\mu) = \zeta(\mu)$.*

Proof. Clearly, there is a permutation $\tau \in S_r$ such that none of the sums $\sum_{i=1}^t \mu_{\tau(i)}$ is 0 for $1 \leq t \leq r$. Consider the special case $r = 2$ (the general case follows easily by induction on r). Recall the addition operation $+$ on $V(\mathbb{K})$ and the canonical projection $\varphi\colon Y(\mathbb{K}) \to V(\mathbb{K})$ from Section 3B. Let $\xi : V(\mathbb{K})\setminus\{0\} \to \Delta(\mathbb{K})$ be the canonical projection. Note that φ and ξ are G-equivariant. Moreover, as g fixes $\zeta(\mu_1)$ and g acts as an isometry, g fixes $\varphi(\mu_1)$, and likewise g fixes $\varphi(\mu_2)$. We have

$$
\begin{aligned}
g \cdot \zeta(\mu) = g \cdot \zeta(\mu_1 + \mu_2) = g \cdot \xi(\varphi(\mu_1 + \mu_2)) &= \xi(g \cdot \varphi(\mu_1 + \mu_2)) \\
&= \xi(g \cdot (\varphi(\mu_1) + \varphi(\mu_2))) = \xi(g \cdot \varphi(\mu_1) + g \cdot \varphi(\mu_2)) \\
&= \xi(\varphi(\mu_1) + \varphi(\mu_2)) = \xi(\varphi(\mu_1 + \mu_2)) = \zeta(\mu_1 + \mu_2) = \zeta(\mu),
\end{aligned}
$$

as required. □

We now have everything in place to prove Theorem 1.1.

Proof of Theorem 1.1. For part (i), suppose $v \in V(k)$ and $G(k_s)\cdot v$ is not cocharacter-closed over k_s. Clearly there is no harm in assuming S is a maximal k-defined torus of G_v, so we shall do this. Since the closed subgroup $\overline{G_v(k_s)}$ generated by $G_v(k_s)$ is k_s-defined and Γ-stable, it is k-defined. Hence S is a maximal torus of $\overline{G_v(k_s)}$; in particular, S is a maximal k_s-defined torus of G_v. Set $H = C_G(S)$. If $\sigma \in Y_k(H)$ and σ destabilizes v but does not fix v then σ properly destabilizes v over k_s for G, by [Bate et al. 2015, Lemma 2.8]. Hence it is enough to prove that such a σ exists.

By Theorem 2.7(ii), $H(k_s)\cdot v$ is not cocharacter-closed over k_s. So we can choose $\mu \in Y_{k_s}(H)$ such that μ properly destabilizes v over k_s for H. If $\mu \in Y_{k_s}(Z(H^0))$ then we are done by Lemma 2.8 and Remark 2.9. So assume otherwise. Then $P_\mu(H^0)$ is a proper subgroup of H^0. By Lemma 4.3, $\Sigma_{v,k_s}(H)$ is a subcomplex of $\Delta_{k_s}(H, \mathbb{K})$. It follows from Lemma 3.5 that $\Sigma_{v,k_s}(H)$ is not completely reducible, since if Q is an opposite parabolic to $P_\mu(H^0)$ in H^0 then there exists $\mu' \in Y_{k_s}(H)$ such that $P_{\mu'} = Q$ and μ' is opposite to μ, which is impossible. Clearly, $\Sigma_{v,k_s}(H)$ is Γ- and $H_v(k_s)$-stable, so by Theorem 3.6 there is a Γ- and $H_v(k_s)$-fixed simplex

$s \in \Sigma_{v,k_s(H)}$, corresponding to some proper parabolic subgroup P of H^0. There exists $\sigma \in Y_k(H)$ such that $P = P_\sigma(H^0)$ [Bate et al. 2013, Lemma 2.5(ii)], and σ destabilizes v by construction. Now $\sigma \notin Y_{k_s}(Z(H^0))$ since P is proper. But every k_s-defined torus of $H_v(k_s)$ is contained in $Z(H^0)$ (since S is contained in $Z(H^0)$), so σ does not fix v. As σ commutes with S, it follows from [Bate et al. 2015, Lemma 2.8] that $v' := \lim_{a\to 0}\sigma(a)\cdot v$ does not lie in $H(k_s)\cdot v$. This completes the proof of (i).

For part (ii), Proposition 2.6 shows that if $G(k')\cdot v$ is cocharacter-closed over k' then $G(k)\cdot v$ is cocharacter-closed over k. For the other direction, suppose that $G(k')\cdot v$ is not cocharacter-closed over k'. Again by Proposition 2.6, we may assume $k' = k_s$. Applying part (i) with $S = 1$, we find $\sigma \in Y_k$ such that σ properly destabilizes v over k_s. In particular, $G(k)\cdot v$ is not cocharacter-closed over k. This finishes part (ii).

Part (iii) of Theorem 1.1 follows using similar arguments to those in the proof of [Bate et al. 2015, Theorem 5.7(ii)]. Let S be a k-defined torus of G_v and let $L = C_G(S)$. First, by the argument of [Bate et al. 2015, Lemma 6.2], we can assume without loss that $v \in V(k_s)$ without changing Σ_{v,k_s}. Second, by [Bate et al. 2015, Lemma 6.3] and the argument of the proof of [Bate et al. 2015, Theorem 6.1], we can pass to a suitable G-variety W and find $w \in W(k)$ such that $\Sigma_{w,k_s} = \bigcap_{\gamma\in\Gamma}\gamma\cdot\Sigma_{v,k_s}$; in particular, Σ_{w,k_s} is a subcomplex of $\Delta_k(\mathbb{Q})$ and $\Sigma_{w,k} = \Sigma_{v,k}$. The arguments of [Bate et al. 2015, Section 6] also show that S fixes w. Hence we can assume without loss that $v \in V(k)$. As before, Lemma 4.3 implies that $\Sigma_{v,k_s}(L)$ is a subcomplex of $\Delta_{k_s}(L, \mathbb{K})$. We may thus apply part (ii) and assume $k = k_s$. But then S is k-split, so the result follows from Theorem 2.7. □

Remark 4.5. We do not know how to prove that $P_\sigma(G^0)$ from Theorem 1.1(i) is normalised by $G_v(k_s)$, but the proof does show that $P_\sigma(G^0)$ is normalised by $H_v(k_s)$.

Proof of Theorem 1.3. For part (i), suppose $v \in V(k)$ and $G(k_s)\cdot v$ is not cocharacter-closed over k_s. Recall that a connected algebraic group is nilpotent if and only if it contains just one maximal torus (see [Humphreys 1975, §21.4 Proposition B]). Let G_i be a simple component of G^0. If $\mathrm{rank}(G_i) = 1$ and $\dim(G_i)_v \geq 2$ then $(G_i)_v^0$ must contain a Borel subgroup B_i of G_i: but then the orbit map $G_i \to G_i\cdot v$ factors through the connected projective variety G_i/B_i and hence is constant, so $(G_i)_v = G_i$.

Let N be the product of the simple components of G^0 that fix v, and let K be the product of the remaining simple components of G^0 together with $Z(G^0)^0$. Then N and K are Γ-stable, so they are k-defined. The next step is to factor out N and reduce to the case when the stabilizer has nilpotent identity component. As in the proof of Theorem 1.1, $\overline{G_v(k_s)}$ is k-defined and we may assume S is a maximal

k-defined torus of G_v and a k_s-defined maximal torus of $\overline{G_v(k_s)}$. We can choose k-defined tori S_0 of K and S_2 of N such that $S = S_0S_2$. Note that K_v^0 is nilpotent — this holds by assumption in case (a), and by the above argument in case (b) — so S_0 is the unique maximal k-defined torus of $\overline{K_v(k_s)}$.

Let $H_0 = N_G(S_0)$, let $H = N_{H_0}(N)$ and let $M = C_K(S_0)$. Then H_0 is k-defined [Conrad et al. 2010, Lemma A.2.9], so H is k-defined as it is Γ-stable and has finite index in H^0. Note that $H^0 = NM = C_G(S_0)^0$, so N is a product of simple components of H^0 and M is the product of $Z(H^0)^0 = S^0Z(G^0)^0$ with the remaining simple components of H^0. The subgroup M of H is normal, so it is Γ-stable and hence k-defined. Now M_v^0 is nilpotent since K_v^0 is, so M_v^0 has a unique maximal torus S' — in particular, $S_0 \subseteq S'$ and S_0 is the unique maximal torus of $\overline{M_v(k_s)}$. Since $G_v(k_s)$ normalises N and K, $G_v(k_s)$ normalises N and S_0, so $G_v(k_s) \subseteq H$; it follows that $H_v(k_s) = G_v(k_s)$.

Let $H_1 = H/NS_0$ and let $\pi : H \to H_1$ be the canonical projection. We wish to find $\lambda_1 \in Y_{k_s}(H_1)$ such that λ_1 properly destabilizes v over k_s and $P_{\lambda_1}(H_1)$ is k-defined. Note that no nontrivial k_s-defined cocharacter of H_1 fixes v; for if $0 \neq \lambda_1 \in Y_{k_s}(H_1)$ fixes v then by Lemma 4.1, there exist $n \in \mathbb{N}$ and $\lambda \in Y_{k_s}(M)$ such that $\pi \circ \lambda = n\lambda_1$, and $\langle \mathrm{Im}(\lambda) \cup S \rangle$ is a k_s-defined torus of $\overline{G_v(k_s)}$ that properly contains S, contradicting the maximality of S. Clearly, $(H_1)_v^0$ is nilpotent with unique maximal torus $S_1' := \pi(S')$. Since $H^0 = C_G(S_0)^0$, $H(k_s) \cdot v$ is not cocharacter-closed over k_s, by Theorem 2.7(ii) and [Bate et al. 2015, Corollary 5.3]. Hence $H_1(k_s) \cdot v$ is not cocharacter-closed over k_s (Lemma 4.1). Let $\lambda_1 \in Y_{k_s}(H_1)$ such that λ_1 destabilizes v. By Lemmas 2.8 and 4.1, we can assume λ_1 does not properly destabilize v over $\bar{k}$. Therefore, there exists $u \in R_u(P_{\lambda_1}(H_1))$ such that $u \cdot \lambda_1$ fixes v; then $u \cdot \lambda_1$ must be a cocharacter of S_1'. It follows that $\Sigma_{v,k_s}(H_1) \subseteq \Delta_{k_s}(T_1, \mathbb{Q})$, where T_1 is a fixed maximal torus of H_1 that contains S_1'. Note that T_1 and S_1' need not be k-defined, or even k_s-defined. As $H_1(k_s) \cdot v$ is not cocharacter-closed over k_s, $\Sigma_{v,k_s}(H_1)$ is not completely reducible (Lemma 3.5(ii)). Now $\Sigma_{v,k_s}(H_1)$ is stabilized by Γ and by $(H_1)_v(k_s)$, so it follows from Theorem 3.9 that $\Sigma_{v,k_s}(H_1)$ contains a Γ-fixed and $(H_1)_v(k_s)$-fixed point x_1. We can write $x_1 = \zeta(\mu_1)$ for some $\mu_1 \in Y_{k_s}(H_1)$. Then μ_1 destabilizes v but does not fix v; moreover, $P_{\mu_1}(H_1^0)$ is Γ-stable and is normalised by $(H_1)_v(k_s)$. In particular, $P_{\mu_1}(H_1^0)$ is k-defined.

By Lemma 4.1, there exist $n \in \mathbb{N}$ and $\mu \in Y_{k_s}(M)$ such that $\pi \circ \mu = n\mu_1$ and μ destabilizes v; note that μ does not fix v, because μ_1 does not. The map π gives a bijection between the parabolic subgroups of M^0 and the parabolic subgroups of H_1^0. So $P_\mu(M^0)$ is Γ-stable — because $P_{\mu_1}(H_1^0)$ is — and hence is defined over k. As $\pi(H_v(k_s))$ is contained in $(H_1)_v(k_s)$ and $(H_1)_v(k_s)$ normalises $P_{\mu_1}(H_1^0)$, $H_v(k_s)$ normalises $P_\mu(M^0)$.

After replacing μ if necessary with an $R_u(P_\mu(H^0))(k_s)$-conjugate of μ, we can assume that μ is a cocharacter of a k-defined maximal torus T of $P_\mu(H^0)$. Let

$\mu^{(1)}, \ldots, \mu^{(r)}$ be the Γ-conjugates of μ. These are cocharacters of T, so they all commute with each other. Set $\sigma = \sum_{i=1}^{r} \mu^{(i)}$, a k-defined cocharacter of T. Note that σ centralizes $S = S_0 S_2$. Now $\pi \circ \sigma$ destabilizes v but does not fix v (since $\pi \circ \sigma \neq 0$), so σ does not fix v. This implies by [Bate et al. 2015, Lemma 2.8] that σ properly destabilizes v over k_s for G. Since $H_v(k_s)$ is Γ-stable and fixes $\zeta(\mu)$, $H_v(k_s)$ fixes $\zeta(\mu^{(i)})$ for all $1 \leq i \leq r$. It follows from Lemma 4.4 that $H_v(k_s)$ fixes $\zeta(\sigma)$: that is, for all $h \in H_v(k_s)$, there exists $u \in R_u(P_\sigma(H^0))(k_s)$ such that $h \cdot \sigma = u \cdot \sigma$. Hence $P_\sigma(G^0)$ is normalised by $H_v(k_s) = G_v(k_s)$. This completes the proof of (i).

Parts (ii) and (iii) now follow as in the proof of Theorem 1.1 (there is no need to reduce to the case when v is a k-point in (iii) because we already assume this). $\square$

Remark 4.6. It can be shown that Theorem 1.3(iii) actually holds without the assumption that v is a k-point. Here is a sketch of the proof. It is enough to prove that Levi ascent holds. Without loss, assume S is a maximal k-defined torus of G_v. As in the proof of Theorem 1.1(iii), we replace v with a k-point w of a k-defined G-variety W, with the property that $\Sigma_{w,k_s} \subseteq \Sigma_{v,k_s}$ and $\Sigma_{w,k_s} = \Sigma_{v,k_s}$. By the arguments of [Bate et al. 2015, Section 6], we can assume that S and N fix w. Suppose $G(k) \cdot v$ is not cocharacter-closed over k. Then $G(k) \cdot w$ is not cocharacter-closed over k, so $L(k_s) \cdot w$ is not cocharacter-closed over k_s, by Galois descent and split Levi ascent. It follows that $H_1(k_s) \cdot w$ is not cocharacter-closed over k_s, where H_1 is defined as in the proof of Theorem 1.3. We do not know whether hypotheses (a) and (b) of Theorem 1.3 hold for w. The key point, however, that makes the proof of Theorem 1.3(i) work is that $\Sigma_{v,k_s}(H_1)$ is contained in a single apartment of $\Delta_{k_s}(H_1, \mathbb{Q})$. The analogous property holds for $\Sigma_{w,k_s}(H_1)$ since $\Sigma_{w,k_s}(H_1) \subseteq \Sigma_{v,k_s}(H_1)$. Hence Galois ascent holds and $H_1(k) \cdot w$ is not cocharacter-closed over k. Then $H_1(k) \cdot v$ is not cocharacter-closed over k, and the result follows.

5. Applications to G-complete reducibility

Many of the constructions in this paper, and in the key references [Bate et al. 2013; 2012; 2015], were inspired originally by the study of Serre's notion of *G-complete reducibility* for subgroups of G. We refer the reader to [Serre 2005] and [Bate et al. 2005] for a thorough introduction to the theory. We simply record the basic definition here:

Definition 5.1. A subgroup H of G is said to be *G-completely reducible over k* if whenever H is contained in a k-defined R-parabolic subgroup P of G, there exists a k-defined R-Levi subgroup L of P containing H. (We do not assume that H is k-defined.)

Theorem 5.2 ([Bate et al. 2015, Theorem 9.3]). *Let H be a subgroup of G and let $\boldsymbol{h} \in H^n$ be a generic tuple of H* (see [Bate et al. 2013, Definition 5.4]). *Then H is G-completely reducible over k if and only if $G(k) \cdot \boldsymbol{h}$ is cocharacter-closed over k, where G acts on G^n by simultaneous conjugation.*

Theorem 5.2 allows us to prove results about G-complete reducibility over k using our results on geometric invariant theory. If G is connected, and $\boldsymbol{h} \in G^n$ is a generic tuple for a subgroup H of G, then $\Sigma_{\boldsymbol{h}}$ is a subcomplex of $\Delta_G(\mathbb{Q})$, since for any $\lambda \in Y$, λ destabilizes $\boldsymbol{h}$ if and only if $H \subseteq P_\lambda$; this means that we are in the territory of Theorem 1.1.

Proof of Theorem 1.4. Let $\boldsymbol{h}$ be a generic tuple of H. Then $\Sigma_{\boldsymbol{h},k_s}$ is a subcomplex of Δ_{G,k_s}, and $C_G(H) = G_{\boldsymbol{h}}$. The result now follows from Theorems 5.2 and 1.1(iii). □

This theory has a counterpart for Lie subalgebras of $\mathfrak{g} := \mathrm{Lie}(G)$. The basic definitions and results were covered for algebraically closed fields in [McNinch 2007] and [Bate et al. 2011, Section 3.3], but the extension to arbitrary fields is straightforward (cf. [Bate et al. 2011, Remark 4.16]).

Definition 5.3. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is *G-completely reducible over k* if whenever P is a k-defined parabolic subgroup of G such that $\mathfrak{h} \subseteq \mathrm{Lie}(P)$, there exists a k-defined Levi subgroup L of P such that $\mathfrak{h} \subseteq \mathrm{Lie}(L)$. (We do not assume that $\mathfrak{h}$ is k-defined.)

The group G acts on $\mathfrak{g}^n$ via the simultaneous adjoint action for any $n \in \mathbb{N}$. The next result follows from [Bate et al. 2011, Lemma 3.8] and the arguments in the proofs of [Bate et al. 2011, Theorems 4.12(iii)] (cf. [Bate et al. 2011, Theorem 3.10(iii)]).

Theorem 5.4. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and let $\boldsymbol{h} \in \mathfrak{h}^n$ such that the components of $\boldsymbol{h}$ generate $\mathfrak{h}$ as a Lie algebra. Then $\mathfrak{h}$ is G-completely reducible over k if and only if $G(k) \cdot \boldsymbol{h}$ is cocharacter-closed over k.*

We now give the applications of our earlier results to G-complete reducibility over k for Lie algebras.

Theorem 5.5. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$.*

(i) *Suppose $\mathfrak{h}$ is k-defined, and let k'/k be a separable algebraic extension. Then $\mathfrak{h}$ is G-completely reducible over k' if and only if $\mathfrak{h}$ is G-completely reducible over k.*

(ii) *Let S be a k-defined torus of $C_G(\mathfrak{h})$ and set $L = C_G(S)$. Then $\mathfrak{h}$ is G-completely reducible over k if and only if $\mathfrak{h}$ is L-completely reducible over k.*

Proof. Pick $\boldsymbol{h} \in \mathfrak{h}^n$ for some $n \in \mathbb{N}$ such that the components of $\boldsymbol{h}$ generate $\mathfrak{h}$ as a Lie algebra. If $\mathfrak{h}$ is k-defined then we can assume that $\boldsymbol{h} \in \mathfrak{h}(k)^n$. Part (i) now follows from Theorems 5.4 and 1.1(ii), and part (ii) from Theorems 5.4 and 1.1(iii). □

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EMBEDDING FUNCTOR FOR CLASSICAL GROUPS AND BRAUER–MANIN OBSTRUCTION

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In memory of Robert Steinberg

Let K be a global field of characteristic not 2. The embedding problem for maximal tori in a classical group G can be described in terms of algebras with involution. The aim of this paper is to give an explicit description of the obstruction group to the Hasse principle in terms of ramification properties of certain commutative étale algebras, and to show that this group is isomorphic to one previously defined by the second author. This builds on our previous work as well as on results of Borovoi. In particular, we show that this explicit obstruction group can be identified with the group of Borovoi (*J. Reine Angew. Math.* 473 (1996), 181–194), where X is the homogeneous space associated to the embedding functor defined by the second author (*Comment. Math. Helv.* 89 (2014), 671–717).

Introduction

Let K be a field of characteristic $\neq 2$, and let G be a reductive linear algebraic group defined over K. The paper [Lee 2014] is concerned with embeddings of maximal tori into G. In particular, if K is a global field, then results of Borovoi [1999] are used to show that the Brauer–Manin obstruction is the only obstruction to the Hasse principle. More precisely, the paper [Lee 2014] defines a homogeneous space X over G having the property that the obstruction to the Hasse principle can be seen as an element of the dual of the group $\text{Б}(X)$, where $\text{Б}(X)$ is the locally trivial subgroup of the algebraic Brauer group of X (see [Borovoi 1999, p. 493, p. 499]).

If G is a classical group, then the above-mentioned embedding problem can be described in terms of embeddings of algebras with involution. The aim of the present paper is to give an explicit description of the obstruction group $\text{Б}(X)$ in terms of ramification properties of certain commutative étale algebras.

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We are not aware of similar descriptions in the case of exceptional groups. Note however that when G is of type G_2, the group Б(X) vanishes; in particular, the Hasse principle holds (see [Beli et al. 2015, Proposition 6.1]).

The paper is structured as follows. In Section 1, we recall from [Lee 2014] the definition of the oriented embedding functor, and we discuss its relationship with embedding questions of algebras with involution. In Sections 2–5, we assume moreover that K is a global field. In these sections we give the description of the obstruction group Б, and prove that Б $\simeq$ Б(X) (see Theorem 2.1). Finally, Section 6 discusses Brauer–Manin obstructions to the Hasse principle, and the relationship of the results of the present paper with those of [Bayer-Fluckiger et al. 2014].

1. Embedding functor, algebras with involution and orientation

1.1. ***The embedding functor.*** Let K be a field of characteristic $\neq 2$, let K_s be a separable closure of K, and let $\Gamma_K = \mathrm{Gal}(K_s/K)$. Let G be a reductive group over K. Let T be a torus and let Ψ be a root datum attached to T (see [Demazure and Grothendieck 2011, Exposé XXI, Definition 1.1.1]). For a maximal torus T' in G, we let $\Phi(G, T')$ be the root datum of G with respect to T'. If $\Phi(G, T')_{K_s}$ and Ψ_{K_s} are isomorphic, then we say that G and Ψ have the same type.

Assume that G and Ψ have the same type. Let $\underline{\mathrm{Isom}}(\Psi, \Phi(G, T'))$ be the scheme of isomorphisms between the root data Ψ and $\Phi(G, T')$. Define

$$\underline{\mathrm{Isomext}}(\Psi, \Phi(G, T')) = \underline{\mathrm{Isom}}(\Psi, \Phi(G, T'))/\mathrm{W}(\Psi),$$

where $\mathrm{W}(\Psi)$ is the Weyl group of Ψ. The scheme $\underline{\mathrm{Isomext}}(\Psi, \Phi(G, T'))$ is independent of the choice of the maximal torus T', and we denote it by $\underline{\mathrm{Isomext}}(\Psi, G)$. An *orientation* is by definition an element of $\underline{\mathrm{Isomext}}(\Psi, G)(K)$.

The *embedding functor* $E(G, \Psi)$ is defined as follows: for any K-algebra C, let $E(G, \Psi)(C)$ be the set of embeddings $f : T_C \to G_C$ such that f is both a closed immersion and a group homomorphism which induces an isomorphism $f^\Psi : \Psi_C \xrightarrow{\sim} \Phi(G_C, f(T_C))$ such that $f^\Psi(\alpha) = \alpha \circ f^{-1}|_{f(T_{C'})}$ for all the C'-roots α in $\Psi_{C'}$ for each C-algebra C' (see [Lee 2014, §2.1]). Given an orientation ν in $\underline{\mathrm{Isomext}}(\Psi, G)(K)$, we define the *oriented embedding functor* as follows (see [Lee 2014, §2.2]): for any K-algebra C, set

$$E(G, \Psi, \nu)(C) = \{f : T_C \hookrightarrow G_C \mid f \in E(G, \Psi)(C) \text{ and the image of } f^\Psi \text{ in } \underline{\mathrm{Isomext}}(\Psi, G)(C) \text{ is } \nu\}.$$

The oriented embedding functor is a homogeneous space under the adjoint action of G. For each root datum Ψ, we can associate a simply connected root datum $\mathrm{sc}(\Psi)$ to it (see [Demazure and Grothendieck 2011, Exposé XXI, §6.5.5 (iii)]). Let $\mathrm{sc}(T)$ be the torus associated to $\mathrm{sc}(\Psi)$.

1.2. ***Algebras with involution and the embedding functor.*** Let L be a field of characteristic $\neq 2$, and let A be a central simple algebra over L. Let τ be an involution of A, and let K be the fixed field of τ in L. Recall that τ is said to be of the *first kind* if $K = L$ and of the *second kind* if $K \neq L$; in this case, L is a quadratic extension of K. Let $\dim_L(A) = n^2$. Let E be a commutative étale algebra of rank n over L, and let $\sigma : E \to E$ be a K-linear involution such that $\sigma|L = \tau|L$. An *embedding* of (E, σ) in (A, τ) is by definition an injective homomorphism $f : E \to A$ such that $\tau(f(e)) = f(\sigma(e))$ for all $e \in E$.

The unitary groups $\mathrm{U}(A, \tau)$ and $\mathrm{U}(E, \sigma)$ are defined as follows. For any commutative K-algebra C, set

$$\mathrm{U}(A, \tau)(C) = \{x \in A \otimes_K C \mid x\tau(x) = 1\}$$

and

$$\mathrm{U}(E, \sigma)(C) = \{x \in E \otimes_K C \mid x\sigma(x) = 1\}.$$

Let $G = \mathrm{U}(A, \tau)^\circ$ be the connected component of $\mathrm{U}(A, \tau)$ containing the neutral element, and let $T = \mathrm{U}(E, \sigma)^\circ$ be the connected component of $\mathrm{U}(E, \sigma)$ containing the neutral element.

Set $F = \{e \in E \mid \sigma(e) = e\}$. If $L \neq K$, then we have $\dim_K(F) = n$ (see for instance [Prasad and Rapinchuk 2010, Proposition 2.1]). If $L = K$, then let us assume that $\dim_K(F) = [(n+1)/2]$.

Then one can associate a root datum Ψ to the torus T such that G is of type Ψ (see [Lee 2014, §2.3.1]). Moreover, except for A of degree 2 with τ orthogonal, there exists a K-embedding from (E, σ) to (A, τ) if and only if there exists an orientation ν such that $E(G, \Psi, \nu)(K)$ is nonempty (see [Lee 2014, Theorem 2.15 and Proposition 2.17]).

1.3. ***Orientations in terms of algebras.*** Let (E, σ) and (A, τ) be as above. Assume moreover that (A, τ) is orthogonal, and that the degree of A is even. Let $\Delta(E)$ be the discriminant of the étale algebra E (see [Knus et al. 1998, Chapter V, §18, p. 290]), and let $Z(A, \tau)$ be the center of the Clifford algebra of (A, τ) (see [Knus et al. 1998, Chapter II (8.7)]). Then an orientation can be thought of as the choice of an isomorphism $\Delta(E) \to Z(A, \tau)$. More precisely:

Proposition 1.3.1. *We have an isomorphism*

$$\underline{\mathrm{Isom}}(\Delta(E), Z(A, \tau)) \simeq \underline{\mathrm{Isomext}}(\Psi, G).$$

Proof. Let E_τ be a maximal τ-invariant étale subalgebra of A. Let $T_\tau = \mathrm{U}(E_\tau, \tau)^\circ$; then T_τ is a maximal torus of G. Let $\Phi(G, T_\tau)$ be the root datum of G with respect to T_τ. Then we have a natural map $\alpha : \underline{\mathrm{Isom}}((E, \sigma), (E_\tau, \tau)) \to \underline{\mathrm{Isom}}(\Psi, \Phi(G, T_\tau))$. Using the identification of $\underline{\mathrm{Aut}}(E, \sigma)$ and $\underline{\mathrm{Aut}}(\Psi)$, we see that α is equivariant under the action of $\underline{\mathrm{Aut}}(E, \sigma)$. Let Γ_0 be the subgroup of $\underline{\mathrm{Aut}}(E, \sigma)$ corresponding to the Weyl group of Ψ under this identification. Note that Γ_0 is the twisted constant scheme which consists of even permutations in $\underline{\mathrm{Aut}}(E,\sigma) \subset \underline{\mathrm{Aut}}(E)$. Indeed, by [Lee 2014, Lemma 2.1.1 (2)] the automorphisms of (E,σ) are in bijection with those of the root datum Ψ. By [Bourbaki 1981, Planche IV, numéro X], these consist of even permutations. Let us consider the following commutative diagram:

$$\begin{array}{ccc} \underline{\mathrm{Isom}}((E,\sigma),(E_\tau,\tau)) & \longrightarrow & \underline{\mathrm{Isom}}(\Psi,\Phi(G,T_\tau)) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Isom}}((E,\sigma),(E_\tau,\tau))/\,\Gamma_0 & \longrightarrow & \underline{\mathrm{Isom}}(\Psi,\Phi(G,T_\tau))/\mathrm{W}(\Psi) \end{array}$$

Recall that $\underline{\mathrm{Isom}}(\Psi,\Phi(G,T_\tau))/\mathrm{W}(\Psi) = \underline{\mathrm{Isomext}}(\Psi,\Phi(G,T_\tau))$, and note that we have $\underline{\mathrm{Isom}}((E,\sigma),(E_\tau,\tau))/\,\Gamma_0 \simeq \underline{\mathrm{Isom}}(\Delta(E),\Delta(E_\tau))$.

If we pick another maximal étale subalgebra E'_τ of A invariant by τ, then the method used for $\underline{\mathrm{Isomext}}(\Psi,\Psi_\tau)$ in [Lee 2014, §2.2.1] shows that we have a canonical isomorphism between $\underline{\mathrm{Isom}}(\Delta(E),\Delta(E'_\tau))$ and $\underline{\mathrm{Isom}}(\Delta(E),\Delta(E_\tau))$.

Let us fix an isomorphism $\Delta(E_\tau) \to Z(A,\tau)$ as in [Bayer-Fluckiger et al. 2014, §2.3]. This gives an isomorphism $\underline{\mathrm{Isom}}(\Delta(E),\Delta(E_\tau)) \to \underline{\mathrm{Isom}}(\Delta(E),Z(A,\tau))$. Hence, we have

$$\underline{\mathrm{Isom}}(\Delta(E),Z(A,\tau)) \simeq \underline{\mathrm{Isomext}}(\Psi,\Phi(G,T_\tau)) = \underline{\mathrm{Isomext}}(\Psi,G). \qquad \square$$

See [Bayer-Fluckiger et al. 2014, §2] for details concerning the construction and properties of orientation in terms of algebras with involution.

2. Obstruction groups

Assume now that K is a global field, let (E,σ), (A,τ) be as in Section 1, and suppose that τ is either orthogonal or unitary. Note that $L = K$ in the first case, and $L \neq K$ in the second case. The aim of this section and the following ones is to recall the definition of the obstruction group to the Hasse principle defined in [Bayer-Fluckiger et al. 2014, §3, §5.1], and show that it is isomorphic to the obstruction group of [Lee 2014] (see Proposition 2.2), as well as to the one considered by Borovoi [1996; 1999] (see Theorem 2.1). As we will see, the group $Ш(E,\sigma)$ is defined in terms of ramification properties of the algebra (E,σ).

Let us denote by Ω_K the set of places of K. For all $v \in \Omega_K$, we denote by K_v the completion of K at v. For all K-algebras B, set $B^v = B \otimes_K K_v$.

The commutative étale algebra E is by definition a product of separable field extensions of L. Let us write $E = E_1 \times \cdots \times E_m$, with $\sigma(E_i) = E_i$ for all $i = 1, \ldots, m$, and such that E_i is either a field stable by σ or a product of two fields exchanged by σ. Recall that $F = E^\sigma$.

Set $I = \{1, \dots, m\}$. We have $F = F_1 \times \cdots \times F_m$, where F_i is the fixed field of σ in E_i for all $i \in I$. Note that either $E_i = F_i = K$ or $E_i = F_i \times F_i$ or E_i is a quadratic field extension of F_i.

Let us write

$$E = E_1 \times \cdots \times E_{m_1} \times E_{m_1+1} \times \cdots \times E_m,$$

where E_i/F_i is a quadratic extension for all $i = 1, \dots, m_1$ and where $E_i = F_i \times F_i$ or $E_i = K$ if $i = m_1 + 1, \dots, m$. Set $E' = E_1 \times \cdots \times E_{m_1}$ and $I' = \{1, \dots, m_1\}$. If $i \in I'$, let Σ_i be the set of places $v \in \Omega_K$ such that all the places of F_i over v split in E_i. Given an m_1-tuple $x = (x_1, \dots, x_{m_1}) \in (\mathbb{Z}/2\mathbb{Z})^{m_1}$, set

$$I_0 = I_0(x) = \{i \mid x_i = 0\}, \quad I_1 = I_1(x) = \{i \mid x_i = 1\}.$$

Note that (I_0, I_1) is a partition of I'. Let S' be the set

$$S' = \left\{(x_1, \dots, x_m) \in (\mathbb{Z}/2\mathbb{Z})^{m_1} \mid \left(\bigcap_{i \in I_0} \Sigma_i\right) \cup \left(\bigcap_{j \in I_1} \Sigma_j\right) = \Omega_K\right\},$$

and set

$$S = S' \cup (0, \dots, 0) \cup (1, \dots, 1).$$

We define an equivalence relation on S by

$$\begin{aligned}(x_1, \dots, x_{m_1}) \sim (x'_1, \dots, x'_{m_1}) \quad &\text{if } (x_1, \dots, x_{m_1}) + (x'_1, \dots, x'_{m_1}) = (1, \dots, 1)\\ &\text{or } (x_1, \dots, x_{m_1}) = (x'_1, \dots, x'_{m_1}).\end{aligned}$$

Let us denote by $Б(E, \sigma)$ the set of equivalence classes of S under the above equivalence relation. It is easy to check that $Б(E, \sigma)$ is a group under componentwise addition (see [Bayer-Fluckiger et al. 2014, Lemma 3.1.1]). Note that in [Bayer-Fluckiger et al. 2014], the group $Б(E, \sigma)$ is denoted by $Ш(E', \sigma)$ (see [Bayer-Fluckiger et al. 2014, §3, §5.1]).

Set $X = E(G, \Psi, u)$. Recall that we are assuming that τ is either orthogonal (and $L = K$) or unitary (and $L \neq K$). The group $Б(X)$ is defined in [Borovoi 1999, §3].

Theorem 2.1. *The groups $Б(E, \sigma)$ and $Б(X)$ are isomorphic.*

This theorem is a consequence of Propositions 2.2 and 2.3 below.

Proposition 2.2. *The groups $Ш^1(K, \mathrm{sc}(\hat{T}))$ and $Б(E, \sigma)$ are isomorphic.*

Proposition 2.3. *The groups $Ш^1(K, \mathrm{sc}(\hat{T}))$ and $Б(X)$ are isomorphic.*

The proofs of these propositions will be given in the next sections. Let us start by introducing some notation that will be used in both proofs. For any finite separable field extension $\mathbb{N}/\mathbb{N}'$ and any discrete $\Gamma_{\mathbb{N}}$-module M, set $\mathrm{I}_{\mathbb{N}/\mathbb{N}'}(M) = \mathrm{Ind}_{\Gamma_{\mathbb{N}}}^{\Gamma_{\mathbb{N}'}}(M)$. Note that $\mathrm{I}_{\mathbb{N}/\mathbb{N}'}(\mathbb{Z})$ is the character group of $\mathrm{R}_{\mathbb{N}/\mathbb{N}'}(\mathbb{G}_m)$. Let $\hat{\mathrm{S}}_{\mathbb{N}/\mathbb{N}'}$ be the character group of the norm-one torus $\mathrm{R}^{(1)}_{\mathbb{N}/\mathbb{N}'}(\mathbb{G}_m)$.

3. Proof of Proposition 2.2 when $L = K$ and τ is orthogonal

We keep the notation of the previous sections, and assume that $L = K$ and that τ is orthogonal. The aim of this section is to prove Proposition 2.2 in this case. The proof of Proposition 2.2 when $L \neq K$ is the subject matter of Section 4.

Let us consider the diagram

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mathrm{R}^{(1)}_{F/K}(\mathbb{G}_m) \arrow[r] \arrow[d] & \mathrm{R}^{(1)}_{E/K}(\mathbb{G}_m) \arrow[r] \arrow[d] & \mathrm{sc}(T) \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{R}_{F/K}(\mathbb{G}_m) \arrow[r] \arrow[d] & \mathrm{R}_{E/K}(\mathbb{G}_m) \arrow[d] & & \\
1 \arrow[r] & \mathbb{G}_m \arrow[r, "\times 2"] \arrow[d] & \mathbb{G}_m \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}
\tag{1}
$$

where the first row (see [Lee 2014, Lemma 3.16]) and the columns are exact. Then consider the corresponding diagram of character groups:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[d] & \\
 & & \mathrm{I}_{E/K}(\mathbb{Z}) \arrow[r, "\pi"] \arrow[d] & \mathrm{I}_{F/K}(\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{sc}(\hat{T}) \arrow[r] & \hat{\mathrm{S}}_{E/K} \arrow[r, "\pi"] \arrow[d] & \hat{\mathrm{S}}_{F/K} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
\tag{2}
$$

Note that we have $\mathrm{I}_{E/K}(\mathbb{Z}) = \bigoplus_{i=1}^m \mathrm{I}_{E_i/K}(\mathbb{Z})$ and $\mathrm{I}_{F/K}(\mathbb{Z}) = \bigoplus_{i=1}^m \mathrm{I}_{F_i/K}(\mathbb{Z})$. The module $\mathrm{I}_{E_i/K}(\mathbb{Z})$ can also be written as $\mathrm{I}_{F_i/K}(\mathrm{I}_{E_i/F_i}(\mathbb{Z}))$. Let d be the degree map from $\mathrm{I}_{E_i/F_i}(\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ to $\mathbb{Z}$, which sends (x, y) to $x + y$. Then on each $\mathrm{I}_{F_i/K}(\mathrm{I}_{E_i/F_i}(\mathbb{Z}))$, the map π is the map induced by the degree map from $\mathrm{I}_{E_i/F_i}(\mathbb{Z})$ to $\mathbb{Z}$.

Set $\Gamma = \Gamma_K$. We derive the following long exact sequence from diagram (2):

$$0 \to \mathrm{sc}(\hat{T})^{\Gamma} \to (\hat{\mathrm{S}}_{E/K})^{\Gamma} \xrightarrow{\pi} (\hat{\mathrm{S}}_{F/K})^{\Gamma} \to \mathrm{H}^1(K, \mathrm{sc}(\hat{T})) \to \mathrm{H}^1(K, \hat{\mathrm{S}}_{E/K}).$$

Thus we have the exact sequence

$$0 \to (\hat{\mathrm{S}}_{F/K})^{\Gamma}/\pi((\hat{\mathrm{S}}_{E/K})^{\Gamma}) \xrightarrow{\delta} \mathrm{H}^1(K, \mathrm{sc}(\hat{T})) \to \mathrm{H}^1(K, \hat{\mathrm{S}}_{E/K}).$$

Note that $\mathrm{H}^2(K, \mathrm{R}^{(1)}_{E/K}(\mathbb{G}_m))$ injects into $\mathrm{H}^2(K, \mathrm{R}_{E/K}(\mathbb{G}_m))$ by Hilbert's Theorem 90. By the Brauer–Hasse–Noether Theorem, $\text{Ш}^2(K, \mathrm{R}_{E/K}(\mathbb{G}_m))$ vanishes, hence so does $\text{Ш}^2(K, \mathrm{R}^{(1)}_{E/K}(\mathbb{G}_m))$. By Poitou–Tate duality, we have

$$\text{Ш}^1(K, \hat{\mathrm{S}}_{E/K}) \simeq \text{Ш}^2(K, \mathrm{R}^{(1)}_{E/K}(\mathbb{G}_m))^* = 0.$$

Therefore, $\text{Ш}^1(K, \mathrm{sc}(\hat{T}))$ is in the image of $(\hat{\mathrm{S}}_{F/K})^{\Gamma}/\pi((\hat{\mathrm{S}}_{E/K})^{\Gamma})$.

Since the F_is are field extensions of K, we have $\mathrm{I}_{F_i/K}(\mathbb{Z})^{\Gamma} \simeq \mathbb{Z}$. Thus, we have $\mathrm{I}_{F/K}(\mathbb{Z})^{\Gamma} \simeq \bigoplus_i^m \mathrm{I}_{F_i/K}(\mathbb{Z})^{\Gamma} \simeq \mathbb{Z}^m$, and $(\hat{\mathrm{S}}_{F/K})^{\Gamma} \simeq \mathbb{Z}^m/(1, \dots, 1)$.

If $E_i = F_i \times F_i$, then π sends $\mathrm{I}_{E_i/K}(\mathbb{Z})^{\Gamma} \simeq \mathrm{I}_{F_i/K}(\mathbb{Z})^{\Gamma} \times \mathrm{I}_{F_i/K}(\mathbb{Z})^{\Gamma}$ surjectively onto $\mathrm{I}_{F_i/K}(\mathbb{Z})^{\Gamma} \simeq \mathbb{Z}$. If $E_i = K$, then $\mathrm{I}_{E_i/K}(\mathbb{Z}) \simeq \mathbb{Z} \simeq \mathrm{I}_{F_i/K}(\mathbb{Z})$. If E_i is a quadratic field extension of F_i, the map π sends $\mathrm{I}_{E_i/K}(\mathbb{Z})^{\Gamma} \simeq \mathbb{Z}$ to $\mathrm{I}_{F_i/K}(\mathbb{Z})^{\Gamma} \simeq \mathbb{Z}$ by multiplication by 2. Recall that $m = m_1 + m_2$, where m_1 is the number of indices i such that E_i is a quadratic field extension of F_i, and m_2 is the number of indices i such that either $E_i = F_i \times F_i$ or $E_i = K$. Then we have

$$(\hat{\mathrm{S}}_{F/K})^{\Gamma}/\pi((\hat{\mathrm{S}}_{E/K})^{\Gamma}) \simeq (\mathbb{Z}/2\mathbb{Z})^{m_1}/(1, \dots, 1).$$

We claim that the map $\delta : (\hat{\mathrm{S}}_{F/K})^{\Gamma}/\pi((\hat{\mathrm{S}}_{E/K})^{\Gamma}) \to \mathrm{H}^1(K, \mathrm{sc}(\hat{T}))$ sends bijectively $\text{Б}(E, \sigma)$ to $\text{Ш}^1(K, \mathrm{sc}(\hat{T}))$.

Let $(I_0, I_1) \in \text{Б}(E, \sigma)$, let a be the corresponding element in

$$(\hat{\mathrm{S}}_{F/K})^{\Gamma}/\pi((\hat{\mathrm{S}}_{E/K})^{\Gamma})$$

and let x be the image of a in $\mathrm{H}^1(K, \mathrm{sc}(\hat{T}))$. We claim that x is in $\text{Ш}^1(K, \mathrm{sc}(\hat{T}))$. It suffices to prove that, for any $v \in \Omega_K$, we have $a^v = 0$.

For a place $v \in \bigcap_{i \in \mathrm{I}_1} \Sigma_i$, we have that E_i^v splits over F_i^v for all $i \in \mathrm{I}_1$. Hence, π maps $\mathrm{I}_{E_i^v/K_v}(\mathbb{Z})^{\Gamma_v} \simeq \mathrm{I}_{F_i^v/K_v}(\mathbb{Z})^{\Gamma_v} \oplus \mathrm{I}_{F_i^v/K_v}(\mathbb{Z})^{\Gamma_v}$ onto $\mathrm{I}_{F_i^v/K_v}(\mathbb{Z})^{\Gamma_v}$ for each $i \in \mathrm{I}_1$, and so $(\hat{\mathrm{S}}_{F/K})^{\Gamma_v}/\pi((\hat{\mathrm{S}}_{E/K})^{\Gamma_v}) = 0$ for each $i \in \mathrm{I}_1$ and $a_i^v = 0$. On the other hand, for each $i \in \mathrm{I}_0$, we have $a_i = 0$ by definition. Therefore, $a^v = 0$.

For a place $v \in \bigcap_{i \in \mathrm{I}_0} \Sigma_i$, we replace $(a_1, \dots, a_{m_1})$ by $(a_1, \dots, a_{m_1}) + (1, \dots, 1)$. Note that $(a_1, \dots, a_{m_1}) + (1, \dots, 1)$ and $(a_1, \dots, a_{m_1})$ represent the same class a in $(\hat{\mathrm{S}}_{F/K})^{\Gamma}/\pi((\hat{\mathrm{S}}_{E/K})^{\Gamma})$. By the same argument as above, we have $a_v = 0$. Since $\left(\bigcap_{i \in \mathrm{I}_0} \Sigma_i\right) \cup \left(\bigcap_{j \in \mathrm{I}_1} \Sigma_j\right) = \Omega_K$, we have $a^v = 0$ for all $v \in \Omega_K$, which proves that x is an element of $\text{Ш}^1(K, \mathrm{sc}(\hat{T}))$.

This proves that δ induces a map $\mathrm{Б}(E,\sigma) \to \text{Ш}^1(K, \mathrm{sc}(\hat{T}))$. We already know that this map is injective. Let us prove that it is also surjective.

Let $0 \neq x \in \text{Ш}^1(K, \mathrm{sc}(\hat{T}))$. Let $a \in (\hat{\mathrm{S}}_{F/K})^\Gamma/\pi((\hat{\mathrm{S}}_{E/K})^\Gamma)$ be the preimage of x, let a^v be the localization of a at the place v, and let $(a_1, \ldots, a_{m_1})$ be a lift of a in $(\mathbb{Z}/2\mathbb{Z})^{m_1}$. Let (I_0, I_1) be the corresponding partition. Now we claim that $\left(\bigcap_{i\in \mathrm{I}_0}\Sigma_i\right) \cup \left(\bigcap_{j\in \mathrm{I}_1}\Sigma_j\right) = \Omega_K$. Suppose that $\left(\bigcap_{i\in \mathrm{I}_0}\Sigma_i\right) \cup \left(\bigcap_{j\in \mathrm{I}_1}\Sigma_j\right) \neq \Omega_K$, and let $v \in \Omega_K \setminus \left(\bigcap_{i\in \mathrm{I}_0}\Sigma_i\right) \cup \left(\bigcap_{j\in \mathrm{I}_1}\Sigma_j\right)$. Therefore, there exist $i_0 \in \mathrm{I}_0$ and $i_1 \in \mathrm{I}_1$ such that $E^v_{i_0}$ is not split over $F^v_{i_0}$ and $E^v_{i_1}$ is not split over $F^v_{i_1}$. Let $F^v_i = \prod_{j=1}^{n_i} L_{i,j}$, where the $L_{i,j}$s are field extensions of K_v. Let $E^v_i = \prod_{j=1}^{n_i} M_{i,j}$, where $M_{i,j}$ is a quadratic étale algebra over $L_{i,j}$. Set $\Gamma_v = \Gamma_{K_v}$. Then we have

$$\mathrm{I}_{F^v_i/K_v}(\mathbb{Z})^{\Gamma_v}/\pi(\mathrm{I}_{E^v_i/K_v}(\mathbb{Z})^{\Gamma_v}) = \bigoplus_{j=1}^{n_i} \mathrm{I}_{L_{i,j}/K_v}(\mathbb{Z})^{\Gamma_v}/\pi(\mathrm{I}_{M_{i,j}/K_v}(\mathbb{Z})^{\Gamma_v}).$$

If $M_{i,j}$ is split over $L_{i,j}$, then

$$\mathrm{I}_{M_{i,j}/K_v}(\mathbb{Z})^{\Gamma_v} = \mathrm{I}_{L_{i,j}\times L_{i,j}/K_v}(\mathbb{Z})^{\Gamma_v} = \mathrm{I}_{L_{i,j}/K_v}(\mathbb{Z})^{\Gamma_v} \oplus \mathrm{I}_{L_{i,j}/K_v}(\mathbb{Z})^{\Gamma_v},$$

so π sends $\mathrm{I}_{M_{i,j}/k_v}(\mathbb{Z})^{\Gamma_v}$ surjectively to $\mathrm{I}_{L_{i,j}/k_v}(\mathbb{Z})^{\Gamma_v}$. On the other hand, if $M_{i,j}$ is a field extension over $L_{i,j}$, then π maps $\mathrm{I}_{M_{i,j}/K_v}(\mathbb{Z})^{\Gamma_v} \simeq \mathbb{Z}$ to $2\mathbb{Z} \subseteq \mathbb{Z} \simeq \mathrm{I}_{L_{i,j}/K_v}(\mathbb{Z})^{\Gamma_v}$ and we have

$$\mathrm{I}_{L_{i,j}/K_v}(\mathbb{Z})^{\Gamma_v}/\pi(\mathrm{I}_{M_{i,j}/K_v}(\mathbb{Z})^{\Gamma_v}) \simeq \mathbb{Z}/2\mathbb{Z}.$$

For $a_i \in \mathrm{I}_{F_i/K}(\mathbb{Z})^\Gamma/\pi(\mathrm{I}_{E_i/K}(\mathbb{Z})^\Gamma) \simeq \mathbb{Z}/2\mathbb{Z}$, the localization map sends a_i diagonally into

$$\mathrm{I}_{F^v_i/K_v}(\mathbb{Z})^{\Gamma_v}/\pi(\mathrm{I}_{E^v_i/K_v}(\mathbb{Z})^{\Gamma_v}) \simeq \bigoplus_{\substack{j \text{ where } M_{i,j} \\ \text{is nonsplit}}} \mathbb{Z}/2\mathbb{Z}.$$

Let a^v_i be the image of a_i in $\mathrm{I}_{F^v_i/K_v}(\mathbb{Z})^{\Gamma_v}/\pi(\mathrm{I}_{E^v_i/K_v}(\mathbb{Z})^{\Gamma_v})$. By our choice of v, we have that $\mathrm{I}_{F^v_{i_0}/K_v}(\mathbb{Z})^{\Gamma_v}/\pi(\mathrm{I}_{E^v_{i_0}/K_v}(\mathbb{Z})^{\Gamma_v})$ (resp. $\mathrm{I}_{F^v_{i_1}/K_v}(\mathbb{Z})^{\Gamma_v}/\mathrm{I}_{E^v_{i_1}/K_v}(\mathbb{Z})^{\Gamma_v}$) is nontrivial. In particular, $a^v_{i_1}$ is nonzero as a_{i_1} is nonzero. Note that

$$\bigoplus_i (\hat{\mathrm{S}}_{F^v_i/K_v})^{\Gamma_v}/\pi((\hat{\mathrm{S}}_{E^v_i/K_v})^{\Gamma_v}) = \frac{\bigoplus_i \mathrm{I}_{F^v_i/K_v}(\mathbb{Z})^{\Gamma_v}/\pi(\mathrm{I}_{E^v_i/K_v}(\mathbb{Z})^{\Gamma_v})}{(\bar{1}, \ldots, \bar{1})},$$

where $\bar{1}$ denotes the image of the diagonal element of $\mathrm{I}_{F^v_i/K_v}(\mathbb{Z})^{\Gamma_v}$ in

$$\mathrm{I}_{F^v_i/K_v}(\mathbb{Z})^{\Gamma_v}/\pi(\mathrm{I}_{E^v_i/K_v}(\mathbb{Z})^{\Gamma_v}).$$

Since $a^v = 0$, we have either $a^v_i = 0 \in \mathrm{I}_{F^v_i/K_v}(\mathbb{Z})^{\Gamma_v}/\pi(\mathrm{I}_{E^v_i/K_v}(\mathbb{Z})^{\Gamma_v})$ for all i, or $a^v_i = \bar{1} \in \mathrm{I}_{F^v_i/K_v}(\mathbb{Z})^{\Gamma_v}/\pi(\mathrm{I}_{E^v_i/K_v}(\mathbb{Z})^{\Gamma_v})$ for all i. In particular, this implies that $a^v_{i_0}$ and $a^v_{i_1}$ are either both 0 or both 1, which is a contradiction. Therefore, we have $\left(\bigcap_{i\in \mathrm{I}_0}\Sigma_i\right) \cup \left(\bigcap_{j\in \mathrm{I}_1}\Sigma_j\right) = \Omega_K$ and $(I_0, I_1) \in \mathrm{Б}(E,\sigma)$. □

4. Proof of Proposition 2.2 when $L \neq K$

We keep the notation of the previous sections and assume that $L \neq K$. The aim of this section is to prove Proposition 2.2 in this case.

In this case, the torus $\mathrm{sc}(T)$ fits in the following exact sequence:

$$1 \to \mathrm{sc}(T) \to \mathrm{R}_{F/K}(\mathrm{R}^{(1)}_{E/F}(\mathbb{G}_m)) \to \mathrm{R}^{(1)}_{L/K}(\mathbb{G}_m) \to 1. \tag{3}$$

We take the dual sequence of exact sequence (3):

$$0 \to \hat{\mathrm{S}}_{L/K} \xrightarrow{\iota} \mathrm{I}_{F/K}(\hat{\mathrm{S}}_{E/F}) \xrightarrow{p} \mathrm{sc}(\hat{T}) \to 0, \tag{4}$$

from which we derive the long exact sequence

(5)

$$\cdots \to \mathrm{H}^1(K, \hat{\mathrm{S}}_{E/K}) \xrightarrow{\iota^1} \mathrm{H}^1(K, \mathrm{I}_{F/K}(\hat{\mathrm{S}}_{E/F})) \xrightarrow{p^1} \mathrm{H}^1(K, \mathrm{sc}(\hat{T})) \to \mathrm{H}^2(K, \hat{\mathrm{S}}_{E/K}).$$

By Poitou–Tate duality, we have $\text{Ш}^2(K, \hat{\mathrm{S}}_{E/K}) \simeq \text{Ш}^1(K, \mathrm{R}^{(1)}_{E/K}(\mathbb{G}_m))^*$. We claim that $\text{Ш}^2(K, \hat{\mathrm{S}}_{E/K}) \simeq \text{Ш}^1(K, \mathrm{R}^{(1)}_{E/K}(\mathbb{G}_m))^* = 0$. To see this, we consider the following exact sequence:

$$1 \to \mathrm{R}^{(1)}_{L/K}(\mathbb{G}_m) \to \mathrm{R}_{L/K}(\mathbb{G}_m) \to \mathbb{G}_m \to 1.$$

By Hilbert Theorem 90, we have $\mathrm{H}^1(K, \mathrm{R}^{(1)}_{L/K}(\mathbb{G}_m)) = K^\times/\mathrm{N}_{L/K}(L^\times)$, where $\mathrm{N}_{L/K}$ is the norm map from L to K. Since the norms of the quadratic extension L over K satisfy the local-global principle, we have $\text{Ш}^1(K, \mathrm{R}^{(1)}_{L/K}(\mathbb{G}_m)) = 0$. Hence $\text{Ш}^2(K, \hat{\mathrm{S}}_{L/K}) = 0$. Therefore, the group $\text{Ш}^1(K, \mathrm{sc}(\hat{T}))$ is contained in the image of $\mathrm{H}^1(K, \mathrm{I}_{F/K}(\hat{\mathrm{S}}_{E/F}))$.

Let us consider the following exact sequence:

$$1 \to \mathbb{G}_m \to \mathrm{R}_{L/K}(\mathbb{G}_m) \xrightarrow{\pi} \mathrm{R}^{(1)}_{L/K}(\mathbb{G}_m) \to 1, \tag{6}$$

where $\pi(x) = x/\sigma(x)$. Considering the dual sequence, we get

$$0 \to \hat{\mathrm{S}}_{L/K} \to \mathrm{I}_{L/K}(\mathbb{Z}) \xrightarrow{d} \mathbb{Z} \to 0, \tag{7}$$

where d is the degree map which maps $(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \simeq \mathrm{I}_{L/K}(\mathbb{Z})$ to $a + b$. Taking the long exact sequence associated to (7), we have

$$\mathrm{I}_{L/K}(\mathbb{Z})^\Gamma \xrightarrow{d} \mathbb{Z} \to \mathrm{H}^1(K, \hat{\mathrm{S}}_{L/K}) \to \mathrm{H}^1(K, \mathrm{I}_{L/K}(\mathbb{Z})) = 0. \tag{8}$$

Since L is a quadratic field extension of K, we obtain

$$\mathrm{H}^1(K, \hat{\mathrm{S}}_{L/K}) \simeq \mathbb{Z}/d(\mathrm{I}_{L/K}(\mathbb{Z})^\Gamma) = \mathbb{Z}/2\mathbb{Z}.$$

Similarly, we have

$$\mathrm{H}^1(K, \mathrm{I}_{F/K}(\hat{\mathrm{S}}_{E/F})) = \mathrm{H}^1(F, \hat{\mathrm{S}}_{E/F}) = \prod_{i=1}^{m} \mathrm{H}^1(F_i, \hat{\mathrm{S}}_{E_i/F_i}).$$

If $E_i = F_i \times F_i$, then $\mathrm{H}^1(F_i, \hat{\mathrm{S}}_{E_i/F_i}) = 0$ since d is surjective. If E_i is a quadratic extension of F_i, then $\mathrm{H}^1(F_i, \hat{\mathrm{S}}_{E_i/F_i}) = \mathbb{Z}/2\mathbb{Z}$. Recall that $m = m_1 + m_2$, where m_1 is the number of indices i such that E_i is a quadratic extension of F_i, and m_2 is the number of indices i such that $E_i = F_i \times F_i$. Then $\mathrm{H}^1(K, \mathrm{I}_{F/K}(\hat{\mathrm{S}}_{E/F})) \simeq (\mathbb{Z}/2\mathbb{Z})^{m_1}$.

The map $\iota^1 : \mathrm{H}^1(K, \hat{\mathrm{S}}_{L/K}) \to \mathrm{H}^1(K, \mathrm{I}_{F/K}(\hat{\mathrm{S}}_{E/F}))$ maps $\mathbb{Z}/2\mathbb{Z}$ diagonally into $(\mathbb{Z}/2\mathbb{Z})^{m_1}$. Therefore, we have

$$\text{Ш}^1(k, \mathrm{sc}(\hat{T})) \subseteq \mathrm{Im}(p^1) \simeq (\mathbb{Z}/2\mathbb{Z})^{m_1}/(1, \dots, 1).$$

Let us show that p^1 maps $\text{Б}(E, \sigma)$ bijectively to $\text{Ш}(K, \mathrm{sc}(\hat{T}))$.

Let (I_0, I_1) be in $\text{Б}(E, \sigma)$, and let a in $\mathrm{H}^1(K, \mathrm{I}_{F/K}(\hat{\mathrm{S}}_{E/F}))$ be the corresponding element. We want to show that $p^1(a)$ is an element of $\text{Ш}^1(K, \mathrm{sc}(\hat{T}))$. Let $v \in \Omega_K$. If $v \in \bigcap_{j \in \mathrm{I}_1} \Sigma_j$, then $a^v = 0$. Hence, it suffices to prove that, for $v \in \Omega_K \setminus \bigcap_{i \in \mathrm{I}_1} \Sigma_i$, we have $a^v = \iota_v^1(1) = \iota^1(1)_v$. Now, since $\left(\bigcap_{i \in \mathrm{I}_0} \Sigma_i\right) \cup \left(\bigcap_{j \in \mathrm{I}_1} \Sigma_j\right) = \Omega_K$, we have $v \in \bigcap_{i \in \mathrm{I}_0} \Sigma_i$. Consequently, $\mathrm{H}^1(F_i, \hat{\mathrm{S}}_{E_i^v/F_i^v}) = 0$ for all $i \in \mathrm{I}_0$, and the projection of $\iota_v^1(1)$ to these components are trivial. For $i \in \mathrm{I}_1$, we have that a_i and the i-th coordinate of $\iota^1(1)$ are both 1, so their images in $\mathrm{H}^1(F_i^v, \hat{\mathrm{S}}_{E_i^v/F_i^v})$ are equal. This proves that $a^v = \iota_v^1(1)$, hence $p^1(a^v) = 0$.

We next show that the restriction of the map p^1 to $\text{Б}(E, \sigma)$ is surjective onto $\text{Ш}^1(K, \mathrm{sc}(\hat{T}))$.

Let $a = (a_1, \dots, a_{m_1}) \in (\mathbb{Z}/2\mathbb{Z})^{m_1} \simeq \mathrm{H}^1(K, \mathrm{I}_{E^\sigma/k}(\hat{\mathrm{S}}_{E/F}))$ and let (I_0, I_1) be the associated partition. If $a = 0$ or $a = (1, \dots, 1)$, then a is in the image of ι^1 and we have $p^1(a) = 0 \in \text{Ш}^1(K, \mathrm{sc}(\hat{T}))$. Hence, we may assume that I_0 and I_1 are nonempty.

We claim that $0 \neq p^1(a) \in \text{Ш}^1(K, \mathrm{sc}(\hat{T}))$ if and only if I_0 and I_1 are nonempty and $\left(\bigcap_{i \in \mathrm{I}_0} \Sigma_i\right) \cup \left(\bigcap_{j \in \mathrm{I}_1} \Sigma_j\right) = \Omega_K$.

Suppose $0 \neq p^1(a) \in \text{Ш}^1(K, \mathrm{sc}(\hat{T}))$. Let $v \in \Omega_K \setminus \bigcap_{i \in \mathrm{I}_0} \Sigma_i$. Then $L^v \not\simeq K^v \times K^v$ and we have $\mathrm{H}^1(L^v, \hat{\mathrm{S}}_{L^v/K_v}) = \mathbb{Z}/2\mathbb{Z}$. Let a^v denote the localization of a at v. Since $p^1(a) \in \text{Ш}^1(K, \mathrm{sc}(\hat{T}))$, we have a^v in the image of ι_v^1, so either $a^v = 0$ or $a^v = \iota_v^1(1)$. It suffices to show that $v \in \bigcap_{i \in \mathrm{I}_1} \Sigma_i$. Consider the i-th component of $(\mathbb{Z}/2\mathbb{Z})^{m_1}$, which corresponds to $\mathrm{H}^1(K, \mathrm{I}_{F_i/K}(\hat{\mathrm{S}}_{E_i/F_i})) = \mathrm{H}^1(F_i, \hat{\mathrm{S}}_{E_i/F_i})$. If E_i splits over F_i at a place $v \in \Omega_K$, then by the exact sequence (8), the map d is surjective and $\mathrm{H}^1(F_i^v, \hat{\mathrm{S}}_{E_i^v/F_i^v}) = 0$, which means that the i-th component vanishes at place v. Since $v \notin \bigcap_{i \in \mathrm{I}_0} \Sigma_i$, there exists an $i \in \mathrm{I}_0$ such that E_i^v is not split over F_i^v. Let $F_i^v = \prod_{j=1}^{n_i} L_{i,j}$, where the $L_{i,j}$s are field extensions of K_v. Let $E_i^v = \prod_{j=1}^{n_i} M_{i,j}$, where $M_{i,j}$ is a quadratic étale algebra over $L_{i,j}$. Then

$$\mathrm{H}^1(F_i^v, \hat{\mathrm{S}}_{E_i^v/F_i^v}) = \prod_j \mathrm{H}^1(L_{i,j}, \hat{\mathrm{S}}_{M_{i,j}/L_{i,j}}).$$

By the choice of i, there is a j such that $M_{i,j}$ is not split over $L_{i,j}$, and hence $\mathrm{H}^1(L_{i,j}, \hat{\mathrm{S}}_{M_{i,j}/L_{i,j}}) \neq 0$. Therefore, the projection of $\iota_v^1(1)$ to $\mathrm{H}^1(L_{i,j}, \hat{\mathrm{S}}_{M_{i,j}/L_{i,j}})$ is 1. On the other hand, the projection of a^v to the same component is 0 since $i \in \mathrm{I}_0$. Therefore, $a^v = 0$ which means that $\mathrm{H}^1(F_i^v, \hat{\mathrm{S}}_{E_i^v/F_i^v}) = 0$ for all $i \in \mathrm{I}_1$, hence $v \in \bigcap_{i \in \mathrm{I}_1} \Sigma_i$. This proves that $a \in \text{Б}(E, \sigma)$. □

5. The proofs of Proposition 2.3 and Theorem 2.1

We keep the notation of the previous sections. As we will see, Theorem 2.1 follows from Propositions 2.2 and 2.3, which we'll now prove.

Proposition 2.3. *The groups* $\text{Ш}^1(K, \mathrm{sc}(\hat{T}))$ *and* $\text{Б}(X)$ *are isomorphic.*

For a connected reductive group H, we denote by H^{tor} the quotient of H by its derived group. Note that H^{tor} is a torus.

Proof of Proposition 2.3. Let $\mathrm{sc}(G)$ be the simply connected cover of G. Recall that $X = E(G, \Psi, u)$. Since X is a homogeneous space under the adjoint action of G, we can view X as a homogeneous space under $\mathrm{sc}(G)$. Let x be in $X(K_s)$ and let $\overline{H} = \mathrm{Stab}_{\mathrm{sc}(G)_{K_s}}(x)$ be the stabilizer of x over K_s. Then $\overline{H}$ is isomorphic to $\mathrm{sc}(T)_{K_s}$ (see [Lee 2014, Lemma 3.9]). Let H^m be the K-form of the multiplicative quotient of $\overline{H}$ constructed in [Borovoi 1999, §§1.1–1.2, pp. 493–494] (note that the hypotheses of [Borovoi 1999, §1.1]. are satisfied: (1.1.1) holds since $\mathrm{sc}(G)$ is simply connected, and (1.1.2) is satisfied since $\overline{H} \simeq \mathrm{sc}(T)_{K_s}$). We have $H^m \simeq \mathrm{sc}(T)$ (see [Lee 2014, Lemma 3.9]). Let $i : H^m \to \mathrm{sc}(G)^{\mathrm{tor}}$ be the morphism of algebraic groups constructed in [Borovoi 1999, §1.2, p. 494]. Let $\hat{H}^m$ (resp. $\mathrm{sc}(\hat{G})^{\mathrm{tor}}$) be the character group of H^m (resp. $\mathrm{sc}(G)^{\mathrm{tor}}$). We view the dual map of i as a complex of finitely generated Galois modules $\mathrm{sc}(\hat{G})^{\mathrm{tor}} \to \hat{H}^m$, where $\mathrm{sc}(\hat{G})^{\mathrm{tor}}$ is in degree 0 and $\hat{H}^m$ is in degree 1. Then we have $\text{Б}(X) = \text{Ш}^2(K, \mathrm{sc}(\hat{G})^{\mathrm{tor}} \to \hat{H}^m)$. This follows from [Borovoi and van Hamel 2012, Theorem 3] (note that the statement was already proved in [Borovoi 1999, Theorem 3.3] under the condition that $X(K_v) \neq \varnothing$ for all $v \in \Omega_K$, and that Theorem 3 of [Borovoi and van Hamel 2012] was conjectured in [Borovoi 1999, Conjecture 3.2]). Since $\mathrm{sc}(G)$ is semisimple, we have $\mathrm{sc}(G)^{\mathrm{tor}} = 1$. Therefore, we have $\text{Ш}^2(K, \mathrm{sc}(\hat{G})^{\mathrm{tor}} \to \hat{H}^m) = \text{Ш}^2(K, 1 \to \hat{H}^m)$. On the other hand, we have $\text{Ш}^2(K, 1 \to \hat{H}^m) = \text{Ш}^1(K, \hat{H}^m)$ by the definition of hypercohomology. Recall that $H^m \simeq \mathrm{sc}(T)$. Therefore, $\text{Б}(X) \simeq \text{Ш}^1(K, \mathrm{sc}(\hat{T}))$. □

Theorem 2.1. *The groups* $\text{Б}(E, \sigma)$ *and* $\text{Б}(X)$ *are isomorphic.*

Proof of Theorem 2.1. By Proposition 2.3 we have $\text{Б}(X) \simeq \text{Ш}^1(K, \mathrm{sc}(\hat{T}))$, and Proposition 2.2 implies that $\text{Ш}^1(K, \mathrm{sc}(\hat{T})) \simeq \text{Б}(E, \sigma)$. □

6. Hasse principle and Brauer–Manin obstruction

We keep the notation of the previous sections and assume that K is a global field. In particular, (E, σ) is an étale algebra with involution and (A, τ) is a central simple algebra with involution, as in Section 2.

The embeddings of (E, σ) into (A, τ) were investigated in several papers; see for instance [Prasad and Rapinchuk 2010; Lee 2014; Bayer-Fluckiger et al. 2014]. In particular, Prasad and Rapinchuk proved in [2010, Theorem 5.1] that the Hasse principle holds if τ is symplectic, and they obtained partial results for τ orthogonal and unitary as well (see the introduction of the same paper).

Since the case where τ is symplectic is covered by the results of Prasad and Rapinchuk, we henceforth assume that τ *is either orthogonal or unitary*.

In [Bayer-Fluckiger et al. 2014] we defined the obstruction group Б(E, σ) (see Section 4 of the present paper; note that this group is denoted by Ш(E', σ) in [Bayer-Fluckiger et al. 2014, §3, §5.1]). Under the hypothesis that (E, σ) can be embedded into (A, τ) everywhere locally, we also defined an element $\bar{f} = \bar{f}((E, \sigma), (A, \tau))$ of Б$(E, \sigma)^*$ which gives a complete obstruction to the Hasse principle:

Theorem 6.1. *(E, σ) can be embedded into (A, τ) if and only if*

$$\bar{f}((E, \sigma), (A, \tau)) = 0.$$

This is proved in [Bayer-Fluckiger et al. 2014, Theorem 4.6.1 and Proposition 5.1.1].

On the other hand, Borovoi [1996] studied the Hasse principle for homogeneous spaces of connected linear algebraic groups with connected or abelian stabilizers. If Y is such a space, he defined a group Б(Y) and, provided $Y(K_v) \neq \varnothing$ for all $v \in \Omega_K$, an element $m_H(Y) \in$ Б$(Y)^*$ such that $Y(K) \neq \varnothing$ if and only if $m_H(Y) = 0$.

Borovoi's results were applied to the embedding problem of algebras with involution in [Lee 2014]. Recall that $G = \mathrm{U}(A, \tau)^\circ$ and $T = \mathrm{U}(E, \sigma)^\circ$ (see Section 1), and that $X = E(G, \Psi, u)$ (see Sections 1 and 4). By Theorem 2.1 we have Б$(E, \sigma) \simeq$ Б(X).

We don't know whether the isomorphism between Б(E, σ) and Б(X) carries $\bar{f}((E, \sigma), (A, \tau))$ to $m_H(X)$. However, these elements vanish simultaneously, and they both provide complete obstructions to the Hasse principle. More precisely:

Theorem 6.2. *Assume that (E, σ) can be embedded into (A, τ) everywhere locally (or, equivalently, that $X(K_v) \neq \varnothing$ for all $v \in \Omega_K$). Then the following assertions are equivalent*:

(i) *(E, σ) can be embedded into (A, τ).*

(ii) *$X(K) \neq \varnothing$.*

(iii) *$\bar{f}((E, \sigma), (A, \tau)) = 0$.*

(iv) $m_H(X) = 0$.

Proof. The equivalence of (i) and (ii) follows from [Lee 2014, Theorem 2.1.5]. The equivalence of (i) and (iii) is proved in [Bayer-Fluckiger et al. 2014, Theorem 4.6.1 and Proposition 5.1.1]. Finally, the equivalence between (ii) and (iv) follows from [Borovoi 1996, Theorem 2.2]. □

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ON MAXIMAL TORI OF ALGEBRAIC GROUPS OF TYPE G_2

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To the memory of Robert Steinberg

Given an octonion algebra C over a field k, its automorphism group is an algebraic semisimple k-group of type G_2. We study the maximal tori of G in terms of the algebra C.

1. Introduction

For classical algebraic groups, and in particular for arithmetic fields, the investigation of maximal tori is an interesting topic in the theory of algebraic groups and arithmetic groups; see [Prasad and Rapinchuk 2009, § 9; 2010] and also [Garibaldi and Rapinchuk 2013]. It is also related to the Galois cohomology of quasisplit semisimple groups by Steinberg's section theorem; that connection is an important ingredient of this paper.

Let k be a field, let k_s be a separable closure and denote by $\Gamma_k = \mathrm{Gal}(k_s/k)$ the absolute Galois group of k. In this paper, we study maximal tori of groups of type G_2. We recall that a semisimple algebraic k-group G of type G_2 is the group of automorphisms of a unique octonion algebra C [Knus et al. 1998, 33.24]. We come now to the following invariant of maximal tori [Gille 2004; Raghunathan 2004]. Given a k-embedding of $i : T \to G$ of a rank -2 torus, we have a natural action of Γ_k on the root system $\Phi(G_{k_s}, i(T_{k_s}))$, and the yoga of twisted forms defines then a cohomology class $\mathbf{type}(T, i) \in H^1(k, W_0)$, which is called the type of the couple (T, i). Here $W_0 \cong \mathbb{Z}/2\mathbb{Z} \times S_3$ is the Weyl group of the Chevalley group of type G_2. By Galois descent [Knus et al. 1998, 29.9], a W_0-torsor is nothing but a couple (k', l), where k' (resp. l) is a quadratic (resp. cubic) étale k-algebra. The main problem is then the following: given an octonion algebra C and such a couple (k', l), under which additional conditions is there a k-embedding $i : T \to G = \mathrm{Aut}(C)$ of type $[(k', l)] \in H^1(k, W_0)$?

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We give a precise answer when the cubic extension l is not a field (Section 4.4). When l is a field, we use subgroups of type A_2 of G to relate with maximal tori of special unitary groups where we can apply results of Knus, Haile, Rost and Tignol [Haile et al. 1996]. This provides a criterion which is quite complicated (see Proposition 5.2.6).

The problem above can be formulated in terms of existence of k-points for a certain homogeneous space X under G associated to k', l; see [Lee 2014, §1] or Section 2.6. We recall here Totaro's general question [2004, Question 0.2].

For a smooth connected affine k-group G over the field k and a homogeneous G-variety Y such that Y has a zero-cycle of degree $d > 0$, does Y necessarily have a closed étale point of degree dividing d?

Starting with Springer's odd extension theorem for quadratic forms, there are several cases where the question has a positive answer, mainly for principal homogeneous spaces (i.e., torsors). We quote here the results by Totaro [2004, Theorem 5.1] and Garibaldi and Hoffmann [2006] for certain exceptional groups, Black [2011] for classical adjoint groups and Black and Parimala [2014] for semisimple simply connected classical groups of rank ≤ 2.

If the base field k is large enough (e.g., $\mathbb{Q}(t)$, $\mathbb{Q}((t))$), we can construct a homogeneous space X under G of the shape above having a quadratic point and a cubic point but no k-point (Theorem 4.5.3). This provides a new class of counterexamples to the question in the case $d = 1$ which are geometrically speaking simpler than those of Florence [2004] and Parimala [2005].

Finally, in case of a number field, we show that this kind of variety satisfies the Hasse principle. In this case, our results are effective; that is, we can describe the type of the maximal tori of a given group of type G_2, for example, for the "compact" G_2 over the rational numbers (see Examples 6.4).

Let us review the contents of the paper. In Section 2, we recall the notion of type and oriented type for a k-embedding $i : T \to G$ of a maximal k-torus in a reductive k-group G. We study then the image of the map $H^1(k, T) \to H^1(k, G)$ of Galois cohomology and relate it, in the quasisplit case, with Steinberg's theorem on Galois cohomology. Section 3 gathers basic facts on octonion algebras which are used in the core of the paper, namely Sections 4 and 5. The number field case is considered in the short Section 6. Finally, the Appendix deals with the Galois cohomology of k-tori and quasisplit reductive k-groups over Laurent series fields.

A. Fiori [2015] investigated independently maximal tori of algebraic groups of type G_2 and their rational conjugacy classes. Though his scope is different, certain tools are common with our paper, for example, the definition and the study of the subgroup of type A_2 attached to a maximal torus (Proposition 5.5 in [loc. cit.], §5.1 here).

2. Maximal tori of reductive groups and image of the cohomology

Let G be a reductive k-group. We are interested in maximal tori of G and also in the images of the map $H^1(k, T) \to H^1(k, G)$. We shall discuss refinements of the application of Steinberg's theorem on rational conjugacy classes to Galois cohomology.

2.1. *Twisted root data.*

2.1.1. *Definition.* In [Lee 2014, 1.3] and [Gille 2014, §6.1], in the spirit of [Demazure and Grothendieck 1970a; 1970b; 1970c], the notion of twisted root data is defined over an arbitrary base scheme S. We focus here on the case of the base field k and use the equivalence of categories between étale sheaves over $\mathrm{Spec}(k)$ and the category of Galois sets, namely sets equipped with a continuous action of the absolute Galois group Γ_k.

We recall from [Springer 1998, §7.4] that a root datum is a quadruple $\Psi = (M, R, M^\vee, R^\vee)$, where M is a lattice, $M^\vee$ its dual, $R \subset M$ a finite subset (the roots), $R^\vee$ a finite subset of $M^\vee$ (the coroots), and a bijection $\alpha \mapsto \alpha^\vee$ of R onto $R^\vee$ which satisfy the next axioms (RD1) and (RD2).

For each $\alpha \in R$, we define endomorphisms s_α of M and $s_\alpha^\vee$ of $M^\vee$ by

$$s_\alpha(m) = m - \langle m, \alpha^\vee\rangle\alpha, \quad s_\alpha^\vee(f) = f - \langle \alpha, f\rangle\alpha^\vee \quad (m \in M, f \in M^\vee).$$

(RD1) For each $\alpha \in R$, $\langle \alpha, \alpha^\vee\rangle = 2$;

(RD2) For each $\alpha \in R$, $s_\alpha(R) = R$ and $s_\alpha^\vee(R^\vee) = R^\vee$.

We denote by $W(\Psi)$ the subgroup of $\mathrm{Aut}(M)$ generated by the s_α; it is called the Weyl group of Ψ.

2.1.2. *Isomorphisms, orientation.* An isomorphism of root data

$$\Psi_1 = (M_1, R_1, M_1^\vee, R_1^\vee) \xrightarrow{\sim} \Psi_2 = (M_2, R_2, {M_2}^\vee, {R_2}^\vee)$$

is an isomorphism $f : M_1 \xrightarrow{\sim} M_2$ such that f induces a bijection $R_1 \xrightarrow{\sim} R_2$ and f induces a dual isomorphism $f^\vee : M_2^\vee \xrightarrow{\sim} M_1^\vee$ such that $f^\vee$ induces a bijection ${R_2}^\vee \xrightarrow{\sim} R_1^\vee$. Let $\mathrm{Isom}(\Psi_1, \Psi_2)$ be the scheme of isomorphisms between Ψ_1 and Ψ_2. We define the quotient $\mathrm{Isomext}(\Psi_1, \Psi_2)$ by $\mathrm{Isomext}(\Psi_1, \Psi_2) = W(\Psi_2)\backslash \mathrm{Isom}(\Psi_1, \Psi_2)$, which is isomorphic to $\mathrm{Isom}(\Psi_1, \Psi_2)/W(\Psi_1)$.

An orientation u between Ψ_1 and Ψ_2 is an element $u \in \mathrm{Isomext}(\Psi_1, \Psi_2)$. We can then define the set $\mathrm{Isomint}_u(\Psi_1, \Psi_2)$ of inner automorphisms with respect to the orientation u as the preimage of u by the projection $\mathrm{Isom}(\Psi_1, \Psi_2) \to \mathrm{Isomext}(\Psi_1, \Psi_2)$.

We denote by $\mathrm{Aut}(\Psi) = \mathrm{Isom}(\Psi, \Psi)$ the group of automorphisms of the root datum Ψ, and we have an exact sequence

$$1 \to W(\Psi) \to \mathrm{Aut}(\Psi) \to \mathrm{Autext}(\Psi) \to 1,$$

where $\mathrm{Autext}(\Psi) = \mathrm{Isomext}(\Psi, \Psi)$ stands for the quotient group of automorphisms of Ψ (called the group of exterior or outer automorphisms of Ψ). The choice of an ordering on the roots permits us to define a set of positive roots Ψ_+, its basis and the Dynkin index $\mathrm{Dyn}(\Psi)$ of Ψ. Furthermore, we have an isomorphism $\mathrm{Aut}(\Psi, \Psi_+) \xrightarrow{\sim} \mathrm{Autext}(\Psi)$ so that the above sequence is split.

2.1.3. *Twisted version.* A twisted root datum is a root datum equipped with a continuous action of Γ_k. To distinguish from the absolute case, we shall use the notation $\underline{\Psi}$. The Weyl group $W(\underline{\Psi})$ is then a finite group equipped with an action of Γ_k. If $\underline{\Psi}_1$, $\underline{\Psi}_2$ are two twisted root data, the sets $\mathrm{Isom}(\underline{\Psi}_1, \underline{\Psi}_2)$, $\mathrm{Isomext}(\underline{\Psi}_1, \underline{\Psi}_2)$ are Galois sets. An orientation between $\underline{\Psi}_1$, $\underline{\Psi}_2$ is an element $u \in \mathrm{Isomext}(\underline{\Psi}_1, \underline{\Psi}_2)(k)$, and the set $\mathrm{Isomint}_u(\underline{\Psi}_1, \underline{\Psi}_2)$ is then a Galois set.

2.2. ***Type of a maximal torus.*** We denote by G_0 the split form of G. We denote by T_0 a maximal k-split torus of G_0 and by $\Psi_0 = \Psi(G_0, T_0)$ the associated root datum. We denote by W_0 the Weyl group of Φ_0 and by $\mathrm{Aut}(\Psi_0)$ its automorphism group.

Let $i : T \to G$ be a k-embedding as a maximal torus. The root datum

$$\underline{\Psi}(G, i(T)) = \Psi(G(T)_{k_s}, i(T)_{k_s})$$

is equipped with an action of the absolute Galois group Γ_k, so it defines a twisted root datum. It is a k-form of the constant root datum $\underline{\Psi}_0$ and we define the type of (T, i) as the isomorphism class of

$$[\underline{\Psi}(G, i(T))] \in H^1(k, \mathrm{Aut}(\Psi_0)).$$

Recall that by Galois descent, those k_s/k-forms are classified by the Galois cohomology pointed set $H^1(k, \mathrm{Aut}(\Psi_0))$.

If two embeddings i, j have the same image, then $\mathbf{type}(T, i) = \mathbf{type}(T, j) \in H^1(k, \mathrm{Aut}(\Psi_0))$. If we compose $i : T \to G$ by an automorphism $f \in \mathrm{Aut}(G)(k)$, we have $\mathbf{type}(T, i) = \mathbf{type}(T, f \circ i) \in H^1(k, \mathrm{Aut}(\Psi_0))$.

Remark 2.2.1. If G is semisimple and has no outer isomorphism (as is the case for groups of type G_2), $W_0 = \mathrm{Aut}(\Psi_0)$ and the next considerations will not add anything.

We would like to have an invariant with value in the Galois cohomology of some Weyl group. The strategy is to "rigidify" by adding an extra data to $i : T \to G$, namely an orientation with respect to a quasisplit form of G.

Given a k-embedding $i : T \to G$, we denote by $\underline{\mathrm{Dyn}}(G, i(T))$ the Dynkin diagram k-scheme of $\underline{\Psi}(G, i(T))$; it is finite étale and then encoded in the Galois set $\mathbf{Dyn}(G_{k_s}, i(T)_{k_s})$. There is a canonical isomorphism: $\underline{\mathrm{Dyn}}(G) \cong \underline{\mathrm{Dyn}}(G, i(T))$ [Demazure and Grothendieck 1970c, XXIV, 3.3].

We denote by G' a quasisplit k-form of G. Let (T', B') be a Killing couple of G', and denote by $\underline{\Psi}' = \underline{\Psi}(G', T')$ the associated twisted root datum and by $W' = N_{G'}(T')/T'$ its Weyl group, which is a twisted constant finite k-group.

Suppose that G is semisimple simply connected or adjoint; in this case, the homomorphism $\mathrm{Autext}(G) \to \mathrm{Aut}_{\mathrm{Dyn}}(\underline{\mathrm{Dyn}}(G))$ is an isomorphism [ibid., XXIV, 3.6]. We fix then an isomorphism $v : \underline{\mathrm{Dyn}}(G') \xrightarrow{\sim} \underline{\mathrm{Dyn}}(G)$. Together with the canonical isomorphism $\underline{\mathrm{Dyn}}(G) \cong \underline{\mathrm{Dyn}}(G, i(T))$, it induces an isomorphism $\tilde{v} : \underline{\mathrm{Dyn}}(G') \xrightarrow{\sim} \underline{\mathrm{Dyn}}(G, i(T))$. For G semisimple simply connected or adjoint, the isomorphism $\tilde{v}$ defines equivalently an orientation

$$u \in \mathrm{Isomext}\big(\underline{\Psi}(G', T')(k), \underline{\Psi}(G, i(T))\big).$$

Then the Galois set $\mathrm{Isomint}_u\big(\underline{\Psi}(G', T'), \underline{\Psi}(G, i(T))\big)$ is a right W'-torsor and its class in $H^1(k, W')$ is called the oriented type of $i : T \to G$ with respect to the orientation v. It is denoted by $\mathbf{type}_v(T, i)$ and we bear in mind that it depends on the choice of G' and on v.

2.3. ***The quasisplit case.*** We deal here with the quasisplit k-group G' and with the exact sequence $1 \to T' \to N_{G'}(T') \xrightarrow{\pi} W' \to 1$. Here we have a canonical isomorphism $\mathrm{id} : \underline{\mathrm{Dyn}}(G') \cong \underline{\mathrm{Dyn}}(G')$ and then a natural way to define an orientation for a k-embedding $j : E \to G'$ of a maximal k-torus. Keeping the notations above, let us state the following result.

Theorem 2.3.1 (Kottwitz). (1) *The map*

$$\mathrm{Ker}\big(H^1(k, N_{G'}(T')) \to H^1(k, G')\big) \xrightarrow{\pi_*} H^1(k, W')$$

is onto.

(2) *For each $\gamma \in H^1(k, W')$, there exists a k-embedding $j : E \to G'$ of a maximal k-torus such that $\mathbf{type}_{\mathrm{id}}((E, j)) = \gamma$.*

In [Kottwitz 1982, Corollary 2.2], this result occurs only as a result on embeddings of maximal tori. It was rediscovered by Raghunathan [2004] and independently by the second author [Gille 2004]. The proof of (1) uses Steinberg's theorem on rational conjugacy classes, and we can explain quickly how one can derive (2) from (1). Given $\gamma \in H^1(k, W')$, assertion (1) provides a principal homogeneous space P under $N' = N_{G'}(T')$ together with a trivialization $\phi : G' \xrightarrow{\sim} P \wedge^{N'} G'$ such that $\pi_*[P] = \gamma$. Then ϕ induces a trivialization at the level of twisted k-groups $\phi_* : G' \xrightarrow{\sim} {}^P G'$. Now if we twist $i' : T' \to G'$ by P, we get a k-embedding

$${}^P i' : {}^P T' \to {}^P G' \overset{\phi_*}{\underset{\sim}{\longleftarrow}} G',$$

and one checks that $\mathbf{type}_{\mathrm{id}}\big({}^P T', {}^P i'\big) = \gamma$.

2.4. ***Image of the cohomology of tori.*** We give now a slightly more precise form of Steinberg's theorem [1965, Theorem 11.1]; see also [Serre 1994, III.2.3].

Theorem 2.4.1. *Let* $[z] \in H^1(k, G')$. *Let* $i : T \to {}_zG'$ *be a maximal* k*-torus of the twisted* k*-group* ${}_zG'$. *Then there exists a* k*-embedding* $j : T \to G'$ *and* $[a] \in H^1(k, T)$ *such that* $j_*[a] = [z]$ *and such that* $\mathbf{type}_{\mathrm{can}}(T, i) = \mathbf{type}_{\mathrm{id}}(T, j)$.

In the result, the first orientation is the canonical one, namely arising from the canonical isomorphism $\underline{\mathrm{Dyn}}(G') \xrightarrow{\sim} \underline{\mathrm{Dyn}}({}_z(G'))$.

Proof. If the base field is finite, there is nothing to do since $H^1(k, G') = 1$ by Lang's theorem. We can then assume that k is infinite. We denote by $P(z)$ the G'-homogeneous space defined by z and by $\phi : G'_{k_s} \xrightarrow{\sim} P(z)_{k_s}$, a trivialization satisfying $z_\sigma = \phi^{-1} \circ \sigma(\phi)$ for each $\sigma \in \Gamma_k$. It induces a trivialization $\varphi : G'_{k_s} \xrightarrow{\sim} ({}_z(G'))_{k_s}$ satisfying $\mathrm{int}(z_\sigma) = \varphi^{-1} \circ \sigma(\varphi)$ for each $\sigma \in \Gamma_k$.

We denote by $(G')^{\mathrm{sc}}$ the simply connected cover of DG' and by $f : (G')^{\mathrm{sc}} \to G'$ the natural k-homomorphism. Let T^{sc} be $({}_zf)^{-1}(i(T))$. Let g^{sc} be a regular element in $T^{\mathrm{sc}}(k)$ and consider the $G'^{\mathrm{sc}}(k_s)$-conjugacy class $\mathcal{C}$ of $\varphi^{-1}(g^{\mathrm{sc}})$ in $(G')^{\mathrm{sc}}(k_s)$. This conjugacy class is rational in the sense that it is stabilized by Γ_k since $(\varphi^{-1}(g^{\mathrm{sc}})) = z_\sigma {}^\sigma(\varphi^{-1}(g^{\mathrm{sc}}))z_\sigma^{-1}$ for each $\sigma \in \Gamma_k$. According to Steinberg [1965, Corollary 10.1] (and [Borel and Springer 1968, 8.6] in the nonperfect case), $\mathcal{C} \cap (G')^{\mathrm{sc}}(k)$ is not empty, so there exist $g_1^{\mathrm{sc}} \in (G')^{\mathrm{sc}}(k)$ and $h^{\mathrm{sc}} \in (G')^{\mathrm{sc}}(k_s)$ such that $\varphi^{-1}(g^{\mathrm{sc}}) = (h^{\mathrm{sc}})^{-1} g_1^{\mathrm{sc}} h^{\mathrm{sc}}$. We put $g = {}_zf(g^{\mathrm{sc}})$, $g_1 = f(g_1^{\mathrm{sc}})$, $h = f(h^{\mathrm{sc}})$, $T_1 = Z_{G'}(g_1)$ and $i_1 : T_1 \to G'$.

Since $g \in ({}_z(G'))(k)$ and $g_1 \in G'(k)$, we have $h^{-1} g_1 h = z_\sigma {}^\sigma(h^{-1} g_1 h) z_\sigma^{-1} = z_\sigma h^{-\sigma} g_1 {}^\sigma h z_\sigma^{-1}$ for each $s \in \Gamma_k$, whence

$$g_1 = a_\sigma g_1 a_\sigma^{-1},$$

where $a_\sigma = h z_\sigma h^{-\sigma}$ is a 1-cocycle cohomologous to z with values in $T_1(k_s) = Z_{G'}(g_1)(k_s)$. It remains to show the equality on the oriented types. By the rigidity trick (see the proof of Proposition 3.2 in [Gille 2004]), up to replacing k by the function field of the T_1-torsor defined by a, we can assume that $[a] = 1 \in H^1(k, T_1)$. We write $a_\sigma = b^{-1} {}^\sigma b$ for some $b \in T_1(k_s)$, and we have that $z_\sigma = (bh)^{-1} {}^\sigma(bh)$ and $\varphi^{-1}(g) = (bh)^{-1} g_1 bh$.

Putting $h_2 = bh \in G'(k_s)$, we have $z_\sigma = h_2^{-1} {}^\sigma h_2$ and $\varphi^{-1}(g) = h_2^{-1} g_1 h_2$. We get k-isomorphisms $\phi_2 = \phi \circ L_{h_2^{-1}} : G' \to P(z)$ and $\varphi_2 = \varphi \circ \mathrm{int}(h_2^{-1}) : G' \xrightarrow{\sim} {}_z(G')$ such that the following diagram commutes

$$\begin{tikzcd} T_1 \arrow[r, "i_1"] \arrow[d, "\varphi_2", "\wr"'] & G' \arrow[d, "\varphi_2", "\wr"'] \\ T \arrow[r, "i"] & {}_z(G') \end{tikzcd}$$

Thus $\mathbf{type}_{\mathrm{can}}(T,i)=\mathbf{type}_{\mathrm{id}}(T_1,i_1)\in H^1(k,W')$. □

2.5. *Image of the cohomology of tori, II.* Recall the following well-known fact.

Lemma 2.5.1. *Let H be a reductive k-group and T be a k-torus of the same rank as H. Let $i,j:T\to H$ be k-embeddings of a maximal k-torus T. If $j=\mathrm{Int}(h)\circ i$ for some $h\in H(k_s)$, then we have $h^{-1}\,{}^{\sigma}h\in i(T)(k_s)$ for all σ in the absolute Galois group Γ_k.*

Proof. For any $\sigma\in\Gamma$ and any $t\in T(k_s)$, we have $j({}^{\sigma}t)={}^{\sigma}h\cdot i({}^{\sigma}t)\cdot{}^{\sigma}h^{-1}$. Therefore, we have $j=\mathrm{Int}({}^{\sigma}h)\circ i=\mathrm{Int}(h)\circ i$, and $h^{-1}\,{}^{\sigma}h$ is a k_s-point of the centralizer $C_H(i(T))=i(T)$. □

Lemma 2.5.2. *Let H be a reductive k-group and let T be a k-torus of the same rank as H. Let v be an orientation of H with respect to a quasisplit form H'. Let $i,j:T\to H$ be k-embeddings of a maximal k-torus T which are $H(k_s)$-conjugate. Then we have* $\mathrm{Im}(i_*)=\mathrm{Im}(j_*)\subseteq H^1(k,H)$ *and* $\mathbf{type}_v(T,i)=\mathbf{type}_v(T,j)$.

Proof. Let $j=\mathrm{Int}(h)\circ i$ for some $h\in H(k_s)$. By Lemma 2.5.1, we have $h^{-1}\,{}^{\sigma}h\in i(T)(k_s)$. Let $[\alpha]\in\mathrm{Im}(j_*)$ and α be a cocycle with values in $j(T(k_s))$ which represents $[\alpha]$. Define $\beta_\sigma=h^{-1}\alpha_\sigma{}^{\sigma}h$. Then β is cohomologous to α and $\beta_\sigma=(h^{-1}\alpha_\sigma h)\cdot(h^{-1}\,{}^{\sigma}h)\in i(T(k_s))$. Hence $[\alpha]=[\beta]\in\mathrm{Im}(i_*)$, which shows that $\mathrm{Im}(i_*)=\mathrm{Im}(j_*)\subseteq H^1(k,H)$.

Let $T_1=i(T)$ and $T_2=j(T)$. Let $\mathrm{Transpt}_G(T_1,T_2)$ be the strict transporter from T_1 to T_2 [Demazure and Grothendieck 1970a, $\mathrm{VI_B}$, Définition 6.1(ii)]. Note that $\mathrm{Transpt}_G(T_1,T_2)$ is a right $N_G(T_1)$-torsor. We have a canonical isomorphism

$$\mathrm{Transpt}_G(T_1,T_2)\wedge\mathrm{Isomint}_v(\underline{\Psi}',\underline{\Psi}(G,T_1))\xrightarrow{\sim}\mathrm{Isomint}_v(\underline{\Psi}',\underline{\Psi}(G,T_2)).$$

Since $j=\mathrm{Int}(h)\circ i$, we have $h\in\mathrm{Transpt}_G(T_1,T_2)(k_s)$ and h defines a trivialization $\phi_h:N_G(T_1)\to\mathrm{Transpt}_G(T_1,T_2)$ which sends the neutral element to h. Let $W_1=N_G(T_1)/T_1$. Since $\phi_h^{-1}\circ\sigma(\phi_h)=h^{-1}\,{}^{\sigma}h\in T_1(k_s)$, the image of the class of $\mathrm{Transpt}_G(T_1,T_2)$ in $H^1(k,W_1)$ is trivial. Hence $\mathrm{Isomint}_v(\underline{\Psi}',\underline{\Psi}(G,T_1))\simeq\mathrm{Isomint}_v(\underline{\Psi}',\underline{\Psi}(G,T_2))$; i.e., $\mathbf{type}_v(T,i)=\mathbf{type}_v(T,j)$. □

Proposition 2.5.3. *Let T be a k-torus of the same rank as G. Let $i_1,i_2:T\to G$ be k-embeddings of T in G. Let v be an orientation of G with respect to a quasisplit form G'. If* $\mathbf{type}_v(T,i_1)=\mathbf{type}_v(T,i_2)\in H^1(k,W')$, *then there is a k-embedding $j:T\to G$ such that $j(T)=i_1(T)$ and j,i_2 are $G(k_s)$-conjugate. In particular, the images of $i_{1,*},i_{2,*},j:H^1(k,T)\to H^1(k,G)$ coincide.*

Proof. Let $T_1=i_1(T)$ and $T_2=i_2(T)$ and again put $W_i=N_G(T_i)/T_i$ for $i=1,2$. Let η denote the class of the $N_G(T_1)$-torsor $\mathrm{Transpt}_G(T_1,T_2)$ in $H^1(k,N_G(T_1))$ and $\bar\eta$ be the image of η in $H^1(k,W_1)$. We have a canonical isomorphism

$$\mathrm{Transpt}_G(T_1,T_2)\wedge\mathrm{Isomint}_v(\underline{\Psi}',\underline{\Psi}(G,T_1))\xrightarrow{\sim}\mathrm{Isomint}_v(\underline{\Psi}',\underline{\Psi}(G,T_2)).$$

Since $\mathbf{type}_v(T, i_1) = \mathbf{type}_v(T, i_2)$, we have

$$\mathrm{Isomint}_v(\underline{\Psi}', \underline{\Psi}(G, T_1)) \simeq \mathrm{Isomint}_v(\underline{\Psi}', \underline{\Psi}(G, T_2)).$$

Hence $\bar{\eta}$ is the trivial class in $H^1(k, W_1)$. Thus the $N_G(T_1)$-torsor $\mathrm{Transpt}_G(T_1, T_2)$ admits a reduction to T_1. More precisely, there exist a T_1-torsor E_1 and an isomorphism $E_1 \wedge^{T_1} N_G(T_1) \xrightarrow{\sim} \mathrm{Transpt}_G(T_1, T_2)$ of $N_G(T_1)$-torsors. We take a point $e_1 \in E_1(k_s)$ and consider its image g in $G(k_s)$ under the mapping

$$E_1 \wedge^{T_1} N_G(T_1) \xrightarrow{\sim} \mathrm{Transpt}_G(T_1, T_2) \hookrightarrow G.$$

Then $h = g^{-1}\, {}^{\sigma}g$ is a k_s-point of the centralizer $C_G(T_1) = T_1$ for all $\sigma \in \Gamma_k$. We define a k-embedding $j : T \to G$ as $j(t) = (\mathrm{Int}(g^{-1}) \circ i_2)(t)$. To see that j is indeed defined over k, we check as follows:

$$\begin{aligned} j({}^{\sigma}t) &= (\mathrm{Int}(g^{-1}) \circ i_2)({}^{\sigma}t) \\ &= \mathrm{Int}(g^{-1})({}^{\sigma}i_2(t)) \\ &= h \cdot {}^{\sigma}((\mathrm{Int}(g^{-1}) \circ i_2)(t)) \cdot h^{-1} \\ &= {}^{\sigma}(j(t)). \end{aligned}$$

By our construction, we have $j(T) = i_1(T)$ and i_2, j are conjugated. Let $f = (j|_{T_1})^{-1} \circ i_1$. Then f is an automorphism of T and $i_1 = j \circ f$. Hence the images of $i_{1,*}$ and j_* coincide. By Lemma 2.5.2, the images of j and $i_{2,*}$ coincide. □

This applies to the quasisplit case and enables us to slightly refine Theorem 2.4.1.

Corollary 2.5.4. *With the notations of Theorem 2.4.1, for each class $\gamma \in H^1(k, W')$, choose (by Theorem 2.3.1) a k-embedding $i(\gamma) : E(\gamma) \to G'$ of oriented type γ. Then the map*

$$\bigsqcup_{\gamma \in H^1(k, W')} H^1(k, E(\gamma)) \xrightarrow{\sqcup i(\gamma)_*} H^1(k, G')$$

is onto.

2.6. ***Varieties of embedding k-tori.*** Let T be a k-torus and $\underline{\Psi}$ be a twisted root datum of Ψ_0 attached to T; i.e., the character group of T is isomorphic to the character group encoded in Ψ. In this section, we will define a k-variety X such that the existence of a k-point of X is equivalent to the existence of a k-embedding of T into G with respect to $\underline{\Psi}$.

We start with a functor. The *embedding functor* $\mathcal{E}(G, \underline{\Psi})$ is defined as follows: for any k-algebra C, $\mathcal{E}(G, \underline{\Psi})(C)$ is the set of all $f : T_C \hookrightarrow G_C$ such that f is both a closed immersion and a group homomorphism which induces an isomorphism $f^{\Psi} : \underline{\Psi}_C \xrightarrow{\sim} \underline{\Psi}(G_C, f(T_C))$ such that $f^{\Psi}(\alpha) = \alpha \circ f^{-1}|_{f(T_{C'})}$ is in $\underline{\Psi}(G_{C'}, f(T_{C'}))$ for all C'-roots α for all C-algebra C'. In fact, the functor $\mathcal{E}(\underline{\Psi}, G)$

is representable by a k-scheme [Lee 2014, Theorem 1.1]. Define the Galois set $\mathrm{Isomext}(\underline{\Psi}, G)$ by $\mathrm{Isomext}(\underline{\Psi}, G) = \mathrm{Isomext}(\underline{\Psi}, \underline{\Psi}(G, E))$, where E stands for an arbitrary maximal k-torus of G. Given an orientation $v \in \mathrm{Isomext}(\underline{\Psi}, G)(k)$, we define the *oriented embedding functor* as follows: for any k-algebra C,

$$\mathcal{E}(G, \underline{\Psi}, v)(C) = \big\{ f : T_C \hookrightarrow G_C \mid f \in \mathcal{E}(G, \underline{\Psi})(C) \text{ and } \\ \text{the image of } f^{\Psi} \text{ in } \mathrm{Isomext}(\underline{\Psi}, G)(C) \text{ is } v \big\}.$$

We have the following result:

Theorem 2.6.1. *In the sense of the étale topology, $\mathcal{E}(G, \underline{\Psi}, v)$ is a left homogeneous space under the adjoint action of G and a torsor over the variety of the maximal tori of G under the right $W(\underline{\Psi})$-action. Moreover, $\mathcal{E}(G, \underline{\Psi}, v)$ is representable by an affine k-scheme.*

Proof. We refer to [Lee 2014, Theorem 1.6]. □

Remark 2.6.2. The definition of varieties of embeddings is quite abstract but is simplified a lot if there is a k-embedding $i : T \to G$ of oriented type isomorphic to $(\underline{\Psi}, v)$. Indeed in this case, the k-variety $\mathcal{E}(G, \underline{\Psi}, v)$ is G-isomorphic to the homogeneous space $G/i(T)$, and we observe that the map $G/i(T) \to G/N_G(i(T))$ is a $W_G(i(T))$-torsor over the variety of maximal tori of G.

Remark 2.6.3. We sketch another way to prove Theorem 2.4.1. With the notations of that result, let $z \in Z^1(k, G')$ and put $G = {}_zG'$. Let T be a maximal k-torus of G and consider the twisted root data $\underline{\Psi} = \underline{\Psi}(G, T)$ attached to T. Let v be the canonical element in $\mathrm{Isomext}(\underline{\Psi}, G)(k)$ and let $v' = c \circ v$, where $c \in \mathrm{Isomext}(G, G')(k)$ corresponds to the canonical orientation $\underline{\mathrm{Dyn}}(G) \cong \underline{\mathrm{Dyn}}(G')$. We denote by X (resp. X') the k-variety of oriented embeddings of T in G (resp. G') with respect to $\underline{\Psi}$ and v (resp. v'). Note that G' acts on X' and we have a natural isomorphism $X \xrightarrow{\sim} {}_zX'$. Theorem 2.3.1(2) shows that $X'(k) \neq \varnothing$ and the choice of a k-point x' of X' defines a G'-equivariant isomorphism $G'/T \xrightarrow{\sim} X'$. In the other hand, the embedding i defines a k-point $x \in X(k)$. Since $X \cong {}_zX'$, we have that ${}_z(G'/T)(k) \neq \varnothing$; hence the class $[z] \in H^1(k, G)$ admits a reduction to $i' : T \hookrightarrow G'$ such that $\mathbf{type}_{\mathrm{can}}(T, i) = \mathbf{type}_{\mathrm{id}}(T, i') \in H^1(k, W')$.

3. Generalities on octonion algebras

Let C be an octonion algebra. We denote by G the automorphism group of C; it is a semisimple k-group of type G_2. We denote by N_C the norm of C; it is a 3-fold Pfister form. In particular, N_C is hyperbolic (equivalently isotropic) if and only if G is split (equivalently isotropic).

3.1. ***Behavior under field extensions.*** If l/k is a field extension of odd degree, the Springer odd extension theorem [Elman et al. 2008, 18.5] implies that C is split if and only if C_l is split. More generally, we have the following criterion.

Lemma 3.1.1. *Let $(k_j)_{j=1,\dots,n}$ be a family of finite field extensions such that* g.c.d.$([k_j : k])$ *is odd. Then C is split if and only if C_{k_j} is split for $j = 1, \dots, n$.* □

Proof. The left implication is obvious. Conversely, assume that C_{k_j} is split for $j = 1, \dots, n$. Then there exists an index j such that $[k_j : k]$ is odd, hence C splits. □

Remark 3.1.2. This is a special case of the following more general result by Garibaldi and Hoffmann [2006, Theorem 0.3] answering positively Totaro's question. Let $(k_j)_{j=1,\dots,n}$ be a family of finite field extensions and put $d = \text{g.c.d.}([k_j : k])$. Let C, C' be Cayley k-algebras such that C_{k_j} and C'_{k_j} are isomorphic for $j = 1, \dots, n$. Then there exists a separable finite field extension K/k of degree dividing d such that C_K is isomorphic to C'_K. This is the case of groups of type G_2 in that theorem which includes also the case of certain groups of type F_4 and E_6.

We recall also the behavior with respect to quadratic étale algebras.

Lemma 3.1.3. *Let k'/k be a quadratic étale algebra. Then the following are equivalent*:

(i) *$C \otimes_k k'$ splits.*

(ii) *There is an isometry $(k', n_{k'/k}) \to (C, N_C)$, where $n_{k'/k} : k' \to k$ stands for the norm map.*

(iii) *There exists an embedding of unital composition k-algebras $k' \to C$.*

Proof. If C is split, all three facts hold so that we can assume that C is not split.

(i) ⇒ (ii): Since C is not split, it follows that k' is a field. Since N_C is split over k', there exists a nontrivial and nondegenerate symmetric bilinear form B such that $B \otimes n_{k'/k}$ is a subform of N_C [Elman et al. 2008, 34.8]. Since N_C is multiplicative, there is an isometry $(k', n_{k'/k}) \to (C, N_C)$.

(ii) ⇒ (iii): Since the orthogonal group $O(N_C)(k)$ acts transitively on the sphere $\{x \in C \mid N_C(x) = 1\}$, we can assume that our isometry $(k', n_{k'/k}) \to (C, N_C)$ maps $1_{k'}$ to 1_C. It is then a map of unital composition k-algebras.

(iii) ⇒ (i): If $k' = k \times k$, then N_C is isotropic and C is split. Hence k' is a field and N_C is k'-isotropic so that $C_{k'}$ is split. □

3.2. ***The Cayley–Dickson process.*** We know that C, up to k-isomorphism, can be obtained by the Cayley–Dickson doubling process; that is, $C \cong C(Q, c) = Q \oplus Qa$, where Q is a k-quaternion algebra and $c \in k^\times$ [Springer and Veldkamp 2000, § 1.5].

We denote by $\sigma_Q = \mathrm{trd}_Q - \mathrm{id}_Q$ the canonical involution of Q and recall that the multiplicativity rule on C, the norm N_C, and the canonical involution σ_C are given by

$$(x+ya)(u+va) = (xu + c\sigma_Q(v)y) + (vx + y\sigma_Q(u))a \quad (x,y,u,v \in Q),$$
$$N_C(x+ya) = N(x) - cN(y),$$
$$\sigma_C(x+ya) = \sigma_Q(x) - ya.$$

Then N_C is isometric to the 3-Pfister form $n_Q \otimes \langle 1, -c\rangle$ and that form determines the octonion algebra [ibid., Corollary 1.7.3]. Also it provides an embedding j of the k-group $H(Q) = (\mathrm{SL}_1(Q) \times_k \mathrm{SL}_1(Q))/\mu_2$ in $\mathrm{Aut}(C(Q,c))$. This map is given by $(g_1, g_2).(q_1, q_2) = (g_1 q_1 g_1^{-1}, g_2 q_2 g_1^{-1})$. Another corollary of the determination of an octonion algebra by its norm is the following well-known fact.

Corollary 3.2.1. *Let C be a octonion k-algebra and let Q be a quaternion algebra. Then the following are equivalent*:

(i) *There exists $c \in k^\times$ such that $C \cong C(Q,c)$.*

(ii) *There exists an isometry $(Q, N_Q) \to (C, N_C)$.*

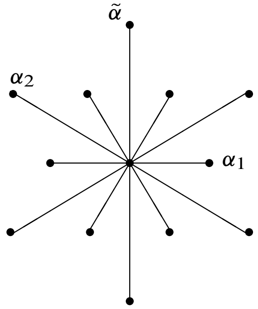
Proof. (i) $\Rightarrow$ (ii) is obvious. Assume that there exists an isometry $(Q, N_Q) \to (C, N_C)$. By the linkage property of Pfister forms [Elman et al. 2008, 24.1(1)], there exists a bilinear 1-Pfister form ϕ such that $N_C \cong N_Q \otimes \phi$. Since N_C represents 1, we can assume that ϕ represents 1 so that $\phi \cong \langle 1, -c\rangle$. Therefore C and $C(Q,c)$ have isometric norms and are isomorphic. □

Remark 3.2.2. In odd characteristic, Hooda provided an alternative proof, see [Hooda 2014, Theorem 4.3] and also a nice generalization [ibid., Proposition 4.2].

Lemma 3.2.3. *Let C be a nonsplit octonion k-algebra. If $D \subseteq C$ is a unital composition subalgebra and $u \in C \setminus D$ then $D \oplus Du$ is a unital composition subalgebra as well.*

Proof. Since C is nonsplit, the corresponding norm map N_C is anisotropic. Let b_C be the polar map of N_C. Since the map $x \mapsto b_C(u,x)$ is linear and the restriction of b_C on $D \times D$ is regular, there is $v \in D$ such that $b_C(v,x) = b_C(u,x)$ for all $x \in D$. Let $u' = u - v$. We have $b_C(u',x) = b_C(v,x) - b_C(u,x) = 0$ for all $x \in D$, so $u' \in D^\perp$. Since $v \in D$ and $u \notin D$, we have $u' \neq 0$, so $N_C(u') \neq 0$. By the doubling process [Springer and Veldkamp 2000, Proposition 1.5.1], we have that $D \oplus Du'$ is a unital composition subalgebra of C. But $u' = u - v$ and $v \in D$, so $D \oplus Du' = D \oplus Du$. □

3.3. ***On the dihedral group, I.*** In this case, $W_0 = \mathrm{Aut}(\Psi_0)$ and $W_0 = D_6 = \mathbb{Z}/6\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} = C_2 \times S_3$ is the dihedral group of order 12. More precisely, $C_2 = \langle c\rangle$ stands for its center. The right way to see it is by its action on the root system $\Psi(G_0, T_0) \subset \widehat{T}_0 = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 = \mathbb{Z}^2$, as provided by the following picture:



where α_1, α_2 stand for a base of the root system G_2 and $\tilde{\alpha} = 3\alpha_1 + 2\alpha_2$.

Let $\{\epsilon_i\}_{i=1}^3$ be an orthonormal basis of $\mathbb{Q}^3$. We can view the root space of G_2 as the hyperplane in $\mathbb{Q}^3$ defined by $\left\{\sum_{i=1}^3 \xi_i \epsilon_i \mid \sum_{i=1}^3 \xi_i = 0\right\}$, and identify α_1, α_2 with $\epsilon_1 - \epsilon_2$ and $-2\epsilon_1 + \epsilon_2 + \epsilon_3$ respectively [Bourbaki 1981, planche IX]. For a root α, let s_α be the reflection orthogonal to α. Under the above identification, the element $c = s_{2\alpha_1+\alpha_2} s_{\alpha_2}$ acts on the roots by $-\mathrm{id}$ and $S_3 = \langle s_{\alpha_1}, s_{2\alpha_1+\alpha_2}\rangle$ acts by permuting the ϵ_i. Note that although $s_{2\alpha_1+\alpha_2} s_{\alpha_2}$ acts on the subspace $\left\{\sum_{i=1}^3 \xi_i \epsilon_i \mid \sum_{i=1}^3 \xi_i = 0\right\}$ by $-\mathrm{id}$, $s_{2\alpha_1+\alpha_2} s_{\alpha_2}$ does not act as $-\mathrm{id}$ on $\{\epsilon_i\}_{i=1}^3$.

Remarks 3.3.1. (a) In the G_2 root system, for any long root β and any short root α orthogonal to β, we have $s_\alpha \circ s_\beta = c$. Also observe that $\widehat{T}_0$ is a sublattice of index 2 of the lattice $\mathbb{Z}\frac{\alpha}{2} \oplus \mathbb{Z}\frac{\beta}{2}$. This is related to the fact that the morphism $\mathrm{SL}_2 \times \mathrm{SL}_2 \to G_0$ defined by the coroots $\alpha^\vee$ and $\beta^\vee$ has kernel equal to the diagonal subgroup μ_2.
(b) The roots α_1, $\tilde{\alpha}$ generate a closed symmetric subsystem of type $A_1 \times A_1$ of G_2. Any subroot system (not necessarily closed) of G_2 which is of type $A_1 \times A_1$ is a W_0-conjugate of the previous one.

3.4. *Subgroups of type $A_1 \times A_1$.* Given an octonion k-algebra C, we relate Cayley–Dickson decomposition to subgroups of $G = \mathrm{Aut}(C)$.

Lemma 3.4.1. *Let H be a semisimple k-subgroup of G of type $A_1 \times A_1$. Then there exists a quaternion algebra Q, $c \in k^\times$, an isomorphism $C \cong C(Q, c)$ and an isomorphism $H \xrightarrow{\sim} H(Q)$ such that the following diagram commutes*:

$$\begin{tikzcd} H \arrow[r, hook] \arrow[d, "\wr"] & G \arrow[d, "\wr"] \\ H(Q) \arrow[r, hook, "j"] & \mathrm{Aut}(C(Q,c)) \end{tikzcd}$$

Proof. We start with a few observations on the split case $G = G_0 = \mathrm{Aut}(C_0)$, where we have the k-subgroup $H_0 = (\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2$ acting on C_0. The root subsystem $\Phi(H_0, T_0)$ is $\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\tilde{\alpha}$ so that the first (resp. the second) factor SL_2 of H_0 corresponds to a short (resp. long) root. We denote by $H_{0,<} \cong \mathrm{SL}_2$ (resp. $H_{0,>}$) the "short" subgroup (resp. the "long" one) of H_0. Taking the decomposition

$C_0 = M_2(k) \oplus M_2(k)_\sharp$, the point is that we have $M_2(k) = (C_0)^{H_{0,>}}$. In other words, we can recover the composition subalgebra $M_2(k)$ of C_0 from H_0.

We come now to our problem. We are given a k-subgroup H of $G = \mathrm{Aut}(C)$ of type $A_1 \times A_1$. Let T be a maximal k-torus of H. Then the root system $\Phi(H_{k_s}, T_{k_s})$ is a subsystem of $\Phi(G_{k_s}, T_{k_s}) \cong \Psi_0$ of type $A_1 \times A_1$; hence W_0-conjugated to the standard one (Remarks 3.3.1(b)). Since the Galois action preserves the length of a root, it follows that we can define by Galois descent the k-subgroups $H_<$ and $H_>$ of H. We define then $Q = (C)^{H_>}$. By Galois descent, it is a quaternion subalgebra of C which is normalized by H. It leads to a Cayley–Dickson decomposition $C = Q \oplus L$, where L is the orthogonal complement of Q in C. Then L is a right Q-module and we choose $a \in L$ such that $L = Qa$. The k-subgroup $H(Q)$ of $\mathrm{Aut}(C)$ is nothing but $\mathrm{Aut}(C, Q)$ [Springer and Veldkamp 2000, §2.1], so we have $H \subseteq H(Q)$. For dimension reasons, we conclude that $H = H(Q)$ as desired. □

4. Embedding a torus in a group of type G_2

We assume that G is a semisimple k-group of type G_2. As in Section 2, we denote its split form by G_0, and T_0, W_0, etc. are defined as before.

4.1. ***On the dihedral group, II.*** We continue to discuss the action of the dihedral group W_0 (of order 12) on the root system of type G_2 started in Section 3.3. Let $\oplus_{i=1}^3 \mathbb{Z}\epsilon_i$ be a W_0-lattice, where the S_3-component of W_0 acts by permuting the ϵ_i and the center acts by $-\mathrm{id}$. Note that G_0 is of type G_2, so G_0 is both adjoint and simply connected and the dual group of G_0 is isomorphic to G_0 itself. Hence we have the following exact sequence of W_0-lattices, where W_0 acts on $\mathbb{Z}$ through its center $\mathbb{Z}/2\mathbb{Z}$ by $-\mathrm{id}$:

$$0 \to \widehat{T}_0 \xrightarrow{f} \oplus_{i=1}^3 \mathbb{Z}\epsilon_i \xrightarrow{\deg} \mathbb{Z} \to 0,$$

where $f(\alpha_1) = \epsilon_1 - \epsilon_2$ and $f(\alpha_2) = -2\epsilon_1 + \epsilon_2 + \epsilon_3$. We also consider its dual sequence

$$0 \to \mathbb{Z} \to \oplus_{i=1}^3 \mathbb{Z}\epsilon_i^\vee \to \widehat{T}_0^\vee \simeq \widehat{T}_0 \to 0.$$

4.2. ***Subtori.*** Keep the notations in Section 3.3. Let us fix an isomorphism

$$\chi : \mathbb{Z}/2\mathbb{Z} \times S_3 \to \langle c \rangle \times \langle s_{\alpha_1}, s_{2\alpha_1+\alpha_2} \rangle = W_0,$$

where $\chi((-1, 1)) = c$, $\chi((1, (12))) = s_{\alpha_1}$ and $\chi((1, (23))) = s_{2\alpha_1+\alpha_2}$.

We identify $\mathbb{Z}/2\mathbb{Z} \times S_3$ with W_0 by χ in the rest of this paper. Under this identification, we have

$$H^1(k, W_0) = H^1(k, \mathbb{Z}/2\mathbb{Z}) \times H^1(k, S_3).$$

Hence a class of $H^1(k, W_0)$ is represented uniquely (up to k-isomorphism) by a couple (k', l), where k' is a quadratic étale algebra of k and l/k is a cubic étale algebra of k.

Given such a couple (k', l), we denote by $\underline{\Psi}_{(k',l)} = [(k', l)] \wedge^{W_0} \Psi_0$ the associated twisted root datum. Let $l' = l \otimes_k k'$ and define the k-torus

$$T^{(k',l)} = \operatorname{Ker}\big(R_{k'/k}(R^1_{l'/k'}(\mathbb{G}_{m,l'})) \xrightarrow{N_{k'/k}} R^1_{l/k}(\mathbb{G}_{m,l})\big).$$

In the following, we prove that the torus encoded in $\underline{\Psi}_{(k',l)}$ is indeed $T^{(k',l)}$. However, we should keep in mind that two nonisomorphic root data $\underline{\Psi}$ may encode the same torus (Remark 4.2.2).

Lemma 4.2.1. *Let T be a k-torus of rank 2 and let $i : T \to G$ be a k-embedding such that* **type**$(T, i) = [(k', l)]$. *Then*:

(1) *The k-torus T is k-isomorphic to $T^{(k',l)}$.*

(2) *If there exists a quadratic étale algebra l_2 such that $l = k \times l_2$, then there is a k-isomorphism*

$$T \cong \big(R^1_{k_1/k}(\mathbb{G}_m) \times_k R^1_{k_2/k}(\mathbb{G}_m)\big)/\mu_2,$$

where k_1, k_2 are quadratic étale algebras such that $k_2 = k'$ and $[k_1] = [k_2] + [l_2] \in H^1(k, \mathbb{Z}/2\mathbb{Z})$.

Proof. (1) We have $W_0 = \mathbb{Z}/2\mathbb{Z} \times S_3$ and from Section 4.1, we have a W_0-resolution

$$0 \to \mathbb{Z} \to \oplus_{i=1}^3 \mathbb{Z}\epsilon_i^\vee \to \widehat{T}_0 \to 0.$$

It follows that $\widehat{T}_0$ is isomorphic to the W_0-module $\oplus_{i=1}^3 \mathbb{Z}\epsilon_i^\vee / \langle (1,1,1) \rangle$.

Let N be the W_0-lattice $\oplus_{i=1}^3 \mathbb{Z}e_i / \langle (1,1,1) \rangle$, where S_3 acts by permuting the indices and $\mathbb{Z}/2\mathbb{Z}$ acts trivially. Note that as $\mathbb{Z}$-lattices, we can identify N with $\widehat{T}_0$. Let $M = N \oplus N$ and equip M with a W_0-action: S_3 acts on N diagonally and $\mathbb{Z}/2\mathbb{Z}$ acts on M by exchanging the two copies of N. Embed N diagonally into M and we get the exact sequence of W_0-modules

$$0 \to N \xrightarrow{f} M = N \oplus N \xrightarrow{g} \widehat{T}_0 \to 0,$$

where $f(x) = (x, x)$ and $g(x, y) = x - y$. After twisting the above exact sequence by the W_0-torsor attached to (k', l) and taking the corresponding tori, we have

$$1 \to T \to R_{k'/k}\big(R^1_{l'/k'}(\mathbb{G}_{m,l'})\big) \xrightarrow{n_{k'/k}} R^1_{l/k}(\mathbb{G}_{m,l}) \to 1.$$

Hence T is the k-torus $T^{(k',l)}$.

(2) If $l = k \times l_2$, then there is an injective homomorphism $\iota : \mathbb{Z}/2\mathbb{Z} \to S_3$ and a class $[z] \in \operatorname{im}(\iota_* : H^1(k, \mathbb{Z}/2\mathbb{Z}) \to H^1(k, S_3))$ such that l corresponds to $[z]$. Let α be a short root such that the corresponding reflection s_α is $\iota(-1)$, and let β be a

long root orthogonal to α. As we mentioned in Remarks 3.3.1(a), the center of W_0 is generated by $s_\alpha \circ s_\beta$. Therefore, the image of the map

$$\mathrm{Id} \times \iota : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/2\mathbb{Z} \times S_3 = W_0$$

is generated by $\{s_\alpha, s_\beta\}$. Let us call it $W^{(k',l_2)}$. Let $H_0 \simeq (\mathrm{SL}_2 \times_k \mathrm{SL}_2)/\mu_2$ be the subgroup of G_0 generated by T_0 and the root groups associated to $\pm\alpha$ and $\pm\beta$. Then H_0 is of type $A_1 \times A_1$ and the Weyl group of H_0 with respect to T_0 is exactly $W^{(k',l_2)}$. Hence there is $[x] \in \mathrm{im}(H^1(k, N_{H_0}(T_0)) \to H^1(k, G_0))$ such that $(G, i(T))$ is isomorphic to ${}_x(G_0, T_0)$. Moreover, the embedding i factorizes through $H = {}_x(H_0)$. Let the first (resp. second) copy of SL_2 of H_0 correspond to the root group $\pm\beta$ (resp. $\pm\alpha$). Let π be the projection from $N_{H_0}(T_0)$ to $N_{H_0}(T_0)/T_0 = W^{(k',l_2)}$. Since

$$([k'], [l_2]) \in H^1(k, \langle s_\beta \circ s_\alpha \rangle) \times H^1(k, \langle s_\alpha \rangle) = H^1(k, W^{(k',l_2)})$$

is equivalent to

$$([k'], [k'] + [l_2]) \in H^1(k, \langle s_\beta \rangle) \times H^1(k, \langle s_\alpha \rangle) = H^1(k, W^{(k',l_2)}),$$

we have

$$\pi_*([x]) = \big([k'] + [l_2], [k']\big) \in H^1(k, \langle s_\alpha \rangle) \times H^1(k, \langle s_\beta \rangle).$$

Therefore,

$$T \simeq {}_x(T_0) \cong \big(R^1_{k_1/k}(\mathbb{G}_m) \times_k R^1_{k_2/k}(\mathbb{G}_m)\big)/\mu_2,$$

where $[k_2] = k'$ and $[k_1] = [k_2] + [l_2]$. □

Remark 4.2.2. A natural question is whether the class of $[(k', l)]$ is determined by the isomorphism class of the torus $T^{(k',l)}$ as a k-torus. It is not the case; there are indeed examples of nonequivalent pairs (k', l) and $(k'_\sharp, l_\sharp)$ such that the k-tori $T^{(k',l)}$ and $T^{(k'_\sharp,l_\sharp)}$ are isomorphic whenever the field k admits a biquadratic field extension $k_1 \otimes_k k_2$. We put then $k_{1,\sharp} = k_2$ and $k_{2,\sharp} = k_1$. With the notations of the proof of Lemma 4.2.1(2), we consider the k-tori

$$T = \big(R^1_{k_1/k}(\mathbb{G}_m) \times_k R^1_{k_2/k}(\mathbb{G}_m)\big)/\mu_2,$$
$$T_\sharp = \big(R^1_{k_{1,\sharp}/k}(\mathbb{G}_m) \times_k R^1_{k_{2,\sharp}/k}(\mathbb{G}_m)\big)/\mu_2.$$

Then the k-tori T and $T_\sharp$ are obviously k-isomorphic. However, the root data $\underline{\Psi}_{(k',l)}$ and $\underline{\Psi}_{(k'_\sharp,l_\sharp)}$ are not isomorphic as $k_2 \not\cong k_{2,\sharp} = k_1$.

Since the pointed set $H^1(k, \mathrm{GL}_2(\mathbb{Z}))$ classifies two-dimensional k-tori, the map $H^1(k, W_0) \to H^1(k, \mathrm{GL}_2(\mathbb{Z}))$ is in this case not injective. It is due to the fact that the normalizer of $C_2 \times (1 \times \mathbb{Z}/2\mathbb{Z})$ in $\mathrm{GL}_2(\mathbb{Z})$ is larger than the normalizer in W_0.

We deal now with the Galois cohomology of those tori.

Lemma 4.2.3. (1) *We have an exact sequence*

$$0 \to \mathrm{Ker}(l^\times \to k^\times)/N_{l'/l}\big(\mathrm{Ker}\big((l')^\times \xrightarrow{n_{l'/k'}} (k')^\times\big)\big) \to H^1(k, T^{(k',l)})$$
$$\to (k')^\times/N_{l'/k'}((l')^\times) \xrightarrow{n_{k'/k}} k^\times/N_{l/k}(l^\times) \to 0,$$

and the map $n_{k'/k}$ admits a section.

(2) *Assume that k' and l are fields. Then $H^1(k, (\widehat{T^{(k',l)}})^0) = 0$.*

Proof. We put $T = T^{(k',l)}$.

(1) The Hilbert theorem 90 produces an isomorphism

$$k^\times/N_{l/k}(l^\times) \xrightarrow{\sim} H^1(k, R^1_{l/k}(\mathbb{G}_{m,l})).$$

Combined with the Shapiro isomorphism, we get an isomorphism

$$(k')^\times/N_{l'/k'}(l'^\times) \xrightarrow{\sim} H^1\big(k', R^1_{l'/k'}(\mathbb{G}_{m,l'})\big) \xrightarrow{\sim} H^1\big(k, R_{k'/k}\big(R^1_{l'/k'}(\mathbb{G}_{m,l'})\big)\big).$$

Putting these two facts together, the long exact sequence of Galois cohomology is

$$\cdots \to \mathrm{Ker}\big((l')^\times \to (k')^\times\big) \xrightarrow{N_{l'/l}} \mathrm{Ker}(l^\times \to k^\times) \to H^1(k, T)$$
$$\to (k')^\times/N_{l'/k'}((l')^\times) \xrightarrow{n_{k'/k}} k^\times/N_{l/k}(l^\times) \to \cdots .$$

Since $k^\times/N_{l/k}(l^\times)$ is of 3-torsion, half of the “diagonal map” $k^\times/N_{l/k}(l^\times) \to (k')^\times/N_{l'/k'}((l')^\times)$ provides a section of $(k')^\times/N_{l'/k'}((l')^\times) \xrightarrow{n_{k'/k}} k^\times/N_{l/k}(l^\times)$.

(2) We have an exact sequence

$$0 \to \widehat{T}^0 \to \mathrm{Coind}_k^{k'}(I_{l'/k'}) \xrightarrow{n_{k'/k}} I_{l/k} \to 0$$

of Galois modules, where $I_{l/k} = \mathrm{Ker}\big(\mathrm{Coind}_k^l(\mathbb{Z}) \to \mathbb{Z}\big)$. It gives rise to the long exact sequence of groups

$$0 \to H^0(k, \widehat{T}^0) \to H^0(k, \mathrm{Coind}_k^{k'}(I_{l'/k'})) \to H^0(k, I_{l/k}) \to \cdots$$
$$\to H^1(k, \widehat{T}^0) \to H^1(k, \mathrm{Coind}_k^{k'}(I_{l'/k'})) \to H^1(k, I_{l/k}) \to \cdots .$$

We consider the exact sequence $0 \to I_{l/k} \to \mathrm{Coind}_k^l(\mathbb{Z}) \to \mathbb{Z} \to 0$ and the corresponding sequence

$$0 \to H^0(k, I_{l/k}) \to H^0(k, \mathrm{Coind}_k^l(\mathbb{Z})) \to H^0(k, \mathbb{Z}) \to \cdots$$
$$\to H^1(k, I_{l/k}) \to H^1(k, \mathrm{Coind}_k^l(\mathbb{Z})) \to H^1(k, \mathbb{Z}).$$

The group $\mathbb{Z} = H^0(k, \mathrm{Coind}_k^l(\mathbb{Z}))$ embeds in $\mathbb{Z}$ by multiplication by 3; also we have $H^1(k, \mathrm{Coind}_k^l(\mathbb{Z})) \xrightarrow{\sim} H^1(l, \mathbb{Z}) = 0$ by Shapiro's isomorphism. The above sequence induces an isomorphism $\mathbb{Z}/3\mathbb{Z} \xrightarrow{\sim} H^1(k, I_{l/k})$. On the other hand, we have $H^1(k, \mathrm{Ind}_k^{k'}(I_{l'/k'})) \xrightarrow{\sim} H^1(k', I_{l'/k'}) \xleftarrow{\sim} \mathbb{Z}/3\mathbb{Z}$. The norm map $n_{k'/k} : H^1(\mathrm{Coind}_k^{k'}(I_{l'/k'})) \to H^1(k, I_{l/k})$ is multiplication by 2 on $\mathbb{Z}/3\mathbb{Z}$. Hence it is injective. By using the starting exact sequence, we conclude that $H^1(k, \widehat{T}^0) = 0$ as desired. $\square$

4.3. ***A necessary condition.*** There is a basic restriction on the types of maximal tori of G.

Proposition 4.3.1. (1) *Let T be a k-torus of rank two and let $i : T \to G$ be a k-embedding such that* $\mathbf{type}(T, i) = [(k', l)]$. *Then $G \times_k k'$ is split.*

(2) *Assume that $l = k \times k \times k$. Then the following are equivalent*:

(i) *There exists a k-embedding $i : T \to G$ of a rank-2 torus T such that* $\mathbf{type}(T, i) = [(k', k^3)]$.

(ii) *$G_{k'}$ splits.*

(iii) *There is an isometry $(k', n_{k'/k}) \hookrightarrow (C, N_C)$.*

Proof. (1) Since G is of type G_2, it is equivalent to show that $G \times_k k'$ is isotropic.

We may assume that $T = T^{(k',l)}$. We consider first the case when $l = k \times l_2$, where l_2 is a quadratic étale k-algebra. Then we have

$$T \times_k k' \xrightarrow{\sim} R^1_{l'/k'}(\mathbb{G}_{m,l'}) \xleftarrow{\sim} R_{l_2 \otimes k'/k'}(\mathbb{G}_{m,l_2 \otimes k'}).$$

Hence $T \times_k k'$ is isotropic.

It remains to consider the case when the cubic k-algebra l is a field. From the first case, we see that $G_{l'}$ is split. In other words, the k'-group $G_{k'}$ is split by the cubic field algebra $l = l \otimes_k k'$ of k'. Hence $C_{k'}$ is split, and hence C splits.

(2) (i) $\Rightarrow$ (ii) follows from (1).

(ii) $\Rightarrow$ (i): If G is split, (i) holds according to Theorem 2.3.1. We may assume that G is not split, and hence is anisotropic. In particular, k is infinite. Since $G_{k'}$ splits, k' is a field and we denote by $\sigma : k' \to k'$ the conjugacy automorphism. We use now a classical trick. Since $G(k')$ is Zariski dense in the Weil restriction $R_{k'/k}(G_{k'})$, there exists a Borel k-subgroup B of $R_{k'/k}(G_{k'})$ such that its conjugate $\sigma(B)$ is opposite to B. The k-group $T = B \cap \sigma(B) \cap G$ of G is then a rank-2 torus. If we write $B = R_{k'/k}(B')$, with B' a Borel k'-subgroup of $G_{k'}$, then $T_{k'}$ is a maximal torus of B'. We denote the natural embedding of the maximal torus T by $i : T \to G$. By seeing $i(T_{k'})$ as a maximal k'-torus of B', it follows that the action of σ on the root system $\Psi(G_{k'}, T')$ is by -1. Thus $\mathbf{type}(T, i) = (k', k^3)$ as desired.

For the equivalence (ii) $\Longleftrightarrow$ (iii), see Lemma 3.1.3. $\square$

Remark 4.3.2. Another proof of (2) is provided by the next Proposition 4.4.1; it is the case $k_1 = k_2$.

4.4. ***The biquadratic case.*** In the dihedral group $D_6 \subset \mathrm{GL}_2(\mathbb{Z})$, it is convenient to change coordinates by considering the diagonal subgroup $(\mathbb{Z}/2\mathbb{Z})^2 = \langle c_1, c_2 \rangle$. The map $(\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{\sim} C_2 \times (1 \times \mathbb{Z}/2\mathbb{Z}) \subset C_2 \times S_3$ is given by $(c_1, c_2) \mapsto (c_1, c_1 c_2)$.

We are interested in the case when the class of (k', l) belongs to the image of $H^1(k, \mathbb{Z}/2\mathbb{Z}) \times H^1(k, \mathbb{Z}/2\mathbb{Z}) \to H^1(k, \mathbb{Z}/2\mathbb{Z}) \times H^1(k, S_3)$. In terms of étale algebras, it rephrases by saying that there are quadratic étale k-algebras $k_1/k, k_2/k$ such that $k' = k_2$ and $l = k \times l_2$, where $[k_2] = [k_1] + [l_2]$. We call that case the biquadratic case. In that case, $T^{(k',l)}$ is k-isomorphic to

$$\big(R^1_{k_1/k}(\mathbb{G}_m) \times R^1_{k_2/k}(\mathbb{G}_m)\big)/\mu_2.$$

Proposition 4.4.1. *Let k_1, k_2 be quadratic étale k-algebras and denote by $\chi_1, \chi_2 \in H^1(k, \mathbb{Z}/2\mathbb{Z})$ their classes. We consider the couple $(k', l) = (k_2, k \times l_2)$, where $[l_2] = [k_1] + [k_2]$. We denote by $\underline{\Psi} = \underline{\Psi}_{(k',l)}$, defined in Section 4.2, and by $X = \mathcal{E}(G, \underline{\Psi})$ the K-variety of embeddings defined in Section 2.6.*

(a) *The following are equivalent*:

(1) $X(k) \neq \varnothing$; *that is, G admits a maximal k-torus of type* $[(k', l)]$.

(2) *$C \otimes_k k_j$ is split for $j = 1, 2$.*

(3) *C admits a quaternion subalgebra Q such that there exists $c \in k^\times$ satisfying*

$$[Q] = \chi_1 \cup (c) = \chi_2 \cup (c) \in {}_2\mathrm{Br}(k).$$

(b) *If the k-variety X has a zero-cycle of odd degree then it has a k-point.*

Proof. (a) If C is split, the statement is trivial since the three assertions hold. We can then assume that C is nonsplit. We choose scalars $a_1, a_2 \in k$ such that $k_j \cong k[t]/t^2 - a_j$ for $j = 1, 2$ if k is of odd characteristic and $k_j \cong k[t]/t^2 + t + a_j$ in the characteristic-two case.

(1) $\Rightarrow$ (2): We assume that $T = T^{k',l} \cong \big(R^1_{k_1/k}(\mathbb{G}_m) \times R^1_{k_2/k}(\mathbb{G}_m)\big)/\mu_2$ embeds in G. Then T_{k_j} is isotropic so that G_{k_j} is isotropic, and hence split for $j = 1, 2$. We conclude that C_{k_j} is split for $j = 1, 2$.

(2) $\Rightarrow$ (3): We shall construct a quaternion subalgebra Q of C which contains k_1 and k_2. Since C_{k_j} splits for $j = 1, 2$, we know that k_j embeds in C as a unital composition subalgebra (Lemma 3.1.3). If $k_1 = k_2$ then Q can be obtained from k_1 by the doubling process from [Springer and Veldkamp 2000, Proposition 1.2.3]. So we can assume that $k_1 \neq k_2$. Let $x \in k_2 \setminus k_1$. Then Lemma 3.2.3 shows that $Q = k_1 \oplus k_1 x$ is a unital composition subalgebra of C. It is of dimension 4, so it is a quaternion subalgebra which contains k_1 and k_2. The common slot lemma yields that there exists $c \in k^\times$ such that $[Q] = \chi_1 \cup (c) = \chi_2 \cup (c) \in \mathrm{Br}(k)$. In odd characteristic, a reference for the common slot lemma is [Lam 2005, Chapter III, Theorem 4.13]. A characteristic-free version is a consequence of a fact on Pfister forms pointed out by Garibaldi and Petersson [2011, Proposition 3.12]. The

1-Pfister quadratic forms $n_{k_1/k}$ and $n_{k_2/k}$ are subforms of the Pfister quadratic form N_Q, so there exists a bilinear quadratic Pfister form $h = \langle 1, c\rangle$ such that $N_Q \cong h \otimes n_{k_1/k} \cong N_Q = h \otimes n_{k_2/k}$. Thus $[Q] = \chi_1 \cup (c) = \chi_2 \cup (c) \in \mathrm{Br}(k)$ according to the characterization of quaternion algebras by their norm forms.

(3) $\Rightarrow$ (1): We have that $C \cong C(Q, c)$, so we get an embedding

$$(\mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q))/\mu_2 \to \mathrm{Aut}(C(Q,c)) \xrightarrow{\sim} G.$$

By embedding k_1 in Q (resp. k_2 in Q), we get an embedding

$$R^1_{k_1/k}(\mathbb{G}_m) \times R^1_{k_2/k}(\mathbb{G}_m) \to \mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q),$$

so that

$$i : (R^1_{k_1/k}(\mathbb{G}_m) \times R^1_{k_2/k}(\mathbb{G}_m))/\mu_2 \to (\mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q))/\mu_2 \to G$$

is an embedding. By the computations of the proof of Lemma 4.2.1(2), it indeed has type $[(k', l)]$.

(b) Assume that X has a 0-cycle of odd degree; i.e., there are finite field extensions $K_1, \ldots, K_r$ of k such that $X(K_i) \neq \varnothing$ for $i = 1, \ldots, r$ and g.c.d.$([K_1 : K], \ldots, [K_r : K])$ is odd. By (a), it follows that $C_{K_i \otimes_k k_1}$ and $C_{K_i \otimes_k k_2}$ are split for $i = 1, \ldots, r$. Then there exists an index i such that $[K_i : k]$ is odd. If $k_1 = k \times k$, then C splits over K_i; it follows that C is split by Lemma 3.1.1, whence $X(k) \neq \varnothing$ by Theorem 2.3.1. We can then assume that k_1 is a field. Then $K_i \otimes_k k_1$ is a field extension of K_j so that $C_{K_j \otimes_k k_1}$ splits; since $[K_i \otimes_k k_1 : k_1]$ is odd, Lemma 3.1.1 shows then that C_{k_1} is split. Similarly C_{k_2} is split, and by (a), we conclude that $X(k) \neq \varnothing$. ☐

In the following, we consider a special case where k' and l have the same discriminant.

Corollary 4.4.2. *Let k'/k be a quadratic étale algebra and let l be a cubic étale k-algebra of discriminant k'. If C admits a maximal k-torus of type* $[(k', l)]$, *then C splits.*

Proof. First, assume that l is not a field, so that $l \cong k \times k'$. Then Proposition 4.4.1 yields that C is split by the quadratic étale k-algebra k_1 which satisfies $[k_1] = [k'] + [l_2] = 0$, whence C is split.

If l is a field, the octonion l-algebra C_l admits a maximal l-torus of type $[(k' \otimes_k l, l \otimes_k l)]$. Since $l \otimes_k l \xrightarrow{\sim} l \times (l \otimes_k k')$, the first case shows that C_l is split. We conclude that C is split by Lemma 3.1.1. ☐

Remark 4.4.3. Take $k = \mathbb{R}$ and let C be the "anisotropic" Cayley algebra (or we simply call it a Cayley algebra). We consider the case where $(k', l) = (\mathbb{C}, \mathbb{R} \times \mathbb{C})$. By Corollary 4.4.2, there is no $\mathbb{R}$-embedding of a maximal torus of type (k', l). However, $G_{k'}$ splits and this example shows that only the direct implication holds in Proposition 4.3.1(1). The only possible type is then $[(\mathbb{C}, \mathbb{R}^3)]$, which is realized according to Proposition 4.3.1(2).

We can now provide a description of such maximal tori.

Proposition 4.4.4. *Let k_1, k_2 be quadratic étale k-algebras. We consider the couple $(k', l) = (k_2, k \times l_2)$, where $[l_2] = [k_1] + [k_2]$, and we assume that C is split by k_1 and k_2. We put $T = (R^1_{k_1/k}(\mathbb{G}_m) \times R^1_{k_2/k}(\mathbb{G}_m))/\mu_2$ and consider a k-embedding $i : T \to G$ of type $[(k', l)]$. Then there exists a quaternion subalgebra Q of C containing k_1 and k_2 and a Cayley–Dickson decomposition $C \cong C(Q, c)$ such that $i : T \to G \cong \mathrm{Aut}(C(Q, c))$ factorizes by the k-subgroup $(\mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q))/\mu_2$ of $\mathrm{Aut}(C(Q, c))$.*

Proof. Consider the case where $k_1 \otimes_k k_2$ is a field. We denote by $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ the Galois group of the biquadratic field extension $k_1 \otimes_k k_2$. This group acts on the root system $\Phi(G_{k_s}, i(T_{k_s}))$ through a W_0-conjugate of the standard subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of W_0 generated by the central symmetry and the symmetry with the horizontal axis (see the figure in Section 3.3). It follows that Γ stabilizes a subroot system Φ_1 of type $A_1 \times A_1$ of $\Phi(G_{k_s}, T_{k_s})$. By Galois descent, the k_s-subgroup generated by the root subgroups of Φ_1 descends to a k-subgroup H of G which is semisimple of type $A_1 \times A_1$. Lemma 3.4.1 shows that there is a Cayley–Dickson decomposition $C = Q \oplus Q.a$ such that $H = H(Q)$. We have then a factorization of $i : T \to G$ by $H(Q) \xrightarrow{\sim} (\mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q))/\mu_2$.

The other cases (k_1 or k_2 split, $k_1 = k_2$) are simpler, of the same flavor, and left to the reader. $\square$

4.5. ***The cubic field case: a first example.*** Beyond the previous "equal discriminant case", the embedding problem for a given octonion algebra C and a couple (k', l) whenever l is a cubic field is much more complicated. The property to carry a maximal torus of "cubic type" encodes information on the relevant k-group, and we shall first investigate specific examples over Laurent series fields. The next fact is inspired by similar considerations on central simple algebras by Chernousov, Rapinchuk and Rapinchuk [Chernousov et al. 2013, §2].

Let us start with a more general setting. Let G_0 be a semisimple Chevalley group defined over $\mathbb{Z}$, equipped with a maximal split subtorus T_0. Denote by Ψ_0 the root datum attached to (G_0, T_0). Let G'/k be a quasisplit form of G_0 and denote by T' a maximal k-torus of G' which is the centralizer of a maximal k-split torus of G'. We denote by $W' = N_{G'}(T')/T'$ the Weyl group of T'.

Lemma 4.5.1. *Let $K = k((t))$. Let E be a W'-torsor defined over k and put $T = E \wedge^{W'} T'$. Assume that $H^1(k, \widehat{T}^0) = 0$, where $\widehat{T}^0$ is the Galois lattice of cocharacters of T. Let $z : \mathrm{Gal}(K_s/K) \to G'(K_s)$ be a Galois cocycle and put $G = {}_zG'/K$. Assume there is an embedding $i : T_K \to G$ satisfying $\mathbf{type}_{\mathrm{can}}(i, T_K) = [E]_K \in$*

$H^1(K, W')$. *Then* $[z]$ *is "unramified"*; *i.e.*, $[z] \in \mathrm{Im}(H^1(k, G') \to H^1(K, G'))$. *In particular*, *there exists a semisimple k-group H such that $G \cong H \times_k K$.*

Proof. By our form of Steinberg's theorem, Theorem 2.4.1, there is a k-embedding $i' : T_K \to G'_K$ such that the class $[z] \in H^1(K, G'_K)$ belongs to the image of $i'_* : H^1(K, T) \to H^1(K, G'_K)$, and furthermore $\mathbf{type}_{\mathrm{can}}(T_K, i) = \mathbf{type}_{\mathrm{id}}(T_K, i') = [E]_K \in H^1(K, W')$.

On the other hand, we know by Theorem 2.3.1 that there exists a k-embedding $j : T \to G'$ such that $\mathbf{type}_{\mathrm{id}}(T, j) = [E]$. By Proposition 2.5.3, the images of $(i')_*$ and $(j_K)_* : H^1(K, T) \to H^1(K, G')$ coincide. It follows that $[z] \in H^1(K, G')$ belongs to the image of $(j_K)_* : H^1(K, T) \to H^1(K, G')$. We appeal now to the localization sequence $0 \to H^1(k, T) \to H^1(K, T) \to H^1(k, \widehat{T}^0) \to 0$ provided by the Appendix (Lemma A.1). Using our vanishing hypothesis $H^1(k, \widehat{T}^0) = 0$ and the commutative diagram

$$\begin{array}{ccccc} H^1(k,T) & \longrightarrow & H^1(K,T) & \longrightarrow & 0 \\ \downarrow{\scriptstyle j_{*,k}} & & \downarrow{\scriptstyle j_{*,K}} & & \\ H^1(k,G') & \longrightarrow & H^1(K,G') & & \end{array}$$

we conclude that $[z]$ comes from $H^1(k, G')$. $\square$

Since every semisimple K-group of type G_2 is an inner form of its split form, the following corollary follows readily.

Corollary 4.5.2. *Let $K = k((t))$ and let G/K be a semisimple k-group of type G_2. Consider a couple (k', l) such that k'/k is a quadratic étale algebra and l/k is a cubic field separable extension. Denote by E/k the W_0-torsor associated to (k', l) and put $T/k = E \wedge^{W_0} T_0$. If the K-torus $T \times_k K$ admits an embedding i in G such that* $\mathbf{type}_{\mathrm{can}}(T_K, i) = [(k', l)]$, *then there exists a semisimple k-group H of type G_2 such that $G \cong H \times_k K$.*

Proof. We can assume that $G = {}_z(G_0)/K$, where $z : \mathrm{Gal}(K_s/K) \to G(K_s)$ is a Galois cocycle. By Lemma 4.2.3(2), we have $H^1(k, \widehat{T}^0) = 0$. The corollary then follows from Lemma 4.5.1 applied to $G' = G_0/k$ and $T' = T_0$. $\square$

Theorem 4.5.3. *Let Q be a quaternion division algebra over k, k' a quadratic étale subalgebra of Q and l/k a Galois cubic field extension. As before, let $K = k((t))$, $K' = k'((t))$, $L = l((t))$. Let $C/K = C(Q_K, t)$ be the octonion algebra built out from the Cayley–Dickson doubling process.*

Let $\underline{\Psi} = \underline{\Psi}_{(K',L)}$ be as defined in Section 4.2, and let $X = \mathcal{E}(G, \underline{\Psi})$ be the K-variety of embeddings defined in Section 2.6. Then $X(K) = \varnothing$, $X(K') \neq \varnothing$ and $X(L) \neq \varnothing$.

Proof. We have $N_C = N_{Q,K} \otimes \langle 1, t\rangle$. Since N_Q is an anisotropic k-form, the quadratic form N_C is anisotropic and cannot be defined over k according to Springer's decomposition theorem [Elman et al. 2008, §19]. It follows that the k-group $G = \mathrm{Aut}(C)$ cannot be defined over k; Lemma 4.5.1 shows there is no embedding of a k-torus with type $[(K', L)]$, and therefore $X(K) = \varnothing$.

Since K' splits C, $G \times_K K'$ is split so that we have $X(K') \neq \varnothing$ by Theorem 2.3.1. It remains to show that $X(L)$ is not empty. We have $[(K', L)] \otimes_K L \cong [K' \otimes_K L, L^3]$. Since K' splits C, $K' \otimes_K L$ splits C and Proposition 4.3.1(2) yields $X(L) \neq \varnothing$. □

Remarks 4.5.4. (a) The requirements on the field k are mild and are satisfied by any local or global field.

(b) Geometrically speaking, the variety X/K is a homogeneous space under a k-group of type G_2 whose geometric stabilizer is a maximal K-torus. As far as we know, it is the simplest example of homogeneous space under a semisimple simply connected group with a 0-cycle of degree one and no rational points; compare with [Florence 2004], where stabilizers are finite and noncommutative, and [Parimala 2005], where stabilizers are parabolic subgroups.

5. Étale cubic algebras and hermitian forms

Our goal is to further investigate the cubic case by using results of Haile, Knus, Rost and Tignol [Haile et al. 1996] on hermitian 3-forms.

Let C be an octonion algebra over k and put $G = \mathrm{Aut}(C)$. Let $i : T \to G$ be a k-embedding of a rank-2 torus, and we denote by $[(k', l)]$ its type.

We denote by $R_{>0}$ the subset of long roots of the root system $R = \Phi(G_{k_s}, i(T_{k_s}))$. Then $R_>$ is a root system of type A_2 and is Γ_k-stable, and hence defines a twisted datum. We consider the k_s-subgroup of G_{k_s} generated by T_{k_s} and the root groups attached to elements of $R_>$; it is semisimple simply connected of type A_2 and descends to a semisimple k-group $J(T, i)$ of G. Our goal is to study such embeddings (T, i) by means of the subgroup $J(T, i)$.

We shall see in the sequel that such a k-group $J(T, i)$ is a special unitary group for some hermitian 3-form for k'/k.

Remarks 5.0.5. (a) J.-P. Serre explained another way to construct the k-subgroup $J(T, i)$. Define the finite k-group of multiplicative type

$$\mu_{T,k_s} = \mathrm{Ker}\left(T_{k_s} \xrightarrow{\prod \alpha} \prod_{\alpha \in R_>} \mathbb{G}_{m,k_s} \right);$$

it descends to a k-subgroup μ_T of T. We claim that

$$J(T, i) = Z_G(\mu_T).$$

For checking that fact, it is harmless to assume that k is algebraically closed. For simplicity, we put $J = J(T, i)$; it is isomorphic to SL_3. Since $\Phi(J, i(T))) = R_>$, we have that $\mu_T = Z(J)$ [Demazure and Grothendieck 1970c, XIX, 1.10.3]; it follows that $\mu_T \cong \mu_3$ and that $J \subseteq Z_G(\mu_T)$. Since J is a semisimple subgroup of maximal rank of G, Borel and de Siebenthal's theorem provides a k-subgroup μ_n of T such that $J = Z_G(\mu_n)$ [Pépin Le Halleur 2012, Proposition 6.6]. Then $\mu_n \subseteq Z(J) \cong \mu_3$ so that $\mu_n = Z(J) = \mu_T$. Thus $J = Z_G(\mu_T)$.

(b) If k is of characteristic 3, we can associate to T another k-subgroup $J_<(T, i)$ of type A_2. Let $R_<$ be the subset of short roots of the root system $R = \Phi(G_{k_s}, i(T_{k_s}))$. It is a 3-closed symmetric subset [Pépin Le Halleur 2012, Lemma 2.4], so the k_s-subgroup of G_{k_s} generated by T_{k_s} and the root groups attached to elements of $R_<$ define a semisimple k_s-subgroup $J_<$ of G_{k_s} [ibid., Theorem 3.1]; furthermore, we have $\Phi(J, i(T_{k_s})) = R_<$. The k_s-group $J_<$ descends to a semisimple k-group $J_<(T, i)$. It is semisimple of type A_2 and adjoint since $R_<$ spans $\widehat{T}(k_s)$.

5.1. ***Rank-3 hermitian forms and octonions.*** Let k'/k be a quadratic étale algebra. From a construction of Jacobson [1958, §5] (see [Knus et al. 1994, §6] for the generalization to an arbitrary base field), we recall that we can attach to a rank-3 hermitian form (E, h) (for k'/k) with trivial (hermitian) discriminant an octonion k-algebra $C(k', E, h) = k' \oplus E$. Furthermore, the k-group $\mathrm{SU}(k', E, h)$ embeds in $\mathrm{Aut}(C(k', E, h))$ by $g.(x, e) = (x, g.e)$. We denote by $J(k', E, h)$ this k-subgroup and we observe that k' is the k-vector subspace of $C(k', E, h)$ of fixed points for the action of $J(k', E, h)$ on $C(k', E, h)$. Also $J(k', E, h)$ is the k-subgroup of $\mathrm{Aut}(C(k', E, h))$ acting trivially on k'.

In a converse way (see [Knus et al. 1998, Exercise 6(b), page 508]), if we are given an embedding of a unital composition k-algebra $k' \to C$, we denote by E the orthogonal subspace of k' for N_C. For any $x, y \in k'$ and $z \in E$, we have

$$0 = \langle xy, z\rangle_C = \langle y, \sigma_C(x)z\rangle_C$$

by using the identity [Springer and Veldkamp 2000, Lemma 1.3.2], so that the multiplication $C \times C \to C$ induces a bilinear k-map $k' \times E \to E$. Then E has a natural k'-structure and the restriction of N_C to E defines a hermitian form h (of trivial discriminant) such that $C = C(k', E, h)$.

Furthermore, if we have two subfields k'_1, k'_2 of C isomorphic to k', the "Skolem–Noether" property [Knus et al. 1998, 33.21] shows that there exists $g \in G(k)$ mapping k'_1 to k'_2. Hence the hermitian forms (E_1, h_1), (E_2, h_2) are isometric.

Remark 5.1.1. Of course, in such a situation, h can be diagonalized as $\langle -b, -c, bc\rangle$ and we have $n_{C(k',E,h)} = n_{k'/k} \otimes \langle\langle b, c\rangle\rangle$. If we take $\langle -1, -1, 1\rangle$, we get one form of the split octonion algebra C_0 and then a k-subgroup $J_0 = \mathrm{SL}_3$ of $\mathrm{Aut}(C_0)$.

Lemma 5.1.2. *In the above setting, we put $G = \mathrm{Aut}(C(k', E, h))$ and $J = J(k', E, h)$.*

(1) *There is a natural exact sequence of algebraic k-groups $1 \to J \to N_G(J) \to \mathbb{Z}/2\mathbb{Z} \to 1$.*

(2) *The map $N_G(J)(k) \to \mathbb{Z}/2\mathbb{Z}$ is onto and the induced action of $\mathbb{Z}/2\mathbb{Z}$ on k' is the Galois action.*

Proof. (1) We consider the commutative exact diagram of k-groups

$$\begin{tikzcd}
 & & & 1 \arrow[d] & \\
1 \arrow[r] & Z(J) \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & J/Z(J) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z_G(J) \arrow[r] & N_G(J) \arrow[r] & \mathrm{Aut}(J) \arrow[r] \arrow[d] & 1 \\
 & & & \mathrm{Autext}(J) = \mathbb{Z}/2\mathbb{Z} \arrow[d] & \\
 & & & 1 &
\end{tikzcd}$$

Let T be a maximal k-torus of J; it is still maximal in G. Then we have $Z_G(J) \subseteq Z_G(T) = T$, and hence $Z_G(J) \subseteq Z(J)$, so that $Z(J) = Z_G(J)$. The diagram provides then an exact sequence $1 \to J \to N_G(J) \to \mathbb{Z}/2\mathbb{Z}$. We postpone the surjectivity.

(2) Now by the "Skolem–Noether property" [Knus et al. 1998, 33.21], the Galois action $\sigma : k' \to k'$ extends to an element $g \in G(k)$. Given $u \in J(k)$, gug^{-1} is an element of $G(k)$ which acts trivially on k', so it belongs to $J(k)$. Since it holds for any field extension of k, we have that $g \in N_G(J)(k)$. We conclude that the map $N_G(J) \to \mathbb{Z}/2\mathbb{Z}$ is surjective and that the induced action of $\mathbb{Z}/2\mathbb{Z}$ on k' is the Galois action. □

Let C be an octonion algebra, put $G = \mathrm{Aut}(C)$ and let J be a semisimple k-subgroup of type A_2 of G. Then J is of maximal rank and we can appeal again to the Borel and de Siebenthal classification theorem [Pépin Le Halleur 2012, Theorem 3.1]. If the characteristic of k is not 3, then J is geometrically conjugated to the standard SL_3 in G_2 and is then simply connected. If the characteristic k is 3, then J may arise as in Remarks 5.0.5(b) from the short roots associated to a maximal k-torus of J; in that case, J is adjoint. We can make a similar statement to Lemma 3.4.1.

Lemma 5.1.3. *Let J be a semisimple simply connected k-subgroup of type A_2 of $G = \mathrm{Aut}(C)$ and we denote by k'/k the quadratic étale algebra attached to the quasisplit form of J. Then there exists a rank-3 hermitian form (E, h) for k'/k, an isomorphism $C \cong C(k', E, h)$, and an isomorphism $J \xrightarrow{\sim} J(k', E, h)$ such that the* *following diagram commutes*

$$\begin{tikzcd} J \arrow[r, hook] \arrow[d, "\wr"] & G \arrow[d, "\wr"] \\ J(E,h) \arrow[r, hook, "j"] & \mathrm{Aut}(C(k',E,h)). \end{tikzcd}$$

Proof. Given a k-maximal torus T of G, we consider the root system $\Psi(G_{k_s}, T_{k_s})$. There are exactly 6 long roots in $\Psi(G_{k_s}, T_{k_s})$ which form an A_2-subsystem of $\Psi(G_{k_s}, T_{k_s})$. Let H be the subgroup of G_{k_s} which is generated by T_{k_s} and the root groups of long roots. Since the Galois action preserves the length of a root, the group H is defined over k. Hence given a k-maximal torus T, there is exactly one subgroup H of G which is a twisted form of SL_3 and contains T. Since all maximal k-split tori are conjugated over k, the split group G_0 of type G_2 has one single conjugacy $G_0(k)$-class of k-subgroups isomorphic to SL_3. It follows that the couple (G, J) is isomorphic over k_s to the couple (G_0, J_0). In particular, by Galois descent, the subspace of fixed points of J on C is an étale subalgebra l of rank 2 which is a unital composition subalgebra of C. We define then the orthogonal subspace E of l in C. Then E has a natural structure of an l-vector space and carries a hermitian form h of trivial (hermitian) discriminant such that $C(l, E, h) = C$ (see [Knus et al. 1998, Exercise 6(b), page 508]). But J acts trivially on l, so that $J \subseteq J(l, E, h)$. For dimension reasons, we conclude that $J = J(l, E, h)$. Then l/k is the discriminant étale algebra of J, and hence $k' = l$. $\square$

Remark 5.1.4. Note that in the above proof, we didn't put any assumption on the characteristic of k. However, in characteristic $\neq 2, 3$, Hooda [2014, Theorem 4.4] proved the above lemma in a quite different way.

5.2. *Embedding maximal tori.* From now on, we assume for simplicity that the characteristic exponent of k is not 2.

Lemma 5.2.1. *Let $G = \mathrm{Aut}(C)$ be a semisimple k-group of type G_2. Let k' (resp. l) be a quadratic (resp. cubic) étale algebra of k. Let $i : T \to G$ be a k-embedding of a maximal k-torus such that* $\mathbf{type}(T, i) = [(k', l)]$ *and denote by $J(T, i)$ the associated k-subgroup of G.*

(1) *The discriminant algebra of $J(T, i)$ is k'/k.*

(2) *By Lemma 5.1.3, we can write $C = C(k', E, h)$ and identify $J(T, i)$ with $J(k', E, h)$. Then there is a k'-embedding $f : k' \otimes_k l \to M_3(k')$ such that $f \circ (\sigma \otimes \mathrm{id}) = \tau_h \circ f$ on $k' \otimes_k l$, where τ_h is the involution on $M_3(k')$ induced by h.*

Proof. (1) We put $J = J(T, i)$. We consider the Galois action on the root system $\Psi(G_{k_s}, i(T)_{k_s})$ and its subroot system $\Psi(J_{k_s}, i(T)_{k_s}) = \Psi(G_{k_s}, i(T)_{k_s})_{>}$. It is given by a map $f : \Gamma_k \to \mathbb{Z}/2\mathbb{Z} \times S_3$ defining $[(k', l)]$. Since the Weyl group of $\Psi(J_{k_s}, i(T)_{k_s})$ is S_3, it follows that the $\star$-action of Γ_k on the Dynkin diagram A_2 is the projection $\Gamma_k \to \mathbb{Z}/2\mathbb{Z}$. Therefore the discriminant algebra of $J(T, i)$ is k'/k.

(2) We have then a k-embedding $i : T \to J = \mathrm{SU}(k', E, h)$. Its type (absolute with respect to J) is $[(k', l)] \in H^1(k, \mathbb{Z}/2\mathbb{Z} \times S_3)$. By [Lee 2014, Theorem 1.15(2)], there is a k'-embedding $k' \otimes_k l \to M_3(k')$ with respect to the conjugacy involution $\sigma \otimes \mathrm{id}$ on $k' \otimes_k l$ and the involution τ_h attached to h. □

Proposition 5.2.2. *Let $G = \mathrm{Aut}(C)$ be a semisimple k-group of type G_2. Let k' (resp. l) be a quadratic (resp. cubic) étale k-algebra. We denote by X the variety of k-embeddings of maximal tori in G attached to the twist of Ψ_0 by (k', l) (seen as a W_0-torsor). The following are equivalent*:

(i) $X(k) \neq \varnothing$; *that is, there exists an embedding $i : T \to G$ of a maximal k-torus of type $[(k', l)]$.*

(ii) *There exists a rank-3 hermitian form (E, h) for k'/k of trivial (hermitian) discriminant such that $C \cong C(k', E, h)$ and such that there exists a k'-embedding of $k' \otimes_k l \to \mathrm{End}_{k'}(E)$ with respect to the conjugacy involution on k' and the involution τ_h attached to h.*

(iii) *There exists a rank-3 hermitian form (E, h) for k'/k of trivial (hermitian) discriminant such that $C \cong C(k', E, h)$ and an element $\lambda \in l^\times$ such that $(l \otimes_k k', \mathrm{t}'_\lambda) \simeq (E, h)$, where $\mathrm{t}'_\lambda(x, y) = \mathrm{tr}_{l \otimes k'/k'}(\lambda x \sigma(y))$.*

Proof. The implication (i) ⇒ (ii) follows from Lemma 5.2.1(2). Conversely, we assume (ii). Then $G \cong \mathrm{Aut}(C(k', E, h))$ admits the k-subgroup $J(k', E, h) \xrightarrow{\sim} \mathrm{SU}(k', E, k)$. By [Lee 2014, Theorem 1.15(2)], there is a k-embedding $i : T \to \mathrm{SU}(k', E, k)$ of a maximal torus whose absolute type (with respect to J) is $[(k', l)]$. The k-embedding $i : T \to \mathrm{SU}(k', E, k) \to G$ also has absolute type $[(k', l)]$.

The equivalence (ii) ⟺ (iii) follows from the embedding criterion of $k' \otimes_k l \to \mathrm{End}_{k'}(E)$ given by [Bayer-Fluckiger et al. 2015, Proposition 1.3.1]. □

Let k', l be as in Proposition 5.2.2. Let $\delta \in k^\times/k^{\times 2}$ be the discriminant of l and $d \in k^\times/k^{\times 2}$ be the discriminant of k'. Let B be a central simple algebra over k' with an involution σ of the second kind. Let Trd be the reduced trace on B. Let $(B, \sigma)_+$ be the k-vector space of σ-symmetric elements of B. Let Q_σ be the quadratic form on $(B, \sigma)_+$ defined by

$$Q_\sigma(x, y) = \mathrm{Trd}(xy).$$

Let us recall some results in [Haile et al. 1996].

Lemma 5.2.3. *Assume that k is not of characteristic 2. Let B be a central simple K-algebra of odd degree $n = 2m - 1$ with involution σ of the second kind. There* *is a quadratic form q_σ of dimension $n(n-1)/2$ and trivial discriminant over k such that*

$$Q_\sigma \simeq \langle 1\rangle \perp \langle 2\rangle \cdot \langle\langle \alpha\rangle\rangle \otimes q_\sigma.$$

Proof. We refer to [ibid., Proposition 4]. □

Theorem 5.2.4. *Assume that k is not of characteristic 2 or 3. Let σ, τ be involutions of the second kind on a central simple algebra B of degree 3. Then σ and τ are isomorphic if and only if Q_σ and Q_τ are isometric.*

Proof. We refer to [ibid., Theorem 15]. □

Let (B, σ) be as in Lemma 5.2.3 with degree $B = 3$ and assume that 6 is invertible in k. Let b_0, $c_0 \in k^\times$ such that $q_\sigma \simeq \langle -b_0, -c_0, b_0c_0\rangle$. Define $\pi(B, \sigma)$ to be the Pfister form $\langle\langle d, b_0, c_0\rangle\rangle$. An involution σ of the second kind is called *distinguished* if $\pi(B, \sigma)$ is hyperbolic. Let (E, h) be a rank-3 hermitian form over k' with trivial (hermitian) discriminant. We can find b, $c \in k^\times$ such that $h \simeq \langle -b, -c, bc\rangle_{k'}$.

Now consider the special case where $(B, \sigma) = (\mathrm{End}_{k'}(E), \tau_h)$. Then we have $q_{\tau_h} = \langle -b, -c, bc\rangle$ and $\pi(\mathrm{End}_{k'}(E), \tau_h) = \langle\langle d, b, c\rangle\rangle$, which is the norm form of the octonion $C(k', E, h)$. It is then possible to recover with that method at least the two following facts.

Remarks 5.2.5. (a) Theorem 2.3.1 for G_2, i.e., all possible types of tori occur in the split case: Given a couple (k', l), we can write the split octonion algebra C as $C(k', E, h)$ for $E = (k')^3$ $h = \langle -1, -1, 1\rangle$. First we note that l can be embedded into $\mathrm{End}_{k'}(E)$ since $\mathrm{End}_{k'}(E)$ is split. As N_C is isotropic, we have that τ_h is distinguished. By [Haile et al. 1996, Corollary 18], every cubic étale algebra l can be embedded as a subalgebra in $\mathrm{End}_{k'}(E)$ with its image in $(\mathrm{End}_{k'}(E), \tau_h)_+$. By Proposition 5.2.2(2), there is an embedding $i : T \to G$ of type $[(k', l)] \in H^1(k, W_0)$.

(b) Corollary 4.4.2 for the "equal discriminant case", i.e., the discriminant algebra of l is k': In this case, there is an embedding $i : T \to G$ of type $[(k', l)]$ if and only if N_C is isotropic. For a proof in the present setting, we assume there is an embedding $i : T \to G$ of type $[(k', l)]$. According to Proposition 5.2.2(2), there exists a 3-hermitian form (E, h) of trivial determinant such that $C \cong C(k', E, h)$ and an embedding $l \otimes_k k' \to \mathrm{End}_{k'}(E)$ with respect to the conjugacy involution on k' and the involution τ_h attached to h. Then $(\mathrm{End}_{k'}(E), \tau_h)_+$ contains a cubic étale algebra isomorphic to l whose discriminant is d. By [ibid., Theorem 16(e)], we have $\pi(\mathrm{End}_{k'}(E), \tau_h) = N_C$ is isotropic. Thus C is split.

Proposition 5.2.6. *Assume that k is not of characteristic 2, 3. Let $G = \mathrm{Aut}(C)$ be a semisimple k-group of type G_2. Let k' (resp. l) be a quadratic (resp. cubic) étale k-algebra. Then there is a k-embedding $i : T \to G$ of type $[(k', l)] \in H^1(k, W_0)$ if and only if the following two conditions both hold*:

(i) *There is a rank-3 k'/k-hermitian form (E, h) of trivial (hermitian) discriminant such that $C \simeq C(k', E, h)$.*

(ii) *Let $b, c \in k^\times$ such that $\langle -b, -c, bc\rangle_{k'}$ is isometric to the form h in* (i). *Then there is $\lambda \in l^\times$ such that $N_{l/k}(\lambda) \in k^{\times 2}$ and the k-quadratic form $\langle\langle d\rangle\rangle \otimes \langle\delta\rangle \cdot t_{l/k}(\langle\lambda\rangle)$ is isometric to $\langle\langle d\rangle\rangle \otimes \langle -b, -c, bc\rangle$, where $t_{l/k}$ denotes the Scharlau transfer with respect to the trace map* $\mathrm{tr} : l \to k$.

Proof. Suppose that there is a k-embedding $i : T \to G$ of type $[(k', l)] \in H^1(k, W_0)$. By Proposition 5.2.2(2), there is a rank-3 (k'/k)-hermitian form (E, h) such that $C \simeq C(k', E, h)$, and there exists an embedding $\iota : k' \otimes_k l \to \mathrm{End}_{k'}(E)$ with respect to the conjugacy involution on k' and the involution τ_h attached to h. By [Haile et al. 1996, Corollary 12], we can find $\lambda \in l^\times$ such that $N_{l/k}(\lambda) \in k^{\times 2}$ and the q_{τ_h} in Lemma 5.2.3 is the k-quadratic form $\langle\delta\rangle \cdot t_{l/k}(\langle\lambda\rangle)$. Since

$$Q_{\tau_h} = 3\langle 1\rangle \perp \langle 2\rangle \cdot \langle\langle d\rangle\rangle \otimes \langle -b, -c, bc\rangle,$$

condition (ii) follows from the Witt cancellation.

Conversely, suppose that (i) and (ii) hold. By Proposition 5.2.2(2), it suffices to prove that there is a k-embedding of l into $(M_3(k'), \tau_h)_+$. Note that every cubic étale k-algebra l can be embedded into $M_3(k')$ as a k-algebra. By [ibid., Corollary 14], for every $\lambda \in l^\times$ such that $N_{l/k}(\lambda) \in k^{\times 2}$, there is an involution σ of the second kind on $M_3(k')$ leaving l elementwise invariant such that

$$Q_\sigma = \langle 1, 1, 1\rangle \perp \langle 2\rangle \cdot \langle\langle d\rangle\rangle \otimes \langle\delta\rangle \cdot t_{l/k}(\langle\lambda\rangle).$$

Condition (ii) implies that we can choose λ so that Q_σ and Q_{τ_h} are isometric. By Theorem 5.2.4, the involutions σ and τ_h are isomorphic, and hence there is a k-embedding of l into $(M_3(k'), \tau_h)_+$. □

6. Hasse principle

We assume that the base field k is a number field.

Proposition 6.1. *Let (k', l) be a couple where k' is a quadratic étale k-algebra and l/k is a cubic étale k-algebra. Let G be a semisimple k-group of type G_2 and let X be the G-homogeneous space of the embeddings of maximal tori with respect to the type $[(k', l)]$. Then X satisfies the Hasse principle.*

Proof. Since G_0 is simply connected, we have $H^1(k_v, G_0) = 1$ for each finite place v of k. The Hasse principle states that the map

$$H^1(k, G_0) \xrightarrow{\sim} \prod_{v \text{ real place}} H^1(k_v, G_0)$$

is bijective. If G is split, $X(k)$ is not empty (Theorem 2.3.1), so we may assume that G is not split. By [Lee 2014, Proposition 2.8], $X(k)$ is not empty if and only if the

Borovoi obstruction $\gamma \in Ш^2(k, T^{(k',l)})$ vanishes. There is a real place v such that G_{k_v} is not split and then is k_v-anisotropic. Since there is a k_v-embedding of $T^{(k',l)}$ in G_{k_v}, the torus $T^{(k',l)}$ is k_v-anisotropic. By a lemma due to Kneser [Sansuc 1981, lemme 1.9.3], we know that $Ш^2(k, T^{(k',l)}) = 0$, so that $\gamma = 0$. Thus $X(k) \neq \varnothing$. □

Remark 6.2. Under the hypothesis of Proposition 6.1, the existence of a k-point on X is controlled by the Borovoi obstruction. It follows from the restriction-corestriction principle in Galois cohomology that X has a k-point if and only if X has a 0-cycle of degree one. In other words, examples like those in Theorem 4.5.3 do not occur over number fields.

Corollary 6.3. *Let k be a number field and k' (resp. l) be quadratic (resp. cubic) étale algebra over k. Let $\delta \in k^\times/k^{\times 2}$ be the discriminant of l and $d \in k^\times/k^{\times 2}$ be the discriminant of k'. Let Σ be the set of (real) places where G is not split. Then $T^{(k',l)}$ can be embedded in G with respect to the type $[(k',l)]$ if and only if $d = -1 \in k_v^\times/k_v^{\times 2}$ and $\delta = 1 \in k_v^\times/k_v^{\times 2}$ for each $v \in \Sigma$.*

Proof. According to Proposition 6.1, $T^{(k',l)}$ can be embedded in G with respect to the type $[(k',l)]$ if and only if this holds everywhere locally or equivalently (by Theorem 2.3.1) if and only if this holds locally on Σ. The problem boils down to the real anisotropic case where the only type is $[(\mathbb{C}, \mathbb{R}^3)]$, according to Remark 4.4.3. □

Examples 6.4. Keep the notations in Corollary 6.3.

(a) Consider the special case where k is the field of rational numbers $\mathbb{Q}$. Suppose that G is anisotropic over $\mathbb{Q}$. Since there is only one real place of $\mathbb{Q}$, by Corollary 6.3, the torus $T^{(k',l)}$ can be embedded in G with respect to type $[(k',l)]$ if and only if k' is imaginary and the discriminant of l is positive.

(b) Let k be a number field. Suppose that G is anisotropic. Note that in this case, k is a real extension over $\mathbb{Q}$. Let k' be an imaginary field extension of k and let the discriminant of l be $[a] \in k^\times/k^{\times 2}$ for some positive $a \in \mathbb{Q}$. Then by Corollary 6.3, the torus $T^{(k',l)}$ can always be embedded in G with respect to type $[(k',l)]$.

Appendix: Galois cohomology of tori and semisimple groups over Laurent series fields

This appendix first provides a reference for a well-known fact on the Galois cohomology of tori in the vein of the short exact sequence computing the tame Brauer group of a Laurent series field. This fact is used in the proof of Lemma 4.5.1. Secondly we apply our version of Steinberg's theorem to Bruhat–Tits theory, answering a question of A. Merkurjev.

We recall that an affine algebraic k-group G is a k-torus if there exists a finite Galois extension k'/k such that $G \times_k k' \xrightarrow{\sim} (\mathbb{G}_{m,k'})^r$. If T is a k-torus, we consider its Galois lattice of characters $\widehat{T} = \mathrm{Hom}_{k_s-gp}(T_{k_s}, \mathbb{G}_{m,k_s})$ and its Galois lattice of cocharacters $\widehat{T}^0 = \mathrm{Hom}_{k_s-gp}(\mathbb{G}_{m,k_s}, T_{k_s})$.

Lemma A.1. *We put $K = k((t))$. Let T/k be an algebraic k-torus. Then we have a natural split exact sequence*

$$0 \to H^1(k,T) \to H^1(K,T) \xrightarrow{\partial} H^1(k,\widehat{T}^0) \to 0.$$

Proof. Let k' be a Galois extension which splits T. We put $\Gamma = \mathrm{Gal}(k'/k)$ and $K' = k'((t))$. We have the exact sequence [Serre 1994, I.2.6(b)]

$$0 \to H^1(\Gamma, T(k')) \to H^1(k,T) \to H^1(k',T).$$

Since $T_{k'}$ is split, Hilbert's theorem 90 shows that $H^1(k',T)=0$, whence there is an isomorphism $H^1(\Gamma, T(k')) \xrightarrow{\sim} H^1(k,T)$. Similarly, we have $H^1(\Gamma, T(K')) \xrightarrow{\sim} H^1(K,T)$. We consider the ($\Gamma$-split) exact sequence

$$1 \to (k'[[t]])^\times \to (K')^\times \to \mathbb{Z} \to 0$$

induced by the valuation. Tensoring with $\widehat{T}^0$, we get a Γ-split exact sequence

$$1 \to T(k'[[t]]) \to T(K') \to \widehat{T}^0 \to 1.$$

It gives rise to a split exact sequence

$$0 \to H^1\big(\Gamma, T(k'[[t]])\big) \to H^1(\Gamma, T(K')) \to H^1(\Gamma, \widehat{T}^0) \to 0.$$

Now we use the filtration argument of [Gille and Szamuely 2006, 6.3.1] by putting

$$U^j = \{x \in k'[[t]]^\times \mid v_t(x-1) \geq j\}$$

for each $j \geq 0$. The $V^j = \widehat{T}^0 \otimes U^j$ filter $T(k'[[t]])$ and each $V^j/V^{j+1} \cong \widehat{T}^0 \otimes_k k'$ is a k'-vector space equipped with a semilinear action, and hence is Γ-acyclic.[1] According to the limit fact [Gille and Szamuely 2006, 6.3.2], we conclude that the specialization map $H^1\big(\Gamma, T(k'[[t]])\big) \to H^1(\Gamma, T(k'))$ is an isomorphism. We have then a split exact sequence

$$0 \to H^1(\Gamma, T(k')) \to H^1(\Gamma, T(K')) \to H^1(\Gamma, \widehat{T}^0) \to 0.$$

Since $H^1(k', \widehat{T}^0) = 0$, we have $H^1(\Gamma, \widehat{T}^0) \xrightarrow{\sim} H^1(k, \widehat{T}^0)$, whence the desired exact sequence. □

Now we relate Bruhat–Tits theory and our version of Steinberg's Theorem 2.4.1. Let G' be a quasisplit semisimple k-group equipped with a maximal k-split subtorus S'. We denote by W' the Weyl group of the maximal torus $T' = C_{G'}(T')$ of G'. Put $K = k((t))$ and denote by K_{nr} the maximal unramified closure of K.

[1] Speiser's lemma shows that $V^j/V^{j+1} = E_j \otimes_k k'$ for a k-vector space E_j on which Γ acts trivially.

Proposition A.2. *Let E be a G'_K-torsor. Then the following are equivalent*:

(i) $E(K_{nr}) \neq \varnothing$.

(ii) *There exists a k-torus embedding $i_0 : T_0 \to G'$ such that $[E]$ belongs to the image of $i_{0,*} : H^1(K, T_0) \to H^1(K, G)$.*

Proof. We denote by $G/K = {}^E G'$ the inner twist of G'_K by E.

(i) $\Rightarrow$ (ii): Then G is split by the extension K_{nr}/K and the technical condition (DE) of Bruhat–Tits theory is satisfied [Bruhat and Tits 1984, Proposition 5.1.6]. It follows that G admits a maximal K-torus $j : T \to G$ which is split over K_{nr} [ibid., Corollary 5.1.2].

In particular, there exists a k-torus T_0 such that $T = T_{0,K}$. We consider now the oriented type $\gamma = \mathbf{type}_{\mathrm{can}}(T, j) \in H^1(K, W')$ provided by the action of the absolute Galois group of K on the root system $\Phi(G_{K_s}, j(T)_{K_s})$. Since T and G are split by K_{nr}, it is given by the action of $\mathrm{Gal}(K_{nr}/K) \cong \mathrm{Gal}(k_s/k)$ on the root system $\Phi(G_{K_{nr}}, j(T)_{K_{nr}})$ and then defines a constant class $\gamma_0 \in H^1(k, W')$ such that $\gamma = (\gamma_0)_K$.

In the other hand, by the Kottwitz embedding (Theorem 2.3.1), there exists a k-embedding $i_0 : T_0 \to G'$ of oriented type γ_0. By Theorem 2.4.1, we conclude that $[E]$ belongs to the image of $i_{0,*} : H^1(K, T_0) \to H^1(K, G')$.

(ii) $\Rightarrow$ (i): We assume there is a k-embedding $i_0 : T_0 \to G'$ such that $[E]$ belongs to the image of $i_{0,*} : H^1(K, T_0) \to H^1(K, G')$. Since $T_{0,K}$ is split by K_{nr}, the Hilbert theorem 90 shows that $H^1(K_{nr}, T_0) = 0$, whence $E(K_{nr}) \neq \varnothing$. □

Remarks A.3. (a) If k is perfect, we have that $\mathrm{cd}(K_{nr}) = 1$ (by Lang, see [Gille and Szamuely 2006, Theorem 6.2.11]) so condition (i) is always satisfied according to Steinberg's theorem.

(b) If k is not perfect, there exist examples when condition (i) is not satisfied, even in the semisimple split simply connected case; see [Gille 2002, Proposition 3 and Theorem 1].

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ON EXTENSIONS OF ALGEBRAIC GROUPS WITH FINITE QUOTIENT

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Consider an exact sequence of group schemes of finite type over a field k,

$$\mathbf{1} \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow \mathbf{1},$$

where Q is finite. We show that Q lifts to a finite subgroup scheme F of G; if Q is étale and k is perfect, then F may be chosen étale as well. As applications, we obtain generalizations of classical results of Arima, Chevalley, and Rosenlicht to possibly nonconnected algebraic groups. We also show that every homogeneous space under such a group has a projective equivariant compactification.

1. Introduction

Consider an extension of algebraic groups, that is, an exact sequence of group schemes of finite type over a field,

$$1 \longrightarrow N \longrightarrow G \xrightarrow{f} Q \longrightarrow 1. \tag{1}$$

Such an extension is generally not split, i.e., f admits no section which is a morphism of group schemes. In this note, we obtain the existence of a splitting in a weaker sense, for extensions with finite quotient group:

Theorem 1.1. *Let G be an algebraic group over a field k, and N a normal subgroup of G with G/N finite. Then there exists a finite subgroup F of G such that $G = N \cdot F$.*

Here $N \cdot F$ denotes, as in [SGA 3_{I} 1970, VIA.5.3.3], the quotient of the semidirect product $N \rtimes F$ by $N \cap F$ embedded as a normal subgroup via $x \mapsto (x, x^{-1})$. If G/N is étale and k is perfect, then the subgroup F may be chosen étale as well. But this fails over any imperfect field k, see Remark 3.3 for details.

In the case where G is smooth and k is perfect, Theorem 1.1 was known to Borel and Serre, and they presented a proof over an algebraically closed field of characteristic 0 (see [Borel and Serre 1964, Lemma 5.11 and footnote on p. 152]). That result was also obtained by Platonov [1966, Lemma 4.14] for smooth linear algebraic groups over perfect fields. In the latter setting, an effective version of

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Theorem 1.1 has been obtained recently by Lucchini Arteche [2015a, Theorem 1.1]; see [Lucchini Arteche 2015b, Proposition 1.1; Chernousov et al. 2008, p. 473; Lötscher et al. 2013, Lemma 5.3] for earlier results in this direction.

Returning to an extension (1) with an arbitrary quotient Q, one may ask whether there exists a subgroup H of G such that $G = N \cdot H$ and $N \cap H$ is finite (when Q is finite, the latter condition is equivalent to the finiteness of H). We then say that (1) is *quasisplit*, and H is a *quasicomplement* of N in G, with *defect group* $N \cap H$.

When Q is smooth and N is an abelian variety, every extension (1) is quasisplit (as shown by Rosenlicht [1956, Theorem 14]; see [Milne 2013, Section 2] for a modern proof). The same holds when Q is reductive (i.e., Q is smooth and affine, and the radical of $Q_{\bar{k}}$ is a torus), N is arbitrary and $\mathrm{char}(k) = 0$, as we will show in Corollary 4.8. On the other hand, the group G of unipotent 3×3 matrices sits in a central extension

$$1 \longrightarrow \mathbb{G}_a \longrightarrow G \longrightarrow \mathbb{G}_a^2 \longrightarrow 1,$$

which is not quasisplit. It would be interesting to determine which classes of groups N, Q yield quasisplit extensions. Another natural problem is to bound the defect group in terms of N and Q. The proof of Theorem 1.1 yields some information in that direction; see Remark 3.4, and [Lucchini Arteche 2015a] for an alternative approach via nonabelian Galois cohomology.

This article is organized as follows. In Section 2, we begin the proof of Theorem 1.1 with a succession of reductions to the case where $Q = G/N$ is étale and N is a smooth connected unipotent group, a torus, or an abelian variety. In Section 3, we show that every class of extensions (1) is torsion in that setting (Lemma 3.1); this quickly implies Theorem 1.1. Section 4 presents some applications of Theorem 1.1 to the structure of algebraic groups: we obtain analogues of classical results of Chevalley, Rosenlicht and Arima on smooth connected algebraic groups (see [Rosenlicht 1956; 1961; Arima 1960]) and of Mostow [1956] on linear algebraic groups in characteristic 0. Finally, we show that every homogeneous space under an algebraic group admits a projective equivariant compactification; this result seems to have been unrecorded so far. It is well known that any such homogeneous space is quasiprojective (see [Raynaud 1970, Corollary VI.2.6]); also, the existence of equivariant compactifications of certain homogeneous spaces having no separable point at infinity has attracted recent interest (see, e.g., [Gabber 2012; Gabber et al. 2014]).

2. Proof of Theorem 1.1: some reductions

We first fix notation and conventions which will be used throughout this article. We consider schemes and their morphisms over a field k, and choose an algebraic closure $\bar{k}$. Given a scheme X and an extension K/k of fields, we denote by X_K the K-scheme obtained from X by the base change $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$.

We use mostly [SGA 3$_{\mathrm{I}}$ 1970; SGA 3$_{\mathrm{II}}$ 1970; SGA 3$_{\mathrm{III}}$ 1970], and occasionally [Demazure and Gabriel 1970], as references for group schemes. Given such a group scheme G, we denote by $e_G \in G(k)$ the neutral element, and by G^0 the neutral component of G, with quotient map $\pi : G \to G/G^0 = \pi_0(G)$. The group law of G is denoted by $\mu : G \times G \to G$, $(x, y) \mapsto xy$.

Throughout this section, we consider an extension (1) and a subgroup F of G. Then the map

$$\nu : N \rtimes F \longrightarrow G, \quad (x, y) \longmapsto xy$$

is a morphism of group schemes with kernel $N \cap F$, embedded in $N \rtimes F$ via $x \mapsto (x, x^{-1})$. Thus, ν factors through a morphism of group schemes

$$\iota : N \cdot F \longrightarrow G.$$

Also, the composition $F \to G \to G/N$ factors through a morphism of group schemes

$$i : F/(N \cap F) \longrightarrow G/N.$$

By [SGA 3$_{\mathrm{I}}$ 1970, VIA.5.4], ι and i are closed immersions of group schemes.

Lemma 2.1. *The following conditions are equivalent*:

(i) *ι is an isomorphism.*

(ii) *i is an isomorphism.*

(iii) *ν is faithfully flat.*

(iv) *For any scheme S and $g \in G(S)$, there exists a faithfully flat morphism of finite presentation $f : S' \to S$ and $x \in N(S')$, $y \in F(S')$ such that $g = xy$ in $G(S')$.*

When G/N is smooth, these conditions are equivalent to:

(v) $G(\bar{k}) = N(\bar{k})F(\bar{k})$.

Proof. (i) $\Leftrightarrow$ (ii): Recall from [SGA 3$_{\mathrm{I}}$ 1970, VIA.5.5.3] that i factors through an isomorphism $F/(N \cap F) \to (N \cdot F)/N$. Thus, we obtain a commutative diagram

$$\begin{array}{ccc} N \cdot F & \xrightarrow{\iota} & G \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle f}\downarrow \\ F/(N \cap F) & \xrightarrow{i} & G/N \end{array}$$

where both vertical arrows are N-torsors for the action of N by right multiplication. As a consequence, this diagram is cartesian. In particular, i is an isomorphism if and only if so is ι; this yields the desired equivalence.

(i) $\Rightarrow$ (iii): Since ν is identified with the quotient map of $N \rtimes F$ by $N \cap F$, the assertion follows from [SGA 3$_{\text{I}}$ 1970, VIA.3.2].

(iii) $\Rightarrow$ (iv): This follows by forming the cartesian square

$$\begin{array}{ccc} S' & \longrightarrow & S \\ \big\downarrow & & \big\downarrow g \\ N \rtimes F & \xrightarrow{\nu} & G \end{array}$$

and observing that ν is of finite presentation, since the schemes G, N and F are of finite type.

(iv) $\Rightarrow$ (i): By our assumption applied to the identity map $G \to G$, there exists a scheme S' and morphisms $x : S' \to N$, $y : S' \to F$ such that the morphism $\nu \circ (x \times y) : N \rtimes F \to G$ is faithfully flat of finite presentation. As a consequence, the morphism of structure sheaves $\mathbb{O}_G \to \nu_*(x \times y)_*(\mathbb{O}_{S'})$ is injective. Thus, so are $\mathbb{O}_G \to \nu_*(\mathbb{O}_{N \rtimes F})$, and hence $\mathbb{O}_G \to i_*(\mathbb{O}_{N \cdot F})$. Since i is a closed immersion, it must be an isomorphism.

(ii) $\Leftrightarrow$ (v): When G/N is smooth, i is an isomorphism if and only if it is surjective on $\bar{k}$-rational points. Since $(G/N)(\bar{k}) = G(\bar{k})/N(\bar{k})$ and likewise for $F/(N \cap F)$, this yields the desired equivalence. $\square$

We assume from now on that the quotient group Q in the extension (1) is *finite*.

Lemma 2.2. *If the exact sequence* $1 \to H^0 \to H \to \pi_0(H) \to 1$ *is quasisplit for any smooth algebraic group* H *such that* $\dim(H) = \dim(G)$, *then* (1) *is quasisplit as well.*

Proof. Consider first the case where G is smooth. Then Q is étale, and hence N contains G^0. By our assumption, there exists a finite subgroup $F \subset G$ such that $G = G^0 \cdot F$. In view of Lemma 2.1 (iv), it follows that $G = N \cdot F$.

If $\operatorname{char}(k) = 0$, then the proof is completed as every algebraic group is smooth (see, e.g., [SGA 3$_{\text{I}}$ 1970, VIB.1.6.1]). So we may assume that $\operatorname{char}(k) = p > 0$. Consider the n-fold relative Frobenius morphism

$$F_G^n : G \longrightarrow G^{(p^n)}$$

and its kernel G_n. Then F_G^n is finite and bijective, so that G_n is an infinitesimal normal subgroup of G. Moreover, the quotient G/G_n is smooth for $n \gg 0$ (see [SGA 3$_{\text{I}}$ 1970, VIIA.8.3]). We may thus choose n so that G/G_n and N/N_n are smooth. The composition $N \to G \to G/G_n$ factors through a closed immersion of group schemes $N/N_n \to G/G_n$ by [SGA 3$_{\text{I}}$ 1970, VIA.5.4] again. Moreover, the image of N/N_n is a normal subgroup of G/G_n, as follows, e.g., from our smoothness assumption and [SGA 3_{I} 1970, VIB.7.3]. This yields an exact sequence

$$1 \longrightarrow N/N_n \longrightarrow G/G_n \longrightarrow Q' \longrightarrow 1,$$

where Q' is a quotient of Q and hence is finite; moreover, $\dim(G/G_n) = \dim(G)$. By our assumption and the first step, there exists a finite subgroup F' of G/G_n such that $G/G_n = (N/N_n) \cdot F'$. In view of [SGA 3_{I} 1970, VIA.5.3.1], there exists a unique subgroup F of G containing G_n such that $F/G_n = F'$; then F is finite as well.

We check that $G = N \cdot F$ by using Lemma 2.1 (iv) again. Let S be a scheme, and $g \in G(S)$. Then there exists a faithfully flat morphism of finite presentation $S' \to S$ and $x' \in (N/N_n)(S')$, $y' \in F'(S')$ such that $F^n_G(g) = x'y'$ in $(G/G_n)(S')$. Moreover, there exists a faithfully flat morphism of finite presentation $S'' \to S'$ and $x'' \in N(S'')$, $y'' \in F(S'')$ such that $F^n_G(x'') = x'$ and $F^n_G(y'') = y'$. Then $y''^{-1}x''^{-1}g \in G_n(S'')$, and hence $g \in N(S'')F(S'')$, since F contains G_n. □

Remark 2.3. With the notation of the proof of Lemma 2.2, there is an exact sequence of quasicomplements

$$1 \longrightarrow G_n \longrightarrow F \longrightarrow F' \longrightarrow 1.$$

When $N = G^0$, so that $G_n \subset N$, we also have an exact sequence of defect groups

$$1 \longrightarrow G_n \longrightarrow N \cap F \longrightarrow (N/G_n) \cap F' \longrightarrow 1.$$

By Lemma 2.2, it suffices to prove Theorem 1.1 when *G is smooth and* $N = G^0$, so that $Q = \pi_0(G)$. We may thus choose a maximal torus T of G (see [SGA 3_{II} 1970, XIV.1.1]). Then the normalizer $N_G(T)$ and the centralizer $Z_G(T)$ are (represented by) subgroups of G (see [SGA 3_{I} 1970, VIB.6.2.5]). Moreover, $N_G(T)$ is smooth by [SGA 3_{II} 1970, XI.2.4]. We now gather further properties of $N_G(T)$:

Lemma 2.4. (i) $G = G^0 \cdot N_G(T)$.

(ii) $N_G(T)^0 = Z_{G^0}(T)$.

(iii) *We have an exact sequence* $1 \to W(G^0, T) \to \pi_0(N_G(T)) \to \pi_0(G) \to 1$, *where* $W(G^0, T) := N_{G^0}(T)/Z_{G^0}(T) = \pi_0(N_{G^0}(T))$ *denotes the Weyl group.*

Proof. (i) By Lemma 2.1 (v), it suffices to show that $G(\bar{k}) = G^0(\bar{k})N_G(T)(\bar{k})$. Let $x \in G(\bar{k})$, then xTx^{-1} is a maximal torus of $G^0(\bar{k})$, and hence $xTx^{-1} = yTy^{-1}$ for some $y \in G^0(\bar{k})$. Thus, $x \in yN_G(T)(\bar{k})$, which yields the assertion.

(ii) We may assume that k is algebraically closed and G is connected (since $N_G(T)^0 = N_{G^0}(T)^0$). Then $Z_G(T)$ is a Cartan subgroup of G, and hence equals its connected normalizer by [SGA 3_{II} 1970, XII.6.6].

(iii) By (i), the natural map $N_G(T)/N_{G^0}(T) \to \pi_0(G)$ is an isomorphism. Combined with (ii), this yields the statement. □

Remark 2.5. If $N_G(T) = N_G(T)^0 \cdot F$ for some subgroup $F \subset N_G(T)$, then by Lemmas 2.1 and 2.4, $G = G^0 \cdot F$. Moreover, the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_G(T)^0 \cap F & \longrightarrow & F & \longrightarrow & \pi_0(N_G(T)) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\ 1 & \longrightarrow & G^0 \cap F & \longrightarrow & F & \longrightarrow & \pi_0(G) & \longrightarrow & 1 \end{array}$$

together with Lemma 2.4 yields the exact sequence

$$1 \longrightarrow Z_{G^0}(T) \cap F \longrightarrow G^0 \cap F \longrightarrow W(G^0, T) \longrightarrow 1.$$

In view of Lemma 2.1 (iv) and Lemma 2.4 (i), it suffices to prove Theorem 1.1 under the additional assumption that T *is normal in* G. Then T is central in G^0, and hence $G^0_{\bar{k}}$ is nilpotent by [SGA 3$_{\text{II}}$ 1970, XII.6.7]. It follows that G^0 *is nilpotent*, in view of [SGA 3$_{\text{I}}$ 1970, VIB.8.3]. To obtain further reductions, we will use the following:

Lemma 2.6. *Let N' be a normal subgroup of G contained in N. Assume that the resulting exact sequence $1 \to N/N' \to G/N' \to Q \to 1$ is quasisplit, and that any exact sequence of algebraic groups $1 \to N' \to G' \to Q' \to 1$, where Q' is finite, is quasisplit as well. Then* (1) *is quasisplit.*

Proof. By assumption, there exists a finite subgroup F' of G/N' for which $G/N' = (N/N') \cdot F'$. Denote by G' the subgroup of G containing N' such that $G'/N' = F'$. By assumption again, there is a finite subgroup F of G' containing N' such that $G' = N' \cdot F$. We check that $G = N \cdot F$ using Lemma 2.1 (iv). Let S be a scheme, and $g \in G(S)$; denote by $f' : G \to G/N'$ the quotient map. Then there exists a faithfully flat morphism of finite presentation $S' \to S$ and $x \in (N/N')(S')$, $y \in F'(S')$ such that $f'(g) = xy$ in $(G/N')(S')$. Moreover, there exists a faithfully flat morphism of finite presentation $S'' \to S'$ and $z \in N(S'')$, $w \in G'(S'')$ such that $f'(z) = x$ and $f'(w) = y$. Then $w^{-1}z^{-1}g \in N'(S'')$, and hence $g \in N(S'')G'(S'')$, as G' contains N'. This shows that $G = N \cdot G' = N \cdot (N' \cdot F)$. We conclude by observing that $N \cdot (N' \cdot F) = N \cdot F$, in view of Lemma 2.1 (iv) again. □

Remark 2.7. With the notation of the proof of Lemma 2.6, we have an exact sequence $1 \to N' \to G' = N' \cdot F \to F' \to 1$, and hence an exact sequence of quasicomplements

$$1 \longrightarrow N' \cap F \longrightarrow F \longrightarrow F' \longrightarrow 1.$$

Moreover, we obtain an exact sequence $1 \to N' \to N \cap G' \to (N/N') \cap F' \to 1$, by using [SGA 3$_{\text{I}}$ 1970, VIA.5.3.1]. Since $N \cap G' = N \cap (N' \cdot F) = N' \cdot (N \cap F)$, where the latter equality follows from Lemma 2.1 (iv), this yields an exact sequence of defect groups

$$1 \longrightarrow N' \cap F \longrightarrow N \cap F \longrightarrow (N/N') \cap F' \longrightarrow 1.$$

Next, we show that it suffices to prove Theorem 1.1 when G^0 is assumed in addition to be *commutative*.

We argue by induction on the dimension of G (assumed to be smooth, with G^0 nilpotent). If $\dim(G) = 1$, then G^0 is either a k-form of $\mathbb{G}_a$ or $\mathbb{G}_m$, or an elliptic curve; in particular, G^0 is commutative. In higher dimensions, the derived subgroup $D(G^0)$ is a smooth, connected normal subgroup of G contained in G^0, and the quotient $G^0/D(G^0)$ is commutative of positive dimension (see [SGA 3_{I} 1970, VIB.7.8, 8.3]). Moreover, $G/D(G^0)$ is smooth, and $\pi_0(G/D(G^0)) = \pi_0(G)$. By the induction assumption, it follows that the exact sequence

$$1 \longrightarrow G^0/D(G^0) \longrightarrow G/D(G^0) \longrightarrow \pi_0(G) \longrightarrow 1$$

is quasisplit. Also, every exact sequence $1 \to D(G^0) \to G' \to Q' \to 1$, where Q' is finite, is quasisplit, by the induction assumption again together with Lemma 2.2. Thus, Lemma 2.6 yields the desired reduction.

We now show that we may further assume G^0 to be *a torus, a smooth connected commutative unipotent group, or an abelian variety*.

Indeed, we have an exact sequence of commutative algebraic groups

$$1 \longrightarrow T \longrightarrow G^0 \longrightarrow H \longrightarrow 1,$$

where T is the maximal torus of G^0, and H is smooth and connected. Moreover, we have an exact sequence

$$1 \longrightarrow H_1 \longrightarrow H \longrightarrow H_2 \longrightarrow 1,$$

where H_1 is a smooth connected affine algebraic group, and H_2 is a pseudoabelian variety in the sense of [Totaro 2013], i.e., H_2 has no nontrivial smooth connected affine normal subgroup. Since H_1 contains no nontrivial torus, it is unipotent; also, H_2 is an extension of a smooth connected unipotent group by an abelian variety A, in view of [Totaro 2013, Theorem 2.1]. Note that T is a normal subgroup of G (the largest subtorus). Also, H_1 is a normal subgroup of G/T (the largest smooth connected affine normal subgroup of the neutral component), and A is a normal subgroup of $(G/T)/H_1$ as well (the largest abelian subvariety). Thus, arguing by induction on the dimension as in the preceding step, with $D(G^0)$ replaced successively by T, H_1 and A, yields our reduction.

When G^0 is unipotent and $\mathrm{char}(k) = p > 0$, we may further assume that G^0 *is killed by* p. Indeed, by [SGA 3_{II} 1970, XVII.3.9], there exists a composition series $\{e_G\} = G_0 \subset G_1 \subset \cdots \subset G_n = G^0$ such that each G_i is normal in G, and each quotient G_i/G_{i-1} is a k-form of some $(\mathbb{G}_a)^{r_i}$; in particular, G_i/G_{i-1} is killed by p. Our final reduction follows by induction on n.

3. Proof of Theorem 1.1: extensions by commutative groups

In this section, we consider smooth algebraic groups Q, N such that Q is finite and N is commutative. Given an extension (1), the action of G on N by conjugation factors through an action of Q by group automorphisms, which we denote by $(x, y) \mapsto y^x$, where $x \in Q$ and $y \in N$. Recall that the isomorphism classes of such extensions with a prescribed Q-action on N form a commutative (abstract) group, which we denote by $\mathrm{Ext}^1(Q, N)$; see [SGA 3_{II} 1970, XVII.App. I] (and [Demazure and Gabriel 1970, III.6.1] for the setting of extensions of group sheaves).

Lemma 3.1. *With the above notation and assumptions, the group* $\mathrm{Ext}^1(Q, N)$ *is torsion.*

Proof. Any extension (1) yields an N-torsor over Q for the étale topology, since Q is finite and étale. This defines a map $\tau : \mathrm{Ext}^1(Q, N) \to H^1_{\text{ét}}(Q, N)$, which is a group homomorphism (indeed, the sum of any two extensions is obtained by taking their direct product, pulling back under the diagonal map $Q \to Q \times Q$, and pushing forward under the multiplication $N \times N \to N$; and the sum of any two torsors is obtained by the analogous operations). The kernel of τ consists of those classes of extensions that admit a section (which is a morphism of schemes). In view of [SGA 3_{II} 1970, XVII.App. I.3.1], this yields an exact sequence

$$0 \longrightarrow HH^2(Q, N) \longrightarrow \mathrm{Ext}^1(Q, N) \xrightarrow{\tau} H^1_{\text{ét}}(Q, N),$$

where HH^i stands for Hochschild cohomology (denoted by H^i in [SGA 3_{I} 1970; SGA 3_{II} 1970; SGA 3_{III} 1970], and by H^i_0 in [Demazure and Gabriel 1970]). Moreover, the group $H^1_{\text{ét}}(Q, N)$ is torsion (as follows, e.g., from [Rosenlicht 1956, Theorem 14]), and $HH^2(Q, N)$ is killed by the order of Q, as a special case of [SGA 3_{II} 1970, XVII.5.2.4]. □

Remark 3.2. The above argument yields that $\mathrm{Ext}^1(Q, N)$ is killed by md if Q is finite étale of order m, and N is a torus split by an extension of k of degree d. Indeed, we just saw that $HH^2(Q, N)$ is killed by m; also, $H^1_{\text{ét}}(Q, N)$ is a direct sum of groups of the form $H^1_{\text{ét}}(\mathrm{Spec}(k'), N)$ for finite separable extensions k' of k, and these groups are killed by d. This yields a slight generalization of [Lucchini Arteche 2015b, Proposition 1.1], via a different approach.

End of the proof of Theorem 1.1. Recall from our reductions in Section 2 that we may assume G^0 to be a smooth commutative unipotent group, a torus, or an abelian variety. We will rather denote G^0 by N, and $\pi_0(G)$ by Q.

We first assume in addition that $\mathrm{char}(k) = 0$ if N is unipotent. Then the n-th power map

$$n_N : N \longrightarrow N, \quad x \longmapsto x^n$$

is an isogeny for any positive integer n. Consider an extension (1) and denote by γ its class in $\mathrm{Ext}^1(Q, N)$. By Lemma 3.1, we may choose n so that $n\gamma = 0$. Also, $n\gamma = (n_N)_*(\gamma)$ (the pushout of γ by n_N); moreover, the exact sequence of commutative algebraic groups

$$1 \longrightarrow N[n] \longrightarrow N \xrightarrow{n_N} N \longrightarrow 1$$

yields an exact sequence

$$\mathrm{Ext}^1(Q, N[n]) \longrightarrow \mathrm{Ext}^1(Q, N) \xrightarrow{(n_N)_*} \mathrm{Ext}^1(Q, N)$$

due to [SGA 3_{II} 1970, XVII.App. I.2.1]. Thus, there is a class $\gamma' \in \mathrm{Ext}^1(Q, N[n])$ with pushout γ, i.e., we have a commutative diagram of extensions

$$\begin{tikzcd} 1 \arrow[r] & N[n] \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\ 1 \arrow[r] & N \arrow[r] & G \arrow[r] & Q \arrow[r] & 1, \end{tikzcd}$$

where the square on the left is cartesian. It follows that G' is a finite subgroup of G, and $G = N \cdot G'$.

Next, we consider the remaining case, where N is unipotent and $\mathrm{char}(k) = p > 0$. In view of our final reduction at the end of Section 2, we may further assume that N is killed by p. Then there exists an étale isogeny $N \to N_1$, where N_1 is a vector group (see [Conrad et al. 2015, Lemma B.1.10]). This yields another commutative diagram of extensions

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\ 1 \arrow[r] & N_1 \arrow[r] & G_1 \arrow[r] & Q \arrow[r] & 1. \end{tikzcd}$$

Assume that there exists a finite subgroup F_1 of G_1 such that $G_1 = N_1 \cdot F_1$. Let F be the pullback of F_1 to G; then F is a finite subgroup, and one checks that $G = N \cdot F$ using Lemma 2.1 (iv). Thus, we may finally assume that N *is a vector group*.

Under that assumption, the N-torsor $G \to Q$ is trivial, since Q is affine. Thus, we may choose a section $s : Q \to G$. Also, we may choose a finite Galois extension of fields K/k such that Q_K is constant. Then s yields a section $s_K : Q_K \to G_K$, equivariant under the Galois group $\Gamma_K := \mathrm{Gal}(K/k)$. So we may view $G(K)$ as the set of the $y\, s(x)$, where $y \in N(K)$ and $x \in Q(K)$, with multiplication

$$y\, s(x)\, y'\, s(x') = y\, y'^{x}\, c(x, x')\, s(xx'),$$

where $c \in Z^2(Q(K), N(K))^{\Gamma_K}$. Consider the (abstract) subgroup $H \subset N(K)$ generated by the $c(x', x'')^x$, where $x, x', x'' \in Q(K)$. Then H is finite, since $N(K)$ is killed by p and $Q(K)$ is finite. Moreover, $Hs(Q(K))$ is a subgroup of $G(K)$, in view of the above formula for the multiplication. Clearly, $Hs(Q(K))$ is finite and stable under Γ_K; thus, it corresponds to a finite (algebraic) subgroup G' of G. Also, we obtain as above that $G = N \cdot G'$. This completes the proof of Theorem 1.1. □

Remark 3.3. If k is perfect, then the subgroup F as in Theorem 1.1 may be chosen étale. Indeed, the reduced subscheme F_{red} is then a subgroup by [SGA 3_{I} 1970, VIA.0.2]. Moreover, $G(\bar{k}) = G^0(\bar{k})F_{\mathrm{red}}(\bar{k})$, and hence $G = G^0 \cdot F_{\mathrm{red}}$ in view of Lemma 2.1 (v).

In contrast, when k is imperfect, there exists a finite group G admitting no étale subgroup F such that $G = G^0 \cdot F$. Consider for example (as in [SGA 3_{I} 1970, VIA.1.3.2]) the subgroup G of $\mathbb{G}_{a,k}$ defined by the additive polynomial $X^{p^2} - tX^p$, where $p := \mathrm{char}(k)$ and $t \in k \setminus k^p$. Then G has order p^2 and G^0 has order p. If $G = G^0 \cdot F$ with F étale, then $G^0 \cap F$ is trivial. Thus, $G \cong G^0 \rtimes F$ and F has order p. Let $K := k(t^{1/p})$. Then F_K is contained in $(G_K)_{\mathrm{red}}$, which is the subgroup of $\mathbb{G}_{a,K}$ defined by the additive polynomial $X^p - t^{1/p}X$. By counting dimensions, it follows that $F_K = (G_K)_{\mathrm{red}}$, which yields a contradiction as $(G_K)_{\mathrm{red}}$ is not defined over k.

Remark 3.4. One may obtain information on the defect group $N \cap F$ by examining the steps in the proof of Theorem 1.1 and combining Remarks 2.3, 2.5 and 2.7. For instance, if G is smooth, then $N \cap F$ is an extension of the Weyl group $W(G^0, T)$ by the nilpotent group $Z_{G^0}(T) \cap F$, where T is a maximal torus of G. If $\mathrm{char}(k) = 0$ (so that G is smooth), then $Z_{G^0}(T) \cap F$ is commutative. Indeed, $Z_{G^0}(T)$ is a connected nilpotent algebraic group, and hence an extension of a semiabelian variety S by a connected unipotent algebraic group U. Thus, $U \cap F$ is trivial, and hence $Z_{G^0}(T) \cap F$ is isomorphic to a subgroup of S.

Remark 3.5. When k is finite, Theorem 1.1 follows readily from our first reduction step (Lemma 2.2) together with a theorem of Lang [1956, Theorem 2]. More specifically, let H be a smooth algebraic group and choose representatives $x_1, \dots, x_m$ of the orbits of the Galois group $\Gamma := \mathrm{Gal}(\bar{k}/k)$ in $\pi_0(H)(\bar{k})$. Denote by $\Gamma_i \subset \Gamma$ the isotropy group of x_i and set $k_i := \bar{k}^{\Gamma_i}$ for $i = 1, \dots, m$. Then $x_i \in \pi_0(H)(k_i)$, and hence the fiber $\pi_{k_i}^{-1}(x_i)$ (a torsor under $H^0_{k_i}$) contains a k_i-rational point. Consider the subfield

$$K := \prod_{i=1}^{n} k_i \subset \bar{k}.$$

Then the finite étale group scheme $\pi_0(H)_K$ is constant, and π is surjective on K-rational points. Thus, $\pi_0(H)$ has a quasicomplement in H: the finite étale group scheme corresponding to the constant, Γ-stable subgroup scheme $H(K)$ of H_K.

4. Some applications

We first recall two classical results on the structure of algebraic groups. The first one is the affinization theorem (see [Demazure and Gabriel 1970, III.3.8] and also [SGA 3_{I} 1970, VIB.12.2]): any algebraic group G has a smallest normal subgroup H such that G/H is affine. Moreover, H is smooth, connected and contained in the center of G^0; we have $\mathbb{O}(H) = k$ (such an algebraic group is called *antiaffine*) and $\mathbb{O}(G/H) = \mathbb{O}(G)$.

Consequently, H is the fiber at e_G of the affinization morphism $G \to \operatorname{Spec} \mathbb{O}(G)$; moreover, the formation of H commutes with arbitrary field extensions. Also, note that H is the largest antiaffine subgroup of G; we will denote H by G_{ant}. The structure of antiaffine groups is described in [Brion 2009] and [Sancho and Sancho 2009].

The second structure result is a version of a theorem of Chevalley, due to Raynaud [1970, Lemma IX.2.7] (see also [Bosch et al. 1990, 9.2 Theorem 1]): any connected algebraic group G has a smallest affine normal subgroup N such that G/N is an abelian variety. Moreover, N is connected; if G is smooth and k is perfect, then N is smooth as well. We will denote N by G_{aff}.

We will also need the following observation:

Lemma 4.1. *Let G be an algebraic group, and N a normal subgroup. Then the quotient map $f : G \to G/N$ yields an isomorphism $G_{\mathrm{ant}}/(G_{\mathrm{ant}} \cap N) \cong (G/N)_{\mathrm{ant}}$.*

Proof. We have a closed immersion of group schemes $G_{\mathrm{ant}}/(G_{\mathrm{ant}} \cap N) \to G/N$; moreover, $G_{\mathrm{ant}}/(G_{\mathrm{ant}} \cap N)$ is antiaffine. So we obtain a closed immersion of commutative group schemes $i : G_{\mathrm{ant}}/(G_{\mathrm{ant}} \cap N) \to (G/N)_{\mathrm{ant}}$. The cokernel of i is antiaffine, as a quotient of $(G/N)_{\mathrm{ant}}$. Also, this cokernel is a subgroup of $(G/N)/(G_{\mathrm{ant}}/(G_{\mathrm{ant}} \cap N))$, which is a quotient of G/G_{ant}. Since the latter is affine, it follows that $\operatorname{Coker}(i)$ is affine as well, by using [SGA 3_{I} 1970, VIB.11.17]. Thus, $\operatorname{Coker}(i)$ is trivial, i.e., i is an isomorphism. $\square$

We now obtain a further version of Chevalley's structure theorem, for possibly nonconnected algebraic groups:

Theorem 4.2. *Any algebraic group G has a smallest affine normal subgroup N such that G/N is proper. Moreover, N is connected.*

Proof. It suffices to show that G admits an affine normal subgroup N such that G/N is proper. Indeed, given another such subgroup N', the natural map

$$G/(N \cap N') \longrightarrow G/N \times G/N'$$

is a closed immersion, and hence $G/(N \cap N')$ is proper. Taking for N a minimal such subgroup, it follows that N is the smallest one. Moreover, the natural morphism $G/N^0 \to G/N$ is finite, since it is a torsor under the finite group N/N^0 (see

[SGA 3_{I} 1970, VIA.5.3.2]). As a consequence, G/N^0 is proper; hence $N = N^0$ by the minimality assumption. Thus, N is connected.

Also, we may reduce to the case where G is smooth by using the relative Frobenius morphism as in the proof of Lemma 2.2.

If in addition G is connected, then we just take $N = G_{\mathrm{aff}}$. In the general case, we consider the (possibly nonnormal) subgroup $H := (G^0)_{\mathrm{aff}}$; then the homogeneous space G/H is proper, since G/G^0 is finite and G^0/H is proper. As a consequence, the automorphism functor of G/H is represented by a group scheme $\mathrm{Aut}_{G/H}$, locally of finite type; in particular, the neutral component $\mathrm{Aut}^0_{G/H}$ is an algebraic group (see [Matsumura and Oort 1967, Theorem 3.7]). The action of G by left multiplication on G/H yields a morphism of group schemes

$$\varphi : G \longrightarrow \mathrm{Aut}_{G/H}.$$

The kernel N of φ is a closed subscheme of H, and hence is affine. To complete the proof, it suffices to show that G/N is proper. In turn, it suffices to check that $(G/N)^0$ is proper. Since $(G/N)^0 \cong G^0/(G^0 \cap N)$, and $G^0 \cap N$ is the kernel of the restriction $G^0 \to \mathrm{Aut}^0_{G/H}$, we are reduced to showing that $\mathrm{Aut}^0_{G/H}$ is proper (by using [SGA 3_{I} 1970, VIA.5.4.1] again).

We claim that $\mathrm{Aut}^0_{G/H}$ is an abelian variety. Indeed, $(G/H)_{\bar{k}}$ is a finite disjoint union of copies of $(G^0/H)_{\bar{k}}$, which is an abelian variety. Also, the natural morphism $A \to \mathrm{Aut}^0_A$ is an isomorphism for any abelian variety A. Thus, $(\mathrm{Aut}^0_{G/H})_{\bar{k}}$ is an abelian variety (a product of copies of $(G^0/H)_{\bar{k}}$); this yields our claim, and completes the proof. □

Remark 4.3. The formation of G_{aff} (for a connected group scheme G) commutes with separable algebraic field extensions, as follows from a standard argument of Galois descent. But this formation does not commute with purely inseparable field extensions, in view of [SGA 3_{II} 1970, XVII.C.5].

Likewise, the formation of N as in Theorem 4.2 commutes with separable algebraic field extensions. As a consequence, $N = (G^0)_{\mathrm{aff}}$ for any smooth group scheme G (since $(G^0)_{\mathrm{aff}}$ is invariant under any automorphism of G, and hence is a normal subgroup of G when k is separably closed). In particular, if k is perfect and G is smooth, then N is smooth as well.

For an arbitrary group scheme G, we may have $N \neq (G^0)_{\mathrm{aff}}$, e.g., when G is infinitesimal: then N is trivial, while $(G^0)_{\mathrm{aff}} = G$.

We do not know if the formations of G_{aff} and N commute with arbitrary separable field extensions.

The structure of proper algebraic groups is easily described as follows:

Proposition 4.4. *Given a proper algebraic group G, there exists an abelian variety A, a finite group F equipped with an action $F \to \mathrm{Aut}_A$ and a normal subgroup*

$D \subset F$ such that D acts faithfully on A by translations and $G \cong (A \rtimes F)/D$, where D is embedded in $A \rtimes F$ via $x \mapsto (x, x^{-1})$. Moreover, $A = G_{\mathrm{ant}}$ and $F/D \cong G/G_{\mathrm{ant}}$ are uniquely determined by G. Finally, G is smooth if and only if F/D is étale.

Proof. Note that G_{ant} is a smooth connected proper algebraic group, and hence an abelian variety. Moreover, the quotient group G/G_{ant} is affine and proper, hence finite. By Theorem 1.1, there exists a finite subgroup $F \subset G$ such that $G = G_{\mathrm{ant}} \cdot F$. In particular, $G \cong (F \ltimes G_{\mathrm{ant}})/(F \cap G_{\mathrm{ant}})$; this implies the existence assertion. For the uniqueness, just note that $\mathcal{O}(G) \cong \mathcal{O}(G/A) \cong \mathcal{O}(F/D)$, and this identifies the affinization morphism to the natural homomorphism $G \to F/D$, with kernel A.

If G is smooth, then so is $G/A \cong F/D$; as F/D is finite, it must be étale. Since the homomorphism $G \to F/D$ is smooth, the converse holds as well. □

Remark 4.5. The simplest examples of proper algebraic groups are the semidirect products $G = A \rtimes F$, where F is a finite group acting on the abelian variety A. If this action is nontrivial (for example, if A is nontrivial and F is the constant group $\mathbb{Z}/2\mathbb{Z}$ acting via $x \mapsto x^{\pm 1}$), then every morphism of algebraic groups $f : G \to H$, where H is connected, has a nontrivial kernel. (Otherwise, A is contained in the center of G by the affinization theorem.) This yields examples of algebraic groups which admit no faithful representation in a connected algebraic group.

Remark 4.6. With the notation and assumptions of Proposition 4.4, consider a subgroup $H \subset G$ and the homogeneous space $X := G/H$. Then there exists an abelian variety B quotient of A, a subgroup $I \subset F$ containing D, and a faithful homomorphism $I \to \mathrm{Aut}_B$ such that the scheme X is isomorphic to the associated fiber bundle $F \times^I B$. Moreover, the schemes F/I and B are uniquely determined by X, and X is smooth if and only if F/I is étale.

Indeed, let $K := A \cdot H$, then $X \cong G \times^K K/H \cong F \times^I K/H$, where $I := F \cap K$. Moreover, $K/H \cong A/(A \cap H)$ is an abelian variety. This shows the existence assertion; those on uniqueness and smoothness are checked as in the proof of Proposition 4.4.

Conversely, given a finite group F and a subgroup $I \subset F$ acting on an abelian variety B, the associated fiber bundle $F \times^I B$ exists (since it is the quotient of the projective scheme $F \times B$ by the finite group I), and is homogeneous whenever F/I is étale (since $(F \times^I B)_{\bar{k}}$ is just a disjoint union of copies of $B_{\bar{k}}$). We do not know how to characterize the homogeneity of $F \times^I B$ when the quotient F/I is arbitrary.

Returning to an arbitrary algebraic group G, we have the "Rosenlicht decomposition" $G = G_{\mathrm{ant}} \cdot G_{\mathrm{aff}}$ when G is smooth and connected (see, e.g., [Brion 2009]). We now extend this result to possibly nonconnected groups:

Theorem 4.7. *Let G be an algebraic group. Then there exists an affine subgroup H of G such that $G = G_{\mathrm{ant}} \cdot H$. If G is smooth and k is perfect, then H may be chosen smooth.*

Proof. By Theorem 4.2, we may choose an affine normal subgroup $N \subset G$ such that G/N is proper. In view of Proposition 4.4, there exists a finite subgroup F of G/N such that $G/N = (G/N)_{\mathrm{ant}} \cdot F$, and $(G/N)_{\mathrm{ant}}$ is an abelian variety. Let H be the subgroup of G containing N such that $G/H = F$. Then H is affine, since it sits in an extension $1 \to N \to H \to F \to 1$. We check that $G = G_{\mathrm{ant}} \cdot H$ by using Lemma 2.1 (iv). Let S be a scheme, and $g \in G(S)$. Denote by g' the image of g in $(G/N)(S)$. Then there exist a faithfully flat morphism of finite presentation $S' \to S$ and $x' \in (G/N)_{\mathrm{ant}}(S')$, $y' \in F(S')$ such that $g' = x'y'$ in $(G/N)(S')$. Moreover, in view of Lemma 4.1, x' lifts to some $x'' \in G_{\mathrm{ant}}(S'')$, where $S'' \to S'$ is faithfully flat of finite presentation. So $gx''^{-1} \in G(S'')$ lifts y', and hence $g \in G_{\mathrm{ant}}(S'')H(S'')$.

If G is smooth and k is perfect, then N may be chosen smooth by Remark 4.3; also, F may be chosen smooth by Remark 3.3. Then H is smooth as well. □

We now derive from Theorem 4.7 a generalization of our main Theorem 1.1, under the additional assumption of characteristic 0 (then reductivity is equivalent to linear reductivity, also known as full reducibility):

Corollary 4.8. *Every extension (1) with reductive quotient group Q is quasisplit when* $\mathrm{char}(k) = 0$.

Proof. Choose an affine subgroup $H \subset G$ such that $G = G_{\mathrm{ant}} \cdot H$ and denote by $R_u(H)$ its unipotent radical. By a result of Mostow [1956, Theorem 6.1], H has a Levi subgroup, i.e., a fully reducible algebraic subgroup L such that $H = R_u(H) \rtimes L$. Note that $R_u(H)$ is normal in G, since it is normalized by H and centralized by G_{ant}. It follows that $G_{\mathrm{ant}} \cdot R_u(H)$ is normal in G, and $G = (G_{\mathrm{ant}} \cdot R_u(H)) \cdot L$. Also, note that the quotient map $f : G \to Q$ sends G_{ant} to e_Q (since Q is affine), and $R_u(H)$ to e_Q as well (since Q is reductive). It follows that the sequence

$$1 \longrightarrow N \cap L \longrightarrow L \xrightarrow{f} Q \longrightarrow 1$$

is exact, where $N = \mathrm{Ker}(f)$. If $N \cap L$ has a quasicomplement H in L, then H is a quasicomplement to N in G (as follows, e.g., from Lemma 2.1 (v)). Thus, we may assume that G is *reductive*. Since every quasicomplement to N^0 in G is a quasicomplement to N, we may further assume that N is *connected*.

We have a canonical decomposition

$$G^0 = D(G^0) \cdot R(G^0),$$

where the derived subgroup $D(G^0)$ is semisimple, the radical $R(G^0)$ is a central torus, and $D(G^0) \cap R(G^0)$ is finite (see, e.g., [SGA 3_{III} 1970, XXII.6.2.4]). Thus, $G = D(G^0) \cdot (R(G^0) \cdot F)$, where $F \subset G$ is a quasicomplement to G^0. Likewise, $N = D(N) \cdot R(N)$, where $D(N) \subset D(G^0)$, $R(N) \subset R(G^0)$ and both are normal in G. Denote by S the neutral component of the centralizer of $D(N)$ in $D(G^0)$. Then S is a normal semisimple subgroup of G, and a quasicomplement to $D(N)$

in $D(G^0)$. If $R(N)$ admits a quasicomplement T in $R(G^0)\cdot F$, then one readily checks that $S\cdot T$ is a quasicomplement to N in G. As a consequence, we may replace G with $R(G^0)\cdot F$, and hence assume that G^0 is a *torus*.

Denote by $X^*(G^0)$ the character group of $G^0_{\bar{k}}$; this is a free abelian group of finite rank, equipped with a continuous action of $F(\bar{k})\rtimes\Gamma$, where Γ denotes the absolute Galois group of k. Moreover, we have a surjective homomorphism $\rho : X^*(G^0)\to X^*(N)$, equivariant for $F(\bar{k})\rtimes\Gamma$. Thus, ρ splits over the rationals, and hence there exists a subgroup $\Lambda\subset X^*(G^0)$, stable by $F(\bar{k})\rtimes\Gamma$, which is mapped isomorphically by ρ to a subgroup of finite index of $X^*(N)$. The quotient $X^*(G^0)/\Lambda$ corresponds to a subtorus $H\subset G^0$, normalized by G, which is a quasicomplement to N in G^0. So $H\cdot F$ is the desired quasicomplement to N in G. □

Remark 4.9. Corollary 4.8 does not extend to positive characteristics, due to the existence of groups without Levi subgroups (see [Conrad et al. 2015, Appendix A.6; McNinch 2010, Section 3.2]). As a specific example, when k is perfect of characteristic $p>0$, there exists a nonsplit extension of algebraic groups

$$1\longrightarrow V\longrightarrow G\xrightarrow{f}\mathrm{SL}_2\longrightarrow 1,$$

where V is a vector group on which SL_2 acts linearly via the Frobenius twist of its adjoint representation. We show that this extension is not quasisplit. Otherwise, let H be a quasicomplement to N in G. Then so is the reduced neutral component of H, and hence we may assume that H is smooth and connected. The quotient map f restricts to an isogeny $H\to\mathrm{SL}_2$, and hence to an isomorphism. Thus, the above extension is split, a contradiction.

Next, we obtain an analogue of the Levi decomposition (see [Mostow 1956] again) for possibly nonlinear algebraic groups:

Corollary 4.10. *Let G be an algebraic group over a field of characteristic* 0. *Then $G=R\cdot S$, where $R\subset G$ is the largest connected solvable normal subgroup, and $S\subset G$ is an algebraic subgroup such that S^0 is semisimple; also, $R\cap S$ is finite.*

Proof. By a standard argument, G has a largest connected solvable normal subgroup R. The quotient G/R is affine, since $R\supset G_{\mathrm{ant}}$. Moreover, R/G_{ant} contains the radical of G/G_{ant}, and hence $(G/R)^0$ is semisimple. In particular, G/R is reductive. So Corollary 4.8 yields the existence of the quasicomplement S. □

Remark 4.11. One may ask for a version of Corollary 4.10 in which the normal subgroup R is replaced with an analogue of the unipotent radical of a linear algebraic group, and the quasicomplement S is assumed to be reductive. But such a version would make little sense when G is an antiaffine semiabelian variety (for example, when G is the extension of an abelian variety A by $\mathbb{G}_m$, associated with an algebraically trivial line bundle of infinite order on A). Indeed, such a group G

has a largest connected reductive subgroup: its maximal torus, which admits no quasicomplement.

Also, recall that the radical R may admit no complement in G, e.g., when $G = \mathrm{GL}_n$ with $n \geq 2$.

Finally, one may also ask for the uniqueness of a minimal quasicomplement in Corollary 4.10, up to conjugacy in $R(k)$ (as for Levi complements, see [Mostow 1956, Theorem 6.2]). But this fails when k is algebraically closed and G is the semidirect product of an abelian variety A with a group F of order 2. Denote by σ the involution of A induced by the nontrivial element of F; then $R = A$, and the complements to R in G are exactly the subgroups generated by the involutions $x\sigma$ where $x \in A^{-\sigma}(k)$, i.e., $\sigma(x) = x^{-1}$. The action of $R(k)$ on complements is given by $yx\sigma y^{-1} = xy\sigma(y)^{-1}\sigma$; moreover, the homomorphism $A \to A^{-\sigma}$, $y \mapsto y\sigma(y)^{-1}$ is generally not surjective. This holds for example when $A = (B \times B)/C$, where B is a nontrivial abelian variety, C is the subgroup of $B \times B$ generated by (x_0, x_0) for some $x_0 \in B(k)$ of order 2, and σ arises from the involution $(x, y) \mapsto (y^{-1}, x^{-1})$ of $B \times B$; then $A^{-\sigma}$ has 2 connected components.

Another consequence of Theorem 4.7 concerns the case where k is finite; then every antiaffine algebraic group is an abelian variety (see [Brion 2009, Proposition 2.2]). This yields readily:

Corollary 4.12. *Let G be an algebraic group over a finite field. Then G sits in an extension of algebraic groups*

$$1 \longrightarrow F \longrightarrow A \times H \longrightarrow G \longrightarrow 1,$$

where F is finite, A is an abelian variety, and H is affine. If G is smooth, then H may be chosen smooth as well.

Returning to an arbitrary base field, we finally obtain the existence of equivariant compactifications of homogeneous spaces:

Theorem 4.13. *Let G be an algebraic group, and H a closed subgroup. Then there exists a projective scheme X equipped with an action of G, and an open G-equivariant immersion $G/H \hookrightarrow X$ with schematically dense image.*

Proof. When G is affine, this follows from a theorem of Chevalley asserting that H is the stabilizer of a line L in a finite-dimensional G-module V (see [SGA 3_{I} 1970, VIB.11.16]). Indeed, one may take for X the closure of the G-orbit of L in the projective space of lines of V; then X satisfies the required properties in view of [Demazure and Gabriel 1970, III.3.5.2]. Note that X is equipped with an ample G-linearized invertible sheaf.

When G is proper, the homogeneous space G/H is proper as well, and hence is projective by [Raynaud 1970, Corollary VI.2.6] (alternatively, this follows from the structure of X described in Remark 4.5).

In the general case, Theorem 4.2 yields an affine normal subgroup N of G such that G/N is proper. Then $N \cdot H$ is a subgroup of G, and $G/(N \cdot H)$ is proper as well, hence projective. It suffices to show the existence of a projective scheme Y equipped with an action of $N \cdot H$, an open immersion $(N \cdot H)/H \to Y$ with schematically dense image, and a $N \cdot H$-linearized ample line bundle: indeed, by [Mumford et al. 1994, Proposition 7.1] applied to the projection $G \times Y \to Y$ and the $N \cdot H$-torsor $G \to G/(N \cdot H)$, the associated fiber bundle $G \times^{N \cdot H} Y$ yields the desired equivariant compactification. In view of Chevalley's theorem used in the first step, it suffices in turn to check that $N \cdot H$ acts on $(N \cdot H)/H$ via an affine quotient group; equivalently, $(N \cdot H)_{\mathrm{ant}} \subset H$.

By Lemma 4.1, $(N \cdot H)_{\mathrm{ant}}$ is a quotient of $(N \rtimes H)_{\mathrm{ant}}$. The latter is the fiber at the neutral element of the affinization morphism $N \rtimes H \to \operatorname{Spec} \mathcal{O}(N \rtimes H)$. Also, $N \rtimes H \cong N \times H$ as schemes, N is affine and the affinization morphism commutes with products; thus, $(N \rtimes H)_{\mathrm{ant}} = H_{\mathrm{ant}}$. As a consequence, $(N \cdot H)_{\mathrm{ant}} = H_{\mathrm{ant}}$; this completes the proof. □

Remark 4.14. If $\mathrm{char}(k) = 0$, the equivariant compactification X of Theorem 4.13 may be taken smooth, as follows from the existence of an equivariant desingularization (see [Kollár 2007, Proposition 3.9.1, Theorem 3.36]).

In arbitrary characteristics, X may be taken normal if G is smooth. Indeed, the G-action on any equivariant compactification X stabilizes the reduced subscheme X_{red} (since $G \times X_{\mathrm{red}}$ is reduced), and lifts to an action on its normalization $\widetilde{X}$ (since $G \times \widetilde{X}$ is normal). But the existence of regular compactifications (equivariant or not) is an open question.

Over any imperfect field k, there exist smooth connected algebraic groups G having no smooth compactification. Indeed, we may take for G the subgroup of $\mathbb{G}_a \times \mathbb{G}_a$ defined by $y^p - y - tx^p = 0$, where $p := \mathrm{char}(k)$ and $t \in k \setminus k^p$. This is a smooth affine curve, and hence has a unique regular compactification X. One checks that X is the curve $(y^p - yz^{p-1} - tx^p = 0) \subset \mathbb{P}^2$, which is not smooth at its point at infinity.

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ESSENTIAL DIMENSION AND ERROR-CORRECTING CODES

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APPENDIX BY ATHENA NGUYEN

To the memory of Robert Steinberg

One of the important open problems in the theory of central simple algebras is to compute the essential dimension of GL_n/μ_m, i.e., the essential dimension of a generic division algebra of degree n and exponent dividing m. In this paper we study the essential dimension of groups of the form

$$G = (\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r})/C,$$

where C is a central subgroup of $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. Equivalently, we are interested in the essential dimension of a generic r-tuple $(A_1, \dots, A_r)$ of central simple algebras such that $\deg(A_i) = n_i$ and the Brauer classes of $A_1, \dots, A_r$ satisfy a system of homogeneous linear equations in the Brauer group. The equations depend on the choice of C via the error-correcting code $\mathrm{Code}(C)$ which we naturally associate to C. We focus on the case where $n_1, \dots, n_r$ are powers of the same prime. The upper and lower bounds on $\mathrm{ed}(G)$ we obtain are expressed in terms of coding-theoretic parameters of $\mathrm{Code}(C)$, such as its weight distribution. Surprisingly, for many groups of the above form the essential dimension becomes easier to estimate when $r \geq 3$; in some cases we even compute the exact value. The Appendix by Athena Nguyen contains an explicit description of the Galois cohomology of groups of the form $(\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r})/C$. This description and its corollaries are used throughout the paper.

1. Introduction

Let k be a base field. Unless otherwise specified, we will assume that every field appearing in this paper contains k and every homomorphism (i.e., inclusion) of fields restricts to the identity map on k.

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We begin by recalling the definition of the essential dimension of a covariant functor $\mathcal{F}$ from the category of fields to the category of sets. Given a field K and an object $\alpha \in \mathcal{F}(K)$, we will say that α *descends* to an intermediate field $k \subset K_0 \subset K$ if α lies in the image of the natural map $\mathcal{F}(K_0) \to \mathcal{F}(K)$. The essential dimension $\mathrm{ed}(\alpha)$ of α is defined as the minimal value of $\mathrm{trdeg}_k(K_0)$ such that α descends to a subfield $k \subset K_0 \subset K$. Given a prime integer p, the essential dimension $\mathrm{ed}_p(\alpha)$ of α at p is defined as the minimal value of $\mathrm{trdeg}_k(K_0)$, where the minimum is taken over all finite field extensions L/K and all intermediate fields $k \subset K_0 \subset L$, such that $[L : K]$ is prime to p and α_L descends to K_0.

The essential dimension $\mathrm{ed}(\mathcal{F})$ (respectively, the essential dimension $\mathrm{ed}_p(\mathcal{F})$ at p) of the functor $\mathcal{F}$ is defined as the maximal value of $\mathrm{ed}(\alpha)$ (respectively of $\mathrm{ed}_p(\alpha)$), where the maximum is taken over all field extensions K/k and all objects $\alpha \in \mathcal{F}(K)$.

Informally speaking, $\mathrm{ed}(\alpha)$ is the minimal number of independent parameters required to define α, $\mathrm{ed}(\mathcal{F})$ is the minimal number of independent parameters required to define any object in $\mathcal{F}$, and $\mathrm{ed}_p(\alpha)$, $\mathrm{ed}_p(\mathcal{F})$ are relative versions of these notions at a prime p. These relative versions are somewhat less intuitive, but they tend to be more accessible and more amenable to computation than $\mathrm{ed}(\alpha)$ and $\mathrm{ed}(\mathcal{F})$. Clearly $\mathrm{ed}(\alpha) \geqslant \mathrm{ed}_p(\alpha)$ for each α, and $\mathrm{ed}(\mathcal{F}) \geqslant \mathrm{ed}_p(\mathcal{F})$. In most cases of interest, $\mathrm{ed}(\alpha)$ is finite for every α. On the other hand, $\mathrm{ed}(\mathcal{F})$ (and even $\mathrm{ed}_p(\mathcal{F})$) can be infinite. For an introduction to the theory of essential dimension, we refer the reader to the surveys [Berhuy and Favi 2003; Reichstein 2010; 2012; Merkurjev 2013].

To every algebraic group G one can associate the functor

$$\mathcal{F}_G := H^1(*, G) : K \mapsto \{\text{isomorphism classes of } G\text{-torsors over } \mathrm{Spec}(K)\}.$$

If G is affine, then the essential dimension of this functor is known to be finite; it is usually denoted by $\mathrm{ed}(G)$, rather than $\mathrm{ed}(\mathcal{F}_G)$. For many specific groups G, $H^1(K, G)$ is in a natural bijective correspondence with the set of isomorphism classes of some algebraic objects defined over K. In such cases, $\mathrm{ed}(G)$ may be viewed as the minimal number of independent parameters required to define any object of this type. This number is often related to classical problems in algebra.

For example, in the case where G is the projective linear group PGL_n, the objects in question are central simple algebras. That is,

$$H^1(K, \mathrm{PGL}_n) = \{\text{isomorphism classes of central simple } K\text{-algebras of degree } n\}. \tag{1}$$

The problem of computing $\mathrm{ed}(\mathrm{PGL}_n)$ is one of the important open problems in the theory of central simple algebras; see [Auel et al. 2011, Section 6]. This problem was first posed by C. Procesi, who showed (using different terminology) that

$$\mathrm{ed}(\mathrm{PGL}_n) \leqslant n^2; \tag{2}$$

see [Procesi 1967, Theorem 2.1]. Stronger (but still quadratic) upper bounds can be found in [Lorenz et al. 2003, Theorem 1.1] and [Lemire 2004, Theorem 1.6].

A more general but closely related problem is computing $\mathrm{ed}(\mathrm{GL}_n/\mu_m)$, where m and n are positive integers and m divides n. Note that

$$(3)\quad H^1(K, \mathrm{GL}_n/\mu_m) = \{\text{isomorphism classes of central simple } K\text{-algebras of degree } n \text{ and exponent dividing } m\}.$$

In particular, $\mathrm{ed}(\mathrm{PGL}_n) = \mathrm{ed}(\mathrm{GL}_n/\mu_n)$. The problem of computing $\mathrm{ed}(\mathrm{GL}_n/\mu_m)$ partially reduces to the case where $m = p^s$ and $n = p^a$ are powers of the same prime p and $1 \leqslant s \leqslant a$.

From now on we will always assume that $\mathrm{char}(k) \neq p$. The inequalities

$$(4)\quad p^{2a-2}+p^{a-s} \geqslant \mathrm{ed}_p(\mathrm{GL}_{p^a}/\mu_{p^s}) \geqslant \begin{cases} (a-1)2^{a-1} & \text{if } p=2 \text{ and } s=1,\\ (a-1)p^a+p^{a-s} & \text{otherwise,} \end{cases}$$

proved in [Baek and Merkurjev 2012] represent a striking improvement on the best previously known bounds. (Here $a \geqslant 2$.) Yet the gap between the lower and upper bounds in (4) remains wide. The gap between the best known upper and lower bounds becomes even wider when $\mathrm{ed}_p(\mathrm{GL}_{p^a}/\mu_{p^s})$ is replaced by $\mathrm{ed}(\mathrm{GL}_{p^a}/\mu_{p^s})$.

These gaps in our understanding of $\mathrm{ed}(\mathrm{GL}_n/\mu_m)$ will not deter us from considering the vastly more general problem of computing the essential dimension of groups of the form

$$(5)\qquad G := (\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r})/C$$

in the present paper. Here $n_1, \ldots, n_r \geqslant 2$ are integers, and $C \subset \mathbb{G}_m^r$ is a central subgroup of $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$.

As usual, we will identify elements $(m_1, \ldots, m_r)$ of $\mathbb{Z}^r$ with characters

$$x\colon \mathbb{G}_m^r \to \mathbb{G}_m, \quad \text{where} \quad x\colon (\tau_1, \ldots, \tau_r) \mapsto \tau_1^{m_1} \cdots \tau_r^{m_r}.$$

The subgroup $C \subset \mathbb{G}_m^r$ is completely determined by the $\mathbb{Z}$-module

$$(6)\quad X(\mathbb{G}_m^r/C) = \{(m_1, \ldots, m_r) \in \mathbb{Z}^r \mid \tau_1^{m_1} \cdots \tau_r^{m_r} = 1 \ \ \forall (\tau_1, \ldots, \tau_r) \in C\},$$

consisting of characters of $\mathbb{G}_m^r$ which vanish on C. The Galois cohomology of G is explicitly described in the Appendix: by Theorem A.1, $H^1(K, G)$ is naturally isomorphic to the set of isomorphism classes of r-tuples $(A_1, \ldots, A_r)$ of central simple K-algebras such that

$$\deg(A_i) = n_i \quad \text{and} \quad A_1^{\otimes m_1} \otimes \cdots \otimes A_r^{\otimes m_r} \text{ is split over } K$$

for every $(m_1, \ldots, m_r) \in X(\mathbb{G}_m^r/C)$. (Note that in the special case where $r = 1$, we recover (1) and (3).) It follows from this description that the essential dimension of G does not change if C is replaced by $C \cap \mu$, where

$$\mu := \mu_{n_1} \times \cdots \times \mu_{n_r}; \tag{7}$$

see Corollary A.2. Thus we will assume throughout that $C \subset \mu$. Unless otherwise specified, we will also assume that $n_1 = p^{a_1}, \ldots, n_r = p^{a_r}$ are powers of the same prime p. Here $a_1, \ldots, a_r \geqslant 1$ are integers. Under these assumptions, instead of $X(\mathbb{G}_m^r/C) \subset \mathbb{Z}^r$, we will consider the subgroup of

$$X(\mu) = (\mathbb{Z}/p^{a_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{a_r}\mathbb{Z})$$

given by

$$\begin{aligned} \mathrm{Code}(C) &:= X(\mu/C) = \\ &\big\{(m_1, \ldots, m_r) \in X(\mu) \mid \tau_1^{m_1} \cdots \tau_r^{m_r} = 1 \;\; \forall (\tau_1, \ldots, \tau_r) \in C\big\}. \end{aligned} \tag{8}$$

In other words, Code(C) consists of those characters of μ which vanish on C. The symbol "Code" indicates that we will view this group as an error-correcting code. In particular, we will define the Hamming weight $\mathrm{w}(y)$ of

$$y = (m_1, \ldots, m_r) \in (\mathbb{Z}/p^{a_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{a_r}\mathbb{Z})$$

as follows. Write $m_i := u_i p^{e_i}$ with $u_i \in (\mathbb{Z}/p^{a_i}\mathbb{Z})^*$ and $0 \le e_i \le a_i$. Then

$$\mathrm{w}(y) := \sum_{i=1}^{r} (a_i - e_i).$$

Our main results relate ed(G) to coding-theoretic invariants of Code(C), such as its weight distribution; see also Corollary A.3. For an introduction to error-correcting coding theory, see [MacWilliams and Sloane 1977].

At this point we should warn the reader that our notions of error-correcting code and Hamming weight are somewhat unusual. In coding-theoretic literature (linear) codes are usually defined as linear subspaces of $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is the field of q elements. In this paper, by a code we will mean an additive subgroup of $(\mathbb{Z}/p^{a_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{a_r}\mathbb{Z})$. Nevertheless, in an important special case, where $a_1 = \cdots = a_r = 1$, our codes are linear codes of length r over $\mathbb{F}_p$ in the usual sense of error-correcting coding theory, and our definition of the Hamming weight coincides with the usual definition.

Theorem 1.1. *Let p be a prime, $G := (\mathrm{GL}_{p^{a_1}} \times \cdots \times \mathrm{GL}_{p^{a_r}})/C$, where $C \subset \mu_{p^{a_1}} \times \cdots \times \mu_{p^{a_r}}$ is a central subgroup, and $y_1, \ldots, y_t$ be a minimal basis for* Code(C); *see Definition* 3.2. *Then*

(a) $\mathrm{ed}_p(G) \geqslant \big(\sum_{i=1}^t p^{\mathrm{w}(y_i)}\big) - p^{2a_1} - \cdots - p^{2a_r} + r - t$,

(b) $\mathrm{ed}(G) \leqslant \big(\sum_{i=1}^t p^{\mathrm{w}(y_i)}\big) - t + \mathrm{ed}(\overline{G})$ *and* $\mathrm{ed}_p(G) \leqslant \big(\sum_{i=1}^t p^{\mathrm{w}(y_i)}\big) - t + \mathrm{ed}_p(\overline{G})$, *where* $\overline{G} := \mathrm{PGL}_{p^{a_1}} \times \cdots \times \mathrm{PGL}_{p^{a_r}}$.

Although the upper and lower bounds of Theorem 1.1 never meet, for many central subgroups $C \subset \mu \subset G$, the term $\sum_{i=1}^{t} p^{\mathrm{w}(y_i)}$ is much larger than any of the other terms appearing in the above inequalities and may be viewed as giving the asymptotic value of $\mathrm{ed}(G)$. In particular, note that in view of (2),

$$(9) \quad \mathrm{ed}_p(\overline{G}) \leqslant \mathrm{ed}(\overline{G}) \leqslant \mathrm{ed}(\mathrm{PGL}_{p^{a_1}}) + \cdots + \mathrm{ed}(\mathrm{PGL}_{p^{a_r}}) \leqslant p^{2a_1} + \cdots + p^{2a_r}.$$

Under additional assumptions on C, we will determine the exact value of $\mathrm{ed}(G)$; see Theorem 1.2.

The fact that we can determine $\mathrm{ed}(G)$ for many choices of C, either asymptotically or exactly, was rather surprising to us, given the wide gap between the best known upper and lower bounds on $\mathrm{ed}(G)$ in the simplest case, where $r = 1$; see (4). Our informal explanation of this surprising phenomenon is as follows. If $\mathrm{Code}(C)$ can be generated by vectors $y_1, \dots, y_t$ of small weight, then $\sum_{i=1}^{t} p^{\mathrm{w}(y_i)}$ no longer dominates the other terms. In particular, this always happens if $r \leqslant 2$. In such cases the value of $\mathrm{ed}(G)$ is controlled by the more subtle "lower order effects", which are poorly understood.

To state our next result, we will need the following terminology. Suppose that $2 \leqslant n_1 \leqslant \cdots \leqslant n_t$ and $z = (z_1, \dots, z_r) \in (\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z})$, where $z_{j_1}, \dots, z_{j_s} \neq 0$ for some $1 \leqslant j_1 < \cdots < j_s \leqslant r$ and $z_j = 0$ for any $j \notin \{j_1, \dots, j_r\}$. We will say that z is *balanced* if

(i) $n_{j_s} \leqslant \frac{1}{2} n_{j_1} n_{j_2} \cdots n_{j_{s-1}}$ and

(ii) $(n_{j_1}, \dots, n_{j_s}) \neq (2,2,2,2), (3,3,3)$ or $(2, n, n)$ for any $n \geqslant 2$.

Note that condition (i) can only hold if $s \geqslant 3$. In particular, $(\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z})$ has no balanced elements if $r \leqslant 2$. In the sequel, we will usually assume that $n_1, \dots, n_r$ are powers of the same prime p. In this situation, condition (ii) is vacuous, unless $p = 2$ or 3.

Theorem 1.2. *Let p be a prime,*

$$G := (\mathrm{GL}_{p^{a_1}} \times \cdots \times \mathrm{GL}_{p^{a_r}})/C,$$

where $a_r \geqslant \cdots \geqslant a_1 \geqslant 1$ are integers, and C is a subgroup of μ, as in (7). Assume that the base field k is of characteristic zero and $\mathrm{Code}(C)$ *has a minimal basis $y_i = (y_{i1}, \dots, y_{ir})$, $i = 1, \dots, t$ satisfying the following conditions*:

(a) *$y_{ij} = -1, 0$ or 1 in $\mathbb{Z}/p^{a_j}\mathbb{Z}$ for every $i = 1, \dots, t$ and $j = 1, \dots, r$.*

(b) *For every $j = 1, \dots, r$, there exists an $i \in \{1, \dots, t\}$ such that y_i is balanced and $y_{ij} \neq 0$.*

Then $\mathrm{ed}(G) = \mathrm{ed}_p(G) = \left(\sum_{i=1}^{t} p^{\mathrm{w}(y_i)}\right) - p^{2a_1} - \cdots - p^{2a_r} + r - t.$

Specializing Theorem 1.2 to the case where $\mathrm{Code}(C)$ is generated by the single element $(1, \dots, 1)$, we obtain the following.

Theorem 1.3. *Let $a_r \geqslant a_{r-1} \geqslant \cdots \geqslant a_1 \geqslant 1$ be integers and $\mathcal{F}\colon \mathsf{Fields}_k \to \mathsf{Sets}$ be the covariant functor where $\mathcal{F}(K)$ is defined as the set of isomorphism classes of r-tuples $(A_1, \dots, A_r)$ of central simple K-algebras such that $\deg(A_i) = p^{a_i}$ for all $i = 1, \dots, r$, and $A_1 \otimes \cdots \otimes A_r$ is split over K.*

(a) *If $a_r \geqslant a_1 + \cdots + a_{r-1}$, then $\mathrm{ed}(\mathcal{F}) = \mathrm{ed}(\mathrm{PGL}_{p^{a_1}} \times \cdots \times \mathrm{PGL}_{p^{a_{r-1}}})$ and*

$$\mathrm{ed}_p(\mathcal{F}) = \mathrm{ed}_p(\mathrm{PGL}_{p^{a_1}} \times \cdots \times \mathrm{PGL}_{p^{a_{r-1}}}).$$

In particular, $\mathrm{ed}(\mathcal{F}) \leqslant p^{2a_1} + \cdots + p^{2a_{r-1}}$.

(b) *Assume that $\mathrm{char}(k) = 0$, $a_r < a_1 + \cdots + a_{r-1}$, and $(p^{a_1}, \dots, p^{a_r})$ is not of the form $(2, 2, 2, 2)$, $(3, 3, 3)$ or $(2, 2^a, 2^a)$ for any $a \geqslant 1$. Then*

$$\mathrm{ed}(\mathcal{F}) = \mathrm{ed}_p(\mathcal{F}) = p^{a_1+\cdots+a_r} - \sum_{i=1}^{r} p^{2a_i} + r - 1. \tag{10}$$

(c) *If $(p^{a_1}, \dots, p^{a_r}) = (2, 2, 2)$, then $\mathrm{ed}(\mathcal{F}) = \mathrm{ed}_2(\mathcal{F}) = 3$.*

Here part (c) treats the smallest of the exceptional cases in part (b). Note that in this case $p = 2$, $r = 3$ and $a_1 = a_2 = a_3 = 1$. Thus

$$p^{a_1+\cdots+a_r} - \sum_{i=1}^{r} p^{2a_i} + r - 1 = -2,$$

and formula (10) fails. The values of $\mathrm{ed}(\mathcal{F})$ and $\mathrm{ed}_p(\mathcal{F})$ in the other exceptional cases, where $(p^{a_1}, \dots, p^{a_r}) = (2, 2, 2, 2)$, $(3, 3, 3)$, or $(2, 2^a, 2^a)$ for some $a \geqslant 2$, remain open.

The results of this paper naturally lead to combinatorial questions, which we believe to be of independent interest but will not address here. For each code (i.e., subgroup) $X \subset (\mathbb{Z}/p^{a_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{a_r}\mathbb{Z})$ of rank t, let $(w_1, \dots, w_t)$ be the minimal profile of X with respect to the Hamming weight function, in the sense of Proposition 3.1. That is, $w_i = \mathrm{w}(y_i)$, where $y_1, \dots, y_t$ is a minimal basis of X. Fixing p, $a_1 \leqslant \cdots \leqslant a_r$ and t, and letting X range over all possible codes with these parameters:

- What is the lexicographically largest profile $(w_1, \dots, w_t)$?
- What is the maximal value of w_t?
- What is the probability that $w_1 = \cdots = w_t$?
- What is the maximal value of $p^{w_1} + \cdots + p^{w_t}$?
- What is the average value of $p^{w_1} + \cdots + p^{w_t}$?
- What is the probability that $w_t > 2a_r$?

Note that the expression $p^{w_1} + \cdots + p^{w_t}$ appears in the formulas given in Theorem 1.1. For large p, the condition that $w_t > 2a_r$ makes $p^{w_1} + \cdots + p^{w_t}$ the dominant term in these formulas. To the best of our knowledge, questions of this type (focusing on the minimal profile of a code, rather than the minimal weight) have not been previously investigated by coding theorists, even in the case where $a_1 = \cdots = a_r = 1$.

The rest of this paper is structured as follows. In Section 2 we prove general bounds on the essential dimension of certain central extensions of algebraic groups. These bounds will serve as the starting point for the proofs of the main theorems. To make these bounds explicit for groups of the form $(\mathrm{GL}_{p^{a_1}} \times \cdots \times \mathrm{GL}_{p^{a_r}})/C$, we introduce and study the notion of a minimal basis in Section 3. Theorems 1.1, 1.2 and 1.3 are then proved in Sections 4, 5 and 6, respectively. The Appendix by Athena Nguyen contains an explicit description of the Galois cohomology of groups of the form (5). This description and its corollaries are used throughout the paper.

2. Essential dimension and central extensions

Let $T = \mathbb{G}_m^r$ be a split k-torus of rank r, and

$$1 \to T \to G \to \overline{G} \to 1 \tag{11}$$

be a central exact sequence of affine algebraic groups. This sequence gives rise to the exact sequence of pointed sets

$$H^1(K, G) \to H^1(K, \overline{G}) \xrightarrow{\partial} H^2(K, T)$$

for any field extension K of the base field k. Any character $x\colon T \to \mathbb{G}_m$, induces a homomorphism $x_*\colon H^2(K, T) \to H^2(K, \mathbb{G}_m)$. We define $\mathrm{ind}^x(G, T)$ as the maximal index of $x_* \circ \partial_K(E) \in H^2(K, T)$, where the maximum is taken over all field extensions K/k and over all $E \in H^1(K, \overline{G})$. In fact, this maximal value is always attained in the case where $E = E_{\mathrm{vers}} \to \mathrm{Spec}(K)$ is a versal G-torsor (for a suitable field K). That is,

$$\mathrm{ind}^x(G, T) = \mathrm{ind}(x_* \circ \partial_K(E_{\mathrm{vers}})) \tag{12}$$

for every $x \in X(T)$; see [Merkurjev 2013, Theorem 6.1]. Finally, we set

$$\mathrm{ind}(G, T) := \min\Big\{ \sum_{i=1}^{r} \mathrm{ind}^{x_i}(G, T) \;\Big|\; x_1, \ldots, x_r \text{ generate } X(T) \Big\}. \tag{13}$$

Our starting point for the proof of the main theorems is the following proposition.

Proposition 2.1. *Assume that the image of every $E \in H^1(K, \overline{G})$ under*

$$\partial\colon H^1(K, \overline{G}) \to H^2(K, T)$$

is p-torsion for every field extension K/k. Then

(a) $\mathrm{ed}_p(G) \geqslant \mathrm{ind}(G,T) - \dim(G)$,

(b) $\mathrm{ed}(G) \leqslant \mathrm{ind}(G,T) + \mathrm{ed}(\overline{G}) - r$ *and* $\mathrm{ed}_p(G) \leqslant \mathrm{ind}(G,T) + \mathrm{ed}_p(\overline{G}) - r$.

These bounds are variants of results that have previously appeared in the literature. Part (a) is a generalization of [Brosnan et al. 2011, Corollary 4.2] (where r is taken to be 1). In the case where T is μ_p^r, rather than $\mathbb{G}_m^r$, a variant of part (a) is proved in [Reichstein 2010, Theorem 4.1] (see also [Merkurjev 2013, Theorem 6.2]) and a variant of part (b) in [Merkurjev 2013, Corollaries 5.8 and 5.12].

Our proof of Proposition 2.1 proceeds along the same lines as these earlier proofs; it relies on the notions of essential and canonical dimension of a gerbe (for which we refer the reader to [Brosnan et al. 2011] and [Merkurjev 2013]), and the computation of the canonical dimension of a product of p-primary Brauer–Severi varieties in [Karpenko and Merkurjev 2008, Theorem 2.1]. In fact, the argument is easier for $T = \mathbb{G}_m^r$ than for μ_p^r. In the former case (which is of interest to us here), the essential dimension of a gerbe banded by T is readily expressible in terms of its canonical dimension (see formula (15) below), while an analogous formula for gerbes banded by μ_p^r requires a far greater effort to prove. (For $r = 1$, compare the proofs of parts (a) and (b) of [Brosnan et al. 2011, Theorem 4.1]. For arbitrary $r \geqslant 1$, see [Karpenko and Merkurjev 2008, Theorem 3.1] or [Merkurjev 2013, Theorem 5.11].)

Proof. If K/k is a field, and $E \in H^1(K,G)$, i.e., $E \to \mathrm{Spec}(K)$ is a $\overline{G}$-torsor, then the quotient stack $[E/G]$ is a gerbe over $\mathrm{Spec}(K)$ banded by T. By [Brosnan et al. 2011, Corollary 3.3] and [Merkurjev 2013, Corollary 5.7],

$$\mathrm{ed}(G) \geqslant \max_{K,E} \mathrm{ed}([E/G]) - \dim(\overline{G}) \quad \text{and} \quad \mathrm{ed}_p(G) \geqslant \max_{K,E} \mathrm{ed}_p([E/G]) - \dim(\overline{G}),$$

where the maximum is taken over all field extensions K/k and all $E \in H^1(K,\overline{G})$. On the other hand, by [Lötscher 2013, Example 3.4(i)],

$$\mathrm{ed}(G) \leqslant \mathrm{ed}(\overline{G}) + \max_{K,E} \mathrm{ed}([E/G]) \quad \text{and} \quad \mathrm{ed}_p(G) \leqslant \mathrm{ed}_p(\overline{G}) + \max_{K,E} \mathrm{ed}_p([E/G]);$$

see also [Merkurjev 2013, Corollary 5.8]. Since $\dim(G) = \dim(\overline{G}) + r$, it remains to show that

$$\max_{K,E} \mathrm{ed}([E/G]) = \max_{K,E} \mathrm{ed}_p([E/G]) = \mathrm{ind}(G,T) - r. \tag{14}$$

Choose a $\mathbb{Z}$-basis $x_1,\dots,x_r$ for the character group $X(T) \simeq \mathbb{Z}^r$ and let $P := P_1 \times \cdots \times P_r$, where P_i is the Brauer–Severi variety associated to $(x_i)_* \circ \partial(E) \in H^2(K,\mathbb{G}_m)$. Since T is a special group (i.e., every T-torsor over every field K/k is split), the set $[E/G](K)$ of isomorphism classes of K-points of $[E/G]$ consists of exactly one element if $P(K) \neq \varnothing$ and is empty otherwise. Thus

$$\mathrm{ed}([E/G]) = \mathrm{cdim}(P) \quad \text{and} \quad \mathrm{ed}_p([E/G]) = \mathrm{cdim}_p(P), \tag{15}$$

where $\mathrm{cdim}(P)$ denotes the canonical dimension of P. (The same argument is used in the proof of [Brosnan et al. 2011, Theorem 4.1(a)] in the case where $r = 1$.) Since we are assuming that $\partial(E)$ is p-torsion, the index of each Brauer–Severi variety P_i is a power of p. Thus by [Karpenko and Merkurjev 2008, Theorem 2.1],

$$\mathrm{cdim}(P) = \mathrm{cdim}_p(P) = \min\Big\{ \sum_{i=1}^{r} \mathrm{ind}\big((x_i)_* \circ \partial_K(E)\big) \,\Big|\, x_1, \dots, x_r \text{ generate } X(T) \Big\} - r;$$

see also [Merkurjev 2013, Theorem 4.14]. Taking $E := E_{\mathrm{vers}}$ to be a versal $\overline{G}$-torsor, we obtain

$$\mathrm{cdim}(P) = \mathrm{cdim}_p(P) = \min\Big\{ \sum_{i=1}^{r} \mathrm{ind}^{x_i}(G, T)) \,\Big|\, x_1, \dots, x_r \text{ generate } X(T) \Big\} - r;$$

see (12). By the definition (13) of $\mathrm{ind}(G, T)$, the last formula can be rewritten as $\mathrm{cdim}(P) = \mathrm{cdim}_p(P) = \mathrm{ind}(G, T) - r$. Combining these equalities with (15), we obtain (14). □

Remark 2.2. Our strategy for proving Theorem 1.1 will be to apply Proposition 2.1 to the exact sequence (11) with $G = (\mathrm{GL}_{p^{a_1}} \times \cdots \times \mathrm{GL}_{p^{a_r}})/C$, and $T := \mathbb{G}_m^r/C$. The only remaining issue is to find an expression for $\mathrm{ind}(G, T)$ in terms of $\mathrm{Code}(C)$.

Usually, the term $\mathrm{ind}(G, T)$ is computed using the formula

$$\mathrm{ind}^x(G, T) = \gcd \dim(\rho),$$

as $\rho\colon G \to \mathrm{GL}(V)$ ranges over all finite-dimensional representations of G such that $\tau \in T$ acts on V via scalar multiplication by $x(\tau)$. See, for example, [Karpenko and Merkurjev 2008, Theorem 4.4] or [Merkurjev 2013, Theorem 6.1] or [Lötscher et al. 2013, Theorem 3.1]. We will not use this approach in the present paper. Instead, we will compute the values of $\mathrm{ind}^x(G, T)$ and $\mathrm{ind}(G, T)$ directly from the definition, using the description of the connecting map $\partial\colon H^1(K, \overline{G}) \to H^2(K, T)$ given by Theorem A.1; see the proof of Proposition 4.1 below.

3. Minimal bases

To carry out the program outlined in Remark 2.2, we will need the notion of a *minimal basis*. This section will be devoted to developing this notion.

The general setting is as follows. Let R be a local ring with maximal ideal $I \subset R$ and A be a finitely generated R-module. We will refer to a generating set $S \subset A$ as a *basis* if no proper subset of S generates A. In the sequel we will specialize R to $\mathbb{Z}/p^a\mathbb{Z}$ and A to a submodule of $(\mathbb{Z}/p^{a_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{a_t}\mathbb{Z})$, where

$a = \max(a_1, \dots, a_r)$. However, in this section it will be convenient for us to work over an arbitrary local ring R.

Let $\pi \colon A \to A/IA$ be the natural projection. We will repeatedly appeal to Nakayama's lemma, which asserts that a subset $S \subset A$ generates A as an R-module if and only if $\pi(S)$ generates A/IA as an R/I-vector space; see [Lang 2002, Section X.4].

By a *weight function* on A we shall mean any function $w : A \to \mathbb{N}$, where $\mathbb{N}$ denotes the set of nonnegative integers. We will fix w throughout and will sometimes refer to $\mathrm{w}(y)$ as *the weight* of $y \in A$. For each basis $B = \{y_1, \dots, y_t\}$ of A, we will define the *profile* of B as

$$\mathrm{w}(B) := (\mathrm{w}(y_1), \dots, \mathrm{w}(y_t)) \in \mathbb{N}^t,$$

where $y_1, \dots, y_t$ are ordered so that $\mathrm{w}(y_1) \leqslant \mathrm{w}(y_2) \leqslant \cdots \leqslant \mathrm{w}(y_t)$. Let $\mathrm{Prof}(A) \subset \mathbb{N}^t$ denote the set of profiles of bases of A.

Proposition 3.1. $\mathrm{Prof}(A)$ *has a unique minimal element with respect to the partial order on* $\mathbb{N}^t$ *given by* $(\alpha_1, \dots, \alpha_t) \preceq (\beta_1, \dots, \beta_t)$ *if* $\alpha_i \leqslant \beta_i$ *for every* $i = 1, \dots, t$.

Note that since every descending chain in $(\mathrm{Prof}(A), \preceq)$ terminates, the unique minimal element is comparable to every element of $\mathrm{Prof}(A)$.

Proof. We argue by contradiction. Set $t := \dim(A/IA)$. Suppose $X = \{x_1, \dots, x_t\}$ and $Y = \{y_1, \dots, y_t\}$ are bases of A such that $\mathrm{w}(X)$ and $\mathrm{w}(Y)$ are distinct minimal elements of $\mathrm{Prof}(A)$. Let us order X and Y so that $\mathrm{w}(x_1) \leqslant \cdots \leqslant \mathrm{w}(x_t)$ and $\mathrm{w}(y_1) \leqslant \cdots \leqslant \mathrm{w}(y_t)$. Since $\mathrm{w}(X) \neq \mathrm{w}(Y)$, there exists an s between 0 and $t-1$ such that

$$\mathrm{w}(x_i) = \mathrm{w}(y_i) \quad \text{for all } i = 1, \dots, s,$$

but $\mathrm{w}(x_{s+1}) \neq \mathrm{w}(y_{s+1})$. After possibly interchanging X and Y, we may assume without loss of generality that $\mathrm{w}(x_{s+1}) < \mathrm{w}(y_{s+1})$.

Let $\pi \colon A \to A/IA$ be the natural projection, as above. By Nakayama's lemma, $\pi(x_1), \dots, \pi(x_{s+1})$ are R/I-linearly independent in A/IA. Choose $t-s-1$ elements of Y, say $y_{j_{s+2}}, \dots, y_{j_t}$, such that $\pi(x_1), \dots, \pi(x_{s+1}), \pi(y_{j_{s+2}}), \dots, \pi(y_{j_t})$ form an R/I-basis of A/IA. After permuting $y_{j_{s+2}}, \dots, y_{j_t}$, we may assume that $\mathrm{w}(y_{j_{s+2}}) \leqslant \cdots \leqslant \mathrm{w}(y_{j_t})$. Applying Nakayama's lemma once again, we see that $Z = \{x_1, \dots, x_{s+1}, y_{j_{s+2}}, \dots, y_{j_t}\}$ is a basis of A.

We claim that $\mathrm{w}(Z) \prec \mathrm{w}(Y)$, where the inequality is strict. Since we assumed that $\mathrm{w}(Y)$ is minimal in $\mathrm{Prof}(A)$, this claim leads to a contradiction, thus completing the proof of Proposition 3.1.

To prove the claim, let $z_1, \dots, z_t$ be the elements of Z, in increasing order of their weight: $\mathrm{w}(z_1) \leqslant \mathrm{w}(z_2) \leqslant \cdots \leqslant \mathrm{w}(z_t)$. It suffices to show that $\mathrm{w}(z_i) \leqslant \mathrm{w}(y_i)$ for every $i = 1, \dots, t$, and $\mathrm{w}(z_{s+1}) < \mathrm{w}(y_{s+1})$. Let us consider three cases.

Case 1: $i \leqslant s$. Since

$$\mathrm{w}(x_1) = \mathrm{w}(y_1) \leqslant \mathrm{w}(x_2) = \mathrm{w}(y_2) \leqslant \cdots \leqslant \mathrm{w}(x_i) = \mathrm{w}(y_i),$$

Z has at least i elements whose weight is at most $\mathrm{w}(y_i)$, namely $x_1, \ldots, x_i$. Thus $\mathrm{w}(z_i) \leqslant \mathrm{w}(y_i)$.

Case 2: $i = s+1$. Z has at least $s+1$ elements, namely $x_1, \ldots, x_{s+1}$ whose weight is at most $\mathrm{w}(x_{s+1})$. Hence, $\mathrm{w}(z_{s+1}) \leqslant \mathrm{w}(x_{s+1}) < \mathrm{w}(y_{s+1})$, as desired.

Case 3: $i > s+1$. Recall that both $y_1, \ldots, y_t$ and $y_{j_{s+2}}, \ldots, y_{j_t}$ are arranged in weight-increasing order. For any $i \geqslant s+2$, there are at least $t-i+1$ elements of Y whose weight is at least $\mathrm{w}(y_{j_i})$, namely $y_{j_i}, y_{j_{i+1}}, \ldots, y_{j_t}$. Thus

$$\mathrm{w}(y_{j_i}) \leqslant \mathrm{w}(y_i)$$

for any $i = s+2, \ldots, t$. Consequently, Z has at least i elements of weight at most $\mathrm{w}(y_i)$, namely $x_1, \ldots, x_{s+1}, y_{j_{s+2}}, \ldots, y_{j_i}$. Hence, $\mathrm{w}(z_i) \leqslant \mathrm{w}(y_i)$, as desired.

This completes the proof of the claim and hence of Proposition 3.1. □

Definition 3.2. We will say that a basis $y_1, \ldots, y_t$ of A is *minimal* if its profile is the minimal element of $\mathrm{Prof}(A)$, as in Proposition 3.1. Note that a minimal basis in A is usually not unique; however, any two minimal bases have the same profile in $\mathbb{N}^t$.

Remark 3.3. We can construct a minimal basis of A using the following "greedy algorithm". Select $y_1 \in A$ of minimal weight, subject to the condition that $\pi(y_1) \neq 0$. Next select y_2 of minimal weight, subject to the condition that $\pi(y_1)$ and $\pi(y_2)$ are R/I-linear independent in A/IA. Then select y_3 of minimal weight, subject to the condition that $\pi(y_1), \pi(y_2)$ and $\pi(y_3)$ are R/I-linear independent in A/IA. Continue recursively. After $t = \dim_{R/I}(A/IA)$ steps, we obtain a minimal basis $y_1, \ldots, y_t$ for A.

Example 3.4. Set $R := \mathbb{F}_p$, $I := (0)$, G a finite p-group, $D := Z(G)[p]$ the subgroup of p-torsion elements of the center $Z(G)$, and $A := X(D)$ the group of characters of D. For $x \in A$, define $\mathrm{w}(x)$ to be the minimal dimension of a representation $G \to \mathrm{GL}(V_x)$, such that D acts on V_x via scalar multiplication by x. If $\{x_1, \ldots, x_t\}$ is a minimal basis of A, then $V_{x_1} \oplus \cdots \oplus V_{x_t}$ is a faithful representation of G of minimal dimension; see [Karpenko and Merkurjev 2008, Remark 4.7].

4. Conclusion of the proof of Theorem 1.1

Recall that we are interested in the essential dimension of the group

$$G = (\mathrm{GL}_{p^{a_1}} \times \cdots \times \mathrm{GL}_{p^{a_r}})/C,$$

where C is a subgroup of $\mu := \mu_{p^{a_1}} \times \cdots \times \mu_{p^{a_r}}$. We will think of the group of characters $X(\mathbb{G}_m^r)$ as $\mathbb{Z}^r$ by identifying the character $x(\tau_1, \ldots, \tau_r) = \tau_1^{m_1} \cdots \tau_r^{m_r}$

with $(m_1, \dots, m_r) \in \mathbb{Z}^r$. Characters of $T := \mathbb{G}_m^r/C$ are identified in this manner with the r-tuples $(m_1, \dots, m_r) \in \mathbb{Z}^r$ such that $\tau_1^{m_1} \cdots \tau_r^{m_r} = 1$ for every $(\tau_1, \dots, \tau_r) \in C$. The relationship among these character groups is illustrated by the following diagram:

$$\begin{tikzcd} & X(\mathbb{G}_m^r/C) \arrow[r, hook] \arrow[d] & X(\mathbb{G}_m^r) \arrow[r, equal] \arrow[d, "\pi"] & \mathbb{Z} \times \cdots \times \mathbb{Z} \ (r \text{ times}) \arrow[d, "\pi"] \\ \mathrm{Code}(C) \arrow[r, equal] & X(\mu/C) \arrow[r, hook] & X(\mu) \arrow[r, equal] & (\mathbb{Z}/p^{a_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{a_r}\mathbb{Z}) \end{tikzcd}$$

Here $\mathrm{Code}(C)$ is as in (8) and π is the natural projection, given by restricting a character from $\mathbb{G}_m^r$ to μ.

Our proof of Theorem 1.1 will be based on the strategy outlined in Remark 2.2. In view of Proposition 2.1 it suffices to establish the following:

Proposition 4.1. *Consider the central exact sequence*

$$1 \to T \to G \to \overline{G} \to 1, \tag{16}$$

where $G = (\mathrm{GL}_{p^{a_1}} \times \cdots \times \mathrm{GL}_{p^{a_r}})/C$, C *is a subgroup of* $\mu := \mu_{p^{a_1}} \times \cdots \times \mu_{p^{a_r}}$, $T := \mathbb{G}_m^r/C$ *and* $\overline{G} := \mathrm{PGL}_{p^{a_1}} \times \cdots \times \mathrm{PGL}_{p^{a_r}}$.

(a) *If* $x \in X(T)$ *and* $y = \pi(x) \in \mathrm{Code}(C)$ *then* $\mathrm{ind}^x(G, T) = p^{\mathrm{w}(y)}$.

(b) $\mathrm{ind}(G, T) = p^{\mathrm{w}(z_1)} + \cdots + p^{\mathrm{w}(z_t)} + r - t$, *where* $z_1, \dots, z_t$ *is a minimal basis of* $\mathrm{Code}(C)$.

Proof of Proposition 4.1(a). Consider the connecting map $\partial\colon H^1(K, \overline{G}) \to H^2(K, T)$ associated to the central exact sequence (16). Given a character $x\colon T \to \mathbb{G}_m$, $x(\tau_1, \dots, \tau_r) = \tau_1^{m_1} \cdots \tau_r^{m_r}$, $\mathrm{ind}^x(G, T)$ is, by definition, the maximal value of $\mathrm{ind}(x_*\partial(E))$, as K ranges over all fields containing k and E ranges over $H^1(K, \overline{G})$. In this case, $\overline{G} = \mathrm{PGL}_{p^{a_1}} \times \cdots \times \mathrm{PGL}_{p^{a_r}}$, and thus $H^1(K, \overline{G})$ is the set of r-tuples $(A_1, \dots, A_r)$ of central simple algebras, where the degree of A_i is p^{a_i}. The group $H^2(K, \mathbb{G}_m)$ is naturally identified with the Brauer group $\mathrm{Br}(K)$, and the map $x_*\partial$ takes an r-tuple $(A_1, \dots, A_r)$, as above, to the Brauer class of $A := A_1^{\otimes m_1} \otimes \cdots \otimes A_r^{\otimes m_r}$.

Since $\deg(A_i) = p^{a_i}$, the Brauer class of A depends only on

$$y = \pi(x) = (m_1 \bmod p^{a_1}, \dots, m_r \bmod p^{a_r}) \in (\mathbb{Z}/p^{a_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{a_r}\mathbb{Z}).$$

Moreover, if $m_i \equiv u_i p^{e_i} \pmod{p^{a_i}}$, where u_i is prime to p and $0 \leqslant e_i \leqslant a_i$, then $\mathrm{ind}(A_i^{\otimes m_i}) \leqslant p^{a_i - e_i}$. Now recall that $\mathrm{w}(y)$ is defined as $\sum_{i=1}^r (a_i - e_i)$. Thus

$$\mathrm{ind}(A) \leqslant \prod_{i=1}^r \mathrm{ind}(A_i^{\otimes m_i}) \leqslant \prod_{i=1}^r p^{a_i - e_i} = p^{\mathrm{w}(y)}.$$

To prove the opposite inequality, we set A_i to be the symbol algebra $(\alpha_i, \beta_i)_{p^{a_i}}$, over the field $K = k(\zeta)(\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r)$, where ζ is a primitive root of unity of degree $p^{\max(a_1,\dots,a_r)}$ and $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r$ are $2r$ independent variables over k. Writing $m_i = u_i p^{e_i}$, as above, we see that $A_i^{\otimes m_i}$ is Brauer equivalent to $B_i := (\alpha_i, \beta_i^{u_i})_{p^{a_i - e_i}}$ over K. An easy valuation-theoretic argument shows that $B := B_1 \otimes_K \cdots \otimes_K B_t$ is a division algebra. (In particular, the norm form of B is a Pfister polynomial and hence, is anisotropic; see [Reichstein 1999, Theorem 3.2 and Proposition 3.4].) Thus

$$\operatorname{ind}(A) = \operatorname{ind}(B) = \operatorname{ind}(B_1) \cdot \cdots \cdot \operatorname{ind}(B_t) = p^{(a_1 - e_1) + \cdots + (a_t - e_t)} = p^{\mathrm{w}(y)},$$

as desired. We conclude that $\operatorname{ind}^x(G, T) \geqslant \operatorname{ind}(A) = p^{\mathrm{w}(y)}$, thus completing the proof of Proposition 4.1(a). □

Our proof of Proposition 4.1(b) will rely on the following elementary lemma.

Lemma 4.2. *Let p be a prime, M be a finite abelian p-group, and $f : \mathbb{Z}^n \to M$ be a surjective $\mathbb{Z}$-module homomorphism for some $n \geqslant 1$. Then for every basis $y_1, \dots, y_t$ of M, there exists a $\mathbb{Z}$-basis $x_1, \dots, x_n$ of $\mathbb{Z}^n$ and an integer c prime to p such that $f(x_1) = cy_1, f(x_2) = y_2, \dots, f(x_t) = y_t$ and $f(x_{t+1}) = \cdots = f(x_n) = 0$.*

Proof. By [Lang 2002, Theorem III.7.8], there exists a basis $e_1, \dots, e_n$ of $\mathbb{Z}^n$ such that $\operatorname{Ker}(f)$ is generated by $p^{d_i} e_i$ for some integers $d_1, \dots, d_t \geqslant 0$. Since M has rank t, we may assume without loss of generality that $d_1, \dots, d_t \geqslant 1$ and $d_{t+1} = \cdots = d_n = 0$. That is, we may identify M with $(\mathbb{Z}/p^{d_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{d_t}\mathbb{Z})$ and assume that

$$f(r_1, \dots, r_n) = (r_1 \bmod p^{d_1}, \dots, r_t \bmod p^{d_t}) \quad \forall (r_1, \dots, r_n) \in \mathbb{Z}^n.$$

It now suffices to lift $cy_1, \dots, y_t \in M$ to a basis $x_1, \dots, x_t$ of $\mathbb{Z}^t$ for a suitable integer c, prime to p. Indeed, if we manage to do this, then we will obtain a basis of $\mathbb{Z}^n$ of the desired form by appending

$$x_{t+1} := e_{t+1}, \dots, x_n := e_n \in \operatorname{Ker}(f)$$

to $x_1, \dots, x_t$. Thus we may assume that $n = t$.

Now observe that $f : \mathbb{Z}^n \to M$, factors as $\mathbb{Z}^n \to (\mathbb{Z}/p^d\mathbb{Z})^n \to M$, where $d := \max(d_1, \dots, d_t)$. Lift each $y_i \in M$ to some $y_i' \in (\mathbb{Z}/p^d\mathbb{Z})^n$. By Nakayama's lemma, $y_1', \dots, y_n'$ form a $\mathbb{Z}/p^d\mathbb{Z}$-basis of $(\mathbb{Z}/p^d\mathbb{Z})^n$. It now suffices to lift $cy_1', y_2', \dots, y_n'$ to a basis of $\mathbb{Z}^n$ for a suitable integer c, prime to p. In other words, we may assume without loss of generality that $M = (\mathbb{Z}/p^d\mathbb{Z})^n$, and $f : \mathbb{Z}^n \to (\mathbb{Z}/p^d\mathbb{Z})^n$ is the natural projection.

Now suppose $y_i = (y_{i1}, \dots, y_{in})$ for some $y_{ij} \in \mathbb{Z}/p^d\mathbb{Z}$. Since $y_1, \dots, y_m$ form a basis of $(\mathbb{Z}/p^d\mathbb{Z})^n$, the matrix $A = (y_{ij})$ is invertible, i.e., $A \in \mathrm{GL}_n(\mathbb{Z}/p^d\mathbb{Z})$.

After rescaling y_1 by $c := \det(A)^{-1}$ in $\mathbb{Z}/p^d\mathbb{Z}$, we may assume that $\det(A) = 1$. The lemma now follows from the surjectivity of the natural projection

$$\mathrm{SL}_t(\mathbb{Z}) \to \mathrm{SL}_t(\mathbb{Z}/p^d\mathbb{Z});$$

see [Shimura 1971, Lemma 1.38]. □

Proof of Proposition 4.1(b). By definition, $\mathrm{ind}(G,T)$ is the minimal value of $\mathrm{ind}^{x_1}(G,T) + \cdots + \mathrm{ind}^{x_r}(G,T)$, as $x_1, \ldots, x_r$ range over the bases of $X(T) \subset \mathbb{Z}^r$. By part (a), we can rewrite this as

$$\mathrm{ind}(G,T) = \min\{p^{\mathrm{w}(\pi(x_1))} + \cdots + p^{\mathrm{w}(\pi(x_r))} \mid x_1, \ldots, x_r \text{is a } \mathbb{Z}\text{-basis of } X(T)\}.$$

Here, as before, $\pi(x_i) \in \mathrm{Code}(C)$ is the restriction of x_i from $T = \mathbb{G}_m^r/C$ to μ/C.

Let $z_1, \ldots, z_t \in \mathrm{Code}(C)$ be a minimal basis, as in the statement of the proposition. We will prove part (b) by showing that

(i) $p^{\mathrm{w}(\pi(x_1))} + \cdots + p^{\mathrm{w}(\pi(x_r))} \geqslant p^{\mathrm{w}(z_1)} + \cdots + p^{\mathrm{w}(z_t)} + r - t$ for every $\mathbb{Z}$-basis $x_1, \ldots, x_r$ of $X(T)$, and

(ii) there exists a particular $\mathbb{Z}$-basis $x_1, \ldots, x_r$ of $X(T)$ such that $p^{\mathrm{w}(\pi(x_1))} + \cdots + p^{\mathrm{w}(\pi(x_r))} = p^{\mathrm{w}(z_1)} + \cdots + p^{\mathrm{w}(z_t)} + r - t$.

To prove (i), note that if $x_1, \ldots, x_r$ form a $\mathbb{Z}$-basis of $X(T)$, then $\pi(x_1), \ldots, \pi(x_r)$ form a generating set for $\mathrm{Code}(C)$. By Nakayama's lemma, every generating set for $\mathrm{Code}(C)$ contains a basis. After renumbering $x_1, \ldots, x_r$, we may assume that $\pi(x_1), \ldots, \pi(x_t)$ is a basis of $\mathrm{Code}(C)$ and $\mathrm{w}(\pi(x_1)) \leqslant \cdots \leqslant \mathrm{w}(\pi(x_t))$. By Proposition 3.1, $\mathrm{w}(z_i) \leqslant \mathrm{w}(\pi(x_i))$ for every $i = 1, \ldots, t$. Thus

$$p^{\mathrm{w}(\pi(x_1))} + \cdots + p^{\mathrm{w}(\pi(x_r))} \geqslant p^{\mathrm{w}(\pi(x_1))} + \cdots + p^{\mathrm{w}(\pi(x_t))} + \underbrace{p^0 + \cdots + p^0}_{r-t \text{ times}} \geqslant p^{\mathrm{w}(z_1)} + \cdots + p^{\mathrm{w}(z_t)} + r - t.$$

To prove (ii), recall that by Lemma 4.2 there exists an integer c, prime to p, and a $\mathbb{Z}$-basis $x_1, \ldots, x_r$ of $X(T)$ such that $\pi(x_1) = cz_1, \pi(x_2) = z_2, \ldots, \pi(x_t) = z_t$, and $\pi(x_{t+1}) = \cdots = \pi(x_r) = 0$. Since c is prime to p, $\mathrm{w}(cz_1) = \mathrm{w}(z_1)$. Thus for this particular choice of $x_1, \ldots, x_r$, we have

$$p^{\mathrm{w}(\pi(x_1))} + \cdots + p^{\mathrm{w}(\pi(x_r))} = p^{\mathrm{w}(cz_1)} + p^{\mathrm{w}(z_2)} + \cdots + p^{\mathrm{w}(z_t)} + \underbrace{p^0 + \cdots + p^0}_{r-t \text{ times}} = p^{\mathrm{w}(z_1)} + \cdots + p^{\mathrm{w}(z_t)} + r - t,$$

as desired. □

5. Proof of Theorem 1.2

Consider the action of a linear algebraic group Γ on an absolutely irreducible algebraic variety X defined over k. We say that a subgroup $S \subset \Gamma$ is a *stabilizer in general position* for this action if there exists a dense open subset $U \subset X$ such that the scheme-theoretic stabilizer $\mathrm{Stab}_\Gamma(x)$ is conjugate to S over $\bar{k}$ for every $x \in U(\bar{k})$. Here, as usual, $\bar{k}$ denotes the algebraic closure of k. In the sequel we will not specify U and will simply say that $\mathrm{Stab}_\Gamma(x)$ is conjugate to S for $x \in X(\bar{k})$ in general position. Note that a stabilizer in general position S for a Γ-action on X does not always exist, and when it does, it is usually not unique. However, over $\bar{k}$, S is unique up to conjugacy.

For the rest of this section we will always assume that $\mathrm{char}(k) = 0$. A theorem of R. W. Richardson [1972] tells us that under this assumption every linear action of a reductive group Γ on a vector space V has a stabilizer $S \subset \Gamma$ in general position. Note that in [Richardson 1972], k is assumed to be algebraically closed. Thus a priori the subgroup S and the open subset $U \subset V$, where all stabilizers are conjugate to S, are only defined over $\bar{k}$. However, U has only finitely many Galois translates. After replacing U by the intersection of all of these translates, we may assume that U is defined over k. Moreover, we may take $S := \mathrm{Stab}_G(x)$ for some k-point $x \in U(k)$ and thus assume that S is defined over k. For a detailed discussion of stabilizers in general position over an algebraically closed field of characteristic zero, see [Popov and Vinberg 1994, Section 7].

We will say that a Γ-action on X is *generically free* if the trivial subgroup $S = \{1_\Gamma\} \subset \Gamma$ is a stabilizer in general position for this action.

Lemma 5.1. *Let Γ be a reductive linear algebraic group and $\rho\colon \Gamma \to \mathrm{GL}(V)$ be a finite-dimensional representation. If $\mathrm{Stab}_\Gamma(v)$ is central in Γ for $v \in V$ in general position, then the induced action of $\Gamma/\operatorname{Ker}(\rho)$ on V is generically free.*

Proof. Let $S \subset \Gamma$ be the stabilizer in general position for the Γ-action on V. Clearly $\operatorname{Ker}(\rho) \subset S$. We claim that, in fact, $\operatorname{Ker}(\rho) = S$; the lemma easily follows from this claim.

To prove the opposite inclusion, $S \subset \operatorname{Ker}(\rho)$, note that under the assumption of the lemma, S is central in Γ. Let $U \subset V$ be a dense open subset such that the stabilizer of every $v \in U(\bar{k})$ is conjugate to S. Since S is central, $\mathrm{Stab}_\Gamma(v)$ is, in fact, equal to S. In other words, S stabilizes every point in U and thus every point in V. That is, $S \subset \operatorname{Ker}(\rho)$, as claimed. □

Our interest in generically free actions in this section has to do with the following fact: if there exists a generically free linear representation $G \to \mathrm{GL}(V)$ then

$$\mathrm{ed}(G) \leqslant \dim(V) - \dim(G); \tag{17}$$

see, e.g., [Reichstein 2010, (2.3)] or [Merkurjev 2013, Proposition 3.13]. This inequality will play a key role in our proof of Theorem 1.2.

Now set $\Gamma := \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ and $\Gamma' := \mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r}$. Let V_i be the natural n_i-dimensional representation, V_i^{-1} be the dual representation, and V_i^{0} be the trivial 1-dimensional representation of GL_{n_i}. For $\epsilon = (\epsilon_1, \dots, \epsilon_r)$, where each ϵ_i is -1, 0 or 1, we define ρ_ϵ to be the natural representation of Γ on the tensor product

$$V_\epsilon = V_1^{\epsilon_1} \otimes \cdots \otimes V_r^{\epsilon_r}. \tag{18}$$

Lemma 5.2. *Suppose $2 \leqslant n_1 \leqslant \cdots \leqslant n_r \leqslant \frac{1}{2} n_1 \cdots n_{r-1}$, and*

$$(n_1, \dots, n_r) \neq (2,2,2,2), (3,3,3) \text{ or } (2,n,n) \quad \text{for any } n \geqslant 2.$$

If $\epsilon = (\epsilon_1, \dots, \epsilon_r) \in \{\pm 1\}^r$, then the induced action of $\Gamma / \operatorname{Ker}(\rho_\epsilon)$ on V_ϵ is generically free.

Proof. By Lemma 5.1 it suffices to prove the following claim: the stabilizer $\mathrm{Stab}_\Gamma(v)$ is central in Γ for $v \in V_\epsilon$ in general position. To prove this claim, we may assume without loss of generality that k is algebraically closed.

We first reduce to the case where $\epsilon = (1, \dots, 1)$. Suppose the claim is true in this case, and let $(\epsilon_1, \dots, \epsilon_r) \in \{\pm 1\}^r$. By choosing bases of $V_1, \dots, V_r$, we can identify V_i with $V_i^{\epsilon_i}$ (we can take the identity map if $\epsilon_i = 1$). Define an automorphism

$$\sigma : \Gamma \to \Gamma,$$
$$(g_1, \dots, g_r) \mapsto (g_1^*, \dots, g_r^*),$$

where

$$g_i^* = \begin{cases} g_i & \text{if } \epsilon_i = 1, \\ (g_i^{-1})^T & \text{if } \epsilon_i = -1. \end{cases}$$

Now $\rho_{(\epsilon_1, \dots, \epsilon_r)}$ is isomorphic to the representation $\rho_{(1,\dots,1)} \circ \sigma$. Since the center of Γ is invariant under σ, we see that the claim holds for ρ_ϵ as well.

From now on we will assume $\epsilon = (1, \dots, 1)$. By [Popov 1987, Theorem 2],

$$\Gamma / Z(\Gamma) = \mathrm{PGL}_{n_1} \times \cdots \times \mathrm{PGL}_{n_r} = \Gamma' / Z(\Gamma')$$

acts generically freely on the projective space $\mathbb{P}(V_\epsilon) = V_\epsilon / Z(\Gamma)$. In other words, for $v \in V_\epsilon$ in general position, the stabilizer in Γ of the associated projective point $[v] \in \mathbb{P}(V_\epsilon)$ is trivial. Hence, the stabilizer of v is contained in $Z(\Gamma)$; see the exact sequence in [Reichstein and Vonessen 2007, Lemma 3.1]. This completes the proof of the claim and thus of Lemma 5.2. $\square$

We are now ready to proceed with the proof of Theorem 1.2. We begin by specializing n_i to p^{a_i} for every $i = 1, \dots, r$, so that Γ becomes $\mathrm{GL}_{p^{a_1}} \times \cdots \times \mathrm{GL}_{p^{a_r}}$. Let

$$y_1, \dots, y_t \in (\mathbb{Z}/p^{a_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{a_r}\mathbb{Z})$$

be a basis of $\mathrm{Code}(C)$ satisfying the conditions of Theorem 1.2. Lift each $y_i = (y_{i1}, \dots, y_{ir})$ to $x_i := (x_{i1}, \dots, x_{ir}) \in \mathbb{Z}^r$ by setting $x_{ij} := -1$, 0 or 1, depending on whether y_{ij} is -1, 0 or 1 in $\mathbb{Z}/p^{a_j}\mathbb{Z}$. (If $p^{a_j} = 2$, then we define each x_{ij} to be 0 or 1.) By Nakayama's lemma, the images of $y_1, \dots, y_t$ are $\mathbb{F}_p$-linearly independent in $\mathrm{Code}(C)/p\,\mathrm{Code}(C)$. Thus the integer vectors $x_1, \dots, x_t$ are $\mathbb{Z}$-linearly independent. (Note that, unlike in the situation of Lemma 4.2, here it will not matter to us whether $x_1, \dots, x_t$ can be completed to a $\mathbb{Z}$-basis of $\mathbb{Z}^r$.) We view each x_i as a character $\mathbb{G}_m^r \to \mathbb{G}_m$ and set

$$\widetilde{C} := \mathrm{Ker}(x_1) \cap \cdots \cap \mathrm{Ker}(x_t) \subset \mathbb{G}_m^r.$$

Since $x_1, \dots, x_t$ are linearly independent,

$$\dim(\widetilde{C}) = r - t. \tag{19}$$

Set $G := \Gamma/C$ and $\widetilde{G} := \Gamma/\widetilde{C}$. By our construction, $\widetilde{C} \cap \mu = C$. Corollary A.2 now tells us that $\mathrm{ed}_p(G) \leqslant \mathrm{ed}(G) = \mathrm{ed}(\widetilde{G})$. By Theorem 1.1(a),

$$\mathrm{ed}(G) \geqslant \mathrm{ed}_p(G) \geqslant \left(\sum_{i=1}^{t} p^{\mathrm{w}(y_i)}\right) - p^{2a_1} - \cdots - p^{2a_r} + r - t.$$

It thus suffices to show that $\mathrm{ed}(\widetilde{G}) \leqslant \left(\sum_{i=1}^{t} p^{\mathrm{w}(y_i)}\right) - p^{2a_1} - \cdots - p^{2a_r} + r - t$, or equivalently,

$$\mathrm{ed}(\widetilde{G}) \leqslant \left(\sum_{i=1}^{t} p^{\mathrm{w}(y_i)}\right) - \dim(\widetilde{G});$$

see (19). By (17), in order to prove the last inequality, it is enough to construct a generically free linear representation of $\widetilde{G}$ of dimension $\sum_{i=1}^{t} p^{\mathrm{w}(y_i)}$. Such a representation is furnished by the lemma below.

Recall that $x_i = (x_{i1}, \dots, x_{ir}) \in \mathbb{Z}^r$, where each $x_{ij} = -1$, 0 or 1, and ρ_{x_i} is the natural representation of $\Gamma := \mathrm{GL}_{p^{a_1}} \times \cdots \times \mathrm{GL}_{p^{a_r}}$ on $V_{x_i} := V_1^{x_{i1}} \otimes \cdots \otimes V_r^{x_{ir}}$, as in (18), with $\dim(V_i) = n_i = p^{a_i}$.

Lemma 5.3. *Let* $V = V_{x_1} \oplus \cdots \oplus V_{x_t}$ *and* $\rho := \rho_{x_1} \oplus \cdots \oplus \rho_{x_t} \colon \Gamma \to \mathrm{GL}(V)$. *Then*

(a) $\dim(V) = p^{\mathrm{w}(y_1)} + \cdots + p^{\mathrm{w}(y_t)}$,

(b) $\mathrm{Ker}(\rho) = \widetilde{C}$, *and*

(c) *the induced action of* $\widetilde{G} = \Gamma/\widetilde{C}$ *on* V *is generically free.*

Proof. For each $i = 1, \dots t$, we have

$$\dim(V_{x_i}) = \prod_{x_{ij} \neq 0} p^{a_j} = \prod_{y_{ij} \neq 0} p^{a_j} = p^{\sum_{y_{ij} \neq 0} a_j}.$$

Since each $y_{ij} = -1$, 0 or 1, $\sum_{y_{ij} \neq 0} a_j = \mathrm{w}(y_i)$. Thus $\dim(V_{x_i}) = p^{\mathrm{w}(y_i)}$, and part (a) follows.

Now choose $v_i \in V_{x_i}$ in general position and set $v := (v_1, \dots, v_r)$. We claim that $\mathrm{Stab}_\Gamma(v)$ is central in Γ.

Suppose for a moment that this claim is established. Since the center $Z(\Gamma) = \mathbb{G}_m^r$ acts on V_{x_i} via scalar multiplication by the character $x_i \colon \mathbb{G}_m^r \to \mathbb{G}_m$, we see that

$$\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho_{|\mathbb{G}_m^r}) = \mathrm{Ker}(x_1) \cap \cdots \cap \mathrm{Ker}(x_t) = \widetilde{C},$$

and part (b) follows. Moreover, by Lemma 5.1, the induced action of $\Gamma / \mathrm{Ker}(\rho)$ on V is generically free. By part (b), $\mathrm{Ker}(\rho) = \widetilde{C}$ and part (c) follows as well.

It remains to prove the claim. Choose $v_i \in V_{x_i}$ in general position and assume that $g = (g_1, \dots, g_r)$ stabilizes $v := (v_1, \dots, v_t)$ in V for some $g_j \in \mathrm{GL}_{p^{a_j}}$. Our goal is to show that g_j is, in fact, central in $\mathrm{GL}_{p^{a_j}}$ for each $j = 1, \dots, r$.

Let us fix j and focus on proving that g_j is central for this particular j. By assumption (b) of Theorem 1.2, there exists an $i = 1, \dots, t$ such that y_i is balanced and $y_{ij} \neq 0$. Let us assume that $y_{ij_1}, \dots, y_{ij_s} = \pm 1$ and $y_{ih} = 0$ for every $h \notin \{j_1, \dots, j_r\}$ and consequently, $x_{ij_1}, \dots, x_{ij_s} = \pm 1$ and $x_{ih} = 0$ for every $h \notin \{j_1, \dots, j_r\}$. By our assumption, $j \in \{j_1, \dots, j_s\}$.

The representation ρ_{x_i} of $\Gamma = \mathrm{GL}_{p^{a_1}} \times \cdots \times \mathrm{GL}_{p^{a_r}}$ on

$$V_{x_i} := V^{x_{i1}} \otimes \cdots \otimes V^{x_{it}} = V^{x_{ij_1}} \otimes \cdots \otimes V^{x_{ij_s}}$$

factors through the projection $\Gamma \to \mathrm{GL}_{p^{a_{j_1}}} \times \cdots \times \mathrm{GL}_{p^{a_{j_s}}}$. Thus if $g = (g_1, \dots, g_r)$ stabilizes $v = (v_1, \dots, v_t) \in V$ then, in particular, g stabilizes v_i and so $(g_{j_1}, \dots, g_{j_s})$ stabilizes v_i.

Since y_i is assumed to be balanced, the conditions of Lemma 5.2 for the action of $\mathrm{GL}_{n_{j_1}} \times \cdots \times \mathrm{GL}_{n_{j_s}}$ on $V_{x_i} = V^{x_{j_1}} \otimes \cdots \otimes V^{x_{j_s}}$ are satisfied. (Recall that here $n_i = p^{a_i}$.) Since $(g_{j_1}, \dots, g_{j_s})$ stabilizes $v_i \in V_{x_i}$ in general position, Lemma 5.2 tells us that $g_{j_1}, \dots, g_{j_s}$ are central in $\mathrm{GL}_{n_{j_1}}, \dots, \mathrm{GL}_{n_{j_s}}$, respectively. In particular, g_j is central in GL_{n_j}, as desired. This completes the proof of Lemma 5.3 and thus of Theorem 1.2. □

6. Proof of Theorem 1.3

Consider the central subgroups $\widetilde{C}$ and C of $\Gamma = \mathrm{GL}_{p^{a_1}} \times \cdots \times \mathrm{GL}_{p^{a_r}}$ given by

$$\widetilde{C} = \{(\tau_1, \dots, \tau_r) \in \mathbb{G}_m^r \mid \tau_1 \cdots \tau_r = 1\} \quad \text{and} \quad C = \{(\tau_1, \dots, \tau_r) \in \mu \mid \tau_1 \cdots \tau_r = 1\}.$$

Set $G := \Gamma / C$ and $\widetilde{G} := \Gamma / \widetilde{C}$. Note that $C = \widetilde{C} \cap \mu$. By Theorem A.1 and Corollary A.2, $H^1(-, G)$ and $H^1(-, \widetilde{G})$ are both isomorphic to the functor $\mathcal{F}$ defined in the statement of Theorem 1.3. In particular, $\mathrm{ed}(\widetilde{G}) = \mathrm{ed}(G) = \mathrm{ed}(\mathcal{F})$ and $\mathrm{ed}_p(\widetilde{G}) = \mathrm{ed}_p(G) = \mathrm{ed}_p(\mathcal{F})$. We are now ready to proceed with the proof of Theorem 1.3.

(a) If $A_1 \otimes \cdots \otimes A_r$ is split over K, then A_r can be recovered from $A_1, \ldots, A_{r-1}$ as the unique central simple K-algebra of degree p^{a_r} which is Brauer-equivalent to

$$(A_1 \otimes \cdots \otimes A_{r-1})^{\mathrm{op}}.$$

(Here B^{op} denotes the opposite algebra of B.) In other words, the morphism of functors

$$\mathcal{F} \to H^1(-, \mathrm{PGL}_{p^{a_1}}) \times \cdots \times H^1(-, \mathrm{PGL}_{p^{a_{r-1}}}) \tag{20}$$

given by $(A_1, \ldots, A_{r-1}, A_r) \to (A_1, \ldots, A_{r-1})$ is injective. We claim that if $a_r \geqslant a_1 + \cdots + a_{r-1}$ (which is our assumption in part (a)), then this morphism is also surjective. Indeed,

$$\deg(A_1 \otimes \cdots \otimes A_{r-1}) = p^{a_1 + \cdots + a_{r-1}}$$

for any choice of central simple K-algebras $A_1, \ldots, A_{r-1}$ such that $\deg(A_i) = p^{a_i}$. Hence, for any such choice, there exists a central simple algebra of degree p^{a_r} which is Brauer-equivalent to $(A_1 \otimes \cdots \otimes A_{r-1})^{\mathrm{op}}$. This proves the claim.

We conclude that if $a_r \geqslant a_1 + \cdots + a_{r-1}$ then (20) is an isomorphism and thus

$$\mathrm{ed}(\widetilde{G}) = \mathrm{ed}(G) = \mathrm{ed}(\mathcal{F}) = \mathrm{ed}(\mathrm{PGL}_{p^{a_1}} \times \cdots \times \mathrm{PGL}_{p^{a_{r-1}}}),$$
$$\mathrm{ed}_p(\widetilde{G}) = \mathrm{ed}_p(G) = \mathrm{ed}_p(\mathcal{F}) = \mathrm{ed}_p(\mathrm{PGL}_{p^{a_1}} \times \cdots \times \mathrm{PGL}_{p^{a_{r-1}}}).$$

The inequality $\mathrm{ed}(\mathcal{F}) \leqslant p^{2a_1} + \cdots + p^{2a_{r-1}}$ now follows from (9).

(b) Now suppose $a_r < a_1 + \cdots + a_{r-1}$. Note that $\mathrm{Code}(C)$ has a minimal basis consisting of the single element $(1, \ldots, 1) \in (\mathbb{Z}/p^{a_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{a_r}\mathbb{Z})$. Moreover, $p^{a_r} \leqslant \frac{1}{2} p^{a_1} \cdots p^{a_{r-1}}$ and consequently, Theorem 1.2 applies. It tells us that if the r-tuple $(p^{a_1}, \ldots, p^{a_r})$ is not of the form $(2, 2, 2, 2)$, $(3, 3, 3)$ or $(2, 2^a, 2^a)$, then

$$\mathrm{ed}(\mathcal{F}) = \mathrm{ed}_p(\mathcal{F}) = \mathrm{ed}(\widetilde{G}) = \mathrm{ed}_p(\widetilde{G}) = \mathrm{ed}(G) = \mathrm{ed}_p(G) = p^{a_1 + \cdots + a_r} - \sum_{i=1}^{r} p^{2a_i} + r - 1,$$

as claimed.

(c) In the case where $(p^{a_1}, \ldots, p^{a_r}) = (2, 2, 2)$, $\mathcal{F}(K)$ is the set of isomorphism classes of triples (A_1, A_2, A_3) of quaternion K-algebras, such that $A_1 \otimes A_2 \otimes A_3$ is split over K. We will show that (i) $\mathrm{ed}(\mathcal{F}) \leqslant 3$ and (ii) $\mathrm{ed}_2(\mathcal{F}) \geqslant 3$.

To prove (i), recall that by a theorem of Albert [Lam 2005, Theorem III.4.8], the condition that $A_1 \otimes A_2 \otimes A_3$ is split over K implies that A_1 and A_2 are linked over K. That is, there exist $a, b, c \in K^*$ such that $A_1 \simeq (a, b)$ and $A_2 \simeq (a, c)$ over K. Hence, the triple $(A_1, A_2, A_3) \in \mathcal{F}(K)$ descends to the triple $(B_1, B_2, B_3) \in \mathcal{F}(K_0)$, where $K_0 = k(a, b, c)$, $B_1 = (a, b)$, $B_2 = (a, c)$ and $B_3 = (a, bc)$ over K_0. Since $\mathrm{trdeg}(K_0/k) \leqslant 3$, assertion (i) follows.

To prove (ii), consider the morphism of functors $f\colon \mathcal{F} \to H^1(-, \mathrm{SO}_4)$ given by

$$f\colon (A_1, A_2, A_3) \mapsto \alpha,$$

where α is a 4-dimensional quadratic form such that

$$\alpha \oplus \mathbb{H} \oplus \mathbb{H} \cong N(A_1) \oplus (-N(A_2)).$$

Here $\mathbb{H}$ denotes the 2-dimensional hyperbolic form $\langle 1, -1\rangle$, $N(A_1)$ denotes the norm form of A_1, and $-N(A_2)$ denotes the opposite norm form of A_2, i.e., the unique 4-dimensional form such that $N(A_2) \oplus (-N(A_2))$ is hyperbolic. Since $N(A_1)$ and $N(A_2)$ are forms of discriminant 1, so is α (this will also be apparent from the explicit computations below). Thus we may view α as an element of the Galois cohomology set $H^1(K, \mathrm{SO}_4)$, which classifies 4-dimensional quadratic forms of discriminant 1 over K, up to isomorphism. Note also that by the Witt cancellation theorem, α is unique up to isomorphism. We conclude that the morphism of functors f is well-defined.

Equivalently, using the definition of the Albert form given in [Lam 2005, p. 69], α is the unique 4-dimensional quadratic form such that $\alpha \oplus \mathbb{H} \cong q$, where q is the 6-dimensional Albert form of A_1 and A_2. Here the Albert form of A_1 and A_2 is isotropic, and hence, can be written as $\alpha \oplus \mathbb{H}$, because A_1 and A_2 are linked; once again, see [Lam 2005, Theorem III.4.8].

Suppose $A_1 = (a, b)$, $A_2 = (a, c)$, and $A_3 = (a, bc)$, as above. Then

$$N(A_1) = \langle\langle -a, -b\rangle\rangle = \langle 1, -a, -b, ab\rangle,$$

and similarly $N(A_2) = \langle 1, -a, -c, ac\rangle$; see, e.g., [Lam 2005, Corollary III.2.2]. Thus

$$N(A_1) \oplus (-N(A_2)) = \langle 1, -1, -a, a, -b, c, ab, -ac\rangle \simeq \langle -b, c, ab, -ac\rangle \oplus \mathbb{H} \oplus \mathbb{H},$$

and we obtain an explicit formula for $\alpha = f(A_1, A_2, A_3)$: $\alpha \cong \langle -b, c, ab, -ac\rangle$.

It is easy to see that any 4-dimensional quadratic form of discriminant 1 over K can be written as $\langle -b, c, ab, -ac\rangle$ for some $a, b, c \in K^*$. In other words, the morphism of functors $f\colon \mathcal{F} \to H^1(-, \mathrm{SO}_4)$ is surjective. Consequently,

$$\mathrm{ed}_2(\mathcal{F}) \geq \mathrm{ed}_2(H^1(-, \mathrm{SO}_4)) = \mathrm{ed}_2(\mathrm{SO}_4);$$

see, e.g., [Berhuy and Favi 2003, Lemma 1.9] or [Reichstein 2010, Lemma 2.2]. On the other hand, $\mathrm{ed}_2(\mathrm{SO}_4) = 3$; see [Reichstein and Youssin 2000, Theorem 8.1(2) and Remark 8.2] or [Reichstein 2010, Corollary 3.6(a)]. Thus

$$\mathrm{ed}_2(\mathcal{F}) \geqslant \mathrm{ed}_2(\mathrm{SO}_4) = 3.$$

This completes the proof of (ii) and thus of part (c) of Theorem 1.3. $\square$

Appendix: Galois cohomology of central quotients of products of general linear groups

Athena Nguyen[1]

In this appendix we will study the Galois cohomology of algebraic groups of the form

$$G := \Gamma/C,$$

where $\Gamma := \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ and $C \subset Z(\Gamma) = \mathbb{G}_m^r$ is a central subgroup. Here $n_1, \ldots, n_r \geqslant 1$ are integers, not necessarily prime powers. Let

$$\overline{G} := G/Z(G) = \mathrm{PGL}_{n_1} \times \cdots \times \mathrm{PGL}_{n_r} = \Gamma/Z(\Gamma).$$

Recall that for any field K/k, $H^1(K, \mathrm{PGL}_n)$ is naturally identified with the set of isomorphism classes of central simple K-algebras of degree n, and

$$H^1(K, \overline{G}) = H^1(K, \mathrm{PGL}_{n_1}) \times \cdots \times H^1(K, \mathrm{PGL}_{n_r})$$

is identified with the set of r-tuples $(A_1, \ldots, A_r)$ of central simple K-algebras such that $\deg(A_i) = n_i$. Denote by ∂_K^i the coboundary map $H^1(K, \mathrm{PGL}_{n_i}) \to H^2(K, \mathbb{G}_m)$ induced by the short exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{GL}_{n_i} \to \mathrm{PGL}_{n_i} \to 1.$$

This map sends a central simple algebra A_i to its Brauer class $[A_i]$ in $H^2(K, \mathbb{G}_m) = \mathrm{Br}(K)$.

Of particular interest to us will be

$$X(\mathbb{G}_m^r/C) = \{(m_1, \ldots, m_r) \in \mathbb{Z}^r \mid \tau_1^{m_1} \cdots \tau_r^{m_r} = 1 \ \ \forall (\tau_1, \ldots, \tau_r) \in \mathbb{G}_m^r\},$$

as in (6). We are now ready to state the main result of this appendix.

Theorem A.1. *Let $\pi\colon G \to \overline{G} := \mathrm{PGL}_{n_1} \times \cdots \times \mathrm{PGL}_{n_r}$ be the natural projection and $\pi_*\colon H^1(K, G) \to H^1(K, \overline{G})$ be the induced map in cohomology. Here K/k is a field extension. Then*:

(a) *The map $\pi_*\colon H^1(K, G) \to H^1(K, \overline{G})$ is injective for every field K/k.*

(b) *The map π_* identifies $H^1(K, G)$ with the set of isomorphism classes of r-tuples $(A_1, \ldots, A_r)$ of central simple K-algebras such that* $\deg(A_i) = n_i$ *and $A_1^{\otimes m_1} \otimes \cdots \otimes A_r^{\otimes m_r}$ is split over K for every $(m_1, \ldots, m_r) \in X(\mathbb{G}_m^r/C)$.*

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[1]This appendix is based on a portion of Nguyen's master's thesis completed at the University of British Columbia. Nguyen gratefully acknowledges the financial support from the University of British Columbia and the Natural Sciences and Engineering Research Council of Canada.

Proof. Throughout, we will identify $H^2(K, \mathbb{G}_m^r)$ with $H^2(K, \mathbb{G}_m)^r$ and $X(\mathbb{G}_m^r)$ with $\mathbb{Z}^n$. A character $x = (m_1, \dots, m_r) \in \mathbb{Z}^n$, i.e., a character $x \colon \mathbb{G}_m^r \to \mathbb{G}_m$ given by $(\tau_1, \dots, \tau_r) \to \tau_1^{m_1} \cdots \tau_r^{m_r}$, induces a map $x_* \colon H^2(K, \mathbb{G}_m)^r \to H^2(K, \mathbb{G}_m)$ in cohomology given by

$$x_*(\alpha_1, \dots, \alpha_r) = \alpha_1^{m_1} \cdot \dots \cdot \alpha_r^{m_r}. \tag{21}$$

Let us now consider the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m^r \arrow[r] \arrow[d, "\eta"'] & \Gamma \arrow[r] \arrow[d] & \prod_{i=1}^r \mathrm{PGL}_{n_i} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{G}_m^r/C \arrow[r] & G \arrow[r, "\pi"] & \prod_{i=1}^r \mathrm{PGL}_{n_i} \arrow[r] & 1
\end{tikzcd}$$

Since $H^1(K, \mathbb{G}_m^r/C) = \{1\}$ by Hilbert's theorem 90, we obtain the following diagram in cohomology with exact rows:

$$\begin{tikzcd}
 & & H^1(K, \prod_{i=1}^r \mathrm{PGL}_{n_i}) \arrow[r, "(\partial_K^1, \dots, \partial_K^r)"] \arrow[d, equal] & H^2(K, \mathbb{G}_m^r) \arrow[d, "\eta_*"] \\
1 \arrow[r] & H^1(K, G) \arrow[r, "\pi_*"] & H^1(K, \prod_{i=1}^r \mathrm{PGL}_{n_i}) \arrow[r, "\partial_K"] & H^2(K, \mathbb{G}_m^r/C)
\end{tikzcd}$$

(a) It follows from [Serre 1997, I.5, Proposition 42] that π_* is injective.

(b) Thus, π_* identifies $H^1(K, G)$ with the set of r-tuples $(A_1, \dots, A_r)$, where $A_i \in H^1(K, \mathrm{PGL}_{n_i})$ is a central simple algebra of degree n_i, and $(\partial_K^1(A_1), \dots, \partial_K^r(A_r)) \in \mathrm{Ker}(\eta_*)$. Recall that ∂_K^i sends a central simple algebra A_i to its Brauer class $[A_i] \in H^2(K, \mathbb{G}_m)$. In the sequel we will use additive notation for the abelian group $H^2(K, \mathbb{G}_m) = \mathrm{Br}(K)$.

Consider an r-tuple $\alpha := ([A_1], \dots, [A_r]) \in H^2(K, \mathbb{G}_m^r)$. Since $\mathbb{G}_m^r/C$ is diagonalizable, $\eta_*(\alpha) = 0$ if and only if $x_*(\eta_*(\alpha)) = 0$ for all $x \in X(\mathbb{G}_m^r/C)$. If $x = (m_1, \dots, m_r) \in X(\mathbb{G}_m^r/C)$, then $x_* \circ \eta_* = (m_1, \dots, m_r) \in X(G_m^r)$. By (21), $x_*(\eta_*(\alpha)) = [A_1^{\otimes m_1} \otimes \dots \otimes A_r^{\otimes m_r}]$, and part (b) follows. □

Corollary A.2. *Let* $\Gamma := \mathrm{GL}_{n_1} \times \dots \times \mathrm{GL}_{n_r}$, C_1, C_2 *be* k*-subgroups of* $Z(\Gamma) = \mathbb{G}_m^r$, $G_1 = \Gamma/C_1$ *and* $G_2 = \Gamma/C_2$. *Denote the central subgroup* $\mu_{n_1} \times \dots \times \mu_{n_r}$ *of* Γ *by* μ.

If $C_1 \cap \mu = C_2 \cap \mu$ *then the Galois cohomology functors* $H^1(-, G_1)$ *and* $H^1(-, G_2)$ *are isomorphic.*

Proof. By Theorem A.1, $H^1(K, G_i)$ is naturally identified with the set of r-tuples $(A_1, \ldots, A_r)$ of central simple algebras such that $\deg(A_i) = n_i$ and

$$A_1^{\otimes m_1} \otimes \cdots \otimes A_r^{\otimes m_r} \text{ is split over } K \text{ for every } (m_1, \ldots, m_r) \in X(\mathbb{G}_m/C_i).$$

Note that since $A_i^{\otimes n_i}$ is split for every i, this condition depends only on the image of $(m_1, \ldots, m_r)$ under the natural projection

$$\pi \colon X(\mathbb{G}_m^r) = \mathbb{Z}^r \to (\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z}) = X(\mu).$$

Our assumption that $C_1 \cap \mu = C_2 \cap \mu$ is equivalent to $X(\mathbb{G}_m^r/C_1)$ and $X(\mathbb{G}_m^r/C_2)$ having the same image under π, and the corollary follows. □

In order to state the second corollary of Theorem A.1, we will need the following definition. By a code we shall mean a subgroup of $X(\mu) = (\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z})$. Given a subgroup $C \subset \mu$, we define the code $\mathrm{Code}(C) := X(\mu/C)$, as in (8).

We will say that two codes are called *equivalent* if one can be obtained from the other by repeatedly performing the following elementary operations:

(1) Permuting entries i and j in every vector of the code, for any i, j with $n_i = n_j$.

(2) Multiplying the i-th entry in every vector of the code by an integer c prime to n_i.

Corollary A.3. *Suppose C_1 and C_2 are subgroups of $\mu := \mu_{n_1} \times \cdots \times \mu_{n_r}$, $G_1 = \Gamma/C_1$ and $G_2 := \Gamma/C_2$. If $\mathrm{Code}(C_1)$ and $\mathrm{Code}(C_2)$ are equivalent, then*

(a) *the Galois cohomology functors $H^1(-, G_1)$, $H^1(-, G_2)$ are isomorphic, and*

(b) *in particular, $\mathrm{ed}(G_1) = \mathrm{ed}(G_2)$ and $\mathrm{ed}_p(G_1) = \mathrm{ed}_p(G_2)$ for every prime p.*

Proof. To prove part (a), it suffices to show that $H^1(-, G_1)$ and $H^1(-, G_2)$ are isomorphic if C_2 is obtained from C_1 by an elementary operation.

(1) Suppose $n_i = n_j$ for some $i, j = 1, \ldots, r$, and $\mathrm{Code}(C_2)$ is obtained from $\mathrm{Code}(C_1)$ by permuting entries i and j in every vector. In this case $C_2 = \alpha(C_1)$, where α is the automorphism of $\Gamma = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ which swaps the i-th and the j-th components. Then α induces an isomorphism between $G_1 = \Gamma/C_1$ and $G_2 = \Gamma/C_2$, and thus an isomorphism between $H^1(-, G_1)$ and $H^1(-, G_2)$.

(2) Now suppose that $\mathrm{Code}(C_1)$ is obtained from $\mathrm{Code}(C_2)$ by multiplying the i-th entry in every vector by some $c \in (\mathbb{Z}/n_i\mathbb{Z})^*$. The description of $H^1(K, G/\mu)$ given by Theorem A.1 now tells us that

$$H^1(K, G_1) \to H^1(K, G_2),$$
$$(A_1, \ldots, A_r) \mapsto (A_1, \ldots, A_{i-1}, [A_i^{\otimes c}]_{n_i}, A_{i+1}, \ldots, A_r),$$

is an isomorphism. Here, by $[A_i^{\otimes c}]_{n_i}$ we mean the unique central simple K-algebra of degree n_i which is Brauer equivalent to $A_i^{\otimes c}$.

Part (b) follows from (a), because $\mathrm{ed}(G)$ and $\mathrm{ed}_p(G)$ are defined entirely in terms of the Galois cohomology functor $H^1(-, G)$. □

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NOTES ON THE STRUCTURE CONSTANTS OF HECKE ALGEBRAS OF INDUCED REPRESENTATIONS OF FINITE CHEVALLEY GROUPS

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This paper is dedicated to the memory of Robert Steinberg.

This paper contains an algorithm for the structure constants of the Hecke algebra of a Gelfand–Graev representation of a finite Chevalley group.

1. Introduction

Let G be a Chevalley group over a finite field $k = F_q$ of characteristic p (as in [Chevalley 1955] or [Steinberg 1968]). Let B be a Borel subgroup of G with $U = O_p(B)$ (the unipotent radical of B), and let T be a maximal torus such that $B = UT$. Let W be the Weyl group of G. Then W is a finite Coxeter group with distinguished generators $S = \{s_1, \dots, s_n\}$.

Let Φ be the root system associated with W, with $\{\alpha_1, \dots, \alpha_n\}$ the set of simple roots corresponding to the generators $s_i \in S$, and $\Phi_\pm$ the set of positive roots (respectively, negative roots) associated with them. For each root α, let U_α be the root subgroup of G corresponding to it. The subgroup U is generated by the root subgroups U_α, $\alpha > 0$.

From [Steinberg 1968, §3], the Chevalley group G has a B,N-pair, with Borel subgroup B, N the subgroup generated by all elements $w_\alpha(t)$, and $B \cap N$ equal to T, the subgroup generated by all elements $h_\alpha(t)$ (see the definitions of $w_\alpha(t)$ and $h_\alpha(t)$ in Section 2). Then $N/T \cong W$. (If the field k contains more than three elements, then N is the normalizer $N = N_G(T)$; see [Steinberg 1968, p. 36]).

By the Bruhat decomposition, the (U, U)-double cosets are parametrized by the elements of N, while the (B, B)-double cosets are parametrized by the elements of W.

We consider induced representations γ of the form ψ^G, for a linear representation ψ of U. Let

$$e = |U|^{-1} \sum_{u \in U} \psi(u^{-1})u$$

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be the primitive idempotent affording ψ in the group algebra $\mathbb{C}U$ of U over the field of complex numbers. Then $\gamma = \psi^G$ is afforded by the left $\mathbb{C}G$-module $\mathbb{C}Ge$. The *Hecke algebra* of γ is the subalgebra $H = e\mathbb{C}Ge$ of $\mathbb{C}G$, and is isomorphic to $(\mathrm{End}_{\mathbb{C}G}\,\mathbb{C}Ge)^{\circ}$. These representations and their Hecke algebras were first investigated by Gelfand and Graev [1962a; 1962b]. In particular, they introduced the important class of *Gelfand–Graev representations* of G, which are the induced representation ψ^G, for a linear representation ψ of U in *general position*, that is, $\psi|U_{\alpha_i} \neq 1$ for each simple root subgroup U_{α_i}, $1 \leq i \leq n$, and $\psi|U_\alpha = 1$ for each positive and not simple root α.

It is known (see [Gelfand and Graev 1962b] for the case of $G = \mathrm{SL}_n(k)$ for a finite field k, and [Steinberg 1968, Theorem 49] for the general case) that the Hecke algebra H of a Gelfand–Graev representation is a commutative algebra, so that a Gelfand–Graev representation is multiplicity-free.

A basis for the Hecke algebra H of a Gelfand–Graev representation ψ^G is given by the nonzero elements of the form ene with $n \in N$. The *standard basis elements* are the nonzero elements of the form $c_n = \mathrm{ind}(n)\, ene$, where $\mathrm{ind}(n) = |U : nUn^{-1} \cap U|$. The structure constants for the standard basis elements, defined by the formulas

$$c_\ell c_m = \sum_n [c_\ell c_m : c_n] c_n,$$

with $\ell, m, n \in N^*$, are algebraic integers (here N^* is the set of elements $n \in N$ such that $ene \neq 0$).

The structure constants of H are given by the formula

$$[c_\ell c_m : c_n] = \sum_{u\ell u_1 = nvm^{-1} \in U\ell U \cap nU_{m^{-1}}m^{-1}} \psi((uu_1)^{-1}v),$$

by [Curtis and Reiner 1981, Proposition 11.30], and the fact that $U\ell U \cap nU_{m^{-1}}m^{-1}$ is a set of representatives of the left U-cosets in $U\ell U \cap nU_{m^{-1}}m^{-1}U$. As in [Curtis 1988; 2009], $U_n = U \cap nU\_n^{-1}$ for $n \in N$. The structure constants are exponential sums involving the linear character ψ of U and combinatorial information about multiplication and intersections patterns of (U, U)-double cosets. The latter information is also given at least partially for the algebraic group $G(\bar{k})$ over the algebraic closure $\bar{k}$ of k corresponding to G, with some questions about the geometry not completely settled at this time. A main result in the paper is an algorithm given in Section 4 for the solutions (u, u_1, v) of the equation $u\ell u_1 = nvm^{-1}$ in the formula above, so that in some sense the structure constants are computable. The approach taken here is based on the theory of *cells* U_τ developed in [Curtis 1988; 2009]. The algorithm for the solutions of the equations is a refined version of an algorithm for them given in [Curtis 2009, Theorem 2.1]. At the end of Section 4, some problems for further research are mentioned.

In case ψ^G is a Gelfand–Graev representation, the values of the irreducible representations of the commutative semisimple algebra H on standard basis elements are obtained as eigenvalues of matrices giving the regular representation of H and whose entries are the structure constants $[c_\ell c_m : c_n]$; see [Curtis 2009, Proposition 1.1].

Formulas for the structure constants based on different algorithms and a different set of representatives of the cosets of U were obtained by Simion [2015].

The irreducible representations of H were obtained in [Curtis 1993] using the results of Deligne and Lusztig [1976] on representations of G defined on the ℓ-adic cohomology of locally closed subsets of the algebraic group $G(\bar{k})$ with Frobenius endomorphism F on which the finite group G acts. The formulas for the irreducible representations of H in [Curtis 1993] involve a homomorphism of algebras $f_T : H \to \mathbb{C}T$ for each F-stable maximal torus T of G, proved using the character formula of Deligne and Lusztig [1976] for the virtual representations $R_{T,\theta}$. The homomorphisms f_T provide an approach to the representations of H, and are of independent interest (see [Bonnafé and Kessar 2008]).

A combinatorial approach to the representations of H based on the structure constants of the Hecke algebra H and the internal structure of the finite Chevalley group G is a main objective of this paper.

Two final sections contain examples in which a combinatorial construction of the homomorphisms f_T is obtained. These include the *Bessel functions over finite fields* of Gelfand and Graev [1962a], for the groups $\mathrm{SL}_2(k)$ and k a finite field of odd characteristic, and a construction of the homomorphisms $f_T : H \to \mathbb{C}T$ for the split torus T in a general Chevalley group.

2. Background and preliminary results

For each root α, there is a homomorphism (see [Steinberg 1968, page 46]) $\varphi = \varphi_\alpha : \mathrm{SL}_2(k) \to G$ such that φ takes

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \to x_\alpha(t), \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \to x_{-\alpha}(t), \quad \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \to w_\alpha(t) \in N, \quad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \to h_\alpha(t) \in T$$

for all $t \in k$. The elements $w_\alpha(t)$ and $h_\alpha(t)$ are given by

$$w_\alpha(t) = x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t), \qquad h_\alpha(t) = w_\alpha(t) w_\alpha(1)^{-1},$$

by [Steinberg 1968, p. 30]. If $w = s_k \cdots s_1$ is a reduced expression of an element $w \in W$ then $\dot{w} = \dot{s}_k \cdots \dot{s}_1$, with $\dot{s}_i = w_{\alpha_i}(t_i)$ for some fixed choice of $t_i \in k^* = k - \{0\}$, is a representative in N of w which is independent of the choice of the reduced expression chosen, by [Steinberg 1968, Lemma 83, p. 242]. In what follows we assume that representatives $\dot{x} \in N$ of all elements $x \in W$ have been chosen in this way, for a fixed choice of representatives $\dot{s}_i$ of the generators $s_i \in S$.

We may assume that
$$\dot{s}_k = \varphi_{\alpha_k}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$
for a simple root α_k, $1 \le k \le n$.

Using the homomorphisms φ_α, we obtain the so-called SL_2-IDENTITY:
$$\dot{s}_k^{-1} x_{\alpha_k}(t)\dot{s}_k = x_{\alpha_k}(-t^{-1})\dot{s}_k h_{\alpha_k}(t) x_{\alpha_k}(-t^{-1}),$$
for a simple root α_k and $t \in k^*$ (cf. [Curtis 2009, Lemma 2.1]).

As in [Deodhar 1985], a *subexpression* τ of a fixed reduced expression $w = s_k \cdots s_1$ is a sequence $\tau = (\tau_k, \dots, \tau_1, \tau_0)$ of elements of W such that $\tau_i \tau_{i-1}^{-1} \in \{1, s_i\}$ for $i = 1, \dots, k$ and $\tau_0 = 1$. Then the set of terminal elements τ_k of subexpressions of $w = s_k \cdots s_1$ coincides with the set of elements $x \in W$ such that $x \le w$ in the Chevalley–Bruhat order. In what follows, the *length* of an element $w \in W$ in terms of the generators $s_i \in S$ is denoted by $\ell(w)$. A subexpression $\tau = (\tau_k, \dots, \tau_1, \tau_0)$ is called a K*-sequence* relative to the triple $w = s_k \cdots s_1, x, y$ of elements of W if it satisfies conditions (2.10)(a-c) of [Kawanaka 1975]. It is understood that a K-sequence for the triple (w, x, y) is always given with reference to a fixed reduced expression $w = s_k \cdots s_1$. Let $J_\tau = \{j : \tau_j \tau_{j-1}^{-1} = s_j\} \cup \{0\}$. Then the defining conditions for a K-sequence state that $\tau_k x = y$ and
$$\ell(s_p \tau_j x) < \ell(\tau_j x)$$
for each $j \in J_\tau$ and p in the interval between j and the next element in J_τ (or simply all $p > j$ if j is the maximal element of J_τ). For each K-sequence τ, set
$$J_\tau^- = \{j \in J_\tau : \ell(s_j \tau_{j'} x) < \ell(\tau_{j'} x)\}$$
where $j' \in J_\tau$ is the predecessor of j, and define a pair of nonnegative integers by
$$a(\tau) = |J_\tau^-|, b(\tau) = k - |J_\tau| + 1 = \mathrm{card}\{j > 0 : \tau_j \tau_{j-1}^{-1} = 1\}.$$

For each element $w \in W$, let $U_w = U \cap^w U_-$ where $U_- =^{w_0} U$ and w_0 is the element of maximal length in W. Then $U = U_w U_{ww_0}$ and $BwB = U_w \dot{w} B$, in both cases with uniqueness of expression. Let $w = s_k \dots s_1$ be a reduced expression of $w \in W$. Then $U_w = U_{\alpha_k} \dot{s}_k U_{s_{k-1} \dots s_1} \dot{s}_k^{-1}$ with uniqueness of expression. An element of U_w expressed in this way, for a fixed reduced expression of w, is said to be in *standard form* (see [Deodhar 1985, Lemma 2.2]), and can be assigned *coordinates* in the field k.

Let w, x, y be elements of W, and $\dot{w}, \dot{x}, \dot{y}$ corresponding elements of N. Let
$$U(w, x, y) = \left\{u \in U_w : u\dot{w}B \cap \dot{y} U_{x^{-1}} \dot{x}^{-1} \ne \varnothing\right\}.$$
Then $U(w, x, y)$ is independent of the choice of representatives $\dot{w}, \dot{x}, \dot{y}$ of w, x, y in N. Moreover, $U_w \dot{w} B \cap \dot{y} U_{x^{-1}} \dot{x}^{-1}$ is a set of representatives of the left B-cosets

in $BwB \cap y(BxB)^{-1}$, and its cardinality is the structure constant $[e_w e_x : e_y]$ of the standard basis elements e_w, e_x, e_y, for $w, x, y \in W$, in the Iwahori Hecke algebra.

The K-sequences were first applied by Kawanaka to prove the following result [Kawanaka 1975, Lemma 2.14b]. For a finite Chevalley group G over $k = F_q$ the nonzero structure constants of the Iwahori Hecke algebra are given by the formula

$$[e_w e_x : e_y] = |B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}| = |U(w, x, y)| = \sum_\tau q^{a(\tau)}(q-1)^{b(\tau)}$$

where the sum is taken over all K-sequences τ for w, x, y, and $a(\tau)$ and $b(\tau)$ are the nonnegative integers defined above.

As a consequence, it follows that $U(w, x, y) \neq \varnothing$ if and only if there exist K-sequences for w, x, y (see also [Borel and Tits 1972, Remark 3.19], where the conditions are stated in a different way).

In [Curtis 1988] a geometric version of Kawanaka's formula was proved. It states that $U(w, x.y)$, viewed as a subset of the algebraic group $G(\bar{k})$, is a disjoint union of subsets U_τ, which we shall call (in this paper) *cells*. The cells U_τ are subsets of $G(\bar{k})$ parametrized by K-sequences τ for w, x, y relative to a fixed reduced expression of the element w, with corresponding subsets U_τ, also called cells (defined in [Curtis 1988]), in the finite Chevalley group $G = G(k)$ (see Lemma 3.3 below for a review of the definition of cells). The result extends Deodhar's decomposition ([Deodhar 1985], and [Curtis 2009, §4]) of the intersection $ByB \cap B_-xB$, viewed as subsets of the flag variety G/B in the algebraic group $G(\bar{k})$, with B_- the Borel subgroup opposite to B. Each cell U_τ is isomorphic (in bijective correspondence as a set, or isomorphic as a variety in $G(\bar{k})$) to a product,

$$U_\tau \cong \prod_\alpha U_\alpha \times \prod_\beta U_\beta^*$$

for certain subsets $\{\alpha\}$ and $\{\beta\}$ of cardinalities $a(\tau)$ and $b(\tau)$ of the positive root subgroups determined by τ and where U_β^* is the set of nonidentity elements in U_β. From the decomposition of $U(w, x, y)$ as a union of cells U_τ, it follows that $U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ can be identified with the set of triples (u, b, v) with $u \in U_\tau$ for some τ, $b \in B$, and $v \in U_{x^{-1}}$ satisfying the equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$ with b and v uniquely determined by u by [Curtis 2009, Lemma 2.4].

3. Relations between cells

Let ℓ, m, n in N^* correspond to elements w, x, y in W. Then ℓ, m, n are multiples by elements of T of representatives $\dot{w}, \dot{x}, \dot{y}$ in N determined as above. The set $U_\ell \ell U \cap nU_{m^{-1}}m^{-1}$ will be obtained by an algorithm based on a fixed reduced expression $w = s_k \cdots s_1$ of the element $w \in W$ in terms of the generators $s_i \in S$, and the theory of cells U_τ associated with K-sequences τ for w, x, y.

As the cells U_τ are contained in the set $U(w, x, y)$ each element $u \in U_\tau$ satisfies a *structure equation*

$$u\dot{w}b = \dot{y}v\dot{x}^{-1}$$

with $b \in B$ and $v \in U_{x^{-1}}$. The subgroup B is a semidirect product $B = UT$, so one has $b = u_1 s$ with $u_1 \in U$, $s \in T$, and it will be important to keep track of these factors in the discussion to follow.

In this section, it will be shown how elements $u \in U_\tau \subseteq U(w, x, y)$, with $\tau = (\tau_k, \dots, \tau_1, \tau_0)$ a K-sequence for w, x, y, are related to elements u' in cells $U_{\tau'}$ with $\tau' = (\tau_{k-1}, \dots, \tau_1, \tau_0)$ a K-sequence for $s_{k-1} \cdots s_1, x', y'$, and how the structure equations for u and u' are related. We keep in mind that $U(w, x, y) \neq \varnothing$ if and only if there exist K-sequences for w, x, y.

Lemma 3.1. *Let $\tau = (\tau_k, \dots, \tau_1, \tau_0)$ be a K-sequence for w, x, y for $k \geq 1$, and consider $\tau' = (\tau_{k-1}, \dots, \tau_0)$.*

(i) *τ' is a K-sequence for $s_k^{-1}w, x, s_k^{-1}y$ if $\tau_k\tau_{k-1}^{-1} = s_k$ and $\ell(s_k y) < \ell(y)$.*

(ii) *τ' is a K-sequence for $s_k^{-1}w, x, y$ if $\tau_k\tau_{k-1}^{-1} = 1$ and $\ell(s_k y) < \ell(y)$.*

(iii) *τ' is a K–sequence for $s_k^{-1}w, x, s_k^{-1}y$ if $\ell(s_k y) > \ell(y)$ and $\tau_k\tau_{k-1}^{-1} = s_k$.*

It is understood that $\tau_0 = 1$ is a K-sequence for $(1, x, x)$ and that $a(\tau_0) = b(\tau_0) = 0$. These sets of conditions are the only possibilities for τ' to be a K-sequence, and one of them must occur.

We first note that either $\ell(s_k y) < \ell(y)$ or $\ell(s_k y) > \ell(y)$, since either $y^{-1}(\alpha_k) \in \Phi_+$ or $y^{-1}(\alpha_k) \in \Phi_-$. The proof then follows immediately from the definition of K-sequence (see the proof of Lemma 2.14 of [Kawanaka 1975]). For example, we verify that the condition $\ell(s_k y) > \ell(y)$ implies $\tau_k \neq \tau_{k-1}$, and hence $\tau_k\tau_{k-1}^{-1} = s_k$. Otherwise $\tau_k = \tau_{k-1}$, $\tau_{k-1}x = y$, and $k \notin J_\tau$. This implies that $\ell(s_k\tau_{k-1}x) < \ell(\tau_{k-1}x)$ by a defining property of K-sequences, and hence $\ell(s_k y) < \ell(y)$, contrary to assumption.

The next result is background for the relation between cells U_τ and $U_{\tau'}$, with τ and τ' as in the preceding lemma. It is a version of Lemma 2.3 of [Curtis 2009]. (Parts (i) and (ii) were misstated in that article and are corrected here. We also take the opportunity to correct the statement on page 220 of [Curtis 2009] that the cells U_τ are invariant under conjugation by elements of T; this was not shown there.)

Lemma 3.2. *Let $w = s_k \cdots s_1$ be a reduced expression with $k \geq 1$ and let $x, y \in W$. Then $U(w, x, y)$ is either empty or is related to sets $U(s_k^{-1}w, x', y')$, with x' and y' depending on the K-sequence τ associated with w, x, y as follows.*

(i) *Let $\ell(s_k y) < \ell(y)$ and assume $\tau_k\tau_{k-1}^{-1} = s_k$. Then $\dot{s}_k U_{s_k^{-1}w}\dot{s}_k^{-1} \cap U(w, x, y)$ is either empty or*

$$\dot{s}_k U_{s_k^{-1}w}\dot{s}_k^{-1} \cap U(w, x, y) = \dot{s}_k U(s_k^{-1}w, x, s_k^{-1}y)\dot{s}_k^{-1}.$$

(ii) *Let* $\ell(s_k y) < \ell(y)$ *and assume* $\tau_k\tau_{k-1}^{-1} = 1$. *Then the part* $U(w, x, y)^\flat$ *of* $U(w, x, y)$ *not in* $\dot{s}_k U_{s_k^{-1}w}\dot{s}_k^{-1}$ *consists of the elements* $u = x_{\alpha_k}(t)\dot{s}_k\tilde{u}\dot{s}_k^{-1}$, *with* $x_{\alpha_k}(t) \in U_{\alpha_k}^*$ *and* $\tilde{u} \in U_{s_k^{-1}w}$ *such that* $\pi(x_{\alpha_k}(-t^{-1})\tilde{u}) \in U(s_k^{-1}w, x, y)$, *and* $t \in k^*$; *here* π *is the projection* $\pi : U \to U_{s_k^{-1}w}$ *accompanying the decomposition* $U = U_{s_k^{-1}w}U_{s_k^{-1}ww_0}$. *The map*

$$u = x_{\alpha_k}(t)\dot{s}_k\tilde{u}\dot{s}_k^{-1} \to \pi(x_{\alpha_k}(-t^{-1})\tilde{u})$$

from $U(w, x, y)^\flat$ *to* $U(s_k^{-1}w, x, y)$ *is surjective. There is a bijection of sets* $U(w, x, y)^\flat \cong U_{\alpha_k}^* \times U(s_k^{-1}w, x, y)$.

(iii) *Let* $\ell(s_k y) > \ell(y)$ *and* $\tau_k\tau_{k-1}^{-1} = s_k$. *Then*

$$U(w, x, y) = U_{\alpha_k}\dot{s}_k U(s_k^{-1}w, x, s_k^{-1}y)\dot{s}_k^{-1}$$

and there is a bijection of sets $U(w, x, y) \cong U_{\alpha_k} \times U(s_k^{-1}w, x, s_k^{-1}y)$.

The proof is included in the proof of Lemma 2.3 of [Curtis 2009].

Lemma 3.3. *Let* w, x, y *be elements of* W *and let* $w = s_k \cdots s_1$ *be a reduced expression for* w. *Let* $\tau = (\tau_k, \dots \tau_1, \tau_0)$ *be a* K*-sequence for* w, x, y *with* $\tau_0 = 1$, *and let* U_τ *be the corresponding cell, viewed as a subset of* $U(w, x, y) \subseteq U_w$. *Let* $\tau' = (\tau_{k-1}, \dots, \tau_1, \tau_0)$ *be a* K*-sequence for* $s_{k-1}\cdots s_1, x', y'$ *as in one of the cases in Lemma 3.1, and let* $U_{\tau'}$ *be the corresponding cell in* $U(s_{k-1}\cdots s_1, x', y')$. *The construction of the cell* U_τ *from* $U_{\tau'}$, *reviewed below, defines a surjective map of sets* $\lambda : U_\tau \to U_{\tau'}$. *Let* $U_\tau(\bar{k})$ *and* $U_{\tau'}(\bar{k})$ *be the corresponding cells in the algebraic group* $G(\bar{k})$ *over the algebraic closure* $\bar{k}$ *of* k. *Then the map* $\lambda : U_\tau(\bar{k}) \to U_{\tau'}(\bar{k})$, *defined as in part* (i), *is a surjective morphism of algebraic sets, defined over* k.

The construction of $U_\tau(\bar{k})$ from $U_{\tau'}(\bar{k})$ was given in the three cases of Lemma 3.1 in the proof of Theorem 1.6 of [Curtis 1988] and in [Curtis 2009, page 220], and will be reviewed here in the case of the algebraic group $G(\bar{k})$. We abbreviate $U_\tau(\bar{k})$ to U_τ, etc.

(i) $\tau_k\tau_{k-1}^{-1} = s_k$ and $\ell(s_k y) < \ell(y)$. In this case, we have $U_\tau \subseteq \dot{s}_k U_{s_k^{-1}w}\dot{s}_k^{-1}$ and $U_{\tau'} \subseteq U(s_k^{-1}w, x, s_k^{-1}y)$, and one has $U_{\tau'} = \dot{s}_k^{-1}U_\tau\dot{s}_k$. The map $\lambda : u \to \dot{s}_k^{-1}u\dot{s}_k$ is clearly a surjective morphism from U_τ to $U_{\tau'}$ and is defined over k because $\dot{s}_k$ belongs to the finite Chevalley group $G(k)$.

(ii) $\tau_k\tau_{k-1}^{-1} = 1$ and $\ell(s_k y) < \ell(y)$. This time U_τ is in the part of $U(w, x, y)$ which is not contained in $\dot{s}_k U_{s_k^{-1}w}\dot{s}_k^{-1}$ and consists of the elements $x_{\alpha_k}(t_k)\dot{s}_k\tilde{u}\dot{s}_k^{-1}$ such that $t_k \neq 0$, $\tilde{u} \in U_{s_k^{-1}w}$ and $\pi(x_{\alpha_k}(-t_k^{-1})\tilde{u}) \in U_{\tau'}$, where π is the projection $U \to U_{s_k^{-1}w}$ associated with the factorization $U = U_{s_k^{-1}w}U_{s_k^{-1}ww_0}$. The map

$$\lambda : x_{\alpha_k}(t_k)\dot{s}_k\tilde{u}\dot{s}_k^{-1} \to \pi(x_{\alpha_k}(-t_k^{-1})\tilde{u})$$

is a surjective morphism defined over k from U_τ to $U_{\tau'}$.

(iii) $\tau_k\tau_{k-1}^{-1} = s_k$ and $\ell(s_k y) > \ell(y)$. In this situation, we have $U_\tau = U_{\alpha_k}\dot{s}_k U_{\tau'}\dot{s}_k^{-1}$, in terms of the factorization: $U_w = U_{\alpha_k}\dot{s}_k U_{s_{k-1}\cdots s_1}\dot{s}_k^{-1}$, and $U_{\tau'} \subseteq U(s_k^{-1}w, x, s_k y)$. Then the map

$$\lambda : x_{\alpha_k}(t_k)\dot{s}_k\tilde{u}\dot{s}^{-1} \to \tilde{u}$$

is a surjective morphism defined over k from U_τ to $U_{\tau'}$ (as the projection from U_w to $\dot{s}_k U_{s_{k-1}\cdots s_1}\dot{s}_k^{-1}$ in the factorization given above, followed by the inner automorphism by an element of $G(k)$). This completes our discussion of the proof of the lemma.

We now have a reduction process for cells, $U_\tau \to U_{\tau'}$, as in the preceding lemma. Let $u \in U_\tau$ correspond to $u' \in U_{\tau'}$ as in the lemma. Then the structure equation $u\dot{w}u_1 s = \dot{y}v\dot{x}^{-1}$ satisfied by u corresponds to the structure equation $u'\dot{s}_k^{-1}\dot{w}u_1's' = \dot{y}'v\dot{x}'^{-1}$ satisfied by u', with uniquely determined factors $\{u, u_1, s, v\}$ and $\{u', u_1', s', v'\}$. The next lemma shows how the elements u', u_1', v' in U and $s' \in T$ are related to u, u_1, v in U and $s \in T$, using the standard form and facts about the multiplicative structure of the Chevalley group such as the decomposition $U = U_x U_{xw_0}$ for elements $x \in W$. It is also shown that the process is reversible, assuming $u \in U_\tau$ is known.

Lemma 3.4. *Suppose that the cell $U_\tau \subseteq U(w, x, y)$ maps onto the cell $U_{\tau'} \subseteq U(s_{k-1}\cdots s_1, x', y')$ as in cases* (i)–(iii) *of Lemma 3.1, and let the structure equation satisfied by $u \in U_\tau$ be $u\dot{w}u_1 s = \dot{y}v\dot{x}^{-1}$ with factors $u \in U_\tau$, $u_1 \in U$, $s \in T$, and $v \in U_{x^{-1}}$ uniquely determined by $u \in U_\tau$. Let $u \to u' = \lambda(u)$ with $u' \in U_{\tau'}$ as in Lemma* 3.3, *and consider the structure equation satisfied by u' with factors $u' \in U$, $s' \in T$, $u_1' \in U$, and $v' \in U_{x^{-1}}$ in each of the cases. Then the factors u', u_1', s' and v' are given as in the proof of the lemma. Conversely, assuming $u \in U_\tau$ is known, u_1, s and v are obtained from u', s', u_1', and v', as shown in the proof of the lemma.*

In case (i), we have $\tau_k\tau_{k-1}^{-1} = s_k$, $\ell(s_k y) < \ell(y)$ and $U_{\tau'} = \dot{s}_k^{-1}U_\tau\dot{s}_k$. Then the equation satisfied by $u \in U_\tau$ is $u\dot{w}u_1 s = \dot{y}v\dot{x}^{-1}$ with $u = \dot{s}_k u'\dot{s}_k^{-1}$ and $u' \in U_{\tau'}$. It becomes the equation for $u' \in U_{\tau'}$ after multiplication by $\dot{s}_k^{-1}$, and the lemma is proved in this case.

For the proof in case (ii) recall that $\tau_k\tau_{k-1}^{-1} = 1$ and $\ell(s_k y) < \ell(y)$. Then, using the standard form for u, the structure equation satisfied by $u = x_{\alpha_k}(t)\dot{s}_k\tilde{u}\dot{s}_k^{-1} \in U_\tau$ is

$$x_{\alpha_k}(t)\dot{s}_k\tilde{u}\dot{s}_k^{-1}\dot{w}u_1 s = \dot{y}v\dot{x}^{-1},$$

with $x_{\alpha_k}(t) \in U_{\alpha_k}^*$, $\tilde{u} \in U_{s_k^{-1}w}$, $s \in T$, $u_1 \in U$, $v \in U_{x^{-1}}$. We want to derive an equation of the form

$$u's_k^{-1}wu_1's' = \dot{y}v'\dot{x}^{-1}$$

with $u' = \pi(x_{\alpha_k}(-t^{-1})\tilde{u}) \in U_{\tau'}$, $s' \in T$, $u_1' \in U$ and $v' \in U_{x^{-1}}$, where π is the projection $U \to U_{s_k^{-1}w}$ as in the proof of Lemma 3.3. As

$$\dot{s}_k x_{\alpha_k}(-t^{-1})\dot{s}_k^{-1} = \dot{s}_k^{-1} x_{\alpha_k}(-t^{-1})\dot{s}_k,$$

we can multiply the equation for $x_{\alpha_k}(t)\dot{s}_k\tilde{u}\dot{s}_k^{-1} \in U_\tau$ by $\dot{s}_k\dot{s}_k{}^{-1}$ and apply the SL_2-IDENTITY from Section 2 to obtain

$$\dot{s}_k x_{\alpha_k}(-t^{-1})\dot{s}_k h_{\alpha_k}(t) x_{\alpha_k}(-t^{-1})\tilde{u}\dot{s}_k^{-1}\dot{w}u_1 s = \dot{y}v\dot{x}^{-1}.$$

One has $x_{\alpha_k}(-t^{-1})\tilde{u} = \pi(x_{\alpha_k}(-t^{-1})\tilde{u})u^*$ for $u^* \in U_{s_k^{-1}ww_0}$ so the equation becomes

$$\pi\big(x_{\alpha_k}(-t^{-1})\tilde{u}\big)\dot{s}_k^{-1}\dot{w}(\dot{s}_k^{-1}\dot{w})^{-1}u^*\dot{s}_k^{-1}\dot{w}u_1 s = \dot{y}\dot{y}^{-1}\big(\dot{s}_k x_{\alpha_k}(-t^{-1})\dot{s}_k^{-1}\dot{s}_k^2 h_{\alpha_k}(t)\big)^{-1}\dot{y}v\dot{x}^{-1}$$

where $(\dot{s}_k^{-1}\dot{w})^{-1}u^*\dot{s}_k^{-1}\dot{w} \in U$ because $u^* \in U_{s_k^{-1}ww_0}$. Note also that $\dot{s}_k^2 h_{\alpha_k}(t) = h_{\alpha_k}(-t)$, and that the right side of the equation is

$$\dot{y}\dot{y}^{-1}\big(h_{\alpha_k}(-t)\big)^{-1}\dot{y}\dot{y}^{-1}\dot{s}_k\big(x_{\alpha_k}(-t^{-1})^{-1}\big)\dot{s}_k^{-1}\dot{y}v\dot{x}^{-1}.$$

Because $\ell(s_k y) < \ell(y)$, one has $\dot{y}^{-1}\dot{s}_k x_{\alpha_k}(-t^{-1})\dot{s}_k^{-1}\dot{y} \in U$, and we consider first the case where $\dot{y}^{-1}\dot{s}_k x_{\alpha_k}(-t^{-1})\dot{s}_k^{-1}\dot{y} \in U_{x^{-1}}$. Then the equation above becomes the structure equation for $u' \in U_{\tau'}$ with $u' = \pi(x_{\alpha_k}(-t^{-1})\tilde{u})$, $u_1' = (\dot{s}_k^{-1}\dot{w})^{-1}u^*\dot{s}_k^{-1}\dot{w}u_1$,

$$s' = s\big(\dot{x}\dot{y}^{-1}(h_{\alpha_k}(-t))^{-1}\dot{y}\dot{x}^{-1}\big)^{-1},$$

and v' is

$$\dot{y}^{-1}\dot{s}_k\big(x_{\alpha_k}(-t^{-1})\big)^{-1}\dot{s}_k^{-1}\dot{y}v \in U_{x^{-1}}$$

conjugated by $\dot{y}^{-1}h_{\alpha_k}(-t)^{-1}\dot{y}$. Note that $\dot{y}^{-1}h_{\alpha_k}(-t)^{-1}\dot{y} \in T$, and that we have conjugated this element past $\dot{y}v\dot{x}^{-1}$ and brought the result to the left-hand side as a factor of s'. We have also used the fact that $U_{x^{-1}}$ is invariant under conjugation by elements of T.

For the reversibility, consider u', u_1', s', v' and $u = x_{\alpha_k}(t)\dot{s}_k\tilde{u}\dot{s}_k^{-1}$ in U_τ. Then $x_{\alpha_k}(-t^{-1})\tilde{u} = u'u^*$, so $u^* = (u')^{-1}\big(x_{\alpha_k}(-t^{-1})\tilde{u}\big)$. Then $s = s'(\dot{x}\dot{y}^{-1}h_{\alpha_k}(-t)\dot{y}\dot{x}^{-1})$, $u_1 = \big((\dot{s}_k^{-1}\dot{w})^{-1}u^*\dot{s}_k^{-1}\dot{w}\big)^{-1}u_1'$ and

$$v = \big(\dot{y}^{-1}\dot{s}_k x_{\alpha_k}(-t^{-1})\dot{s}_k^{-1}\dot{y}\big)^{-1}\dot{y}^{-1}h_{\alpha_k}(-t)^{-1}\dot{y}v'\dot{y}^{-1}h_{\alpha_k}(-t)\dot{y} \in U_{x^{-1}},$$

completing the proof of reversibility in this case, using the fact again that $U_{x^{-1}}$ is invariant under conjugation by elements of T.

Now we have to discuss the case $\dot{y}^{-1}\dot{s}_k x_{\alpha_k}(-t^{-1})\dot{s_k^{-1}}\dot{y} \notin U_{x^{-1}}$. Then we obtain a new formula for v' as follows. We have

$$\dot{y}^{-1}\dot{s}_k x_{\alpha_k}(-t^{-1})^{-1}\dot{s_k^{-1}}\dot{y}v = v'v'',$$

with uniquely determined factors $v' \in U_{x^{-1}}$ and $v'' \in U_{x^{-1}w_0}$. Then $v''\dot{x}^{-1} = \dot{x}^{-1}\dot{x}v''\dot{x}^{-1}$ with $\dot{x}v''\dot{x}^{-1} \in U$. Then the structure equation in this case for $u' =$

$\pi(x_{\alpha_k}(-t^{-1})\tilde{u})$ has as factors $u_1' = (\dot{s}_k^{-1}\dot{w})^{-1}u^*\dot{s}_k^{-1}\dot{w}u_1 s'(\dot{x}v''\dot{x}^{-1})^{-1}(s')^{-1}$, s' as in the first case, and v' as defined at the beginning of this paragraph, conjugated by $\dot{y}^{-1}h_{\alpha_k}(-t)^{-1}\dot{y}$.

For the reversibility in this case, suppose we have $u = x_{\alpha_k}(t)\dot{s}_k\tilde{u}\dot{s}_k^{-1} \in U_\tau$, $u' = \pi(x_{\alpha_k}(-t^{-1})\tilde{u})$, and u_1', s', and v' as above. We have to solve for u_1, s, and v. Then, after reversing the conjugation by $\dot{y}^{-1}h_{\alpha_k}(-t)^{-1}\dot{y}$, we obtain

$$\left(\dot{y}^{-1}\dot{s}_k x_{\alpha_k}(-t^{-1})\dot{s}_k^{-1}\dot{y}\right)^{-1}v' = v(v'')^{-1}.$$

Then v is the projection of the left-hand side in $U_{x^{-1}}$. Then from

$$u_1' = (\dot{s}_k^{-1}\dot{w})^{-1}u^*\dot{s}_k^{-1}\dot{w}u_1 s'(\dot{x}v''\dot{x}^{-1})^{-1}(s')^{-1},$$

we can express u^* as the projection of $x_{\alpha_k}(-t^{-1})\tilde{u}$ in $U_{s_k^{-1}ww_0}$ and $(\dot{x}v''\dot{x}^{-1})^{-1}$ from $x_{\alpha_k}(-t^{-1})$ and v', so we recover u_1. Finally the element s is computed as in the first case. Therefore we have obtained u_1, s, and v, completing the proof of reversibility in this case.

In case (iii), the structure equation satisfied by $u = x_{\alpha_k}(t)\dot{s}_k\tilde{u}\dot{s}_k^{-1} \in U_\tau$ is

$$x_{\alpha_k}(t)\dot{s}_k\tilde{u}\dot{s}_k^{-1}\dot{w}u_1 s = \dot{y}v\dot{x}^{-1}$$

as in case (ii). This time $\ell(s_k y) > \ell(y)$, so $\dot{y}^{-1}x_{\alpha_k}(t)\dot{y} \in U$ and the structure equation for $\tilde{u} \in U_{\tau'}$ becomes

$$\tilde{u}\dot{s}_k^{-1}\dot{w}u_1 s = \dot{s}_k^{-1}\dot{y}\dot{y}^{-1}x_{\alpha_k}(t)^{-1}\dot{y}v\dot{x}^{-1},$$

and the rest of the proof is handled as in case (ii), depending on whether $\dot{y}^{-1}x_{\alpha_k}(t)\dot{y} \in U_{x^{-1}}$ or not. This completes the proof of the lemma.

4. Solution of the structure equation

Let $\ell, m, n \in N^*$ be representatives of w, x, y in W, and let $w = s_k \cdots s_1$ be a fixed reduced expression of w. From the Introduction, the structure constants of standard basis elements c_ℓ, c_m, c_n of the Hecke algebra H of an induced representation $\gamma = \psi^G$ of G are given by the formula

$$[c_\ell c_m : c_n] = \sum_{u\ell u_1 = nvm^{-1} \in U\ell U \cap nU_{m^{-1}}m^{-1}} \psi((uu_1)^{-1}v),$$

where $U\ell U \cap nU_{m^{-1}}m^{-1}$ is the set of elements $u \in U$, $u_1 \in U$, $v \in U_{m^{-1}}$ satisfying the equation

$$u\ell u_1 = nvm^{-1}.$$

The elements ℓ, m, n are multiples of $\dot{w}$, $\dot{x}$, $\dot{y}$ by elements of T, $\ell = \dot{w}s$, $m = \dot{x}s'$, etc. for elements s, s', s'' in T. The equations $u\ell u_1 = nvm^{-1}$ as above can be rewritten in the form $u\dot{w}\hat{u}_1\hat{s} = \dot{y}\hat{v}\dot{x}^{-1}$ for $u \in U$, $\hat{s} = s(\dot{x}s''(s')^{-1}\dot{x}^{-1})^{-1} \in T$, $\hat{u}_1 = su_1s^{-1} \in U$, $\hat{v} = s''v(s'')^{-1} \in U_{x^{-1}}$, using the fact that the subsets U_x, for $x \in W$, are invariant under conjugation by elements of T. Each element $u \in U$ satisfying the equation above belongs to the set $U(w, x, y)$, and consequently $u \in U_\tau$ for a cell U_τ defined by a K-sequence τ for the elements $\{w, x, y\}$ in W. An algorithm for the solutions u, $\hat{u}_1$, $\hat{v}$ of these equations is the main result of this section.

As $u \in U_\tau$ is known in terms of the root subgroups from the main result of [Curtis 1988], the problem is to calculate $\hat{u}_1$ and $\hat{v}$ in terms of a given expression of u.

The following remarks may throw some light on these problems. Let τ be a K-sequence for w, x, y in W, and let U_τ be the corresponding cell in $U(w, x, y)$. Each element $u \in U_\tau$ satisfies a structure equation

$$u\dot{w}b = \dot{y}v\dot{x}^{-1}$$

with $b \in B$ and $v \in U_{x^{-1}}$. It is known that the elements b and v in the structure equation are uniquely determined by u [Curtis 2009, Lemma 2.4]. A main theorem in [Curtis 2009] was an inductive construction of the solutions of the structure equation. Theorem 4.1 below gives more information, and in a sense, calculates b and v from an expression of u in standard form using a fixed reduced expression of w. In particular, this result determines, for each element $u \in U_\tau$, the solutions (u_1, v), of the equations $u\ell u_1 = nvm^{-1}$ with $u \in U_\tau$, $u_1 \in U$, and $v \in U_{x^{-1}}$, needed for the structure constants of H.

Theorem 4.1. *Let w, x, y be elements of W, and let U_τ be a cell associated with a K-sequence τ for w, x, y, and a fixed reduced expression $w = s_k \cdots s_1$ for w. The algorithm given below determines the set of elements $s \in T$, $u_1 \in U$ and $v \in U_{x^{-1}}$ satisfying the equation $u\dot{w}u_1s = \dot{y}v\dot{x}^{-1}$, for a given element $u \in U_\tau$. The possibility that the set of solutions is empty is not excluded.*

Let w, x, y, the cell U_τ, and $w = s_k \cdots s_1$ be as in the hypothesis of the theorem. Let u be a fixed element of U_τ. With these as a starting point, the algorithm gives the elements $s \in T$, $u_1 \in U$ and $v \in U_{x^{-1}}$ satisfying the equation stated in the theorem. It is proved by induction on $\ell(w)$.

We begin with the case $\ell(w) = 1$, so $\dot{w} = \dot{s}_1$ and let $\tau = (\tau_1, \tau_0)$ be a K-sequence for (s_1, x, y) corresponding to one of the three cases in Lemma 3.1.

Case (i). $\tau_1 = s_1$, $\ell(s_1y) < \ell(y)$. Then $U_{\tau_0} = 1$, $U_\tau = \dot{s}_1U_{\tau_0}\dot{s}_1^{-1} = 1$, $s_1x = y$ and it is easily proved using Lemma 83 of [Steinberg 1968] that $\dot{s}_1\dot{x} = \dot{y}$. Then there is a unique solution, namely $(1, 1, 1)$, of the structure equation $u\dot{s}_1b = \dot{s}_1\dot{x}v(\dot{x})^{-1}$.

Case (ii). $\tau_1 = \tau_0$, and $\ell(s_1y) < \ell(y)$. In this case the definition of K-sequence implies $x = y$. We also have (by part (ii) of the proof of Lemma 3.3) $U_\tau = U^*_{\alpha_1}$. First assume $\ell(x) = 1$. Then the assumptions imply that $x = s_1$, $\dot{x} = \dot{y} = \dot{s}_1$, and for each element $u \in U^*_{\alpha_1}$ there is a unique solution of the structure equation $u\dot{s}_1b = \dot{s}_1v\dot{s}_1^{-1}$

by the SL_2-IDENTITY, so that quadruples (u, s, u_1, v) with $u \in U_\tau$, $b = u_1 s \in B$, and $v \in U_{x^{-1}}$ satisfying the equation exist, and are known. Now let $\ell(x) > 1$; then $\ell(s_1 x) < \ell(x)$ implies $\dot{x} = \dot{s}_1 \dot{x}_1$ with $\ell(s_1 x_1) > \ell(x_1)$. For each $v \in U^*_{\alpha_1}$ one has $\dot{x}_1^{-1} v \dot{x}_1 \in U$ because $\ell(s_1 x_1) > \ell(x_1)$. Moreover

$$\dot{s}_1 \dot{x}_1 (\dot{x}_1^{-1} v \dot{x}_1) \dot{x}_1^{-1} \dot{s}_1^{-1} = \dot{s}_1 v \dot{s}_1^{-1} \in U_-$$

so $\dot{x}_1^{-1} v \dot{x}_1 \in U_{x^{-1}}$. The unique solution of the structure equation $u \dot{s}_1 b = \dot{s}_1 v \dot{s}_1^{-1}$ with $u \in U^*_{\alpha_1}$ from the case $\ell(x) = 1$ now yields the unique solution $(u, b, \dot{x}_1^{-1} v \dot{x}_1)$ of the structure equation for (s_1, x, y), namely

$$u \dot{s}_1 b = \dot{y} \dot{x}_1^{-1} v \dot{x}_1 \dot{x}^{-1}$$

as $\dot{y} \dot{x}_1^{-1} = \dot{s}_1$ and $\dot{x}_1 \dot{x}^{-1} = \dot{s}_1^{-1}$. Note that in case (ii) there is no solution of the structure equation $u \dot{s}_1 b = \dot{y} v \dot{x}^{-1}$ in case $u = 1$ and $x = y$ as this would contradict the fact that $B \dot{y}^{-1} \dot{s}_1 B \neq B \dot{x}^{-1} B$ by the uniqueness part of the Bruhat decomposition.

Case (iii). $\tau_1 = s_1$, and $\ell(s_1 y) > \ell(y)$. In this case $U_\tau = U_{\alpha_1}$ and the unique solution of the structure equation $u \dot{s}_1 b = \dot{y} v \dot{x}^{-1}$ for $u \in U_{\alpha_1}$ is

$$(u, h_{\alpha_1}(-1), \dot{y}^{-1} u \dot{y}),$$

for each $u \in U_{\alpha_1}$, noting that $\dot{y}^{-1} u \dot{y} \in U_{x^{-1}}$, and $\dot{y} \dot{x}^{-1} = \dot{s}_1^{-1} = \dot{s}_1 h_{\alpha_1}(-1)$ by [Steinberg 1968, Lemma 83] again. This completes the discussion of the solutions of the structure equation for the case $\ell(w) = 1$.

We now proceed to the general case, with $\ell(w) > 1$. Let τ be a K-sequence for w, x, y and let $u \in U_\tau$. Let u correspond to $u' = \lambda(u) \in U_{\tau'}$ (as in Lemma 3.3) for the K-sequence τ' for $s_{k-1} \dots s_1, x', y'$ in one of the three cases of Lemma 3.1, and let

$$u' s_{k-1} \dots s_1 u'_1 s' = \dot{y}' v' \dot{x}'$$

be the structure equation satisfied by u'. By the induction hypothesis, the factors u'_1, s', and v' of the structure equation for u' are determined by u'. As $u' = \lambda(u)$, the elements u_1, s, and v satisfying the equation $u \dot{w} b = u \dot{w} u_1 s = \dot{y} v \dot{x}^{-1}$ are determined by u, using Lemma 3.4. This completes the proof of the theorem.

At the beginning of the section, it was explained how the solutions $u \in U$, $u_1 \in U$, and $v \in U_{x^{-1}}$ of the equations $u \ell u_1 = n v m^{-1}$, for $\ell, m, n \in N^*$, required for the formulas for the structure constants are obtained from the solutions $u \in U_\tau$, $u_1 \in U$, $s \in T$, $v \in U_{x^{-1}}$ of the equations $u \dot{w} u_1 s = \dot{y} v \dot{x}^{-1}$ solved by the algorithm in Theorem 4.1. For this step, it is necessary to determine the elements t, t', t'' in T such that $\ell = \dot{w} t$, $m = \dot{x} t'$, $n = \dot{y} t''$, in order to transform the first set of equations to the second by the algorithms for multiplication the Chevalley group. This information can be obtained from Steinberg's proof of Theorem 49 in [Steinberg 1968, §14], in case ψ^G is a Gelfand–Graev representation. The theorem states that the Hecke algebra H of a Gelfand–Graev representation is a commutative algebra, and the proof is obtained by constructing a certain antiautomorphism f of the Chevalley group G whose extension to the group algebra is at the same time an antiautomorphism of the group algebra and whose restriction to the Hecke algebra H is the identity. As shown in [Steinberg 1968], the representatives $\ell \in N^*$ of the basis elements of H have the form $t\dot{w}$ for elements $w \in W$ such that $w = w_0 w_\pi$, where w_0 is the element of maximal length in W and w_π is the element of maximal length in the subgroup of the Weyl group generated by the reflections taken from a subset π of the set of simple roots, and t is an element of T such that $t\dot{w}$ is fixed by the antiautomorphism f. From the discussion on page 262 of [Steinberg 1968], it follows directly that $\dot{w}t$, with $w = w_0 w_\pi$ as above, represents a basis element of H, fixed by the antiautomorphism f, whenever t commutes with w_π.

We recall the connection between the solutions of the equation $u\ell u_1 = nvm^{-1}$ and the solutions of the equation $u\dot{w}\hat{u}_1\hat{s} = \dot{y}\hat{v}\dot{x}^{-1}$ with $\ell = \dot{w}s$, $m = \dot{x}s'$, $n = \dot{y}s''$, $s, s', s'' \in T$ and $\hat{u}_1 = su_1s^{-1} \in U$, $\hat{v} = s''v(s'')^{-1} \in U_{x^{-1}}$, for $u \in U$. For a solution $u \in U$, we have $u \in U(w, x, y)$ so $u \in U_\tau$ for a K-sequence τ for w, x, y. We can now state a formula for the structure constants $[c_\ell c_m : c_n]$ based on Theorem 4.1.

Corollary 4.2. *The structure constants are given by the formula*

$$[c_\ell c_m : c_n] = \sum_{\tau} \sum_{u \in U_\tau} \psi((uu_1)^{-1}v)$$

where for each K-sequence τ for w, x, y, the sum is taken over solutions of the equation $u\dot{w}\hat{u}_1\hat{s} = \dot{y}\hat{v}\dot{x}^{-1}$ obtained by Theorem 4.1, with $u \in U_\tau$ and $\hat{u}_1$, $\hat{v}$, $\hat{s}$ satisfying the conditions $\hat{u}_1 = su_1s^{-1} \in U$, $\hat{v} = s''v(s'')^{-1} \in U_{x^{-1}}$ and $\hat{s} = s(\dot{x}s''(s')^{-1}\dot{x}^{-1})^{-1} \in T$. If there are no solutions satisfying these conditions, then the structure constant is zero.

We end this section with two problems for further research.

1. The first problem is to apply the algorithm obtained in Theorem 4.1 and Corollary 4.2 to obtain formulas for the structure constants $[c_\ell c_m : c_n]$ which can be used to give a combinatorial proof of the existence of the homomorphisms f_T mentioned in the Introduction.

2. The second problem is to develop a theory of cells for Chevalley groups over a p-adic field K, using the Bruhat decomposition for these groups obtained by Iwahori and Matsumoto [1965].

5. Example: application to $\mathbf{SL_2}(k)$

Let G be the Chevalley group $\mathrm{SL}_2(k)$ for a finite field k of odd characteristic, and let H be the Hecke algebra of a Gelfand–Graev representation of G. Gelfand and

Graev [1962a] stated formulas for the structure constants of the standard basis of H, and calculated the irreducible representations of H.

As an application of the ideas in Section 4, we shall calculate the structure constants of H relative to the standard basis of the Hecke algebra, and apply them to give a self-contained proof, different from the one obtained by Gelfand and Graev, of formulas for the irreducible representations of H (for another approach, using the Deligne-Lusztig character formula, see [Curtis 1993, §5]).

We begin with a Gelfand–Graev character ψ^G of $G = \mathrm{SL}_2(k)$, for a linear character ψ of U in general position. Then we may assume that

$$\psi\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix} = \chi(\alpha), \quad \begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix} \in U, \quad \alpha \in k$$

for a nontrivial additive character χ on k. The standard basis elements of the Hecke algebra H of ψ^G are the elements

$$c_\lambda = q e_\psi n_\lambda e_\psi, \quad n_\lambda = \begin{pmatrix}0 & \lambda\\ -\lambda^{-1} & 0\end{pmatrix}, \quad \lambda \neq 0, \quad q = |k|$$

together with the identity element e_ψ and one other basis element $e_{-1} = e_\psi\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)e_\psi$, where

$$e_\psi = |U|^{-1}\sum_{u\in U}\psi(u^{-1})u$$

as in the Introduction.

Lemma 5.1. *The algebra H is commutative with identity element e_ψ. One has $e_{-1}^2 = e_\psi$, and $e_{-1}c_\lambda = c_{-\lambda}$ for each $\lambda \neq 0$. The other nonzero structure constants of H for the standard basis elements are as follows. For c_λ, c_μ, c_ν as above one has*

$$[c_\lambda c_\mu : c_\nu] = \chi(\lambda\mu\nu^{-1} + \lambda\mu^{-1}\nu + \lambda^{-1}\mu\nu),$$

and

$$[c_\lambda c_\lambda : e_{-1}] = q, \quad [c_\lambda c_{-\lambda} : e_\psi] = q,$$

for $\lambda, \mu, \nu \neq 0$ in k, and $q = |k|$.

The structure constants are computed using the formula at the beginning of §4 and the solutions of the structure equation

$$u\ell u_1 = nv(m)^{-1}$$

with $\ell, m, n \in N$ (see [Curtis 2009, §3] for more details).

The group $G = \mathrm{SL}_2(k)$ can be viewed as the group of fixed points by the usual Frobenius endomorphism F of the semisimple algebraic group $\mathrm{SL}_2(\bar{k})$, over the algebraic closure $\bar{k}$ of k. There are two conjugacy classes of F-stable maximal tori in $\mathrm{SL}_2(\bar{k})$ with representatives in the finite group G given by the *split torus*

T_0 consisting of the matrices $\left(\begin{smallmatrix}\mu & 0\\ 0 & \mu^{-1}\end{smallmatrix}\right)$ with $\mu \neq 0$ in k, and the *Coxeter torus* T_1, isomorphic to the set C of elements ξ of norm 1 in the quadratic extension of k, that is, $\xi^{q+1} = 1$. The main theorem on the representations of the Hecke algebra H of a Gelfand–Graev representation of G states that the irreducible representations of H factor through the group algebra of one of the maximal tori of G. More precisely, one has:

Theorem 5.2. *Each irreducible representation f of the Hecke algebra H of a Gelfand–Graev representation of G can be factored as*

$$f = \theta \circ f_T,$$

where f_T is a homomorphism, independent of θ, of H into the group algebra of a maximal torus T of G, and θ is an irreducible representation of the group algebra of the maximal torus. The homomorphisms from H into the group algebras of the two types of maximal tori are given as follows. For the split torus T_0, consisting of diagonal matrices with entries in k^*, the homomorphism $f_{T_0} : H \to \mathbb{C}T_0$ is given by

$$f_{T_0}(c_\lambda) = \sum_t \chi(\lambda(t+t^{-1}))\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}, \quad t \in k^*, \quad \text{and} \quad f_{T_0}(e_{-1}) = \begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix},$$

where c_λ is a standard basis element of H as above. For the Coxeter torus, the homomorphism $f_{T_1} : H \to \mathbb{C}C$ is given by

$$f_{T_1}(c_\lambda)(\xi) = -\chi(\lambda(\xi+\xi^{-1})), \quad \xi \in C, \quad \text{and} \quad f_{T_1}(e_{-1}) = \xi_{-1},$$

where ξ_{-1} is the unique element in C of order two.

Lemma 5.3. *Let $a, b \in k$. Then*:

(i) $$\sum_{t\in k^*} \chi(at) = -1 + q\delta_{a,0}.$$

(ii) $$\sum_{t\in k^*} \chi(at+bt^{-1}) = \sum_{t\in k^*} \chi(t+abt^{-1}) + q\delta_{a,0}\delta_{b,0}.$$

(iii) *For the Coxeter torus C, we first note that $\xi+\xi^{-1} \in k$ because $\xi^{q+1} = 1$ for $\xi \in C$ implies that $\xi+\xi^{-1} = \xi+\xi^q \in k$. Let $\xi \in C$, $\eta \in F_{q^2}$. Then*

$$\sum_{\xi\in C} \chi(\xi\eta+\xi^q\eta^q) = -\sum_{t\in k^*} \chi(t+\eta\eta^q t^{-1}) + \delta_{\eta,0}q.$$

We refer to [Chang 1976, Lemma 1.2]. Part (iii) is proved using an analysis of quadratic equations over k. The result, and extensions of it to F-stable maximal tori in general finite reductive groups, are suggested by the Davenport–Hasse Theorem on Gauss sums.

For the proof that f_{T_0} is a homomorphism, let c_λ, c_μ, c_ν be standard basis elements of H as above. Then

$$c_\lambda c_\mu = \sum_\nu [c_\lambda c_\mu : c_\nu] c_\nu + \delta_{\lambda,\mu} q e_{-1} + \delta_{\lambda,-\mu} q e_\psi,$$

with the structure constants as in Lemma 5.1. We have

$$\begin{aligned} f_{T_0}(c_\lambda) f_{T_0}(c_\mu) &= \sum_{t\in k^*}\sum_{s\in k^*} \chi\big(\lambda(ts+t^{-1}s^{-1})\big)\chi\big(\mu(s+s^{-1})\big)\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \\ &= \sum_t \sum_s \chi\big((\lambda t+\mu)s + (\lambda t^{-1}+\mu)s^{-1}\big)\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \\ &= \sum_t \sum_{s'} \chi\big(s' + (\lambda^2+\mu^2+\lambda\mu(t+t^{-1})(s')^{-1})\big)\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \end{aligned}$$

by Lemma 5.3(ii), and have to show this is equal to

$$\begin{aligned} &\sum_t \sum_\nu [c_\lambda c_\mu : c_\nu] f_{T_0}(c_\nu)(t)\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} + \delta_{\lambda,\mu} q \begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix} + \delta_{\lambda,-\mu} q \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \\ &\qquad = \sum_t \sum_\nu \chi\big(\lambda\mu\nu^{-1} + \lambda\mu^{-1}\nu + \lambda^{-1}\mu\nu + \nu(t+t^{-1})\big)\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}, \end{aligned}$$

etc. The result is clear, by another application of Lemma 5.3(ii), in case $\lambda \neq \pm\mu$.

Now let $\lambda = \mu$. The expressions to be checked agree except possibly at $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1\end{smallmatrix}\right)$. At $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1\end{smallmatrix}\right)$, the first expression becomes

$$\sum_s \chi\big(\lambda(t+1)s + \lambda(t^{-1}+1)s^{-1}\big)$$

with $t=-1$, which is $q-1$ by Lemma 5.3. The second expression at $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1\end{smallmatrix}\right)$ becomes

$$\sum_\nu \chi\big(\nu + (2\lambda^2 + (-2\lambda^2))\nu^{-1}\big) + q = -1+q$$

by Lemma 5.3 again, completing the proof in this case. The proof in case $\lambda = -\mu$ is similar and will be omitted.

For the homomorphism from H into the group algebra of C, we first have

$$\begin{aligned} f_{T_1}(c_\lambda) f_{T_1}(c_\mu) &= \sum_{\xi\in C}\sum_{\eta\in C} \chi\big(\lambda(\xi\eta + (\xi\eta)^{-1}) + \mu(\eta+\eta^{-1})\big)\xi \\ &= \sum_\xi \sum_\eta \chi\big((\lambda\xi+\mu)\eta + (\lambda\xi^{-1}+\mu)\eta^{-1}\big)\xi \\ &= -\sum_\xi \sum_{t\in k^*} \chi\big(t + (\lambda\xi+\mu)(\lambda\xi^q+\mu)t^{-1}\big)\xi \end{aligned}$$

by Lemma 5.3(ii). We have to show that this is equal to

$$\sum_{\xi\in C}\sum_{\nu\in k^*}\chi\left(\lambda\mu\nu^{-1}+\lambda\mu^{-1}\nu+\lambda^{-1}\mu\nu\right)f_{T_1}(c_\nu))\xi+\delta_{\lambda,\mu}qf_{T_1}(e_{-1})+\delta_{\lambda,-\mu}qf_{T_1}(e_\psi)$$
$$=-\sum_{\xi\in C}\sum_{\nu\in k^*}\chi\big(\nu(\xi+\xi^{-1}+\lambda\mu^{-1}+\lambda^{-1}\mu)+\nu^{-1}\lambda\mu\big)\xi+\delta_{\lambda,\mu}qf_{T_1}(e_{-1})+\delta_{\lambda,-\mu}qf_{T_1}(e_\psi)$$
$$=-\sum_{\xi}\sum_{\nu'}\big(\chi(\nu'+\nu'^{-1}(\xi+\xi^{q})\lambda\mu+\lambda^2+\mu^2)\big)\xi+\delta_{\lambda,\mu}qf_{T_1}(e_{-1})+\delta_{\lambda,-\mu}qf_{T_1}(e_\psi),$$

where we have used Lemma 5.3(ii) and Lemma 5.1 for the structure constant formula. Together, these formulas prove the multiplication formula in case $\lambda\neq\pm\mu$.

In case $\lambda=\mu$, it is only necessary to check the expressions at $\xi=\xi_{-1}$, where ξ_{-1} is the unique element of C such that $\xi_{-1}^2=1$, $\xi_{-1}\neq 1$, so $\xi_{-1}=-1$ in F_{q^2}. The contribution from the first expression is

$$\sum_{\eta}\chi\left(\lambda(\xi_{-1}+1)\eta+\lambda(\xi_{-1}^{-1}+1)\eta^{-1}\right)=-\sum_{t\in k^*}\chi\left(t+(2\lambda^2+\lambda^2(\xi_{-1}+\xi_{-1}^{q}))t^{-1}\right)+q$$

by Lemma 5.3(iii). As $\xi_{-1}+\xi_{-1}^{q}=-2$ in F_{q^2}, this expression is equal to $1+q$ by Lemma 5.3. For the second expression at ξ_{-1} we obtain

$$-\sum_{\nu'}\chi\left(\nu'+(\nu')^{-1}((\xi_{-1}+\xi_{-1}^{q})\lambda^2+2\lambda^2)\right)+q=q+1,$$

completing the proof in this case.

For the remaining case $\lambda=-\mu$ it is only necessary to check both expressions at $\xi=1$ and this is immediate.

Corollary 5.4. *The formulas for the irreducible representations of the Hecke algebra H are*

$$f(c_\lambda)=\theta\circ f_T(c_\lambda)=\sum_{t\in T}\chi\left(\lambda(t+t^{-1})\right)\theta(t),\qquad f(e_{-1})=\theta(-1)$$

for the split torus T, and an irreducible representation θ of T, and

$$f(c_\lambda)=\pi\circ f_{T_1}(c_\lambda)=-\sum_{\xi\in C}\chi\left(\lambda(\xi+\xi^{-1})\right)\pi(\xi),\qquad f(e_{-1})=\pi(-1)$$

for the Coxeter torus T_1 represented by C, and an irreducible representation π of C.

The fact that all the irreducible representations of H are obtained in this way follows by a counting argument.

Gelfand and Graev [1962a] obtained these formulas, and pointed out that they are similar to the integral formulas for Bessel functions over $\mathbb{C}$ (see [Whittaker and Watson 1927, Chapter XVII]). They mentioned that the formulas in Corollary 5.4 can be called *Bessel functions over finite fields*.

The group $G = \mathrm{SL}_2(k)$, for $q = |k|$ odd, has two classes of Gelfand–Graev representations. For a determination of the irreducible characters of G, and how they appear in the Gelfand–Graev representations, including for example the subtle cases of those of degree $\frac{1}{2}(q+1)$ and $\frac{1}{2}(q-1)$, see [Gelfand and Graev 1962a, §4] and [Digne and Michel 1991, §15.9].

6. Example: the homomorphisms f_T associated with principal series representations of finite Chevalley groups

We return to the set-up described in the Introduction, with G a Chevalley group over a finite field k, with a Borel subgroup $B = UT$ containing the torus T, and Weyl group W. Let ψ^G be a Gelfand–Graev representation of G, and $H = e\mathbb{C}Ge$ the Hecke algebra associated with it, with

$$e = |U|^{-1} \sum_{u \in U} \psi(u^{-1})u$$

(remember that H is a commutative algebra!). In this section, we give a character theoretic construction of a homomorphism $f_T : H \to \mathbb{C}T$ and the resulting irreducible representations of H. An open problem is to find a combinatorial construction of homomorphisms f_T for twisted tori T. Such a result would define a family of functions associated with the Hecke algebra H and maximal tori in G, starting from the Bessel functions over k in the case of $\mathrm{SL}_2(k)$ and the Coxeter torus C. A proof would require information about the structure constants, and extensions of Lemma 5.3(iii), which would be of independent interest. Homomorphisms $f_T : H \to \mathbb{C}T$ from a Gelfand–Graev Hecke algebra H to the group algebra of a maximal torus are known to exist, for a connected reductive algebraic group G defined over a finite field, with Frobenius endomorphism F ([Curtis 1993] and [Bonnafé and Kessar 2008]), and are derived using the trace formula in ℓ-adic cohomology.

We are concerned with the *principal series* representations of G. These are the induced representations λ^G, where λ is a linear character of the Borel subgroup B with U in its kernel, and the irreducible representations of G which occur as constituents of λ^G for some choice of lambda. We require the following result of Kilmoyer [1978, Proposition 6.1].

Lemma 6.1. *Let ψ^G be a fixed Gelfand–Graev character of G. Each induced character λ^G, as above, contains a unique irreducible constituent ξ_λ which appears with multiplicity one in both λ^G and the Gelfand–Graev character ψ^G.*

As in [Curtis 1993], we introduce the notation $\boldsymbol{a}$ for the element of the group algebra $\mathbb{C}G$ given by

$$\boldsymbol{a} = \sum_{g \in G} \alpha(g^{-1})g$$

for a complex valued function α on G. If α is an irreducible character of G, then $\boldsymbol{a}$ is a multiple of the central primitive idempotent in the group algebra associated with α.

Theorem 6.2. *Let $\boldsymbol{l}_\lambda$ and $\boldsymbol{x}_\lambda$ be the elements of the group algebra corresponding to the induced character λ^G and the irreducible character ξ_λ as in Lemma 6.1. Then*

$$e\boldsymbol{l}_\lambda = e\boldsymbol{x}_\lambda \neq 0$$

and affords an irreducible representation $f_{\xi_\lambda} : H \to \mathbb{C}^*$ of H of degree one, such that

$$h(e\boldsymbol{x}_\lambda) = f_{\xi_\lambda}(h)e\boldsymbol{x}_\lambda, \quad h \in H.$$

The representation f_{ξ_λ} is the restriction to H of the unique irreducible character of the group algebra $\mathbb{C}G$ obtained from the character ξ_λ of G as in Lemma 6.1.

By Lemma 6.1, the class function $\boldsymbol{l}_\lambda = \boldsymbol{x}_\lambda + \boldsymbol{y}$ where $\boldsymbol{y}$ is a linear combination of central primitive idempotents corresponding to irreducible characters of G which do not appear in the Gelfand–Graev representation ψ^G. Then $e\boldsymbol{y} = 0$, so $e\boldsymbol{l}_\lambda = e\boldsymbol{x}_\lambda \neq 0$ and $e\boldsymbol{x}_\lambda$ is a nonzero multiple of the primitive central idempotent in H affording the irreducible representation f_{ξ_λ} of H of degree one, as in the statement of the theorem, by [Curtis and Reiner 1981, Corollary 11.26 and Theorem 11.25]. The last statement of the theorem also follows from [Curtis and Reiner 1981, Theorem 11.25].

Theorem 6.3. *Let λ be an irreducible character of B with U in its kernel. Let ξ_λ be the irreducible character of G which appears with multiplicity one in λ^G and in the Gelfand–Graev character ψ^G, and let $f_{\xi_\lambda} : H \to \mathbb{C}$ be the irreducible representation of the Hecke algebra H of the Gelfand–Graev representation ψ^G defined in Theorem 6.2. There exists a unique homomorphism of algebras $f_T : H \to \mathbb{C}T$, independent of λ, such that, for each linear character λ of T, one has*

$$f_{\xi_\lambda}(h) = \tilde{\lambda} \circ f_T(h), h \in H,$$

where $\tilde{\lambda}$ is the extension of $\lambda : T \to \mathbb{C}$ to the group algebra $\mathbb{C}T$. The homomorphism f_T is given by the formula $f_T(c_n) = \sum_{t \in T} f_T(c_n)(t)t$, where

$$f_T(c_n)(t) = \operatorname{ind} n |B|^{-1}|U|^{-1} \sum_{g \in G, u \in U, gung^{-1} = tu'} \psi(u^{-1}),$$

for a standard basis element c_n of H and $gung^{-1} = tu'$, $t \in T$, $u' \in U$, is an element of B which projects onto the element $t \in T$ by the homomorphism $B \to T$. If there are no solutions to the equation $gung^{-1} = tu'$, then $f_T(c_n)(t) = 0$.

By the proof of Theorem 6.2, the representation f_{ξ_λ} of the Hecke algebra H is the restriction to H of the unique irreducible character (see Lemma 6.1) ξ_λ extended to the group algebra $\mathbb{C}G$. Moreover, the proof of Theorem 6.2 shows that

$$f_{\xi_\lambda}(h) = \lambda^G(h), \quad h \in H,$$

with λ^G extended to the group algebra, because $\boldsymbol{e}\boldsymbol{l}_\lambda = \boldsymbol{e}\boldsymbol{x}_\lambda$ and $\boldsymbol{e}\boldsymbol{x}_\lambda$ affords the representation f_{ξ_λ} of $H = e\mathbb{C}Ge \subseteq \mathbb{C}G$. In more detail, h lies in H, and is viewed as an element of the group algebra $\mathbb{C}G$. Then $\lambda^G(h)$ is the trace of the action of h on a module M affording the induced character λ^G. As $he = h$, the trace is computed on the module eM, which is one dimensional, and affords the representation f_{ξ_λ} of H, by Theorem 6.2.

For a standard basis element c_n of H, we have

$$c_n = \operatorname{ind} n\, ene = |U|^{-1} \sum_{u_1 n u_2 \in UnU} \psi(u_1^{-1}u_2^{-1}) u_1 n u_2,$$

by [Curtis and Reiner 1981, Proposition 11.30(i)]. Then, with λ^G extended to the group algebra, we obtain

$$\lambda^G(c_n) = |U|^{-1} \sum_{u_1 n u_2 \in UnU} \psi(u_1^{-1}u_2^{-1}) \lambda^G(u_1 n u_2) = \operatorname{ind}\, n |U|^{-1} \sum_{u \in U} \psi(u^{-1}) \lambda^G(un).$$

We have used the fact that the double coset UnU contains $\operatorname{ind} n$ one sided cosets. For the induced character we have, by [Curtis and Reiner 1981, 10.3],

$$\lambda^G(un) = |B|^{-1} \sum_{g \in G} \dot{\lambda}(g^{-1}ung),$$

where $\dot{\lambda}(x) = 0$ if $x \notin B$. Then $\dot{\lambda}(g^{-1}ung) \neq 0$ only if $g^{-1}ung = u't$ with $u' \in U$ and $t \in T$, and in that case, $\dot{\lambda}(g^{-1}ung) = \lambda(t)$. Therefore

$$\lambda^G(c_n) = \operatorname{ind} n |B|^{-1} |U|^{-1} \sum_{t \in T} \sum_{g^{-1}ung = u't} \psi(u^{-1}) \lambda(t).$$

Then, for $t \in T$,

$$f_T(c_n)(t) = \operatorname{ind} n |B|^{-1} |U|^{-1} \sum_{g^{-1}ung = u't} \psi(u^{-1})$$

is independent of λ, and we have

$$f_{\xi_\lambda}(h) = \tilde{\lambda} \circ f_T(h), \quad h \in H.$$

The facts that $f_T : H \to \mathbb{C}T$ is a homomorphism of algebras and is a uniquely determined linear map with the factorization property stated in the theorem both follow from the orthogonality relations for the linear characters λ of T. This completes the proof of the theorem.

It is a nice exercise to derive the formula for the homomorphism $f_{T_0} : H \to \mathbb{C}T_0$ given in Theorem 5.2 from the statement of the preceding theorem.

Theorem 6.3, for principal series representations of finite Chevalley groups, is a special case of Theorem 4.2 in [Curtis 1993] for representations $R_{T,\theta}$ of connected reductive algebraic groups defined over finite fields. The point of including it here is that in the special case of principal series representations, it is possible to give a combinatorial proof of the existence of the homomorphisms f_T.

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COMPLEMENTS ON DISCONNECTED REDUCTIVE GROUPS

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Dedicated to the memory of Robert Steinberg

We present several results on disconnected reductive groups, in particular, on the characteristic-zero representation theory of finite groups of Lie type coming from disconnected reductive groups in positive characteristic. We generalize slightly the setting of our 1994 paper on that subject and show how most of our earlier results extend to the new situation. In particular, we give a classification of quasi-semisimple conjugacy classes over an arbitrary algebraically closed field, and over finite fields; we generalize a formula of Steinberg on the number of unipotent classes to disconnected groups and a formula for the tensor product of the Steinberg character with a Lusztig induced character.

1. Introduction

Let $\boldsymbol{G}$ be a (possibly disconnected) linear algebraic group over an algebraically closed field. We assume that the connected component $\boldsymbol{G}^0$ is reductive, and then call $\boldsymbol{G}$ a (possibly disconnected) reductive group. This situation was studied by Steinberg [1968] where he introduced the notion of quasi-semisimple elements.

Assume now that $\boldsymbol{G}$ is over an algebraic closure $\overline{\mathbb{F}}_q$ of the finite field $\mathbb{F}_q$, defined over $\mathbb{F}_q$ with corresponding Frobenius endomorphism F. Let $\boldsymbol{G}^1$ be an F-stable connected component of $\boldsymbol{G}$. We want to study $(\boldsymbol{G}^0)^F$-class functions on $(\boldsymbol{G}^1)^F$; if $\boldsymbol{G}^1$ generates $\boldsymbol{G}$, they coincide with $\boldsymbol{G}^F$-class functions on $(\boldsymbol{G}^1)^F$.

This setting, adopted here, is also taken up by Lusztig in his series of papers on disconnected groups [Lusztig 2003; 2004a; 2004b; 2004c; 2004d; 2004e; 2005; 2006b; 2006a; 2009] and is slightly more general than the setting of our paper "Groupes réductifs non connexes", which we will refer to as [DM 1994], where we assumed that $\boldsymbol{G}^1$ contains an F-stable quasicentral element. A detailed comparison of both situations is done in the next section.

As the title says, this paper contains a series of complements to [DM 1994] which are mostly straightforward developments that various people have asked us about and that, except when mentioned otherwise (see the introductions to Sections 4 and

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8) have not appeared in the literature, as far as we know; we thank in particular Olivier Brunat, Gerhard Hiss, Cheryl Praeger and Karine Sorlin for asking these questions.

In Section 2 we show how quite a few results of [DM 1994] are still valid in our more general setting.

In Section 3 we take a "global" viewpoint to give a formula for the scalar product of two Deligne–Lusztig characters on the whole of $\boldsymbol{G}^F$.

In Section 4 we show how to extend to disconnected groups the formula of Steinberg [1968, 15.1] counting unipotent elements.

In Section 5 we extend the theorem that tensoring Lusztig induction with the Steinberg character gives ordinary induction.

In Section 6 we give a formula for the characteristic function of a quasi-semi-simple class, extending the case of a quasicentral class which was treated in [DM 1994].

In Section 7 we show how to classify quasi-semisimple conjugacy classes, first for a (possibly disconnected) reductive group over an arbitrary algebraically closed field, and then over $\mathbb{F}_q$.

Finally, in Section 8 we extend to our setting previous results on Shintani descent.

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2. Preliminaries

In this paper, we consider a (possibly disconnected) algebraic group $\boldsymbol{G}$ over $\overline{\mathbb{F}}_q$ (except at the beginning of Section 7 where we accept an arbitrary algebraically closed field), defined over $\mathbb{F}_q$ with corresponding Frobenius endomorphism F. If $\boldsymbol{G}^1$ is an F-stable component of $\boldsymbol{G}$, we define the class functions on $(\boldsymbol{G}^1)^F$ to be the complex-valued functions invariant under $(\boldsymbol{G}^0)^F$-conjugacy (or equivalently under $(\boldsymbol{G}^1)^F$-conjugacy). Note that if $\boldsymbol{G}^1$ does not generate $\boldsymbol{G}$, there may be less functions invariant by $\boldsymbol{G}^F$-conjugacy than by $(\boldsymbol{G}^1)^F$-conjugacy; but the propositions we prove will apply *in particular* to the $\boldsymbol{G}^F$-invariant functions so we do not lose any generality. The class functions on $(\boldsymbol{G}^1)^F$ are provided with the scalar product

$$\langle f, g\rangle_{(\boldsymbol{G}^1)^F} = |(\boldsymbol{G}^1)^F|^{-1} \sum_{h\in(\boldsymbol{G}^1)^F} f(h)\overline{g(h)}.$$

We call $\boldsymbol{G}$ reductive when $\boldsymbol{G}^0$ is reductive.

When $\boldsymbol{G}$ is reductive, following [Steinberg 1968], we call an element quasi-semi-simple if it normalizes a pair $\boldsymbol{T}^0 \subset \boldsymbol{B}^0$ of a maximal torus of $\boldsymbol{G}^0$ and a Borel subgroup of $\boldsymbol{G}^0$. Following [DM 1994, définition-théorème 1.15], we call a quasi-semisimple element σ quasicentral if it has maximal dimension of the centralizer $C_{\boldsymbol{G}^0}(\sigma)$ (that we will also denote by $\boldsymbol{G}^{0\sigma}$) amongst all quasi-semisimple elements of $\boldsymbol{G}^0\cdot\sigma$.

In the sequel, we fix a reductive group $\boldsymbol{G}$ and (except in the next section where we take a "global" viewpoint) an F-stable connected component $\boldsymbol{G}^1$ of $\boldsymbol{G}$. In most of [DM 1994] we assumed that $(\boldsymbol{G}^1)^F$ contained a quasicentral element. Here we do not assume this. Note however that by [DM 1994, proposition 1.34], $\boldsymbol{G}^1$ contains an element σ which induces an F-stable quasicentral automorphism of $\boldsymbol{G}^0$. Such an element will be enough for our purpose, and we fix one from now on.

By [DM 1994, proposition 1.35], if $H^1(F, Z\boldsymbol{G}^0) = 1$ then $(\boldsymbol{G}^1)^F$ contains quasi-central elements. Here is an example where $(\boldsymbol{G}^1)^F$ does not contain quasicentral elements.

Example 2.1. Take $s = \left(\begin{smallmatrix}\xi & 0\\ 0 & 1\end{smallmatrix}\right)$, where ξ is a generator of $\mathbb{F}_q^\times$, take $\boldsymbol{G}^0 = \mathrm{SL}_2$ and let $\boldsymbol{G} = <\boldsymbol{G}^0, s> \subset \mathrm{GL}_2$ endowed with the standard Frobenius endomorphism on GL_2, so that s is F-stable and $\boldsymbol{G}^F = \mathrm{GL}_2(\mathbb{F}_q)$. We take $\boldsymbol{G}^1 = \boldsymbol{G}^0 \cdot s$. Here quasicentral elements are central and coincide with $\boldsymbol{G}^0 \cdot s \cap Z\boldsymbol{G}$, which is nonempty since if $\eta \in \mathbb{F}_{q^2}$ is a square root of ξ then $\left(\begin{smallmatrix}\eta & 0\\ 0 & \eta\end{smallmatrix}\right) \in \boldsymbol{G}^0 \cdot s \cap Z\boldsymbol{G}$; but $\boldsymbol{G}^0 \cdot s$ does not meet $(Z\boldsymbol{G})^F$. □

In the above example $\boldsymbol{G}^1/\boldsymbol{G}^0$ is a semisimple element of $\boldsymbol{G}/\boldsymbol{G}^0$. No such example exists when $\boldsymbol{G}^1/\boldsymbol{G}^0$ is unipotent:

Lemma 2.2. *Let $\boldsymbol{G}^1$ be an F-stable connected component of $\boldsymbol{G}$ such that $\boldsymbol{G}^1/\boldsymbol{G}^0$ is a unipotent element of $\boldsymbol{G}/\boldsymbol{G}^0$. Then $(\boldsymbol{G}^1)^F$ contains unipotent quasicentral elements.*

Proof. Let $\boldsymbol{T}^0 \subset \boldsymbol{B}^0$ be a pair of an F-stable maximal torus of $\boldsymbol{G}^0$ and an F-stable Borel subgroup of $\boldsymbol{G}^0$. Then $N_{\boldsymbol{G}^F}(\boldsymbol{T}^0 \subset \boldsymbol{B}^0)$ meets $(\boldsymbol{G}^1)^F$, since any two F-stable pairs $\boldsymbol{T}^0 \subset \boldsymbol{B}^0$ are $(\boldsymbol{G}^0)^F$-conjugate. Let su be the Jordan decomposition of an element of $N_{(\boldsymbol{G}^1)^F}(\boldsymbol{T}^0 \subset \boldsymbol{B}^0)$. Then $s \in \boldsymbol{G}^0$ since $\boldsymbol{G}^1/\boldsymbol{G}^0$ is unipotent, and u is F-stable, unipotent and still in $N_{(\boldsymbol{G}^1)^F}(\boldsymbol{T}^0 \subset \boldsymbol{B}^0)$ thus quasi-semisimple, and so is quasicentral by [DM 1994, corollaire 1.33]. □

Note, however, that there may exist a unipotent quasicentral element σ which is rational as an automorphism but such that there is no rational element inducing the same automorphism.

Example 2.3. We give an example in $\boldsymbol{G} = \mathrm{SL}_5 \rtimes <\sigma'>$, where $\boldsymbol{G}^0 = \mathrm{SL}_5$ has the standard rational structure over a finite field $\mathbb{F}_q$ of characteristic 2 with $q \equiv 1 \bmod 5$ and σ' is the automorphism of $\boldsymbol{G}^0$ given by $g \mapsto J\,{}^t g^{-1} J$ where J is the antidiagonal matrix with all nonzero entries equal to 1, so that σ' stabilizes the pair $\boldsymbol{T}^0 \subset \boldsymbol{B}^0$, where $\boldsymbol{T}^0$ is the maximal torus of diagonal matrices and $\boldsymbol{B}^0$ is the Borel subgroup of upper triangular matrices; hence σ' is quasi-semisimple. Let t be the diagonal matrix with entries (a, a, a^{-4}, a, a), where a^{q-1} is a nontrivial 5-th root of unity $\zeta \in \mathbb{F}_q$. We claim that $\sigma = t\sigma'$ is as announced: it is still quasi-semisimple; we have $\sigma^2 = t\sigma'(t) = tt^{-1} = 1$ so that σ is unipotent; we have ${}^F\sigma = {}^F t t^{-1}\sigma = \zeta\sigma$, so that

σ is rational as an automorphism but not rational. Moreover a rational element inducing the same automorphism must be of the form $z\sigma$ with z central in $\boldsymbol{G}^0$ and $z\cdot{}^F z^{-1}=\zeta\,\mathrm{Id}$; but the center $Z\boldsymbol{G}^0$ is generated by $\zeta\,\mathrm{Id}$ and for any $z=\zeta^k\,\mathrm{Id}\in Z\boldsymbol{G}^0$, we have $z\cdot{}^F z^{-1}=\zeta^{k(q-1)}\,\mathrm{Id}=\mathrm{Id}\neq\zeta\,\mathrm{Id}$. $\square$

As in [DM 1994] we call a *Levi* of $\boldsymbol{G}$ a subgroup $\boldsymbol{L}$ of the form $N_{\boldsymbol{G}}(\boldsymbol{L}^0\subset\boldsymbol{P}^0)$ where $\boldsymbol{L}^0$ is a Levi subgroup of the parabolic subgroup $\boldsymbol{P}^0$ of $\boldsymbol{G}^0$. A particular case is a "torus" $N_{\boldsymbol{G}}(\boldsymbol{T}^0,\boldsymbol{B}^0)$ where $\boldsymbol{T}^0\subset\boldsymbol{B}^0$ is a pair of a maximal torus of $\boldsymbol{G}^0$ and a Borel subgroup of $\boldsymbol{G}^0$; note that a "torus" meets all connected components of $\boldsymbol{G}$, while (contrary to what is stated erroneously after [DM 1994, définition 1.4]) this may not be the case for a Levi.

We define a Levi of $\boldsymbol{G}^1$ to be a set of the form $\boldsymbol{L}^1=\boldsymbol{L}\cap\boldsymbol{G}^1$, where $\boldsymbol{L}$ is a Levi of $\boldsymbol{G}$ and the intersection is nonempty; note that if $\boldsymbol{G}^1$ does not generate $\boldsymbol{G}$, there may exist several Levis of $\boldsymbol{G}$ which have same intersection with $\boldsymbol{G}^1$. Nevertheless $\boldsymbol{L}^1$ determines $\boldsymbol{L}^0$ as the identity component of $<\boldsymbol{L}^1>$.

We assume now that $\boldsymbol{L}^1$ is an F-stable Levi of $\boldsymbol{G}^1$ of the form $N_{\boldsymbol{G}^1}(\boldsymbol{L}^0\subset\boldsymbol{P}^0)$. If $\boldsymbol{U}$ is the unipotent radical of $\boldsymbol{P}^0$, we define $\boldsymbol{Y}^0_{\boldsymbol{U}}=\{x\in\boldsymbol{G}^0\mid x^{-1}\cdot{}^F x\in\boldsymbol{U}\}$ on which $(g,l)\in\boldsymbol{G}^F\times\boldsymbol{L}^F$ such that $gl\in\boldsymbol{G}^0$ acts by $x\mapsto gxl$, where $\boldsymbol{L}=N_{\boldsymbol{G}}(\boldsymbol{L}^0,\boldsymbol{P}^0)$. Along the same lines as [DM 1994, proposition 2.10] we define the following:

Definition 2.4. Let $\boldsymbol{L}^1$ be an F-stable Levi of $\boldsymbol{G}^1$ of the form $N_{\boldsymbol{G}^1}(\boldsymbol{L}^0\subset\boldsymbol{P}^0)$ and let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{P}^0$. For λ a class function on $(\boldsymbol{L}^1)^F$ and $g\in(\boldsymbol{G}^1)^F$, we set

$$R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}(\lambda)(g)=|(\boldsymbol{L}^1)^F|^{-1}\sum_{l\in(\boldsymbol{L}^1)^F}\lambda(l)\,\mathrm{Trace}\big((g,l^{-1})\mid H^*_c(\boldsymbol{Y}^0_{\boldsymbol{U}})\big),$$

and for γ a class function on $(\boldsymbol{G}^1)^F$ and $l\in(\boldsymbol{L}^1)^F$, we set

$${}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}(\gamma)(l)=|(\boldsymbol{G}^1)^F|^{-1}\sum_{g\in(\boldsymbol{G}^1)^F}\gamma(g)\,\mathrm{Trace}\big((g^{-1},l)\mid H^*_c(\boldsymbol{Y}^0_{\boldsymbol{U}})\big).$$

In the above, H^*_c denotes the ℓ-adic cohomology with compact support, where we have chosen once and for all a prime number $\ell\neq p$. In order to consider the virtual character $\mathrm{Trace}(x\mid H^*_c(\boldsymbol{X}))=\sum_i(-1)^i\,\mathrm{Trace}(x\mid H^i_c(\boldsymbol{X},\overline{\mathbb{Q}}_\ell))$ as a complex character we chose once and for all an embedding $\overline{\mathbb{Q}}_\ell\hookrightarrow\mathbb{C}$.

Writing $R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}$ and ${}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}$ is an abuse of notation: the definition needs the choice of a $\boldsymbol{P}^0$ such that $\boldsymbol{L}^1=N_{\boldsymbol{G}^1}(\boldsymbol{L}^0\subset\boldsymbol{P}^0)$. Our subsequent statements will use an implicit choice. Under certain assumptions, we will prove a Mackey formula (Theorem 2.6) which when true implies that $R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}$ and ${}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}$ are independent of the choice of $\boldsymbol{P}^0$.

By the same arguments as for [DM 1994, proposition 2.10] (using that $(\boldsymbol{L}^1)^F$ is nonempty and [DM 1994, proposition 2.3]) Definition 2.4 agrees with the restriction to $(\boldsymbol{G}^1)^F$ and $(\boldsymbol{L}^1)^F$ of [DM 1994, définition 2.2].

The two maps $R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}$ and ${}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}$ are adjoint with respect to the scalar products on $(\boldsymbol{G}^1)^F$ and $(\boldsymbol{L}^1)^F$.

We note the following variation on [DM 1994, proposition 2.6] where, for u (resp. v) a unipotent element of $\boldsymbol{G}$ (resp. $\boldsymbol{L}$), we set

$$Q_{\boldsymbol{L}^0}^{\boldsymbol{G}^0}(u,v)=\begin{cases}\operatorname{Trace}\big((u,v)\mid H_c^*(\boldsymbol{Y}_{\boldsymbol{U}}^0)\big) & \text{if } uv\in\boldsymbol{G}^0,\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 2.5. *Let su be the Jordan decomposition of an element of $(\boldsymbol{G}^1)^F$ and λ a class function on $(\boldsymbol{L}^1)^F$.*

(i) *If s is central in $\boldsymbol{G}$, we have*

$$(R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}\lambda)(su)=|(\boldsymbol{L}^0)^F|^{-1}\sum_{v\in(\boldsymbol{L}^0\cdot u)^F_{\mathrm{unip}}}Q_{\boldsymbol{L}^0}^{\boldsymbol{G}^0}(u,v^{-1})\lambda(sv).$$

(ii) *In general,*

$$(R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}\lambda)(su)=\sum_{\{h\in(\boldsymbol{G}^0)^F\mid {}^h\boldsymbol{L}\ni s\}}\frac{|{}^h\boldsymbol{L}^0\cap C_{\boldsymbol{G}}(s)^{0F}|}{|(\boldsymbol{L}^0)^F||C_{\boldsymbol{G}}(s)^{0F}|}R_{{}^h\boldsymbol{L}^1\cap C_{\boldsymbol{G}}(s)^0\cdot su}^{C_{\boldsymbol{G}}(s)^0\cdot su}({}^h\lambda)(su).$$

(iii) *If tv is the Jordan decomposition of an element of $(\boldsymbol{L}^1)^F$ and γ a class function on $(\boldsymbol{G}^1)^F$, we have*

$$({}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}\gamma)(tv)=|(\boldsymbol{G}^{t0})^F|^{-1}\sum_{u\in(\boldsymbol{G}^{t0}\cdot v)^F_{\mathrm{unip}}}Q_{\boldsymbol{L}^{t0}}^{\boldsymbol{G}^{t0}}(u,v^{-1})\gamma(tu).$$

In the above we abused notation to write ${}^h\boldsymbol{L}\ni s$ for $<\boldsymbol{L}^1>\ni {}^{h^{-1}}s$.

Proof. Part (i) results from [DM 1994, proposition 2.6(i)] using the same arguments as the proof of [DM 1994, propsition 2.10]; we then get (ii) by plugging (i) into [DM 1994, proposition 2.6(i)]. □

In our setting the Mackey formula [DM 1994, définition 3.1] is still valid in the cases where we proved it: théorèmes 3.2 and 4.5 in [DM 1994]. Before stating it we remark that [DM 1994, proposition 1.40] remains true without the assumption that $(\boldsymbol{G}^1)^F$ contains quasicentral elements; we need only replace $(\boldsymbol{G}^0)^F.\sigma$ with $(\boldsymbol{G}^1)^F$ in the proof. Thus any F-stable Levi of $\boldsymbol{G}^1$ is $(\boldsymbol{G}^0)^F$-conjugate to a Levi containing σ. This explains why we only state the Mackey formula in the case of Levis containing σ.

Theorem 2.6. *If $\boldsymbol{L}^1$ and $\boldsymbol{M}^1$ are two F-stable Levis of $\boldsymbol{G}^1$ containing σ then under one of the following assumptions*:

- $\boldsymbol{L}^0$ (*resp.* $\boldsymbol{M}^0$) *is a Levi subgroup of an F-stable parabolic subgroup normalized by* $\boldsymbol{L}^1$ (*resp.* $\boldsymbol{M}^1$),
- *one of* $\boldsymbol{L}^1$ *and* $\boldsymbol{M}^1$ *is a "torus",*

we have

$$ {}^*R^{\boldsymbol{G}^1}_{\boldsymbol{L}^1} R^{\boldsymbol{G}^1}_{\boldsymbol{M}^1} = \sum_{x\in[\boldsymbol{L}^{\sigma 0F}\backslash \mathcal{S}_{\boldsymbol{G}^{\sigma 0}}(\boldsymbol{L}^{\sigma 0},\boldsymbol{M}^{\sigma 0})^F/\boldsymbol{M}^{\sigma 0F}]} R^{\boldsymbol{L}^1}_{(\boldsymbol{L}^1\cap {}^x\boldsymbol{M}^1)} {}^*R^{{}^x\boldsymbol{M}^1}_{(\boldsymbol{L}^1\cap {}^x\boldsymbol{M}^1)} \operatorname{ad} x, $$

where $\mathcal{S}_{\boldsymbol{G}^{\sigma 0}}(\boldsymbol{L}^{\sigma 0}, \boldsymbol{M}^{\sigma 0})$ is the set of elements $x \in \boldsymbol{G}^{\sigma 0}$ such that $\boldsymbol{L}^{\sigma 0}\cap {}^x\boldsymbol{M}^{\sigma 0}$ contains a maximal torus of $\boldsymbol{G}^{\sigma 0}$.

Proof. We first prove the theorem in the case of F-stable parabolic subgroups $\boldsymbol{P}^0 = \boldsymbol{L}^0 \ltimes \boldsymbol{U}$ and $\boldsymbol{Q}^0 = \boldsymbol{M}^0 \ltimes \boldsymbol{V}$ following the proof of [DM 1994, théorème 3.2]. The difference is that the variety we consider here is the intersection with $\boldsymbol{G}^0$ of the variety considered in [loc. cit.]. Here, the left-hand side of the Mackey formula is given by $\overline{\mathbb{Q}}_\ell[(\boldsymbol{U}^F\backslash(\boldsymbol{G}^0)^F/\boldsymbol{V}^F)^\sigma]$ instead of $\overline{\mathbb{Q}}_\ell[(\boldsymbol{U}^F\backslash(\boldsymbol{G}^0)^F.<\sigma>/\boldsymbol{V}^F)^\sigma]$. Nevertheless we can use [DM 1994, lemme 3.3], which remains valid with the same proof. As for [DM 1994, lemme 3.5], we have to replace it with the following:

Lemma 2.7. *For any $x \in \mathcal{S}_{\boldsymbol{G}^{\sigma 0}}(\boldsymbol{L}^{\sigma 0}, \boldsymbol{M}^{\sigma 0})^F$, the map*

$$ \big(l(\boldsymbol{L}^0 \cap {}^x\boldsymbol{V}^F), ({}^x\boldsymbol{M}^0 \cap \boldsymbol{U}^F)\cdot {}^x m\big) \mapsto \boldsymbol{U}^F l x m \boldsymbol{V}^F $$

is an isomorphism from $(\boldsymbol{L}^0)^F/(\boldsymbol{L}^0 \cap {}^x\boldsymbol{V}^F) \times_{(\boldsymbol{L}^0\cap {}^x\boldsymbol{M}^0)^F} ({}^x\boldsymbol{M}^0 \cap \boldsymbol{U}^F)\backslash {}^x(\boldsymbol{M}^0)^F$ to $\boldsymbol{U}^F\backslash(\boldsymbol{P}^0)^F x(\boldsymbol{Q}^0)^F/\boldsymbol{V}^F$, compatible with the following action of $(\boldsymbol{L}^1)^F\times((\boldsymbol{M}^1)^F)^{-1}$: $(\lambda, \mu^{-1}) \in (\boldsymbol{L}^1)^F \times ((\boldsymbol{M}^1)^F)^{-1}$ acts by mapping $(l(\boldsymbol{L}^0 \cap {}^x\boldsymbol{V}^F), ({}^x\boldsymbol{M}^0 \cap \boldsymbol{U}^F)\cdot {}^x m)$ to the class of $(\lambda l \nu^{-1}(\boldsymbol{L}^0\cap {}^x\boldsymbol{V}^F), ({}^x\boldsymbol{M}^0\cap\boldsymbol{U}^F)\cdot \nu {}^x m \mu^{-1})$ with $\nu \in (\boldsymbol{L}^1)^F \cap {}^x(\boldsymbol{M}^1)^F$ (independent of ν).

Proof. The isomorphism of the lemma involves only connected groups and is a known result (see, e.g., [Digne and Michel 1991, 5.7]). The compatibility with the actions is straightforward. □

This allows us to complete the proof in the first case.

We now prove the second case following Section 4 of [DM 1994]. We first notice that the statement and proof of lemme 4.1 in [DM 1994] don't use the element σ but only its action. In lemmes 4.2, 4.3 and 4.4 there is no σ involved but only the action of the groups $\boldsymbol{L}^F$ and $\boldsymbol{M}^F$ on the pieces of a variety depending only on $\boldsymbol{L}$, $\boldsymbol{M}$ and the associated parabolics. This gives the second case. □

We now rephrase [DM 1994, proposition 4.8] and [DM 1994, proposition 4.11] in our setting, specializing the Mackey formula to the case of two "tori". Let $\mathcal{T}_1$ be the set of "tori" of $\boldsymbol{G}^1$; if $\boldsymbol{T}^1 = N_{\boldsymbol{G}^1}(\boldsymbol{T}^0, \boldsymbol{B}^0) \in \mathcal{T}_1^F$ then $\boldsymbol{T}^0$ is F-stable. We define $\operatorname{Irr}((\boldsymbol{T}^1)^F)$ as the set of restrictions to $(\boldsymbol{T}^1)^F$ of extensions to $<(\boldsymbol{T}^1)^F>$ of elements of $\operatorname{Irr}((\boldsymbol{T}^0)^F)$.

Proposition 2.8. *If $\boldsymbol{T}^1, \boldsymbol{T}'^1 \in \mathcal{T}_1^F$ and $\theta \in \operatorname{Irr}((\boldsymbol{T}^1)^F)$, $\theta' \in \operatorname{Irr}((\boldsymbol{T}'^1)^F)$ then*

$$ \langle R^{\boldsymbol{G}^1}_{\boldsymbol{T}^1}(\theta), R^{\boldsymbol{G}^1}_{\boldsymbol{T}'^1}(\theta')\rangle_{(\boldsymbol{G}^1)^F} = 0 $$

unless $(\boldsymbol{T}^1,\theta)$ *and* $(\boldsymbol{T}'^1,\theta')$ *are* $(\boldsymbol{G}^0)^F$*-conjugate.*

Additionally,

(i) *if for some* $n\in N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^1)$ *and* $\zeta\neq 1$ *we have* ${}^n\theta=\zeta\theta$ *then* $R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)=0$;

(ii) *otherwise* $\langle R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta), R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)\rangle_{(\boldsymbol{G}^1)^F}=\bigl|\{n\in N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^1)\mid {}^n\theta=\theta\}\bigr|/|(\boldsymbol{T}^1)^F|$.

If $\boldsymbol{T}^1=\boldsymbol{T}'^1$, *the above can be written as*

$$\langle R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta), R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta')\rangle_{(\boldsymbol{G}^1)^F}=\langle \operatorname{Ind}_{(\boldsymbol{T}^1)^F}^{N_{\boldsymbol{G}^1}(\boldsymbol{T}^0)^F}\theta, \operatorname{Ind}_{(\boldsymbol{T}^1)^F}^{N_{\boldsymbol{G}^1}(\boldsymbol{T}^0)^F}\theta'\rangle_{N_{\boldsymbol{G}^1}(\boldsymbol{T}^0)^F},$$

where when $A^1\subset B^1$ *are cosets of finite groups* $A^0\subset B^0$ *and* χ *is an* A^0*-class function on* A^1 *for* $x\in B^1$, *we set* $\operatorname{Ind}_{A^1}^{B^1}\chi(x)=|A^0|^{-1}\sum_{\{y\in B^0\mid {}^yx\in A^1\}}\chi({}^yx)$.

Proof. As noticed above Theorem 2.6, we may assume that $\boldsymbol{T}^1$ and $\boldsymbol{T}'^1$ contain σ. By [DM 1994, proposition 1.39], if $\boldsymbol{T}^1$ and $\boldsymbol{T}'^1$ contain σ, they are $(\boldsymbol{G}^0)^F$-conjugate if and only if they are conjugate under $\boldsymbol{G}^{\sigma 0F}$. The Mackey formula shows then that the scalar product vanishes when $\boldsymbol{T}^1$ and $\boldsymbol{T}'^1$ are not $(\boldsymbol{G}^0)^F$-conjugate.

Otherwise we may assume $\boldsymbol{T}^1=\boldsymbol{T}'^1$ and the Mackey formula gives

$$\langle R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta), R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)\rangle_{(\boldsymbol{G}^1)^F}=|(\boldsymbol{T}^{\sigma 0})^F|^{-1}\sum_{n\in N_{\boldsymbol{G}^{\sigma 0}}(\boldsymbol{T}^{\sigma 0})^F}\langle\theta, {}^n\theta\rangle_{(\boldsymbol{T}^1)^F}.$$

The term $\langle\theta,{}^n\theta\rangle_{(\boldsymbol{T}^1)^F}$ is 0 unless ${}^n\theta=\zeta_n\theta$ for some constant ζ_n and, in this last case, $\langle\theta,{}^n\theta\rangle_{(\boldsymbol{T}^1)^F}=\bar{\zeta}_n$. If ${}^{n'}\theta=\zeta_{n'}\theta$ then ${}^{nn'}\theta=\zeta_{n'}{}^n\theta=\zeta_{n'}\zeta_n\theta$, and thus the ζ_n form a group; if this group is not trivial, that is, some ζ_n is not equal to 1, we have

$$\langle R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta), R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)\rangle_{(\boldsymbol{G}^1)^F}=0,$$

which implies that in this case $R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)=0$. This gives (i) since by [DM 1994, proposition 1.39], if $\boldsymbol{T}^1\ni\sigma$ then $N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^1)=N_{\boldsymbol{G}^{\sigma 0}}(\boldsymbol{T}^{\sigma 0})^F\cdot(\boldsymbol{T}^0)^F$, so that if there exists n as in (i), there exists an $n\in N_{\boldsymbol{G}^{\sigma 0}}(\boldsymbol{T}^{\sigma 0})^F$ with same action on θ since $(\boldsymbol{T}^0)^F$ has trivial action on θ.

In case (ii), for each nonzero term we have ${}^n\theta=\theta$ and we have to check that the value $|((\boldsymbol{T}^\sigma)^0)^F|^{-1}\bigl|\{n\in N_{\boldsymbol{G}^{\sigma 0}}(\boldsymbol{T}^{\sigma 0})^F\mid {}^n\theta=\theta\}\bigr|$ given by the Mackey formula is equal to the stated value. This results again from [DM 1994, proposition 1.39], written as $N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^1)=N_{\boldsymbol{G}^{\sigma 0}}(\boldsymbol{T}^1)^F\cdot(\boldsymbol{T}^0)^F$, and from $N_{\boldsymbol{G}^{\sigma 0}}(\boldsymbol{T}^1)^F\cap(\boldsymbol{T}^0)^F=((\boldsymbol{T}^\sigma)^0)^F$.

We now prove the final remark. By definition we have

$$\begin{aligned}&\langle \operatorname{Ind}_{(\boldsymbol{T}^1)^F}^{N_{\boldsymbol{G}^1}(\boldsymbol{T}^0)^F}\theta, \operatorname{Ind}_{(\boldsymbol{T}^1)^F}^{N_{\boldsymbol{G}^1}(\boldsymbol{T}^0)^F}\theta'\rangle_{N_{\boldsymbol{G}^1}(\boldsymbol{T}^0)^F}\\ &\qquad=|N_{\boldsymbol{G}^1}(\boldsymbol{T}^0)^F|^{-1}|(\boldsymbol{T}^1)^F|^{-2}\sum_{x\in N_{\boldsymbol{G}^1}(\boldsymbol{T}^0)^F}\ \sum_{\{n,n'\in N_{\boldsymbol{G}^1}(\boldsymbol{T}^0)^F\mid {}^nx,{}^{n'}x\in\boldsymbol{T}^1\}}\theta({}^nx)\overline{\theta({}^{n'}x)}.\end{aligned}$$

Doing the summation over $t={}^nx$ and $n''=n'n^{-1}\in N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F$, we get

$$|N_{\boldsymbol{G}^1}(\boldsymbol{T}^0)^F|^{-1}|(\boldsymbol{T}^1)^F|^{-2}\sum_{t\in(\boldsymbol{T}^1)^F}\ \sum_{n\in N_{\boldsymbol{G}^1}(\boldsymbol{T}^0)^F}\ \sum_{\{n''\in N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F\mid {}^{n''}t\in\boldsymbol{T}^1\}}\theta(t)\overline{\theta({}^{n''}t)}.$$

The condition $n'' \in N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F$, together with ${}^{n''}t \in \boldsymbol{T}^1$, is equivalent to $n'' \in N_{\boldsymbol{G}^0}(\boldsymbol{T}^1)^F$, so that we get

$$|(\boldsymbol{T}^1)^F|^{-1} \sum_{n'' \in N_{\boldsymbol{G}^0}(\boldsymbol{T}^1)^F} \langle \theta, {}^{n''}\theta \rangle_{(\boldsymbol{T}^1)^F}.$$

As explained in the first part of the proof, the scalar product $\langle \theta, {}^{n''}\theta \rangle_{(\boldsymbol{T}^1)^F}$ is zero unless ${}^{n''}\theta = \zeta_{n''}\theta$ for some root of unity $\zeta_{n''}$ and arguing as in the first part of the proof, we find that the above sum is zero if there exists n'' such that $\zeta_{n''} \neq 1$ and is equal to $|(\boldsymbol{T}^1)^F|^{-1}|\{n \in N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^1) \mid {}^n\theta = \theta\}|$ otherwise. $\square$

Remark 2.9. In the context of Proposition 2.8, if σ is F-stable then we may apply θ to it and for any $n \in N_{\boldsymbol{G}^{\sigma 0}}(\boldsymbol{T}^{\sigma 0})^F$, we have $\theta({}^n\sigma) = \theta(\sigma)$, so for any $n \in N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^1)$ and ζ such that ${}^n\theta = \zeta\theta$, we have $\zeta = 1$. When $H^1(F, Z\boldsymbol{G}^0) = 1$, we may choose σ to be F-stable so that $\zeta \neq 1$ never happens.

Here is an example where $\zeta_n = -1$, and thus $R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta) = 0$: we take again the context of Example 2.1 and take $\boldsymbol{T}^0 = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\right\}$ and let $\boldsymbol{T}^1 = \boldsymbol{T}^0 \cdot s$; let us define θ on $ts \in (\boldsymbol{T}^1)^F$ by $\theta(ts) = -\lambda(t)$, where λ is the nontrivial order-2 character of $(\boldsymbol{T}^0)^F$ (Legendre symbol); then for any $n \in N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^1) \backslash \boldsymbol{T}^0$, we have ${}^n\theta = -\theta$. $\square$

We define *uniform* functions as the class functions on $(\boldsymbol{G}^1)^F$ which are linear combinations of the $R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)$ for $\theta \in \operatorname{Irr}((\boldsymbol{T}^1)^F)$. Proposition 4.11 in [DM 1994] extends as follows to our context:

Corollary 2.10 (of Proposition 2.8). *Let $p^{\boldsymbol{G}^1}$ be the projector to uniform functions on $(\boldsymbol{G}^1)^F$. We have*

$$p^{\boldsymbol{G}^1} = |(\boldsymbol{G}^1)^F|^{-1} \sum_{\boldsymbol{T}^1 \in \mathcal{T}_1^F} |(\boldsymbol{T}^1)^F| R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1} \circ {}^*R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}.$$

Proof. We need only check that for any $\theta \in \operatorname{Irr}((\boldsymbol{T}^1)^F)$ such that $R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta) \neq 0$ and any class function χ on $(\boldsymbol{G}^1)^F$, we have $\langle p^{\boldsymbol{G}^1}\chi, R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)\rangle_{(\boldsymbol{G}^1)^F} = \langle \chi, R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)\rangle_{(\boldsymbol{G}^1)^F}$. By Proposition 2.8, to evaluate the left-hand side we may restrict the sum to tori conjugate to $\boldsymbol{T}^1$, so we get

$$\begin{aligned}\langle p^{\boldsymbol{G}^1}\chi, R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)\rangle_{(\boldsymbol{G}^1)^F} &= |N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^1)|^{-1}|(\boldsymbol{T}^1)^F|\langle R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1} \circ {}^*R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}\chi, R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)\rangle_{(\boldsymbol{G}^1)^F} \\ &= |N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^1)|^{-1}|(\boldsymbol{T}^1)^F|\langle \chi, R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1} \circ {}^*R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1} \circ R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)\rangle_{(\boldsymbol{G}^1)^F}.\end{aligned}$$

The equality to be proved is true if $R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta) = 0$; otherwise by Proposition 2.8, we have ${}^*R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1} \circ R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta) = |(\boldsymbol{T}^1)^F|^{-1} \sum_{n \in N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^1)} {}^n\theta$, whence in that case

$$R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1} \circ {}^*R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1} \circ R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta) = |(\boldsymbol{T}^1)^F|^{-1}|N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^1)| R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)$$

since $R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}({}^n\theta) = R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\theta)$, and hence the result. $\square$

We now adapt the definition of duality to our setting.

Definition 2.11.
- For a connected reductive group $\boldsymbol{G}$, we define the $\mathbb{F}_q$-rank as the maximal dimension of a split torus, and define $\varepsilon_{\boldsymbol{G}} = (-1)^{\mathbb{F}_q\text{-rank of } \boldsymbol{G}}$ and $\eta_{\boldsymbol{G}} = \varepsilon_{\boldsymbol{G}/\operatorname{rad}\boldsymbol{G}}$.
- For an F-stable connected component $\boldsymbol{G}^1$ of a (possibly disconnected) reductive group, we define $\varepsilon_{\boldsymbol{G}^1} = \varepsilon_{\boldsymbol{G}^{\sigma 0}}$ and $\eta_{\boldsymbol{G}^1} = \eta_{\boldsymbol{G}^{\sigma 0}}$, where $\sigma \in \boldsymbol{G}^1$ induces an F-stable quasicentral automorphism of $\boldsymbol{G}^0$.

Let us see that these definitions agree with [DM 1994]: in [DM 1994, définition 3.6(i)], we define $\varepsilon_{\boldsymbol{G}^1}$ to be $\varepsilon_{\boldsymbol{G}^{0\tau}}$, where τ is any quasi-semisimple element of $\boldsymbol{G}^1$ which induces an F-stable automorphism of $\boldsymbol{G}^0$ and lies in a "torus" of the form $N_{\boldsymbol{G}^1}(\boldsymbol{T}_0 \subset \boldsymbol{B}_0)$, where both $\boldsymbol{T}^0$ and $\boldsymbol{B}^0$ are F-stable; by [DM 1994, proposition 1.36(ii)], a σ as above is such a τ.

We fix an F-stable pair $(\boldsymbol{T}_0 \subset \boldsymbol{B}_0)$ and define duality on $\operatorname{Irr}((\boldsymbol{G}^1)^F)$ by

$$D_{\boldsymbol{G}^1} = \sum_{\boldsymbol{P}^0 \supset \boldsymbol{B}^0} \eta_{\boldsymbol{L}^1} R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1} \circ {}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}, \tag{2.12}$$

where in the sum, $\boldsymbol{P}^0$ runs over F-stable parabolic subgroups containing $\boldsymbol{B}^0$ such that $N_{\boldsymbol{G}^1}(\boldsymbol{P}^0)$ is nonempty, and $\boldsymbol{L}^1$ denotes $N_{\boldsymbol{G}^1}(\boldsymbol{L}^0 \subset \boldsymbol{P}^0)$, where $\boldsymbol{L}^0$ is the Levi subgroup of $\boldsymbol{P}^0$ containing $\boldsymbol{T}^0$. The duality thus defined coincides with the duality defined in [DM 1994, définition 3.10] when σ is in $(\boldsymbol{G}^1)^F$.

In our context we can define $\operatorname{St}_{\boldsymbol{G}^1}$ similarly to [DM 1994, définition 3.16], as $D_{\boldsymbol{G}^1}(\operatorname{Id}_{\boldsymbol{G}^1})$, and [DM 1994, proposition 3.18] remains true:

Proposition 2.13. $\operatorname{St}_{\boldsymbol{G}^1}$ *vanishes outside quasi-semisimple elements, and if* $x \in (\boldsymbol{G}^1)^F$ *is quasi-semisimple, we have*

$$\operatorname{St}_{\boldsymbol{G}^1}(x) = \varepsilon_{\boldsymbol{G}^1} \varepsilon_{(\boldsymbol{G}^x)^0} |(\boldsymbol{G}^x)^0|_p.$$

3. A global formula for the scalar product of Deligne–Lusztig characters

In this section we give a result of a different flavor, where we do not restrict our attention to a connected component $\boldsymbol{G}^1$.

Definition 3.1. For any character θ of $\boldsymbol{T}^F$, we define $R_{\boldsymbol{T}}^{\boldsymbol{G}}$ as in [DM 1994, définition 2.2]. If for a "torus" $\boldsymbol{T}$ and $\alpha = g\boldsymbol{G}^0 \in \boldsymbol{G}/\boldsymbol{G}^0$ we denote by $\boldsymbol{T}^{[\alpha]}$ or $\boldsymbol{T}^{[g]}$ the unique connected component of $\boldsymbol{T}$ which meets $g\boldsymbol{G}^0$, this is equivalent to

$$R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)(g) = |(\boldsymbol{T}^0)^F|/|\boldsymbol{T}^F| \sum_{\{a \in [\boldsymbol{G}^F/(\boldsymbol{G}^0)^F] \mid {}^a g \in \boldsymbol{T}^F(\boldsymbol{G}^0)^F\}} R_{\boldsymbol{T}^{[{}^a g]}}^{\boldsymbol{G}^{[{}^a g]}}(\theta)({}^a g)$$

for $g \in \boldsymbol{G}^F$, where the right-hand side is defined by Definition 2.4 (see [DM 1994, proposition 2.3]).

We deduce from Proposition 2.8 the following formula for the whole group $\boldsymbol{G}$:

Proposition 3.2. *Let $\boldsymbol{T}$, $\boldsymbol{T}'$ be two "tori" of $\boldsymbol{G}$ and let $\theta \in \operatorname{Irr}(\boldsymbol{T}^F)$, $\theta' \in \operatorname{Irr}(\boldsymbol{T}'^F)$. Then $\langle R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta), R_{\boldsymbol{T}'}^{\boldsymbol{G}}(\theta')\rangle_{\boldsymbol{G}^F} = 0$ if $\boldsymbol{T}^0$ and $\boldsymbol{T}'^0$ are not $\boldsymbol{G}^F$-conjugate, and if $\boldsymbol{T}^0 = \boldsymbol{T}'^0$, we have*

$$\langle R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta), R_{\boldsymbol{T}'}^{\boldsymbol{G}}(\theta')\rangle_{\boldsymbol{G}^F} = \big\langle \operatorname{Ind}_{\boldsymbol{T}^F}^{N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F}(\theta), \operatorname{Ind}_{\boldsymbol{T}'^F}^{N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F}(\theta')\big\rangle_{N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F}.$$

Proof. Definition 3.1 can be written

$$R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)(g) = |(\boldsymbol{T}^0)^F|/|\boldsymbol{T}^F| \sum_{\{a\in[\boldsymbol{G}^F/(\boldsymbol{G}^0)^F] \mid {}^a g\in \boldsymbol{T}^F(\boldsymbol{G}^0)^F\}} R_{({}^{a^{-1}}\boldsymbol{T})^{[g]}}^{\boldsymbol{G}^{[g]}}({}^{a^{-1}}\theta)(g).$$

So the scalar product we want to compute is equal to

$$\begin{aligned}\langle R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta), R_{\boldsymbol{T}'}^{\boldsymbol{G}}(\theta')\rangle_{\boldsymbol{G}^F} &= \frac{1}{|\boldsymbol{G}^F|}\frac{|(\boldsymbol{T}^0)^F|}{|\boldsymbol{T}^F|}\frac{|(\boldsymbol{T}'^0)^F|}{|\boldsymbol{T}'^F|}\\ &\quad\times \sum_{\substack{\alpha\in\boldsymbol{G}^F/\boldsymbol{G}^{0F}\\ g\in(\boldsymbol{G}^0)^F.\alpha}} \sum R_{({}^{a^{-1}}\boldsymbol{T})^{[\alpha]}}^{\boldsymbol{G}.\alpha}({}^{a^{-1}}\theta)(g)\overline{R_{({}^{a'^{-1}}\boldsymbol{T}')^{[\alpha]}}^{\boldsymbol{G}.\alpha}({}^{a'^{-1}}\theta')(g)},\end{aligned}$$

where the inner sum runs over $a \in [\boldsymbol{G}^F/(\boldsymbol{G}^0)^F]$ such that ${}^a\alpha \in \boldsymbol{T}^F(\boldsymbol{G}^0)^F$ and $a' \in [\boldsymbol{G}^F/(\boldsymbol{G}^0)^F]$ such that ${}^{a'}\alpha \in \boldsymbol{T}'^F(\boldsymbol{G}^0)^F$. This product can be written

$$\begin{aligned}\langle R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta), R_{\boldsymbol{T}'}^{\boldsymbol{G}}(\theta')\rangle_{\boldsymbol{G}^F} &= \frac{|(\boldsymbol{G}^0)^F|}{|\boldsymbol{G}^F|}\frac{|(\boldsymbol{T}^0)^F|}{|\boldsymbol{T}^F|}\frac{|(\boldsymbol{T}'^0)^F|}{|\boldsymbol{T}'^F|}\\ &\quad\times \sum_{\alpha\in\boldsymbol{G}^F/\boldsymbol{G}^{0F}} \sum \big\langle R_{({}^{a^{-1}}\boldsymbol{T})^{[\alpha]}}^{\boldsymbol{G}.\alpha}({}^{a^{-1}}\theta), R_{({}^{a'^{-1}}\boldsymbol{T}')^{[\alpha]}}^{\boldsymbol{G}.\alpha}({}^{a'^{-1}}\theta')\big\rangle_{(\boldsymbol{G}^0)^F.\alpha},\end{aligned}$$

where the inner sum is as above. By Proposition 2.8 the scalar product on the right-hand side is zero unless $({}^{a^{-1}}\boldsymbol{T})^{[\alpha]}$ and $({}^{a'^{-1}}\boldsymbol{T}')^{[\alpha]}$ are $(\boldsymbol{G}^0)^F$-conjugate, which implies that $\boldsymbol{T}^0$ and $\boldsymbol{T}'^0$ are $(\boldsymbol{G}^0)^F$-conjugate. So we can assume that $\boldsymbol{T}^0 = \boldsymbol{T}'^0$. Moreover for each a' indexing a nonzero summand, there is a representative $y \in a'^{-1}(\boldsymbol{G}^0)^F$ such that $({}^y\boldsymbol{T}')^{[\alpha]} = ({}^{a^{-1}}\boldsymbol{T})^{[\alpha]}$. This last equality and the condition on a imply the condition ${}^{a'}\alpha \in \boldsymbol{T}'^F(\boldsymbol{G}^0)^F$ since this condition can be written $({}^y\boldsymbol{T}')^{[\alpha]} \neq \varnothing$. Thus we can do the summation over all such $y \in \boldsymbol{G}^F$, provided we divide by $|N_{(\boldsymbol{G}^0)^F}(({}^{a^{-1}}\boldsymbol{T})^{[\alpha]})|$. So we get, applying Proposition 2.8, that the above expression is equal to

$$\begin{aligned}&\frac{|(\boldsymbol{G}^0)^F|}{|\boldsymbol{G}^F|}\frac{|(\boldsymbol{T}^0)^F|^2}{|\boldsymbol{T}^F||\boldsymbol{T}'^F|}\sum_{\alpha\in\boldsymbol{G}^F/\boldsymbol{G}^{0F}}\ \sum_{\{a\in[\boldsymbol{G}^F/(\boldsymbol{G}^0)^F] \mid {}^a\alpha\in\boldsymbol{T}^F(\boldsymbol{G}^0)^F\}} |N_{(\boldsymbol{G}^0)^F}(({}^{a^{-1}}\boldsymbol{T})^{[\alpha]})|^{-1}\\ &\qquad\times \sum_{\{y\in\boldsymbol{G}^F \mid ({}^y\boldsymbol{T}')^{[\alpha]}=({}^{a^{-1}}\boldsymbol{T})^{[\alpha]}\}} \big\langle \operatorname{Ind}_{({}^{a^{-1}}\boldsymbol{T})^{[\alpha]}}^{N_{\boldsymbol{G}^0.\alpha}({}^{a^{-1}}\boldsymbol{T}^0)^F}\ {}^{a^{-1}}\theta, \operatorname{Ind}_{({}^{a^{-1}}\boldsymbol{T})^{[\alpha]}}^{N_{\boldsymbol{G}^0.\alpha}({}^{a^{-1}}\boldsymbol{T}^0)^F}\ {}^y\theta'\big\rangle_{N_{\boldsymbol{G}^0.\alpha}(\boldsymbol{T}^0)^F}.\end{aligned}$$

We now conjugate everything by a, take ay as new variable y, set $b = {}^a\alpha$ and get

$$\tag{3.3} \frac{|(\boldsymbol{T}^0)^F|^2}{|\boldsymbol{T}^F||\boldsymbol{T}'^F|} \sum_{b\in\boldsymbol{T}^F/(\boldsymbol{T}^0)^F} |N_{(\boldsymbol{G}^0)^F}(\boldsymbol{T}^{[b]})|^{-1} \\ \times \sum_{\{y\in\boldsymbol{G}^F\mid({}^y\boldsymbol{T}')^{[b]}=\boldsymbol{T}^{[b]}\}} \left\langle \operatorname{Ind}_{\boldsymbol{T}^{[b]F}}^{N_{\boldsymbol{G}^0.b}(\boldsymbol{T}^0)^F}\theta, \operatorname{Ind}_{\boldsymbol{T}^{[b]F}}^{N_{\boldsymbol{G}^0.b}(\boldsymbol{T}^0)^F}{}^y\theta'\right\rangle_{N_{\boldsymbol{G}^0.b}(\boldsymbol{T}^0)^F}$$

since for $b \in \boldsymbol{T}^F/(\boldsymbol{T}^0)^F$, any choice of $a \in \boldsymbol{G}^F/(\boldsymbol{G}^0)^F$ gives an $\alpha = {}^{a^{-1}}b$ which satisfies the condition ${}^a\alpha \in \boldsymbol{T}^F(\boldsymbol{G}^0)^F$.

Let us now transform the right-hand side of Proposition 3.2. Using the definition we have

$$\begin{aligned}
&\langle \operatorname{Ind}_{\boldsymbol{T}^F}^{N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F}(\theta), \operatorname{Ind}_{\boldsymbol{T}'^F}^{N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F}(\theta)\rangle_{N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F} \\
&\quad = |\boldsymbol{T}^F|^{-1}|\boldsymbol{T}'^F|^{-1}|N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F|^{-1} \sum_{\{n,x,x'\in N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F\mid {}^xn\in\boldsymbol{T},{}^{x'}n\in\boldsymbol{T}'\}} \theta({}^xn)\overline{\theta'({}^{x'}n)} \\
&\quad = |\boldsymbol{T}^F|^{-1}|\boldsymbol{T}'^F|^{-1}|N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F|^{-1} \\
&\qquad \times \sum_{b,a,a'\in[N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F/N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F]} \sum_{\left\{\begin{smallmatrix} n\in N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F b \\ x_0,x_0'\in N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F\end{smallmatrix}\middle| \begin{smallmatrix} {}^{x_0}n\in({}^{a^{-1}}\boldsymbol{T})^{[b]} \\ {}^{x_0'}n\in({}^{a'^{-1}}\boldsymbol{T}')^{[b]}\end{smallmatrix}\right\}} {}^{a^{-1}}\theta({}^{x_0}n)\overline{{}^{a'^{-1}}\theta'({}^{x_0'}n)} \\
&\quad = \frac{|(\boldsymbol{T}^0)^F|}{|\boldsymbol{T}^F|}\frac{|(\boldsymbol{T}'^0)^F|}{|\boldsymbol{T}'^F|}\frac{|N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F|}{|N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F|} \\
&\qquad \times \sum_{b,a,a'\in[N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F/N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F]} \langle \operatorname{Ind}_{({}^{a^{-1}}\boldsymbol{T})^{[b]F}}^{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F\cdot b}{}^{a^{-1}}\theta, \operatorname{Ind}_{({}^{a'^{-1}}\boldsymbol{T}')^{[b]F}}^{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F\cdot b}{}^{a'^{-1}}\theta'\rangle_{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F b}.
\end{aligned}$$

We may simplify the sum by conjugating the terms in the scalar product by a to get

$$\left\langle \operatorname{Ind}_{\boldsymbol{T}^{[{}^ab]F}}^{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F.{}^ab}\theta, \operatorname{Ind}_{({}^{aa'^{-1}}\boldsymbol{T}')^{[{}^ab]F}}^{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F.{}^ab}{}^{aa'^{-1}}\theta'\right\rangle_{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F\,{}^ab}.$$

Then we may take, given a, the conjugate ${}^ab$ as new variable b, and aa'^{-1} as the new variable a' to get

$$\frac{|(\boldsymbol{T}^0)^F|}{|\boldsymbol{T}^F|}\frac{|(\boldsymbol{T}'^0)^F|}{|\boldsymbol{T}'^F|} \sum_{b,a'\in\left[\frac{N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F}{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F}\right]} \left\langle \operatorname{Ind}_{\boldsymbol{T}^{[b]F}}^{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F\cdot b}\theta, \operatorname{Ind}_{({}^{a'}\boldsymbol{T}')^{[b]F}}^{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F\cdot b}{}^{a'}\theta'\right\rangle_{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F b}.$$

Now, by Frobenius reciprocity, for the inner scalar product not to vanish, there must be some element $x \in N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F$ such that ${}^x({}^{a'}\boldsymbol{T}')^{[b]F}$ meets $\boldsymbol{T}^{[b]F}$ which, considering the definitions, implies that $({}^{xa'}\boldsymbol{T}')^{[b]} = \boldsymbol{T}^{[b]}$. We may then conjugate the term

$$\operatorname{Ind}_{({}^{a'}\boldsymbol{T}')^{[b]F}}^{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F\cdot b}\ {}^{a'}\theta'$$

by such an x to get

$$\operatorname{Ind}_{\boldsymbol{T}^{[b]F}}^{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F\cdot b}\ {}^{xa'}\theta'$$

and take $y = xa'$ as a new variable, provided we count the number of x for a given a', which is $|N_{\boldsymbol{G}^0}(\boldsymbol{T}^{[b]})^F|$. We get

$$\text{(3.4)}\quad \frac{|(\boldsymbol{T}^0)^F|}{|\boldsymbol{T}^F|}\frac{|(\boldsymbol{T}'^0)^F|}{|\boldsymbol{T}'^F|}\sum_{b\in[N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F/N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F]}|N_{\boldsymbol{G}^0}(\boldsymbol{T}^{[b]})^F|^{-1}$$
$$\times\sum_{\{y\in N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F\mid({}^y\boldsymbol{T}')^{[b]}=\boldsymbol{T}^{[b]}\}}\left\langle\operatorname{Ind}_{\boldsymbol{T}^{[b]F}}^{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F\cdot b}\theta,\operatorname{Ind}_{\boldsymbol{T}^{[b]F}}^{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F\cdot b}\ {}^y\theta'\right\rangle_{N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F b}.$$

Since any $b\in[N_{\boldsymbol{G}}(\boldsymbol{T}^0)^F/N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)^F]$ such that $\boldsymbol{T}^{[b]F}$ is not empty has a representative in $\boldsymbol{T}^F$, we can do the first summation over $b\in[\boldsymbol{T}^F/(\boldsymbol{T}^0)^F]$ so that (3.3) is equal to (3.4). □

4. Counting unipotent elements in disconnected groups

A proof of the following result appeared recently in [Lawther et al. 2014, Theorem 1.1]; our proof given below, that we wrote in February 1994 in answer to a question of Cheryl Praeger, is much shorter and case-free.

Proposition 4.1. *Assume $\boldsymbol{G}^1/\boldsymbol{G}^0$ is unipotent and take $\sigma\in\boldsymbol{G}^1$ unipotent F-stable and quasicentral (see Lemma 2.2). Then the number of unipotent elements of $(\boldsymbol{G}^1)^F$ is given by $|(\boldsymbol{G}^{\sigma 0})^F|_p^2\,|\boldsymbol{G}^{0F}|/|(\boldsymbol{G}^{\sigma 0})^F|$.*

Proof. Let $\chi_{\mathcal{U}}$ be the characteristic function of the set of unipotent elements of $(\boldsymbol{G}^1)^F$. Then $|(\boldsymbol{G}^1)^F_{\text{unip}}|=|(\boldsymbol{G}^1)^F|\,\langle\chi_{\mathcal{U}},\mathrm{Id}\rangle_{(\boldsymbol{G}^1)^F}$ and

$$\langle\chi_{\mathcal{U}},\mathrm{Id}\rangle_{(\boldsymbol{G}^1)^F}=\langle\mathrm{D}_{\boldsymbol{G}^1}(\chi_{\mathcal{U}}),\mathrm{D}_{\boldsymbol{G}^1}(\mathrm{Id})\rangle_{(\boldsymbol{G}^1)^F}=\langle\mathrm{D}_{\boldsymbol{G}^1}(\chi_{\mathcal{U}}),\mathrm{St}_{\boldsymbol{G}^1}\rangle_{(\boldsymbol{G}^1)^F},$$

where the first equality holds since $\mathrm{D}_{\boldsymbol{G}^1}$ is an isometry by [DM 1994, corollaire 3.12]. According to [DM 1994, proposition 2.11], for any σ-stable and F-stable Levi subgroup $\boldsymbol{L}^0$ of a σ-stable parabolic subgroup of $\boldsymbol{G}^0$, setting $\boldsymbol{L}^1=\boldsymbol{L}^0.\sigma$, we have $R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}(\pi.\chi_{\mathcal{U}}|_{(\boldsymbol{L}^1)^F})=R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}(\pi).\chi_{\mathcal{U}}$ and ${}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}(\varphi).\chi_{\mathcal{U}}|_{(\boldsymbol{L}^1)^F}={}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}(\varphi.\chi_{\mathcal{U}})$. Thus, by (2.12), $\mathrm{D}_{\boldsymbol{G}^1}(\pi.\chi_{\mathcal{U}})=\mathrm{D}_{\boldsymbol{G}^1}(\pi).\chi_{\mathcal{U}}$; in particular, $\mathrm{D}_{\boldsymbol{G}^1}(\chi_{\mathcal{U}})=\mathrm{D}_{\boldsymbol{G}^1}(\mathrm{Id}).\chi_{\mathcal{U}}=\mathrm{St}_{\boldsymbol{G}^1}.\chi_{\mathcal{U}}$. Now, by Proposition 2.13, the only unipotent elements on which $\mathrm{St}_{\boldsymbol{G}^1}$ does not vanish are the quasi-semisimple (thus quasicentral) ones; by [DM 1994, corollaire 1.37], all such elements are in the $\boldsymbol{G}^{0F}$-class of σ and, again by Proposition 2.13, we have

$\mathrm{St}_{G^1}(\sigma) = |(G^{\sigma 0})^F|_p$. We get

$$\begin{aligned}|(G^1)^F|\langle \mathrm{D}_{G^1}(\chi_{\mathcal{U}}), \mathrm{St}_{G^1}\rangle_{(G^1)^F} &= |(G^1)^F|\langle \mathrm{St}_{G^1} .\chi_{\mathcal{U}}, \mathrm{St}_{G^1}\rangle_{(G^1)^F} \\ &= \left|\{G^{0F}\text{-class of } \sigma\}\right| \left|(G^{\sigma 0})^F\right|_p^2,\end{aligned}$$

whence the proposition. □

Example 4.2. The formula of Proposition 4.1 applies in the following cases where σ induces a diagram automorphism of order 2 and q is a power of 2:

- $G^0 = \mathrm{SO}_{2n}$, $(G^{\sigma 0})^F = \mathrm{SO}_{2n-1}(\mathbb{F}_q)$;
- $G^0 = \mathrm{GL}_{2n}$, $(G^{\sigma 0})^F = \mathrm{Sp}_{2n}(\mathbb{F}_q)$;
- $G^0 = \mathrm{GL}_{2n+1}$, $(G^{\sigma 0})^F = \mathrm{SO}_{2n+1}(\mathbb{F}_q) \simeq \mathrm{Sp}_{2n}(\mathbb{F}_q)$;
- $G^0 = E_6$, $(G^{\sigma 0})^F = F_4(\mathbb{F}_q)$;

And it applies to the case where $G^0 = \mathrm{Spin}_8$, where σ induces a diagram automorphism of order 3 and q is a power of 3, in which case $(G^{\sigma 0})^F = G_2(\mathbb{F}_q)$.

5. Tensoring by the Steinberg character

Proposition 5.1. *Let L^1 be an F-stable Levi of G^1. Then, for any class function γ on $(G^1)^F$, we have*

$$^*R_{L^1}^{G^1}(\gamma \cdot \varepsilon_{G^1} \mathrm{St}_{G^1}) = \varepsilon_{L^1} \mathrm{St}_{L^1} \mathrm{Res}_{(L^1)^F}^{(G^1)^F} \gamma.$$

Proof. Let su be the Jordan decomposition of a quasi-semisimple element of G^1 with s semisimple. We claim that u is quasicentral in G^s. Indeed su, being quasi-semisimple, is in a "torus" T; thus s and u also are in T. By [DM 1994, théorème 1.8(iii)], the intersection of $T \cap G^s$ is a "torus" of G^s; thus u is quasi-semisimple in G^s, and hence quasicentral since unipotent.

Let tv be the Jordan decomposition of an element $l \in (L^1)^F$, where t is semisimple. Since St_{L^1} vanishes outside quasi-semisimple elements, the right-hand side of the proposition vanishes on l unless it is quasi-semisimple, which by our claim means that v is quasicentral in L^t. By the character formula Proposition 2.5 the left-hand side of the proposition evaluates at l to

$$^*R_{L^1}^{G^1}(\gamma \cdot \varepsilon_{G^1} \mathrm{St}_{G^1})(l) = |(G^{t0})^F|^{-1} \sum_{u \in (G^{t0} \cdot v)_{\mathrm{unip}}^F} Q_{L^{t0}}^{G^{t0}}(u, v^{-1}) \gamma(tu) \varepsilon_{G^1} \mathrm{St}_{G^1}(tu).$$

By the same argument as above, applied to St_{G^1}, the only nonzero terms in the above sum are for u quasicentral in G^t. For such u, by [DM 1994, proposition 4.16], $Q_{L^{t0}}^{G^{t0}}(u, v^{-1})$ vanishes unless u and v are $(G^{t0})^F$-conjugate. Hence both sides of the equality-to-prove vanish unless u and v are quasicentral and $(G^{t0})^F$-conjugate. In that case, by [DM 1994, proposition 4.16] and [Digne and Michel 1991, (**),

p. 98], we have $Q_{\boldsymbol{L}^{t0}}^{\boldsymbol{G}^{t0}}(u, v^{-1}) = Q_{\boldsymbol{L}^{l0}}^{\boldsymbol{G}^{l0}}(1,1) = \varepsilon_{\boldsymbol{G}^{l0}}\varepsilon_{\boldsymbol{L}^{l0}}|(\boldsymbol{G}^{l0})^F|_{p'}|(\boldsymbol{L}^{l0})^F|_p$. Taking into account that the $(\boldsymbol{G}^{t0})^F$-class of v has cardinality $|(\boldsymbol{G}^{t0})^F|/|(\boldsymbol{G}^{l0})^F|$ and that by Proposition 2.13 we have $\mathrm{St}_{\boldsymbol{G}^1}(l) = \varepsilon_{\boldsymbol{G}^{\sigma 0}}\varepsilon_{\boldsymbol{G}^{l0}}|(\boldsymbol{G}^{l0})^F|_p$, the left-hand side of the proposition reduces to $\gamma(l)\varepsilon_{\boldsymbol{L}^{l0}}|(\boldsymbol{L}^{l0})^F|_p$, which is also the value of the right-hand side by applying Proposition 2.13 in $\boldsymbol{L}^1$. □

By adjunction, we get the following:

Corollary 5.2. *For any class function λ on $(\boldsymbol{L}^1)^F$, we have*

$$R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}(\lambda)\varepsilon_{\boldsymbol{G}^1}\mathrm{St}_{\boldsymbol{G}^1} = \mathrm{Ind}_{(\boldsymbol{L}^1)^F}^{(\boldsymbol{G}^1)^F}(\varepsilon_{\boldsymbol{L}^1}\mathrm{St}_{\boldsymbol{L}^1}\lambda).$$

6. Characteristic functions of quasi-semisimple classes

One of the goals of this section is Proposition 6.4 where we give a formula for the characteristic function of a quasi-semisimple class which shows, in particular, that it is uniform; this generalizes the case of quasicentral elements given in [DM 1994, proposition 4.14].

If $x \in (\boldsymbol{G}^1)^F$ has Jordan decomposition $x = su$, we will denote by d_x the map from class functions on $(\boldsymbol{G}^1)^F$ to class functions on $(C_{\boldsymbol{G}}(s)^0 \cdot u)^F$ given by

$$(d_x f)(v) = \begin{cases} f(sv) & \text{if } v \in (C_{\boldsymbol{G}}(s)^0\cdot u)^F \text{ is unipotent,} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.1. *Let $\boldsymbol{L}^1$ be an F-stable Levi of $\boldsymbol{G}^1$. If $x = su$ is the Jordan decomposition of an element of $(\boldsymbol{L}^1)^F$, we have $d_x \circ {}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1} = {}^*R_{C_{\boldsymbol{L}}(s)^0\cdot u}^{C_{\boldsymbol{G}}(s)^0\cdot u} \circ d_x$.*

Proof. For v unipotent in $(C_{\boldsymbol{G}}(s)^0\cdot u)^F$ and f a class function on $(\boldsymbol{G}^1)^F$, we have

$$(d_x {}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1} f)(v) = ({}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1} f)(sv) = ({}^*R_{C_{\boldsymbol{L}}(s)^0\cdot su}^{C_{\boldsymbol{G}}(s)^0\cdot su} f)(sv) = ({}^*R_{C_{\boldsymbol{L}}(s)^0\cdot u}^{C_{\boldsymbol{G}}(s)^0\cdot u} d_x f)(v),$$

where the second equality is by [DM 1994, corollaire 2.9] and the last is by the character formula Proposition 2.5(iii). □

Proposition 6.2. *If $x = su$ is the Jordan decomposition of an element of $(\boldsymbol{G}^1)^F$, we have $d_x \circ p^{\boldsymbol{G}^1} = p^{C_{\boldsymbol{G}}(s)^0\cdot u} \circ d_x$.*

Proof. Let f be a class function on $(\boldsymbol{G}^1)^F$. For $v \in (C_{\boldsymbol{G}}(s)^0\cdot u)^F$ unipotent, we have

$$(d_x p^{\boldsymbol{G}^1} f)(v) = p^{\boldsymbol{G}^1} f(sv) = |(\boldsymbol{G}^1)^F|^{-1} \sum_{\boldsymbol{T}^1 \in \mathcal{T}_1^F} |(\boldsymbol{T}^1)^F| (R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1} \circ {}^*R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1} f)(sv),$$

where the last equality is by Corollary 2.10, and which by Proposition 2.5(ii) is

$$\sum_{\boldsymbol{T}^1\in\mathcal{T}_1^F} \sum_{\{h \in (\boldsymbol{G}^0)^F \mid {}^h\boldsymbol{T} \ni s\}} \frac{|{}^h\boldsymbol{T}^0 \cap C_{\boldsymbol{G}}(s)^{0F}|}{|(\boldsymbol{G}^0)^F||C_{\boldsymbol{G}}(s)^{0F}|} \big(R_{{}^h\boldsymbol{T}\cap C_{\boldsymbol{G}}(s)^0\cdot su}^{C_{\boldsymbol{G}}(s)^0\cdot su} \circ {}^{h*}R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1} f\big)(sv).$$

Using that ${}^{h*}R^{\boldsymbol{G}^1}_{\boldsymbol{T}^1} f = {}^*R^{\boldsymbol{G}^1}_{{}^h\boldsymbol{T}^1} f$ and summing over the ${}^h\boldsymbol{T}^1$, this becomes

$$\sum_{\{\boldsymbol{T}^1\in\mathcal{T}_1^F\mid\boldsymbol{T}\ni s\}} \frac{|\boldsymbol{T}^0\cap C_{\boldsymbol{G}}(s)^{0F}|}{|C_{\boldsymbol{G}}(s)^{0F}|}\big(R^{C_{\boldsymbol{G}}(s)^0\cdot su}_{\boldsymbol{T}^1\cap C_{\boldsymbol{G}}(s)^0\cdot su}\circ {}^*R^{\boldsymbol{G}^1}_{\boldsymbol{T}^1} f\big)(sv).$$

Using that by Proposition 2.5(i) for any class function χ on $\boldsymbol{T}^1\cap C_{\boldsymbol{G}}(s)^0\cdot su^F$,

$$\begin{aligned}(R^{C_{\boldsymbol{G}}(s)^0\cdot su}_{\boldsymbol{T}^1\cap C_{\boldsymbol{G}}(s)^0\cdot su}\chi)(sv) &= |\boldsymbol{T}^0\cap C_{\boldsymbol{G}}(s)^{0F}|^{-1}\sum_{v'\in(\boldsymbol{T}\cap C_{\boldsymbol{G}}(s)^0\cdot u)^F_{\text{unip}}} Q^{(\boldsymbol{G}^s)^0}_{(\boldsymbol{T}^s)^0}(v,v'^{-1})\chi(sv')\\ &= R^{C_{\boldsymbol{G}}(s)^0\cdot u}_{\boldsymbol{T}\cap C_{\boldsymbol{G}}(s)^0\cdot u}(d_x\chi)(v),\end{aligned}$$

and using Lemma 6.1, we get

$$|C_{\boldsymbol{G}}(s)^0\cdot su^F|^{-1}\sum_{\{\boldsymbol{T}^1\in\mathcal{T}_1^F\mid\boldsymbol{T}\ni s\}}|(\boldsymbol{T}^s)^{0F}|\big(R^{C_{\boldsymbol{G}}(s)^0\cdot u}_{\boldsymbol{T}\cap C_{\boldsymbol{G}}(s)^0\cdot u}\circ {}^*R^{C_{\boldsymbol{G}}(s)^0\cdot u}_{\boldsymbol{T}\cap C_{\boldsymbol{G}}(s)^0\cdot u} d_x f\big)(v),$$

which is the desired result if we apply Corollary 2.10 in $C_{\boldsymbol{G}}(s)^0\cdot u$ and remark that by [DM 1994, théorème 1.8(iv)], the map $\boldsymbol{T}^1\mapsto \boldsymbol{T}\cap C_{\boldsymbol{G}}(s)^0\cdot u$ induces a bijection between $\{\boldsymbol{T}^1\in\mathcal{T}_1^F\mid\boldsymbol{T}\ni s\}$ and F-stable "tori" of $C_{\boldsymbol{G}}(s)^0\cdot u$. □

Corollary 6.3. *A class function f on $(\boldsymbol{G}^1)^F$ is uniform if and only if for every $x\in(\boldsymbol{G}^1)^F$, the function $d_x f$ is uniform.*

Proof. Indeed, $f = p^{\boldsymbol{G}^1} f$ if and only if for any $x\in(\boldsymbol{G}^1)^F$, we have $d_x f = d_x p^{\boldsymbol{G}^1} f = p^{C_{\boldsymbol{G}}(s)^0\cdot u} d_x f$, where the last equality holds by Proposition 6.2. □

For $x\in(\boldsymbol{G}^1)^F$, we consider the class function $\pi^{\boldsymbol{G}^1}_x$ on $(\boldsymbol{G}^1)^F$ defined by

$$\pi^{\boldsymbol{G}^1}_x(y) = \begin{cases}0 & \text{if } y \text{ is not conjugate to } x,\\ |C_{\boldsymbol{G}^0}(x)^F| & \text{if } y = x.\end{cases}$$

Proposition 6.4. *For a quasi-semisimple $x\in(\boldsymbol{G}^1)^F$, the function $\pi^{\boldsymbol{G}^1}_x$ is uniform, given by*

$$\begin{aligned}\pi^{\boldsymbol{G}^1}_x &= \varepsilon_{\boldsymbol{G}^{x0}}|C_{\boldsymbol{G}}(x)^0|_p^{-1}\sum_{\{\boldsymbol{T}^1\in\mathcal{T}_1^F\mid\boldsymbol{T}^1\ni x\}}\varepsilon_{\boldsymbol{T}^1}R^{\boldsymbol{G}^1}_{\boldsymbol{T}^1}(\pi^{\boldsymbol{T}^1}_x)\\ &= |W^0(x)|^{-1}\sum_{w\in W^0(x)}\dim R^{C_{\boldsymbol{G}}(x)^0}_{\boldsymbol{T}_w}(\mathrm{Id})\, R^{\boldsymbol{G}^1}_{C_{\boldsymbol{G}^1}(\boldsymbol{T}_w)}(\pi_x^{C_{\boldsymbol{G}^1}(\boldsymbol{T}_w)}),\end{aligned}$$

where in the second equality $W^0(x)$ denotes the Weyl group of $C_{\boldsymbol{G}}(x)^0$ and $\boldsymbol{T}_w$ denotes an F-stable torus of type w of this last group.

Proof. First, using Corollary 6.3 we prove that $\pi^{\boldsymbol{G}^1}_x$ is uniform. Let su be the Jordan decomposition of x. For $y\in(\boldsymbol{G}^1)^F$, the function $d_y\pi^{\boldsymbol{G}^1}_x$ is zero unless the semisimple part of y is conjugate to s. Hence it is sufficient to evaluate $d_y\pi^{\boldsymbol{G}^1}_x(v)$ for elements y whose semisimple part is equal to s. For such elements, $d_y\pi^{\boldsymbol{G}^1}_x(v)$

is up to a coefficient equal to $\pi_u^{C_G(s)^0\cdot u}$. This function is uniform by [DM 1994, proposition 4.14], since u being the unipotent part of a quasi-semisimple element is quasicentral in $C_{\boldsymbol{G}}(s)$ (see the beginning of the proof of Proposition 5.1).

Thus we have $\pi_x^{\boldsymbol{G}^1}=p^{\boldsymbol{G}^1}\pi_x^{\boldsymbol{G}^1}$. We use this to get the formula of the proposition. We start by using Proposition 2.13 to write $\pi_x^{\boldsymbol{G}^1}\operatorname{St}_{\boldsymbol{G}^1}=\varepsilon_{\boldsymbol{G}^1}\varepsilon_{\boldsymbol{G}^{x0}}|(\boldsymbol{G}^{x0})^F|_p\pi_x^{\boldsymbol{G}^1}$, or equivalently, $\pi_x^{\boldsymbol{G}^1}=\varepsilon_{\boldsymbol{G}^1}\varepsilon_{\boldsymbol{G}^{x0}}|(\boldsymbol{G}^{x0})^F|_p^{-1}p^{\boldsymbol{G}^1}(\pi_x^{\boldsymbol{G}^1}\operatorname{St}_{\boldsymbol{G}^1})$. Using Corollary 2.10 and that by Proposition 5.1 we have ${}^*R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\pi_x^{\boldsymbol{G}^1}\operatorname{St}_{\boldsymbol{G}^1})=\varepsilon_{\boldsymbol{G}^1}\varepsilon_{\boldsymbol{T}^1}\operatorname{St}_{\boldsymbol{T}^1}\operatorname{Res}_{(\boldsymbol{T}^1)^F}^{(\boldsymbol{G}^1)^F}(\pi_x^{\boldsymbol{G}^1})$, we get

$$p^{\boldsymbol{G}^1}(\pi_x^{\boldsymbol{G}^1}\operatorname{St}_{\boldsymbol{G}^1})=\varepsilon_{\boldsymbol{G}^1}|(\boldsymbol{G}^1)^F|^{-1}\sum_{\boldsymbol{T}^1\in\mathcal{T}_1^F}|(\boldsymbol{T}^1)^F|\varepsilon_{\boldsymbol{T}^1}R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}\big(\operatorname{St}_{\boldsymbol{T}^1}\operatorname{Res}_{(\boldsymbol{T}^1)^F}^{(\boldsymbol{G}^1)^F}(\pi_x^{\boldsymbol{G}^1})\big).$$

The function $\operatorname{St}_{\boldsymbol{T}^1}$ is constant equal to 1. Now we have

$$\operatorname{Res}_{(\boldsymbol{T}^1)^F}^{(\boldsymbol{G}^1)^F}\pi_x^{\boldsymbol{G}^1}=|(\boldsymbol{T}^0)^F|^{-1}\sum_{\{g\in(\boldsymbol{G}^0)^F\mid{}^gx\in\boldsymbol{T}^1\}}\pi_{{}^gx}^{\boldsymbol{T}^1}.$$

To see this, do the scalar product with a class function f on $(\boldsymbol{T}^1)^F$:

$$\big\langle\operatorname{Res}_{(\boldsymbol{T}^1)^F}^{(\boldsymbol{G}^1)^F}\pi_x^{\boldsymbol{G}^1},f\big\rangle_{(\boldsymbol{T}^1)^F}=\langle\pi_x^{\boldsymbol{G}^1},\operatorname{Ind}_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}f\rangle_{(\boldsymbol{G}^1)^F}=|(\boldsymbol{T}^0)^F|^{-1}\sum_{\{g\in(\boldsymbol{G}^0)^F\mid{}^gx\in\boldsymbol{T}^1\}}f({}^gx).$$

Using that $|(\boldsymbol{T}^0)^F|=|(\boldsymbol{T}^1)^F|$, we then get

$$p^{\boldsymbol{G}^1}(\pi_x^{\boldsymbol{G}^1}\operatorname{St}_{\boldsymbol{G}^1})=\varepsilon_{\boldsymbol{G}^1}|(\boldsymbol{G}^1)^F|^{-1}\sum_{\boldsymbol{T}^1\in\mathcal{T}_1^F}\sum_{\{g\in(\boldsymbol{G}^0)^F\mid{}^gx\in\boldsymbol{T}^1\}}\varepsilon_{\boldsymbol{T}^1}R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\pi_{{}^gx}^{\boldsymbol{T}^1}).$$

Taking ${}^{g^{-1}}\boldsymbol{T}^1$ as summation index, we get

$$p^{\boldsymbol{G}^1}(\pi_x^{\boldsymbol{G}^1}\operatorname{St}_{\boldsymbol{G}^1})=\varepsilon_{\boldsymbol{G}^1}\sum_{\{\boldsymbol{T}^1\in\mathcal{T}_1^F\mid\boldsymbol{T}^1\ni x\}}\varepsilon_{\boldsymbol{T}^1}R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\pi_x^{\boldsymbol{T}^1}),$$

and hence

$$\pi_x^{\boldsymbol{G}^1}=\varepsilon_{\boldsymbol{G}^{x0}}|(\boldsymbol{G}^{x0})^F|_p^{-1}\sum_{\{\boldsymbol{T}^1\in\mathcal{T}_1^F\mid\boldsymbol{T}^1\ni x\}}\varepsilon_{\boldsymbol{T}^1}R_{\boldsymbol{T}^1}^{\boldsymbol{G}^1}(\pi_x^{\boldsymbol{T}^1}),$$

which is the first equality of the proposition.

For the second equality of the proposition, we first use [DM 1994, théorème 1.8(iii) and (iv)] to sum over tori of $C_{\boldsymbol{G}}(x)^0$: the $\boldsymbol{T}^1\in\mathcal{T}_1^F$ containing x are in bijection with the maximal tori of $C_{\boldsymbol{G}}(x)^0$ by $\boldsymbol{T}^1\mapsto(\boldsymbol{T}^{1x})^0$ and conversely $\boldsymbol{S}\mapsto C_{\boldsymbol{G}^1}(\boldsymbol{S})$. This bijection satisfies $\varepsilon_{\boldsymbol{T}^1}=\varepsilon_{\boldsymbol{S}}$ by the definition of ε.

We then sum over $(C_{\boldsymbol{G}}(x)^0)^F$-conjugacy classes of maximal tori, which are parameterized by F-conjugacy classes of $W^0(x)$. We then have to multiply by $|(C_{\boldsymbol{G}}(x)^0)^F|/|N_{(C_{\boldsymbol{G}}(x)^0)}(\boldsymbol{S})^F|$ the term indexed by the class of $\boldsymbol{S}$. Then we sum over the elements of $W^0(x)$. We then have to multiply the term indexed by w by $|C_{W^0(x)}(wF)|/|W^0(x)|$. Using $|N_{(C_{\boldsymbol{G}}(x)^0)}(\boldsymbol{S})^F|=|\boldsymbol{S}^F||C_{W^0(x)}(wF)|$, and the formula for $\dim R_{\boldsymbol{T}_w}^{C_{\boldsymbol{G}}(x)^0}(\mathrm{Id})$ we get the result. □

7. Classification of quasi-semisimple classes

The first items of this section, before Proposition 7.7, apply for algebraic groups over an arbitrary algebraically closed field k.

We denote by $\mathcal{C}(\boldsymbol{G}^1)$ the set of conjugacy classes of $\boldsymbol{G}^1$, that is, the orbits under $\boldsymbol{G}^0$-conjugacy, and denote by $\mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}}$ the set of quasi-semisimple classes.

Proposition 7.1. *For $\boldsymbol{T}^1 \in \mathcal{T}_1$, write $\boldsymbol{T}^1 = \boldsymbol{T}^0 \cdot \sigma$, where σ is quasicentral. Then $\mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}}$ is in bijection with the set of $N_{\boldsymbol{G}^0}(\boldsymbol{T}^1)$-orbits in $\boldsymbol{T}^1$, which itself is in bijection with the set of W^σ-orbits in $\mathcal{C}(\boldsymbol{T}^1)$, where $W = N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)/\boldsymbol{T}^0$. We have $\mathcal{C}(\boldsymbol{T}^1) \simeq \boldsymbol{T}^1/\mathcal{L}_\sigma(\boldsymbol{T}^0)$, where $\mathcal{L}_\sigma$ is the map $t \mapsto t^{-1}.{}^\sigma t$.*

Proof. By definition, every quasi-semisimple element of $\boldsymbol{G}^1$ is in some $\boldsymbol{T}^1 \in \mathcal{T}_1$ and $\mathcal{T}_1$ is a single orbit under $\boldsymbol{G}^0$-conjugacy. It is thus sufficient to find how classes of $\boldsymbol{G}^1$ intersect $\boldsymbol{T}^1$. By [DM 1994, proposition 1.13], two elements of $\boldsymbol{T}^1$ are $\boldsymbol{G}^0$-conjugate if and only if they are conjugate under $N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)$. We can replace $N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)$ by $N_{\boldsymbol{G}^0}(\boldsymbol{T}^1)$ since if ${}^g(\sigma t) = \sigma t'$, where $g \in N_{\boldsymbol{G}^0}(\boldsymbol{T}^0)$, then the image of g in W lies in W^σ. By [DM 1994, définition-théorème 1.15(iii)], elements of W^σ have representatives in $\boldsymbol{G}^{\sigma 0}$. Write $g = s\dot{w}$, where $\dot{w}$ is such a representative and $s \in \boldsymbol{T}^0$. Then ${}^{s\dot{w}}(t\sigma) = \mathcal{L}_\sigma(s^{-1})\,{}^w t\sigma$, whence the proposition. $\square$

Lemma 7.2.
$$\boldsymbol{T}^0 = \boldsymbol{T}^{\sigma 0}.\mathcal{L}_\sigma(\boldsymbol{T}^0).$$

Proof. This is proved in [DM 1994, corollaire 1.33] when σ is unipotent (and then the product is direct). We proceed similarly to that proof: $\boldsymbol{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\boldsymbol{T}^0)$ is finite since its exponent divides the order of σ (if $\sigma(t^{-1}\,{}^\sigma t) = t^{-1}\,{}^\sigma t$ then $(t^{-1}\,{}^\sigma t)^n = t^{-1}\,{}^{\sigma^n} t$ for all $n \geq 1$), and $\dim(\boldsymbol{T}^{\sigma 0}) + \dim(\mathcal{L}_\sigma(\boldsymbol{T}^0)) = \dim(\boldsymbol{T}^0)$ as the exact sequence $1 \to \boldsymbol{T}^{0\sigma} \to \boldsymbol{T}^0 \to \mathcal{L}_\sigma(\boldsymbol{T}^0) \to 1$ shows, using that $\dim(\boldsymbol{T}^{\sigma 0}) = \dim \boldsymbol{T}^{0\sigma}$. $\square$

It follows that $\boldsymbol{T}^0/\mathcal{L}_\sigma(\boldsymbol{T}^0) \simeq \boldsymbol{T}^{\sigma 0}/(\boldsymbol{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\boldsymbol{T}^0))$; since the set $\mathcal{C}(\boldsymbol{G}^{\sigma 0})_{\mathrm{ss}}$ of semisimple classes of $\boldsymbol{G}^{\sigma 0}$ identifies with the set of W^σ-orbits on $\boldsymbol{T}^{\sigma 0}$, this induces a surjective map $\mathcal{C}(\boldsymbol{G}^{\sigma 0})_{\mathrm{ss}} \to \mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}}$.

Example 7.3. We will describe the quasi-semisimple classes of $\boldsymbol{G}^0 \cdot \sigma$, where $\boldsymbol{G}^0 = \mathrm{GL}_n(k)$ and σ is the quasicentral automorphism given by $\sigma(g) = J\,{}^t g^{-1} J^{-1}$, where, if n is even, J is the matrix $\left(\begin{smallmatrix} 0 & -J_0 \\ J_0 & 0 \end{smallmatrix}\right)$ with
$$J_0 = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix},$$
and if n is odd, J is the antidiagonal matrix
$$J = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

(any outer algebraic automorphism of GL_n is equal to σ up to an inner automorphism).

The automorphism σ normalizes the pair $\boldsymbol{T}^0 \subset \boldsymbol{B}^0$, where $\boldsymbol{T}^0$ is the diagonal torus and $\boldsymbol{B}^0$ the group of upper triangular matrices. Then $\boldsymbol{T}^1 = N_{\boldsymbol{G}^1}(\boldsymbol{T}^0 \subset \boldsymbol{B}^0) = \boldsymbol{T}^0 \cdot \sigma$ is in $\mathcal{T}_1$. For $\mathrm{diag}(x_1, \dots, x_n) \in \boldsymbol{T}^0$, where $x_i \in k^\times$, we have $\sigma(\mathrm{diag}(x_1, \dots, x_n)) = \mathrm{diag}(x_n^{-1}, \dots, x_1^{-1})$. It follows that $\mathcal{L}_\sigma(\boldsymbol{T}^0) = \{\mathrm{diag}(x_1, x_2, \dots, x_2, x_1)\}$ — here x_{m+1} is a square when $n = 2m+1$ but this is not a condition since k is algebraically closed. As suggested above, we could take as representatives of $\boldsymbol{T}^0/\mathcal{L}_\sigma(\boldsymbol{T}^0)$ the set $\boldsymbol{T}^{\sigma 0}/(\boldsymbol{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\boldsymbol{T}^0))$, but since $\boldsymbol{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\boldsymbol{T}^0)$ is not trivial (it consists of the diagonal matrices with entries ± 1 placed symmetrically), it is more convenient to take for representatives of the quasi-semisimple classes, the set $\{\mathrm{diag}(x_1, x_2, \dots, x_{\lfloor n/2 \rfloor}, 1, \dots, 1)\}\sigma$. In this model, the action of W^σ is generated by the permutations of the $\lfloor n/2 \rfloor$ first entries, and by the maps $x_i \mapsto x_i^{-1}$, so the quasi-semisimple classes of $\boldsymbol{G}^0 \cdot \sigma$ are parameterized by the quasi-semisimple classes of $\boldsymbol{G}^{\sigma 0}$.

We continue the example, computing group of components of centralizers.

Proposition 7.4. *Let $s\sigma = \mathrm{diag}(x_1, x_2, \dots, x_{\lfloor n/2 \rfloor}, 1, \dots, 1)\sigma$ be a quasi-semisimple element as above. If $\mathrm{char}\, k = 2$ then $C_{\boldsymbol{G}^0}(s\sigma)$ is connected. Otherwise, if n is odd, $A(s\sigma) := C_{\boldsymbol{G}^0}(s\sigma)/C_{\boldsymbol{G}^0}(s\sigma)^0$ is of order two, generated by $-1 \in Z\boldsymbol{G}^0 = Z\,\mathrm{GL}_n(k)$. If n is even, $A(s\sigma) \neq 1$ if and only if for some i, we have $x_i = -1$; then $x_i \mapsto x_i^{-1}$ is an element of W^σ which has a representative in $C_{\boldsymbol{G}^0}(s\sigma)$ generating $A(s\sigma)$, which is of order 2.*

Proof. We will use that for a group G and an automorphism σ of G, we have an exact sequence (see, for example, [Steinberg 1968, 4.5])

$$1 \to (ZG)^\sigma \to G^\sigma \to (G/ZG)^\sigma \to (\mathcal{L}_\sigma(G) \cap ZG)/\mathcal{L}_\sigma(ZG) \to 1. \tag{7.5}$$

If we take $G = \boldsymbol{G}^0 = \mathrm{GL}_n(k)$ in (7.5) and $s\sigma$ for σ, since on $Z\boldsymbol{G}^0$ the map $\mathcal{L}_\sigma = \mathcal{L}_{s\sigma}$ is $z \mapsto z^2$, hence surjective, we get that $\boldsymbol{G}^{0^{s\sigma}} \to \mathrm{PGL}_n^{s\sigma}$ is surjective and has kernel $(Z\boldsymbol{G}^0)^\sigma = \{\pm 1\}$.

Assume n odd and take $G = \mathrm{SL}_n(k)$ in (7.5). We have $Z\,\mathrm{SL}_n^\sigma = \{1\}$ so that we get the following diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathrm{GL}_n^{s\sigma} \arrow[r] & \mathrm{PGL}_n^{s\sigma} \arrow[r] \arrow[d, equal] & 1 \\
 & 1 \arrow[r] & \mathrm{SL}_n^{s\sigma} \arrow[r] \arrow[u, hook] & \mathrm{PGL}_n^{s\sigma} \arrow[r] & 1
\end{tikzcd}$$

This shows that $\mathrm{GL}_n^{s\sigma} / \mathrm{SL}_n^{s\sigma} \simeq \{\pm 1\}$; by [Steinberg 1968, 8.1], $\mathrm{SL}_n^{s\sigma}$ is connected, hence $\mathrm{PGL}_n^{s\sigma}$ is connected. Thus $\mathrm{GL}_n^{s\sigma} = (\mathrm{GL}_n^{s\sigma})^0 \times \{\pm 1\}$ is connected if and only if $\mathrm{char}\, k = 2$.

Assume now that n is even; then $(\boldsymbol{T}^0)^\sigma$ is connected, and hence $-1 \in (\mathrm{GL}_n^{s\sigma})^0$ for all $s \in \boldsymbol{T}^0$. Using this, the exact sequence $1 \to \{\pm 1\} \to \mathrm{GL}_n^{s\sigma} \to \mathrm{PGL}_n^{s\sigma} \to 1$ implies $A(s\sigma) = \boldsymbol{G}^{s\sigma}/\boldsymbol{G}^{0s\sigma} = \mathrm{GL}_n^{s\sigma}/(\mathrm{GL}_n^{s\sigma})^0 \simeq \mathrm{PGL}_n^{s\sigma}/(\mathrm{PGL}_n^{s\sigma})^0$. To compute this group we use (7.5) with $\mathrm{SL}_n(k)$ for G and $s\sigma$ for σ:

$$1 \to \{\pm 1\} \to \mathrm{SL}_n^{s\sigma} \to \mathrm{PGL}_n^{s\sigma} \to (\mathcal{L}_{s\sigma}(\mathrm{SL}_n) \cap Z\,\mathrm{SL}_n)/\mathcal{L}_\sigma(Z\,\mathrm{SL}_n) \to 1,$$

which, since $\mathrm{SL}_n^{s\sigma}$ is connected, implies that

$$A(s\sigma) = (\mathcal{L}_{s\sigma}(\mathrm{SL}_n) \cap Z\,\mathrm{SL}_n)/\mathcal{L}_\sigma(Z\,\mathrm{SL}_n)$$

is nontrivial (of order 2) if and only if $\mathcal{L}_{s\sigma}(\mathrm{SL}_n) \cap Z\,\mathrm{SL}_n$ contains an element which is not a square in $Z\,\mathrm{SL}_n$; thus $A(s\sigma)$ is trivial if $\operatorname{char} k = 2$. We assume now that $\operatorname{char} k \neq 2$. Then a nonsquare is of the form $\operatorname{diag}(z, \dots, z)$ with $z^m = -1$ if we set $m = n/2$.

The following lemma is a transcription of [Steinberg 1968, 9.5].

Lemma 7.6. *Let σ be a quasicentral automorphism of the connected reductive group $\boldsymbol{G}$ which stabilizes the pair $\boldsymbol{T} \subset \boldsymbol{B}$ of a maximal torus and a Borel subgroup*; *let W be the Weyl group of $\boldsymbol{T}$ and let $s \in \boldsymbol{T}$. Then*

$$\boldsymbol{T} \cap \mathcal{L}_{s\sigma}(\boldsymbol{G}) = \{\mathcal{L}_w(s^{-1}) \mid w \in W^\sigma\} \cdot \mathcal{L}_\sigma(\boldsymbol{T}).$$

Proof. Assume $t = \mathcal{L}_{s\sigma}(x)$ for $t \in \boldsymbol{T}$, or equivalently $xt = {}^{s\sigma}x$. Then if x is in the Bruhat cell $\boldsymbol{B}w\boldsymbol{B}$, we must have $w \in W^\sigma$. Taking for w a σ-stable representative $\dot{w}$ and writing the unique Bruhat decomposition $x = u_1\dot{w}t_1u_2$, where $u_2 \in \boldsymbol{U}$, $t_1 \in \boldsymbol{T}$ and $u_1 \in \boldsymbol{U} \cap {}^w\boldsymbol{U}^-$, where $\boldsymbol{U}$ is the unipotent radical of $\boldsymbol{B}$ and $\boldsymbol{U}^-$ the unipotent radical of the opposite Borel, the equality $xt = {}^{s\sigma}x$ implies that $\dot{w}t_1t = {}^{s\sigma}(\dot{w}t_1)$, or equivalently, $t = \mathcal{L}_{w^{-1}}(s^{-1})\mathcal{L}_\sigma(t_1)$, whence the lemma. $\square$

We apply this lemma taking SL_n for $\boldsymbol{G}$ and $\boldsymbol{T}'^0 = \boldsymbol{T}^0 \cap \mathrm{SL}_n$ for $\boldsymbol{T}$: we get $\mathcal{L}_{s\sigma}(\mathrm{SL}_n) \cap Z\,\mathrm{SL}_n = \{\mathcal{L}_w(s^{-1}) \mid w \in W^\sigma\} \cdot \mathcal{L}_\sigma(\boldsymbol{T}'^0) \cap Z\,\mathrm{SL}_n$. The element

$$\operatorname{diag}(x_1, x_2, \dots, x_m, 1, \dots, 1)\sigma$$

is conjugate to

$$s\sigma = \operatorname{diag}(y_1, y_2, \dots, y_m, y_m^{-1}, \dots, y_1^{-1})\sigma \in (\boldsymbol{T}'^0)^\sigma \cdot \sigma,$$

where $y_i^2 = x_i$. It will have a nonconnected centralizer if and only if for some $w \in W^\sigma$ and some $t \in \boldsymbol{T}'^0$, we have $\mathcal{L}_w(s^{-1}) \cdot \mathcal{L}_\sigma(t) = \operatorname{diag}(z, \dots, z)$ with $z^m = -1$, and then an appropriate representative of w (multiplying if needed by an element of $Z\,\mathrm{GL}_n$) will be in $C_{\boldsymbol{G}^0}(s\sigma)$ and have a nontrivial image in $A(s\sigma)$. Since s and w are σ-fixed, we have $\mathcal{L}_w(s) \in (\boldsymbol{T}'^0)^\sigma$; thus it is of the form $\operatorname{diag}(a_1, \dots, a_m, a_m^{-1}, \dots a_1^{-1})$. Since

$$\mathcal{L}_\sigma(\boldsymbol{T}'^0) = \{\operatorname{diag}(t_1, \dots, t_m, t_m, \dots, t_1) \mid t_1t_2 \cdots t_m = 1\},$$

we get $z = a_1t_1 = a_2t_2 = \cdots = a_mt_m = a_m^{-1}t_m = \cdots = a_1^{-1}t_1$; in particular, $a_i = \pm 1$ for all i and $a_1a_2\cdots a_m = -1$. We can take w up to conjugacy in W^σ since $\mathcal{L}_{vwv^{-1}}(s^{-1}) = {}^v\mathcal{L}_w({}^{v^{-1}}s^{-1})$ and $\mathcal{L}_\sigma(\boldsymbol{T}'^0)$ is invariant under W^σ-conjugacy. We see W^σ as the group of permutations of $\{1, 2, \ldots, m, -m, \ldots, -1\}$ which preserves the pairs $\{i, -i\}$. A nontrivial cycle of w has, up to conjugacy, either the form $(1, -1)$ or $(1, -2, 3, \ldots, (-1)^{i-1}i, -(i+1), -(i+2), \ldots, -k, -1, 2, -3, \ldots, k)$ with $0 \le i \le k \le n$ and i odd, or $(1, -2, 3, \ldots, (-1)^{i-1}i, i+1, i+2, \ldots, k)$ with $0 \le i \le k \le n$ and i even (the case $i = 0$ meaning that there is no sign change). The contribution to $a_1\cdots a_m$ of the orbit $(1, -1)$ is $a_1 = y_1^2$, and hence is 1 except if $y_1^2 = x_1 = -1$. Let us consider an orbit of the second form. The k first coordinates of $\mathcal{L}_w(s^{-1})$ are $(y_1y_2, \ldots, y_iy_{i+1}, y_{i+1}/y_{i+2}, \ldots, y_k/y_1)$. Hence there must exist signs ε_j such that $y_2 = \varepsilon_1/y_1$, $y_3 = \varepsilon_2/y_2, \ldots, y_{i+1} = \varepsilon_i/y_i$ and $y_{i+2} = \varepsilon_{i+1}y_{i+1}, \ldots, y_k = \varepsilon_{k-1}y_{k-1}$, $y_1 = \varepsilon_k y_k$. This gives

$$y_1 = \begin{cases} \varepsilon_1\cdots\varepsilon_k y_1 & \text{if } i \text{ is even,} \\ \varepsilon_1\cdots\varepsilon_k/y_1 & \text{if } i \text{ is odd.} \end{cases}$$

The contribution of the orbit to $a_1\cdots a_m$ is $\varepsilon_1\cdots\varepsilon_k$ and thus is 1 if i is even and $x_1 = y_1^2$ if i is odd. Again, we see that one of the x_i must equal -1 to get $a_1\cdots a_m = -1$. Conversely if $x_1 = -1$, for any z such that $z^m = -1$, choosing t such that $\mathcal{L}_\sigma(t) = \operatorname{diag}(-z, z, z, \ldots, z, -z)$ and taking $w = (1, -1)$, we get $\mathcal{L}_w(s^{-1})\mathcal{L}_\sigma(t) = \operatorname{diag}(z, \ldots, z)$ as desired. □

We now go back to the case where $k = \overline{\mathbb{F}}_q$, and in the context of Proposition 7.1, we now assume that $\boldsymbol{T}^1$ is F-stable and that σ induces an F-stable automorphism of $\boldsymbol{G}^0$.

Proposition 7.7. *Let* $\boldsymbol{T}^{1\mathrm{rat}} = \{s \in \boldsymbol{T}^1 \mid \exists n \in N_{\boldsymbol{G}^0}(\boldsymbol{T}^1), {}^{nF}s = s\}$; *then* $\boldsymbol{T}^{1\mathrm{rat}}$ *is stable by* $\boldsymbol{T}^0$*-conjugacy, which gives a meaning to* $\mathcal{C}(\boldsymbol{T}^{1\mathrm{rat}})$*. Then* $c \mapsto c \cap \boldsymbol{T}^1$ *induces a bijection between* $(\mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}})^F$ *and the* W^σ*-orbits on* $\mathcal{C}(\boldsymbol{T}^{1\mathrm{rat}})$*.*

Proof. A class $c \in \mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}}$ is F-stable if and only if given $s \in c$, we have ${}^Fs \in c$. If we take $s \in c \cap \boldsymbol{T}^1$ then ${}^Fs \in c \cap \boldsymbol{T}^1$, which as observed in the proof of Proposition 7.1 implies that ${}^Fs$ is conjugate to s under $N_{\boldsymbol{G}^0}(\boldsymbol{T}^1)$, that is, $s \in \boldsymbol{T}^{1\mathrm{rat}}$. Thus c is F-stable if and only if $c \cap \boldsymbol{T}^1 = c \cap \boldsymbol{T}^{1\mathrm{rat}}$. The proposition then results from Proposition 7.1, observing that $\boldsymbol{T}^{1\mathrm{rat}}$ is stable under $N_{\boldsymbol{G}^0}(\boldsymbol{T}^1)$-conjugacy and that the corresponding orbits are the W^σ-orbits on $\mathcal{C}(\boldsymbol{T}^{1\mathrm{rat}})$. □

Example 7.8. When $\boldsymbol{G}^1 = \mathrm{GL}_n(\overline{\mathbb{F}}_q)\cdot\sigma$ with σ as in Example 7.3, the map

$$\operatorname{diag}(x_1, x_2, \ldots, x_{\lfloor n/2\rfloor}, 1, \ldots, 1) \mapsto \operatorname{diag}(x_1, x_2, \ldots, x_{\lfloor n/2\rfloor}, \dagger, x_{\lfloor n/2\rfloor}^{-1}, \ldots, x_2^{-1}, x_1^{-1}),$$

where † represents 1 if n is odd and an omitted entry otherwise, is compatible with the action of W^σ as described in Example 7.3 on the left-hand side and the natural action on the right-hand side. This map induces a bijection from $\mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}}$ to the semisimple classes of $(\mathrm{GL}_n^\sigma)^0$ which restricts to a bijection from $(\mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}})^F$ to the F-stable semisimple classes of $(\mathrm{GL}_n^\sigma)^0$.

We now compute the cardinality of $(\mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}})^F$.

Proposition 7.9. *Let f be a function on $(\mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}})^F$. Then*

$$\sum_{c\in(\mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}})^F} f(c) = |W^\sigma|^{-1} \sum_{w\in W^\sigma} \tilde{f}(w),$$

where $\tilde{f}(w) := \sum_s f(s)$, where s runs over representatives of $\boldsymbol{T}^{1^{wF}}/\mathcal{L}_\sigma(\boldsymbol{T}^0)^{wF}$ in $\boldsymbol{T}^{1^{wF}}$.

Proof. We have

$$\mathcal{C}(\boldsymbol{T}^{1\mathrm{rat}}) = \bigcup_{w\in W^\sigma} \left\{ s\mathcal{L}_\sigma(\boldsymbol{T}^0) \in \boldsymbol{T}^1/\mathcal{L}_\sigma(\boldsymbol{T}^0) \mid s\mathcal{L}_\sigma(\boldsymbol{T}^0) \text{ is } wF\text{-stable} \right\}.$$

The conjugation by $v \in W^\sigma$ sends a wF-stable coset $s\mathcal{L}_\sigma(\boldsymbol{T}^0)$ to a $vwFv^{-1}$-stable coset; and the number of w such that $s\mathcal{L}_\sigma(\boldsymbol{T}^0)$ is wF-stable is equal to $N_{W^\sigma}(s\mathcal{L}_\sigma(\boldsymbol{T}^0))$. It follows that

$$\sum_{c\in(\mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}})^F} f(c) = |W^\sigma|^{-1} \sum_{w\in W^\sigma} \sum_{s\mathcal{L}_\sigma(\boldsymbol{T}^0)\in(\boldsymbol{T}^1/\mathcal{L}_\sigma(\boldsymbol{T}^0))^{wF}} f(s\mathcal{L}_\sigma(\boldsymbol{T}^0)).$$

The proposition follows since, because $\mathcal{L}_\sigma(\boldsymbol{T}^0)$ is connected, we have

$$(\boldsymbol{T}^1/\mathcal{L}_\sigma(\boldsymbol{T}^0))^{wF} = \boldsymbol{T}^{1^{wF}}/\mathcal{L}_\sigma(\boldsymbol{T}^0)^{wF}. \qquad \square$$

Corollary 7.10. *We have $|(\mathcal{C}(\boldsymbol{G}^1)_{\mathrm{qss}})^F| = |(\mathcal{C}(\boldsymbol{G}^{\sigma 0})_{ss})^F|$.*

Proof. Let us take $f = 1$ in Proposition 7.9. We need to sum over $w \in W^\sigma$ the value $|\boldsymbol{T}^{1^{wF}}/\mathcal{L}_\sigma(\boldsymbol{T}^0)^{wF}|$. First note that $|\boldsymbol{T}^{1^{wF}}/\mathcal{L}_\sigma(\boldsymbol{T}^0)^{wF}| = |\boldsymbol{T}^{0^{wF}}/\mathcal{L}_\sigma(\boldsymbol{T}^0)^{wF}|$. By Lemma 7.2 we have the exact sequence

$$1 \to \boldsymbol{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\boldsymbol{T}^0) \to \boldsymbol{T}^{\sigma 0} \times \mathcal{L}_\sigma(\boldsymbol{T}^0) \to \boldsymbol{T}^0 \to 1,$$

whence the Galois cohomology exact sequence is

$$1 \to (\boldsymbol{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\boldsymbol{T}^0))^{wF} \to \boldsymbol{T}^{\sigma 0^{wF}} \times (\mathcal{L}_\sigma(\boldsymbol{T}^0))^{wF} \\ \to \boldsymbol{T}^{0^{wF}} \to H^1(wF, (\boldsymbol{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\boldsymbol{T}^0))) \to 1.$$

Using that for any automorphism τ of a finite group G we have $|G^\tau| = |H^1(\tau, G)|$, we have

$$|(\boldsymbol{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\boldsymbol{T}^0))^{wF}| = |H^1(wF, (\boldsymbol{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\boldsymbol{T}^0)))|.$$

Together with the above exact sequence, this implies that $|\boldsymbol{T}^{0^{wF}}/\mathcal{L}_\sigma(\boldsymbol{T}^0)^{wF}| = |\boldsymbol{T}^{\sigma 0^{wF}}|$, whence

$$|(\mathcal{C}(\boldsymbol{G}^1)_{\text{qss}})^F| = |W^\sigma|^{-1}\sum_{w\in W^\sigma}|\boldsymbol{T}^{\sigma 0^{wF}}|.$$

The corollary follows by either applying the same formula for the connected group $\boldsymbol{G}^{\sigma 0}$, or referring to [Lehrer 1992, Proposition 2.1]. □

8. Shintani descent

We now look at Shintani descent in our context; we will show it commutes with Lusztig induction when $\boldsymbol{G}^1/\boldsymbol{G}^0$ is semisimple and the characteristic is good for $\boldsymbol{G}^{\sigma 0}$. We should mention previous work on this subject: Eftekhari [1996, II.3.4] has the same result for Lusztig induction from a torus; he does not need to assume p good but needs q to be large enough to apply results of Lusztig, identifying Deligne–Lusztig induction with induction of character sheaves; Digne [1999, 1.1] has the result in the same generality as here apart from the assumption that $\boldsymbol{G}^1$ contains an F-stable quasicentral element; however, a defect of his proof is the use without proof of the property given in Lemma 8.4 below.

As above, $\boldsymbol{G}^1$ denotes an F-stable connected component of $\boldsymbol{G}$ of the form $\boldsymbol{G}^0\cdot\sigma$, where σ induces a quasicentral automorphism of $\boldsymbol{G}^0$ commuting with F.

Applying Lang's theorem, one can write any element of $\boldsymbol{G}^1$ as $x\cdot{}^{\sigma F}x^{-1}\sigma$ for some $x\in\boldsymbol{G}^0$, or as $\sigma\cdot{}^F x^{-1}\cdot x$ for some $x\in\boldsymbol{G}^0$. Using that σ, as an automorphism, commutes with F, it is easy to check that the correspondence $x\cdot{}^{\sigma F}x^{-1}\sigma\mapsto\sigma\,{}^F x^{-1}\cdot x$ induces a bijection $n_{F/\sigma F}$ from the $(\boldsymbol{G}^0)^F$-conjugacy classes of $(\boldsymbol{G}^1)^F$ to the $\boldsymbol{G}^{0^{\sigma F}}$-conjugacy classes of $(\boldsymbol{G}^1)^{\sigma F}$ and that $|\boldsymbol{G}^{0^{\sigma F}}|\,|c|=|(\boldsymbol{G}^0)^F|\,|n_{F/\sigma F}(c)|$ for any $(\boldsymbol{G}^0)^F$-class c in $(\boldsymbol{G}^1)^F$. It follows that the operator $\operatorname{sh}_{F/\sigma F}$ from $(\boldsymbol{G}^0)^F$-class functions on $(\boldsymbol{G}^1)^F$ to $\boldsymbol{G}^{0^{\sigma F}}$-class functions on $(\boldsymbol{G}^1)^{\sigma F}$ defined by $\operatorname{sh}_{F/\sigma F}(\chi)(n_{F/\sigma F}x)=\chi(x)$ is an isometry.

The remainder of this section is devoted to the proof of the following:

Proposition 8.1. *Let $\boldsymbol{L}^1 = N_{\boldsymbol{G}^1}(\boldsymbol{L}^0\subset\boldsymbol{P}^0)$ be a Levi of $\boldsymbol{G}^1$ containing σ, where $\boldsymbol{L}^0$ is F-stable; we have $\boldsymbol{L}^1=\boldsymbol{L}^0\cdot\sigma$. Assume that σ is semisimple and that the characteristic is good for $\boldsymbol{G}^{\sigma 0}$. Then*

$$\operatorname{sh}_{F/\sigma F}\circ{}^*R^{\boldsymbol{G}^1}_{\boldsymbol{L}^1} = {}^*R^{\boldsymbol{G}^1}_{\boldsymbol{L}^1}\circ\operatorname{sh}_{F/\sigma F}\quad\textit{and}\quad \operatorname{sh}_{F/\sigma F}\circ R^{\boldsymbol{G}^1}_{\boldsymbol{L}^1} = R^{\boldsymbol{G}^1}_{\boldsymbol{L}^1}\circ\operatorname{sh}_{F/\sigma F}.$$

Proof. The second equality follows from the first by adjunction, using that the adjoint of $\operatorname{sh}_{F/\sigma F}$ is $\operatorname{sh}^{-1}_{F/\sigma F}$. Let us prove the first equality.

Let χ be a $(\boldsymbol{G}^0)^F$-class function on $\boldsymbol{G}^1$ and let $\sigma l u = u\sigma l$ be the Jordan decomposition of an element of $(\boldsymbol{L}^1)^{\sigma F}$ with u unipotent and σl semisimple. By the character formula Proposition 2.5(iii) and the definition of $Q^{\boldsymbol{G}^{t0}}_{\boldsymbol{L}^{t0}}$ for $t=\sigma l$, we have

$$\left({}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}\,\mathrm{sh}_{F/\sigma F}(\chi)\right)(\sigma l u) = |(\boldsymbol{G}^{\sigma l})^{0\sigma F}|^{-1} \sum_{v\in(\boldsymbol{G}^{\sigma l})^{0\sigma F}_{\mathrm{unip}}} \mathrm{sh}_{F/\sigma F}(\chi)(\sigma l v)\,\mathrm{Trace}\big((v,u^{-1})|H^*_c(Y_{\boldsymbol{U},\sigma F})\big),$$

where v (resp. u) acts by left- (resp. right-) translation on

$$Y_{\boldsymbol{U},\sigma F} = \{x\in(\boldsymbol{G}^{\sigma l})^0 \mid x^{-1}\cdot{}^{\sigma F}x\in\boldsymbol{U}\},$$

where $\boldsymbol{U}$ denotes the unipotent radical of $\boldsymbol{P}^0$; in the summation, v is in the identity component of $\boldsymbol{G}^{\sigma l}$ since, σ being semisimple, u is in $\boldsymbol{G}^0$ and hence in $(\boldsymbol{G}^{\sigma l})^0$ by [DM 1994, théorème 1.8(i)] since σl is semisimple.

Let us write $l={}^F\lambda^{-1}\cdot\lambda$ with $\lambda\in\boldsymbol{L}^0$, so that $\sigma l = n_{F/\sigma F}(l'\sigma)$, where $l'=\lambda\cdot{}^{\sigma F}\lambda^{-1}$.

Lemma 8.2. *For $v\in(\boldsymbol{G}^{\sigma l})^{0\sigma F}_{\mathrm{unip}}$, we have*

$$\sigma l v = n_{F/\sigma F}((\sigma l\cdot v')^{\sigma^F\lambda^{-1}}),$$

where $v' = n_{\sigma F/\sigma F} v\in(\boldsymbol{G}^{\sigma l})^{0\sigma F}$ is defined by writing $v={}^{\sigma F}\eta\cdot\eta^{-1}$, where $\eta\in(\boldsymbol{G}^{\sigma t})^0$ and setting $v'=\eta^{-1}\cdot{}^{\sigma F}\eta$.

Proof. We have

$$\sigma l v = \sigma l^{\sigma F}\eta\cdot\eta^{-1} = {}^{\sigma F}\eta\sigma l\eta^{-1} = \sigma^F(\eta\lambda^{-1})\lambda\eta^{-1};$$

thus $\sigma l v = n_{F/\sigma F}((\lambda\eta^{-1})\cdot{}^{\sigma F}(\eta\lambda^{-1})\sigma)$. And we have

$$(\lambda\eta^{-1})\cdot{}^{\sigma F}(\eta\lambda^{-1})\sigma = \lambda v'^{\sigma F}\lambda^{-1}\sigma = {}^F\lambda l v'\sigma^F\lambda^{-1} = (\sigma l v')^{\sigma^F\lambda^{-1}};$$

thus $\mathrm{sh}_{F/\sigma F}(\chi)(\sigma l v) = \chi((\sigma l v')^{\sigma^F\lambda^{-1}})$. □

Lemma 8.3. (i) *We have $(\sigma l)^{\sigma^F\lambda^{-1}} = l'\sigma$.*

(ii) *The conjugation $x\mapsto x^{\sigma^F\lambda^{-1}}$ maps $\boldsymbol{G}^{\sigma l}$ and the action of σF on it to $\boldsymbol{G}^{l'\sigma}$ with the action of F on it; in particular, it induces bijections*

$$(\boldsymbol{G}^{\sigma l})^{0\sigma F}\xrightarrow{\sim}(\boldsymbol{G}^{l'\sigma})^{0F} \quad\text{and}\quad Y_{\boldsymbol{U},\sigma F}\xrightarrow{\sim}Y_{\boldsymbol{U},F},$$

where $Y_{\boldsymbol{U},F}=\{x\in(\boldsymbol{G}^{l'\sigma})^0\mid x^{-1\,F}x\in\boldsymbol{U}\}$.

Proof. Part (i) is an obvious computation and shows that if $x\in\boldsymbol{G}^{\sigma l}$ then $x^{\sigma^F\lambda^{-1}}\in\boldsymbol{G}^{l'\sigma}$. To prove (ii), it remains to show that if $x\in\boldsymbol{G}^{\sigma l}$ then ${}^F(x^{\sigma^F\lambda^{-1}}) = ({}^{\sigma F}x)^{\sigma^F\lambda^{-1}}$. From $x^\sigma = x^{l^{-1}} = x^{\lambda^{-1}.{}^F\lambda}$, we get $x^{\sigma^F\lambda^{-1}} = x^{\lambda^{-1}}$, whence ${}^F(x^{\sigma^F\lambda^{-1}}) = ({}^Fx)^{{}^F\lambda^{-1}} = (({}^{\sigma F}x)^\sigma)^{{}^F\lambda^{-1}} = ({}^{\sigma F}x)^{\sigma^F\lambda^{-1}}$. □

Applying Lemmas 8.2 and 8.3 we get

$$({}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}\,\mathrm{sh}_{F/\sigma F}(\chi))(\sigma l u) = |(\boldsymbol{G}^{\sigma l})^{0\sigma F}|^{-1}\sum_{v\in(\boldsymbol{G}^{\sigma l})^{0\sigma F}_{\mathrm{unip}}}\chi((\sigma l v')^{\sigma^F\lambda^{-1}})\,\mathrm{Trace}\big(v^{\sigma^F\lambda^{-1}},(u^{\sigma^F\lambda^{-1}})^{-1})\mid H^*_c(Y_{\boldsymbol{U},F})\big).$$

Lemma 8.4. *Assume that the characteristic is good for $\boldsymbol{G}^{\sigma 0}$, where σ is a quasi-central element of $\boldsymbol{G}$. Then it is also good for $(\boldsymbol{G}^s)^0$, where s is any quasi-semisimple element of $\boldsymbol{G}^0 \cdot \sigma$.*

Proof. Let Σ_σ (resp. Σ_s) be the root system of $\boldsymbol{G}^{\sigma 0}$ (resp. $(\boldsymbol{G}^s)^0$). By definition, a characteristic p is good for a reductive group if for no closed subsystem of its root system the quotient of the generated lattices has p-torsion. The system Σ_s is not a closed subsystem of Σ_σ in general, but the relationship is expounded in [Digne and Michel 2002]: let Σ be the root system of $\boldsymbol{G}^0$ with respect to a σ-stable pair $\boldsymbol{T} \subset \boldsymbol{B}$ of a maximal torus and a Borel subgroup of $\boldsymbol{G}^0$. Up to conjugacy, we may assume that s also stabilizes that pair. Let $\overline{\Sigma}$ denote the set of sums of the σ-orbits in Σ, and Σ' the set of averages of the same orbits. Then Σ' is a nonnecessarily reduced root system, but Σ_σ and Σ_s are subsystems of Σ' and are reduced. The system $\overline{\Sigma}$ is reduced, and the set of sums of orbits whose average is in Σ_σ (resp. Σ_s) is a closed subsystem that we denote by $\overline{\Sigma}_\sigma$ (resp. $\overline{\Sigma}_s$).

We now need a generalization of [Bourbaki 1981, chapitre VI, §1.1, lemme 1]:

Lemma 8.5. *Let $\mathcal{L}$ be a finite set of lines generating a vector space V over a field of characteristic* 0; *then two reflections of V which stabilize $\mathcal{L}$ and have a common eigenvalue $\zeta \neq 1$ with ζ-eigenspace the same line of $\mathcal{L}$ are equal.*

Proof. Here we mean by reflection an element $s \in \mathrm{GL}(V)$ such that $\ker(s-1)$ is a hyperplane. Let s and s' be reflections as in the statement. The product $s^{-1}s'$ stabilizes $\mathcal{L}$, so it has a power which fixes $\mathcal{L}$, and thus is semisimple. On the other hand, $s^{-1}s'$ by assumption fixes one line $L \in \mathcal{L}$ and induces the identity on V/L, and thus is unipotent. Being semisimple and unipotent, it has to be the identity. □

It follows from Lemma 8.5 that two root systems with proportional roots have the same Weyl group, and thus the same good primes; therefore:

- Σ_s and $\overline{\Sigma}_s$ have the same good primes, as well as Σ_σ and $\overline{\Sigma}_\sigma$.
- The bad primes for $\overline{\Sigma}_s$ are a subset of those for $\overline{\Sigma}$, since it is a closed subsystem.

It only remains to show that the good primes for $\overline{\Sigma}$ are the same as for $\overline{\Sigma}_\sigma$, which can be checked case by case: we can reduce to the case where Σ is irreducible, in which case these systems coincide except when Σ is of type A_{2n}; but in this case, $\overline{\Sigma}$ is of type B_n and Σ_σ is of type B_n or C_n, which have the same set $\{2\}$ of bad primes. □

Since the characteristic is good for $\boldsymbol{G}^{\sigma 0}$, hence also for $(\boldsymbol{G}^{\sigma l})^0$ by Lemma 8.4, the elements v' and v are conjugate in $(\boldsymbol{G}^{\sigma l})^{0\sigma F}$ (see [Digne and Michel 1985, IV, corollaire 1.2]). By Lemma 8.3(ii), the element $v^{\sigma^F \lambda^{-1}}$ runs over the unipotent elements of $(\boldsymbol{G}^{l'\sigma})^{0^F}$ when v runs over $(\boldsymbol{G}^{\sigma l})^{0\sigma F}_{\mathrm{unip}}$. Moreover, using the equality

$|(\boldsymbol{G}^{\sigma l})^{0\sigma F}| = |(\boldsymbol{G}^{l'\sigma})^{0^F}|$, we get

$$(*)\qquad ({}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}\,\mathrm{sh}_{F/\sigma F}(\chi))(\sigma l u) = \frac{1}{|(\boldsymbol{G}^{l'\sigma})^{0^F}|}\sum_{u_1\in(\boldsymbol{G}^{l'\sigma})^{0^F}_{\mathrm{unip}}} \chi(u_1 l'\sigma)\,\mathrm{Trace}\big((u_1,(u^{\sigma^F\lambda^{-1}})^{-1})\,|\,H_c^*(Y_{\boldsymbol{U},F})\big).$$

On the other hand, by Lemma 8.2 applied with $v = u$, we have

$$\begin{aligned}\big(\mathrm{sh}_{F/\sigma F}\,{}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}(\chi)\big)(\sigma l u) &= {}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}(\chi)((\sigma l u)^{\sigma^F\lambda^{-1}})\\ &= {}^*R_{\boldsymbol{L}^1}^{\boldsymbol{G}^1}(\chi)(l'\sigma\cdot u^{\sigma^F\lambda^{-1}}),\end{aligned}$$

where the second equality holds by Lemma 8.3(i). By the character formula this is equal to the right-hand side of formula (*). □

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EXTENDING HECKE ENDOMORPHISM ALGEBRAS

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We dedicate this paper to the memory of Robert Steinberg.

The (Iwahori–)Hecke algebra in the title is a q-deformation $\mathcal{H}$ of the group algebra of a finite Weyl group W. The algebra $\mathcal{H}$ has a natural enlargement to an endomorphism algebra $\mathcal{A} = \mathrm{End}_{\mathcal{H}}(\mathcal{T})$ where $\mathcal{T}$ is a q-permutation module. In type A_n (i.e., $W \cong \mathfrak{S}_{n+1}$), the algebra $\mathcal{A}$ is a q-Schur algebra which is quasi-hereditary and plays an important role in the modular representation of the finite groups of Lie type. In other types, $\mathcal{A}$ is not always quasi-hereditary, but the authors conjectured 20 year ago that $\mathcal{T}$ can be enlarged to an $\mathcal{H}$-module $\mathcal{T}^+$ so that $\mathcal{A}^+ = \mathrm{End}_{\mathcal{H}}(\mathcal{T}^+)$ is at least standardly stratified, a weaker condition than being quasi-hereditary, but with "strata" corresponding to Kazhdan–Lusztig two-sided cells.

The main result of this paper is a "local" version of this conjecture in the equal parameter case, viewing $\mathcal{H}$ as defined over $\mathbb{Z}[t, t^{-1}]$, with the localization at a prime ideal generated by a cyclotomic polynomial $\Phi_{2e}(t)$, $e \neq 2$. The proof uses the theory of rational Cherednik algebras (also known as RDAHAs) over similar localizations of $\mathbb{C}[t, t^{-1}]$. In future papers, the authors hope to prove global versions of the conjecture, maintaining these localizations.

1. Introduction

Let $\mathscr{G} = \{G(q)\}$ be a family of finite groups of Lie type having irreducible (finite) Coxeter system (W, S) [Curtis and Reiner 1987, (68.22)]. The pair (W, S) remains fixed throughout this paper. Let $B(q)$ be a Borel subgroup of $G(q)$. There are index parameters $c_s \in \mathbb{Z}$, $s \in S$, defined by

$$[B(q) : {}^{s}B(q) \cap B(q)] = q^{c_s}, \quad s \in S.$$

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The generic Hecke algebra $\mathcal{H}$ over the ring $\mathcal{Z}=\mathbb{Z}[t,t^{-1}]$ of Laurent polynomials associated to $\mathscr{G}$ has basis T_w, $w\in W$, subject to relations

$$T_sT_w=\begin{cases}T_{sw}, & sw>w,\\ t^{2c_s}T_{sw}+(t^{2c_s}-1)T_w, & sw<w,\end{cases}\tag{1.1}$$

for $s\in S$, $w\in W$. This algebra is defined just using t^2, but it is convenient to have its square root t available. We call $\mathcal{H}$ a Hecke algebra of Lie type over $\mathcal{Z}$. It is related to the representation theory of the groups in $\mathscr{G}$ as follows: for any prime power q, let R be any field (we will shortly allow R to be a ring) in which the integer q is invertible and has a square root $\sqrt{q}$. Let $\mathcal{H}_R=\mathcal{H}\otimes_{\mathcal{Z}}R$ be the algebra obtained by base change through the map $\mathcal{Z}\to R$, $t\mapsto\sqrt{q}$. Then $\mathcal{H}_R\cong\mathrm{End}_{G(q)}\big(\mathrm{ind}_{B(q)}^{G(q)}R\big)$. Thus, the generic Hecke algebra $\mathcal{H}$ is the quantumization (in the sense of [Deng et al. 2008, §0.4]) of an infinite family of important endomorphism algebras.

In type A_n, i.e., when $\mathscr{G}=\{\mathrm{GL}_{n+1}(q)\}$, one can also consider the q-Schur algebras, viz., algebras Morita equivalent to

$$S_R:=\mathrm{End}_{\mathcal{H}_R}\Big(\bigoplus_{J\subseteq S}\mathrm{ind}_{\mathcal{H}_{J,R}}^{\mathcal{H}_R}\mathrm{IND}_J\Big).\tag{1.2}$$

In this case, S_R is a quasi-hereditary algebra whose representation theory is closely related to that of the quantum general linear groups. The q-Schur algebras have historically played an important role in representation theory of the finite general linear groups, thanks to the work of Dipper, James, and others. More generally, although the definition (1.2) makes sense in all types, less is known about its properties or the precise role it plays in the representation theory or homological algebra of the corresponding groups in $\mathscr{G}$. The purpose of this paper, and its sequels, is to enhance S_R in a way described below, so that it does become relevant to these questions.

1A. ***Stratifying systems.*** At this point, it will be useful to review the notion of a *strict stratifying system* for an algebra. These systems provide a framework for studying algebras similar to quasi-hereditary algebras. They appear in the statement of the first main theorem. Although the algebras in Theorem 5.6 below are shown later to be quasi-hereditary, the theory of stratifying systems is useful both in providing a framework and as a tool in obtaining the final results.

First, recall that a preorder on a set X is a transitive and reflexive relation $\leq$. The associated equivalence relation $\sim$ on X is defined by setting, for $x,y\in X$,

$$x\sim y\iff x\leq y\ \text{and}\ y\leq x.$$

A preorder induces an evident partial order, still denoted $\leq$, on the set $\overline{X}$ of equivalence classes of $\sim$. In this paper, a set X with a preorder is called a quasi-poset. Also, if $x\in X$, let $\bar{x}\in\overline{X}$ be its associated equivalence class.

Now let R be a Noetherian commutative ring, and let A be an R-algebra, finitely generated and projective as an R-module. Let Λ be a finite quasi-poset. For each $\lambda \in \Lambda$, it is required that there is given a finitely generated A-module $\Delta(\lambda)$ and a finitely generated projective A-module $P(\lambda)$ together with a fixed surjective morphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ of A-modules. The following conditions are required:

(1) For $\lambda, \mu \in \Lambda$,

$$\mathrm{Hom}_A(P(\lambda), \Delta(\mu)) \neq 0 \quad \Longrightarrow \quad \lambda \leq \mu.$$

(2) Every irreducible A-module L is a homomorphic image of some $\Delta(\lambda)$.

(3) For $\lambda \in \Lambda$, the A-module $P(\lambda)$ has a finite filtration by A-submodules with top section $\Delta(\lambda)$ and other sections of the form $\Delta(\mu)$ with $\bar{\mu} > \bar{\lambda}$.

When these conditions all hold, the data $\{\Delta(\lambda)\}_{\lambda\in\Lambda}$ is a *strict stratifying system* for A-mod. It is also clear that $\Delta(\lambda)_{R'}, P(\lambda)_{R'}, \ldots$ is a strict stratifying system for $A_{R'}$-mod *for any base change* $R \to R'$, provided R' is a Noetherian commutative ring. (Notice that condition (2) is redundant if it is known that the direct sum of the projective modules in (3) is a progenerator — a property preserved by base change.)

An ideal J in an R-algebra A as above is called a *stratifying ideal* provided that J is an R-direct summand of A (or equivalently, the inclusion $J \hookrightarrow A$ is R-split), and for A/J-modules M, N inflation defines an isomorphism

$$\mathrm{Ext}^n_{A/J}(M, N) \xrightarrow{\sim} \mathrm{Ext}^n_A(M, N), \quad \text{for all } n \geq 0. \tag{1A.1}$$

of Ext-groups. A *standard stratification* of length n of A is a sequence $0 = J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_n = A$ of stratifying ideals of A such that each J_i/J_{i-1} is a projective A/J_{i-1}-module. If A-mod has a strict stratifying system with quasi-poset Λ, then it has a standard stratification of length $n = |\bar{\Lambda}|$; see [Du et al. 1998, Theorem 1.2.8].

In the case of a finite dimensional algebra A over a field k, the notion of a strict stratifying system $\{\Delta(\lambda)\}_{\lambda\in\Lambda}$ for A-mod simplifies somewhat. In this case, it can be assumed that each $\Delta(\lambda)$ has an irreducible head $L(\lambda)$, that $\lambda \neq \mu \Longrightarrow L(\lambda) \ncong L(\mu)$, and that $P(\lambda)$ is indecomposable. Two caveats are in order, however: (i) it may be necessary to enlarge the base set Λ to be able to index all the irreducible modules, though $\bar{\Lambda}$ can remain the same; (ii) it may be easier to verify (1), (2), and (3) over a larger ring and then base change. The irreducible head versions of the $\Delta(\lambda)$ can then be obtained as direct summands of the base-changed versions.

When the algebra A arises as an endomorphism algebra $A = \mathrm{End}_B(T)$, there is a useful theory for obtaining a strict stratifying system for A-mod. In fact, this is how such stratifying systems initially arose; see [Cline et al. 1996; Du et al. 1998]. This approach is followed in the proof of the main theorem in this paper. For convenience, we summarize the sufficient conditions that will be used, all taken from [Du et al. 1998, Theorem 1.2.10].

Theorem 1.1. *Let B be a finitely generated projective R-algebra and let T be a finitely generated right B-module which is projective over R. Define $A := \mathrm{End}_B(T)$. Assume that $T = \oplus_{\lambda\in\Lambda} T_\lambda$, where Λ is a finite quasi-poset. For $\lambda \in \Lambda$, assume there is given a fixed R-submodule $S_\lambda \subseteq T_\lambda$ and an increasing filtration $F_\lambda^\bullet : 0 = F_\lambda^0 \subseteq F_\lambda^1 \subseteq \cdots \subseteq F_\lambda^{t(\lambda)} = T_\lambda$ satisfying the following conditions*:

(1) *For $\lambda \in \Lambda$, $F_\lambda^\bullet$ has bottom section $F_\lambda^1/F_\lambda^0 \cong S_\lambda$ and higher sections $F_\lambda^{i+1}/F_\lambda^i$ $(1 \le i \le t(\lambda)-1)$ of the form S_ν with $\bar\nu > \bar\lambda$.*

(2) *For $\lambda, \mu \in \Lambda$, $\mathrm{Hom}_B(S_\mu, T_\lambda) \neq 0 \implies \lambda \le \mu$.*

(3) *For $\lambda \in \Lambda$, $\mathrm{Ext}^1_B(T_\lambda/F_\lambda^i, T) = 0 \quad$ for all i.*

Let $A = \mathrm{End}_B(T)$ and, for $\lambda \in \Lambda$, define $\Delta(\lambda) := \mathrm{Hom}_B(S_\lambda, T) \in A$-mod. Assume that each $\Delta(\lambda)$ is R-projective. Then $\{\Delta(\lambda)\}_{\lambda\in\Lambda}$ is a strict stratifying system for A-mod.

It is interesting to note that these sufficient conditions are not, in general, preserved under base change, though the resulting strict stratifying systems are preserved (becoming strict stratifying systems for the base-changed version of the algebra A).

1B. ***Cells and q-permutation modules.*** We assume familiarity with Kazhdan–Lusztig cell theory for the Coxeter systems (W, S). See, for instance, [Deng et al. 2008; Lusztig 2003]. In Conjecture 1.2 below and in Theorem 5.6, the set Λ will be the set Ω of left Kazhdan–Lusztig cells for (W, S). For each $\omega \in \Omega$, let

$$S(\omega) := \mathcal{H}^{\le_L \omega}/\mathcal{H}^{<_L \omega} \in \mathcal{H}\text{-mod} \tag{1B.1}$$

be the corresponding left cell module. It is known that $S(\omega)$ is a free $\mathcal{Z}$-module with basis corresponding to certain Kazhdan–Lusztig basis elements C'_x, $x \in \omega$; see Section 2. The corresponding dual left cell module is defined

$$S_\omega := \mathrm{Hom}_{\mathcal{Z}}(S(\omega), \mathcal{Z}) \in \text{mod-}\mathcal{H}. \tag{1B.2}$$

It is regarded as a *right* $\mathcal{H}$-module. Because $S(\omega)$ and hence S_ω are free over $\mathcal{Z}$, if R is a commutative $\mathcal{Z}$-module, we can define

$$\begin{cases} S_R(\omega) := S(\omega) \otimes_{\mathcal{Z}} R, \\ S_{\omega,R} := S_\omega \otimes_{\mathcal{Z}} R = \mathrm{Hom}_R(S_R(\omega), R). \end{cases}$$

For the special choice $R = \mathscr{Q}$ — see (1C.1) below for the definition of $\mathscr{Q}$ — we also use the notations

$$\begin{cases} \tilde S(\omega) := S_{\mathscr{Q}}(\omega), \\ \tilde S_\omega := S_{\omega,\mathscr{Q}}, \qquad \omega \in \Omega. \end{cases} \tag{1B.3}$$

In addition, for $\lambda \subseteq S$, let W_λ be the parabolic subgroup of W generated by the $s \in \lambda$, and put $x_\lambda = \sum_{w\in W_\lambda} T_w$, with T_w as in (1.1) above. The induced modules

$x_\lambda\mathcal{H}$ (also called q-permutation modules) have an increasing filtration with sections S_ω for various $\omega \in \Omega$ (precisely, those left cells ω whose right set $\mathscr{R}(\omega)$ contains λ).

Let $\mathcal{T} = \bigoplus_\lambda x_\lambda\mathcal{H}$, and $\mathcal{A} := \mathrm{End}_{\mathcal{H}}(\mathcal{T})$. For $\omega \in \Omega$, put $\Delta(\omega) := \mathrm{Hom}_{\mathcal{H}}(S_\omega, \mathcal{T}) \in \mathcal{A}$-mod. The algebra $\mathcal{A}$ is very well behaved in type A, a q-Schur algebra; a theme of [Du et al. 1998] was that suitable enlargements, appropriately compatible with two-sided cell theory, should have similar good properties for all types.

Each two-sided cell may be identified with the set of left cells it contains, and the resulting collection $\overline{\Omega}$ of sets of left cells is a partition of Ω. There are various natural preorders on Ω, but we will be mainly interested in those whose associated equivalence relation has precisely the set $\overline{\Omega}$ as its associated partition. We call such a preorder *strictly compatible* with $\overline{\Omega}$.

1C. ***A conjecture.*** Now we are ready to state the following conjecture, which is a variation (see the Appendix) on [op. cit., Conjecture 2.5.2]. We informally think of the algebra $\mathcal{A}^+$ in the conjecture as an extension of $\mathcal{A}$ as a Hecke endomorphism algebra (justifying the title of the paper).

Conjecture 1.2. *There exists a preorder $\leq$ on the set Ω of left cells in W, strictly compatible with its partition $\overline{\Omega}$ into two-sided cells, and a right $\mathcal{H}$-module $\mathcal{X}$ such that the following statements hold*:

(1) *$\mathcal{X}$ has an finite filtration with sections of the form S_ω, $\omega \in \Omega$.*

(2) *Let $\mathcal{T}^+ := \mathcal{T} \oplus \mathcal{X}$ and put*

$$\begin{cases} \mathcal{A}^+ := \mathrm{End}_{\mathcal{H}}(\mathcal{T}^+), \\ \Delta^+(\omega) := \mathrm{Hom}_{\mathcal{H}}(S_\omega, \mathcal{T}^+), \quad \textit{for any } \omega \in \Omega. \end{cases}$$

Then, for any commutative, Noetherian $\mathcal{Z}$-algebra R, the set $\{\Delta^+(\omega)_R\}_{\omega\in\Omega}$ is a strict stratifying system for $\mathcal{A}^+_R$-mod relative to the quasi-poset $(\Omega, \leq)$.

The main result of this paper, given in Theorem 5.6, establishes a special "local case" of this conjecture. A more detailed description of this theorem requires some preliminary notation. Throughout this paper, e is positive integer ($\neq 2$ in our main results). Let $\Phi_{2e}(t)$ denote the (cyclotomic) minimum polynomial for a primitive $2e$th root of unity $\sqrt{\zeta} = \exp(2\pi i/2e) \in \mathbb{C}$. Fix a modular system $(K, \mathscr{Q}, k)$ by letting

$$\begin{cases} \mathscr{Q} := \mathbb{Q}[t, t^{-1}]_{\mathfrak{p}}, & \text{where } \mathfrak{p} = (\Phi_{2e}(t)); \\ K := \mathbb{Q}(t), & \text{the fraction field of } \mathscr{Q}; \\ k := \mathscr{Q}/\mathfrak{m} \cong \mathbb{Q}(\sqrt{\zeta}), & \text{the residue field of } \mathscr{Q}. \end{cases} \tag{1C.1}$$

Here $\mathfrak{m}$ denotes the maximal ideal of the DVR $\mathscr{Q}$. With some abuse of notation, we sometimes identify $\sqrt{\zeta}$ with its image in k. (Without passing to an extension or completion, the ring $\mathscr{Q}$ might not have such a root of unity in it.) The algebra $\mathcal{H}_{\mathbb{Q}(t)}$ is split semisimple, with irreducible modules corresponding to the irreducible modules of the group algebra $\mathbb{Q}W$. The $\mathscr{Q}$-algebra

$$\widetilde{\mathcal{H}} := \mathcal{H} \otimes_{\mathcal{Z}} \mathscr{Q} \tag{1C.2}$$

has a presentation by elements $T_w \otimes 1$ (which will still be denoted T_w, $w \in W$) completely analogous to (1.1). Similar remarks apply to $\mathcal{H}_k$, replacing t^2 by ζ. Then Theorem 5.6 establishes that there exists an $\widetilde{\mathcal{H}}$-module $\widetilde{\mathcal{X}}$ which is filtered by dual left cell modules $\tilde{S}_\omega$ such that the analogues of conditions (1) and (2) over $\mathscr{Q}$ in Conjecture 1.2 hold. The preorder used in Theorem 5.6 is constructed as in [Ginzburg et al. 2003] from a "sorting function" f, and is discussed in detail in the next section.

With more work, it can be shown, when $e \neq 2$, that the $\mathscr{Q}$-algebra $\tilde{\mathcal{A}}^+ := \mathcal{A}^+_{\mathscr{Q}}$ is quasi-hereditary. This is done in Theorem 6.4. Then Theorem 6.5 identifies the module category for a base-changed version of this algebra with a RDAHA-category $\mathcal{O}$ in [Ginzburg et al. 2003]. Such an identification in type A was conjectured in [loc. cit.], and then proved by Rouquier in [2008] (when $e \neq 2$).

Generally speaking, this paper focuses on the "equal parameter" case, i.e., all $c_s = 1$ in (1.1), which covers the Hecke algebras relevant to all untwisted finite Chevalley groups. We will assume this condition unless explicitly stated otherwise, avoiding a number of complications involving Kazhdan–Lusztig basis elements and Lusztig's algebra $\mathcal{J}$. In this context, the critical Proposition 3.1 depends on results of [Ginzburg et al. 2003] which, in part, were only determined in the equal parameter case. Nevertheless, much of our discussion applies in the unequal parameter cases. In particular, we mention that the elementary, but important, Lemma 4.3 is stated and proved using unequal parameter notation. This encourages the authors to believe the main results are also provable in the unequal parameter case, though this has not yet been carried out. Note that all the rank 2 cases are treated in [Du et al. 1998], leaving the quasisplit cases with rank > 2. All these quasisplit cases have parameters confined to the set $\{1, 2, 3\}$.

2. Some preliminaries

This section recalls some mostly well-known facts and fixes notation regarding cell theory. Let W be a finite Weyl group associated to a finite root system Φ with a fixed set of simple roots Π. Let $S := \{s_\alpha \mid \alpha \in \Pi\}$. Let $\mathcal{H}$ be a Hecke algebra over $\mathcal{Z}$ defined by (1.1). We assume (unless explicitly noted otherwise) that each $c_s = 1$ for $s \in S$. Thus, (W, S) corresponds, in the language of the introduction, to some types of split Chevalley groups, though we will have no further need of that context.

Let

$$C'_w = t^{-l(w)} \sum_{y \leq w} P_{y,w} T_y,$$

where the $P_{y,w}$ is a Kazhdan–Lusztig polynomial in $\mathfrak{q} := t^2$. Then $\{C'_w\}_{w\in W}$ is a Kazhdan–Lusztig (or canonical) basis for $\mathcal{H}$. The element C'_x is denoted c_x in [Lusztig 2003], a reference we frequently quote. Let $h_{x,y,z} \in \mathcal{Z}$ denote the structure constants. In other words,

$$C'_x C'_y = \sum_{z\in W} h_{x,y,z} C'_z.$$

Using the preorders $\leq_L$ and $\leq_R$ on W, the positivity (see [Deng et al. 2008, §7.8]) of the coefficients of the $h_{x,y,z}$ implies

$$h_{x,y,z} \neq 0 \quad \Longrightarrow \quad z \leq_L y \text{ and } z \leq_R x. \tag{2.3}$$

The Lusztig function $a : W \to \mathbb{N}$ is defined as follows. For $z \in W$, let $a(z)$ be the smallest nonnegative integer such that $t^{a(z)} h_{x,y,z} \in \mathbb{N}[t]$ for all $x, y \in W$. It may equally be defined as the smallest nonnegative integer such that $t^{-a(x)} h_{x,y,z} \in \mathbb{N}[t^{-1}]$, as used in [Lusztig 2003] (or see [Deng et al. 2008, §7.8]). In fact, each $h_{x,y,z}$ is invariant under the automorphism $\mathcal{Z} \to \mathcal{Z}$ sending t to t^{-1}. It is not difficult to see that $a(z) = a(z^{-1})$. For $x, y, z \in W$, let $\gamma_{x,y,z}$ be the coefficient of $t^{-a(z)}$ in $h_{x,y,z^{-1}}$. Also, by [Lusztig 2003, Conjectures 14.2(P8) and 15.6],

$$\gamma_{x,y,z} \neq 0 \quad \Longrightarrow \quad x \sim_L y^{-1},\ y \sim_L z^{-1},\ z \sim_L x^{-1}. \tag{2.4}$$

The function a is constant on two-sided cells in W, and so can be regarded as a function with values in $\mathbb{N}$ on (a) the set of two-sided cells; (b) the set of left (or right) cells; and (c) the set $\mathrm{Irr}(\mathbb{Q}W)$ of irreducible $\mathbb{Q}W$-modules.[1] In addition, a is related to the generic degrees d_E, $E \in \mathrm{Irr}(\mathbb{Q}W)$. For $E \in \mathrm{Irr}(\mathbb{Q}W)$, let $d_E = bt^{a_E} + \cdots + ct^{A_E}$, with $a_E \leq A_E$ and $bc \neq 0$, so that t^{a_E} and t^{A_E} are the smallest and largest powers of t appearing nontrivially in d_E, respectively. Then $a_E = a(E)$; see [Lusztig 2003, Proposition 20.6]. Also, as noted in [Ginzburg et al. 2003, §6], $A_E = N - a(E \otimes \det)$, where N is the number of positive roots in Φ. Following [Ginzburg et al. 2003, §6], we will use the "sorting function" $f : \mathrm{Irr}(\mathbb{Q}W) \to \mathbb{N}$ defined by

$$f(E) = a_E + A_E = a(E) + N - a(E \otimes \det). \tag{2.5}$$

The function f is also constant on two-sided cells: if E is an irreducible $\mathbb{Q}W$-module associated to a two-sided cell $\boldsymbol{c}$, then $E \otimes \det$ is an irreducible module associated to the two-sided cell $w_0\boldsymbol{c}$. See [Lusztig 1984, Lemma 5.14(iii)].

The function f is used in [Ginzburg et al. 2003] to define various order structures on the set $\mathrm{Irr}(\mathbb{Q}W)$ of irreducible $\mathbb{Q}W$-modules. Put $E <_f E'$ (our notation) provided

[1] It is well known that $\mathbb{Q}$ is a splitting field for W [Benard 1971].

$f(E) < f(E')$. There are at least two natural ways to extend $<_f$ to a preorder. The first way, which is only in the background for us, is to set $E \preceq_f E' \iff E \cong E'$ or $E <_f E'$. This gives a poset structure, and is used, in effect, by [Ginzburg et al. 2003] for defining a highest weight category $\mathcal{O}$; see [op. cit., §2.5, §6.2.1].

We use $<_f$ here to define a preorder $\leq_f$ on the set Ω of left cells: First, observe that the function f above is constant on irreducible modules associated to the same left cell (or even the same two-sided cell) and so may be viewed as a function on Ω. We can now define the (somewhat subtle) preorder $\leq_f$ on Ω by setting $\omega \leq_f \omega'$ (for $\omega, \omega' \in \Omega$) if and only if either $f(\omega) < f(\omega')$, or ω and ω' *lie in the same two-sided cell.* Note that the "equivalence classes" of the preorder $\leq_f$ identify with the set of two-sided cells — thus, $\leq_f$ is strictly compatible with the set of two-sided cells in the sense of Section 1. Also,

$$E <_{LR} E' \quad \Longrightarrow \quad E' <_f E; \tag{2.6}$$

see [op. cit., Lemma 6.6]. Here E, E' are in $\mathrm{Irr}(\mathbb{Q}W)$, and the notation $E <_{LR} E'$ means that the two-sided cell associated with E is strictly smaller than that associated with E', with respect to the Kazhdan–Lusztig order on two-sided cells. A property similar to (2.6) holds if $<_{LR}$ is replaced with $<_L$, defined similarly, but using left cells. In terms of Ω, this left cell version reads:

$$\omega, \omega' \in \Omega,\ \omega <_L \omega' \quad \Longrightarrow \quad f(\omega) > f(\omega'). \tag{2.7}$$

Notice that (2.7) follows from (2.6) using [Lusztig 1987a, Corollary 1.9(c)]. (The latter result implies that ω, ω' on the left in (2.7) cannot belong to the same two-sided cell.) Thus, the preorder $\leq_f$ is a refinement of the preorder $\leq_L^{\mathrm{op}}$ on Ω, and $\leq_f$ induces on the set of two-sided cells a refinement of the partial order $\leq_{LR}^{\mathrm{op}}$. For further discussion, see the Appendix.

3. (Dual) Specht modules of Ginzburg–Guay–Opdam–Rouquier

The asymptotic form $\mathcal{J}$ of $\mathcal{H}$ is a ring with $\mathbb{Z}$-basis $\{j_x \mid x \in W\}$ and multiplication

$$j_x j_y = \sum_z \gamma_{x,y,z^{-1}} j_z.$$

This ring was originally introduced in [Lusztig 1987a], though we follow [Lusztig 2003, §18.3], using a slightly different notation.

3A. ***The mapping ϖ and its properties.*** As per [op. cit., §18.9], define a $\mathcal{Z}$-algebra homomorphism

$$\varpi : \mathcal{H} \to \mathcal{J}_{\mathcal{Z}} = \mathcal{J} \otimes \mathcal{Z}, \quad C'_w \mapsto \sum_{z \in W} \sum_{\substack{d \in \boldsymbol{D} \\ a(d)=a(z)}} h_{w,d,z} j_z, \tag{3A.1}$$

where $\boldsymbol{D}$ is the set of distinguished involutions in W. Also, for any $\mathcal{Z}$-algebra R, there is an algebra homomorphism $\varpi_R : \mathcal{H}_R = \mathcal{H} \otimes_{\mathcal{Z}} R \to \mathcal{J}_R = \mathcal{J}_{\mathcal{Z}} \otimes_{\mathcal{Z}} R$, obtained by base change. In obvious cases, we often drop the subscript R from ϖ_R.

In particular, $\varpi_{\mathbb{Q}(t)}$ becomes an isomorphism

$$\varpi = \varpi_{\mathbb{Q}(t)} : \mathcal{H}_{\mathbb{Q}(t)} \xrightarrow{\sim} \mathcal{J}_{\mathbb{Q}(t)}. \tag{3A.2}$$

See [Lusztig 1987a]. Also, ϖ induces a monomorphism

$$\varpi = \varpi_{\mathbb{Q}[t,t^{-1}]} : \mathcal{H}_{\mathbb{Q}[t,t^{-1}]} \hookrightarrow \mathcal{J}_{\mathbb{Q}[t,t^{-1}]} = \mathcal{J}_{\mathbb{Q}} \otimes \mathbb{Q}[t, t^{-1}]. \tag{3A.3}$$

Moreover, base change to $\mathbb{Q}[t, t^{-1}]/(t-1)$ induces an isomorphism

$$\varpi = \varpi_{\mathbb{Q}} : \mathbb{Q}W \xrightarrow{\sim} \mathcal{J}_{\mathbb{Q}} \tag{3A.4}$$

(compare [Lusztig 1987b, Proposition 1.7]). This allows us to identify irreducible $\mathbb{Q}W$-modules with irreducible $\mathcal{J}_{\mathbb{Q}}$-modules.[2]

For the irreducible (left) $\mathcal{J}_{\mathbb{Q}}$-module identified with $E \in \mathrm{Irr}(\mathbb{Q}W)$, the (left) $\mathcal{H}_{\mathbb{Q}[t,t^{-1}]}$-module

$$S(E) := \varpi^*(E \otimes \mathbb{Q}[t, t^{-1}]) = \varpi^*(E_{\mathbb{Q}[t,t^{-1}]})$$

is called here a *dual Specht module* for $\mathcal{H}_{\mathbb{Q}[t,t^{-1}]}$; compare [Ginzburg et al. 2003, Corollary 6.10].[3] Note that $S(E) \cong E_{\mathbb{Q}[t,t^{-1}]}$ as a $\mathbb{Q}[t, t^{-1}]$-module. Therefore, $S(E)$ is a free $\mathbb{Q}[t, t^{-1}]$-module. Putting $S_E = \mathrm{Hom}_{\mathbb{Q}[t,t^{-1}]}(S(E), \mathbb{Q}[t, t^{-1}])$, define

$$\begin{cases} \tilde{S}(E) := S_{\mathcal{Q}}(E), \\ \tilde{S}_E := S_{E,\mathcal{Q}}, \end{cases} \tag{3A.5}$$

where, in general, for base change to a commutative, Noetherian $\mathbb{Q}[t, t^{-1}]$-algebra R,

$$\begin{cases} S_R(E) := S(E) \otimes_{\mathbb{Q}[t,t^{-1}]} R, \\ S_{E,R} := S_E \otimes_{\mathbb{Q}[t,t^{-1}]} R \cong \mathrm{Hom}_R(S_R(E), R). \end{cases}$$

[2]The map ϖ is the composition $\phi \circ \dagger$, where ϕ and $\dagger$ are defined in [Lusztig 2003, §18.9] and [op. cit., §3.5], respectively. The numbers $\hat{n}_z$ appearing there (which are ± 1 by definition in [op. cit., §18.8]) are all equal to 1, because of the positivity (see [op. cit., §7.8]) of the structure constants appearing in [op. cit., 14.1]. This ϖ is not the same one as defined in [Ginzburg et al. 2003, p. 647], where the C-basis was used. Nevertheless, the arguments of [op. cit., §6] go through, using the C'-basis and our ϖ (see Remark 5.2 below), so [op. cit., Theorem 6.8] guarantees the modules $S_{\mathbb{C}}(E)$ defined below using our setup are the same, at least up to a (two-sided cell preserving) permutation of the isomorphism types labeled by the E, as the modules $S(E)$ defined in [op. cit., Definition 6.1] with $R = \mathbb{C}$. The proof of [op. cit., Theorem 6.8] also establishes such an identification of the various modules $S_R(E)$ when R is a completion of $\mathbb{C}[t, t^{-1}]$.

[3]In [Ginzburg et al. 2003, Definition 6.1], the module $S(E)$ there is called a standard module. Our choice of terminology is justified by the discussion following the proof of Lemma 5.1 below.

The following proposition is proved using RDAHAs, and it is the only ingredient in the proof of Theorem 5.6 where these algebras are used.

Proposition 3.1. *Assume that $e \neq 2$. Suppose E, E' are irreducible $\mathbb{Q}W$-modules. If $E \ncong E'$ and*

$$\mathrm{Hom}_{\mathcal{H}_k}(S_k(E), S_k(E')) \neq 0,$$

then $f(E) < f(E')$. Also, $\mathrm{Hom}_{\mathcal{H}_k}(S_k(E), S_k(E)) \cong k$.

Proof. Without loss, we replace k in the statement of the proposition by $\mathbb{C}$, using the analogous definitions of $S_{\mathbb{C}}(E)$. In addition, the statement of the proposition is invariant under any two-sided cell preserving permutation of the labeling of the irreducible modules. After applying such a permutation on the right (say) we may assume, by [Ginzburg et al. 2003, Theorem 6.8] and taking into account note 1 on page 235, that

$$\mathrm{KZ}(\Delta(E)) \cong S_{\mathbb{C}}(E),$$

where

(1) $\Delta(E)$ is the standard module for a highest weight category $\mathcal{O}$ given in [op. cit.], having partial order $\leq_f$ (see [op. cit., Lemma 2.9, §6.2.1]) on its set of isomorphism classes of irreducible modules, which are indexed by isomorphism classes of irreducible $\mathbb{Q}W$-modules. We take $k_{H,1} = 1/e > 0$ in [op. cit.] above Theorem 6.8 and in Remark 3.2 there.

(2) The functor $\mathrm{KZ}: \mathcal{O} \to \bar{\mathcal{O}}$ is naturally isomorphic to the quotient map $M \mapsto \overline{M}$ in [op. cit., Proposition 5.9 and Theorem 5.14], the quotient category there identifying with $\mathcal{H}_{\mathbb{C}}$-mod.

Using [Ginzburg et al. 2003, Proposition 5.9], which requires $e \neq 2$, we have, for any irreducible $\mathbb{C}W$-modules E and E',

$$\mathrm{Hom}_{\mathcal{O}}(\Delta(E), \Delta(E')) \cong \mathrm{Hom}_{\bar{\mathcal{O}}}(\bar{\Delta}(E), \bar{\Delta}(E')) \cong \mathrm{Hom}_{\mathcal{H}_{\mathbb{C}}}(S_{\mathbb{C}}(E), S_{\mathbb{C}}(E')).$$

If $E \ncong E'$, then $\Delta(E) \ncong \Delta(E')$ and $\mathrm{Hom}_{\mathcal{O}}(\Delta(E), \Delta(E')) \neq 0$ implies that $E <_f E'$, i.e., $f(E) < f(E')$.

On the other hand, if $E \cong E'$, then $\mathrm{Hom}_{\mathcal{O}}(\Delta(E), \Delta(E')) \cong \mathbb{C}$. This implies

$$\mathrm{Hom}_{\mathcal{H}_{\mathbb{C}}}(S_{\mathbb{C}}(E), S_{\mathbb{C}}(E')) \cong \mathbb{C}.$$

Returning to the original $k = \mathbb{Q}(\sqrt{\zeta})$, we may conclude the same isomorphism holds in the original setting as well. □

Corollary 3.2. *Assume $e \neq 2$. Let E, E' be irreducible $\mathbb{Q}W$-modules. Then*

$$\mathrm{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E')) \neq 0 \quad \Longrightarrow \quad f(E) < f(E').$$

In particular, $\mathrm{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E)) = 0$.

Proof. In (1C.1) let $\pi = \Phi_{2e}(t)$ be the generator of the maximal ideal $\mathfrak{m}$ of $\mathscr{Q}$, and consider the short exact sequence

$$0 \longrightarrow \tilde{S}(E') \xrightarrow{\pi} \tilde{S}(E') \longrightarrow S_k(E') \longrightarrow 0.$$

By the long exact sequence of Ext, there is an exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E')) \xrightarrow{\pi} \operatorname{Hom}_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E')) &\longrightarrow \operatorname{Hom}_{\tilde{\mathcal{H}}_k}(S_k(E), S_k(E')) \\ &\longrightarrow \operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E')) \xrightarrow{\pi} \operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E')) \\ &\longrightarrow \operatorname{Ext}^1_{\tilde{\mathcal{H}}_k}(S_k(E), S_k(E')). \end{aligned}$$

Because $\mathcal{H}_{\mathbb{Q}(t)} = \tilde{\mathcal{H}}_{\mathbb{Q}(t)}$ is semisimple,

$$\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E'))_{\mathbb{Q}(t)} \cong \operatorname{Ext}^1_{\mathcal{H}_{\mathbb{Q}(t)}}(S(E)_{\mathbb{Q}(t)}, S(E')_{\mathbb{Q}(t)}) = 0.$$

In other words, if it is nonzero, $\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E'))$ is a torsion module, so the map

$$\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E')) \xrightarrow{\pi} \operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E'))$$

is not injective. Thus, it suffices to prove that when $f(E) \neq f(E')$, the map

$$\operatorname{Hom}_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E')) \longrightarrow \operatorname{Hom}_{\tilde{\mathcal{H}}_k}(S_k(E), S_k(E')) \tag{3A.6}$$

is surjective. If $E \ncong E'$, Proposition 3.1 gives $\operatorname{Hom}_{\tilde{\mathcal{H}}_k}(S_k(E), S_k(E')) = 0$ implying the surjectivity of (3A.6) trivially. On the other hand, if $E \cong E'$, the proposition gives $\operatorname{Hom}_{\mathcal{H}_k}(S_k(E), S_k(E')) \cong k$. This also gives surjectivity of the map in (3A.6), since it becomes surjective upon restriction to $\mathscr{Q} \subseteq \operatorname{Hom}_{\tilde{\mathcal{H}}}(\tilde{S}(E), \tilde{S}(E'))$ (taking $E' = E$). $\square$

4. Two preliminary lemmas

Let R be a commutative ring and let $\mathscr{C}$ be an abelian R-category. For $A, B \in \mathscr{C}$, let $\operatorname{Ext}^1_{\mathscr{C}}(A, B)$ denote the Yoneda group of extensions of A by B. (We do not require the higher Ext-groups in this section.) Let $M, Y \in \mathscr{C}$, and suppose that $\operatorname{Ext}^1_{\mathscr{C}}(M, Y)$ is generated as an R-module by elements $\epsilon_1, \dots, \epsilon_m$. Let $\chi := \oplus_i \epsilon_i \in \operatorname{Ext}^1_{\mathscr{C}}(M^{\oplus m}, Y)$ correspond to the short exact sequence $0 \to Y \to X \to M^{\oplus m} \to 0$.

Lemma 4.1. *The map* $\operatorname{Ext}^1_{\mathscr{C}}(M, Y) \longrightarrow \operatorname{Ext}^1_{\mathscr{C}}(M, X)$, *induced by the inclusion* $Y \hookrightarrow X$, *is the zero map.*

Proof. Using the "long" exact sequence of $\operatorname{Ext}^\bullet$, it suffices to show that the map δ in the sequence

$$\operatorname{Hom}_{\mathscr{C}}(M, X) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(M, M^{\oplus m}) \xrightarrow{\delta} \operatorname{Ext}^1_{\mathscr{C}}(M, Y)$$

is surjective — equivalently, that each $\epsilon_i \in \mathrm{Ext}^1_{\mathscr{C}}(M, Y)$ lies in the image of δ. Let $0 \to Y \to X_i \to M \to 0$ correspond to $\epsilon_i \in \mathrm{Ext}^1_{\mathscr{C}}(M, Y)$. By construction, ϵ_i is the image of χ under the natural map

$$j_i^* : \mathrm{Ext}^1_{\mathscr{C}}(M^{\oplus m}, Y) \longrightarrow \mathrm{Ext}^1_{\mathscr{C}}(M, Y),$$

which is the pull-back of the inclusion j_i of M into the i-th summand of $M^{\oplus m}$. So there is a natural commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & X \arrow[r] & M^{\oplus m} \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u] & X_i \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "j_i"'] & 0
\end{tikzcd}$$

There is a corresponding commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(M, X) \arrow[r] & \mathrm{Hom}_{\mathscr{C}}(M, M^{\oplus m}) \arrow[r, "\delta"] & \mathrm{Ext}^1_{\mathscr{C}}(M, Y) \\
\mathrm{Hom}_{\mathscr{C}}(M, X_i) \arrow[r] \arrow[u] & \mathrm{Hom}_{\mathscr{C}}(M, M) \arrow[r, "\delta_i"] \arrow[u] & \mathrm{Ext}^1_{\mathscr{C}}(M, Y) \arrow[u, equal]
\end{tikzcd} \tag{4.7}$$

where each row is part of a "long" exact sequence. Then $\delta_i(1_M) = \epsilon_i$. Therefore, the commutativity of the right hand square in (4.7) immediately says that ϵ_i lies in the image of δ. □

This lemma together with the additivity of the functor $\mathrm{Ext}^1_{\mathscr{C}}$ gives immediately the following.

Corollary 4.2. *Maintain the setup above. If* $\mathrm{Ext}^1_{\mathscr{C}}(M, M) = 0$, *then* $\mathrm{Ext}^1_{\mathscr{C}}(M, X) = 0$.

Next, let R be a commutative ring which is a $\mathcal{Z}$-algebra and write $q = t^2 \cdot 1$, the image in R of $t^2 \in \mathcal{Z}$. We allow general parameters c_s and $s \in S$ in (1.1) for the rest of this section.

Lemma 4.3. *Let* $\mathfrak{N} \subseteq \mathfrak{M}$ *be left ideals in* $\mathcal{H}_R$, *with each spanned by the Kazhdan–Lusztig basis elements* C'_y *that they contain. Let* $s \in S$ *be a simple reflection and assume either* $\mathfrak{N} = 0$ *or that* $q^{c_s} + 1$ *is not a zero divisor in* R. *Suppose* $0 \neq x \in \mathfrak{M}/\mathfrak{N}$ *satisfies*

$$T_s \cdot x = q^{c_s} x. \tag{4.8}$$

Then x *is represented in* $\mathfrak{M}$ *by an* R*-linear combination of Kazhdan–Lusztig basis elements* C'_y *with* $sy < y$.

Proof. Let $[m]$ denote the image in $\mathfrak{M}/\mathfrak{N}$ of $m \in \mathfrak{M}$. Note that $\mathfrak{M}$, $\mathfrak{N}$, and $\mathfrak{M}/\mathfrak{N}$ are all R-free, since the C'_y which belong to $\mathfrak{M}$ and $\mathfrak{N}$ form a basis for $\mathfrak{M}$ and $\mathfrak{N}$, respectively. The R-module $\mathfrak{M}/\mathfrak{N}$ has a basis consisting of all $[C'_y] \neq 0$ with $C'_y \in \mathfrak{M}$.

Write $x = \sum_y a_y[C'_y]$ with $a_y[C'_y] \neq 0$ and $C'_y \in \mathfrak{M}$. Observe that, for $y \in W$, $s \in S$,

$$sy < y \Longrightarrow T_s C'_y = q^{c_s} C'_y. \tag{4.9}$$

Therefore, in the above expression for x, it may also be assumed that $sy > y$ for each nonzero term $a_y[C'_y]$. Let $a_w[C'_w] \neq 0$ be chosen with w maximal among these y. In general, for $sy > y$, we have

$$T_s C'_y = -C'_y + C'_{sy} + \sum_{\substack{z<y \\ sz<z}} b_z C'_z$$

for various $b_z \in R$. Equating coefficients of $[C'_w]$ gives by (4.8) that $(q^{c_s}+1)a_w = 0$, since C'_w does not appear with any coefficient in the expressions $T_s C'_y$ with $y \neq w$ and $sy > y$. Now the hypothesis on zero divisors forces $a_w = 0$, a contradiction. $\square$

Remark 4.4. As observed in (4.9) above, elements $x \in \mathfrak{M}/\mathfrak{N}$ satisfying the conclusion of Lemma 4.3 also satisfy its hypothesis (4.8). Next, suppose that $\lambda \subseteq S$ and $\mathfrak{L}$ is any $\mathcal{H}_R$-module. By Frobenius reciprocity, the R-module $\mathrm{Hom}_{\mathcal{H}_R}(\mathcal{H}_R x_\lambda, \mathfrak{L})$ identifies with the R-submodule $\mathcal{X} \subseteq \mathfrak{L}$ consisting of all $x \in \mathfrak{L}$ satisfying (4.8) for all $s \in \lambda$. Suppose $\mathfrak{L}$ can be realized as $\mathfrak{L} = \mathfrak{M}/\mathfrak{N}$, with $\mathfrak{M}$, $\mathfrak{N}$ as in the statement of Lemma 4.3. If $q^{c_s}+1$ is invertible in R for all $s \in \lambda$, then the lemma implies that $\mathcal{X}$ has an R-basis consisting of all nonzero $[C'_y]$ in $\mathfrak{L}$ with $sy < y$ for all $s \in \lambda$.

Thus, if R' is an R-algebra, then the R'-module $\mathrm{Hom}_{\mathcal{H}_{R'}}(\mathcal{H}_{R'} x_\lambda, \mathfrak{L}_{R'})$ has essentially the "same basis." This fact will be used in proving the following corollary.

In the result below, we allow $c_s \neq 1$. In case $c_s = 1$, assumption (2) is satisfied for $R = \mathcal{Q}$ if and only if $e \neq 2$.

Corollary 4.5. *Suppose R is a commutative domain with fraction field F, and assume that R is also a $\mathcal{Z}$-algebra. Let $\lambda \subseteq S$. Assume that*

(1) *$\mathcal{H}_F$ is semisimple*;

(2) *$q^{c_s}+1$ is invertible in R, for each $s \in \lambda$.*

Then, for any dual left cell module $S_{\omega,R}$ over R,

$$\mathrm{Ext}^1_{\mathcal{H}_R}(S_{\omega,R}, x_\lambda \mathcal{H}_R) = 0.$$

Proof. Put $\mathcal{S} := S_{\omega,R}$. Using condition (1) and [Du et al. 1998, Lemma 1.2.13], it suffices to prove, for each $R' = R/\langle d \rangle$ ($d \in R$), that the map

$$\mathrm{Hom}_{\mathcal{H}_R}(\mathcal{S}, x_\lambda \mathcal{H}_R) \longrightarrow \mathrm{Hom}_{\mathcal{H}_{R'}}(\mathcal{S}_{R'}, x_\lambda \mathcal{H}_{R'})$$

is surjective. Here $\mathcal{S}_{R'} = \mathcal{S} \otimes_R R'$.

By [op. cit., Lemma 2.1.9], the left $\mathcal{H}_R$-module $(x_\lambda\mathcal{H}_R)^* := \mathrm{Hom}_R(x_\lambda\mathcal{H}_R, R)$ is naturally isomorphic to $\mathcal{H}_R x_\lambda$. By hypothesis, $\mathcal{S} = \mathcal{L}^*$ is the dual of a left cell module $\mathcal{L}$, R-free by definition. Thus, $\mathcal{L} \cong \mathcal{S}^*$; also, $(\mathcal{H}_R x_\lambda)^* \cong x_\lambda\mathcal{H}_R$. There are similar isomorphisms for analogous R'-modules (for which we use the same notation $(-)^*$). The functor $(-)^*$ provides a contravariant equivalence from the category of finitely generated R-free left $\mathcal{H}_R$-modules and the corresponding right $\mathcal{H}_R$-module category. A similar statement holds with R replaced by R'. Finally, there is a natural isomorphism $(-)^* \otimes_R R' \xrightarrow{\sim} (- \otimes_R R')^*$.

Consequently, it is sufficient to prove that

$$\mathrm{Hom}_{\mathcal{H}_R}(\mathcal{H}_R x_\lambda, \mathcal{L}) \longrightarrow \mathrm{Hom}_{\mathcal{H}_{R'}}(\mathcal{H}_{R'} x_\lambda, \mathcal{L}_{R'})$$

is surjective. (Here $\mathcal{L}_{R'}$ denotes the left cell module in $\mathcal{H}_{R'}$ defined by the same left cell as $\mathcal{L}$ for $\mathcal{H}$.) However, viewing $\mathcal{L}$ and $\mathcal{L}_{R'}$ as cell modules (over $\mathcal{H}_R$ and $\mathcal{H}_{R'}$, respectively), hypothesis (2), Lemma 4.3, and Remark 4.4 give the "same basis" (over R and R', respectively). □

5. The construction of $\tilde{X}_\omega$ and the main theorem

In this section, we prove the main result of the paper (Theorem 5.6).

Let $\mathscr{Q}$ be as in (1C.1). Recall that $\tilde{\mathcal{H}}$ denotes the $\mathscr{Q}$-algebra $\mathcal{H} \otimes_{\mathcal{Z}} \mathscr{Q}$. In general, modules for $\tilde{\mathcal{H}}$ are decorated with a "tilde" (e.g., $\tilde{X}$). In particular, we recall from (1B.3) the notations $\tilde{S}(\omega)$ and $\tilde{S}_\omega$.

5A. *Preliminaries.* Consider a left cell ω and let $\mathcal{J}_\omega = \sum_{y\in\omega} \mathbb{Z} j_y$. Then (2.4) implies that $\mathcal{J}_\omega$ is a left $\mathcal{J}$-module. Using the monomorphism ϖ in Section 3, form the left $\mathcal{H}$-module $\varpi^*(\mathcal{J}_\omega \otimes \mathcal{Z})$, the restriction of the $\mathcal{J}_{\mathcal{Z}}$-module $\mathcal{J}_\omega \otimes \mathcal{Z}$ to $\mathcal{H}$.

Lemma 5.1. *There is an $\mathcal{H}$-module isomorphism*

$$\sigma : \varpi^*(\mathcal{J}_\omega \otimes \mathcal{Z}) \longrightarrow S(\omega) := \mathcal{H}^{\leq_L \omega}/\mathcal{H}^{<_L \omega}$$

induced by the map $\sigma : \mathcal{J}_{\mathcal{Z}} \to \mathcal{H},\ j_y \mapsto C'_y$. In particular, $\tilde{S}(\omega)$ is a direct sum of modules $\tilde{S}(E)$ for some $E \in \mathrm{Irr}(\mathbb{Q}W)$.

Proof. This is a refinement of [Lusztig 2003, §18.10]. We first observe that the map σ clearly induces a $\mathcal{Z}$-module isomorphism. It remains to check for $y \in \omega$ that

$$\sigma(\varpi(C'_x) j_y) \equiv C'_x C'_y \mod \mathcal{H}^{<_L \omega}, \quad (x \in W)$$

The proof of [op. cit, §18.10(a)][4] gives the left hand equality in the expression

$$\text{(5A.1)}\quad \sigma(\varpi(C'_x) j_y) = \sigma\Bigg(\sum_{\substack{u\\ a(y)=a(u)}} h_{x,y,u} j_u\Bigg) = \sum_{\substack{u\\ a(y)=a(u)}} h_{x,y,u} C'_u \equiv C'_x C'_y \mod \mathcal{H}^{<_L \omega}.$$

[4]The main ingredient is [op. cit, §18.9(b)]. As previously noted, the $\hat{n}_z$ may be set equal to 1.

The middle equality is just the definition of σ. Finally, the right hand congruence follows from the fact that, when $h_{x,y,u}C'_u$ is nonzero mod $\mathcal{H}^{<_L\omega}$, u must belong to the same left cell ω as y, and hence have the same a-value. □

If W is of type A and ω is the left cell containing the longest word $w_{0,\lambda}$ for a partition λ. Then $\varpi^*(\mathcal{J}_\omega\otimes\mathcal{Z})$ is isomorphic to the left cell module whose dual is the Specht module S_λ. So $\tilde{S}(E)$ above could be called a "dual Specht module," with $\tilde{S}(E)^*$ a "Specht module." The modules $\tilde{S}_\omega$ are also candidates for the name "Specht module" [Du et al. 1998, p. 198].

Remark 5.2. A completely analogous result to Lemma 5.1 holds if the Kazhdan–Lusztig C-basis (instead of the C'-basis here) is used, as in [Ginzburg et al. 2003]. First, it follows from [Lusztig 1985, (3.2)] that the map (which we call τ) $\mathcal{Z}\to\mathcal{Z}$, sending $t\mapsto -t$, takes the coefficients $h_{x,y,z}$ to analogous coefficients for the C-basis. Extend τ to an automorphism, still denoted τ, of $\mathcal{J}_\mathcal{Z}$, taking j_x to its C-analogue; we may put $\tau(j_x)=(-1)^{\ell(x)}j_x$. Thus, any expression $h_{x,y,z}j_z$ is sent to a C-basis analogue. In particular, $\varpi(C'_x)$ is sent to $\varpi(C_x)$, where the latter ϖ is taken in the C-basis set-up. Now it is clear from (5A.1) that the analogue of Lemma 5.1 holds in the C-basis set-up. Note the resulting left cell modules in $\mathcal{H}$ do not depend on which canonical basis is used. This allows an identification of the module $S(\omega)$ in Lemma 5.1 with its C-basis counterpart.

An analogous result holds for two-sided cells, e.g., the $\mathcal{H}$-module $\varpi^*(\mathcal{J}_{\underline{c}}\otimes_\mathbb{Z}\mathcal{Z})$ in [Ginzburg et al. 2003, Corollary 6.4] does not depend on the whether the C'-basis is used (as in this paper) or the C-basis is used (as in [op. cit]). We do not know, however, if the base-change of the automorphism τ to $\mathcal{J}_{\mathbb{Q}(t)}$ preserves the isomorphism types of irreducible $\mathcal{J}_{\mathbb{Q}(t)}$-modules, though their associated two-sided cells are preserved. This leads to the "permutation" language used in footnote 2. In particular, we do not know if the bijection noted below [Ginzburg et al. 2003, Definition 6.1] depends on the choice of C- or C'-basis set-up, and could result in one choice leading to an identification which is a (two-sided cell preserving) permutation of the other.

Corollary 5.3. *Assume that $e\neq 2$. For left cells ω, ω', we have*

$$\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_\omega,\tilde{S}_{\omega'})\neq 0 \quad\Longrightarrow\quad f(\omega)>f(\omega').$$

Proof. By Lemma 5.1 and Corollary 3.2 (which requires $e\neq 2$),

$$\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}(\omega'),\tilde{S}(\omega))\neq 0 \quad\Longrightarrow\quad f(\omega)>f(\omega').$$ □

For $\lambda\subseteq S$, the induced (right) $\mathcal{H}$-module $x_\lambda\mathcal{H}$ (see Section 1B) has an increasing filtration

$$F^\bullet_\lambda:\quad 0=F^0_\lambda\subseteq F^1_\lambda\subseteq\cdots\subseteq F^{m_\lambda}_\lambda \tag{5A.2}$$

with sections $F_\lambda^{i+1}/F_\lambda^i \cong S_{\omega_i}$ and bottom section $F_\lambda^1 = F_\lambda^1/F_\lambda^0 \cong S_{\omega_1}$, where ω_1 is the left cell containing the longest word $w_{\lambda,0}$ in the parabolic subgroup W_λ. If $i > 1$, then $\omega_1 >_L \omega_i$; see [Du et al. 1998, (2.3.7)]. The indexing ω_i of (some of) the left cells depends on λ, and is formally "opposite" (in reverse order) to that used in [op. cit]. We write $\omega_\lambda := \omega_1$ to denote its dependence of the latter cell on λ.

Lemma 5.4. *In the filtration (5A.2), if $i > 1$, then $f(\omega_i) > f(\omega_\lambda)$.*

Proof. This follows from (2.7), since $\omega_1 >_L \omega_i$ for all $2 \le i \le m_\lambda$, as noted above. □

5B. ***First construction of a module $\tilde{X}_\omega$.*** Let $\omega \in \Omega$ be a fixed left cell. The construction of $\tilde{X}_\omega$ relies on Corollary 5.3.

We iteratively construct an $\tilde{\mathcal{H}}$-module $\tilde{X}_\omega$, filtered by dual left cell modules, such that $\tilde{S}_\omega \subseteq \tilde{X}_\omega$ is the lowest nonzero filtration term, and

$$\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_{\omega'}, \tilde{X}_\omega) = 0 \quad \text{for all left cells } \omega'.$$

It will also be a consequence of the construction that every other filtration term $\tilde{S}_\nu$, $\nu \in \Omega$, satisfies $f(\nu) > f(\omega)$.

For $j \in \mathbb{N}$, let

$$\Omega_j = \{\nu \in \Omega \mid f(\nu) = j\}.$$

Fix $i = f(\omega)$. Suppose $\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_\tau, \tilde{S}_\omega) \neq 0$ for some $\tau \in \Omega$. Then, by the Corollary 5.3, $f(\tau) > f(\omega) = i$. Assume $f(\tau) = j$ is minimal with this property. Since $\mathcal{Q}$ is a DVR and $\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_\tau, \tilde{S}_\omega)$ is finitely generated, it follows that $\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_\tau, \tilde{S}_\omega)$ is a direct sum of $m_\tau (\ge 0)$ nonzero cyclic $\mathcal{Q}$-modules. Let $\tilde{Y}_\tau$ be the extension of $\tilde{S}_\tau^{\oplus m_\tau}$ by $\tilde{S}_\omega$, constructed as above Lemma 4.1 (using generators for the cyclic modules). Then by Lemma 4.1, Corollary 4.2, and Corollary 5.3,

$$\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_\tau, \tilde{Y}_\tau) = 0.$$

Let

$$\Omega_{j,\omega} = \{\nu \in \Omega_j \mid \operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_\nu, \tilde{S}_\omega) \neq 0\}.$$

If $\nu \in \Omega_{j,\omega} \setminus \{\tau\}$, then $\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_\nu, \tilde{S}_\omega) \cong \operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_\nu, \tilde{Y}_\tau)$ by Corollary 5.3, together with the long exact sequence for Ext.[5]

Thus, if $\tilde{Y}_{\tau,\nu}$ denotes the corresponding extension of $\tilde{S}_\nu^{\oplus m_\nu}$ by $\tilde{Y}_\tau$ (again using the construction above Lemma 4.1), then

$$\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_{\omega'}, \tilde{Y}_{\tau,\nu}) = 0 \quad \text{for } \omega' = \tau, \nu.$$

[5] We also use the fact that $f(\nu) \neq f(\tau)$ implies that $\operatorname{Hom}_{\mathcal{H}}(\widetilde{S}_\nu, \widetilde{S}_\tau) = 0$ since $\operatorname{Hom}_{\mathcal{Q}}(\widetilde{S}_\nu, \widetilde{S}_\tau)$ and hence $\operatorname{Hom}_{\mathcal{H}}(\widetilde{S}_\nu, \widetilde{S}_\tau))$ are free $\mathcal{O}$-modules. Thus, if $\operatorname{Hom}_{\mathcal{H}}(\widetilde{S}_\nu, \widetilde{S}_\tau)) \neq 0$, then it remains nonzero upon base change to K. This is impossible since ν and τ belong to different two-sided cells and $\tilde{\mathcal{H}}_K$ is semisimple.

From the general identity

$$\mathrm{Ext}^1_{\tilde{\mathcal{H}}}(A, C) \oplus \mathrm{Ext}^1_{\tilde{\mathcal{H}}}(B, C) \cong \mathrm{Ext}^1_{\tilde{\mathcal{H}}}(A \oplus B, C),$$

one sees that $\tilde{Y}_{\tau,\nu}$ is isomorphic to the "sum" extension of $\tilde{S}_\tau^{\oplus m_\tau} \oplus \tilde{S}_\nu^{\oplus m_\nu}$ by $\tilde{S}_\omega$. Continuing this process, we obtain an extension $\tilde{Y}_j$ of $\bigoplus_{\tau\in\Omega_{j,\omega}} \tilde{S}_\tau^{\oplus m_\tau}$ by $\tilde{S}_\omega$, with

$$\mathrm{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_{\omega'}, \tilde{Y}_j) = 0 \quad \text{for all } \omega' \in \bigcup_{\ell\le j} \Omega_\ell.$$

Thus, $\mathrm{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_{\omega'}, \tilde{Y}_j) \neq 0$ implies that $f(\omega') > j$.

Continuing the above construction with the role of $\tilde{S}_\omega$ replaced by $\tilde{Y}_{j_1}$ with $j_1 = j$, we obtain a module $\tilde{Y}_{j_1,j_2}$ such that $j_1 < j_2$ and

$$\mathrm{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_{\omega'}, \tilde{Y}_{j_1,j_2}) = 0 \quad \text{for all } \omega' \in \bigcup_{\ell\le j_2} \Omega_\ell.$$

Let m be the maximal f-value. This construction will stop after a finite number $r = r(\omega)$ of steps, resulting in an $\tilde{\mathcal{H}}$-module $\tilde{X}_\omega := \tilde{Y}_{j_1,j_2,\ldots,j_r}$ such that

$$f(\omega) < j_1 < j_2 < \cdots < j_r \le m, \qquad \text{and } \mathrm{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_{\omega'}, \tilde{X}_\omega) = 0 \text{ for all } \omega' \in \Omega.$$

5C. ***A second construction of a module $\tilde{X}_\omega$.*** The construction will generally lead to a larger module $\tilde{X}_\omega$, so is not as "efficient" as the first construction above, in some sense. Nevertheless, the construction has similar properties, is cleaner, and has the very considerable advantage that it first builds an $\mathcal{H}$-module X_ω, then sets $\tilde{X}_\omega = X_{\omega,\mathcal{Q}} := (X_\omega)_{\mathcal{Q}}$. Both X_ω and $\tilde{X}_\omega$ are built with the requirement $e \neq 2$, this condition being needed in the supporting Proposition 5.5(3) below.

As before, Ω denotes the set of all left cells of W, and $\Omega_i = \{\omega \in \Omega \mid f(\omega) = i\}$ for $i \in \mathbb{N}$.

Fix $\omega \in \Omega$, and put $i_0 = f(\omega)$. For each $i \in \mathbb{Z}$, put $X_{\omega,i} = 0$ if $i < i_0$ (we use these terms only as a notational convenience), and put $X_{\omega,i_0} = S_\omega$. Next, we give a recursive definition of $X_{\omega,j}$ for all $j \ge i_0$, with the case $j = i_0$ just given. If $X_{\omega,j}$ has been defined, define $X_{\omega,j+1}$ as follows: Let M denote the direct sum (possibly zero) of all $\mathcal{H}$-modules S_τ with $f(\tau) = j+1$. Using the category $\mathcal{H}$-mod for $\mathscr{C}$ in the construction above Lemma 4.1, and $Y = X_{\omega,j}$, put $X_{\omega,j+1} = X$ in that construction (making some choice for the generators $\mathrm{Ext}^1_{\mathcal{H}}(M, Y)$ that are used). For j sufficiently large, we have $\Omega_i = 0$ for all $i > j$, and so $X_{\omega,i} = X_{\omega,j}$. Thus, we set $X_\omega := X_{\omega,j}$ for any such sufficiently large j.

Proposition 5.5. *The $\mathcal{H}$-module X_ω and the increasing filtration $\{X_{\omega,i}\}_{i\in\mathbb{Z}}$, constructed above, have the following properties*:

(1) *The smallest index of a nonzero section $X_{\omega,i}/X_{\omega,i-1}$ is $i = f(\omega) = i_0$, and the section is S_ω in that case.*

(2) *All sections* $X_{\omega,i}/X_{\omega,i-1}$ *are direct sums of modules* S_τ, $\tau \in \Omega$, *with varying multiplicities* (*possibly* 0), *and with* $f(\tau) = i$.

(3) *If* $e \neq 2$, *then* $\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(S_{\nu,\mathscr{Q}}, X_{\omega,\mathscr{Q}}) = 0$ *for all* $\nu, \omega \in \Omega$.

Proof. Properties (1) and (2) are immediate from the construction of X_ω.

To prove (3), fix ν and $\omega \in \Omega$. We will apply Corollary 5.3 several times. First, it shows the vanishing in (3) holds section by section of $X_{\omega,\mathscr{Q}}$, unless $f(\nu) > f(\omega)$. So assume that $f(\nu) > f(\omega)$.

Put $j = f(\nu) - 1$ and let M be the $\mathcal{H}$-module used above in the construction of $X_{\omega,j+1}$ from $Y = X_{\omega,j}$. Lemma 4.1 implies the map $\operatorname{Ext}^1_{\mathcal{H}}(M, Y) \to \operatorname{Ext}^1_{\mathcal{H}}(M, X_{\omega,j+1})$ is the zero map. Applying the flat base change from $\mathscr{Z}$ to $\mathscr{Q}$, we find that the map $\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(M_{\mathscr{Q}}, Y_{\mathscr{Q}}) \to \operatorname{Ext}^1_{\tilde{\mathcal{H}}}(M_{\mathscr{Q}}, X_{\mathscr{Q}})$ is zero, with $X = X_{\omega,j+1}$. However, Corollary 5.3 implies $\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(M_{\mathscr{Q}}, M_{\mathscr{Q}}) = 0$. Now the long exact sequence argument of Corollary 4.2 shows that $\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(M_{\mathscr{Q}}, X_{\mathscr{Q}}) = 0$. Since S_ν is a direct summand of M (by construction, since $f(\nu) = j + 1$), it follows that $\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(S_{\nu,\mathscr{Q}}, X_{\mathscr{Q}}) = 0$.

However, $X_\omega/X_{\omega,j+1}$ is filtered by modules S_τ with $f(\tau) > j+1 = f(\nu)$. So

$$\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(S_{\nu,\mathscr{Q}}, (X_\omega/X_{\omega,j+1})_{\mathscr{Q}}) = 0$$

by Corollary 4.2 again. Together with the conclusion of the previous paragraph, this gives the required vanishing $\operatorname{Ext}^1_{\tilde{\mathcal{H}}}(S_{\nu,\mathscr{Q}}, X_{\omega,\mathscr{Q}}) = 0$. □

To complete the second construction, set $\tilde{X}_\omega = X_{\omega,\mathscr{Q}}$.

5D. *The main result.* Let Ω' be the set of all left cells that do not contain the longest element of a parabolic subgroup. Put

$$\tilde{\mathcal{T}} = \bigoplus_{\lambda \subseteq S} x_\lambda \tilde{\mathcal{H}} \quad \text{and} \quad \tilde{\mathcal{X}} = \bigoplus_{\omega \in \Omega'} \tilde{X}_\omega.$$

Here and in the theorem below, objects (modules, algebras, etc.) are decorated with a tilde $\sim$ because they are taken over the DVR $\mathscr{Q}$ in (1C.1).

We are now ready to prove the following main result of the paper.

Theorem 5.6. *Assume that* $e \neq 2$. *Let* $\tilde{\mathcal{T}}^+ = \tilde{\mathcal{T}} \oplus \tilde{\mathcal{X}}$, $\tilde{\mathcal{A}}^+ = \operatorname{End}_{\tilde{\mathcal{H}}}(\tilde{\mathcal{T}}^+)$, *and* $\tilde{\Delta}(\omega) = \operatorname{Hom}_{\tilde{\mathcal{H}}}(\tilde{S}_\omega, \tilde{\mathcal{T}}^+)$ *for* $\omega \in \Omega$. *Then* $\{\tilde{\Delta}(\omega)\}_{\omega\in\Omega}$ *is a strict stratifying system for the category* $\tilde{\mathcal{A}}^+$*-mod with respect to the quasi-poset* $(\Omega, \leq_f)$.

Proof. For each left cell ω, put $\tilde{T}_\omega = x_\lambda \tilde{\mathcal{H}}$ if ω contains the longest element $w_{\lambda,0}$ of W_λ, where $\lambda \subseteq S$. If there is no such λ for ω, put $\tilde{T}_\omega = \tilde{X}_\omega$ as constructed in Section 5B. (One can use the $\tilde{X}_\omega$ from Section 5C with slight adjustments, left to the reader.) In the first case, $\tilde{T}_\omega$ has a filtration by dual left cell modules, and $\tilde{S}_\omega$ appears at the bottom. Moreover, $f(\omega) < f(\omega')$ for any other filtration section $\tilde{S}_{\omega'}$, by Lemma 5.4. This same property holds also in the case $\tilde{T}_\omega = \tilde{X}_\omega$ by construction.

Put $\tilde{T} = \bigoplus_\omega \tilde{T}_\omega$ and note $\tilde{\mathcal{T}}^+ = \tilde{T}$. We will apply Theorem 1.1 to $\tilde{T}$ and the various $\tilde{T}_\omega$, where $\tilde{\mathcal{H}}$ plays the role of the algebra B there, $\mathscr{Q}$ plays the role of R there, $\tilde{S}_\omega$ is S_λ, etc. We are required to the check three conditions (1), (2), (3) in Theorem 1.1. The construction in Section 5B of dual left cell filtrations of the various $\tilde{T}_\omega$ is precisely what is required for the verification of (1).

Condition (2) translates directly to the requirement

$$\mathrm{Hom}_{\tilde{\mathcal{H}}}(\tilde{S}_\mu, \tilde{T}_\omega) \neq 0 \quad \Longrightarrow \quad \omega \leq_f \mu$$

for given μ, ω. However, if $\mathrm{Hom}_{\tilde{\mathcal{H}}}(\tilde{S}_\mu, \tilde{T}_\omega) \neq 0$, then there must be a nonzero $\mathrm{Hom}_{\tilde{\mathcal{H}}}(\tilde{S}_\mu, \tilde{S}_{\omega'})$ for some filtration section $\tilde{S}_{\omega'}$ of $\tilde{T}_\omega$. In particular, $f(\omega') \geq f(\omega)$. Also, $(\tilde{S}_\mu)_K$ and $(\tilde{S}_{\omega'})_K$ must have a common irreducible constituent, forcing the two-sided cells containing μ and ω' to agree. This gives $f(\mu) = f(\omega') \geq f(\omega)$; so (2) holds.

Finally,

$$\mathrm{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_\mu, \tilde{T}_\omega) = 0 \quad \text{for all } \mu, \omega. \tag{5D.1}$$

This follows from the construction Section 5B for $\tilde{T}_\omega = \tilde{X}_\omega$ and by Corollary 4.5 in case $\tilde{T}_\omega = x_\lambda \tilde{\mathcal{H}}$. The conclusion of Theorem 1.1 now immediately gives the theorem we are proving here. □

6. Identification of $\tilde{\mathcal{A}}^+ = \mathrm{End}_{\tilde{\mathcal{H}}}(\tilde{\mathcal{T}}^+)$

The constructions in Section 5B of the modules $\tilde{X}_\omega$ in the previous section work just as well using the modules $\tilde{S}_E := \tilde{S}(E)^*$ for $E \in \mathrm{Irr}(\mathbb{Q}W)$ defined in (3A.5) to replace the dual left cell modules $\tilde{S}_\omega$. This results in right $\mathcal{H}$-modules $\tilde{X}_E$. As in the case of $\tilde{X}_\omega$, we have the following property, with the same proof. In the statement of the following proposition, $\tilde{X}_E$ can be defined using either of the two constructions.

Proposition 6.1. *If $e \neq 2$, then $\mathrm{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_{E'}, \tilde{X}_E) = 0$ for all $E, E' \in \mathrm{Irr}(\mathbb{Q}W)$.*

If we use the first construction given in Section 5B, the modules $\tilde{X}_E$ have strong indecomposability properties, which the modules $\tilde{X}_\omega$, $\omega \in \Omega$, generally do not have with either construction. In the following proposition, we assume that $\tilde{X}_E$ is defined by the first construction Section 5B.

The following result can be argued without using RDAHAs, but it is faster to quote Rouquier's 1-faithful covering theory, especially [Rouquier 2008, Theorem 5.3], which applies to our $e \neq 2$ case, over $\mathscr{R}$, where

$$\mathscr{R} := (\mathbb{C}[t, t^{-1}]_{(t-\sqrt{\zeta})})^\wedge$$

is the completion of the localization $\mathbb{C}[t, t^{-1}]_{(t-\sqrt{\zeta})}$ at the maximal ideal $(t-\sqrt{\zeta})$. Note that $\mathscr{R}$ is a $\mathscr{Q}$-module via the natural ring homomorphism $\mathscr{Q} \to \mathscr{R}$. Note also that the set $\mathrm{Irr}(\mathbb{Q}W)$ corresponds naturally to the set $\mathrm{Irr}(W) := \mathrm{Irr}(\mathbb{C}W)$ in [op. cit].

Proposition 6.2. *Assume that $e \neq 2$. The right $\tilde{\mathcal{H}}$-modules $\tilde{X}_E$ are indecomposable, as is each $\tilde{X}_E \otimes k$. The endomorphism algebras of all these modules are local with radical quotient k.*

Proof. It is clear that $\tilde{X}_{E,\mathscr{R}} = \tilde{X}_E \otimes_{\mathscr{Q}} \mathscr{R}$ can be constructed from $\tilde{S}_{E,\mathscr{R}}$ in the same way that $\tilde{X}_E$ is constructed from $\tilde{S}_E$, again using the method of Section 5B. Also, the proof of [op. cit, Theorem 6.8] shows that the $\mathscr{R}$-dual of $\tilde{S}_{E,\mathscr{R}}$ is the KZ-image of the standard module $\Delta_{\mathscr{R}}(E)$ in the $\mathscr{R}$-version of $\mathcal{O}$. (Recall the issues in footnote 2.)

Consequently, by the 1-faithful property, $(\tilde{X}_{E,\mathscr{R}})^*$ is the image of a dually constructed module P under the functor KZ, filtered by standard modules, and with $\mathrm{Ext}^1_{\mathcal{O}}(P, -)$ vanishing on all standard modules. Such a module P is projective in $\mathcal{O}$, by [op. cit, Lemma 4.22]. (We remark that both $\mathcal{O}$ and KZ would be given a subscript $\mathscr{R}$ in [Ginzburg et al. 2003] though not in [Rouquier 2008].)

If we knew P were indecomposable, we could say $\tilde{X}_{E,\mathscr{R}}$ is indecomposable. However, the indecomposability of P requires proof.[6] We do this by showing P is the projective cover in $\mathcal{O}$ of the standard module $\Delta(E) = \Delta_{\mathcal{O}}(E)$. We can, instead, inductively show the truncation P_i, associated to the poset ideal of all $E' \in \mathrm{Irr}(\mathbb{Q}W)$ with $f(E') \leq i$, is the projective cover of $\Delta(E)$ in the associated truncation $\mathcal{O}_i$ of $\mathcal{O}$. This requires $\Delta(E)$ to be an object of $\mathcal{O}_i$, or equivalently $f(E) \leq i$.

If $f(E) = i$, then $P_i = \Delta(E)$ is trivially the projective cover of $\Delta(E)$. Inductively, P_{i-1} is the projective cover of $\Delta(E)$ in $\mathcal{O}_{i-1}$ for some $i > f(E)$. Let P' denote the projective cover of $\Delta(E)$ in $\mathcal{O}_i$. The truncation $(P')_{i-1}$ to $\mathcal{O}_{i-1}$ of P' — that is, its largest quotient which is an object of $\mathcal{O}_{i-1}$ — is clearly isomorphic to P_{i-1}. Let $\theta : P' \to P_i$ be a homomorphism extending a given isomorphism $\psi : (P')_{i-1} \to P_{i-1}$ and let $\tau : P_i \to P'$ be a homomorphism extending ψ^{-1}. Let M, M' denote the kernels of the natural surjections $P_i \twoheadrightarrow P_{i-1}$ and $P' \twoheadrightarrow (P')_{i-1}$. The map $\tau\theta : P' \to P'$ is surjective and, consequently, it is an isomorphism. It induces the identity on $(P')_{i-1}$. Therefore, the induced map

$$\tau|_M \theta|_{M'} : M' \longrightarrow M'$$

is an isomorphism, and $M = M' \oplus M''$ for some object M'' in $\mathcal{O}$. By construction, M is a direct sum of objects $\Delta(E')$, with $f(E') = i$, each appearing with multiplicity $m_{E'} = \mathrm{rank}\,\mathrm{Ext}^1_{\mathcal{O}}(P_i, \Delta(E'))$. However,

$$\mathrm{Ext}^1_{\mathcal{O}}(P_{i-1}, \Delta(E')) \cong \mathrm{Hom}_{\mathcal{O}}(M', \Delta(E')).$$

[6] A similar point should be made regarding the uniqueness claim in [Rouquier 2008, Proposition 4.45], which is false without a minimality assumption on $Y(M)$ there.

It follows that $M'' = 0$ and $P_i \cong P'$ is indecomposable.

In particular, P is indecomposable and consequently $\tilde{X}_{E,\mathscr{R}}$ is indecomposable, as noted. In turn, this implies $\tilde{X}_E$ is indecomposable. The 0-faithfulness (or just the covering property itself) of the cover given by $\mathcal{O}$ and KZ imply

$$\mathrm{End}_{\tilde{\mathcal{H}}_{\mathscr{R}}}(\tilde{X}_{E,\mathscr{R}})^{\mathrm{op}} \cong \mathrm{End}_{\tilde{\mathcal{H}}_{\mathscr{R}}}(\tilde{X}^*_{E,\mathscr{R}})^{\mathrm{op}} \cong \mathrm{End}_{\mathcal{O}}(P).$$

Thus, the base-changed module $P \otimes_{\mathscr{R}} \mathbb{C}$ has endomorphism ring

$$\mathrm{End}_{\mathcal{O}_{\mathbb{C}}}(P \otimes_{\mathscr{R}} \mathbb{C}) \cong \mathrm{End}_{\mathcal{O}}(P) \otimes_{\mathscr{R}} \mathbb{C},$$

where $\mathcal{O}_{\mathbb{C}}$ is the $\mathbb{C}$-version of $\mathcal{O}$. This is a standard consequence of the projectivity of P. By [Rouquier 2008, Theorem 5.3], the $\mathbb{C}$ versions of KZ and $\mathcal{O}$ give a cover for $\tilde{\mathcal{H}}_{\mathscr{R}} \otimes \mathbb{C}$. So $\mathrm{End}_{\tilde{\mathcal{H}}_{\mathbb{C}}}(\tilde{X}_{E,\mathscr{R}} \otimes \mathbb{C})^{\mathrm{op}} \cong \mathrm{End}_{\mathcal{O}_{\mathbb{C}}}(P \otimes \mathbb{C})$ is local, with radical quotient $\mathbb{C}$.

However, we have

$$(\tilde{X}_E \otimes_{\mathscr{Q}} k) \otimes_k \mathbb{C} \cong \tilde{X}_{E,\mathscr{R}} \otimes \mathbb{C}.$$

In particular, $\tilde{X}_E \otimes_{\mathscr{Q}} k$ is indecomposable since (by endomorphism ring considerations) the $\tilde{\mathcal{H}}_{\mathscr{R}} \otimes \mathbb{C}$-module $\tilde{X}_{E,\mathscr{R}} \otimes \mathbb{C}$ is indecomposable. So the endomorphism ring of $\tilde{X}_E \otimes_{\mathscr{Q}} k$ over the finite dimensional algebra $\tilde{\mathcal{H}} \otimes_{\mathscr{Q}} k$ is local. The radical quotient is a division algebra D over k with base change $- \otimes_k \mathbb{C}$ to a semisimple quotient of $\mathrm{End}_{\tilde{\mathcal{H}}_{\mathbb{C}}}(\tilde{X}_{E,\mathscr{R}} \otimes \mathbb{C})$, which could only be $\mathbb{C}$ itself. Consequently, $D = k$.

Finally, the vanishing $\mathrm{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{X}_E, \tilde{X}_E) = 0$ implies

$$\mathrm{End}_{\tilde{\mathcal{H}}}(\tilde{X}_E) \otimes_{\mathscr{Q}} k \cong \mathrm{End}_{\tilde{\mathcal{H}}_k}(\tilde{X}_E \otimes_{\mathscr{Q}} k).$$

So the ring $\mathrm{End}_{\tilde{\mathcal{H}}}(\tilde{X}_E)$ is local with radical quotient k. This completes the proof. □

Lemma 6.3. *Assume $e \neq 2$. Let $E \in \mathrm{Irr}(\mathbb{Q}W)$. Then $\tilde{X}_E$ is a direct summand of $\tilde{\mathcal{T}}^+$.*

Proof. Suppose first that $\tilde{S}(E)$ is a direct summand of a left cell module $\tilde{S}(\omega) \cong (\tilde{S}_\omega)^*$, where ω contains the longest element of a parabolic subgroup W_λ, for $\lambda \subseteq S$. This implies $\tilde{S}_\omega$ is the lowest term in the dual left cell module filtration of $x_\lambda \tilde{\mathcal{H}}$. Consequently, there is an inclusion $\psi : \tilde{S}_E \hookrightarrow x_\lambda \tilde{\mathcal{H}}$ with cokernel filtered by (sections) $\tilde{S}_{E'}$, $E' \in \mathrm{Irr}(\mathbb{Q}W)$. Thus, $\psi^{-1} : \psi(\tilde{S}_E) \to \tilde{X}_E$ may be extended to a map $\phi : x_\lambda \tilde{\mathcal{H}} \to \tilde{X}_E$ of $\tilde{\mathcal{H}}$-modules. Similarly (using $e \neq 2$ and Corollary 4.5), there is a map $\tau : \tilde{X}_E \to x_\lambda \tilde{\mathcal{H}}$ extending ψ. The composite $\tau\phi$ restricts to the identity on $\tilde{S}_E \subseteq \tilde{X}_E$.

On the other hand, restriction from $\tilde{X}_E$ to $\tilde{S}_E$ defines a homomorphism

$$\mathrm{End}_{\tilde{\mathcal{H}}}(\tilde{X}_E) \longrightarrow \mathrm{End}_{\tilde{\mathcal{H}}}(\tilde{S}_E)$$

since $(\tilde{S}_E)_K$ is a unique summand of the (completely reducible) $\tilde{\mathcal{H}} \otimes_{\mathscr{Q}} K$-module $\tilde{X}_E \otimes_{\mathscr{Q}} K$. (Observe $\tilde{S}_E = \tilde{X}_E \cap (\tilde{S}_E)_K$, since the $\mathscr{Q}$-torsion module $(\tilde{X}_E \cap (\tilde{S}_E)_K)/\tilde{S}_E$

must be zero in the $\mathscr{Q}$-torsion free module $\tilde{X}_E/\tilde{S}_E$.) Thus, $\tau\phi$ is a unit in the local endomorphism ring $\mathrm{End}_{\tilde{\mathcal{H}}}(\tilde{X}_E)$, so $\tilde{X}_E$ is a summand of $x_\lambda\tilde{\mathcal{H}}$, and hence of $\tilde{\mathcal{T}}$.

Next consider the case in which $\tilde{S}_E$ is a summand of a dual left cell module $\tilde{S}_\omega$ (this always happens for some ω), but ω does not contain the longest element of any parabolic subgroup. In this case, $\tilde{X}_\omega$ is one of the summands of $\tilde{\mathcal{X}}$ by construction. The argument above may be repeated with $\tilde{X}_\omega$ playing the role of $x_\lambda\tilde{\mathcal{H}}$. In the same way, $\tilde{X}_E$ is a direct summand of $\tilde{X}_\omega$, and thus of $\tilde{\mathcal{X}}$.

In both cases, we conclude that $\tilde{X}_E$ is a direct summand of $\tilde{\mathcal{T}}\oplus\tilde{\mathcal{X}}=\tilde{\mathcal{T}}^+$. □

Theorem 6.4. *Assume that $e\neq 2$. The $\mathscr{Q}$-algebra $\tilde{\mathcal{A}}^+$ is quasi-hereditary, with standard modules $\tilde{\Delta}(E)=\mathrm{Hom}_{\tilde{\mathcal{H}}}(\tilde{S}_E,\tilde{\mathcal{T}}^+)$, $E\in\mathrm{Irr}(\mathbb{Q}W)$, and partial order $<_f$.*

Proof. We have already seen that this algebra is standardly stratified with strict stratifying system $\{\tilde{\Delta}(\omega)\}_{\omega\in\Omega}$. Clearly, $\tilde{\Delta}(\omega)$ is a direct sum of various $\tilde{\Delta}(E)$, and every $\tilde{\Delta}(E)$ arises as such a summand.

Put $\tilde{P}(E)=(\tilde{X}_E)^\diamond:=\mathrm{Hom}_{\tilde{\mathcal{H}}}(\tilde{X}_E,\tilde{\mathcal{T}}^+)$, $E\in\mathrm{Irr}(\mathbb{Q}W)$. Then $\tilde{P}(E)$ is a direct summand of $\tilde{\mathcal{A}}^+=\mathrm{End}_{\tilde{\mathcal{H}}}(\tilde{\mathcal{T}}^+)$, viewed as a left module over itself. Thus, $\tilde{P}(E)$ is projective as an $\tilde{\mathcal{A}}^+$-module, and $\tilde{P}(E)^\diamond:=\mathrm{Hom}_{\tilde{\mathcal{A}}^+}(\tilde{P}(E),\tilde{\mathcal{T}}^+)$ is naturally isomorphic to $\tilde{X}_E$. In particular, the contravariant functor $(-)^\diamond$ gives an isomorphism

$$\mathrm{End}_{\tilde{\mathcal{A}}^+}(\tilde{P}(E))\cong(\mathrm{End}_{\tilde{\mathcal{H}}}(\tilde{X}_E))^{\mathrm{op}}.$$

Consequently, $\tilde{P}(E)$ also has a local endomorphism ring with radical quotient k, as does $\mathrm{End}_{\tilde{\mathcal{A}}^+_k}(\tilde{P}(E)\otimes_{\mathscr{Q}}k)$. It follows that $\tilde{P}(E)$ is an indecomposable projective $\tilde{\mathcal{A}}^+$-module with a irreducible head. (The arguments in this paragraph are largely standard, many taken from [Du et al. 1998].)

By (5D.1), $\mathrm{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_\omega,\tilde{\mathcal{T}}^+)=0$ for all dual left cell module $\tilde{S}_\omega$. Consequently, a similar vanishing holds with $\tilde{S}_\omega$ replaced by any module $\tilde{S}_{E'}$, $E'\in\mathrm{Irr}(\mathbb{Q}W)$. It follows that the restriction map

$$\tilde{P}(E)=\mathrm{Hom}_{\tilde{\mathcal{H}}}(\tilde{X}_E,\tilde{\mathcal{T}}^+)\longrightarrow\mathrm{Hom}_{\tilde{\mathcal{H}}}(\tilde{S}_E,\tilde{\mathcal{T}}^+)=\tilde{\Delta}(E)$$

is surjective. Hence, $\tilde{\Delta}(E)$ has an irreducible head. Also, repeating the argument for filtered submodules of $\tilde{X}_E$, we find that the kernel of the above map has a filtration with sections $\tilde{\Delta}(E')$, $E'\in\mathrm{Irr}(\mathbb{Q}W)$ (rather than $\tilde{X}_E$ itself), satisfying $f(E')>f(E)$.

Next, we claim that $\tilde{\Delta}(E)^\diamond:=\mathrm{Hom}_{\tilde{\mathcal{A}}^+}(\tilde{\Delta}(E),\tilde{\mathcal{T}}^+)$ is naturally isomorphic to $\tilde{S}_E$. More precisely, we claim that the natural map $\mathrm{ev}:\tilde{S}_E\to(\tilde{S}_E)^{\diamond\diamond}$ is an isomorphism. We showed above that the sequence

$$0\longrightarrow(\tilde{X}_E/\tilde{S}_E)^\diamond\longrightarrow(\tilde{X}_E)^\diamond\longrightarrow(\tilde{S}_E)^\diamond\longrightarrow 0$$

is exact. Applying $(-)^\diamond$ once more, we get an injection

$$0\longrightarrow(\tilde{S}_E)^{\diamond\diamond}\longrightarrow(\tilde{X}_E)^{\diamond\diamond}$$

with $\tilde{X}_E \xrightarrow{\text{ev}} (\tilde{X}_E)^{\diamond\diamond}$ an isomorphism. This gives inclusions

$$\tilde{S}_E \cong \text{ev}(\tilde{S}_E) \subseteq (\tilde{S}_E)^{\diamond\diamond} \subseteq (\tilde{X}_E)^{\diamond\diamond} \cong \tilde{X}_E.$$

If $(-) \otimes_{\mathscr{Q}} K$ is applied, the first inclusion becomes an isomorphism. This gives

$$(\tilde{S}_E)^{\diamond\diamond} \subseteq (\tilde{X}_E)^{\diamond\diamond} \cap (\tilde{S}_E)_K = \tilde{S}_E$$

identifying $\tilde{X}_E$ with $(\tilde{X}_E)^{\diamond\diamond}$ and $\tilde{S}_E$ with its image in $(\tilde{X}_E)^{\diamond\diamond}$. Consequently, $\text{ev}(\tilde{S}_E) = (\tilde{S}_E)^{\diamond\diamond}$, proving the claim.

Finally, we suppose $E \not\cong E' \in \text{Irr}(\mathbb{Q}W)$ and $\text{Hom}_{\tilde{\mathcal{A}}^+}(\tilde{P}(E'), \tilde{\Delta}(E)) \neq 0$. Using the identifications $\tilde{P}(E') = (\tilde{X}_{E'})^\diamond$, $\tilde{\Delta}(E) = (\tilde{S}_E)^\diamond$, $\tilde{P}(E')^\diamond \cong \tilde{X}_{E'}$, and $\tilde{\Delta}(E)^\diamond \cong \tilde{S}_E$, we have

$$0 \neq \text{Hom}_{\tilde{\mathcal{A}}^+}(\tilde{P}(E'), \tilde{\Delta}(E)) \cong \text{Hom}_{\tilde{\mathcal{H}}}(\tilde{S}_E, \tilde{X}_{E'}) \subseteq \text{Hom}_{\tilde{\mathcal{H}}_K}(\tilde{S}_E \otimes_{\mathscr{Q}} K, \tilde{X}_{E'} \otimes_{\mathscr{Q}} K).$$

This implies $f(E') < f(E)$. It follows now from [Du et al. 1998, Theorem 1.2.8] (in the context of stratified algebras), [Du and Scott 1994, Corollary 2.5], or [Rouquier 2008, Theorem 4.16] that $\tilde{\mathcal{A}}^+$ is quasi-hereditary over $\mathscr{Q}$. □

We are now ready to establish the category equivalence mentioned in the introduction. Again, we use the covering theory of [Rouquier 2008].

Theorem 6.5. *Assume that $e \neq 2$. The category of left modules over the base-changed algebra*

$$\tilde{\mathcal{A}}^+_{\mathscr{R}} := \tilde{\mathcal{A}}^+ \otimes_{\mathscr{Q}} \mathscr{R}$$

is equivalent to the $\mathscr{R}$-category $\mathcal{O}$ of modules, as defined in [Rouquier 2008] for the RDAHA associated to W over $\mathscr{R}$.

Proof. Continuing the proof of the theorem above, the projective indecomposable $\tilde{\mathcal{A}}^+$-modules are the various $\tilde{P}(E) = (\tilde{X}_E)^\diamond$. Consequently, $\tilde{\mathcal{T}}^+ = (\tilde{\mathcal{A}}^+)^\diamond$ is the direct sum of the modules, $\tilde{X}_E$, each with nonzero multiplicities. The modules $\tilde{X}_{E,\mathscr{R}}$ remain indecomposable, as observed in the proof of the indecomposability of the modules $\tilde{X}_E$ above. By construction, $\text{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_{E'}, \tilde{X}_E) = 0$ for all $E, E' \in \text{Irr}(\mathbb{Q}W)$. Thus, there is a similar vanishing for $\tilde{S}_{E',\mathscr{R}}$ and $\tilde{X}_{E,\mathscr{R}}$, and — in the reverse order — for their $\mathscr{R}$-linear duals. Observe that $(\tilde{S}_{E',\mathscr{R}})^* \cong \tilde{S}(E') \otimes_{\mathscr{Q}} \mathscr{R}$ is $\text{KZ}(\Delta(E'))$, taking $\Delta(E') = \Delta_{\mathcal{O}}(E')$ to be the standard module for the category $\mathcal{O}$ over $\mathscr{R}$ as discussed in [loc. cit.] together with KZ for this category.

Put

$$Y = \bigoplus_E (\tilde{X}_{E,\mathscr{R}})^*$$

and set $Y(\tilde{S}^*_{E,\mathscr{R}}) = (\tilde{X}_{E,\mathscr{R}})^*$. This notation imitates that of [op. cit, Proposition 4.45]. The first part of this proposition is missing a necessary minimality assumption on the rank of $Y(M)$, in the terminology there.[7] However, this is satisfied for $M = (\tilde{S}_{E,\mathscr{R}})^*$ and $Y(M) = (\tilde{X}_{E,\mathscr{R}})^*$ because $(\tilde{X}_{E,\mathscr{R}})^*$ is indecomposable. Several other corrections, in addition to the minimality requirement, should be made to [op. cit, Proposition 4.45]:

- A' should be redefined as $\mathrm{End}_B(Y)^{\mathrm{op}}$;
- P' should be redefined as $\mathrm{Hom}_B(Y, B)^{\mathrm{op}}$.

In addition, B in [op. cit, §4.2.1] should be redefined as $\mathrm{End}_A(P)^{\mathrm{op}}$. The instances of "op" here and above insure action on the left, and consistency with [Ginzburg et al. 2003, Theorems 5.14 and 5.15]. The definition of P' is given to be consistent with the basis covering property $\mathrm{End}_{A'}(P')^{\mathrm{op}} \cong B$, as in [loc. cit.] — we do not need this fact below.

With these changes, [Rouquier 2008, Theorem 5.3, Proposition 4.45, and Corollary 4.46] guarantees that $\mathcal{A}'$-mod is equivalent to $\mathcal{O}$, where $\mathcal{A}' = \mathrm{End}_{\tilde{\mathcal{H}}_{\mathscr{R}}}(Y)$. (All we really need for this are the 0- and 1-faithfulness of the $\mathcal{O}$ version of the KZ functor.) However, $\mathrm{End}_{\tilde{\mathcal{H}}_{\mathscr{R}}}(Y) \cong \mathrm{End}_{\tilde{\mathcal{H}}_{\mathscr{R}}}(Y^*)^{\mathrm{op}}$, and Y^* is the direct sum $\bigoplus_E \tilde{X}_{E,\mathscr{R}}$. Hence,

$$Y^{*\diamond} \cong \bigoplus_E (\tilde{X}_{E,\mathscr{R}})^{\diamond} \cong \bigoplus_E \tilde{P}(E) \otimes_{\mathscr{Q}} \mathscr{R}.$$

Recall that $(\tilde{X}_{E,\mathscr{R}})^{\diamond\diamond} \cong \tilde{X}_{E,\mathscr{R}}$, so that the analogous property holds for Y^*. Thus,

$$\mathrm{End}_{\tilde{\mathcal{H}}_{\mathscr{R}}}(Y^*)^{\mathrm{op}} \cong \mathrm{End}_{\tilde{\mathcal{A}}^+_{\mathscr{R}}}(Y^{*\diamond}).$$

Since the module $Y^{*\diamond}$ as displayed above is clearly a projective generator for $\tilde{\mathcal{A}}^+_{\mathscr{R}}$, there is a Morita equivalence over $\mathscr{R}$ of $\tilde{\mathcal{A}}^+_{\mathscr{R}}$ with $\mathcal{A}'$. Hence, $\tilde{\mathcal{A}}^+_{\mathscr{R}}$-mod is equivalent to $\mathcal{O}$, as $\mathscr{R}$-categories. □

Appendix: comparison with [Du et al. 1998, Conjecture 2.5.2]

Conjecture 1.2 in this paper retains the most essential features of [Du et al. 1998, Conjecture 2.5.2], but is more flexible. In particular:

(1) Conjecture 1.2 does not specify the preorder $\leq$, only requiring that it be strictly compatible with the partition of Ω into two-sided cells. This allows the use of the preorder $\leq_f$, defined in Section 2 above. [Du et al. 1998] specifies for $\leq$ the preorder $\leq_{LR}^{\mathrm{op}}$ built from the preorder $\leq_{LR}$ originally used by Kazhdan–Lusztig to define the two-sided cells. In both cases, the set $\overline{\Omega}$ of "strata" is the same, identifying with the set of two sided cells.

[7] The proposition claims uniqueness for a pair $(Y(M), p_m)$. However, one gets another pair by adding a direct summand $F(P)$ to the kernel of p_m, where P is any finitely generated module in the highest weight category $\mathscr{C}$.

(2) Conjecture 1.2 concerns the Hecke algebra $\mathcal{H}$ (defined by the relations (1.1) over $\mathcal{Z} = \mathbb{Z}[t, t^{-1}]$, whereas [Du et al. 1998, Conj. 2.5.2] uses Hecke algebras over $\mathbb{Z}[t^2, t^{-2}]$. Largely, this change has been made to conform to the literature, which most often uses the former ring. There is an additional advantage that the quotient field $\mathbb{Q}(t)$ is almost always a splitting field for the Hecke algebra $\mathcal{H}_{\mathbb{Q}(t)}$. Note that $\mathbb{Q}(t)$ is always a splitting field in case the rank is greater than 2. In the rank 2 case of ${}^2F_4$, $\mathcal{H}_{\mathbb{Q}(t)}$ splits after $\sqrt{2}$ is adjoined. The conjecture in all rank 2 cases follows from [Du et al. 1998, §3.5].

(3) The role of $\mathcal{A}_R^+$ in Conjecture 1.2 is played by $\mathrm{End}_{\mathcal{H}_R}(\mathcal{T}_R^+)$ in [Du et al. 1998, Conjecture 2.5.2]. The two R-algebras are the same whenever R is flat over $\mathcal{Z} = \mathbb{Z}[t, t^{-1}]$. While it is an interesting question as to whether or not such a base change property holds for any $\mathcal{Z}$-algebra R, it seems best to separate this issue from the main stratification proposal of the conjecture.

Finally, we mention that the original conjecture [Du et al. 1998, Conjecture 2.5.2] was checked in that paper for all rank two types (in both the equal and unequal parameter cases), and checked later in type A for all ranks; see [op. cit]. These verifications show also that Conjecture 1.2 is true in these cases.

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PRODUCTS OF PARTIAL NORMAL SUBGROUPS

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In memory of Robert Steinberg.

We show that the product of two partial normal subgroups of a locality (in the sense of Chermak) is again a partial normal subgroup. This generalizes a theorem of Chermak and fits into the context of building a local theory of localities.

1. Introduction

Localities were introduced by Andrew Chermak [2013], in the context of his proof of the existence and uniqueness of centric linking systems. Roughly speaking, localities are group-like structures which are essentially the "same" as the transporter systems of Oliver and Ventura [2007]; see the appendix to [Chermak 2013]. As centric linking systems are special cases of transporter systems, the existence of centric linking systems implies that there is a locality attached to every fusion system. It is work in progress of Chermak to build a local theory of localities similar to the local theory of fusion systems as developed by Aschbacher [2008; 2011]. In fact, it seems often an advantage to work inside of localities, where some group theoretical concepts and constructions can be expressed more naturally than in fusion systems. Thus, one can hope to improve the local theory of fusion systems, once a way of translating between fusion systems and localities is established. The results of this paper can be considered as first evidence that some constructions are easier in the world of localities. We prove that the product of partial normal subgroups of a locality is itself a partial normal subgroup, whereas in fusion systems the product of normal subsystems has only been defined in special cases; see [Aschbacher 2011, Theorem 3]. It is work in progress of Chermak to show that there is a one-to-one correspondence between the normal subsystems of a saturated fusion system $\mathcal{F}$ and the partial normal subgroups of a linking locality attached to $\mathcal{F}$ in the sense of [Henke 2015, Definition 2]. This is one reason why our result seems particularly important in the case of linking localities. Another reason is that the concept of a linking locality generalizes properties of localities corresponding to centric linking

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systems and is thus interesting for studying the homotopy theory of fusion systems; see [Broto et al. 2003; 2005; 2007; Henke 2015]. It is however crucial for our proof that we work with arbitrary localities, since our arguments rely heavily on the theory of quotient localities introduced by Chermak [2015], and a quotient of a linking locality is not necessarily a linking locality again. Thus, we feel that the method of our proof gives evidence for the value of studying localities in general rather than restricting attention only to the special case of linking localities.

To describe the results of this paper in more detail, let $\mathcal{L}$ be a partial group as defined in [Chermak 2013, Definition 2.1; 2015, Definition 1.1]. Thus, there is an involutory bijection $\mathcal{L} \to \mathcal{L}$, $f \mapsto f^{-1}$, called an "inversion", and a multivariable product Π which is only defined on certain words in $\mathcal{L}$. Let $\boldsymbol{D}$ be the domain of the product; i.e., $\boldsymbol{D}$ is a set of words in $\mathcal{L}$ and Π is a map $\boldsymbol{D} \to \mathcal{L}$. Following Chermak, we call a nonempty subset $\mathcal{H}$ of $\mathcal{L}$ a *partial subgroup* of $\mathcal{L}$ if $h^{-1} \in \mathcal{H}$ for all $h \in \mathcal{H}$ and $\Pi(v) \in \mathcal{H}$ for all words v in the alphabet $\mathcal{H}$ with $v \in \boldsymbol{D}$. A partial subgroup $\mathcal{N}$ is called a *partial normal subgroup* if $x^f := \Pi(f^{-1}, x, f) \in \mathcal{N}$ for all $x \in \mathcal{N}$ and $f \in \mathcal{L}$ for which $(f^{-1}, x, f) \in \boldsymbol{D}$. Given two subsets $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{L}$, the product $\mathcal{M}\mathcal{N}$ is naturally defined by

$$\mathcal{M}\mathcal{N} = \{\Pi(m, n) : m \in \mathcal{M},\ n \in \mathcal{N},\ (m, n) \in \boldsymbol{D}\}.$$

The problem is however to show that this is again a partial normal subgroup if $\mathcal{M}$ and $\mathcal{N}$ are partial normal subgroups. Indeed, as we show in Example 2.3, this is not true in general if $\mathcal{L}$ is an arbitrary partial group. It is true however in the important case that $(\mathcal{L}, \Delta, S)$ is a locality. Chermak [2015, Theorem 5.1] proved this in a special case and we build on his result to prove the general case. More precisely, we prove the following theorem:

Theorem 1. *Let $(\mathcal{L}, \Delta, S)$ be a locality and let $\mathcal{M}, \mathcal{N}$ be partial normal subgroups of $\mathcal{L}$. Then $\mathcal{M}\mathcal{N} = \mathcal{N}\mathcal{M}$ is a partial normal subgroup of $\mathcal{L}$ and $(\mathcal{M}\mathcal{N}) \cap S = (\mathcal{M} \cap S)(\mathcal{N} \cap S)$. Moreover, for every $g \in \mathcal{M}\mathcal{N}$ there exists $m \in \mathcal{M}$ and $n \in \mathcal{N}$ such that $(m, n) \in \boldsymbol{D}$, $g = \Pi(m, n)$, and $S_g = S_{(m,n)}$.*

To understand the technical conditions stated in the last sentence of the theorem, we recall from [Chermak 2013; 2015] that

$$S_g = \{s \in S :\ (g^{-1}, s, g) \in \boldsymbol{D} \text{ and } s^g \in S\}$$

for any $g \in \mathcal{L}$. Moreover, for a word $v = (g_1, \dots, g_n)$ in $\mathcal{L}$, S_v is the set of all $s \in S$ such that there exist $x_0, \dots, x_n \in S$ with $s = x_0$, $(g_i^{-1}, x_{i-1}, g_i) \in \boldsymbol{D}$ and $x_{i-1}^{g_i} = x_i$ for $i = 1, \dots, n$. By [op. cit, Proposition 2.6 and Corollary 2.7], the sets S_g and S_v are subgroups of S, $S_g \in \Delta$ for any $g \in \mathcal{L}$, and $S_v \in \Delta$ if and only if $v \in \boldsymbol{D}$. Therefore, the condition $S_g = S_{(m,n)}$ stated in the theorem is crucial for proving that certain products are defined in $\mathcal{L}$. This is particularly important for the proof of our next theorem which concerns products of more than two partial normal subgroups. Given subsets $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_l$ of $\mathcal{L}$ define their product via

$$\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l := \{\Pi(n_1, n_2, \cdots, n_l) : (n_1, n_2, \ldots, n_l) \in \boldsymbol{D},\ n_i \in \mathcal{N}_i \text{ for } 1 \le i \le l\}.$$

We prove:

Theorem 2. *Let $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_l$ be partial normal subgroups of a locality $(\mathcal{L}, \Delta, S)$. Then $\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l$ is a partial normal subgroup of $\mathcal{L}$. Moreover, the following hold:*

(1) $\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l = (\mathcal{N}_1 \cdots \mathcal{N}_k)(\mathcal{N}_{k+1} \cdots \mathcal{N}_l)$ *for every* $1 \le k < l$.

(2) $\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l = \mathcal{N}_{1\sigma}\mathcal{N}_{2\sigma} \cdots \mathcal{N}_{l\sigma}$ *for every permutation* $\sigma \in S_l$.

(3) *For every* $g \in \mathcal{N}_1 \ldots \mathcal{N}_l$ *there exists* $(n_1, \ldots, n_l) \in \boldsymbol{D}$ *with* $n_i \in \mathcal{N}_i$ *for every* $i = 1, \ldots, l,\ g = \Pi(n_1, \ldots, n_l)$, *and* $S_g = S_{(n_1,\ldots,n_l)}$.

As already mentioned above, it is work in progress of Andrew Chermak to show that for every fusion system $\mathcal{F}$ and a linking locality $(\mathcal{L}, \Delta, S)$ attached to $\mathcal{F}$ there is a one-to-one correspondence between the normal subsystems of $\mathcal{F}$ and the partial normal subgroups of $\mathcal{L}$. When this work is complete, our results will imply the existence of a product of an arbitrary finite number of normal subsystems of $\mathcal{F}$.

In this text only relatively few demands will be made on understanding the concepts introduced in [Chermak 2013; 2015]. In Section 2, we point the reader to the few general results needed about partial groups, give a concise definition of a locality and review some basic facts about localities. In Section 3, we summarize what is needed about partial normal subgroups and quotient localities.

2. Partial groups and localities

We refer the reader to [Chermak 2013, Definition 2.1] or [Chermak 2015, Definition 1.1] for the precise definition of a partial group, and to the elementary properties of partial groups stated in [2013, Lemma 2.2] or [2015, Lemma 1.4]. Adapting Chermak's notation we write $\boldsymbol{W}(\mathcal{L})$ for the set of words in a set $\mathcal{L}$, $\varnothing$ for the empty word, and $v_1 \circ v_2 \circ \cdots \circ v_n$ for the concatenation of words $v_1, \ldots, v_n \in \boldsymbol{W}(\mathcal{L})$.

For the remainder of this text let $\mathcal{L}$ be a partial group with product $\Pi : \boldsymbol{D} \to \mathcal{L}$ defined on the domain $\boldsymbol{D} \subseteq \boldsymbol{W}(\mathcal{L})$.

Again following Chermak's notation, we set $\mathbf{1} = \Pi(\varnothing)$. Moreover, given a word $v = (f_1, \ldots, f_n) \in \boldsymbol{D}$, we write $f_1 f_2 \ldots f_n$ for the product $\Pi(v)$. Recall the definitions of partial subgroups and partial normal subgroups from the introduction. Note that a partial subgroup of $\mathcal{L}$ is always a partial group itself whose product is the restriction of the product Π to $\boldsymbol{W}(\mathcal{H}) \cap \boldsymbol{D}$. Observe furthermore that $\mathcal{L}$ forms a group in the usual sense if $\boldsymbol{W}(\mathcal{L}) = \boldsymbol{D}$; see [op. cit., Lemma 1.3]. So it makes sense to call a partial subgroup $\mathcal{H}$ of $\mathcal{L}$ a *subgroup of* $\mathcal{L}$ if $\boldsymbol{W}(\mathcal{H}) \subseteq \boldsymbol{D}$. In particular, we can talk about *p-subgroups of* $\mathcal{L}$ meaning subgroups of $\mathcal{L}$ whose order is a power of p.

We will need the Dedekind lemma [Chermak 2015, Lemma 1.10] in the following slightly more general form:

2.1 (Dedekind lemma). *Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{A}$ be subsets of $\mathcal{L}$ such that $\mathcal{A}$ is a partial subgroups of $\mathcal{L}$ and $\mathcal{K} \subseteq \mathcal{A}$. Then $\mathcal{A} \cap (\mathcal{H}\mathcal{K}) = (\mathcal{A} \cap \mathcal{H})\mathcal{K}$ and $\mathcal{A} \cap (\mathcal{K}\mathcal{H}) = \mathcal{K}(\mathcal{A} \cap \mathcal{H})$.*

Proof. Clearly, $(\mathcal{A} \cap \mathcal{H})\mathcal{K} \subseteq \mathcal{A} \cap (\mathcal{H}\mathcal{K})$. Taking $h \in \mathcal{H}$ and $k \in \mathcal{K}$ with $(h, k) \in \boldsymbol{D}$ and $hk \in \mathcal{A}$, we have $(h, k, k^{-1}) \in \boldsymbol{D}$ by [op. cit, Lemma 1.4(d)] and then $h = h(kk^{-1}) = (hk)k^{-1} \in \mathcal{A}$ as $\mathcal{K} \subseteq \mathcal{A}$ and $\mathcal{A}$ is a partial subgroup. Hence, $h \in \mathcal{A} \cap \mathcal{H}$ and $hk \in (\mathcal{A} \cap \mathcal{H})\mathcal{K}$. The second equation follows similarly. □

Before we continue with more definitions, we illustrate the concepts we mentioned so far with examples. For this purpose we say that two groups G_1 and G_2 *form an amalgam*, if the set-theoretic intersection $G_1 \cap G_2$ is a subgroup of both G_1 and G_2, and the restriction of the multiplication on G_1 to a multiplication on $G_1 \cap G_2$ is the same as the restriction of the multiplication on G_2 to a multiplication on $G_1 \cap G_2$.

Example 2.2. Let G_1 and G_2 be groups which form an amalgam. Set $\mathcal{L} = G_1 \cup G_2$ and $\boldsymbol{D} = \boldsymbol{W}(G_1) \cup \boldsymbol{W}(G_2)$. Define a partial product $\Pi : \boldsymbol{D} \to \mathcal{L}$ by sending $v = (f_1, \dots, f_n) \in \boldsymbol{W}(G_i)$ to the product $f_1 \dots f_n$ in the group G_i for $i = 1, 2$. Define an inversion $\mathcal{L} \to \mathcal{L}$ by sending $f \in G_i$ to the inverse of f in the group G_i for $i = 1, 2$. Then $\mathcal{L}$ with these structures forms a partial group. (For readers familiar with the concept of an objective partial group as introduced in [Chermak 2013, Definition 2.6] or [Chermak 2015, Definition 2.1] we mention that, setting $\Delta := \{G_1, G_2\}$, $(\mathcal{L}, \Delta)$ is an objective partial group if $G_1 \cap G_2$ is properly contained in G_1 and G_2.)

Let $\mathcal{K}$ be a subset of $\mathcal{L}$. Then $\mathcal{K}$ is a partial subgroup of $\mathcal{L}$ if and only if $\mathcal{K} \cap G_i$ is a subgroup of G_i for each $i = 1, 2$. The subset $\mathcal{K}$ is a subgroup of $\mathcal{L}$ if and only if $\mathcal{K}$ is a subgroup of G_i for some $i = 1, 2$. Moreover, $\mathcal{K}$ is a partial normal subgroup of $\mathcal{L}$ if and only if $(\mathcal{K} \cap G_i) \trianglelefteq G_i$ for $i = 1, 2$.

We use the construction method introduced in the previous example to show that the product of two partial normal subgroups of a partial group is not in general itself a partial normal subgroup.

Example 2.3. Let $G_1 \cong C_2 \times C_4$ and let G_2 be a dihedral group of order 16. Choose G_1 and G_2 such that G_1 and G_2 form an amalgam with $G_1 \cap G_2 \cong C_2 \times C_2$ and $\Phi(G_1) = Z(G_2)$. Let $\mathcal{M}$ and $\mathcal{N}$ be the two cyclic subgroups of G_1 of order 4. Form the locality $\mathcal{L}$ as in Example 2.2. As G_1 is abelian, a subgroup $\mathcal{K}$ of G_1 is normal in G_1 and thus a partial normal subgroup of $\mathcal{L}$ if and only if $\mathcal{K} \cap G_2 \trianglelefteq G_2$. As $G_1 \cap G_2 \cong C_2 \times C_2$ and $\mathcal{M}$ and $\mathcal{N}$ are cyclic of order 4, we have $\mathcal{M} \cap G_2 =$

$\mathcal{N} \cap G_2 = \Phi(G_1) = Z(G_2) \trianglelefteq G_2$. Thus $\mathcal{M}$ and $\mathcal{N}$ are partial normal subgroups of $\mathcal{L}$. The product $\mathcal{M}\mathcal{N}$ in $\mathcal{L}$ is the same as the product $\mathcal{M}\mathcal{N}$ in G_1 and thus equal to G_1. However, as G_2 does not have a normal fours subgroup, $G_1 \cap G_2$ is not normal in G_2 and thus $\mathcal{M}\mathcal{N} = G_1$ is not a partial normal subgroup of $\mathcal{L}$.

The previous example shows that the concept of a partial group (and even the concept of an objective partial group) is too general for our purposes. Therefore, we will focus on localities. We give a definition of a locality which, in contrast to the definition given by Chermak [2013; 2015], does not require the reader to be familiar with the definition of an objective partial group and can easily seen to be equivalent to Chermak's definition. For any $g \in \mathcal{L}$, $\boldsymbol{D}(g)$ denotes the set of $x \in \mathcal{L}$ with $(g^{-1}, x, g) \in \boldsymbol{D}$. Thus, $\boldsymbol{D}(g)$ denotes the set of elements $x \in \mathcal{L}$ for which the conjugation $x^g := \Pi(g^{-1}, x, g)$ is defined. If $g \in \mathcal{L}$ and $X \subseteq \boldsymbol{D}(g)$ we set $X^g := \{x^g : x \in X\}$. If we write X^g for some $g \in \mathcal{L}$ and some subset $X \subseteq \mathcal{L}$, we will always implicitly mean that $X \subseteq \boldsymbol{D}(g)$.

Definition 2.4. We say that $(\mathcal{L}, \Delta, S)$ is a *locality* if the partial group $\mathcal{L}$ is finite as a set, S is a p-subgroup of $\mathcal{L}$, Δ is a nonempty set of subgroups of S, and the following conditions hold:

(L1) S is maximal with respect to inclusion among the p-subgroups of $\mathcal{L}$.

(L2) A word $(f_1, \dots, f_n) \in \boldsymbol{W}(\mathcal{L})$ is an element of $\boldsymbol{D}$ if and only if there exist $P_0, \dots, P_n \in \Delta$ such that

$$(*) \qquad P_{i-1} \subseteq \boldsymbol{D}(f_i) \quad \text{and} \quad P_{i-1}^{f_i} = P_i.$$

(L3) For any subgroup Q of S, for which there exist $P \in \Delta$ and $g \in \mathcal{L}$ with $P \subseteq \boldsymbol{D}(g)$ and $P^g \leq Q$, we have $Q \in \Delta$.

If $(\mathcal{L}, \Delta, S)$ is a locality and $v = (f_1, \dots, f_n) \in \boldsymbol{W}(\mathcal{L})$, then we say that $v \in \boldsymbol{D}$ via $P_0, \dots, P_n$ (or $v \in \boldsymbol{D}$ via P_0), if $P_0, \dots, P_n \in \Delta$ and $(*)$ holds.

From now on let $(\mathcal{L}, \Delta, S)$ ***be a locality.***

Note that $P = P^{\mathbf{1}} \leq S$ for all $P \in \Delta$. As $\Delta \neq \emptyset$, property (L3) implies thus $S \in \Delta$. For any $g \in \mathcal{L}$, write c_g for the conjugation map

$$c_g : \boldsymbol{D}(g) \to \mathcal{L},\ x \mapsto x^g.$$

Recall the definitions of S_g and S_v from the introduction. Note that $S_g \subseteq \boldsymbol{D}(g)$. For any subgroup X of $\mathcal{L}$ set

$$N_{\mathcal{L}}(X) := \{f \in \mathcal{L} :\ X \subseteq \boldsymbol{D}(f),\ X^f = X\}.$$

2.5 (Important properties of localities). *The following hold*:

(a) $N_{\mathcal{L}}(P)$ *is a subgroup of* $\mathcal{L}$ *for each* $P \in \Delta$.

(b) *Let $P \in \Delta$ and $g \in \mathcal{L}$ with $P \subseteq S_g$. Then $Q := P^g \in \Delta$ (so in particular Q is a subgroup of S). Moreover, $N_{\mathcal{L}}(P) \subseteq \boldsymbol{D}(g)$ and*

$$c_g : N_{\mathcal{L}}(P) \to N_{\mathcal{L}}(Q)$$

is an isomorphism of groups.

(c) *Let $w = (g_1, \dots, g_n) \in \boldsymbol{D}$ via $(X_0, \dots, X_n)$. Then*

$$c_{g_1} \circ \cdots \circ c_{g_n} = c_{\Pi(w)}$$

is a group isomorphism $N_{\mathcal{L}}(X_0) \to N_{\mathcal{L}}(X_n)$.

(d) *For every $g \in \mathcal{L}$, $S_g \in \Delta$. In particular, S_g is a subgroup of S.*

(e) *For any $w \in \boldsymbol{W}(\mathcal{L})$, S_w is a subgroup of $S_{\Pi(w)}$, and $S_w \in \Delta$ if and only if $w \in \boldsymbol{D}$.*[1]

Proof. Properties (a)–(c) correspond to statements in [Chermak 2015, Lemma 2.3] except for the fact stated in (b) that $Q \in \Delta$. This is however true by [op. cit., Proposition 2.6(c)]. Property (d) is true by [op. cit., Proposition 2.6(a)] and property (e) is stated in [op. cit., Corollary 2.7]. □

3. Partial normal subgroups and quotient localities

In this section we continue to assume that $(\mathcal{L}, \Delta, S)$ is a locality. The following theorem is a special case of Theorem 1 and will be used to prove the more general theorem.

Theorem 3.1. *Let $\mathcal{M}, \mathcal{N}$ be partial normal subgroups of $\mathcal{L}$ such that $\mathcal{M} \cap \mathcal{N} = 1$. Then $\mathcal{M}\mathcal{N} = \mathcal{N}\mathcal{M}$ is a partial normal subgroup of $\mathcal{L}$. Moreover, for any $f \in \mathcal{M}\mathcal{N}$ there exists $m \in \mathcal{M}$ and $n \in \mathcal{N}$ such that $(m, n) \in \boldsymbol{D}$, $f = mn$, and $S_f = S_{(m,n)}$.*

Proof. As $\mathcal{M} \cap \mathcal{N} \subseteq S$, it follows from [Chermak 2015, Lemma 5.3] that $\mathcal{M}$ normalizes $\mathcal{N} \cap S$ and $\mathcal{N}$ normalizes $\mathcal{M} \cap S$. So by [op. cit., Theorem 5.1], $\mathcal{M}\mathcal{N} = \mathcal{N}\mathcal{M}$ is a partial normal subgroup of $\mathcal{L}$. Moreover, by [op. cit., Lemma 5.2], for any $f \in \mathcal{M}\mathcal{N}$ there exist $m \in \mathcal{M}$ and $n \in \mathcal{N}$ such that $(m, n) \in \boldsymbol{D}$, $f = mn$, and $S_f = S_{(m,n)}$. □

To deduce Theorem 1 from Theorem 3.1, we need the theory of quotient localities developed in [Chermak 2015]; see also [Chermak 2013, Sections 3 and 4]. For the convenience of the reader we quickly summarize this theory here. After that we state some more specialized lemmas needed in our proof.

Throughout let $\mathcal{K}$ be a partial normal subgroup of $\mathcal{L}$ and $T = S \cap \mathcal{K}$.

3.2. (a) *T is strongly closed in $(\mathcal{L}, \Delta, S)$; that is, $t^g \in T$ for every $g \in \mathcal{L}$ and every $t \in T \cap S_g$. In particular, $T^g = T$ for any $g \in \mathcal{L}$ with $T \subseteq S_g$.*

(b) *T is maximal in the poset of p-subgroups of $\mathcal{N}$.*

[1] Recall the definition of $S_{(g_1,\dots,g_n)}$ from the introduction.

Proof. Let $g \in \mathcal{L}$ and $t \in T \cap S_g$. Then $t^g \in S$ and, as $\mathcal{N}$ is a partial normal subgroup, $t^g \in \mathcal{N}$. Hence, $t^g \in S \cap \mathcal{N} = T$. This proves (a). Property (b) is proved in [Chermak 2015, Lemma 3.1(c)]. □

We write $\uparrow_{\mathcal{K}}$ for the relation $\uparrow$ introduced in [op. cit., Definition 3.6], but with the partial normal subgroup $\mathcal{N}$ replaced by $\mathcal{K}$. Thus $\uparrow_{\mathcal{K}}$ is a relation on the set $\mathcal{L} \circ \Delta$ of pairs $(f, P) \in \mathcal{L} \times \Delta$ with $P \leq S_f$. For $(f, P), (g, Q) \in \mathcal{L} \circ \Delta$, we have $(f, P) \uparrow_{\mathcal{K}} (g, Q)$ if there exist $x \in N_{\mathcal{K}}(P, Q)$ and $y \in N_{\mathcal{K}}(P^f, Q^g)$ such that $xg = fy$. We say then $(f, P) \uparrow_{\mathcal{K}} (g, Q)$ via (x, y). One easily sees that $\uparrow_{\mathcal{K}}$ is reflexive and transitive. Moreover, $(f, P) \uparrow_{\mathcal{K}} (f, S_f)$ via $(\mathbf{1}, \mathbf{1})$. An element $f \in \mathcal{L}$ is called *$\uparrow_{\mathcal{K}}$-maximal* if (f, S_f) is maximal with respect to the relation $\uparrow_{\mathcal{K}}$ (i.e., if $(f, S_f) \uparrow_{\mathcal{K}} (g, Q)$ implies $(g, Q) \uparrow_{\mathcal{K}} (f, S_f)$ for any $(g, Q) \in \mathcal{L} \circ \Delta$). We summarize some important technical properties of the relation $\uparrow_{\mathcal{K}}$ in the following lemma.

3.3. *The following hold*:

(a) *Every element of $N_{\mathcal{L}}(S)$ is $\uparrow_{\mathcal{K}}$-maximal. In particular, every element of S is $\uparrow_{\mathcal{K}}$-maximal.*

(b) *If $f \in \mathcal{L}$ is $\uparrow_{\mathcal{K}}$-maximal, then $T \leq S_f$.*

(c) *(Stellmacher's splitting lemma) Let $(x, f) \in \boldsymbol{D}$ such that $x \in \mathcal{K}$ and f is $\uparrow_{\mathcal{K}}$-maximal. Then $S_{(x,f)} = S_{xf}$.*

Proof. Property (a) is [Chermak 2015, Lemma 3.7(a)], (b) is [op. cit., Proposition 3.9], and (c) is [op. cit., Lemma 3.12]. □

The relation $\uparrow_{\mathcal{K}}$ is crucial for defining a quotient locality $\mathcal{L}/\mathcal{K}$ somewhat analogously to quotients of groups. A *coset* of $\mathcal{K}$ in $\mathcal{L}$ is of the form

$$\mathcal{K}f = \{kf : k \in \mathcal{K},\ (k, f) \in \boldsymbol{D}\}$$

for some $f \in \mathcal{L}$. A *maximal coset* of $\mathcal{K}$ is a coset which is maximal with respect to inclusion among the cosets of $\mathcal{K}$ in $\mathcal{L}$. The set of these maximal cosets is denoted by $\mathcal{L}/K$.

3.4. *The following hold*:

(a) *$f \in \mathcal{L}$ is $\uparrow_{\mathcal{K}}$-maximal if and only if $\mathcal{K}f$ is a maximal coset.*

(b) *The maximal cosets of $\mathcal{K}$ form a partition of $\mathcal{L}$.*

Proof. This is [op. cit., Proposition 3.14(b),(c),(d)]. □

The reader might note that what we call a coset would be more precisely called a right coset. The distinction does however not matter very much, since we are mostly interested in the maximal cosets and, by [op. cit., Proposition 3.14(a)], we have $\mathcal{K}f = f\mathcal{K}$ for any $\uparrow_{\mathcal{K}}$-maximal element $f \in \mathcal{L}$. By 3.4(b), we can define a map

$$\rho : \mathcal{L} \to \bar{\mathcal{L}} := \mathcal{L}/\mathcal{K}$$

sending $f \in \mathcal{L}$ to the unique maximal coset of $\mathcal{K}$ containing f. This should be thought of as a "quotient map". We adopt the bar notation similarly as used for groups. Thus, if X is an element or a subset of $\mathcal{L}$, then $\overline{X}$ denotes the image of X under ρ. Furthermore, if X is an element or a subset of $\boldsymbol{W}(\mathcal{L})$ then $\overline{X}$ denotes the image of X under ρ^*, where ρ^* denotes the map $\boldsymbol{W}(\mathcal{L}) \to \boldsymbol{W}(\mathcal{L})$ with $(f_1, \dots, f_n)\rho^* = (f_1\rho, \dots, f_n\rho)$. In particular,

$$\overline{\boldsymbol{D}} = \boldsymbol{D}\rho^*.$$

We note:

3.5. *Let $f, g \in \mathcal{L}$ such that $\bar{g} = \bar{f}$ and f is $\uparrow_{\mathcal{K}}$-maximal. Then $g \in \mathcal{K}f$.*

Proof. By 3.4(a), $\mathcal{K}f$ is a maximal coset, so $\bar{g} = \bar{f} = \mathcal{K}f$ by the definition of ρ. Hence, again by the definition of ρ, $g \in \mathcal{K}f$. □

Recall the definition of a homomorphism of a partial groups from [Chermak 2013, Definition 3.1] and [Chermak 2015, Definition 1.11]. By [op. cit., Lemma 3.16], there is a unique mapping $\overline{\Pi} : \overline{\boldsymbol{D}} \to \bar{\mathcal{L}}$ and a unique involutory bijection $\bar{f} \mapsto \bar{f}^{-1}$ such that $\bar{\mathcal{L}}$ with these structures is a partial group and ρ is a homomorphism of partial groups. Since ρ is a homomorphism, we have $\overline{\Pi}(\bar{v}) = \overline{\Pi}(v\rho^*) = \Pi(v)\rho = \overline{\Pi(v)}$ for $v \in \boldsymbol{D}$ and $\bar{f}^{-1} = \overline{f^{-1}}$ by the definition of a homomorphism of partial groups and by [op. cit., Lemma 1.13]. In particular, $\bar{\boldsymbol{1}} = \overline{\Pi}(\varnothing)$ is the identity element in $\bar{\mathcal{L}}$. So ρ has kernel $\ker(\rho) = \{f \in \mathcal{L} : \bar{f} = \bar{\boldsymbol{1}}\} = \mathcal{K}\boldsymbol{1} = \mathcal{K}$. By [op. cit., Proposition 4.2], $(\bar{\mathcal{L}}, \bar{\Delta}, \bar{S})$ is a locality for $\bar{\Delta} := \{\overline{P} : P \in \Delta\}$. We will use this important fact throughout without further reference. We remark:

3.6. *Let $v = (f_1, \dots, f_n) \in \boldsymbol{W}(\mathcal{L})$ such that each f_i is $\uparrow_{\mathcal{K}}$-maximal and $\bar{v} \in \overline{\boldsymbol{D}}$. Then $v \in \boldsymbol{D}$ and $\overline{\Pi}(\bar{v}) = \overline{\Pi(v)}$.*

Proof. As $\bar{v} \in \overline{\boldsymbol{D}}$, there is $u = (g_1, \dots, g_n) \in \boldsymbol{D}$ such that $\bar{u} = \bar{v}$. Then $\overline{g_i} = \overline{f_i}$ for $i = 1, \dots, n$, i.e., $g_i \in \mathcal{K}f_i$ by 3.5. Now by [op. cit., Proposition 3.14(e)], $v \in \boldsymbol{D}$. As seen above, since ρ is a homomorphism of partial groups, $\overline{\Pi}(\bar{v}) = \overline{\Pi(v)}$. □

There is a nice correspondence between the partial subgroups of $\mathcal{L}$ containing $\mathcal{K}$ and the partial subgroups of $\bar{\mathcal{L}}$.

3.7. *Let $\mathfrak{H}$ be the set of partial subgroups of $\mathcal{L}$ containing $\mathcal{K}$.*

(a) *Let $\mathcal{H} \in \mathfrak{H}$. Then the maximal cosets of $\mathcal{K}$ contained in $\mathcal{H}$ form a partition of $\mathcal{H}$.*

(b) *Write $\overline{\mathfrak{H}}$ for the set of partial subgroups of $\bar{\mathcal{L}}$. Then the map $\mathfrak{H} \to \overline{\mathfrak{H}}$ with $\mathcal{H} \mapsto \overline{\mathcal{H}}$ is well defined and a bijection. Moreover, for any $\mathcal{H} \in \mathfrak{H}$, we have $\overline{\mathcal{H}} \trianglelefteq \bar{\mathcal{L}}$ if and only if $\mathcal{H} \trianglelefteq \mathcal{L}$.*

Proof. Property (a) is [Chermak 2015, Lemma 3.15]. The map ρ is a homomorphism of partial groups and $(\bar{\mathcal{L}}, \bar{\Delta}, \bar{S})$ is a locality. From the way $\bar{\boldsymbol{D}}$ and $\bar{\Delta}$ are defined, it follows that ρ is a projection in the sense of [op. cit., Definition 4.5]. Hence, property (b) is a reformulation of [op. cit., Proposition 4.8]. $\square$

3.8. *For any subset X of $\mathcal{L}$ and for any partial subgroup $\mathcal{H}$ of $\mathcal{L}$ containing $\mathcal{K}$, $\overline{X} \cap \overline{\mathcal{H}} = \overline{X \cap \mathcal{H}}$.*

Proof. Clearly, $\overline{X \cap \mathcal{H}} \subseteq \overline{X} \cap \overline{\mathcal{H}}$. Let now $x \in X$ such that $\bar{x} \in \overline{\mathcal{H}}$. Then there exists $h \in \mathcal{H}$ such that $\bar{x} = \bar{h}$ and, by 3.7(a), we may choose h such that $\mathcal{K}h$ is a maximal coset. By the definition of ρ, this means $x \in \mathcal{K}h \subseteq \mathcal{H}$ and hence $x \in X \cap \mathcal{H}$. Thus $\bar{x} \in \overline{X \cap \mathcal{H}}$, proving $\overline{X} \cap \overline{\mathcal{H}} \subseteq \overline{X \cap \mathcal{H}}$. $\square$

3.9. *Let $R \leq S$. Then $\{f \in \mathcal{L} : \bar{f} \in \bar{R}\} = \mathcal{K}R$.*

Proof. Clearly, $\bar{f} \in \bar{R}$ for any $f \in \mathcal{K}R$, as $\mathcal{K}$ is the kernel of ρ. Let now $f \in \mathcal{L}$ and $r \in R$ with $\bar{f} = \bar{r}$. As every element of S is $\uparrow_{\mathcal{K}}$-maximal by 3.3(a), it follows from 3.5 that $f \in \mathcal{K}r \subseteq \mathcal{K}R$. This proves the assertion. $\square$

3.10. *Let $T \leq R \leq S$. Then $R = \{s \in S : \bar{s} \in \bar{R}\}$ and $N_{\bar{S}}(\bar{R}) = \overline{N_S(R)}$.*

Proof. By 3.9 and the Dedekind lemma (2.1), we have $\{s \in S : \bar{s} \in \bar{R}\} = S \cap (\mathcal{K}R) = (S \cap \mathcal{K})R = TR = R$. Moreover, for any element $t \in S$ with $\bar{t} \in N_{\bar{S}}(\bar{R})$ and any $r \in R$, we have $\overline{r^t} = \bar{r}^{\bar{t}} \in \bar{R}$, so $r^t \in \{s \in S : \bar{s} \in \bar{R}\} = R$. Hence, $N_{\bar{S}}(\bar{R}) \leq \overline{N_S(R)}$. As ρ is a homomorphism of partial groups, $\overline{N_S(R)} \subseteq N_{\bar{S}}(\bar{R})$, so the assertion holds. $\square$

3.11. *For every $f \in \mathcal{L}$ such that f is $\uparrow_{\mathcal{K}}$-maximal, we have $\overline{S_f} = \bar{S}_{\bar{f}}$*

Proof. Set $P = S_f$ and $Q = P^f$. As ρ is a homomorphism of partial groups, one easily observes that $\bar{P} \subseteq \bar{S}_{\bar{f}}$. As $(\bar{\mathcal{L}}, \bar{\Delta}, \bar{S})$ is a locality, $\bar{S}_{\bar{f}}$ is a p-group. So assuming the assertion is wrong, there exists $a \in S$ such that $\bar{a} \in N_{\bar{S}_{\bar{f}}}(\bar{P}) \backslash \bar{P}$. As f is $\uparrow$-maximal, $T \leq P = S_f$ by 3.3(b). Hence, by 3.10 applied with P in the role of R, $\bar{a} \in \overline{N_S(P)}$. So by 3.10 now applied with $N_S(P)$ in the role of R, $a \in N_S(P)$. Using 2.5(a),(b), we conclude that $A := P\langle a \rangle$ is a p-subgroup of the group $N_{\mathcal{L}}(P)$ and that A^f is a p-subgroup of the group $N_{\mathcal{L}}(Q)$. As $\overline{A^f} \subseteq \bar{S}$, we have $\overline{A^f} \subseteq N_{\bar{S}}(\bar{Q})$. By 3.2(a), $T = T^f \leq Q$. Thus, by 3.10, $\overline{A^f} \subseteq \overline{N_S(Q)}$. Now 3.9 yields $A^f \subseteq (\mathcal{K}N_S(Q)) \cap N_{\mathcal{L}}(Q) = N_{\mathcal{K}}(Q)N_S(Q)$, where the last equality uses the Dedekind lemma (2.1). Recall that $N_{\mathcal{L}}(Q)$ is a finite group. Clearly, $N_{\mathcal{K}}(Q)$ is a normal subgroup of $N_{\mathcal{L}}(Q)$. It follows from 3.2(b) that $T \in \mathrm{Syl}_p(N_{\mathcal{K}}(Q))$. So $N_S(Q) \in \mathrm{Syl}_p(N_{\mathcal{K}}(Q)N_S(Q))$ and by Sylow's theorem, there exists $c \in N_{\mathcal{K}}(Q)$ such that $A^{fc} \leq N_S(Q)$. Then $(f, P) \uparrow_{\mathcal{K}} (fc, A)$ via $(\mathbf{1}, c)$ contrary to f being $\uparrow_{\mathcal{K}}$-maximal. $\square$

3.12. *Suppose that $f, g \in \mathcal{L}$ such that $\bar{f} = \bar{g}$, $S_f = S_g$, and f is $\uparrow_{\mathcal{K}}$-maximal. Then g is $\uparrow_{\mathcal{K}}$-maximal and $\mathcal{K}f = \mathcal{K}g$.*

Proof. As f is $\uparrow_{\mathcal{K}}$-maximal and $\bar{f} = \bar{g}$, we have $g \in \mathcal{K}f$ by 3.5, i.e., there exists $k \in \mathcal{K}$ with $(k, f) \in \boldsymbol{D}$ and $g = kf$. By Stellmacher's splitting lemma 3.3(c), we have $S_g = S_{kf} = S_{(k,f)}$. Hence, $S_f = S_g = S_{(k,f)}$ and thus $k \in N_{\mathcal{L}}(S_f)$. By 2.5(c), $k^{-1} \in N_{\mathcal{L}}(S_f)$ and $(k^{-1}, k, f) \in \boldsymbol{D}$ as via S_f. Hence, $(k^{-1}, g) = (k^{-1}, kf) \in \boldsymbol{D}$, $k^{-1}g = k^{-1}(kf) = k^{-1}kf = (k^{-1}k)f = f$ and $S_f^g = S_f^f$. This shows that $(f, S_f) \uparrow_{\mathcal{K}} (g, S_f)$ via $(k^{-1}, \mathbf{1})$. We conclude that g is $\uparrow_{\mathcal{K}}$-maximal as f is $\uparrow_{\mathcal{K}}$-maximal and $\uparrow_{\mathcal{K}}$ is transitive. By 3.4, $\mathcal{K}g$ and $\mathcal{K}f$ are both maximal cosets, and the maximal cosets of $\mathcal{K}$ form a partition of $\mathcal{L}$. So it follows that $\mathcal{K}f = \mathcal{K}g$. □

4. Proof of Theorem 1

Throughout this section assume the hypothesis of Theorem 1. Set

$$\mathcal{K} := \mathcal{M} \cap \mathcal{N}.$$

Observe that $\mathcal{K}$ is a partial normal subgroup of $\mathcal{L}$. As in Section 3, let

$$\rho : \mathcal{L} \to \bar{\mathcal{L}} := \mathcal{L}/\mathcal{K}$$

be the quotient map sending $f \in \mathcal{L}$ to the unique maximal coset of $\mathcal{K}$ containing f, and use the bar notation as introduced there. Set

$$T := \mathcal{K} \cap S.$$

4.1. $\overline{\mathcal{M}} \cap \overline{\mathcal{N}} = 1$.

Proof. As $\mathcal{K}$ is contained in $\mathcal{M}$ and $\mathcal{N}$, this is a special case of 3.8. □

4.2. *We have $\overline{\mathcal{M}}\,\overline{\mathcal{N}} = \overline{\mathcal{N}}\,\overline{\mathcal{M}}$, and $\overline{\mathcal{M}}\,\overline{\mathcal{N}}$ is a partial normal subgroup of $\bar{\mathcal{L}}$. Moreover, for any $x \in \overline{\mathcal{M}}\,\overline{\mathcal{N}}$, there exist $\overline{m} \in \overline{\mathcal{M}}$ and $\bar{n} \in \overline{\mathcal{N}}$ such that $(\overline{m}, \bar{n}) \in \bar{\boldsymbol{D}}$, $x = \overline{m}\,\bar{n}$ and $\bar{S}_x = \bar{S}_{(\overline{m},\bar{n})}$.*

Proof. By 3.7(b), $\overline{\mathcal{M}}$ and $\overline{\mathcal{N}}$ are partial normal subgroups of $\bar{\mathcal{L}}$. By 4.1, $\overline{\mathcal{M}} \cap \overline{\mathcal{N}} = 1$. Hence, the assertion follows from Theorem 3.1. □

4.3. *Let $x \in \overline{\mathcal{M}}\,\overline{\mathcal{N}}$. Then there exist $m \in \mathcal{M}$ and $n \in \mathcal{N}$ with $(m, n) \in \boldsymbol{D}$ such that m, n, and mn are $\uparrow_{\mathcal{K}}$-maximal, $x = \overline{m}\,\bar{n} = \overline{mn}$ and $S_{mn} = S_{(m,n)}$.*

Proof. By 4.2, there exist $\overline{m} \in \overline{\mathcal{M}}$, $\bar{n} \in \overline{\mathcal{N}}$ such that $(\overline{m}, \bar{n}) \in \bar{\boldsymbol{D}}$, $x = \overline{m}\,\bar{n}$, and $\bar{S}_x = \bar{S}_{(\overline{m},\bar{n})}$. By 3.7(a), we may furthermore choose preimages $m \in \mathcal{M}$ and $n \in \mathcal{N}$ of $\overline{m}$ and $\bar{n}$ such that m and n are $\uparrow_{\mathcal{K}}$-maximal. Then, by 3.2(a) and 3.3(b), $m, n \in N_{\mathcal{L}}(T)$. By 3.6, $(m, n) \in \boldsymbol{D}$ and $\overline{m}\,\bar{n} = \overline{mn}$. It remains to prove that $S_{mn} = S_{(m,n)}$ and that mn is $\uparrow_{\mathcal{K}}$-maximal. As an intermediate step we prove the following two properties:

(4-1) $\quad S_f \subseteq S_{(m,n)}$ for every $f \in \mathcal{L}$ with $\bar{f} = x$.

(4-2) $\quad S_f = S_{(m,n)}$ for every $\uparrow_{\mathcal{K}}$-maximal element $f \in \mathcal{L}$ with $\bar{f} = x$.

For the proof of (4-1) and (4-2) note first that, by 3.11, $\bar{S}_{\bar{m}} = \overline{S_m}$ and $\bar{S}_{\bar{n}} = \overline{S_n}$ as m and n are $\uparrow_{\mathcal{K}}$-maximal. Hence,

$$\bar{S}_x = \bar{S}_{(\bar{m},\bar{n})} = \{\bar{s} : \bar{s} \in \bar{S}_{\bar{m}},\ \bar{s}^{\bar{m}} \in \bar{S}_{\bar{n}}\} = \{\bar{s} : s \in S_m,\ \bar{s}^{\bar{m}} \in \overline{S_n}\}.$$

If $s \in S_m$ then, by definition of S_m, $(m^{-1}, s, m) \in \boldsymbol{D}$ and $s^m \in S$. Moreover, as ρ is a homomorphism of partial groups, $\overline{s^m} = \bar{s}^{\bar{m}}$. So $\bar{s}^{\bar{m}} \in \overline{S_n}$ is equivalent to $s^m \in S_n$ by 3.10 since $T \leq S_n$. Hence,

$$\bar{S}_x = \{\bar{s} : s \in S_m,\ s^m \in S_n\} = \overline{S_{(m,n)}}.$$

As $m, n \in N_{\mathcal{L}}(T)$, $T \leq S_{(m,n)}$. Clearly, $\overline{S_f} \subseteq \bar{S}_x$ for every $f \in \mathcal{L}$ with $\bar{f} = x$. If such f is in addition $\uparrow_{\mathcal{K}}$-maximal, then $\overline{S_f} = \bar{S}_x$ and $T \leq S_f$ by 3.11 and 3.3(b). Now (4-1) and (4-2) follow from 3.10. As $\overline{mn} = \bar{m}\bar{n} = x$, (4-1) yields in particular $S_{mn} \subseteq S_{(m,n)}$ and thus $S_{mn} = S_{(m,n)}$ by 2.5(e). Choosing $f \in \mathcal{L}$ to be $\uparrow_{\mathcal{K}}$-maximal with $\bar{f} = x$, we obtain from (4-2) that $S_f = S_{(m,n)} = S_{mn}$. So mn is $\uparrow_{\mathcal{K}}$-maximal by 3.12 completing the proof. □

4.4. *Let $f \in \mathcal{L}$ with $\bar{f} \in \overline{\mathcal{M}}\overline{\mathcal{N}}$. Then $f \in \mathcal{M}\mathcal{N}$ and there exist $m \in \mathcal{M}$, $n \in \mathcal{N}$ with $(m, n) \in \boldsymbol{D}$, $f = mn$, and $S_f = S_{(m,n)}$.*

Proof. By 4.3, we can choose $m \in \mathcal{M}$ and $n \in \mathcal{N}$ with $(m, n) \in \boldsymbol{D}$ such that mn is $\uparrow_{\mathcal{K}}$-maximal, $\bar{f} = \overline{mn}$ and $S_{mn} = S_{(m,n)}$. Then there exists $k \in \mathcal{K}$ with $(k, mn) \in \boldsymbol{D}$ and $f = k(mn)$. As $S_{mn} = S_{(m,n)}$, it follows that $S_{(k,mn)} = S_{(k,m,n)}$ and $(k, m, n) \in \boldsymbol{D}$ by 2.5(e). Hence, $(km, n) \in \boldsymbol{D}$ and $f = (km)n$ by the axioms of a partial group. As $\mathcal{K} \subseteq \mathcal{M}$, we have $km \in \mathcal{M}$ and so $f = (km)n \in \mathcal{M}\mathcal{N}$. It is now sufficient to show that $S_{(km,n)} = S_f$. As mn is $\uparrow_{\mathcal{K}}$-maximal, it follows from Stellmacher's splitting lemma 3.3(c) that $S_f = S_{k(mn)} = S_{(k,mn)} = S_{(k,m,n)} \subseteq S_{(km,n)}$. By 2.5(e), $S_{(km,n)} \subseteq S_{(km)n} = S_f$. So $S_f = S_{(km,n)}$, proving the assertion. □

Proof of Theorem 1. By 4.2 and 3.7(b), there exists a partial normal subgroup $\mathcal{H}$ of $\mathcal{L}$ containing $\mathcal{K}$ such that $\overline{\mathcal{H}} = \overline{\mathcal{M}}\overline{\mathcal{N}} = \overline{\mathcal{N}}\overline{\mathcal{M}}$. Then for any $f \in \mathcal{L}$ with $\bar{f} \in \overline{\mathcal{M}}\overline{\mathcal{N}}$, there exists $h \in \mathcal{H}$ with $\bar{f} = \bar{h}$. By 3.7(a), we can choose h to by $\uparrow_{\mathcal{K}}$-maximal. So by 3.5, $f \in \mathcal{K}h \subseteq \mathcal{H}$. This shows $\mathcal{H} = \{f \in \mathcal{L} : \bar{f} \in \overline{\mathcal{M}}\overline{\mathcal{N}}\}$.

We need to prove that $\mathcal{H} = \mathcal{M}\mathcal{N} = \mathcal{N}\mathcal{M}$. As the situation is symmetric in $\mathcal{M}$ and $\mathcal{N}$, it is enough to prove that $\mathcal{H} = \mathcal{M}\mathcal{N}$. Since ρ is a homomorphism, for any $m \in \mathcal{M}$ and $n \in \mathcal{N}$ with $(m, n) \in \boldsymbol{D}$, we have $\overline{mn} = \bar{m}\bar{n} \in \overline{\mathcal{M}}\overline{\mathcal{N}}$ and thus $mn \in \mathcal{H}$. Hence, $\mathcal{M}\mathcal{N} \subseteq \mathcal{H}$. By 4.4, we have $\mathcal{H} \subseteq \mathcal{M}\mathcal{N}$, so $\mathcal{H} = \mathcal{M}\mathcal{N}$. Moreover, 4.4 shows that for every $f \in \mathcal{M}\mathcal{N}$, there exists $m \in \mathcal{M}$ and $n \in \mathcal{N}$ such that $(m, n) \in \boldsymbol{D}$, $f = mn$, and $S_f = S_{(m,n)}$. So it only remains to prove that $S \cap (\mathcal{M}\mathcal{N}) = (S \cap \mathcal{M})(S \cap \mathcal{N})$. Clearly, $(S \cap \mathcal{M})(S \cap \mathcal{N}) \subseteq S \cap (\mathcal{M}\mathcal{N})$. Let now $s \in S \cap (\mathcal{M}\mathcal{N})$. By what we just said, there exists $m \in \mathcal{M}$ and $n \in \mathcal{N}$ with $(m, n) \in \boldsymbol{D}$, $s = mn$, and $S_s = S_{(m,n)}$. As $S_s = S$, it follows that $m, n \in G := N_{\mathcal{L}}(S)$. By 2.5(a), G is a subgroup of $\mathcal{L}$. Furthermore, property (L1) in the definition of a locality implies that S is a Sylow

p-subgroup of G. Note that $X := G \cap \mathcal{M}$ and $Y := G \cap \mathcal{N}$ are normal subgroups of G. Hence, $s = mn \in (XY) \cap S = (X \cap S)(Y \cap S) = (\mathcal{M} \cap S)(\mathcal{N} \cap S)$ completing the proof. □

5. The Proof of Theorem 2

Throughout, let $(\mathcal{L}, \Delta, S)$ be a locality with partial normal subgroups $\mathcal{N}_1, \dots, \mathcal{N}_l$. We prove Theorem 2 in a series of lemmas.

5.1. *Let $1 \leq k < l$ such that the products $\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_k$ and $\mathcal{N}_{k+1}\mathcal{N}_{k+2} \cdots \mathcal{N}_l$ are partial normal subgroups. Suppose furthermore that for any $f \in \mathcal{N}_1 \cdots \mathcal{N}_k$ and any $g \in \mathcal{N}_{k+1} \cdots \mathcal{N}_l$ there exist $u = (n_1, \dots, n_k)$, $v = (n_{k+1}, \dots, n_l) \in \boldsymbol{D}$ such that $n_i \in \mathcal{N}_i$ for $i = 1, \dots, l$, $f = \Pi(u)$, $g = \Pi(v)$, $S_f = S_u$ and $S_g = S_v$. Then*

$$\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l = (\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_k)(\mathcal{N}_{k+1}\mathcal{N}_{k+2} \cdots \mathcal{N}_l)$$

is a partial normal subgroup of $\mathcal{L}$, and for every $h \in \mathcal{N}_1, \dots, \mathcal{N}_l$ there exists $w = (n_1, \dots, n_l) \in \boldsymbol{D}$ such that $n_i \in \mathcal{N}_i$ for $i = 1, \dots, l$, $h = \Pi(w)$, and $S_h = S_w$.

Proof. By Theorem 1, $(\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_k)(\mathcal{N}_{k+1}\mathcal{N}_{k+2} \cdots \mathcal{N}_l)$ is a partial normal subgroup of $\mathcal{L}$. If $w = (n_1, \dots, n_l) \in \boldsymbol{D}$ with $n_i \in \mathcal{N}_i$ for $i = 1, \dots, l$, then $u = (n_1, \dots, n_k)$, $v = (n_{k+1}, \dots, n_l) \in \boldsymbol{D}$, and

$$\Pi(w) = \Pi(\Pi(u), \Pi(v)) \in (\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_k)(\mathcal{N}_{k+1}\mathcal{N}_{k+2} \cdots \mathcal{N}_l).$$

This proves that $\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l \subseteq (\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_k)(\mathcal{N}_{k+1}\mathcal{N}_{k+2} \cdots \mathcal{N}_l)$. To prove the converse inclusion, let

$$h \in (\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_k)(\mathcal{N}_{k+1}\mathcal{N}_{k+2} \cdots \mathcal{N}_l).$$

Then by Theorem 1, there exist $f \in \mathcal{N}_1 \cdots \mathcal{N}_k$ and $g \in \mathcal{N}_{k+1} \cdots \mathcal{N}_l$ such that $(f, g) \in \boldsymbol{D}$, $h = fg$, and $S_h = S_{(f,g)}$. By assumption, there exist $u = (n_1, \dots, n_k)$ and $v = (n_{k+1}, \dots, n_l) \in \boldsymbol{D}$ such that $n_i \in \mathcal{N}_i$ for $i = 1, \dots, l$, $f = \Pi(u)$, $g = \Pi(v)$, $S_f = S_u$, and $S_g = S_v$. Then $S_h = S_{(f,g)} = S_{u \circ v}$, $u \circ v \in \boldsymbol{D}$ via S_h, and

$$h = fg = \Pi(\Pi(u), \Pi(v)) = \Pi(u \circ v) \in \mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l,$$

proving the assertion. □

5.2. (*a*) *The product $\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l$ is a partial normal subgroup, and for every $f \in \mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l$ there exists $w = (n_1, \dots, n_l) \in \boldsymbol{D}$ such that $n_i \in \mathcal{N}_i$ for $i = 1, \dots, l$, $f = \Pi(w)$, and $S_f = S_w$.*

(*b*) *For every $1 \leq k < l$, we have*

$$\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l = (\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_k)(\mathcal{N}_{k+1}\mathcal{N}_{k+2} \cdots \mathcal{N}_l).$$

Proof. We prove this by induction on l. Clearly, the claim is true for $l = 1$. Assume now $l > 1$. Then there exists always $1 \le k < l$. For any such k, it follows by induction (one time applied with $\mathcal{N}_1, \ldots, \mathcal{N}_k$ and one time applied with $\mathcal{N}_{k+1}, \ldots, \mathcal{N}_l$ in place of $\mathcal{N}_1, \ldots, \mathcal{N}_l$) that the hypothesis of 5.1 is fulfilled, so the assertion follows. $\square$

5.3. *Let $\sigma \in S_l$ be a permutation. Then $\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l = \mathcal{N}_{1\sigma}\mathcal{N}_{2\sigma} \cdots \mathcal{N}_{l\sigma}$.*

Proof. We may assume that $\sigma = (i, i+1)$ for some $1 \le i < l$, as S_l is generated by transpositions of this form. Note that $\mathcal{N}_1 \cdots \mathcal{N}_{i-1}$, $\mathcal{N}_i\mathcal{N}_{i+1}$ and $\mathcal{N}_{i+2} \cdots \mathcal{N}_l$ are partial normal subgroups by 5.2(a), where it is understood that $\mathcal{N}_r \cdots \mathcal{N}_s = \{\mathbf{1}\}$ if $r > s$. By Theorem 1, we have $\mathcal{M}\mathcal{N} = \mathcal{N}\mathcal{M}$ for any two partial normal subgroups. Using this fact and 5.2(b) repeatedly, we obtain

$$\begin{aligned}
\mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_l &= (\mathcal{N}_1 \cdots \mathcal{N}_{i-1})(\mathcal{N}_i\mathcal{N}_{i+1})(\mathcal{N}_{i+2} \cdots \mathcal{N}_l) \\
&= (\mathcal{N}_1 \cdots \mathcal{N}_{i-1})(\mathcal{N}_{i+1}\mathcal{N}_i)(\mathcal{N}_{i+2} \cdots \mathcal{N}_l) \\
&= \mathcal{N}_{1\sigma}\mathcal{N}_{2\sigma} \cdots \mathcal{N}_{l\sigma}.
\end{aligned}$$
$\square$

Proof of Theorem 2. It follows from 5.2(a) that $\mathcal{N}_1 \cdots \mathcal{N}_l$ is a partial normal subgroup and that (c) holds. Property (a) is 5.2(b), and property (b) is 5.3. $\square$

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LUSZTIG INDUCTION AND ℓ-BLOCKS OF FINITE REDUCTIVE GROUPS

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To the memory of Robert Steinberg

We present a unified parametrisation of ℓ-blocks of quasisimple finite groups of Lie type in nondefining characteristic via Lusztig's induction functor in terms of e-Jordan-cuspidal pairs and e-Jordan quasicentral cuspidal pairs.

1. Introduction

The work of Fong and Srinivasan for classical matrix groups and of Schewe for certain blocks of groups of exceptional type exhibited a close relation between the ℓ-modular block structure of groups of Lie type and the decomposition of Lusztig's induction functor, defined in terms of ℓ-adic cohomology. This connection was extended to unipotent blocks of arbitrary finite reductive groups and large primes ℓ by Broué–Malle–Michel [1993], to all unipotent blocks by Cabanes–Enguehard [1994] and Enguehard [2000], to arbitrary blocks for primes $\ell \geq 7$ by Cabanes–Enguehard [1999], to nonquasi-isolated blocks by Bonnafé–Rouquier [2003] and to quasi-isolated blocks of exceptional groups at bad primes by the authors [2013].

It is the main purpose of this paper to unify and extend all of the preceding results in particular from [Cabanes and Enguehard 1999] so as to establish a statement in its largest possible generality, without restrictions on the prime ℓ, the type of group or the type of block, in terms of e-Jordan quasicentral cuspidal pairs (see Section 2 for the notation used).

Theorem A. *Let $\boldsymbol{H}$ be a simple algebraic group of simply connected type with a Frobenius endomorphism $F : \boldsymbol{H} \to \boldsymbol{H}$ endowing $\boldsymbol{H}$ with an $\mathbb{F}_q$-rational structure. Let $\boldsymbol{G}$ be an F-stable Levi subgroup of $\boldsymbol{H}$. Let ℓ be a prime not dividing q and set $e = e_\ell(q)$.*

(a) *For any e-Jordan-cuspidal pair $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ such that $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$, there exists a unique ℓ-block $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}^F$ such that all irreducible constituents of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ lie in $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$.*

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(b) *The map* $\Xi : (\boldsymbol{L}, \lambda) \mapsto b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ *is a surjection from the set of* $\boldsymbol{G}^F$*-conjugacy classes of e-Jordan-cuspidal pairs* $(\boldsymbol{L}, \lambda)$ *of* $\boldsymbol{G}$ *such that* $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ *to the set of* ℓ*-blocks of* $\boldsymbol{G}^F$.

(c) *The map* Ξ *restricts to a surjection from the set of* $\boldsymbol{G}^F$*-conjugacy classes of e-Jordan* quasicentral *cuspidal pairs* $(\boldsymbol{L}, \lambda)$ *of* $\boldsymbol{G}$ *such that* $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ *to the set of* ℓ*-blocks of* $\boldsymbol{G}^F$.

(d) *For* $\ell \geq 3$ *the map* Ξ *restricts to a bijection between the set of* $\boldsymbol{G}^F$*-conjugacy classes of e-Jordan quasicentral cuspidal pairs* $(\boldsymbol{L}, \lambda)$ *of* $\boldsymbol{G}$ *with* $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ *and the set of* ℓ*-blocks of* $\boldsymbol{G}^F$.

(e) *The map* Ξ *itself is bijective if* $\ell \geq 3$ *is good for* $\boldsymbol{G}$, *and moreover* $\ell \neq 3$ *if* $\boldsymbol{G}^F$ *has a factor* ${}^3D_4(q)$.

The restrictions in (d) and (e) are necessary (see Remark 3.15 and Example 3.16).

In fact, part (a) of the preceding result is a special case of the following characterisation of the ℓ'-characters in a given ℓ-block in terms of Lusztig induction:

Theorem B. *In the setting of Theorem A let* b *be an* ℓ*-block of* $\boldsymbol{G}^F$ *and denote by* $\mathcal{L}(b)$ *the set of e-Jordan cuspidal pairs* $(\boldsymbol{L}, \lambda)$ *of* $\boldsymbol{G}$ *such that* $\big\{\chi \in \mathrm{Irr}(b) \mid \langle \chi, R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)\rangle \neq 0\big\} \neq \varnothing$. *Then*

$$\mathrm{Irr}(b) \cap \mathcal{E}(\boldsymbol{G}^F, \ell') = \big\{\chi \in \mathcal{E}(\boldsymbol{G}^F, \ell') \mid \exists\, (\boldsymbol{L}, \lambda) \in \mathcal{L}(b) \text{ with } (\boldsymbol{L}, \lambda) \ll_e (\boldsymbol{G}, \chi)\big\}.$$

Note that at present, it is not known whether Lusztig induction $R_{\boldsymbol{L}}^{\boldsymbol{G}}$ is independent of the parabolic subgroup containing the Levi subgroup $\boldsymbol{L}$ used to define it. Our proofs will show, though, that in our case $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ is defined unambiguously.

An important motivation for this work comes from the recent reductions of most long-standing famous conjectures in modular representation theory of finite groups to questions about quasisimple groups. Among the latter, the quasisimple groups of Lie type form the by far most important part. A knowledge and suitable inductive description of the ℓ-blocks of these groups is thus of paramount importance for an eventual proof of those central conjectures. Our results are specifically tailored for use in an inductive approach by considering groups that occur as Levi subgroups inside groups of Lie type of simply connected type, that is, inside quasisimple groups.

Our paper is organised as follows; in Section 2, we set up e-Jordan (quasicentral) cuspidal pairs and discuss some of their properties. In Section 3 we prove Theorem A (see Theorem 3.14) on parametrising ℓ-blocks by e-Jordan-cuspidal and e-Jordan quasicentral cuspidal pairs and Theorem B (see Theorem 3.6) on characterising ℓ'-characters in blocks. The crucial case turns out to be when $\ell = 3$. In particular, the whole section on pages 287–289 is devoted to the situation of extra-special defect groups of order 27, excluded in [Cabanes and Enguehard 1999], which eventually turns out to behave just as the generic case. An important ingredient of Section 3 is Theorem 3.4, which shows that the distribution of ℓ'-characters in ℓ-blocks is preserved under Lusztig induction from e-split Levi subgroups. Finally, in Section 4 we collect some results relating e-Jordan-cuspidality and usual e-cuspidality.

2. Cuspidal pairs

Throughout this section, $\boldsymbol{G}$ is a connected reductive linear algebraic group over the algebraic closure of a finite field of characteristic p, and $F : \boldsymbol{G} \to \boldsymbol{G}$ is a Frobenius endomorphism endowing $\boldsymbol{G}$ with an $\mathbb{F}_q$-structure for some power q of p. By $\boldsymbol{G}^*$ we denote a group in duality with $\boldsymbol{G}$ with respect to some fixed F-stable maximal torus of $\boldsymbol{G}$, with corresponding Frobenius endomorphism also denoted by F.

e-Jordan-cuspidality. Let e be a positive integer. We will make use of the terminology of Sylow e-theory (see for instance [Broué et al. 1993]). For an F-stable maximal torus $\boldsymbol{T}$, $\boldsymbol{T}_e$ denotes its Sylow e-torus. Then a Levi subgroup $\boldsymbol{L} \le \boldsymbol{G}$ is called *e-split* if $\boldsymbol{L} = C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L})_e)$, and $\lambda \in \mathrm{Irr}(\boldsymbol{L}^F)$ is called *e-cuspidal* if ${}^*R^{\boldsymbol{L}}_{\boldsymbol{M}\le\boldsymbol{P}}(\lambda) = 0$ for all proper e-split Levi subgroups $\boldsymbol{M} < \boldsymbol{L}$ and any parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{L}$ containing $\boldsymbol{M}$ as Levi complement. (It is expected that Lusztig induction is in fact independent of the ambient parabolic subgroup. This would follow for example if the Mackey formula holds for $R^{\boldsymbol{G}}_{\boldsymbol{L}}$, and has been proved whenever $\boldsymbol{G}^F$ does not have any component of type ${}^2E_6(2)$, $E_7(2)$ or $E_8(2)$, see [Bonnafé and Michel 2011]. All the statements made in this section using $R^{\boldsymbol{G}}_{\boldsymbol{L}}$ are valid independent of the particular choice of parabolic subgroup — we will make clarifying remarks at points where there might be any ambiguity.)

Definition 2.1. Let $s \in \boldsymbol{G}^{*F}$ be semisimple. Following [Cabanes and Enguehard 1999, Section 1.3] we say that $\chi \in \mathcal{E}(\boldsymbol{G}^F, s)$ is *e-Jordan-cuspidal*, or *satisfies condition (J)* with respect to some $e \ge 1$ if

(J_1) $Z^\circ(C^\circ_{\boldsymbol{G}^*}(s))_e = Z^\circ(\boldsymbol{G}^*)_e$, and

(J_2) χ corresponds under Jordan decomposition (see [Digne and Michel 1991, Theorem 13.23]) to the $C_{\boldsymbol{G}^*}(s)^F$-orbit of an e-cuspidal unipotent character of $C^\circ_{\boldsymbol{G}^*}(s)^F$.

If $\boldsymbol{L} \le \boldsymbol{G}$ is e-split and $\lambda \in \mathrm{Irr}(\boldsymbol{L}^F)$ is e-Jordan-cuspidal, then $(\boldsymbol{L}, \lambda)$ is called an *e-Jordan-cuspidal pair*.

It is shown in [Cabanes and Enguehard 1999, Proposition 1.10] that χ is e-Jordan-cuspidal if and only if it satisfies the *uniform criterion*

(U): for every F-stable maximal torus $\boldsymbol{T} \le \boldsymbol{G}$ with $\boldsymbol{T}_e \not\le Z(\boldsymbol{G})$ we have ${}^*R^{\boldsymbol{G}}_{\boldsymbol{T}}(\chi) = 0$.

Remark 2.2. By [Cabanes and Enguehard 1999, Proposition 1.10(ii)] it is known that e-cuspidality implies e-Jordan-cuspidality; moreover e-Jordan-cuspidality and e-cuspidality agree at least in the following situations:

(1) when $e = 1$;

(2) for unipotent characters (see [Broué et al. 1993, Corollary 3.13]);

(3) for characters lying in an ℓ'-series where ℓ is an odd prime, good for $\boldsymbol{G}$, e is the order of q modulo ℓ and either $\ell \geq 5$ or $\ell = 3 \in \Gamma(\boldsymbol{G}, F)$ as defined in [Cabanes and Enguehard 1994, Notation 1.1] (see [Cabanes and Enguehard 1999, Theorem 4.2 and Remark 5.2]); and

(4) for characters lying in a quasi-isolated ℓ'-series of an exceptional type simple group for ℓ a bad prime (this follows by inspection of the explicit results in [Kessar and Malle 2013]).

To see the first point, assume that χ is 1-Jordan-cuspidal. Suppose if possible that χ is not 1-cuspidal. Then there exists a proper 1-split Levi subgroup $\boldsymbol{L}$ of $\boldsymbol{G}$ such that ${}^*R_{\boldsymbol{L}}^{\boldsymbol{G}}(\chi)$ is nonzero. Then ${}^*R_{\boldsymbol{L}}^{\boldsymbol{G}}(\chi)(1) \neq 0$ as ${}^*R_{\boldsymbol{L}}^{\boldsymbol{G}}$ is ordinary Harish-Chandra restriction. Hence the projection of ${}^*R_{\boldsymbol{L}}^{\boldsymbol{G}}(\chi)$ to the space of uniform functions of $\boldsymbol{L}^F$ is nonzero in contradiction to the uniform criterion (U).

It seems reasonable to expect (and that is formulated as a conjecture in [Cabanes and Enguehard 1999, Section 1.11]) that e-cuspidality and e-Jordan-cuspidality agree in general. See Section 4 below for a further discussion of this.

We first establish conservation of e-Jordan-cuspidality under some natural constructions:

Lemma 2.3. *Let $\boldsymbol{L}$ be an F-stable Levi subgroup of $\boldsymbol{G}$ and $\lambda \in \mathrm{Irr}(\boldsymbol{L}^F)$. Let $\boldsymbol{L}_0 = \boldsymbol{L} \cap [\boldsymbol{G}, \boldsymbol{G}]$ and let λ_0 be an irreducible constituent of $\mathrm{Res}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda)$. Let $e \geq 1$. Then $(\boldsymbol{L}, \lambda)$ is an e-Jordan-cuspidal pair for $\boldsymbol{G}$ if and only if $(\boldsymbol{L}_0, \lambda_0)$ is an e-Jordan-cuspidal pair for $[\boldsymbol{G}, \boldsymbol{G}]$.*

Proof. Note that $\boldsymbol{L}$ is e-split in $\boldsymbol{G}$ if and only if $\boldsymbol{L}_0$ is e-split in $\boldsymbol{G}_0$. Let $\iota : \boldsymbol{G} \hookrightarrow \tilde{\boldsymbol{G}}$ be a regular embedding. It is shown in the proof of [Cabanes and Enguehard 1999, Proposition 1.10] that condition (J) with respect to $\boldsymbol{G}$ is equivalent to condition (J) with respect to $\tilde{\boldsymbol{G}}$. Since ι restricts to a regular embedding $[\boldsymbol{G}, \boldsymbol{G}] \hookrightarrow \tilde{\boldsymbol{G}}$, the same argument shows that condition (J) with respect to $\tilde{\boldsymbol{G}}$ is equivalent to that condition with respect to $[\boldsymbol{G}, \boldsymbol{G}]$. □

Proposition 2.4. *Let $s \in \boldsymbol{G}^{*F}$ be semisimple, and $\boldsymbol{G}_1 \leq \boldsymbol{G}$ an F-stable Levi subgroup with $\boldsymbol{G}_1^*$ containing $C_{\boldsymbol{G}^*}(s)$. For $(\boldsymbol{L}_1, \lambda_1)$ an e-Jordan-cuspidal pair of $\boldsymbol{G}_1$ below $\mathcal{E}(\boldsymbol{G}_1^F, s)$ define $\boldsymbol{L} := C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L}_1)_e)$ and $\lambda := \epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{L}_1} R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda_1)$. Then $Z^\circ(\boldsymbol{L}_1)_e = Z^\circ(\boldsymbol{L})_e$, and $(\boldsymbol{L}_1, \lambda_1) \mapsto (\boldsymbol{L}, \lambda)$ defines a bijection $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ between the set of e-Jordan-cuspidal pairs of $\boldsymbol{G}_1$ below $\mathcal{E}(\boldsymbol{G}_1^F, s)$ and the set of e-Jordan-cuspidal pairs of $\boldsymbol{G}$ below $\mathcal{E}(\boldsymbol{G}^F, s)$.*

We note that the character λ and hence the bijection $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ above are independent of the choice of parabolic subgroup. This is explained in the proof below.

Proof. We first show that $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ is well-defined. Let $(\boldsymbol{L}_1, \lambda_1)$ be e-Jordan-cuspidal in $\boldsymbol{G}_1$ below $\mathcal{E}(\boldsymbol{G}_1^F, s)$, so $s \in \boldsymbol{L}_1^*$. Then $\boldsymbol{L}^* := C_{\boldsymbol{G}^*}(Z^\circ(\boldsymbol{L}_1^*)_e)$ clearly is an e-split Levi subgroup of $\boldsymbol{G}^*$. Moreover we have

$$\boldsymbol{L}_1^* = C_{\boldsymbol{G}_1^*}(Z^\circ(\boldsymbol{L}_1^*)_e) = C_{\boldsymbol{G}^*}(Z^\circ(\boldsymbol{L}_1^*)_e) \cap \boldsymbol{G}_1^* = \boldsymbol{L}^* \cap \boldsymbol{G}_1^*.$$

Now $s \in \boldsymbol{L}_1^*$ by assumption, so

$$\boldsymbol{L}_1^* = \boldsymbol{L}^* \cap \boldsymbol{G}_1^* \geq \boldsymbol{L}^* \cap C_{\boldsymbol{G}^*}(s) = C_{\boldsymbol{L}^*}(s).$$

In particular, $\boldsymbol{L}_1^*$ and $\boldsymbol{L}^*$ have a maximal torus in common, so $\boldsymbol{L}_1^*$ is a Levi subgroup of $\boldsymbol{L}^*$. Thus, passing to duals, $\boldsymbol{L}_1$ is a Levi subgroup of $\boldsymbol{L} = C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L}_1)_e)$.

We clearly have $Z^\circ(\boldsymbol{L}_1)_e \leq Z^\circ(\boldsymbol{L})_e$. For the reverse inclusion, observe that $Z^\circ(\boldsymbol{L})_e \leq \boldsymbol{L}_1$, as $\boldsymbol{L}_1$ is a Levi subgroup in $\boldsymbol{L}$, so indeed $Z^\circ(\boldsymbol{L})_e \leq Z^\circ(\boldsymbol{L}_1)_e$.

Hence by [Digne and Michel 1991, Theorem 13.25], $\lambda := \epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{L}_1} R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda_1)$ is irreducible since, as we saw above, $\boldsymbol{L}_1^* \geq C_{\boldsymbol{L}^*}(s)$. By [Digne and Michel 1991, Remark 13.28], λ is independent of the choice of parabolic subgroup of $\boldsymbol{L}$ containing $\boldsymbol{L}_1$ as Levi subgroup. Let's argue that λ is e-Jordan-cuspidal. Indeed, for any F-stable maximal torus $\boldsymbol{T} \leq \boldsymbol{L}$ we have by the Mackey-formula (which holds as one of the Levi subgroups is a maximal torus by a result of Deligne–Lusztig, see [Bonnafé and Michel 2011, Theorem 2]) that $\epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{L}_1} {}^*R_{\boldsymbol{T}}^{\boldsymbol{L}}(\lambda) = {}^*R_{\boldsymbol{T}}^{\boldsymbol{L}} R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda_1)$ is a sum of $\boldsymbol{L}^F$-conjugates of ${}^*R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\lambda_1)$. As λ_1 is e-Jordan-cuspidal, this vanishes if $\boldsymbol{T}_e \not\leq Z^\circ(\boldsymbol{L}_1)_e = Z^\circ(\boldsymbol{L})_e$. So λ satisfies condition (U), hence is e-Jordan-cuspidal, and $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ is well-defined.

It is clearly injective, since if $(\boldsymbol{L}, \lambda) = \Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}(\boldsymbol{L}_2, \lambda_2)$ for some e-cuspidal pair $(\boldsymbol{L}_2, \lambda_2)$ of $\boldsymbol{G}_1$, then $Z^\circ(\boldsymbol{L}_1)_e = Z^\circ(\boldsymbol{L})_e = Z^\circ(\boldsymbol{L}_2)_e$, whence $\boldsymbol{L}_1 = C_{\boldsymbol{G}_1}(Z^\circ(\boldsymbol{L}_1)_e) = C_{\boldsymbol{G}_1}(Z^\circ(\boldsymbol{L}_2)_e) = \boldsymbol{L}_2$, and then the bijectivity of $R_{\boldsymbol{L}_1}^{\boldsymbol{L}}$ on $\mathcal{E}(\boldsymbol{L}_1^F, s)$ shows that $\lambda_1 = \lambda_2$ as well.

We now construct an inverse map. For this, let $(\boldsymbol{L}, \lambda)$ be an e-Jordan-cuspidal pair of $\boldsymbol{G}$ below $\mathcal{E}(\boldsymbol{G}^F, s)$, and $\boldsymbol{L}^* \leq \boldsymbol{G}^*$ dual to $\boldsymbol{L}$. Set

$$\boldsymbol{L}_1^* := C_{\boldsymbol{G}_1^*}(Z^\circ(\boldsymbol{L}^*)_e) = C_{\boldsymbol{G}^*}(Z^\circ(\boldsymbol{L}^*)_e) \cap \boldsymbol{G}_1^* = \boldsymbol{L}^* \cap \boldsymbol{G}_1^*,$$

an e-split Levi subgroup of $\boldsymbol{G}_1^*$. Note that $s \in \boldsymbol{L}^*$, so there exists some maximal torus $\boldsymbol{T}^*$ of $\boldsymbol{G}^*$ with $\boldsymbol{T}^* \leq C_{\boldsymbol{G}^*}(s) \leq \boldsymbol{G}_1^*$, whence $\boldsymbol{L}_1^*$ is a Levi subgroup of $\boldsymbol{L}^*$. Now again

$$\boldsymbol{L}_1^* = \boldsymbol{L}^* \cap \boldsymbol{G}_1^* \geq \boldsymbol{L}^* \cap C_{\boldsymbol{G}^*}(s) = C_{\boldsymbol{L}^*}(s).$$

So the dual $\boldsymbol{L}_1 := C_{\boldsymbol{G}_1}(Z^\circ(\boldsymbol{L})_e)$ is a Levi subgroup of $\boldsymbol{L}$ such that $\epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R_{\boldsymbol{L}_1}^{\boldsymbol{L}}$ preserves irreducibility on $\mathcal{E}(\boldsymbol{L}_1^F, s)$. We define λ_1 to be the unique constituent of ${}^*R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda)$ in the series $\mathcal{E}(\boldsymbol{L}_1^F, s)$. Then λ_1 is e-Jordan-cuspidal. Indeed, for any

F-stable maximal torus $\boldsymbol{T} \le \boldsymbol{L}_1$ with $\boldsymbol{T}_e \not\le Z^\circ(\boldsymbol{L})_e = Z^\circ(\boldsymbol{L}_1)_e$ we get that ${}^*R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\lambda_1)$ is a constituent of ${}^*R_{\boldsymbol{T}}^{\boldsymbol{L}}(\lambda) = 0$ by e-Jordan-cuspidality of λ. Here note that the set of constituents of ${}^*R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\eta)$, where η is a constituent of ${}^*R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda)$ different from λ_1, is disjoint from the set of irreducible constituents of ${}^*R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\lambda_1)$.

Thus we have obtained a well-defined map ${}^*\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ from e-Jordan-cuspidal pairs in $\boldsymbol{G}$ to e-Jordan-cuspidal pairs in $\boldsymbol{G}_1$, both below the series s. As the map $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ preserves the e-part of the centre, ${}^*\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}} \circ \Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ is the identity. It remains to prove that $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ is surjective. For this, let $(\boldsymbol{M}, \mu)$ be any e-Jordan-cuspidal pair of $\boldsymbol{G}$ below $\mathcal{E}(\boldsymbol{G}^F, s)$, let $(\boldsymbol{L}_1, \lambda_1) = {}^*\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}(\boldsymbol{M}, \mu)$ and $(\boldsymbol{L}, \lambda) = \Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}(\boldsymbol{L}_1, \lambda_1)$. Then we have $Z^\circ(\boldsymbol{M})_e \le Z^\circ(\boldsymbol{L}_1)_e = Z^\circ(\boldsymbol{L})_e$, so $\boldsymbol{L} = C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L})_e) \le C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{M})_e) = \boldsymbol{M}$ is an e-split Levi subgroup of $\boldsymbol{M}$. As $\boldsymbol{L}_1 \le \boldsymbol{L} \le \boldsymbol{M}$ and $\epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{M}} R_{\boldsymbol{L}_1}^{\boldsymbol{M}}$ is a bijection from $\mathcal{E}(\boldsymbol{L}_1^F, s)$ to $\mathcal{E}(\boldsymbol{M}^F, s)$, it follows that $\epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{M}} R_{\boldsymbol{L}}^{\boldsymbol{M}}$ is a bijection between $\mathcal{E}(\boldsymbol{L}^F, s)$ and $\mathcal{E}(\boldsymbol{M}^F, s)$. As λ and μ are e-Jordan-cuspidal, (J$_1$) implies that $Z^\circ(\boldsymbol{M}^*)_e = Z^\circ(\boldsymbol{L}^*)_e$, so $\boldsymbol{M} = \boldsymbol{L}$, that is, $(\boldsymbol{M}, \mu)$ is in the image of $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$. The proof is complete. □

The above bijection also preserves relative Weyl groups.

Lemma 2.5. *In the situation and notation of Proposition 2.4 let* $(\boldsymbol{L}, \lambda) = \Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}(\boldsymbol{L}_1, \lambda_1)$. *Then* $N_{\boldsymbol{G}_1^F}(\boldsymbol{L}_1, \lambda_1) \le N_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ *and this inclusion induces an isomorphism of relative Weyl groups* $W_{\boldsymbol{G}_1^F}(\boldsymbol{L}_1, \lambda_1) \cong W_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$.

Proof. Let $g \in N_{\boldsymbol{G}_1^F}(\boldsymbol{L}_1, \lambda_1)$. Then g normalises $Z^\circ(\boldsymbol{L}_1)_e$ and hence also $\boldsymbol{L} = C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L}_1)_e)$. Thus,

$$ {}^g\lambda = \epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R_{{}^g\boldsymbol{L}_1}^{{}^g\boldsymbol{L}}({}^g\lambda_1) = \epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda_1) = \lambda $$

and the first assertion follows.

For the second assertion, let $g \in N_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ and let $\boldsymbol{T}$ be an F-stable maximal torus of $\boldsymbol{L}_1$ and θ an irreducible character of $\boldsymbol{T}^F$ such that λ_1 is a constituent of $R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\theta)$. Since $\lambda_1 \in \mathcal{E}(\boldsymbol{L}_1^F, s)$, $(\boldsymbol{T}, \theta)$ corresponds via duality (between $\boldsymbol{L}_1$ and $\boldsymbol{L}_1^*$) to the $\boldsymbol{L}_1^{*F}$-class of s, and all constituents of $R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\theta)$ are in $\mathcal{E}(\boldsymbol{L}_1^F, s)$. Consequently, $R_{\boldsymbol{L}_1}^{\boldsymbol{L}}$ induces a bijection between the set of constituents of $R_{\boldsymbol{T}}^{\boldsymbol{L}_1}(\theta)$ and the set of constituents of $R_{\boldsymbol{T}}^{\boldsymbol{L}}(\theta)$. In particular, λ is a constituent of $R_{\boldsymbol{T}}^{\boldsymbol{L}}(\theta)$. Since g stabilises λ, λ is also a constituent of $R_{{}^g\boldsymbol{T}}^{\boldsymbol{L}}({}^g\theta)$. Hence $(\boldsymbol{T}, \theta)$ and ${}^g(\boldsymbol{T}, \theta)$ are geometrically conjugate in $\boldsymbol{L}$. Let $l \in \boldsymbol{L}$ geometrically conjugate ${}^g(\boldsymbol{T}, \theta)$ to $(\boldsymbol{T}, \theta)$. Since $C_{\boldsymbol{G}^*}(s) \le \boldsymbol{G}_1^*$, we have $lg \in \boldsymbol{G}_1$ (see for instance [Kessar and Malle 2013, Lemma 7.5]). Hence $F(l)l^{-1} = F(lg)(lg)^{-1} \in \boldsymbol{G}_1 \cap \boldsymbol{L} = \boldsymbol{L}_1$. By the Lang–Steinberg theorem applied to $\boldsymbol{L}_1$, there exists $l_1 \in \boldsymbol{L}_1$ such that $l_1 l \in \boldsymbol{L}^F$. Also, since $l_1 \in \boldsymbol{G}_1$ and $g \in \boldsymbol{G}^F$, $l_1 lg \in \boldsymbol{G}_1^F$. Thus, up to replacing g by $l_1 lg$, we may assume that $g \in \boldsymbol{G}_1^F$.

Since $\boldsymbol{L}_1 = C_{\boldsymbol{G}_1}(Z^\circ(\boldsymbol{L})_e)$, it follows that $g \in N_{\boldsymbol{G}_1^F}(\boldsymbol{L}_1)$, and thus

$$ \epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda_1) = \lambda = {}^g\lambda = \epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R_{\boldsymbol{L}_1}^{\boldsymbol{L}}({}^g\lambda_1). $$

Since $R_{L_1}^{L}$ induces a bijection between the set of characters in the geometric Lusztig series of L_1^F corresponding to s (the union of series $\mathcal{E}(L_1^F, t)$, where t runs over the semisimple elements of L_1^{*F} which are L_1-conjugate to s) and the set of characters in the geometric Lusztig series of L^F corresponding to s, it suffices to prove that ${}^g\lambda_1 \in \mathcal{E}(L_1^F, t)$ for some $t \in L_1^{*F}$ which is L_1^{*F}-conjugate to s. Let T, θ and l be as above. Since $lg \in G_1$ and $g \in G_1$, it follows that $l \in G_1 \cap L = L_1$. Hence ${}^g(T, \theta)$ and (T, θ) are geometrically conjugate in L_1. The claim follows as ${}^g\lambda_1$ is a constituent of $R_{{}^gT}^{L_1}({}^g\theta)$. □

e-Jordan-cuspidality and ℓ-blocks. We next investigate the behaviour of ℓ-blocks with respect to the map $\Psi_{G_1}^{G}$. For this, let $\ell \neq p$ be a prime. We set

$$e_\ell(q) := \text{order of } q \text{ modulo } \begin{cases} \ell & \text{if } \ell \neq 2, \\ 4 & \text{if } \ell = 2. \end{cases}$$

For a semisimple ℓ'-element s of G^{*F}, we denote by $\mathcal{E}_\ell(G^F, s)$ the union of all Lusztig series $\mathcal{E}(G^F, st)$, where $t \in G^{*F}$ is an ℓ-element commuting with s. We recall that the set $\mathcal{E}_\ell(G^F, s)$ is a union of ℓ-blocks. Further, if $G_1 \leq G$ is an F-stable Levi subgroup such that G_1^* contains $C_{G^*}(s)$, then $\epsilon_{G_1}\epsilon_G R_{G_1}^{G}$ induces a bijection, which we refer to as the *Jordan correspondence*, between the ℓ-blocks in $\mathcal{E}(G_1^F, s)$ and the ℓ-blocks in $\mathcal{E}(G^F, s)$, see [Broué 1990, §2A].

Proposition 2.6. *Let $\ell \neq p$ be a prime, $s \in G^{*F}$ a semisimple ℓ'-element and $G_1 \leq G$ an F-stable Levi subgroup with G_1^* containing $C_{G^*}(s)$. Assume that b is an ℓ-block in $\mathcal{E}_\ell(G^F, s)$, and c its Jordan corresponding block in $\mathcal{E}_\ell(G_1^F, s)$. Let $e := e_\ell(q)$.*

(a) *Let (L_1, λ_1) be e-Jordan-cuspidal in G_1 and set $(L, \lambda) = \Psi_{G_1}^{G}(L_1, \lambda_1)$. If all constituents of $R_{L_1}^{G_1}(\lambda_1)$ lie in c, then all constituents of $R_L^G(\lambda)$ lie in b.*

(b) *Let (L, λ) be e-Jordan-cuspidal in G and set $(L_1, \lambda_1) = {}^*\Psi_{G_1}^{G}(L, \lambda)$. If all constituents of $R_L^G(\lambda)$ lie in b, then all constituents of $R_{L_1}^{G_1}(\lambda_1)$ lie in c.*

Proof. Note that the hypothesis of part (a) means that for any parabolic subgroup P of G_1 containing L_1 as Levi subgroup, all constituents of $R_{L_1 \leq P}^{G_1}(\lambda_1)$ lie in c. A similar remark applies to the conclusion, as well as to part (b).

For (a), note that by the definition of $\Psi_{G_1}^{G}$ we have that all constituents of

$$\epsilon_L \epsilon_{L_1} R_L^G(\lambda) = R_{L_1}^G(\lambda_1) = R_{G_1}^G R_{L_1}^{G_1}(\lambda_1)$$

are contained in $R_{G_1}^G(c)$, hence in b by Jordan correspondence.

In (b), suppose that η is a constituent of $R_{L_1}^{G_1}(\lambda_1)$ not lying in c. Then by Jordan correspondence, $R_{G_1}^G(\eta)$ does not belong to b, whence $R_{L_1}^G(\lambda_1)$ has a constituent not lying in b, contradicting our assumption that all constituents of $R_{L_1}^G(\lambda_1) = R_L^G R_{L_1}^L(\lambda_1) = \epsilon_L \epsilon_{L_1} R_L^G(\lambda)$ are in b. □

***e*-quasicentrality.** For a prime ℓ not dividing q, we denote by $\mathcal{E}(\boldsymbol{G}^F, \ell')$ the set of irreducible characters of $\boldsymbol{G}^F$ lying in a Lusztig series $\mathcal{E}(\boldsymbol{G}^F, s)$, where $s \in \boldsymbol{G}^{*F}$ is a semisimple ℓ'-element. Recall from [Kessar and Malle 2013, Definition 2.4] that a character $\chi \in \mathcal{E}(\boldsymbol{G}^F, \ell')$ is said to be of *central ℓ-defect* if the ℓ-block of $\boldsymbol{G}^F$ containing χ has a central defect group and χ is said to be of *quasicentral ℓ-defect* if some (and hence any) character of $[\boldsymbol{G}, \boldsymbol{G}]^F$ covered by χ is of central ℓ-defect.

Lemma 2.7. *Let $\boldsymbol{L}$ be an F-stable Levi subgroup of $\boldsymbol{G}$, and set $\boldsymbol{L}_0 = \boldsymbol{L} \cap [\boldsymbol{G}, \boldsymbol{G}]$. Let $\ell \neq p$ be a prime.*

(a) *If $\boldsymbol{L}_0 = C_{[\boldsymbol{G},\boldsymbol{G}]}(Z(\boldsymbol{L}_0)^F_\ell)$, then $\boldsymbol{L} = C_{\boldsymbol{G}}(Z(\boldsymbol{L})^F_\ell)$.*

(b) *Let $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ and let λ_0 be an irreducible constituent of $\mathrm{Res}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda)$. Then λ_0 is of quasicentral ℓ-defect if and only if λ is of quasicentral ℓ-defect.*

Proof. Since $\boldsymbol{G} = Z^\circ(\boldsymbol{G})[\boldsymbol{G}, \boldsymbol{G}]$ and $Z^\circ(\boldsymbol{G}) \le \boldsymbol{L}$, we have that $\boldsymbol{L} = Z^\circ(\boldsymbol{G})\boldsymbol{L}_0$. Hence if $\boldsymbol{L}_0 = C_{[\boldsymbol{G},\boldsymbol{G}]}(Z(\boldsymbol{L}_0)^F_\ell)$, then $\boldsymbol{L} = C_{\boldsymbol{G}}(Z(\boldsymbol{L}_0)^F_\ell) \supseteq C_{\boldsymbol{G}}(Z(\boldsymbol{L})^F_\ell) \supseteq \boldsymbol{L}$. This proves (a). In (b), since λ is in an ℓ'-Lusztig series, the index in $\boldsymbol{L}^F$ of the stabiliser in $\boldsymbol{L}^F$ of λ_0 is prime to ℓ and on the other hand, λ_0 extends to a character of the stabiliser in $\boldsymbol{L}^F$ of λ_0. Thus, $\lambda(1)_\ell = \lambda_0(1)_\ell$. Since $[\boldsymbol{L}_0, \boldsymbol{L}_0] = [\boldsymbol{L}, \boldsymbol{L}]$, the assertion follows by [Kessar and Malle 2013, Proposition 2.5(a)]. $\square$

Remark 2.8. The converse of assertion (a) of Lemma 2.7 fails in general, even when we restrict to $e_\ell(q)$-split Levi subgroups: let ℓ be odd and $\boldsymbol{G} = \mathrm{GL}_\ell$ with F such that $\boldsymbol{G}^F = \mathrm{GL}_\ell(q)$ with $\ell \mid (q-1)$. Let $\boldsymbol{L}$ a 1-split Levi subgroup of type $\mathrm{GL}_{\ell-1} \times \mathrm{GL}_1$. Then $Z(\boldsymbol{L})^F_\ell \cong C_\ell \times C_\ell$ and $\boldsymbol{L} = C_{\boldsymbol{G}}(Z(\boldsymbol{L})^F_\ell)$. But $Z(\boldsymbol{L}_0)^F_\ell \cong C_\ell \cong Z([\boldsymbol{G}, \boldsymbol{G}])^F_\ell$, hence $C_{[\boldsymbol{G},\boldsymbol{G}]}(Z(\boldsymbol{L}_0)^F_\ell) = [\boldsymbol{G}, \boldsymbol{G}]$.

One might hope for further good properties of the bijection of Proposition 2.6 with respect to (quasi-)centrality. In this direction, we observe the following:

Lemma 2.9. *In the situation of Proposition 2.4, if $(\boldsymbol{L}, \lambda)$ is of central ℓ-defect for a prime ℓ with $e_\ell(q) = e$, then so is $(\boldsymbol{L}_1, \lambda_1) = {}^*\Psi^{\boldsymbol{G}}_{\boldsymbol{G}_1}(\boldsymbol{L}, \lambda)$, and we have $Z(\boldsymbol{L})^F_\ell = Z(\boldsymbol{L}_1)^F_\ell$.*

Proof. By assumption, we have that $\lambda(1)_\ell = |\boldsymbol{L}^F : Z(\boldsymbol{L})^F|_\ell$. Now $Z(\boldsymbol{L})$ lies in every maximal torus of $\boldsymbol{L}$, hence in $\boldsymbol{L}_1$, so we have that $Z(\boldsymbol{L})^F_\ell \le Z(\boldsymbol{L}_1)^F_\ell$. As $\lambda = \epsilon_{\boldsymbol{L}_1}\epsilon_{\boldsymbol{L}} R^{\boldsymbol{L}}_{\boldsymbol{L}_1}(\lambda_1)$, we obtain $\lambda(1)_\ell = \lambda_1(1)_\ell |\boldsymbol{L}^F : \boldsymbol{L}_1^F|_\ell$, whence

$$\lambda_1(1)_\ell = \lambda(1)_\ell |\boldsymbol{L}^F : \boldsymbol{L}_1^F|_\ell^{-1} = |\boldsymbol{L}_1^F|_\ell |Z(\boldsymbol{L})^F|_\ell^{-1} \ge |\boldsymbol{L}_1^F : Z(\boldsymbol{L}_1)^F|_\ell.$$

But clearly $\lambda_1(1)_\ell \le |\boldsymbol{L}_1^F : Z(\boldsymbol{L}_1)^F|_\ell$, so we have equality throughout, as claimed. $\square$

Example 2.10. The converse of Lemma 2.9 does not hold in general. To see this, let $\boldsymbol{G} = \mathrm{PGL}_\ell$ with $\boldsymbol{G}^F = \mathrm{PGL}_\ell(q)$, $\boldsymbol{L} = \boldsymbol{G}$, and $\boldsymbol{G}_1 \le \boldsymbol{G}$ an F-stable maximal torus such that $\boldsymbol{G}_1^F$ is a Coxeter torus of $\boldsymbol{G}^F$, of order Φ_ℓ. Assume that $\ell \mid (q-1)$

(so $e = 1$). Then $\boldsymbol{L}_1 = \boldsymbol{G}_1$. Here, any $\lambda_1 \in \mathrm{Irr}(\boldsymbol{L}_1^F)$ is e-(Jordan-)cuspidal, and certainly of central ℓ-defect, and $|Z(\boldsymbol{L}_1)_\ell^F| = (\Phi_\ell)_\ell = \ell$ for $\ell \geq 3$, while clearly $Z(\boldsymbol{L})_\ell^F = Z(\boldsymbol{G})_\ell^F = 1$. Furthermore

$$\lambda(1)_\ell = \lambda_1(1)_\ell[\boldsymbol{L}^F : \boldsymbol{L}_1^F]_\ell = [\boldsymbol{L}^F : \boldsymbol{L}_1^F]_\ell,$$

since λ_1 is linear. Since $|Z(\boldsymbol{L}^F)|_\ell = 1$ and $|\boldsymbol{L}_1^F|_\ell > 1$, it follows that

$$\lambda(1)_\ell |Z(\boldsymbol{L}^F)|_\ell < |\boldsymbol{L}^F|_\ell,$$

hence λ is not of central ℓ-defect (and not even of quasicentral ℓ-defect).

Example 2.11. We also recall that e-(Jordan-)cuspidal characters are not always of central ℓ-defect, even when ℓ is a good prime: let $\boldsymbol{G}^F = \mathrm{SL}_{\ell^2}(q)$ with $\ell \mid (q-1)$, so $e = 1$. Then for $\boldsymbol{T}$ a Coxeter torus and $\theta \in \mathrm{Irr}(\boldsymbol{T}^F)$ in general position, $R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)$ is e-(Jordan-) cuspidal but not of quasicentral ℓ-defect.

For the next definition note that the property of being of (quasi)-central ℓ-defect is invariant under automorphisms of $\boldsymbol{G}^F$.

Definition 2.12. Let $\ell \neq p$ be a prime and $e = e_\ell(q)$. A character $\chi \in \mathcal{E}(\boldsymbol{G}^F, \ell')$ is called *e-Jordan quasicentral cuspidal* if χ is e-Jordan cuspidal and the $C_{\boldsymbol{G}^*}(s)^F$-orbit of unipotent characters of $C^\circ_{\boldsymbol{G}^*}(s)^F$ which corresponds to χ under Jordan decomposition consists of characters of quasicentral ℓ-defect, where $s \in \boldsymbol{G}^{*F}$ is a semisimple ℓ'-element such that $\chi \in \mathcal{E}(\boldsymbol{G}^F, s)$. An *e-Jordan quasicentral cuspidal pair of* $\boldsymbol{G}$ is a pair $(\boldsymbol{L}, \lambda)$ such that $\boldsymbol{L}$ is an e-split Levi subgroup of $\boldsymbol{G}$ and $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ is an e-Jordan quasicentral cuspidal character of $\boldsymbol{L}^F$.

We note that the set of e-Jordan quasicentral cuspidal pairs of $\boldsymbol{G}$ is closed under $\boldsymbol{G}^F$-conjugation. Also, note that Lemma 2.3 remains true upon replacing the e-Jordan-cuspidal property by the e-Jordan quasicentral cuspidal property. This is because, with the notation of Lemma 2.3, the orbit of unipotent characters corresponding to λ under Jordan decomposition is a subset of the orbit of unipotent characters corresponding to λ_0 under Jordan decomposition. Finally we note that the bijection $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ of Proposition 2.6 preserves e-quasicentrality since, with the notation of the proposition, λ_1 and λ correspond to the same orbit of unipotent characters under Jordan decomposition.

3. Lusztig induction and ℓ-blocks

Here we prove our main results on the parametrisation of ℓ-blocks in terms of e-Harish-Chandra series, in arbitrary Levi subgroups of simple groups of simply connected type. As in Section 2, $\ell \neq p$ will be prime numbers, q a power of p and $e = e_\ell(q)$.

Preservation of ℓ-blocks by Lusztig induction. We first extend [Cabanes and Enguehard 1999, Theorem 2.5]. The proof will require three auxiliary results:

Lemma 3.1. *Let $\boldsymbol{G}$ be connected reductive with a Frobenius endomorphism F endowing $\boldsymbol{G}$ with an $\mathbb{F}_q$-rational structure. Let $\boldsymbol{M}$ be an e-split Levi of $\boldsymbol{G}^F$ and c an ℓ-block of $\boldsymbol{M}^F$. Suppose that*

(1) *the set $\{d^{1,\boldsymbol{M}^F}(\mu) \mid \mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')\}$ is linearly independent*; *and*

(2) *there exists a subgroup $Z \le Z(\boldsymbol{M})^F_\ell$ and a block d of $C^\circ_{\boldsymbol{G}}(Z)^F$ such that all irreducible constituents of $R^{C^\circ_{\boldsymbol{G}}(Z)}_{\boldsymbol{M}}(\mu)$, where $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$, lie in the block d.*

Then there exists a block b of $\boldsymbol{G}^F$ such that all irreducible constituents of $R^{\boldsymbol{G}}_{\boldsymbol{M}}(\mu)$, where $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$, lie in the block b.

Proof. We adapt the argument of [Kessar and Malle 2013, Proposition 2.16]. Let $\chi \in \mathrm{Irr}(\boldsymbol{G}^F, \ell')$ be such that $\langle R^{\boldsymbol{G}}_{\boldsymbol{M}}(\mu), \chi\rangle \neq 0$ for some $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$. Then $\langle \mu, {}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)\rangle \neq 0$. In particular, $c.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi) \neq 0$. All constituents of ${}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)$ lie in $\mathcal{E}(\boldsymbol{M}^F, \ell')$, so by assumption (1) it follows that $d^{1,\boldsymbol{M}^F}(c.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)) \neq 0$. Since $d^{1,\boldsymbol{M}^F}(c.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi))$ vanishes on ℓ-singular elements of $\boldsymbol{M}^F$, we have that

$$\langle d^{1,\boldsymbol{M}^F}(c.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)), c.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)\rangle = \langle d^{1,\boldsymbol{M}^F}(c.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)), d^{1,\boldsymbol{M}^F}(c.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi))\rangle \neq 0.$$

If φ and φ' are irreducible ℓ-Brauer characters of $\boldsymbol{M}^F$ lying in different ℓ-blocks of $\boldsymbol{M}^F$, then $\langle \varphi, \varphi'\rangle = 0$ (see for instance [Nagao and Tsushima 1989, Chapter 3, Exercise 6.20(ii)]). Thus,

$$\langle d^{1,\boldsymbol{M}^F}(c.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)), c'.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)\rangle = \langle d^{1,\boldsymbol{M}^F}(c.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)), d^{1,\boldsymbol{M}^F}(c'.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi))\rangle = 0$$

for all blocks c' of $\boldsymbol{M}^F$ different from c. So, $\langle d^{1,\boldsymbol{M}^F}(c.{}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)), {}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)\rangle \neq 0$ from which it follows that $\langle d^{1,\boldsymbol{M}^F}(\mu'), {}^*R^{\boldsymbol{G}}_{\boldsymbol{M}}(\chi)\rangle \neq 0$ for some $\mu' \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$.

Continuing as in the proof of [Kessar and Malle 2013, Proposition 2.12] gives the required result. Note that condition (1) of this proposition is not necessarily met as stated, since μ' may be different from μ. However, μ and μ' are in the same block of $\boldsymbol{M}^F$ which is sufficient to obtain the conclusion of the lemma. □

Lemma 3.2. *Let $\boldsymbol{G}$ be connected reductive with a Frobenius endomorphism F. Suppose that $\boldsymbol{G}$ has connected centre and $[\boldsymbol{G}, \boldsymbol{G}]$ is simply connected. Let $\boldsymbol{G} = \boldsymbol{XY}$ such that either $\boldsymbol{X}$ is an F-stable product of components of $[\boldsymbol{G}, \boldsymbol{G}]$ and $\boldsymbol{Y}$ is the product of the remaining components with $Z(\boldsymbol{G})$, or vice versa. Suppose further that $\boldsymbol{G}^F/\boldsymbol{X}^F\boldsymbol{Y}^F$ is an ℓ-group. Let $\boldsymbol{N}$ be an F-stable Levi subgroup of $\boldsymbol{Y}$ and set $\boldsymbol{M} = \boldsymbol{XN}$. Let c be an ℓ-block of $\boldsymbol{M}^F$ and let c' be an ℓ-block of $\boldsymbol{N}^F$ covered by c. Suppose that there exists a block b' of $\boldsymbol{Y}^F$ such that every irreducible constituent of $R^{\boldsymbol{Y}}_{\boldsymbol{N}}(\tau)$ where $\tau \in \mathrm{Irr}(c') \cap \mathcal{E}(\boldsymbol{N}^F, \ell')$ lies in b'. Then there exists a block b of $\boldsymbol{G}^F$ such that every irreducible constituent of $R^{\boldsymbol{G}}_{\boldsymbol{M}}(\mu)$ where $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$ lies in b.*

Proof. We will use the extension of Lusztig induction to certain disconnected groups as in [Cabanes and Enguehard 1999, Section 1.1]. Let

$$\begin{aligned} \boldsymbol{G}_0 &= [\boldsymbol{G}, \boldsymbol{G}] = [\boldsymbol{X}, \boldsymbol{X}] \times [\boldsymbol{Y}, \boldsymbol{Y}], \\ \boldsymbol{M}_0 &= \boldsymbol{G}_0 \cap \boldsymbol{M} = [\boldsymbol{X}, \boldsymbol{X}] \times ([\boldsymbol{Y}, \boldsymbol{Y}] \cap \boldsymbol{N}). \end{aligned}$$

Then, $\boldsymbol{G}_0^F \subseteq \boldsymbol{X}^F\boldsymbol{Y}^F$ and $\boldsymbol{M}_0^F \subseteq \boldsymbol{X}^F\boldsymbol{N}^F$. Let $\boldsymbol{T}$ be an F-stable maximal torus of $\boldsymbol{M}$. Since $\boldsymbol{G}$ and hence also $\boldsymbol{M}$ has connected centre, $\boldsymbol{M} = \boldsymbol{M}_0^F\boldsymbol{T}^F$ and $\boldsymbol{G}^F = \boldsymbol{G}_0^F\boldsymbol{T}^F$. Further, $A := \boldsymbol{X}^F\boldsymbol{Y}^F \cap \boldsymbol{T}^F = \boldsymbol{X}^F\boldsymbol{N}^F \cap \boldsymbol{T}^F$ and $\boldsymbol{X}^F\boldsymbol{Y}^F = \boldsymbol{G}_0^F A = (\boldsymbol{G}_0 A)^F$, $\boldsymbol{X}^F\boldsymbol{N}^F = \boldsymbol{M}_0^F A = (\boldsymbol{M}_0 A)^F$. As in [Cabanes and Enguehard 1999, Section 1.1], we denote by $\mathcal{E}(\boldsymbol{X}^F\boldsymbol{Y}^F, \ell')$ the set of irreducible characters of $\boldsymbol{X}^F\boldsymbol{Y}^F$ that appear in the restriction of elements of $\mathcal{E}(\boldsymbol{G}^F, \ell')$ to $\boldsymbol{X}^F\boldsymbol{Y}^F$.

Let $\chi \in \mathcal{E}(\boldsymbol{G}^F, \ell')$. Since $\boldsymbol{G}^F/\boldsymbol{X}^F\boldsymbol{Y}^F$ is an ℓ-group, by [Cabanes and Enguehard 1999, Proposition 1.3(i)], $\mathrm{Res}^{\boldsymbol{G}^F}_{\boldsymbol{X}^F\boldsymbol{Y}^F}(\chi)$ is irreducible. Now if $\chi' \in \mathrm{Irr}(\boldsymbol{G}^F)$ has the same restriction to $\boldsymbol{X}^F\boldsymbol{Y}^F$ as χ, then again since $\boldsymbol{G}^F/\boldsymbol{X}^F\boldsymbol{Y}^F$ is an ℓ-group, either $\chi' = \chi$ or $\chi' \notin \mathcal{E}(\boldsymbol{G}^F, \ell')$. In other words, the restriction from $\mathbb{Z}\mathcal{E}(\boldsymbol{G}^F, \ell')$ to $\mathbb{Z}\mathcal{E}(\boldsymbol{X}^F\boldsymbol{Y}^F, \ell')$ is a bijection. Similarly, the restriction from $\mathbb{Z}\mathcal{E}(\boldsymbol{M}^F, \ell')$ to $\mathbb{Z}\mathcal{E}(\boldsymbol{X}^F\boldsymbol{N}^F, \ell')$ is a bijection.

In particular, every block of $\boldsymbol{G}^F$ covers a unique block of $\boldsymbol{X}^F\boldsymbol{Y}^F$. Since $\boldsymbol{G}^F/\boldsymbol{X}^F\boldsymbol{Y}^F$ is an ℓ-group, there is a bijection (through covering) between the set of blocks of $\boldsymbol{G}^F$ and the set of blocks of $\boldsymbol{X}^F\boldsymbol{Y}^F$. Hence, by the injectivity of restriction from $\mathbb{Z}\mathcal{E}(\boldsymbol{G}^F, \ell')$ to $\mathbb{Z}\mathcal{E}(\boldsymbol{X}^F\boldsymbol{Y}^F, \ell')$, it suffices to prove that there is a block b_0 of $\boldsymbol{X}^F\boldsymbol{Y}^F$ such that every irreducible constituent of $\mathrm{Res}^{\boldsymbol{G}^F}_{\boldsymbol{X}^F\boldsymbol{Y}^F} R^{\boldsymbol{G}}_{\boldsymbol{M}}(\mu)$ as μ ranges over $\mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$ lies in b_0.

Following [Cabanes and Enguehard 1999, Section 1.1], we have $\mathrm{Res}^{\boldsymbol{G}^F}_{\boldsymbol{X}^F\boldsymbol{Y}^F} R^{\boldsymbol{G}}_{\boldsymbol{M}} = R^{\boldsymbol{G}_0 A}_{\boldsymbol{M}_0 A} \mathrm{Res}^{\boldsymbol{M}^F}_{\boldsymbol{X}^F\boldsymbol{N}^F}$ on $\mathrm{Irr}(\boldsymbol{M}^F)$ (where here $R^{\boldsymbol{G}_0 A}_{\boldsymbol{M}_0 A}$ is Lusztig induction in the disconnected setting). Thus, it suffices to prove that there is a block b_0 of $\boldsymbol{X}^F\boldsymbol{Y}^F$ such that every irreducible constituent of $R^{\boldsymbol{G}_0 A}_{\boldsymbol{M}_0 A} \mathrm{Res}^{\boldsymbol{M}^F}_{\boldsymbol{X}^F\boldsymbol{N}^F}(\mu)$ as μ ranges over $\mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$ is contained in b_0.

By the above arguments applied to $\boldsymbol{M}^F$ and $\boldsymbol{X}^F\boldsymbol{N}^F$, there is a unique block c_0 of $\boldsymbol{X}^F\boldsymbol{N}^F$ covered by c. The surjectivity of restriction from $\mathbb{Z}\mathcal{E}(\boldsymbol{M}^F, \ell')$ to $\mathbb{Z}\mathcal{E}(\boldsymbol{X}^F\boldsymbol{N}^F, \ell')$ implies that it suffices to prove that there is a block b_0 of $\boldsymbol{X}^F\boldsymbol{Y}^F$ such that every irreducible constituent of $R^{\boldsymbol{G}_0 A}_{\boldsymbol{M}_0 A}(\mu)$ for $\mu \in \mathrm{Irr}(c_0) \cap \mathcal{E}(\boldsymbol{X}^F\boldsymbol{N}^F, \ell')$ is contained in b_0.

The group $I := \{(x, x^{-1}) \mid x \in \boldsymbol{X}^F \cap \boldsymbol{Y}^F\} \le \boldsymbol{X} \times \boldsymbol{Y}$ is the kernel of the multiplication map $\boldsymbol{X}^F \times \boldsymbol{Y}^F \to \boldsymbol{X}^F\boldsymbol{Y}^F$. Identifying $\boldsymbol{X}^F\boldsymbol{Y}^F$ with $\boldsymbol{X}^F \times \boldsymbol{Y}^F/I$ through multiplication, $\mathrm{Irr}(\boldsymbol{X}^F\boldsymbol{Y}^F)$ is the subset of $\mathrm{Irr}(\boldsymbol{X}^F \times \boldsymbol{Y}^F)$ consisting of characters whose kernel contains I. Since $\boldsymbol{X}^F \cap \boldsymbol{Y}^F \le \boldsymbol{X} \cap \boldsymbol{Y} \le Z(\boldsymbol{G}) \le \boldsymbol{M}$, I is also the kernel of the multiplication map $\boldsymbol{X}^F \times \boldsymbol{N}^F \to \boldsymbol{X}^F\boldsymbol{N}^F$ and we may identify $\mathrm{Irr}(\boldsymbol{X}^F\boldsymbol{Y}^F)$ with the subset of $\mathrm{Irr}(\boldsymbol{X}^F \times \boldsymbol{N}^F)$ consisting of characters whose kernel contains I.

Any parabolic subgroup of $\boldsymbol{G}_0$ containing $\boldsymbol{M}_0$ as Levi subgroup is of the form $[\boldsymbol{X},\boldsymbol{X}]\boldsymbol{P}$, where $\boldsymbol{P}$ is a parabolic subgroup of $[\boldsymbol{Y},\boldsymbol{Y}]$ containing $\boldsymbol{N}\cap[\boldsymbol{Y},\boldsymbol{Y}]$ as Levi subgroup. Let $\boldsymbol{U}:=R_u(\boldsymbol{X}\boldsymbol{P})=R_u(\boldsymbol{P})\leq[\boldsymbol{Y},\boldsymbol{Y}]$ and denote by $\mathcal{L}^{-1}(\boldsymbol{U})$ the inverse image of $\boldsymbol{U}$ under the Lang map $\boldsymbol{G}\to\boldsymbol{G}$ given by $g\mapsto g^{-1}F(g)$.

The Deligne–Lusztig variety associated to $R_{\boldsymbol{M}_0A}^{\boldsymbol{G}_0A}$ (with respect to $\boldsymbol{X}\boldsymbol{P}$) is

$$\mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{G}_0A.$$

Since $\boldsymbol{T}=(\boldsymbol{T}\cap\boldsymbol{M}_0)Z(\boldsymbol{G})$, $\boldsymbol{U}$ is normalised by $\boldsymbol{T}$ and in particular by A. Hence,

$$\begin{aligned}\mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{G}_0A=(\mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{G}_0)A&=[\boldsymbol{X},\boldsymbol{X}]^F(\mathcal{L}^{-1}(\boldsymbol{U})\cap[\boldsymbol{Y},\boldsymbol{Y}])A\\&=[\boldsymbol{X},\boldsymbol{X}]^F(A\cap\boldsymbol{X}^F)(\mathcal{L}^{-1}(\boldsymbol{U})\cap[\boldsymbol{Y},\boldsymbol{Y}])(A\cap\boldsymbol{Y}^F).\end{aligned}$$

For the last equality, note that

$$A=\boldsymbol{X}^F\boldsymbol{Y}^F\cap\boldsymbol{T}=(\boldsymbol{X}^F\cap\boldsymbol{T})(\boldsymbol{Y}^F\cap\boldsymbol{T})=(\boldsymbol{X}^F\cap A)(\boldsymbol{Y}^F\cap A).$$

Now, $\mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{Y}=(\mathcal{L}^{-1}(\boldsymbol{U})\cap[\boldsymbol{Y},\boldsymbol{Y}])\boldsymbol{S}^F$ for any F-stable maximal torus $\boldsymbol{S}$ of $\boldsymbol{Y}$. Applying this with $\boldsymbol{S}=\boldsymbol{T}\cap\boldsymbol{Y}$, we have $(\mathcal{L}^{-1}(\boldsymbol{U})\cap[\boldsymbol{Y},\boldsymbol{Y}])(A\cap\boldsymbol{Y}^F)=\mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{Y}$. Also, $[\boldsymbol{X},\boldsymbol{X}]^F(A\cap\boldsymbol{X}^F)=\boldsymbol{X}^F$. Altogether this gives $\mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{G}_0A=\boldsymbol{X}^F(\mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{Y})$. Further, $\mathcal{L}^{-1}(\boldsymbol{U})\cap\boldsymbol{Y}$ is the variety underlying $R_{\boldsymbol{N}}^{\boldsymbol{Y}}$ (with respect to the parabolic subgroup $\boldsymbol{P}Z(\boldsymbol{G})$). Hence, for any $\tau_1\in\mathrm{Irr}(\boldsymbol{X}^F)$, $\tau_2\in\mathrm{Irr}(\boldsymbol{Y}^F)$ such that I is in the kernel of $\tau_1\tau_2$, we have

$$R_{\boldsymbol{M}_0A}^{\boldsymbol{G}_0A}(\tau_1\tau_2)=\tau_1R_{\boldsymbol{N}}^{\boldsymbol{Y}}(\tau_2).$$

Further, $\tau_1\tau_2\in\mathcal{E}(\boldsymbol{X}^F\boldsymbol{N}^F,\ell')$ if and only if $\tau_1\in\mathcal{E}(\boldsymbol{X}^F,\ell')$ and $\tau_2\in\mathcal{E}(\boldsymbol{N}^F,\ell')$.

To conclude note that c' is the unique block of $\boldsymbol{N}^F$ covered by c_0 and $c_0=dc'$, where d is a block $\boldsymbol{X}^F$. Let b' be the block of $\boldsymbol{Y}^F$ in the hypothesis. Then, setting $b_0=db'$ gives the desired result. □

We will also make use of the following well-known extension of [Enguehard 2008, Proposition 1.5].

Lemma 3.3. *Suppose that q is odd. Let $\boldsymbol{G}$ be connected reductive with a Frobenius endomorphism F. Suppose that all components of $\boldsymbol{G}$ are of classical type A, B, C or D and that $Z(\boldsymbol{G})/Z^\circ(\boldsymbol{G})$ is a 2-group. Let $s\in\boldsymbol{G}^{*F}$ be semisimple of odd order. Then all elements of $\mathcal{E}(\boldsymbol{G}^F,s)$ lie in the same 2-block of $\boldsymbol{G}^F$.*

Proof. Since s has odd order and $Z(\boldsymbol{G})/Z^\circ(\boldsymbol{G})$ is a 2-group, $C_{\boldsymbol{G}^*}(s)$ is connected. On the other hand, since all components of $\boldsymbol{G}^*$ are of classical type and s has odd order, $C^\circ_{\boldsymbol{G}^*}(s)$ is a Levi subgroup of $\boldsymbol{G}$. Thus, $C_{\boldsymbol{G}^*}(s)$ is a Levi subgroup of $\boldsymbol{G}^*$ and by Jordan correspondence the set of 2-blocks of $\boldsymbol{G}^F$ which contain a character of $\mathcal{E}(\boldsymbol{G}^F,s)$ is in bijection with the set of unipotent 2-blocks of $\boldsymbol{C}^F$, where $\boldsymbol{C}$ is a Levi subgroup of $\boldsymbol{G}$ in duality with $C_{\boldsymbol{G}^*}(s)$. Since all components of $\boldsymbol{C}$ are also of classical type, the claim follows by [Enguehard 2008, Proposition 1.5(a)]. □

We now have the following extension of [Cabanes and Enguehard 1999, Theorem 2.5] to all primes.

Theorem 3.4. *Let $\boldsymbol{H}$ be a simple algebraic group of simply connected type with a Frobenius endomorphism $F : \boldsymbol{H} \to \boldsymbol{H}$ endowing $\boldsymbol{H}$ with an $\mathbb{F}_q$-rational structure. Let $\boldsymbol{G}$ be an F-stable Levi subgroup of $\boldsymbol{H}$. Let ℓ be a prime not dividing q and set $e = e_\ell(q)$. Let $\boldsymbol{M}$ be an e-split Levi subgroup of $\boldsymbol{G}$ and let c be a block of $\boldsymbol{M}^F$. Then there exists a block b of $\boldsymbol{G}^F$ such that every irreducible constituent of $R_{\boldsymbol{M}}^{\boldsymbol{G}}(\mu)$ for every $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$ lies in b.*

Proof. Let $\dim(\boldsymbol{G})$ be minimal such that the claim of the theorem does not hold. Let $s \in \boldsymbol{M}^{*F}$ be a semisimple ℓ'-element with $\mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell') \subseteq \mathcal{E}(\boldsymbol{M}^F, s)$. Then all irreducible constituents of $R_{\boldsymbol{M}}^{\boldsymbol{G}}(\mu)$ where $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$ are in $\mathcal{E}(\boldsymbol{G}^F, s)$.

First suppose that s is not quasi-isolated and let $\boldsymbol{G}_1$ be a proper F-stable Levi subgroup of $\boldsymbol{G}$ whose dual contains $C_{\boldsymbol{G}^*}(s)$. Let $\boldsymbol{M}^*$ be a Levi subgroup of $\boldsymbol{G}^*$ in duality with $\boldsymbol{M}$ and set $\boldsymbol{M}_1^* = C_{\boldsymbol{G}_1^*}(Z^\circ(\boldsymbol{M}^*)_e)$. Then, as in the proof of Proposition 2.4, $\boldsymbol{M}_1^*$ is an e-split Levi subgroup of $\boldsymbol{G}_1^*$ and letting $\boldsymbol{M}_1$ be the dual of $\boldsymbol{M}_1^*$ in $\boldsymbol{G}$, $\boldsymbol{M}_1$ is an e-split Levi subgroup of $\boldsymbol{G}_1$. Further, $\boldsymbol{M}_1^* \geq C_{\boldsymbol{M}^*}(s)$. Hence there exists a unique block say c_1 of $\boldsymbol{M}_1^F$ such that $\mathrm{Irr}(c_1) \cap \mathcal{E}(\boldsymbol{M}_1^F, \ell') \subseteq \mathcal{E}(\boldsymbol{M}_1^F, s)$ and such that c_1 and c are Jordan corresponding blocks.

By induction our claim holds for $\boldsymbol{G}_1$ and the block c_1 of $\boldsymbol{M}_1$. Let b_1 be the block of $\boldsymbol{G}_1^F$ such that every irreducible constituent of $R_{\boldsymbol{M}_1}^{\boldsymbol{G}_1}(\mu)$ where $\mu \in \mathrm{Irr}(c_1) \cap \mathcal{E}(\boldsymbol{M}_1^F, \ell')$ lies in b_1 and let b be the Jordan correspondent of b_1 in $\boldsymbol{G}^F$.

Now let $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, s)$ and let χ be an irreducible constituent of $R_{\boldsymbol{M}}^{\boldsymbol{G}}(\mu)$. Let μ_1 be the unique character in $\mathrm{Irr}(\boldsymbol{M}_1^F, s)$ with $\mu = \pm R_{\boldsymbol{M}_1}^{\boldsymbol{M}}(\mu_1)$. Then, $\mu_1 \in \mathrm{Irr}(c_1)$ and

$$R_{\boldsymbol{M}}^{\boldsymbol{G}}(\mu) = R_{\boldsymbol{M}}^{\boldsymbol{G}}\big(R_{\boldsymbol{M}_1}^{\boldsymbol{M}}(\mu_1)\big) = R_{\boldsymbol{G}_1}^{\boldsymbol{G}}\big(R_{\boldsymbol{M}_1}^{\boldsymbol{G}_1}(\mu_1)\big).$$

All irreducible constituents of $R_{\boldsymbol{M}_1}^{\boldsymbol{G}_1}(\mu_1)$ lie in b_1. Hence, by the above equation and by the Jordan decomposition of blocks, χ lies in b, a contradiction.

So, we may assume from now on that s is quasi-isolated in $\boldsymbol{G}^*$. By [Cabanes and Enguehard 1999, Theorem 2.5], we may assume that ℓ is bad for $\boldsymbol{G}$ and hence for $\boldsymbol{H}$. So $\boldsymbol{H}$ is not of type A. If $\boldsymbol{H}$ is of type B, C or D, then $\ell = 2$ and we have a contradiction by Lemma 3.3.

Thus $\boldsymbol{H}$ is of exceptional type. Suppose that $s = 1$. By [Broué et al. 1993, Theorem 3.2] $\boldsymbol{G}^F$ satisfies an e-Harish-Chandra theory above each unipotent e-cuspidal pair $(\boldsymbol{L}, \lambda)$ and by [Enguehard 2000, Theorems A and A.bis], all irreducible constituents of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ lie in the same ℓ-block of $\boldsymbol{G}^F$.

So we may assume that $s \neq 1$. We consider the case that $\boldsymbol{G} = \boldsymbol{H}$. Then by [Kessar and Malle 2013, Theorem 1.4], $\boldsymbol{G}^F$ satisfies an e-Harish-Chandra theory above each e-cuspidal pair $(\boldsymbol{L}, \lambda)$ below $\mathcal{E}(\boldsymbol{G}^F, s)$ and by [Kessar and Malle 2013, Theorem 1.2], all irreducible constituents of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ lie in the same ℓ-block of $\boldsymbol{G}^F$.

So, we may assume that $\boldsymbol{G}$ is proper in $\boldsymbol{H}$. If $\boldsymbol{H}$ is of type G_2, F_4 or E_6, then $\ell = 2$, all components of $\boldsymbol{G}$ are of classical type. For G_2 and F_4 we have that $Z(\boldsymbol{H})$ and therefore $Z(\boldsymbol{G})$ is connected. If $\boldsymbol{H}$ is of type E_6, since 2 is bad for $\boldsymbol{G}$, $\boldsymbol{G}$ has a component of type D_n, $n \geq 4$. By rank considerations, $[\boldsymbol{G}, \boldsymbol{G}]$ is of type D_4 or D_5. Since $|Z(\boldsymbol{H})/Z^\circ(\boldsymbol{H})| = 3$ it follows again that $Z(\boldsymbol{G})$ is connected. In either case we get a contradiction by Lemma 3.3.

So, $\boldsymbol{H}$ is of type E_7 or E_8. Since $\boldsymbol{G}$ is proper in $\boldsymbol{H}$, 5 is good for $\boldsymbol{G}$, hence $\ell = 3$ or 2. Also, we may assume that at least one of the two assumptions of Lemma 3.1 fails to hold for $\boldsymbol{G}$, $\boldsymbol{M}$ and c.

Suppose that $\ell = 3$. Since $\boldsymbol{G}$ is proper in $\boldsymbol{H}$ and 3 is bad for $\boldsymbol{G}$, either $[\boldsymbol{G}, \boldsymbol{G}]$ is of type E_6, or $\boldsymbol{H}$ is of type E_8 and $[\boldsymbol{G}, \boldsymbol{G}]$ is of type $E_6 + A_1$ or of type E_7. In all cases, $Z(\boldsymbol{G})$ is connected (note that if $\boldsymbol{H}$ is of type E_7, then $[\boldsymbol{G}, \boldsymbol{G}]$ is of type E_6, whence the order of $Z(\boldsymbol{G})/Z^\circ(\boldsymbol{G})$ divides both 2 and 3). If $\boldsymbol{G} = \boldsymbol{M}$, there is nothing to prove, so we may assume that $\boldsymbol{M}$ is proper in $\boldsymbol{G}$. Let

$$\boldsymbol{C} := C^\circ_{\boldsymbol{G}}(Z(\boldsymbol{M})^F_3) \geq \boldsymbol{M}.$$

We claim that there is a block, say d, of $\boldsymbol{C}^F$ such that for all $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\boldsymbol{M}^F, \ell')$, every irreducible constituent of $R^{\boldsymbol{C}}_{\boldsymbol{M}}(\mu)$ lies in d. Indeed, since $\boldsymbol{M}$ is proper in $\boldsymbol{G}$ and since $Z(\boldsymbol{G})$ is connected, by [Cabanes and Enguehard 1993, Proposition 2.1] $\boldsymbol{C}$ is proper in $\boldsymbol{G}$. Also, by direct calculation either $\boldsymbol{C}$ is a Levi subgroup of $\boldsymbol{G}$ or 3 is good for $\boldsymbol{C}$. In the first case, the claim follows by the inductive hypothesis since $\boldsymbol{M}$ is also e-split in $\boldsymbol{C}$. In the second case, we are done by [Cabanes and Enguehard 1999, Theorem 2.5].

Thus, we may assume that assumption (1) of Lemma 3.1 does not hold. Hence, by [Cabanes and Enguehard 1999, Theorem 1.7], 3 is bad for $\boldsymbol{M}$. Consequently, $\boldsymbol{M}$ has a component of nonclassical type. Since $\boldsymbol{M}$ is proper in $\boldsymbol{G}$, this means that $[\boldsymbol{G}, \boldsymbol{G}]$ is of type $E_6 + A_1$ or of type E_7 and $[\boldsymbol{M}, \boldsymbol{M}]$ is of type E_6. Suppose $[\boldsymbol{G}, \boldsymbol{G}]$ is of type $E_6 + A_1$. Since $[\boldsymbol{M}, \boldsymbol{M}]$ is of type E_6, and since 3 is good for groups of type A, the result follows from Lemma 3.2, applied with $\boldsymbol{X}$ being the component of $\boldsymbol{G}$ of type E_6, and [ibid., Theorem 2.5].

So we have $[\boldsymbol{G}, \boldsymbol{G}]$ of type E_7 and $[\boldsymbol{M}, \boldsymbol{M}]$ of type E_6. Suppose that s is not quasi-isolated in $\boldsymbol{M}^*$. Then c is in Jordan correspondence with a block, say c' of a proper F-stable Levi subgroup, say $\boldsymbol{M}'$ of $\boldsymbol{M}$. The prime 3 is good for any proper Levi subgroup of $\boldsymbol{M}$, hence by [ibid., Theorem 1.7] condition (1) of Lemma 3.1 holds for the group $\boldsymbol{M}'$ and the block c'. By Jordan decomposition of blocks, this condition also holds for $\boldsymbol{M}$ and c, a contradiction. So, s is quasi-isolated in $\boldsymbol{M}^*$. Since as pointed out above, $\boldsymbol{G}$ has connected centre, so does $\boldsymbol{M}$ whence s is isolated in $\boldsymbol{M}^*$. Also, note that since s is also quasi-isolated in $\boldsymbol{G}^*$, by the same reasoning s is isolated in $\boldsymbol{G}^*$. Inspection shows that the only possible case for this is when s has order three with $C_{\boldsymbol{G}^*}(s)$ of type $A_5 + A_2$, $C_{\boldsymbol{M}^*}(s)$ of type $3A_2$. Since s is supposed to be a $3'$-element, this case does not arise here.

Now suppose that $\ell = 2$. Since $Z(\boldsymbol{H})/Z^\circ(\boldsymbol{H})$ has order dividing 2, by Lemma 3.3 we may assume that $\boldsymbol{G}$ has at least one nonclassical component, that is we are in one of the cases $[\boldsymbol{G}, \boldsymbol{G}] = E_6$, or $\boldsymbol{H} = E_8$ and $[\boldsymbol{G}, \boldsymbol{G}] = E_6 + A_1$ or E_7. Again, in all cases, $Z(\boldsymbol{G})$ is connected and consequently $C_{\boldsymbol{G}^*}(s)$ is connected and s is isolated.

Suppose first that $[\boldsymbol{G}, \boldsymbol{G}] = E_7$. We claim that all elements of $\mathcal{E}(\boldsymbol{G}^F, s)$ lie in the same 2-block. Indeed, let $\bar{s}$ be the image of s under the surjective map $\boldsymbol{G}^* \to [\boldsymbol{G}, \boldsymbol{G}]^*$ induced by the regular embedding of $[\boldsymbol{G}, \boldsymbol{G}]$ in $\boldsymbol{G}$. By [Kessar and Malle 2013, Table 4], all elements of $\mathcal{E}([\boldsymbol{G}, \boldsymbol{G}]^F, \bar{s})$ lie in the same 2-block, say d of $[\boldsymbol{G}, \boldsymbol{G}]^F$. So, any block of $\boldsymbol{G}^F$ which contains a character in $\mathcal{E}(\boldsymbol{G}^F, s)$ covers d. By general block theoretical reasons, there are at most $|\boldsymbol{G}^F/[\boldsymbol{G}, \boldsymbol{G}]^F|_{2'}$ 2-blocks of $\boldsymbol{G}^F$ covering a given d. Now since s is a $2'$-element, $C_{[\boldsymbol{G},\boldsymbol{G}]^*}(\bar{s})$ is connected. Thus, if $\mu \in \mathcal{E}([\boldsymbol{G}, \boldsymbol{G}]^F, \bar{s})$, then there are $|\boldsymbol{G}^F/[\boldsymbol{G}, \boldsymbol{G}]^F|_{2'}$ different $2'$-Lusztig series of $\boldsymbol{G}^F$ containing an irreducible character covering μ. Since characters in different $2'$-Lusztig series lie in different 2-blocks, the claim follows.

By the claim above, we may assume that either $[\boldsymbol{G}, \boldsymbol{G}] = E_6$ or $[\boldsymbol{G}, \boldsymbol{G}] = E_6 + A_1$. Since s is isolated of odd order in $\boldsymbol{G}^*$, by [Kessar and Malle 2013, Table 1] all components of $C_{\boldsymbol{G}^*}(s)$ are of type A_2 or A_1. Consequently, all components of $C_{\boldsymbol{M}^*}(s)$ are of type A. Suppose first that $\boldsymbol{M}$ has a nonclassical component. Then $[\boldsymbol{M}, \boldsymbol{M}]$ is of type E_6, and $[\boldsymbol{G}, \boldsymbol{G}] = E_6 + A_1$. This may be ruled out by Lemma 3.2, applied with $\boldsymbol{X}$ equal to the product of the component of type E_6 with $Z(\boldsymbol{G})$ and $\boldsymbol{Y}$ equal to the component of type A_1.

So finally suppose that all components of $\boldsymbol{M}$ are of classical type. Then, $C_{\boldsymbol{M}^*}(s) = C^\circ_{\boldsymbol{M}^*}(s)$ is a Levi subgroup of $\boldsymbol{M}$ with all components of type A. Hence, the first hypothesis of Lemma 3.1 holds by the Jordan decomposition of blocks and [Cabanes and Enguehard 1999, Theorem 1.7]. So, we may assume that the second hypothesis of Lemma 3.1 does not hold. Let

$$\boldsymbol{C} := C^\circ_{\boldsymbol{G}}(Z(\boldsymbol{M}^F)_2).$$

Since $\boldsymbol{M}$ is a proper e-split Levi subgroup of $\boldsymbol{G}$, and since $Z(\boldsymbol{G})$ is connected, by [Cabanes and Enguehard 1993, Proposition 2.1] $\boldsymbol{C}$ is proper in $\boldsymbol{G}$. By induction, we may assume that $\boldsymbol{C}$ is not a Levi subgroup of $\boldsymbol{G}$. In particular, the intersection of $\boldsymbol{C}$ with the component of type E_6 of $\boldsymbol{G}$ is proper in that component and hence all components of $\boldsymbol{C}$ are of type A or D. If all components of $\boldsymbol{C}$ are of type A, then 2 is good for $\boldsymbol{C}$ and the second hypothesis of Lemma 3.1 holds by [Cabanes and Enguehard 1999, Theorem 2.5]. Thus we may assume that $\boldsymbol{C}$ has a component of type D. Since all components of $\boldsymbol{C}$ are classical, by Lemma 3.3, we may assume that $Z(\boldsymbol{C})/Z^\circ(\boldsymbol{C})$ is not a 2-group and consequently $\boldsymbol{C}$ has a component of type A_n, with $n \equiv 2 \pmod 3$. But by the Borel–de Siebenthal algorithm, a group of type E_6 has no subsystem subgroup of type $D_m + A_n$ with $n \geq 1$ and $m \geq 4$. $\square$

Characters in ℓ-blocks. Using the results collected so far, it is now easy to characterise all characters in ℓ'-series inside a given ℓ-block in terms of Lusztig induction.

Definition 3.5. As in [Cabanes and Enguehard 1999, Section 1.11] (see also [Broué et al. 1993, Definition 3.1]) for e-split Levi subgroups $\boldsymbol{M}_1, \boldsymbol{M}_2$ of $\boldsymbol{G}$ and $\mu_i \in \mathrm{Irr}(\boldsymbol{M}_i^F)$, we write $(\boldsymbol{M}_1, \mu_1) \leq_e (\boldsymbol{M}_2, \mu_2)$ if $\boldsymbol{M}_1 \leq \boldsymbol{M}_2$ and μ_2 is a constituent of $R_{\boldsymbol{M}_1}^{\boldsymbol{M}_2}(\mu_1)$ (with respect to some parabolic subgroup of $\boldsymbol{M}_2$ with Levi subgroup $\boldsymbol{M}_1$). We let $\ll_e$ denote the transitive closure of the relation $\leq_e$.

As pointed out in [Cabanes and Enguehard 1999, Section 1.11] it seems reasonable to expect that the relations $\leq_e$ and $\ll_e$ coincide. While this is known to hold for unipotent characters (see [Broué et al. 1993, Theorem 3.11]), it is open in general.

We put ourselves in the situation and notation of Theorem A.

Theorem 3.6. *Let b be an ℓ-block of $\boldsymbol{G}^F$ and denote by $\mathcal{L}(b)$ the set of e-Jordan-cuspidal pairs $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ such that there is $\chi \in \mathrm{Irr}(b)$ with $\langle \chi, R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)\rangle \neq 0$. Then*

$$\mathrm{Irr}(b) \cap \mathcal{E}(\boldsymbol{G}^F, \ell') = \big\{\chi \in \mathcal{E}(\boldsymbol{G}^F, \ell') \mid \exists\, (\boldsymbol{L}, \lambda) \in \mathcal{L}(b) \textit{ with } (\boldsymbol{L}, \lambda) \ll_e (\boldsymbol{G}, \chi)\big\}.$$

Proof. Let b be as in the statement and first assume that $\chi \in \mathrm{Irr}(b) \cap \mathcal{E}(\boldsymbol{G}^F, \ell')$. If χ is not e-Jordan-cuspidal, then it is not e-cuspidal, so there exists a proper e-split Levi subgroup $\boldsymbol{M}_1$ such that χ occurs in $R_{\boldsymbol{M}_1}^{\boldsymbol{G}}(\mu_1)$ for some $\mu_1 \in \mathcal{E}(\boldsymbol{M}_1^F, \ell')$. Thus inductively we obtain a chain of e-split Levi subgroups $\boldsymbol{M}_r \lneq \ldots \lneq \boldsymbol{M}_1 \lneq \boldsymbol{M}_0 := \boldsymbol{G}$ and characters $\mu_i \in \mathcal{E}(\boldsymbol{M}_i^F, \ell')$ (with $\mu_0 := \chi$) such that $(\boldsymbol{M}_r, \mu_r)$ is e-Jordan cuspidal and such that $(\boldsymbol{M}_i, \mu_i) \leq_e (\boldsymbol{M}_{i-1}, \mu_{i-1})$ for $i = 1, \ldots, r$, whence $(\boldsymbol{M}_r, \mu_r) \ll_e (\boldsymbol{G}, \chi)$. Let b_r be the ℓ-block of $\boldsymbol{M}_r^F$ containing μ_r. Now Theorem 3.4 yields that for each i there exists a block, say b_i, of $\boldsymbol{M}_i^F$ such that all constituents of $R_{\boldsymbol{M}_i}^{\boldsymbol{M}_{i-1}}(\zeta_i)$ lie in b_{i-1} for all $\zeta_i \in \mathrm{Irr}(b_i) \cap \mathcal{E}(\boldsymbol{M}_i^F, \ell')$. In particular, χ lies in b_0, so $b_0 = b$, and thus $(\boldsymbol{M}_r, \mu_r) \in \mathcal{L}(b)$.

For the reverse inclusion, let $(\boldsymbol{L}, \lambda) \in \mathcal{L}(b)$ and $\chi \in \mathrm{Irr}(\boldsymbol{G}^F, \ell')$ such that $(\boldsymbol{L}, \lambda) \ll_e (\boldsymbol{G}, \chi)$. Thus there exists a chain of e-split Levi subgroups $\boldsymbol{L} = \boldsymbol{M}_r \lneq \ldots \lneq \boldsymbol{M}_0 = \boldsymbol{G}$ and characters $\mu_i \in \mathrm{Irr}(\boldsymbol{M}_i^F)$ with $(\boldsymbol{M}_i, \mu_i) \leq_e (\boldsymbol{M}_{i-1}, \mu_{i-1})$. Again, an application of Theorem 3.4 allows us to conclude that $\chi \in \mathrm{Irr}(b)$. □

ℓ-blocks and derived subgroups. In the following two results, which will be used in showing that the map Ξ in Theorem A is surjective, $\boldsymbol{G}$ is connected reductive with Frobenius endomorphism F, and $\boldsymbol{G}_0 := [\boldsymbol{G}, \boldsymbol{G}]$. Here, in the cases that the Mackey formula is not known to hold we assume that $R_{\boldsymbol{L}_0}^{\boldsymbol{G}_0}$ and $R_{\boldsymbol{L}}^{\boldsymbol{G}}$ are with respect to a choice of parabolic subgroups $\boldsymbol{P}_0 \geq \boldsymbol{L}_0$ and $\boldsymbol{P} \geq \boldsymbol{L}$ such that $\boldsymbol{P}_0 = \boldsymbol{G}_0 \cap \boldsymbol{P}$.

Lemma 3.7. *Let b be an ℓ-block of $\boldsymbol{G}^F$ and let b_0 be an ℓ-block of $\boldsymbol{G}_0^F$ covered by b. Let $\boldsymbol{L}$ be an F-stable Levi subgroup of $\boldsymbol{G}$, $\boldsymbol{L}_0 = \boldsymbol{L} \cap \boldsymbol{G}_0$ and let $\lambda_0 \in \mathrm{Irr}(\boldsymbol{L}_0^F)$. Suppose that every irreducible constituent of $R_{\boldsymbol{L}_0}^{\boldsymbol{G}_0}(\lambda_0)$ is contained in b_0. Then* *there exists $\lambda \in \operatorname{Irr}(\boldsymbol{L}^F)$ and $\chi \in \operatorname{Irr}(b)$ such that λ_0 is an irreducible constituent of $\operatorname{Res}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda)$ and χ is an irreducible constituent of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$.*

Proof. Since $\boldsymbol{G} = Z^\circ(\boldsymbol{G})\boldsymbol{G}_0$, by [Bonnafé 2006, Proposition 10.10] we have that

$$R_{\boldsymbol{L}}^{\boldsymbol{G}} \operatorname{Ind}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda_0) = \operatorname{Ind}^{\boldsymbol{G}^F}_{\boldsymbol{G}_0^F} R^{\boldsymbol{G}_0}_{\boldsymbol{L}_0}(\lambda_0).$$

Note that the result in [Bonnafé 2006] is only stated for the case that $\boldsymbol{G}$ has connected centre but the proof does not use this hypothesis. The right hand side of the above equality evaluated at 1 is nonzero. Let $\chi' \in \operatorname{Irr}(\boldsymbol{G}^F)$ be a constituent of the left hand side of the equality. There exists $\lambda \in \operatorname{Irr}(\boldsymbol{L}^F)$ and χ_0 in $\operatorname{Irr}(\boldsymbol{G}_0^F)$ such that λ is an irreducible constituent of $\operatorname{Ind}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda_0)$, χ' is an irreducible constituent of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$, χ_0 is an irreducible constituent of $R^{\boldsymbol{G}_0}_{\boldsymbol{L}_0}(\lambda_0)$ and χ' is an irreducible constituent of $\operatorname{Ind}^{\boldsymbol{G}^F}_{\boldsymbol{G}_0^F}(\chi_0)$. Since $\chi_0 \in \operatorname{Irr}(b_0)$, χ' lies in a block, say b', of $\boldsymbol{G}^F$ which covers b_0. Since b also covers b_0 and since $\boldsymbol{G}^F/\boldsymbol{G}_0^F$ is abelian, there exists a linear character, say θ of $\boldsymbol{G}^F/\boldsymbol{G}_0^F$ such that $b = b' \otimes \theta$ (see [Kessar and Malle 2013, Lemma 2.2]). Now the result follows from [Bonnafé 2006, Proposition 10.11] with $\chi = \chi' \otimes \theta$. □

Lemma 3.8. *Let b be an ℓ-block of $\boldsymbol{G}^F$ and let $\boldsymbol{L}$ be an F-stable Levi subgroup of $\boldsymbol{G}$ and $\lambda \in \operatorname{Irr}(\boldsymbol{L}^F)$ such that every irreducible constituent of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ is contained in b. Let $\boldsymbol{L}_0 = \boldsymbol{L} \cap \boldsymbol{G}_0$ and let $\lambda_0 \in \operatorname{Irr}(\boldsymbol{L}_0^F)$ be an irreducible constituent of $\operatorname{Res}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda)$. Then there exists an ℓ-block b_0 of $\boldsymbol{G}_0^F$ covered by b and an irreducible character χ_0 of $\boldsymbol{G}_0^F$ in the block b_0 such that χ_0 is a constituent of $R^{\boldsymbol{G}_0}_{\boldsymbol{L}_0}(\lambda_0)$.*

Proof. Arguing as in the proof of Lemma 3.7, there exist $\chi \in \operatorname{Irr}(\boldsymbol{G}^F)$, $\lambda' \in \operatorname{Irr}(\boldsymbol{L}^F)$ and χ_0 in $\operatorname{Irr}([\boldsymbol{G},\boldsymbol{G}]^F)$ such that λ' is an irreducible constituent of $\operatorname{Ind}^{\boldsymbol{L}^F}_{\boldsymbol{L}_0^F}(\lambda_0)$, χ is an irreducible constituent of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda')$, χ_0 is an irreducible constituent of $R^{[\boldsymbol{G},\boldsymbol{G}]}_{\boldsymbol{L}_0}(\lambda_0)$ and χ is an irreducible constituent of $\operatorname{Ind}^{\boldsymbol{G}^F}_{[\boldsymbol{G},\boldsymbol{G}]^F}(\chi_0)$. Now, $\lambda = \theta \otimes \lambda'$ for some linear character θ of $\boldsymbol{L}^F/\boldsymbol{L}_0^F$. By [Bonnafé 2006, Proposition 10.11], $\theta \otimes \chi$ is an irreducible constituent of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$, and hence $\theta \otimes \chi \in \operatorname{Irr}(b)$. Further, $\theta \otimes \chi$ is also a constituent of $\operatorname{Ind}^{\boldsymbol{G}^F}_{[\boldsymbol{G},\boldsymbol{G}]^F}(\chi_0)$, hence b covers the block of $[\boldsymbol{G},\boldsymbol{G}]^F$ containing χ_0. □

Unique maximal abelian normal subgroups. A crucial ingredient for proving injectivity of the map in parts (d) and (e) of Theorem A is a property related to the nonfailure of factorisation phenomenon of finite group theory, which holds for the defect groups of many blocks of finite groups of Lie type and which was highlighted by Cabanes [1994]: for a prime ℓ, an ℓ-group is said to be *Cabanes* if it has a unique maximal abelian normal subgroup.

Now first consider the following setting: let $\boldsymbol{G}$ be connected reductive. For $i = 1, 2$, let $\boldsymbol{L}_i$ be an F-stable Levi subgroup of $\boldsymbol{G}$ with $\lambda_i \in \mathcal{E}(\boldsymbol{L}_i^F, \ell')$, and let u_i denote the ℓ-block of $\boldsymbol{L}_i^F$ containing λ_i. Suppose that $C_{\boldsymbol{G}}(Z(\boldsymbol{L}_i^F)_\ell) = \boldsymbol{L}_i$ and that λ_i is of quasicentral ℓ-defect. Then by [Kessar and Malle 2013, Propositions

2.12, 2.13, 2.16] there exists a block b_i of $\boldsymbol{G}^F$ such that all irreducible characters of $R_{\boldsymbol{L}_i}^{\boldsymbol{G}}(\lambda_i)$ lie in b_i and $(Z(\boldsymbol{L}_i^F)_\ell, u_i)$ is a b_i-Brauer pair.

Lemma 3.9. *In the above situation, assume further that for $i = 1, 2$ there exists a maximal b_i-Brauer pair (P_i, c_i) such that $(Z(\boldsymbol{L}_i^F)_\ell, u_i) \trianglelefteq (P_i, c_i)$, and such that P_i is Cabanes with $Z(\boldsymbol{L}_i^F)_\ell$ as the unique maximal abelian normal subgroup of P_i. If $b_1 = b_2$ then the pairs $(\boldsymbol{L}_1, \lambda_1)$ and $(\boldsymbol{L}_2, \lambda_2)$ are $\boldsymbol{G}^F$-conjugate.*

Proof. Suppose that $b_1 = b_2$. Since maximal b_1-Brauer pairs are $\boldsymbol{G}^F$-conjugate, it follows that ${}^g(Z(\boldsymbol{L}_2^F)_\ell, u_2) \le {}^g(P_2, c_2) = (P_1, c_1)$ for some $g \in \boldsymbol{G}^F$. By transport of structure, ${}^gZ(\boldsymbol{L}_2^F)_\ell$ is a maximal normal abelian subgroup of P_1, hence ${}^gZ(\boldsymbol{L}_2^F)_\ell = Z(\boldsymbol{L}_1^F)_\ell$. By the uniqueness of inclusion of Brauer pairs it follows that ${}^g(Z(\boldsymbol{L}_2^F)_\ell, u_2) = (Z(\boldsymbol{L}_1)_\ell^F, u_1)$. Since $\boldsymbol{L}_i = C_{\boldsymbol{G}}(Z(\boldsymbol{L}_i^F)_\ell)$, this means that ${}^g\boldsymbol{L}_2 = \boldsymbol{L}_1$. Further, since λ_i is of quasicentral ℓ-defect, by [Kessar and Malle 2013, Proposition 2.5(f)], λ_i is the unique element of $\mathcal{E}(\boldsymbol{L}_i^F, \ell') \cap \mathrm{Irr}(u_i)$. Thus ${}^gu_2 = u_1$ implies that ${}^g\lambda_2 = \lambda_1$ and $(\boldsymbol{L}_1, \lambda_1)$ and $(\boldsymbol{L}_2, \lambda_2)$ are $\boldsymbol{G}^F$-conjugate as required. □

By the proof of Theorems 4.1 and 4.2 of [Cabanes and Enguehard 1999] we also have:

Proposition 3.10. *Let $\boldsymbol{G}$ be connected reductive with simply connected derived subgroup. Suppose that $\ell \ge 3$ is good for $\boldsymbol{G}$, and $\ell \ne 3$ if $\boldsymbol{G}^F$ has a factor ${}^3D_4(q)$. Let b be an ℓ-block of $\boldsymbol{G}^F$ such that the defect groups of b are Cabanes. If $(\boldsymbol{L}, \lambda)$ and $(\boldsymbol{L}', \lambda')$ are e-Jordan-cuspidal pairs of $\boldsymbol{G}$ such that $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$, $\lambda' \in \mathcal{E}(\boldsymbol{L}'^F, \ell')$ with $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda) = b = b_{\boldsymbol{G}^F}(\boldsymbol{L}', \lambda')$, then $(\boldsymbol{L}, \lambda)$ and $(\boldsymbol{L}', \lambda')$ are $\boldsymbol{G}^F$-conjugate.*

Proof. This is essentially contained in Section 4 of [Cabanes and Enguehard 1999]; all references in this proof are to this paper. Indeed, let $(\boldsymbol{L}, \lambda)$ be an e-Jordan-cuspidal pair of $\boldsymbol{G}$ such that $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$. Let $\boldsymbol{T}^*$, $\boldsymbol{T}$, $\boldsymbol{K} = C_{\boldsymbol{G}}^\circ(Z(\boldsymbol{L})_\ell^F)$, $\boldsymbol{K}^*$, $\boldsymbol{M}$ and $\boldsymbol{M}^*$ be as in the notation before Lemma 4.4. Let $Z = Z(\boldsymbol{M})_\ell^F$ and let $\lambda_{\boldsymbol{K}}$ and $\lambda_{\boldsymbol{M}}$ be as in Definition 4.6, with λ replacing ζ. Then $Z \le \boldsymbol{T}$ and by Lemma 4.8, $\boldsymbol{M} = C_{\boldsymbol{G}}^\circ(Z)$. The simply connected hypothesis and the restrictions on ℓ imply that $C_{\boldsymbol{G}}(Z) = C_{\boldsymbol{G}}^\circ(Z) = \boldsymbol{M}$. Let $b_Z = \hat{b}_Z$ be the ℓ-block of $\boldsymbol{M}^F$ containing $\lambda_{\boldsymbol{M}}$. Then by Lemma 4.13, (Z, b_Z) is a self centralising Brauer pair and $(1, b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)) \le (Z, b_Z)$. Further, by Lemma 4.16 there exists a maximal b-Brauer pair (D, b_D) such that $(Z, b_Z) \le (D, b_D)$, Z is normal in D and $C_D(Z) = Z$. Note that the first three conclusions of Lemma 4.16 hold under the conditions we have on ℓ (it is only the fourth conclusion which requires $\ell \in \Gamma(\boldsymbol{G}, F)$). By Lemma 4.10 and its proof, we also have

$$(1, b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)) \le (Z(\boldsymbol{L})_\ell^F, b_{\boldsymbol{K}^F}(\boldsymbol{L}, \lambda)) \le (Z, b_Z).$$

Suppose that $\boldsymbol{N}$ is a proper e-split Levi subgroup of $\boldsymbol{G}$ containing $C_{\boldsymbol{G}}^\circ(z) = C_{\boldsymbol{G}}(z)$ for some $1 \ne z \in Z(D)\boldsymbol{G_a} \cap \boldsymbol{G_b}$. Then $\boldsymbol{N}$ contains $\boldsymbol{L}$, $\boldsymbol{M}$ and Z by Lemma 4.15(b). Since $\boldsymbol{L} \cap \boldsymbol{G_b} = \boldsymbol{K} \cap \boldsymbol{G_b}$ by Lemma 4.4(iii), it follows that $\boldsymbol{N}$ also contains $\boldsymbol{K}$ and

$\boldsymbol{K} = C_{\boldsymbol{N}}(Z(\boldsymbol{L}^F))$. Thus, replacing $\boldsymbol{G}$ with $\boldsymbol{N}$ in Lemma 4.13 we get that

$$(1, b_{\boldsymbol{N}^F}(\boldsymbol{L}, \lambda)) \leq (Z(\boldsymbol{L})_\ell^F, b_{\boldsymbol{K}^F}(\boldsymbol{L}, \lambda)) \leq (D, b_D).$$

Let $(\boldsymbol{L}', \lambda')$ be another e-Jordan-cuspidal pair of $\boldsymbol{G}$ with $\lambda' \in \mathcal{E}(\boldsymbol{L}'^F, \ell')$ such that $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda) = b = b_{\boldsymbol{G}^F}(\boldsymbol{L}', \lambda')$. Denote by $\boldsymbol{K}'$, $\boldsymbol{M}'$, D' etc. the corresponding groups and characters for $(\boldsymbol{L}', \lambda')$. Up to replacing by a $\boldsymbol{G}^F$-conjugate, we may assume that $(D', b_{D'}) = (D, b_D)$.

Suppose first that there is a $1 \neq z \in Z(D)\boldsymbol{G_a} \cap \boldsymbol{G_b}$. By Lemma 4.15(b), there is a proper e-split Levi subgroup $\boldsymbol{N}$ containing $C_{\boldsymbol{G}}(z)$. Moreover, $\boldsymbol{N}$ contains D, $\boldsymbol{L}'$, $\boldsymbol{M}'$, $\boldsymbol{K}'$ and $\boldsymbol{G_a}$ and we also have

$$(1, b_{\boldsymbol{N}^F}(\boldsymbol{L}', \lambda')) \leq (Z(\boldsymbol{L}')_\ell^F, b_{\boldsymbol{K}'^F}(\boldsymbol{L}', \lambda')) \leq (D, b_D).$$

By the uniqueness of inclusion of Brauer pairs it follows that $b_{\boldsymbol{N}^F}(\boldsymbol{L}, \lambda) = b_{\boldsymbol{N}^F}(\boldsymbol{L}', \lambda')$. Also D is a defect group of $b_{\boldsymbol{N}^F}(\boldsymbol{L}, \lambda)$. Thus, in this case we are done by induction.

So, we may assume that $Z(D) \leq \boldsymbol{G_a}$ hence $D \leq \boldsymbol{G_a}$. From here on, the proof of Lemma 4.17 goes through without change, the only property that is used being that Z is the unique maximal abelian normal subgroup of D. $\square$

We will also need the following observation:

Lemma 3.11. *Let $P = P_1 \times P_2$ where P_1 and P_2 are Cabanes. Suppose that P_0 is a normal subgroup of P such that $\pi_i(P_0) = P_i$, $i = 1, 2$, where $\pi_i : P_1 \times P_2 \to P_i$ denote the projection maps. Then P_0 is Cabanes with maximal normal abelian subgroup $(A_1 \times A_2) \cap P_0$, where A_i is the unique maximal normal abelian subgroup of P_i, $i = 1, 2$.*

Proof. Let $A = A_1 \times A_2$. The group $A \cap P_0$ is abelian and normal in P_0. Let S be a normal abelian subgroup of P_0. Since $\pi_i(P_0) = P_i$, $\pi_i(S)$ is normal in P_i and since S is abelian, so is $\pi_i(S)$. Thus, $\pi_i(S)$ is a normal abelian subgroup of P_i and is therefore contained in A_i. So, $S \leq (\pi_1(S) \times \pi_2(S)) \cap P_0 \leq (A_1 \times A_2) \cap P_0 = A \cap P_0$ and the result is proved. $\square$

Linear and unitary groups at $\ell = 3$. The following will be instrumental in the proof of statement (e) of Theorem A.

Lemma 3.12. *Let q be a prime power such that $3 \mid (q-1)$ (respectively $3 \mid (q+1)$). Let $G = \mathrm{SL}_n(q)$ (respectively $\mathrm{SU}_n(q)$) and let P be a Sylow 3-subgroup of G. Then P is Cabanes unless $n = 3$ and $3 \parallel (q-1)$ (respectively $3 \parallel (q+1)$). In particular, if P is not Cabanes, then P is extra-special of order 27 and exponent 3. In this case $N_G(P)$ acts transitively on the set of subgroups of order 9 of P.*

Proof. Embed $P \leq \mathrm{SL}_n(q) \leq \mathrm{GL}_n(q)$. A Sylow 3-subgroup of $\mathrm{GL}_n(q)$ is contained in the normaliser $C_{q-1} \wr \mathfrak{S}_n$ of a maximally split torus. According to [Cabanes 1994, Lemme 4.1], the only case in which $\mathfrak{S}_n$ has a quadratic element on $(C_{q-1}^n)_3 \cap \mathrm{SL}_n(q)$ is when $n = 3$ and $3 \parallel (q-1)$. If there is no quadratic element in this action, then

P is Cabanes by [Cabanes 1994, Proposition 2.3]. In the case of $\mathrm{SU}_n(q)$, the same argument applies with the normaliser $C_{q+1} \wr \mathfrak{S}_n$ of a Sylow 2-torus inside $\mathrm{GU}_n(q)$.

Now assume we are in the exceptional case. Clearly $|P| = 27$. Let $P_1, P_2 \leq P$ be subgroups of order 9, and let $u_i \in P_i$ be noncentral. Then u_i is G-conjugate to $\mathrm{diag}(1, \zeta, \zeta^2)$, where ζ is a primitive 3rd-root of unity in $\mathbb{F}_q$ (respectively $\mathbb{F}_{q^2}$). In particular, there exists $g \in G$ such that ${}^g u_1 = u_2$. Let $\bar{\ }: G \to G/Z(G)$ denote the canonical map. Then ${}^{\bar{g}}(\bar{u}_1) = \bar{u}_2$. Since the Sylow 3-subgroup $\bar{P}$ of $\bar{G}$ is abelian, there exists $\bar{h} \in N_{\bar{G}}(\bar{P})$ with ${}^{\bar{h}}(\bar{u}_1) = \bar{u}_2$. Then $h \in N_G(P)$ and ${}^h P_1 = P_2$ as $P_i = \langle Z(G), u_i \rangle$. $\square$

Lemma 3.13. *Suppose that* $3 \parallel n$ *and* $3 \parallel (q-1)$ (*respectively* $3 \parallel (q+1)$). *Let* $\tilde{\boldsymbol{G}} = \mathrm{GL}_n$, $\boldsymbol{G} = \mathrm{SL}_n$ *and suppose that* $\tilde{\boldsymbol{G}}^F = \mathrm{GL}_n(q)$ (*respectively* $\mathrm{GU}_n(q)$). *Let s be a semisimple $3'$-element of $\tilde{\boldsymbol{G}}^F$ such that a Sylow 3-subgroup D of $C_{\boldsymbol{G}^F}(s)$ is extra-special of order 27 and let $P_1, P_2 \leq D$ have order 9. There exists $g \in N_{\boldsymbol{G}^F}(D) \cap C_{\boldsymbol{G}^F}(C_{\boldsymbol{G}^F}(D))$ such that ${}^g P_1 = P_2$.*

Proof. Set $d = \frac{n}{3}$. Identify $\tilde{\boldsymbol{G}}$ with the group of linear transformations of an n-dimensional $\mathbb{F}_q$-vector space V with chosen basis $\{e_{i,r} \mid 1 \leq i \leq d, 1 \leq r \leq 3\}$. For $g \in \tilde{\boldsymbol{G}}$, write $a(g)_{i,r,j,s}$ for the coefficient of $e_{i,r}$ in $g(e_{j,s})$. Let $w \in \tilde{\boldsymbol{G}}$ be defined by $w(e_{i,r}) = e_{i+1,r}$, $1 \leq i \leq d$, $1 \leq r \leq 3$. For $1 \leq i \leq d$ let V_i be the span of $\{e_{i,1}, e_{i,2}, e_{i,3}\}$ and $\tilde{\boldsymbol{G}}_i = \mathrm{GL}(V_i)$ considered as a subgroup of $\tilde{\boldsymbol{G}}$ through the direct sum decomposition $V = \bigoplus_{1 \leq i \leq d} V_i$.

Up to conjugation in $\tilde{\boldsymbol{G}}$ we may assume $F = \mathrm{ad}_w \circ F_0$, where F_0 is the standard Frobenius morphism which raises every matrix entry to its q-th power in the linear case, respectively the composition of the latter by the transpose inverse map in the unitary case. Note that then each $\tilde{\boldsymbol{G}}_i$ is F_0-stable.

Thus, given the hypothesis on the structure of D, we may assume the following up to conjugation: s has d distinct eigenvalues $\delta_1, \ldots, \delta_d$ with $\delta_{i+1} = \delta_i^q$ (respectively δ_i^{-q}); V_i is the δ_i-eigenspace of s, and $C_{\tilde{\boldsymbol{G}}}(s) = \prod_{i=1}^d \tilde{\boldsymbol{G}}_i$. Further, $F(\tilde{\boldsymbol{G}}_i) = \tilde{\boldsymbol{G}}_{i+1}$ and denoting by $\Delta : \tilde{\boldsymbol{G}}_1 \to \prod_{i=1}^d \tilde{\boldsymbol{G}}_i$, $x \mapsto xF(x) \cdots F^{d-1}(x)$, the twisted diagonal map we have $C_{\tilde{\boldsymbol{G}}^F}(s) = \Delta(\tilde{\boldsymbol{G}}_1^{F^d})$. Here, $\tilde{\boldsymbol{G}}_1^{F^d} = \tilde{\boldsymbol{G}}_1^{F_0^d}$ is isomorphic to either $\mathrm{GL}_3(q^d)$ or $\mathrm{GU}_3(q^d)$. Note that $\mathrm{GU}_3(q^d)$ occurs only if d is odd.

Consider $\tilde{\boldsymbol{G}}_1^{F_0} \leq \tilde{\boldsymbol{G}}_1^{F_0^d}$. Let U_1 be the Sylow 3-subgroup of the diagonal matrices in $\tilde{\boldsymbol{G}}_1^{F_0}$ of determinant 1 and let $\sigma_1 \in \tilde{\boldsymbol{G}}_1^{F_0}$ be defined by $\sigma_1(e_{1,r}) = e_{1,r+1}$ for $1 \leq r \leq 3$. Then $D_1 := \langle U_1, \sigma_1 \rangle$ is a Sylow 3-subgroup of $\tilde{\boldsymbol{G}}_1^{F_0}$. Since by hypothesis the Sylow 3-subgroups of $C_{\boldsymbol{G}^F}(s)$ have order 27, $D := \Delta(D_1)$ is a Sylow 3-subgroup of $C_{\boldsymbol{G}^F}(s)$ with $\Delta(U_1) \cong U_1$ elementary abelian of order 9. Note that $\Delta(\sigma_1)(e_{i,r}) = e_{i,r+1}$ for $1 \leq i \leq d$ and $1 \leq r \leq 3$.

Let $\zeta \in \overline{\mathbb{F}}_q$ be a primitive third root of unity. Let $u_1 \in U_1$ be such that $u_1(e_{1,r}) = \zeta^r e_{1,r}$ for $1 \leq r \leq 3$. For $1 \leq r \leq 3$, let W_r be the span of $\{e_{1,r}, \ldots, e_{d,r}\}$. Then W_r is the ζ^r-eigenspace of $\Delta(u_1)$, whence

$$C_{\tilde{\boldsymbol{G}}}(D) \leq C_{\tilde{\boldsymbol{G}}}(\Delta(U_1)) = C_{\tilde{\boldsymbol{G}}}(\Delta(u_1)) = \prod_{1\leq r\leq 3} \mathrm{GL}(W_r).$$

Since $\Delta(\sigma_1)(W_r) = W_{r+1}$, and $\Delta(\sigma_1)$ acts on $C_{\tilde{\boldsymbol{G}}}(\Delta(U_1))$, it follows that $C_{\tilde{\boldsymbol{G}}}(D) = \Delta'(\mathrm{GL}(W_1))$, where $\Delta' : \mathrm{GL}(W_1) \to \prod_{1\leq r\leq 3} \mathrm{GL}(W_r)$, $x \mapsto x\,{}^{\sigma}x\,{}^{\sigma^2}x$, is the twisted diagonal.

We claim that $\Delta(\tilde{\boldsymbol{G}}_1^{F_0})$ centralises $C_{\tilde{\boldsymbol{G}}}(D)$. Indeed, note that $g \in \Delta(\tilde{\boldsymbol{G}}_1^{F_0})$ if and only if $a(g)_{i,r,j,s} = 0$ if $i \neq j$ and $a(g)_{i,r,i,s} = a(F_0^{i-1}(g))_{1,r,1,s} = a(g)_{1,r,1,s}$ for all i and all r, s. Also, $h \in C_{\tilde{\boldsymbol{G}}}(D)$ if and only if $a(h)_{i,r,j,s} = 0$ if $r \neq s$ and $a(h)_{i,r,j,r} = a(h)_{i,1,j,1}$ for all i, j and all r. The claim follows from an easy matrix multiplication.

Let $H = [\tilde{\boldsymbol{G}}_1^{F_0}, \tilde{\boldsymbol{G}}_1^{F_0}]$ and note that $D_1 \leq H$. By Lemma 3.12 applied to H any two subgroups of D_1 of order 9 are conjugate by an element of $N_H(D_1)$. The lemma follows from the claim above. $\square$

Parametrising ℓ-blocks. We can now prove our main theorem, Theorem A, which we restate. Recall Definition 2.1 of e-Jordan (quasicentral) cuspidal pairs.

Theorem 3.14. *Let $\boldsymbol{H}$ be a simple algebraic group of simply connected type with a Frobenius endomorphism $F : \boldsymbol{H} \to \boldsymbol{H}$ endowing $\boldsymbol{H}$ with an $\mathbb{F}_q$-rational structure. Let $\boldsymbol{G}$ be an F-stable Levi subgroup of $\boldsymbol{H}$. Let ℓ be a prime not dividing q and set $e = e_\ell(q)$.*

(a) *For any e-Jordan-cuspidal pair $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ such that $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$, there exists a unique ℓ-block $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}^F$ such that all irreducible constituents of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ lie in $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$.*

(b) *The map $\Xi : (\boldsymbol{L}, \lambda) \mapsto b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ is a surjection from the set of $\boldsymbol{G}^F$-conjugacy classes of e-Jordan-cuspidal pairs $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ with $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ to the set of ℓ-blocks of $\boldsymbol{G}^F$.*

(c) *The map Ξ restricts to a surjection from the set of $\boldsymbol{G}^F$-conjugacy classes of e-Jordan quasicentral cuspidal pairs $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ with $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ to the set of ℓ-blocks of $\boldsymbol{G}^F$.*

(d) *For $\ell \geq 3$ the map Ξ restricts to a bijection between the set of $\boldsymbol{G}^F$-conjugacy classes of e-Jordan quasicentral cuspidal pairs $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$ with $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ and the set of ℓ-blocks of $\boldsymbol{G}^F$.*

(e) *The map Ξ itself is bijective if $\ell \geq 3$ is good for $\boldsymbol{G}$, and moreover $\ell \neq 3$ if $\boldsymbol{G}^F$ has a factor ${}^3D_4(q)$.*

Remark 3.15. Note that (e) is best possible. See [Enguehard 2000; Kessar and Malle 2013] for counterexamples to the conclusion for bad primes, and [Enguehard 2000, p. 348] for a counterexample in the case $\ell = 3$ and $\boldsymbol{G}^F = {}^3D_4(q)$. Counterexamples in the case $\ell = 2$ and $\boldsymbol{G}$ of type A_n occur in the following situation. Let

$\boldsymbol{G}^F = \mathrm{SL}_n(q)$ with $4 \mid (q+1)$. Then $e = 2$ and the unipotent 2-(Jordan-)cuspidal pairs of $\boldsymbol{G}^F$ correspond to 2-cores of partitions of $n-1$ (see [Broué et al. 1993, §3A]). On the other hand, by [Cabanes and Enguehard 1993, Theorem 13], $\boldsymbol{G}^F$ has a unique unipotent 2-block.

Also, part (d) is best possible as the next example shows.

Example 3.16. Consider $\boldsymbol{G} = \mathrm{SL}_n$ with $n > 1$ odd, $\tilde{\boldsymbol{G}} = \mathrm{GL}_n$, and let $\boldsymbol{G}^F = \mathrm{SL}_n(q)$ be such that $q \equiv 1 \pmod n$ and $4 \mid (q+1)$. Then for $\ell = 2$ we have $e = e_2(q) = 2$, and $\mathbb{F}_q$ contains a primitive n-th root of unity, say ζ. Let $\tilde{s} = \mathrm{diag}(1, \zeta, \dots, \zeta^{n-1}) \in \tilde{\boldsymbol{G}}^{*F}$ and let s be its image in $\boldsymbol{G}^* = \mathrm{PGL}_n$. Then $C^\circ_{\boldsymbol{G}^*}(s)$ is the maximal 1-torus consisting of the image of the diagonal torus of $\tilde{\boldsymbol{G}}^*$. Thus, $(C^\circ_{\boldsymbol{G}^*}(s))_2 = 1 = Z^\circ(\boldsymbol{G}^*)_2$.

As $|C_{\boldsymbol{G}^*}(s)^F : C^\circ_{\boldsymbol{G}^*}(s)^F| = n$ we have $|\mathcal{E}(\boldsymbol{G}^F, s)| = n$, and all of these characters are 2-Jordan quasicentral cuspidal. We claim that all elements of $\mathcal{E}(\boldsymbol{G}^F, s)$ lie in the same 2-block of $\boldsymbol{G}^F$, so do not satisfy the conclusion of Theorem 3.14(d).

Let $\tilde{\boldsymbol{T}}$ be a maximal torus of $\tilde{\boldsymbol{G}}$ in duality with $C_{\tilde{\boldsymbol{G}}^*}(s)$ and let $\tilde{\theta} \in \mathrm{Irr}(\tilde{\boldsymbol{T}}^F)$ in duality with $\tilde{s}$. Let $\boldsymbol{T} = \tilde{\boldsymbol{T}} \cap \boldsymbol{G}$, and let $\theta = \tilde{\theta}|_{\boldsymbol{T}^F}$. Since $\tilde{s}$ is regular, $\tilde{\lambda} := R^{\tilde{\boldsymbol{G}}}_{\tilde{\boldsymbol{T}}}(\theta) \in \mathrm{Irr}(\tilde{\boldsymbol{G}}^F)$, and $\mathcal{E}(\tilde{\boldsymbol{G}}^F, \tilde{s}) = \{\tilde{\lambda}\}$. Further, $\tilde{\lambda}$ covers every element of $\mathcal{E}(\boldsymbol{G}^F, s)$. By [Bonnafé 2005, Proposition 10.10(b*)],

$$R^{\boldsymbol{G}}_{\boldsymbol{T}}(\theta) = \mathrm{Res}^{\tilde{\boldsymbol{G}}^F}_{\boldsymbol{G}^F} R^{\tilde{\boldsymbol{G}}}_{\tilde{\boldsymbol{T}}}(\tilde{\theta}) = \mathrm{Res}^{\tilde{\boldsymbol{G}}^F}_{\boldsymbol{G}^F}(\tilde{\lambda}).$$

Thus, every element of $\mathcal{E}(\boldsymbol{G}^F, s)$ is a constituent of $R^{\boldsymbol{G}}_{\boldsymbol{T}}(\theta)$. On the other hand, since $\tilde{\boldsymbol{T}}$ is the torus of diagonal matrices, we have $\boldsymbol{T} = C_{\boldsymbol{G}}(\boldsymbol{T}_2^F)$ by explicit computation. Hence by [Kessar and Malle 2013, Propositions 2.12, 2.13(1), 2.16(1)], all constituents of $R^{\boldsymbol{G}}_{\boldsymbol{T}}(\theta)$ lie in a single 2-block of $\boldsymbol{G}^F$.

Proof of Theorem 3.14. Parts (a) and (b) are immediate from Theorem 3.4 and the proof of Theorem 3.6. We next consider part (e), where it remains to show injectivity under the given assumptions. By [Cabanes and Enguehard 1999, Theorem 4.1 and Remark 5.2] only $\ell = 3$ and $\boldsymbol{G}$ of (possibly twisted) type A_n remains to be considered. Note that the claim holds if $3 \in \Gamma(\boldsymbol{G}, F)$ by [Cabanes and Enguehard 1999, Section 5.2]. Thus we may assume that the ambient simple algebraic group $\boldsymbol{H}$ of simply connected type is either SL_m or E_6, and $3 \notin \Gamma(\boldsymbol{G}, F)$. By Proposition 3.10 the claim holds for all blocks whose defect groups are Cabanes.

Let first $\boldsymbol{H} = \mathrm{SL}_m$ and $\boldsymbol{G} \le \boldsymbol{H}$ be an F-stable Levi subgroup. As $3 \notin \Gamma(\boldsymbol{G}, F)$ we have $3 \mid (q-1)$ when F is untwisted. We postpone the twisted case for a moment. Embed $\boldsymbol{H} \hookrightarrow \tilde{\boldsymbol{H}} = \mathrm{GL}_m$. Then $\tilde{\boldsymbol{G}} = \boldsymbol{G}Z(\boldsymbol{H})$ is an F-stable Levi subgroup of $\tilde{\boldsymbol{H}}$, so has connected centre. Moreover, as $\tilde{\boldsymbol{H}}$ is self-dual, so is its Levi subgroup $\tilde{\boldsymbol{G}}$. In particular, $3 \in \Gamma(\tilde{\boldsymbol{G}}, F)$. Now let b be a 3-block of $\boldsymbol{G}^F$ in $\mathcal{E}_3(\boldsymbol{G}^F, s)$, with $s \in \boldsymbol{G}^{*F}$ a semisimple $3'$-element. Let $\tilde{b}$ be a block of $\tilde{\boldsymbol{G}}$ covering b, contained in $\mathcal{E}_3(\tilde{\boldsymbol{G}}^F, \tilde{s})$, where $\tilde{s}$ is a preimage of s under the induced map $\tilde{\boldsymbol{G}}^* \to \boldsymbol{G}^*$. Since $3 \mid (q-1)$, $C_{\tilde{\boldsymbol{G}}}(\tilde{s})^F$ has a single unipotent 3-block, and so by [Cabanes and Enguehard 1999, Proposition 5.1] a Sylow 3-subgroup $\tilde{D}$ of $C_{\tilde{\boldsymbol{G}}}(\tilde{s})^F$ is a defect group of $\tilde{b}$. Thus, $D := \tilde{D} \cap \boldsymbol{G} = \tilde{D} \cap \boldsymbol{H}$ is a defect group of b.

Now $C_{\tilde{\boldsymbol{G}}}(\tilde{s})$ is an F-stable Levi subgroup of $\tilde{\boldsymbol{G}}$, so also an F-stable Levi subgroup of $\tilde{\boldsymbol{H}} = \mathrm{GL}_m$. As such, it is a direct product of factors GL_{m_i} with $\sum_i m_i = m$. Assume that there is more than one F-orbit on the set of factors. Then by Lemma 3.11 the Sylow 3-subgroup $\tilde{D}$ of $C_{\tilde{\boldsymbol{G}}}(\tilde{s})^F$ has the property that $D = \tilde{D} \cap \boldsymbol{H}$ is Cabanes and we are done. Hence, we may assume that F has just one orbit on the set of factors of $C_{\tilde{\boldsymbol{G}}}(\tilde{s})$. But this is only possible if F has only one orbit on the set of factors of $\tilde{\boldsymbol{G}}$. This implies that $\tilde{\boldsymbol{G}}^F \cong \mathrm{GL}_n(q^{m/n})$ and $\boldsymbol{G}^F \cong \mathrm{SL}_n(q^{m/n})$ for some $n \mid m$.

Exactly the same arguments apply when F is twisted, except that now $3 \mid (q+1)$. So replacing q by $q^{m/n}$ we may now suppose that $\boldsymbol{G} = \mathrm{SL}_n$ with $3 \notin \Gamma(\boldsymbol{G}, F)$. Assume that the defect groups of b are not Cabanes. Let $(\boldsymbol{L}, \lambda)$ be an e-Jordan-cuspidal pair for b with $\lambda \in \mathcal{E}(\boldsymbol{L}^F, s)$ and let $\tilde{\boldsymbol{L}} = Z^\circ(\tilde{\boldsymbol{G}})\boldsymbol{L}$. There exists an irreducible character $\tilde{\lambda}$ of $\tilde{\boldsymbol{L}}^F$ covering λ, an irreducible constituent $\tilde{\chi}$ of $R_{\tilde{\boldsymbol{L}}}^{\tilde{\boldsymbol{G}}}(\tilde{\lambda})$ and an irreducible constituent, say χ of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ such that $\tilde{\chi}$ covers χ. By Lemma 2.3, $(\tilde{\boldsymbol{L}}, \tilde{\lambda})$ is e-Jordan-cuspidal. Let $\tilde{b}$ be the block of $\tilde{\boldsymbol{G}}^F$ associated to $(\tilde{\boldsymbol{L}}, \tilde{\lambda})$, contained in $\mathcal{E}_3(\tilde{\boldsymbol{G}}^F, \tilde{s})$. So, $\tilde{b}$ covers b.

As seen above $C_{\tilde{\boldsymbol{G}}}(\tilde{s})^F$ has a single unipotent 3-block and a Sylow 3-subgroup $\tilde{D}$ of $C_{\tilde{\boldsymbol{G}}}(\tilde{s})^F$ is a defect group of $\tilde{b}$ and $D := \tilde{D} \cap \boldsymbol{G}$ is a defect group of b. Moreover F has a single orbit on the set of factors of $C_{\tilde{\boldsymbol{G}}}(\tilde{s})$. By Lemma 3.12, $C_{\tilde{\boldsymbol{G}}}(\tilde{s})^F = \mathrm{GL}_3(q^{\frac{n}{3}})$ or $\mathrm{GU}_3(q^{\frac{n}{3}})$, 3 does not divide $\frac{n}{3}$ and D is extra-special of order 27 and exponent 3. Also, $\tilde{\boldsymbol{L}}$ is an e-split Levi subgroup isomorphic to a direct product of 3 copies of $\mathrm{GL}_{\frac{n}{3}}$.

Let $U = Z(\boldsymbol{L})_3^F$ and let c be the 3-block of $\boldsymbol{L}^F$ containing λ. From the structure of $\tilde{\boldsymbol{L}}$ given above, $|U| = 9$ and $\boldsymbol{L} = C_{\boldsymbol{G}}(U)$. Thus, by [Cabanes and Enguehard 1999, Theorem 2.5], (U, c) is a b-Brauer pair. Let (D, f) be a maximal b-Brauer pair such that $(U, c) \leq (D, f)$.

Let $(\boldsymbol{L}', \lambda')$ be another e-Jordan-cuspidal pair for b with $\lambda' \in \mathcal{E}({\boldsymbol{L}'}^F, s)$. Let $U' = Z(\boldsymbol{L}')_3^F$ and let c' be the 3-block of ${\boldsymbol{L}'}^F$ containing λ', so $|U'| = 9$ and (U', c') is also a b-Brauer pair. Since all maximal b-Brauer pairs are $\boldsymbol{G}^F$-conjugate, there exists $h \in \boldsymbol{G}^F$ such that ${}^h(U', c') \leq (D, f)$. Thus, U and ${}^hU'$ are subgroups of order 9 of D. By Lemma 3.13, there exists $g \in N_{\boldsymbol{G}^F}(D) \cap C_{\boldsymbol{G}^F}(C_{\boldsymbol{G}^F}(D))$ such that ${}^{gh}U' = U$. Since g centralises $C_{\boldsymbol{G}^F}(D)$, ${}^g f = f$ and since g normalises D, ${}^g D = D$. Hence

$$(U, {}^{gh}c') = {}^{gh}(U', c') \leq {}^g(D, f) = (D, f).$$

By the uniqueness of inclusion of Brauer pairs we get that ${}^{gh}(U', c') = (U, c)$. Thus ${}^{gh}\boldsymbol{L}' = \boldsymbol{L}$ and ${}^{gh}c' = c$. Since U is abelian of maximal order in D, (U, c) is a self-centralising Brauer pair. In particular, there is a unique irreducible character in c with U in its kernel. Since $\lambda \in \mathcal{E}(\boldsymbol{L}^F, \ell')$, U is contained in the kernel of λ. Hence ${}^{gh}\lambda' = \lambda$ and injectivity is proved for type A.

Finally suppose that $\boldsymbol{H}$ is of type E_6. By our preliminary reductions we may assume that $\boldsymbol{G}$ has only factors of type A and $3 \notin \Gamma(\boldsymbol{G}, F)$. Thus $\boldsymbol{G}$ must have at least one factor of type A_2 or A_5. The remaining possibilities hence are: $\boldsymbol{G}$ is of type A_5, $2A_2 + A_1$, or $2A_2$. Note that for $\boldsymbol{G}$ of type $2A_2 + A_1$, the A_1-factor of the derived subgroup $[\boldsymbol{G}, \boldsymbol{G}]$ splits off, and that $2A_2$ is a Levi subgroup of A_5. So it suffices to show the claim for Levi subgroups of this particular Levi subgroup $\boldsymbol{G}$ of type A_5. Since $\boldsymbol{H}$ is simply connected, $[\boldsymbol{G}, \boldsymbol{G}] \cong \mathrm{SL}_6$ and thus virtually the same arguments as for the case of $\boldsymbol{G} = \mathrm{SL}_n$ apply. This completes the proof of (e).

Part (d) follows whenever $\ell \geq 3$ is good for $\boldsymbol{G}$, and $\ell \neq 3$ if $\boldsymbol{G}^F$ has a factor ${}^3D_4(q)$, since then by (e) there is a unique e-Jordan-cuspidal pair for any ℓ-block, and its (unipotent) Jordan correspondent has quasicentral ℓ-defect by [Cabanes and Enguehard 1994, Proposition 4.3] and Remark 2.2. So now assume that either $\ell \geq 3$ is bad for $\boldsymbol{G}$, or that $\ell = 3$ and $\boldsymbol{G}^F$ has a factor ${}^3D_4(q)$.

Note that it suffices to prove the statement for quasi-isolated blocks, since then it follows tautologically for all others using the Jordan correspondence, Proposition 2.4 and the remarks after Definition 2.12. Here note that by Lemma 2.5 the bijections of Proposition 2.4 extend to conjugacy classes of pairs. We first prove surjectivity. For this, by Lemma 3.7, Lemma 2.7 and by parts (a) and (b), we may assume that $\boldsymbol{G} = [\boldsymbol{G}, \boldsymbol{G}]$. Further, since $[\boldsymbol{G}, \boldsymbol{G}]$ is simply connected, hence a direct product of its components, we may assume that $\boldsymbol{G}$ is simple. Then surjectivity for unipotent blocks follows from [Enguehard 2000, Theorems A and A.bis], while for all other quasi-isolated blocks it is shown in [Kessar and Malle 2013, Theorem 1.2] (these also include the case that $\boldsymbol{G}^F = {}^3D_4(q)$).

Now we prove injectivity. If $\boldsymbol{G} = \boldsymbol{H}$, then the claim for unipotent blocks follows from [Enguehard 2000, Theorems A and A.bis], while for all other quasi-isolated blocks it is shown in [Kessar and Malle 2013, Theorem 1.2] (these also include the case that $\boldsymbol{G}^F = {}^3D_4(q)$). Note that in Table 4 of [Kessar and Malle 2013], each of the lines 6, 7, 10, 11, 14 and 20 give rise to two e-cuspidal pairs and so to two e-Harish-Chandra series, but each e-Jordan cuspidal pair $(\boldsymbol{L}, \lambda)$ which corresponds to these lines has the Cabanes property of Lemma 3.9, so they give rise to different blocks.

So, we may assume that $\boldsymbol{G} \neq \boldsymbol{H}$, and thus $\ell = 3$. Suppose first that $\boldsymbol{G}^F$ has a factor ${}^3D_4(q)$. Then $\boldsymbol{H}$ is of type E_6, E_7 or E_8, there is one component of $[\boldsymbol{G}, \boldsymbol{G}]$ of type D_4 and all other components are of type A. Denote by $\boldsymbol{G}_2$ the component of type D_4, and by $\boldsymbol{G}_1$ the product of the remaining components with $Z^\circ(\boldsymbol{G})$. We note that $Z(\boldsymbol{G}_1)/Z^\circ(\boldsymbol{G}_1)$ is a $3'$-group. Indeed, if $\boldsymbol{H}$ is of type E_7 or E_8, then $Z(\boldsymbol{G})/Z^\circ(\boldsymbol{G})$ is of order prime to 3, hence the same is true of $Z(\boldsymbol{G}_1)/Z^\circ(\boldsymbol{G}_1)$ and if $\boldsymbol{H}$ is of type E_6, then $\boldsymbol{G}_1 = Z^\circ(\boldsymbol{G})$.

Now, $\boldsymbol{G}^F = \boldsymbol{G}_1^F \times \boldsymbol{G}_2^F$. So, the map $((\boldsymbol{L}_1, \lambda_1), (\boldsymbol{L}_2, \lambda_2)) \to (\boldsymbol{L}_1\boldsymbol{L}_2, \lambda_1\lambda_2)$ is a bijection between pairs of e-Jordan cuspidal pairs for $\boldsymbol{G}_1^F$ and $\boldsymbol{G}_2^F$ and e-Jordan cuspidal pairs for $\boldsymbol{G}^F$. The bijection preserves conjugacy and quasicentrality. All components of $\boldsymbol{G}_1$ are of type A and as noted above 3 does not divide the order of $Z(\boldsymbol{G}_1)/Z^\circ(\boldsymbol{G}_1)$, hence by [Cabanes and Enguehard 1999, Section 5.2] we may assume that $\boldsymbol{G} = \boldsymbol{G}_2$, in which case we are done by [Enguehard 2000, Theorem A] and [Kessar and Malle 2013, Lemma 6.13].

Thus, $\boldsymbol{G}^F$ has no factor ${}^3D_4(q)$. Set $\boldsymbol{G}_0 := [\boldsymbol{G}, \boldsymbol{G}]$. Since 3 is bad for $\boldsymbol{G}$, and $\boldsymbol{G}$ is proper in $\boldsymbol{H}$, we are in one of the following cases: $\boldsymbol{H}$ is of type E_7 and $\boldsymbol{G}_0$ is simple of type E_6, or $\boldsymbol{G}$ is of type E_8 and $\boldsymbol{G}_0$ is of type E_6, $E_6 + A_1$ or E_7. In all cases, note that $Z(\boldsymbol{G})$ is connected,

Let $s \in \boldsymbol{G}^{*F}$ be a quasi-isolated semisimple $3'$-element. Let $\bar{s}$ be the image of s under the surjection $\boldsymbol{G}^* \to \boldsymbol{G}_0^*$. Since $Z(\boldsymbol{G})$ is connected, s is isolated in $\boldsymbol{G}^*$ and consequently $\bar{s}$ is isolated in $\boldsymbol{G}_0^*$. In particular, if $\boldsymbol{G}_0$ has a component of type A_1, then the projection of $\bar{s}$ into that factor is the identity. Since s has order prime to 3, this means that if $\boldsymbol{G}_0$ has a component of type E_6, then $C_{\boldsymbol{G}_0^*}(\bar{s})$ is connected. We will use this fact later. Also, we note here that $\bar{s} \neq 1$ as otherwise the result would follow from [Enguehard 2000] and the standard correspondence between unipotent blocks and blocks lying in central Lusztig series. Finally, we note that by [Kessar and Malle 2013, Theorem 1.2] the conclusion of parts (a) and (d) of the theorem holds for $\boldsymbol{G}_0^F$ as all components of $\boldsymbol{G}_0$ are of different type (so e is the same for the factors of $\boldsymbol{G}_0^F$ as for $\boldsymbol{G}^F$).

Let b be a 3-block of $\boldsymbol{G}^F$ in the series s and $(\boldsymbol{L}, \lambda)$ be an e-Jordan quasicentral cuspidal pair for b such that $s \in \boldsymbol{L}^{*F}$ and $\lambda \in \mathcal{E}(\boldsymbol{L}^F, s)$. Let $\boldsymbol{L}_0 = \boldsymbol{L} \cap \boldsymbol{G}_0$ and let λ_0 be an irreducible constituent of the restriction of λ to $\boldsymbol{L}_0^F$. By Lemma 3.8 there exists a block b_0 of $\boldsymbol{G}_0^F$ covered by b, and such that all irreducible constituents of $R_{\boldsymbol{L}_0}^{\boldsymbol{G}_0}(\lambda_0)$ belong to b. By Lemma 2.3 and the remarks following Definition 2.12, $(\boldsymbol{L}_0, \lambda_0)$ is an e-Jordan quasicentral cuspidal pair of $\boldsymbol{G}_0^F$ for b_0.

First suppose that $C_{\boldsymbol{G}_0}(\bar{s})$ is connected. Then all elements of $\mathcal{E}(\boldsymbol{G}_0^F, \bar{s})$ are $\boldsymbol{G}^F$-stable and in particular, b_0 is $\boldsymbol{G}^F$-stable. Now let $(\boldsymbol{L}', \lambda')$ be another e-Jordan quasicentral cuspidal pair for b. Let $\boldsymbol{L}_0' = \boldsymbol{L}' \cap \boldsymbol{G}_0$ and λ_0' be an irreducible constituent of the restriction of λ' to $\boldsymbol{L}_0'^F$. Then, as above $(\boldsymbol{L}_0', \lambda_0')$ is an e-Jordan quasicentral cuspidal pair for b_0. But there is a unique e-Jordan quasicentral cuspidal pair for b_0 up to $\boldsymbol{G}_0^F$-conjugacy. So, up to replacing by a suitable $\boldsymbol{G}_0^F$-conjugate we may assume that $(\boldsymbol{L}_0, \lambda_0) = (\boldsymbol{L}_0', \lambda_0')$, hence $\boldsymbol{L} = \boldsymbol{L}'$, and λ and λ' cover the same character $\lambda_0 = \lambda_0'$ of $\boldsymbol{L}_0^F = {\boldsymbol{L}_0}'^F$.

If $\mu \in \mathcal{E}(\boldsymbol{G}_0^F, \bar{s})$, then there are $|\boldsymbol{G}^F/\boldsymbol{G}_0^F|_{3'}$ different $3'$-Lusztig series of $\boldsymbol{G}^F$ containing an irreducible character covering μ. Since characters in different $3'$-Lusztig series lie in different 3-blocks, there are at least $|\boldsymbol{G}^F/\boldsymbol{G}_0^F|_{3'}$ different blocks of $\boldsymbol{G}^F$ covering b_0. Moreover, if b' is a block of $\boldsymbol{G}^F$ covering b_0, then there exists a linear character, say θ of $\boldsymbol{G}^F/\boldsymbol{G}_0^F \cong \boldsymbol{L}^F/\boldsymbol{L}_0^F$ of $3'$-degree such that $(\boldsymbol{L}, \theta \otimes \lambda)$ is an e-Jordan quasicentral cuspidal pair for b' and λ_0 appears in the restriction of $\theta \otimes \lambda$ to $\boldsymbol{L}_0^F$. Since there are at most $|\boldsymbol{L}^F/\boldsymbol{L}_0^F|_{3'} = |\boldsymbol{G}^F/\boldsymbol{G}_0^F|_{3'}$ irreducible characters of $\boldsymbol{L}^F$ in $3'$-series covering λ_0, it follows that $\lambda = \lambda'$.

Thus, we may assume that $C_{\boldsymbol{G}_0}(\bar{s})$ is not connected. Hence, by the remarks above $\boldsymbol{G}_0$ is simple of type E_7. Further $\bar{s}$ corresponds to one of the lines 5, 6, 7, 12, 13, or 14 of Table 4 of [Kessar and Malle 2013] (note that $\bar{s}$ is isolated and that e-Jordan (quasi-)central cuspidality in this case is the same as e-(quasi-)central cuspidality).

By [Kessar and Malle 2013, Lemma 5.2], $\boldsymbol{L}_0 = C_{\boldsymbol{G}_0}(Z(\boldsymbol{L}_0^F)_3)$. In other words, $(\boldsymbol{L}_0, \lambda_0)$ is a good pair for b_0 in the sense of [Kessar and Malle 2013, Definition 7.10]. In particular, there is a maximal b_0-Brauer pair (P_0, c_0) such that $(Z(\boldsymbol{L}_0^F)_3, b_{\boldsymbol{L}_0^F}(\lambda_0)) \trianglelefteq (P_0, c_0)$. Here for a finite group X and an irreducible character η of X, we denote by $b_X(\eta)$ the ℓ-block of X containing η. By inspection of the relevant lines of Table 4 of [Kessar and Malle 2013] (and the proof of [Kessar and Malle 2013, Theorem 1.2]), one sees that the maximal Brauer pair (P_0, c_0) can be chosen so that $Z(\boldsymbol{L}_0^F)_3$ is the unique maximal abelian normal subgroup of P_0.

By [Kessar and Malle 2013, Theorem 7.11] there exists a maximal b-Brauer pair (P, c) and $\nu \in \mathcal{E}(\boldsymbol{L}^F, \ell')$ such that ν covers λ_0, $P_0 \le P$ and we have an inclusion of b-Brauer pairs $(Z(\boldsymbol{L}^F)_3, b_{\boldsymbol{L}^F}(\nu)) \trianglelefteq (P, c)$. Since λ also covers λ_0, $\lambda = \tau \otimes \nu$ for some linear character τ of $\boldsymbol{L}^F/\boldsymbol{L}_0^F \cong \boldsymbol{G}^F/\boldsymbol{G}_0^F$. Since tensoring with linear characters preserves block distribution and commutes with Brauer pair inclusion, replacing c with the block of $C_{\boldsymbol{G}^F}(P_0)$ whose irreducible characters are of the form $\tau \otimes \varphi$, $\varphi \in \mathrm{Irr}(c)$, we get that there exists a maximal b-Brauer pair (P, c) such that $P_0 \le P$ and $(Z(\boldsymbol{L}^F)_3, b_{\boldsymbol{L}^F}(\lambda)) \trianglelefteq (P, c)$.

Being normal in $\boldsymbol{G}^F$, $Z(\boldsymbol{G}^F)_3$ is contained in the defect groups of every block of $\boldsymbol{G}^F$, and in particular $Z(\boldsymbol{G}^F)_3 \le P$. On the other hand, since $\boldsymbol{G}_0$ has centre of order 2, $P_0 Z(\boldsymbol{G}^F)_3$ is a defect group of b whence P is a direct product of P_0 and $Z(\boldsymbol{G}^F)_3$. Now, $Z(\boldsymbol{L}_0^F)_3$ is the unique maximal abelian normal subgroup of P_0. Hence, $Z(\boldsymbol{L}^F)_3 = Z(\boldsymbol{G}^F)_3 \times Z(\boldsymbol{L}_0^F)_3$ is the unique maximal normal abelian subgroup of P (see Lemma 3.11). Finally note that by Lemma 2.7, λ is also of quasicentral ℓ-defect. By Lemma 3.9 it follows that up to conjugacy $(\boldsymbol{L}, \lambda)$ is the unique e-Jordan quasicentral cuspidal pair of $\boldsymbol{G}^F$ for b.

Finally, we show (c). In view of the part (d) just proved above, it remains to consider the prime $\ell = 2$ only. Suppose first that all components of $\boldsymbol{G}$ are of classical type. Let $s \in \boldsymbol{G}^{*F}$ be semisimple of odd order and let b be a 2-block of $\boldsymbol{G}^F$ in series s. By Lemma 3.17 below there is an e-torus, say $\boldsymbol{S}$ of $C^\circ_{\boldsymbol{G}^*}(s)$ such that $\boldsymbol{T}^* := C_{C^\circ_{\boldsymbol{G}^*}(s)}(\boldsymbol{S})$ is a maximal torus of $C^\circ_{\boldsymbol{G}^*}(s)$. Let $\boldsymbol{L}^* = C_{\boldsymbol{G}^*}(\boldsymbol{S})$ and let $\boldsymbol{L}$ be a Levi subgroup of $\boldsymbol{G}$ in duality with $\boldsymbol{L}^*$. Then $\boldsymbol{L}$ is an e-split subgroup of $\boldsymbol{G}$ and $\boldsymbol{T}^* = C^\circ_{\boldsymbol{L}^*}(s)$. Let $\lambda \in \mathrm{Irr}(\boldsymbol{L}^F, s)$ correspond via Jordan decomposition to the trivial character of $\boldsymbol{T}^{*F}$. Then $(\boldsymbol{L}, \lambda)$ is an e-Jordan quasicentral cuspidal pair of $\boldsymbol{G}$.

Let $\boldsymbol{G} \hookrightarrow \tilde{\boldsymbol{G}}$ be a regular embedding. By part (a), Lemmas 3.3 and 3.8, there exists $g \in \tilde{\boldsymbol{G}}^F$ such that $b = b_{\boldsymbol{G}^F}({}^g\boldsymbol{L}, {}^g\lambda)$. Now since $(\boldsymbol{L}, \lambda)$ is e-Jordan quasicentral cuspidal, so is $({}^g\boldsymbol{L}, {}^g\lambda)$. In order to see this, first note that, up to multiplication by a suitable element of $\boldsymbol{G}^F$ and by an application of the Lang–Steinberg theorem, we may assume that g is in some F-stable maximal torus of $Z^\circ(\tilde{\boldsymbol{G}})\boldsymbol{L}$. Thus ${}^g\boldsymbol{L} = \boldsymbol{L}$, and λ and ${}^g\lambda$ correspond to the same $C_{\boldsymbol{L}^*}(s)^F$ orbit of unipotent characters of $C^\circ_{\boldsymbol{L}^*}(s)^F$.

Now suppose that $\boldsymbol{G}$ has a component of exceptional type. Then we can argue just as in the proof of surjectivity for bad ℓ in part (d). □

Lemma 3.17. *Let $\boldsymbol{G}$ be connected reductive with a Frobenius morphism $F : \boldsymbol{G} \to \boldsymbol{G}$. Let $e \in \{1, 2\}$ and let $\boldsymbol{S}$ be a Sylow e-torus of $\boldsymbol{G}$. Then $C_{\boldsymbol{G}}(\boldsymbol{S})$ is a torus.*

Proof. Let $\boldsymbol{C} := [C_{\boldsymbol{G}}(\boldsymbol{S}), C_{\boldsymbol{G}}(\boldsymbol{S})]$ and assume that $\boldsymbol{C}$ has semisimple rank at least one. Let $\boldsymbol{T}$ be a maximally split torus of $\boldsymbol{C}$. Then the Sylow 1-torus of $\boldsymbol{T}$, hence of $\boldsymbol{C}$ is nontrivial. Similarly, the reductive group $\boldsymbol{C}'$ with complete root datum obtained from that of $\boldsymbol{C}$ by replacing the automorphism on the Weyl group by its negative, again has a nontrivial Sylow 1-torus. But then $\boldsymbol{C}$ also has a nontrivial Sylow 2-torus. Thus in any case $\boldsymbol{C}$ has a noncentral e-torus, which is a contradiction to its definition. □

4. Jordan decomposition of blocks

Lusztig induction induces Morita equivalences between Jordan corresponding blocks. We show that this also behaves nicely with respect to e-cuspidal pairs and their corresponding e-Harish-Chandra series.

Jordan decomposition and e-cuspidal pairs. Throughout this subsection, $\boldsymbol{G}$ is a connected reductive algebraic group with a Frobenius endomorphism $F : \boldsymbol{G} \to \boldsymbol{G}$ endowing $\boldsymbol{G}$ with an $\mathbb{F}_q$-structure for some power q of p. Our results here are valid for all groups $\boldsymbol{G}^F$ satisfying the Mackey-formula for Lusztig induction. At present this is known to hold unless $\boldsymbol{G}$ has a component $\boldsymbol{H}$ of type E_6, E_7 or E_8 with $\boldsymbol{H}^F \in \{{}^2E_6(2), E_7(2), E_8(2)\}$, see Bonnafé–Michel [2011]. The following is in complete analogy with Proposition 2.4:

Proposition 4.1. *Assume that $\boldsymbol{G}^F$ has no factor ${}^2E_6(2)$, $E_7(2)$ or $E_8(2)$. Let $s \in \boldsymbol{G}^{*F}$, and $\boldsymbol{G}_1 \le \boldsymbol{G}$ an F-stable Levi subgroup with $\boldsymbol{G}_1^*$ containing $C_{\boldsymbol{G}^*}(s)$. For $(\boldsymbol{L}_1, \lambda_1)$ an e-cuspidal pair of $\boldsymbol{G}_1$ below $\mathcal{E}(\boldsymbol{G}_1^F, s)$ define $\boldsymbol{L} := C_{\boldsymbol{G}}(Z^\circ(\boldsymbol{L}_1)_e)$ and $\lambda := \epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{L}_1} R^{\boldsymbol{L}}_{\boldsymbol{L}_1}(\lambda_1)$. Then $(\boldsymbol{L}_1, \lambda_1) \mapsto (\boldsymbol{L}, \lambda)$ defines a bijection $\Psi^{\boldsymbol{G}}_{\boldsymbol{G}_1}$ between the set of e-cuspidal pairs of $\boldsymbol{G}_1$ below $\mathcal{E}(\boldsymbol{G}_1^F, s)$ and the set of e-cuspidal pairs of $\boldsymbol{G}$ below $\mathcal{E}(\boldsymbol{G}^F, s)$.*

Proof. We had already seen in the proof of Proposition 2.4 that $\boldsymbol{L}$ is e-split and $Z^\circ(\boldsymbol{L}_1)_e = Z^\circ(\boldsymbol{L})_e$. For the well-definedness of $\Psi^{\boldsymbol{G}}_{\boldsymbol{G}_1}$ it remains to show that λ is e-cuspidal. For any e-split Levi subgroup $\boldsymbol{X} \le \boldsymbol{L}$ the Mackey formula [Bonnafé and Michel 2011, Theorem] gives

$$\epsilon_{\boldsymbol{L}}\epsilon_{\boldsymbol{L}_1} {}^*R^{\boldsymbol{L}}_{\boldsymbol{X}}(\lambda) = {}^*R^{\boldsymbol{L}}_{\boldsymbol{X}} R^{\boldsymbol{L}}_{\boldsymbol{L}_1}(\lambda_1) = \sum_g R^{\boldsymbol{X}}_{\boldsymbol{X}\cap {}^g\boldsymbol{L}_1} {}^*R^{{}^g\boldsymbol{L}_1}_{\boldsymbol{X}\cap {}^g\boldsymbol{L}_1}(\lambda_1^g)$$

where the sum runs over a suitable set of double coset representatives $g \in \boldsymbol{L}^F$. Here, $\boldsymbol{X} \cap {}^g\boldsymbol{L}_1$ is e-split in $\boldsymbol{L}_1$ since $\boldsymbol{L}_1 \cap \boldsymbol{X}^g = \boldsymbol{L}_1 \cap C_{\boldsymbol{L}}(Z^\circ(\boldsymbol{X}^g)_e) = C_{\boldsymbol{L}_1}(Z^\circ(\boldsymbol{X}^g)_e)$. The e-cuspidality of λ_1 thus shows that the only nonzero terms in the above sum are those for which $\boldsymbol{L}_1 \cap \boldsymbol{X}^g = \boldsymbol{L}_1$, i.e., those with $\boldsymbol{L}_1 \leq \boldsymbol{X}^g$. But then $Z^\circ(\boldsymbol{L})_e = Z^\circ(\boldsymbol{L}_1)_e = Z^\circ(\boldsymbol{X}^g)_e$, and as $\boldsymbol{X}$ is e-split in $\boldsymbol{L}$ we deduce that necessarily $\boldsymbol{X} = \boldsymbol{L}$ if ${}^*R_{\boldsymbol{X}}^{\boldsymbol{L}}(\lambda) \neq 0$. So λ is indeed e-cuspidal, and $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ is well-defined.

Injectivity was shown in the proof of Proposition 2.4, where we had constructed an inverse map with $\boldsymbol{L}_1^* := \boldsymbol{L}^* \cap \boldsymbol{G}_1^*$ and λ_1 the unique constituent of ${}^*R_{\boldsymbol{L}_1}^{\boldsymbol{L}}(\lambda)$ in $\mathcal{E}(\boldsymbol{L}_1^F, s)$. We claim that λ_1 is e-cuspidal. Indeed, for any e-split Levi subgroup $\boldsymbol{X} \leq \boldsymbol{L}_1$ let $\boldsymbol{Y} := C_{\boldsymbol{L}}(Z^\circ(\boldsymbol{X})_e)$, an e-split Levi subgroup of $\boldsymbol{L}$. Then ${}^*R_{\boldsymbol{X}}^{\boldsymbol{L}_1}(\lambda_1)$ is a constituent of

$$ {}^*R_{\boldsymbol{X}}^{\boldsymbol{L}}(\lambda) = {}^*R_{\boldsymbol{X}}^{\boldsymbol{Y}}\,{}^*R_{\boldsymbol{Y}}^{\boldsymbol{L}}(\lambda) = 0 $$

by e-cuspidality of λ, unless $\boldsymbol{Y} = \boldsymbol{L}$, whence $\boldsymbol{X} = \boldsymbol{Y} \cap \boldsymbol{L}_1 = \boldsymbol{L} \cap \boldsymbol{L}_1 = \boldsymbol{L}_1$.

Thus we have obtained a well-defined map ${}^*\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ from e-cuspidal pairs in $\boldsymbol{G}$ to e-cuspidal pairs in $\boldsymbol{G}_1$, both below the series s. The rest of the proof is again as for Proposition 2.4. □

Jordan decomposition, e-cuspidal pairs and ℓ-blocks. We next remove two of the three possible exceptions in Proposition 4.1 for characters in ℓ'-series:

Lemma 4.2. *The assertions of Proposition 4.1 remain true for $\boldsymbol{G}^F$ having no factor $E_8(2)$ whenever $s \in \boldsymbol{G}^{*F}$ is a semisimple ℓ'-element, where $e = e_\ell(q)$. In particular, $\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}$ exists.*

Proof. Let s be a semisimple ℓ'-element. Then by [Cabanes and Enguehard 1999, Theorem 4.2] we may assume that $\ell \leq 3$, so in fact $\ell = 3$. The character table of $\boldsymbol{G}^{*F} = {}^2E_6(2).3$ is known; there are 12 classes of nontrivial elements $s \in \boldsymbol{G}^{*F}$ of order prime to 6. Their centralisers $C_{\boldsymbol{G}^*}(s)$ only have factors of type A, and are connected. Thus all characters in those series $\mathcal{E}(\boldsymbol{G}^F, s)$ are uniform, so the Mackey-formula is known for them with respect to any Levi subgroup. Thus, the argument in Proposition 4.1 is applicable to those series. For $\boldsymbol{G}^F = E_7(2)$, the conjugacy classes of semisimple elements can be found in [Lübeck]. From this one verifies that again all nontrivial semisimple $3'$-elements have centraliser either of type A, or of type ${}^2D_4(q)A_1(q)\Phi_4$, or ${}^3D_4(q)\Phi_1\Phi_3$. In the latter two cases, proper Levi subgroups are either direct factors, or again of type A, and so once more the Mackey-formula is known to hold with respect to any Levi subgroup. □

Remark 4.3. The assertion of Lemma 4.2 can be extended to most ℓ'-series of $\boldsymbol{G}^F = E_8(2)$. Indeed, again by [Cabanes and Enguehard 1999, Theorem 4.2] we only need to consider $\ell \in \{3, 5\}$. For $\ell = 3$ there are just two types of Lusztig series for $3'$-elements which can not be treated by the arguments above, with corresponding centraliser $E_6(2)\Phi_3$ respectively ${}^2D_6(2)\Phi_4$. For $\ell = 5$, there are five types of Lusztig series, with centraliser ${}^2E_6(2){}^2A_2(2)$, $E_7(2)\Phi_2$, ${}^2D_7(2)\Phi_2$, $E_6(2)\Phi_3$ and ${}^2D_5(2)\Phi_2\Phi_6$ respectively. Note that the first one is isolated, so the assertion can be checked using [Kessar and Malle 2013].

Proposition 4.4. *Assume that $\boldsymbol{G}^F$ has no factor $E_8(2)$. Let $s \in \boldsymbol{G}^{*F}$ be a semisimple ℓ'-element, and $\boldsymbol{G}_1 \leq \boldsymbol{G}$ an F-stable Levi subgroup with $\boldsymbol{G}_1^*$ containing $C_{\boldsymbol{G}^*}(s)$. Assume that b is an ℓ-block in $\mathcal{E}_\ell(\boldsymbol{G}^F, s)$, and c is its Jordan correspondent in $\mathcal{E}_\ell(\boldsymbol{G}_1^F, s)$. Let $e = e_\ell(q)$.*

(a) *Let $(\boldsymbol{L}_1, \lambda_1)$ be e-cuspidal in $\boldsymbol{G}_1$, where $(\boldsymbol{L}, \lambda) = \Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}(\boldsymbol{L}_1, \lambda_1)$. If all constituents of $R_{\boldsymbol{L}_1}^{\boldsymbol{G}_1}(\lambda_1)$ lie in c, then all constituents of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ lie in b.*

(b) *Let $(\boldsymbol{L}, \lambda)$ be e-cuspidal in $\boldsymbol{G}$, where $(\boldsymbol{L}_1, \lambda_1) = {}^*\Psi_{\boldsymbol{G}_1}^{\boldsymbol{G}}(\boldsymbol{L}, \lambda)$. If all constituents of $R_{\boldsymbol{L}}^{\boldsymbol{G}}(\lambda)$ lie in b, then all constituents of $R_{\boldsymbol{L}_1}^{\boldsymbol{G}_1}(\lambda_1)$ lie in c.*

The proof is identical to the one of Proposition 2.6, using Proposition 4.1 and Lemma 4.2 in place of Proposition 2.4.

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FREE RESOLUTIONS OF SOME SCHUBERT SINGULARITIES

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Dedicated to the memory of Professor Robert Steinberg.

In this paper we construct free resolutions of a class of closed subvarieties of affine spaces (the so-called "opposite big cells" of Grassmannians). Our class covers the determinantal varieties, whose resolutions were first constructed by A. Lascoux (*Adv. in Math.* 30:3 (1978), 202–237). Our approach uses the geometry of Schubert varieties. An interesting aspect of our work is its connection to the computation of the cohomology of homogeneous bundles (that are not necessarily completely reducible) on partial flag varieties.

1. Introduction

A classical problem in commutative algebra and algebraic geometry is to describe the syzygies of the defining ideals of interesting varieties. Let $k \leq n \leq m$ be positive integers. The space D_k of $m \times n$ matrices (over a field $\Bbbk$) of rank at most k is a closed subvariety of the mn-dimensional affine space of all $m \times n$ matrices. When $\Bbbk = \mathbb{C}$, a minimal free resolution of the coordinate ring $\Bbbk[D_k]$ as a module over the coordinate ring of the mn-dimensional affine space (i.e., the mn-dimensional polynomial ring) was constructed by A. Lascoux [1978]; see also [Weyman 2003, Chapter 6].

In this paper, we construct free resolutions for a larger class of singularities, *viz.*, Schubert singularities, i.e., the intersection of a singular Schubert variety and the "opposite big cell" inside a Grassmannian. The advantage of our method is that it is algebraic group-theoretic, and is likely to work for Schubert singularities in more general flag varieties. In this process, we have come up with a method to compute the cohomology of certain homogeneous vector bundles (which are not completely reducible) on flag varieties. We will work over $\Bbbk = \mathbb{C}$.

Let $N = m + n$. Let $\mathrm{GL}_N = \mathrm{GL}_N(\mathbb{C})$ be the group of $N \times N$ invertible matrices. Let B_N be the Borel subgroup of all upper-triangular matrices and B_N^- the *opposite*

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Borel subgroup of all lower-triangular matrices in GL_N. Let P be the maximal parabolic subgroup corresponding to omitting the simple root α_n, i.e., the subgroup of GL_N comprising the matrices in which the (i, j)-th entry (i.e., in row i and column j) is zero, if $n+1 \le i \le N$ and $1 \le j \le n$; in other words,

$$P = \left\{ \begin{bmatrix} A_{n\times n} & C_{n\times m} \\ 0_{m\times n} & E_{m\times m} \end{bmatrix} \in \mathrm{GL}_N \right\}.$$

We have a canonical identification of the Grassmannian of n-dimensional subspaces of $\Bbbk^N$ with GL_N/P. Let W and W_P be the Weyl groups of GL_N and of P, respectively; note that $W = S_N$ (the symmetric group) and $W_P = S_n \times S_m$. For $w \in W/W_P$, let $X_P(w) \subseteq \mathrm{GL}_N/P$ be the Schubert variety corresponding to w (i.e., the closure of the B_N-orbit of the coset $wP \in \mathrm{GL}_N/P$, equipped with the canonical reduced scheme structure). The B_N^--orbit of the coset $(\mathrm{id} \cdot P)$ in GL_N/P is denoted by $O^-_{\mathrm{GL}_N/P}$, and is usually called the *opposite big cell* in GL_N/P; it can be identified with the mn-dimensional affine space. (See Section 2.2.)

Write W^P for the set of minimal representatives (under the Bruhat order) in W for the elements of W/W_P. For $1 \le r \le n-1$, we consider certain subsets $\mathcal{W}_r$ of W^P (Definition 3.3); there is $w \in \mathcal{W}_{n-k}$ such that $D_k = X_P(w) \cap O^-_{\mathrm{GL}_N/P}$. Note that for any $w \in W^P$, $X_P(w) \cap O^-_{\mathrm{GL}_N/P}$ is a closed subvariety of $O^-_{\mathrm{GL}_N/P}$. Our main result is a description of the minimal free resolution of the coordinate ring of $X_P(w) \cap O^-_{\mathrm{GL}_N/P}$ as a module over the coordinate ring of $O^-_{\mathrm{GL}_N/P}$ for every $w \in \mathcal{W}_r$. This latter ring is a polynomial ring. We now outline our approach.

First we recall the Kempf–Lascoux–Weyman "geometric technique" of constructing minimal free resolutions. Suppose that we have a commutative diagram of varieties

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "q'"'] & \mathbb{A} \times V \arrow[r] \arrow[d, "q"] & V \\ Y \arrow[r, hook] & \mathbb{A} & \end{tikzcd} \tag{1-1}$$

where $\mathbb{A}$ is an affine space, Y a closed subvariety of $\mathbb{A}$ and V a projective variety and q is the projection to the first factor. Suppose further that the (necessarily proper) map q' is birational, and that the inclusion $Z \hookrightarrow \mathbb{A} \times V$ is a subbundle (over V) of the trivial bundle $\mathbb{A} \times V$. Let ξ be the dual of the quotient bundle on V corresponding to Z. Then the derived direct image $\boldsymbol{R}q'_*\mathbb{O}_Z$ is quasi-isomorphic to a minimal complex $F_\bullet$ with

$$F_i = \bigoplus_{j \ge 0} \mathrm{H}^j(V, \textstyle\bigwedge^{i+j} \xi) \otimes_{\mathbb{C}} R(-i-j).$$

Here R is the coordinate ring of $\mathbb{A}$; it is a polynomial ring and $R(k)$ refers to twisting with respect to its natural grading. If q' is such that the natural map

$\mathcal{O}_Y \to \boldsymbol{R}q'_*\mathcal{O}_Z$ is a quasi-isomorphism, (for example, if q' is a desingularization of Y and Y has rational singularities), then $F_\bullet$ is a minimal free resolution of $\mathbb{C}[Y]$ over the polynomial ring R.

The difficulty in applying this technique in any given situation is two-fold: one must find a suitable morphism $q' : Z \to Y$ such that the map $\mathcal{O}_Y \to \boldsymbol{R}q'_*\mathcal{O}_Z$ is a quasi-isomorphism and such that Z is a vector bundle over a projective variety V; and, one must be able to compute the necessary cohomology groups. We overcome this for opposite cells in a certain class (which includes the determinantal varieties) of Schubert varieties in a Grassmannian, in two steps.

As the first step, we need to establish the existence of a diagram as above. This is done using the geometry of Schubert varieties. We take $\mathbb{A} = O^-_{\mathrm{GL}_N/P}$ and

$$Y = Y_P(w) := X_P(w) \cap O^-_{\mathrm{GL}_N/P}.$$

Let $\tilde{P}$ be a parabolic subgroup with $B_N \subseteq \tilde{P} \subsetneq P$. The inverse image of $O^-_{\mathrm{GL}_N/P}$ under the natural map $\mathrm{GL}_N/\tilde{P} \to \mathrm{GL}_N/P$ is $O^-_{\mathrm{GL}_N/P} \times P/\tilde{P}$. Let $\tilde{w}$ be the representative of the coset $w\tilde{P}$ in $W^{\tilde{P}}$. Then $X_{\tilde{P}}(\tilde{w}) \subseteq \mathrm{GL}_N/\tilde{P}$ (the Schubert subvariety of $\mathrm{GL}_N/\tilde{P}$ associated to $\tilde{w}$) maps properly and birationally onto $X_P(w)$. We may choose $\tilde{P}$ to ensure that $X_{\tilde{P}}(\tilde{w})$ is smooth. Let $Z_{\tilde{P}}(\tilde{w})$ be the preimage of $Y_P(w)$ in $X_{\tilde{P}}(\tilde{w})$. We take $Z = Z_{\tilde{P}}(\tilde{w})$. Then V, which is the image of Z under the second projection, is a smooth Schubert subvariety of $P/\tilde{P}$. The vector bundle ξ on V that we obtain is the restriction of a homogeneous bundle on $P/\tilde{P}$. Thus we get:

(1-2)
$$\begin{tikzcd} Z_{\tilde{P}}(\tilde{w}) \arrow[r, hook] \arrow[d, "q'"] & O^-_{\mathrm{GL}_N/P} \times V \arrow[r] \arrow[d, "q"] & V \\ Y_P(w) \arrow[r, hook] & O^- & \end{tikzcd}$$

See Theorem 3.7 and Corollary 3.9. In this diagram, q' is a desingularization of $Y_P(w)$. Since it is known that Schubert varieties have rational singularities, we have that the map $\mathcal{O}_Y \to \boldsymbol{R}q'_*\mathcal{O}_Z$ is a quasi-isomorphism, so $F_\bullet$ is a minimal resolution.

As the second step, we need to determine the cohomology of the homogeneous bundles $\bigwedge^t \xi$ over V. There are two ensuing issues: computing cohomology of homogeneous vector bundles over Schubert subvarieties of flag varieties is difficult, and furthermore, these bundles are not usually completely reducible, so one cannot apply the Borel–Weil–Bott theorem directly. We address the former issue by restricting our class; if $w \in \mathcal{W}_r$ (for some r) then V will equal $P/\tilde{P}$. Regarding the latter issue, we inductively replace $\tilde{P}$ by larger parabolic subgroups (still inside P), such that at each stage, the computation reduces to that of the cohomology of completely reducible bundles on Grassmannians; using various spectral sequences, we are able to determine the cohomology groups that determine the minimal free resolution. See Proposition 5.5 for the key inductive step. In contrast, in Lascoux's construction of the resolution of determinantal ideals, one comes across only completely reducible bundles; therefore, one may use the Borel–Weil–Bott theorem to compute the cohomology of the bundles $\bigwedge^t \xi$. The idea of using $\mathbb{P}^1$-fibrations for the computation of cohomology on flag varieties and their Schubert varieties goes back to M. Demazure [1968; 1974]; see also the related "one-step construction" of Kempf [1976].

Computing cohomology of homogeneous bundles, in general, is difficult, and is of independent interest; we hope that our approach would be useful in this regard. The best results, as far as we know, are due to G. Ottaviani and E. Rubei [2006], which deal with general homogeneous bundles on Hermitian symmetric spaces. The only Hermitian symmetric spaces in Type A are the Grassmannians, so their results do not apply to our situation.

Since the opposite big cell $O^-_{\mathrm{GL}_N/P}$ intersects every B_N-orbit of GL_N/P, the variety $Y_P(w)$ captures all the singularities of $X_P(w)$ for every $w \in W$. In this paper, we describe a construction of a minimal free resolution of $\mathbb{C}[Y_P(w)]$ over $\mathbb{C}[O^-_{\mathrm{GL}_N/P}]$. We hope that our methods could shed some light on the problem of construction of a locally free resolution of $\mathbb{O}_{X_P(w)}$ as an $\mathbb{O}_{\mathrm{GL}_N/P}$-module.

The paper is organized as follows. Section 2 contains notations and conventions (Section 2.1) and the necessary background material on Schubert varieties (Section 2.2) and homogeneous bundles (Section 2.3). In Section 3, we discuss properties of Schubert desingularization, including the construction of Diagram (1-2). Section 4 is devoted to a review of the Kempf–Lascoux–Weyman technique and its application to our problem. Section 5 explains how the cohomology of the homogeneous bundles on certain partial flag varieties can be computed; Section 6 gives some examples. Finally, in Section 7, we describe Lascoux's resolution in terms of our approach and describe the multiplicity and Castelnuovo–Mumford regularity of $\mathbb{C}[Y_P(w)]$.

2. Preliminaries

In this section, we collect various results about Schubert varieties, homogeneous bundles, and the Kempf–Lascoux–Weyman geometric technique.

2.1. ***Notation and conventions.*** We collect the symbols used and the conventions adopted in the rest of the paper here. For details on algebraic groups and Schubert varieties, the reader may refer to [Borel 1991; Jantzen 2003; Billey and Lakshmibai 2000; Seshadri 2007].

Let $m \geq n$ be positive integers and $N = m + n$. We denote by GL_N (respectively, B_N, B_N^-) the group of all (respectively, upper-triangular, lower-triangular) invertible $N \times N$ matrices over $\mathbb{C}$. The Weyl group W of GL_N is isomorphic to the group

S_N of permutations of N symbols and is generated by the *simple reflections* s_i, for $1 \le i \le N-1$, which correspond to the transpositions $(i, i+1)$. For $w \in W$, its *length* is the smallest integer l such that $w = s_{i_1} \cdots s_{i_l}$ as a product of simple reflections. For every $1 \le i \le N-1$, there is a *minimal parabolic subgroup* P_i containing s_i (thought of as an element of GL_N) and a *maximal parabolic subgroup* $P_{\hat{\imath}}$ *not containing* s_i. Any parabolic subgroup can be written as $P_{\hat{A}} := \bigcap_{i \in A} P_{\hat{\imath}}$ for some $A \subset \{1, \ldots, N-1\}$. On the other hand, for $A \subseteq \{1, \ldots, N-1\}$ write P_A for the subgroup of GL_N generated by P_i, $i \in A$. Then P_A is a parabolic subgroup and $P_{\{1,\ldots,N-1\}\setminus A} = P_{\hat{A}}$.

The following is fixed for the rest of this paper:

(a) P is the maximal parabolic subgroup $P_{\hat{n}}$ of GL_N;

(b) for $1 \le s \le n-1$, $\tilde{P}_s$ is the parabolic subgroup $P_{\{1,\ldots,s-1,n+1,\ldots,N-1\}} = \cap_{i=s}^{n} P_{\hat{\imath}}$ of GL_N;

(c) for $1 \le s \le n-1$, Q_s is the parabolic subgroup $P_{\{1,\ldots,s-1\}} = \cap_{i=s}^{n-1} P_{\hat{\imath}}$ of GL_n.

We write the elements of W in *one-line* notation: $(a_1, \ldots, a_N)$ is the permutation $i \mapsto a_i$. For any $A \subseteq \{1, \ldots, N-1\}$, define W_{P_A} to be the subgroup of W generated by $\{s_i : i \in A\}$. By W^{P_A} we mean the subset of W consisting of the minimal representatives (under the Bruhat order) in W of the elements of W/W_{P_A}. For $1 \le i \le N$, we represent the elements of $W^{P_{\hat{\imath}}}$ by sequences $(a_1, \ldots, a_i)$ with $1 \le a_1 < \cdots < a_i \le N$ since under the action of the group $W_{P_{\hat{\imath}}}$, every element of W can be represented minimally by such a sequence.

For $w = (a_1, a_2, \ldots, a_n) \in W^P$, let $r(w)$ be the integer r such that $a_r \le n < a_{r+1}$.

We identify $\mathrm{GL}_N = \mathrm{GL}(V)$ for some N-dimensional vector-space V. Let $A := \{i_1 < i_2 < \cdots < i_r\} \subseteq \{1, \ldots, N-1\}$. Then $\mathrm{GL}_N / P_{\hat{A}}$ is the set of all flags

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_r \subsetneq V$$

of subspaces V_j of dimension i_j inside V. We call $\mathrm{GL}_N / P_{\hat{A}}$ a *flag variety*. If $A = \{1, \ldots, N-1\}$ (i.e., $P_{\hat{A}} = B_N$), then we call the flag variety a *full flag variety*; otherwise, a *partial flag variety*. The *Grassmannian* $\mathrm{Gr}_{i,N}$ of i-dimensional subspaces of V is $\mathrm{GL}_N / P_{\hat{\imath}}$.

Let $\tilde{P}$ be any parabolic subgroup containing B_N and $\tau \in W$. The *Schubert variety* $X_{\tilde{P}}(\tau)$ is the closure inside $\mathrm{GL}_N / \tilde{P}$ of $B_N \cdot e_\tau$ where e_τ is the coset $\tau \tilde{P}$, endowed with the canonical reduced scheme structure. Hereafter, when we write $X_{\tilde{P}}(\tau)$, we mean that τ is the representative in $W^{\tilde{P}}$ of its coset. The *opposite big cell* $O^-_{\mathrm{GL}_N/\tilde{P}}$ in $\mathrm{GL}_N / \tilde{P}$ is the B_N^--orbit of the coset $(\mathrm{id} \cdot \tilde{P})$ in $\mathrm{GL}_N / \tilde{P}$. Let $Y_{\tilde{P}}(\tau) := X_{\tilde{P}}(\tau) \cap O^-_{\mathrm{GL}_N/\tilde{P}}$; we refer to $Y_{\tilde{P}}(\tau)$ as the *opposite cell* of $X_{\tilde{P}}(\tau)$.

We will write R^+, R^-, $R^+_{\tilde{P}}$, $R^-_{\tilde{P}}$, to denote, respectively, positive and negative roots for GL_N and for $\tilde{P}$. We denote by ϵ_i the character that sends the invertible diagonal matrix with $t_1, \ldots, t_n$ on the diagonal to t_i.

2.2. ***Précis on*** $\mathbf{GL}_n$ ***and Schubert varieties.*** Let $\tilde{P}$ be a parabolic subgroup of GL_N with $B_N \subseteq \tilde{P} \subseteq P$. We will use the following proposition extensively in the sequel.

Proposition 2.2.1. *Write $U_{\tilde{P}}^-$ for the negative unipotent radical of $\tilde{P}$.*

(a) *$O^-_{\mathrm{GL}_N/\tilde{P}}$ can be naturally identified with $U_{\tilde{P}}^- \tilde{P}/\tilde{P}$.*

(b) *For*

$$z = \begin{bmatrix} A_{n\times n} & C_{n\times m} \\ D_{m\times n} & E_{m\times m} \end{bmatrix} \in \mathrm{GL}_N,$$

$zP \in O^-_{\mathrm{GL}_N/P}$ if and only if A is invertible.

(c) *For $1 \le s \le n-1$, the inverse image of $O^-_{\mathrm{GL}_N/P}$ under the natural map $\mathrm{GL}_N/\tilde{P}_s \to \mathrm{GL}_N/P$ is isomorphic to $O^-_{\mathrm{GL}_N/P} \times P/\tilde{P}_s$ as schemes. Every element of $O^-_{\mathrm{GL}_N/P} \times P/\tilde{P}_s$ is of the form*

$$\begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \bmod \tilde{P}_s \in \mathrm{GL}_N/\tilde{P}_s.$$

Moreover, two matrices

$$\begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A'_{n\times n} & 0_{n\times m} \\ D'_{m\times n} & I_m \end{bmatrix}$$

in GL_N represent the same element modulo $\tilde{P}_s$ if and only if there exists a matrix $q \in Q_s$ such that $A' = Aq$ and $D' = Dq$.

(d) *For $1 \le s \le n-1$, $P/\tilde{P}_s$ is isomorphic to GL_n/Q_s. In particular, the projection map $O^-_{\mathrm{GL}_N/P} \times P/\tilde{P} \to P/\tilde{P}_s$ is given by*

$$\begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \bmod \tilde{P}_s \longmapsto A \bmod \tilde{Q} \in \mathrm{GL}_n/Q \simeq P/\tilde{P}_s.$$

Proof. (a) Note that $U_{\tilde{P}}^-$ is the subgroup of GL_N generated by the (one-dimensional) root subgroups $U_\alpha, \alpha \in R^- \setminus R_{\tilde{P}}^-$ and that $U_{\tilde{P}}^- \tilde{P}/\tilde{P} = B_N^- \tilde{P}/\tilde{P}$. Hence under the canonical projection $\mathrm{GL}_N \to \mathrm{GL}_N/P$, $g \mapsto gP$, the subgroup U_P^- is mapped onto $O^-_{\mathrm{GL}_N/\tilde{P}}$. It is easy to check that this is an isomorphism.

(b) Suppose that $zP \in O^-_{\mathrm{GL}_N/P}$. By (a), we see that there exist matrices $A'_{n\times n}$, $C'_{n\times m}$, $D'_{m\times n}$, and $E'_{m\times m}$ such that

$$z_1 := \begin{bmatrix} I_n & 0_{n\times m} \\ D'_{m\times n} & I_m \end{bmatrix} \in U_P^-, \quad z_2 := \begin{bmatrix} A'_{n\times n} & C'_{n\times m} \\ 0_{m\times n} & E'_{m\times m} \end{bmatrix} \in P$$

$$\text{and } z = \begin{bmatrix} A_{n\times n} & C_{n\times m} \\ D_{m\times n} & E_{m\times m} \end{bmatrix} = z_1 z_2.$$

Hence $A = A'$ is invertible. Conversely, if A is invertible, then we may write $z = z_1 z_2$ where

$$z_1 := \begin{bmatrix} I_n & 0 \\ DA^{-1} & I_m \end{bmatrix} \in U_P^- \quad \text{and} \quad z_2 := \begin{bmatrix} A & C \\ 0 & E - DA^{-1}C \end{bmatrix}.$$

Since $z \in \mathrm{GL}_N$, $z_2 \in P$.

(c) Let $z \in U_P^- P \subseteq \mathrm{GL}_N$. Then we can write $z = z_1 z_2$ uniquely with $z_1 \in U_P^-$ and $z_2 \in P$. For, if

$$\begin{bmatrix} I_n & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \begin{bmatrix} A_{n\times n} & C_{n\times m} \\ 0_{m\times n} & E_{m\times m} \end{bmatrix} = \begin{bmatrix} I_n & 0_{n\times m} \\ D'_{m\times n} & I_m \end{bmatrix} \begin{bmatrix} A'_{n\times n} & C'_{n\times m} \\ 0_{m\times n} & E'_{m\times m} \end{bmatrix},$$

then $A = A'$, $C = C'$, $DA = D'A'$, and $DC + E = D'C' + E'$, which yields that $D' = D$ (since $A = A'$ is invertible, by (b)) and $E = E'$. Hence $U_P^- \times_{\mathbb{C}} P = U_P^- P$. Therefore, for any parabolic subgroup $P' \subseteq P$, $U_P^- \times_{\mathbb{C}} P/P' = U_P^- P/P'$. The asserted isomorphism now follows by taking $P' = \tilde{P}_s$.

For the next statement, let

$$\begin{bmatrix} A_{n\times n} & C_{n\times m} \\ D_{m\times n} & E_{m\times m} \end{bmatrix} \in \mathrm{GL}_N$$

with A invertible (which we may assume by (b)). Then we have a decomposition (in GL_N)

$$\begin{bmatrix} A & C \\ D & E \end{bmatrix} = \begin{bmatrix} A & 0_{n\times m} \\ D & I_m \end{bmatrix} \begin{bmatrix} I_n & A^{-1}C \\ 0_{m\times n} & E - DA^{-1}C \end{bmatrix}.$$

Hence

$$\begin{bmatrix} A & C \\ D & E \end{bmatrix} \equiv \begin{bmatrix} A & 0_{n\times m} \\ D & I_m \end{bmatrix} \bmod \tilde{P}_s.$$

Finally,

$$\begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \equiv \begin{bmatrix} A'_{n\times n} & 0_{n\times m} \\ D'_{m\times n} & I_m \end{bmatrix} \bmod \tilde{P}_s$$

if and only if there exist matrices $q \in Q_s$, $q'_{n\times m}$, and $\tilde{q}_{n\times n}$ in GL_m such that

$$\begin{bmatrix} A' & 0 \\ D' & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ D & I \end{bmatrix} \begin{bmatrix} q & q' \\ 0_{m\times n} & \tilde{q} \end{bmatrix},$$

which holds if and only if $q' = 0$, $\tilde{q} = I_m$, $A' = Aq$, and $D' = Dq$ (since A and A' are invertible).

(d) There is a surjective morphism of $\mathbb{C}$-group schemes $P \to \mathrm{GL}_n$,

$$\begin{bmatrix} A_{n\times n} & C_{n\times m} \\ 0_{m\times n} & E_{m\times m} \end{bmatrix} \longmapsto A.$$

This induces the required isomorphism. Notice that the element

$$\begin{bmatrix} A_{n\times n} & C_{n\times m} \\ D_{m\times n} & E_{m\times m} \end{bmatrix} \bmod \tilde{P}_s \in O^-_{\mathrm{GL}_N/P} \times P/\tilde{P}_s$$

decomposes (uniquely) as

$$\begin{bmatrix} I_n & 0 \\ DA^{-1} & I_m \end{bmatrix} \left(\begin{bmatrix} A & C \\ 0 & E \end{bmatrix} \bmod \tilde{P}_s \right)$$

Hence it is mapped to $A \bmod Q_s \in \mathrm{GL}_n/Q_s$. Now use (c). □

Discussion 2.2.2. Let $\tilde{P} = P_{\{\widehat{i_1,\dots,i_t}\}}$ with $1 \le i_1 < \cdots < i_t \le N-1$. Then using Proposition 2.2.1(a) and its proof, $O^-_{\mathrm{GL}_N/\tilde{P}}$ can be identified with the affine space of lower-triangular matrices with possible nonzero entries x_{ij} at row i and column j where (i, j) is such that there exists $l \in \{i_1, \dots, i_t\}$ such that $j \le l < i \le N$. To see this, note (from the proof of Proposition 2.2.1(a)) that we are interested in those (i, j) such that the root $\epsilon_i - \epsilon_j$ belongs to $R^- \setminus R^-_{\tilde{P}}$. Since $R^-_{\tilde{P}} = \bigcap_{k=1}^{t} R^-_{P_{\widehat{i_k}}}$, we see that we are looking for (i, j) such that $\epsilon_i - \epsilon_j \in R^- \setminus R^-_{P_{\hat{l}}}$ for some $l \in \{i_1, \dots, i_t\}$. For the maximal parabolic group $P_{\hat{l}}$, we have $R^- \setminus R^-_{P_{\hat{l}}} = \{\epsilon_i - \epsilon_j \mid 1 \le j \le l < i \le N\}$. Hence $\dim O^-_{\mathrm{GL}_N/\tilde{P}} = |R^- \setminus R^-_{\tilde{P}}|$.

Let $\alpha = \epsilon_i - \epsilon_j \in R^- \setminus R^-_{\tilde{P}}$ and $l \in \{i_1, \dots, i_t\}$. Then the Plücker coordinate $p^{(l)}_{s_\alpha}$ on the Grassmannian $\mathrm{GL}_N/P_{\hat{l}}$ lifts to a regular function on $\mathrm{GL}_N/\tilde{P}$, which we denote by the same symbol. Its restriction to $O^-_{G/\tilde{P}}$ is the $l \times l$-minor with column indices $\{1, 2, \dots, l\}$ and row indices $\{1, \dots, j-1, j+1, \dots, l, i\}$. In particular,

$$x_{ij} = p^{(j)}_{s_\alpha}|_{O^-_{G/\tilde{P}}} \quad \text{for every } \alpha = \epsilon_i - \epsilon_j \in R^- \setminus R^-_{\tilde{P}}. \tag{2.2.3}$$

Example 2.2.4. Figure 1 shows the shape of $O^-_{\mathrm{GL}_N/\tilde{P}_s}$ for some $1 \le s \le n-1$. The rectangular region labelled with a circled D is $O^-_{\mathrm{GL}_N/P}$. The trapezoidal region labelled with a circled A is $O^-_{P/\tilde{P}_s}$. In this case, the x_{ij} appearing in (2.2.3) are exactly those in the regions labelled A and B.

Remark 2.2.5. $X_{\tilde{P}}(w)$ is an irreducible (and reduced) variety of dimension equal to the length of w. (Here we use that w is the representative in $W^{\tilde{P}}$.) It can be seen easily that under the natural projection $\mathrm{GL}_N/B_N \to \mathrm{GL}_N/\tilde{P}$, $X_{B_N}(w)$ maps birationally onto $X_{\tilde{P}}(w)$ for every $w \in W^{\tilde{P}}$. It is known that Schubert varieties are normal, Cohen–Macaulay and have rational singularities; see, e.g., [Brion and Kumar 2005, Section 3.4].

2.3. ***Homogeneous bundles and representations.*** Let Q be a parabolic subgroup of GL_n. We collect here some results about homogeneous vector bundles on GL_n/Q. Most of these results are well known, but for some of them, we could not find a reference, so we give a proof here for the sake of completeness. Online notes of Ottaviani [1995] and of D. Snow [1994] discuss the details of many of these results.

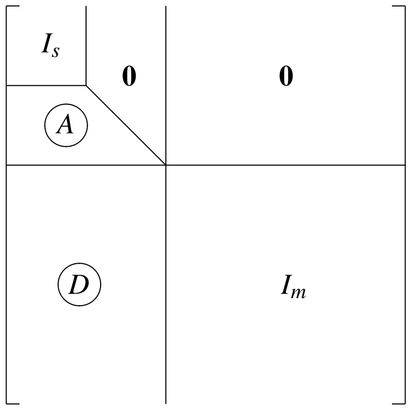

Figure 1. Shape of $O^-_{\mathrm{GL}_N/\tilde{P}_s}$.

Let L_Q and U_Q be respectively the Levi subgroup and the unipotent radical of Q. Let E be a finite-dimensional vector-space on which Q acts on the right; the vector-spaces that we will encounter have natural right action.

Definition 2.3.1. Define $\mathrm{GL}_n \times^Q E := (\mathrm{GL}_n \times E)/\sim$, where $\sim$ is the equivalence relation $(g, e) \sim (gq, eq)$ for every $g \in \mathrm{GL}_n$, $q \in Q$, and $e \in E$. Then $\pi_E : \mathrm{GL}_n \times^Q E \to \mathrm{GL}_n/Q$, $(g, e) \mapsto gQ$, is a vector bundle called the *vector bundle associated to* E (and the principal Q-bundle $\mathrm{GL}_n \to \mathrm{GL}_n/Q$). For $g \in \mathrm{GL}_n$, $e \in E$, we write $[g, e] \in \mathrm{GL}_n \times^Q E$ for the equivalence class of $(g, e) \in \mathrm{GL}_n \times E$ under $\sim$. We say that a vector bundle $\pi : \boldsymbol{E} \to \mathrm{GL}_n/Q$ is *homogeneous* if $\boldsymbol{E}$ has a GL_n-action and π is GL_n-equivariant, i.e., for every $y \in \boldsymbol{E}$, $\pi(g \cdot y) = g \cdot \pi(y)$.

In this section, we abbreviate $\mathrm{GL}_n \times^Q E$ as $\tilde{E}$. It is known that $\boldsymbol{E}$ is homogeneous if and only if $\boldsymbol{E} \simeq \tilde{E}$ for some Q-module E. (If this is the case, then E is the fibre of $\boldsymbol{E}$ over the coset Q.) A homogeneous bundle $\tilde{E}$ is said to be *irreducible* (respectively, *indecomposable*, *completely reducible*) if E is an irreducible (respectively indecomposable, completely reducible) Q-module. It is known that E is completely reducible if and only if U_Q acts trivially and that E is irreducible if and only if additionally it is irreducible as a representation of L_Q. See [Snow 1994, Section 5] or [Ottaviani 1995, Section 10] for the details.

Let $\sigma : \mathrm{GL}_n/Q \to \tilde{E}$ be a section of π_E. Let $g \in \mathrm{GL}_n$; write $[h, f] = \sigma(gQ)$. There exists a unique $q \in Q$ such that $h = gq$. Let $e = fq^{-1}$. Then $[g, e] = [h, f]$. If $[h, f'] = [h, f]$, then $f' = f$, so the assignment $g \mapsto e$ defines a function $\phi : \mathrm{GL}_n \to E$. This is Q-equivariant in the following sense:

$$\phi(gq) = \phi(g)q, \quad \text{for every } q \in Q \text{ and } g \in \mathrm{GL}_n. \tag{2.3.2}$$

Conversely, any such map defines a section of π_E. The set of sections $\mathrm{H}^0(\mathrm{GL}_n/Q, \tilde{E})$ of π_E is a finite-dimensional vector-space with $(\alpha\phi)(g) = \alpha(\phi(g))$ for every $\alpha \in \mathbb{C}$, ϕ a section of π_E, and $g \in \mathrm{GL}_n$.

Note that GL_n acts on GL_n/Q by multiplication on the left; setting $h \cdot [g, e] = [hg, e]$ for $g, h \in \mathrm{GL}_n$ and $e \in E$, we extend this to $\tilde{E}$. We can also define a natural GL_n-action on $\mathrm{H}^0(\mathrm{GL}_n/Q, \tilde{E})$ as follows. For any map $\phi : \mathrm{GL}_n \to E$, set $h \circ \phi$ to be the map $g \mapsto \phi(h^{-1}g)$. If ϕ satisfies (2.3.2), then for every $q \in Q$ and $g \in \mathrm{GL}_n$, $(h \circ \phi)(gq) = \phi(h^{-1}gq) = (\phi(h^{-1}g))q = ((h \circ \phi)(g))q$, so $h \circ \phi$ also satisfies (2.3.2). The action of GL_n on the sections is on the left:

$$(h_2h_1) \circ \phi = [g \mapsto \phi(h_1^{-1}h_2^{-1}g)] = [g \mapsto (h_1 \circ \phi)(h_2^{-1}g)] = h_2 \circ (h_1 \circ \phi).$$

In fact, $\mathrm{H}^i(\mathrm{GL}_n/Q, \tilde{E})$ is a GL_n-module for every i.

Suppose now that E is one-dimensional. Then Q acts on E by a character λ; we denote the associated line bundle on GL_n/Q by L_λ.

Discussion 2.3.3. Let $Q = P_{\widehat{i_1,\dots,i_t}}$, with $1 \le i_1 < \cdots < i_t \le n-1$. A weight λ is said to be *Q-dominant* if when we write $\lambda = \sum_{i=1}^n a_i\omega_i$ in terms of the fundamental weights ω_i, we have, $a_i \ge 0$ for all $i \notin \{i_1, \dots, i_t\}$, or equivalently, the associated line bundle (defined above) L_λ on Q/B_n has global sections. If we express λ as $\sum_{i=1}^n \lambda_i\epsilon_i$, then λ is Q-dominant if and only if for every $0 \le j \le t$, $\lambda_{i_j+1} \ge \lambda_{i_j+2} \ge \cdots \ge \lambda_{i_{j+1}}$ where we set $i_0 = 0$ and $i_{r+1} = n$. We will write $\lambda = (\lambda_1, \dots, \lambda_n)$ to mean that $\lambda = \sum_{i=1}^n \lambda_i\epsilon_i$. Every finite-dimensional irreducible Q-module is of the form $\mathrm{H}^0(Q/B_n, L_\lambda)$ for a Q-dominant weight λ. Hence the irreducible homogeneous vector bundles on GL_n/Q are in correspondence with Q-dominant weights. We describe them now. If $Q = P_{\widehat{n-i}}$, then $\mathrm{GL}_n/Q = \mathrm{Gr}_{i,n}$. (Recall that, for us, the GL_n-action on $\mathbb{C}^n$ is on the right.) On $\mathrm{Gr}_{i,n}$, we have the *tautological sequence*

$$0 \to \mathcal{R}_i \to \mathbb{C}^n \otimes \mathcal{O}_{\mathrm{Gr}_{i,n}} \to \mathcal{Q}_{n-i} \to 0 \tag{2.3.4}$$

of homogeneous vector bundles. The bundle $\mathcal{R}_i$ is called the *tautological subbundle* (of the trivial bundle $\mathbb{C}^n$) and $\mathcal{Q}_{n-i}$ is called the *tautological quotient bundle*. Every irreducible homogeneous bundle on $\mathrm{Gr}_{i,n}$ is of the form $\mathrm{S}_{(\lambda_1,\dots,\lambda_{n-i})}\,\mathcal{Q}^*_{n-i} \otimes \mathrm{S}_{(\lambda_{n-i+1},\dots,\lambda_n)}\,\mathcal{R}^*_i$ for some $P_{\widehat{n-i}}$-dominant weight λ. Here S_μ denotes the *Schur functor* associated to the partition μ. Now suppose that $Q = P_{\widehat{i_1,\dots,i_t}}$ with $1 \le i_1 < \cdots < i_t \le n-1$. Since the action is on the right, GL_n/Q projects to $\mathrm{Gr}_{n-i,n}$ precisely when $i = i_j$ for some $1 \le j \le t$. For each $1 \le j \le t$, we can take the pullback of the tautological bundles $\mathcal{R}_{n-i_j}$ and $\mathcal{Q}_{i_j}$ to GL_n/Q from $\mathrm{GL}_n/P_{\hat{i}_j}$. The irreducible homogeneous bundle corresponding to a Q-dominant weight λ is

$$\mathrm{S}_{(\lambda_1,\dots,\lambda_{i_1})}\,\mathcal{U}_{i_1} \otimes \mathrm{S}_{(\lambda_{i_1+1},\dots,\lambda_{i_2})}(\mathcal{R}_{n-i_1}/\mathcal{R}_{n-i_2})^* \otimes \\ \cdots \otimes \mathrm{S}_{(\lambda_{i_{t-1}+1},\dots,\lambda_{i_t})}(\mathcal{R}_{n-i_{t-1}}/\mathcal{R}_{n-i_t})^* \otimes \mathrm{S}_{(\lambda_{i_t+1},\dots,\lambda_{i_n})}(\mathcal{R}_{n-i_t})^*.$$

See [Weyman 2003, Section 4.1]. Hereafter, we will write $\mathcal{U}_i = \mathcal{Q}_i^*$. Moreover, abusing notation, we will use $\mathcal{R}_i$, $\mathcal{Q}_i, \mathcal{U}_i$ etc. for these vector bundles on any (partial) flag variety on which they would make sense.

A Q-dominant weight is called $(i_1, \dots, i_r)$-*dominant* in [op. cit., p. 114]. Although our definition looks like Weyman's definition, we should keep in mind that our action is on the right. We only have to be careful when we apply the Borel–Weil–Bott theorem (more specifically, Bott's algorithm). In this paper, our computations are done only on Grassmannians. If μ and ν are partitions, then (μ, ν) will be Q-dominant (for a suitable Q), and will give us the vector bundle $\mathrm{S}_\mu \mathcal{Q}^* \otimes \mathrm{S}_\nu \mathcal{R}^*$ (this is where the right-action of Q becomes relevant) and to compute its cohomology, we will have to apply Bott's algorithm to the Q-dominant weight (ν, μ). (In [op. cit.], one would get $\mathrm{S}_\mu \mathcal{R}^* \otimes \mathrm{S}_\nu \mathcal{Q}^*$ and would apply Bott's algorithm to (μ, ν).) See, for example, the proof of Proposition 5.4 or the examples that follow it.

Proposition 2.3.5. *Let $Q_1 \subseteq Q_2$ be parabolic subgroups and E a Q_1-module. Let $f : \mathrm{GL}_n/Q_1 \to \mathrm{GL}_n/Q_2$ be the natural map. Then for every $i \geq 0$,*

$$R^i f_*(\mathrm{GL}_n \times^{Q_1} E) = \mathrm{GL}_n \times^{Q_2} \mathrm{H}^i(Q_2/Q_1, \mathrm{GL}_n \times^{Q_1} E).$$

Proof. For Q_2 (respectively, Q_1), the category of homogeneous vector bundles on GL_n/Q_2 (respectively, GL_n/Q_1) is equivalent to the category of finite-dimensional Q_2-modules (respectively, finite-dimensional Q_1-modules). Now, the functor f^* from the category of homogeneous vector bundles over GL_n/Q_2 to that over GL_n/Q_1 is equivalent to the restriction functor $\mathrm{Res}_{Q_1}^{Q_2}$. Hence their corresponding right-adjoint functors f_* and the induction functor $\mathrm{Ind}_{Q_1}^{Q_2}$ are equivalent; one may refer to [Hartshorne 1977, II.5, p. 110] and [Jantzen 2003, I.3.4, "Frobenius Reciprocity"] to see that these are indeed adjoint pairs. Hence, for homogeneous bundles on GL_n/Q_1, $R^i f_*$ can be computed using $R^i \mathrm{Ind}_{Q_1}^{Q_2}$. On the other hand, note that $\mathrm{Ind}_{Q_1}^{Q_2}(-)$ is the functor $\mathrm{H}^0(Q_2/Q_1, \mathrm{GL}_n \times^{Q_1} -)$ on Q_1-modules, which follows from [op. cit., I.3.3, Equation (2)]. The proposition now follows. $\square$

3. Properties of Schubert desingularization

This section is devoted to proving some results on smooth Schubert varieties in partial flag varieties. In Theorem 3.5, we show that opposite cells of certain smooth Schubert varieties in $\mathrm{GL}_N/\tilde{P}$ are linear subvarieties of the affine variety $O^-_{\mathrm{GL}_N/\tilde{P}}$, where $\tilde{P} = \tilde{P}_s$ for some $1 \leq s \leq n-1$. Using this, we show in Theorem 3.7 that if $X_P(w) \in \mathrm{GL}_N/P$ is such that there exists a parabolic subgroup $\tilde{P} \subsetneq P$ such that the birational model $X_{\tilde{P}}(\tilde{w}) \subseteq \mathrm{GL}_N/\tilde{P}$ of $X_P(w)$ is smooth (we say that $X_P(w)$ has a *Schubert desingularization* if this happens) then the inverse image of $Y_P(w)$

inside $X_{\tilde{P}}(\tilde{w})$ is a vector bundle over a Schubert variety in $P/\tilde{P}$. This will give us a realisation of Diagram (1-2).

Recall the following result about the tangent space of a Schubert variety; see [Billey and Lakshmibai 2000, Chapter 4] for details.

Proposition 3.1. *Let $\tau \in W^{\tilde{P}}$. Then the dimension of the tangent space of $X_{\tilde{P}}(\tau)$ at e_{id} is*

$$\#\{s_\alpha \mid \alpha \in R^- \setminus R^-_{\tilde{P}} \text{ and } \tau \geq s_\alpha \text{ in } W/W_{\tilde{P}}\}.$$

In particular, $X_{\tilde{P}}(\tau)$ is smooth if and only if

$$\dim X_{\tilde{P}}(\tau) = \#\{s_\alpha \mid \alpha \in R^- \setminus R^-_{\tilde{P}} \text{ and } \tau \geq s_\alpha \text{ in } W/W_{\tilde{P}}\}.$$

Notation 3.2. We adopt the following notation: Let $w = (a_1, a_2, \dots, a_n) \in W^P$. Let $r = r(w)$, i.e., the index r such that $a_r \leq n < a_{r+1}$. Let $1 \leq s \leq r$. We write $\tilde{P} = \tilde{P}_s$. Let $\tilde{w}$ be the minimal representative of w in $W^{\tilde{P}}$. Let $c_{r+1} > \cdots > c_n$ be such that $\{c_{r+1}, \dots, c_n\} = \{1, \dots, n\} \setminus \{a_1, \dots, a_r\}$; let $w' := (a_1, \dots, a_r, c_{r+1}, \dots, c_n) \in S_n$, the Weyl group of GL_n.

Our concrete descriptions of free resolutions will be for the following class of Schubert varieties.

Definition 3.3. Let $1 \leq r \leq n-1$. Let

$$\mathcal{W}_r = \{(n-r+1, \dots, n, a_{r+1}, \dots, a_{n-1}, N) \in W^P : n < a_{r+1} < \cdots < a_{n-1} < N\}.$$

The determinantal variety of $(m \times n)$ matrices of rank at most k can be realised as $Y_P(w)$, $w = (k+1, \dots, n, N-k+1, \dots N) \in \mathcal{W}_{n-k}$ [Seshadri 2007, Section 1.6].

Proposition 3.4. *$X_{\tilde{P}_s}(\tilde{w})$ is smooth in the following situations*:

(a) *$w \in W^P$ arbitrary and $s = 1$* [Kempf 1971].

(b) *$w \in \mathcal{W}_r$ for some $1 \leq r \leq n-1$ and $s = r$.*

Proof. For both (a) and (b): Let $w_{\max} \in W (= S_N)$ be the maximal representative of $\tilde{w}$. We claim that

$$w_{\max} = (a_s, a_{s-1}, \dots, a_1, a_{s+1}, a_{s+2}, \dots, a_n, b_{n+1}, \dots, b_N) \in W.$$

Assume the claim. Then $w_{\max}$ is a 4231- and 3412-avoiding element of W; hence $X_{B_N}(w_{\max})$ is smooth; see [Lakshmibai and Sandhya 1990; Billey and Lakshmibai 2000, 8.1.1]. Since $w_{\max}$ is the maximal representative (in W) of $\tilde{w}\tilde{P}_s$, we see that $X_{B_N}(w_{\max})$ is a fibration over $X_{\tilde{P}_s}(\tilde{w})$ with smooth fibres $\tilde{P}_s/B_N$; therefore $X_{\tilde{P}_s}(\tilde{w})$ is smooth.

To prove the claim, we need to show that $X_{P_{\hat{i}}}(w_{\max}) = X_{P_{\hat{i}}}(\tilde{w})$ for every $s \leq i \leq n$ and that $w_{\max}$ is the maximal element of W with this property. This follows, since for every $\tau := (c_1, \dots, c_N) \in W$ and for every $1 \leq i \leq N$, $X_{P_{\hat{i}}}(\tau) = X_{P_{\hat{i}}}(\tau')$ where $\tau' \in W^{P_{\hat{i}}}$ is the element with $c_1, \dots, c_i$ written in the increasing order. □

Theorem 3.5. *Identify $O^-_{G/\tilde{P}}$ with $O^-_{G/P} \times O^-_{P/\tilde{P}}$ as in Figure 1, with $O^-_{G/P}$ thought of as $M_{m,n}$, the space of all $m \times n$ matrices. If $w \in W^P$ is arbitrary and $s = 1$ (see Proposition 3.4(a)) then we have an identification of $Y_{\tilde{P}}(\tilde{w})$ with $\mathcal{V}_w \times \mathcal{V}'_w$, where $\mathcal{V}_w$ is the linear subspace of $O^-_{G/P}$ given by*

$$x_{ij} = 0 \quad \textit{if} \quad \begin{cases} 1 \le j \le r(w), \quad \textit{or} \\ r(w)+1 \le j \le n-1 \ \textit{ and } \ a_j - n < i \le m. \end{cases}$$

and $\mathcal{V}'_w$ is the linear subspace of $O^-_{P/\tilde{P}}$ given by

$$x_{ij} = 0 \quad \textit{for every} \ \ 1 \le j \le r(w) \ \ \textit{and for every} \ \ i \ge \max\{a_j+1, s+1\}.$$

On the other hand, if $w \in \mathcal{W}_r$ for some $1 \le r \le n-1$ and $s = r$ (see Proposition 3.4(b)) then we have an identification of $Y_{\tilde{P}}(\tilde{w})$ with $\mathcal{V}_w \times O^-_{P/\tilde{P}}$, where $\mathcal{V}_w$ is the linear subspace of $O^-_{G/P}$ given by

$$x_{ij} = 0 \quad \textit{if} \quad \begin{cases} 1 \le j \le r, \quad \textit{or} \\ r+1 \le j \le n-1 \ \textit{ and } \ a_j - n < i \le m. \end{cases}$$

Proof. Consider the first case: w arbitrary and $s = 1$. Since $a_1 < \cdots < a_n$, we see that for every $j \le n$ and for every $i \ge \max\{a_j+1, s+1\}$, the reflection (i, j) equals $(1, 2, \ldots, j-1, i)$ in $W/W_{P_{\hat{j}}}$, while $\tilde{w}$ equals $(a_1, \ldots, a_j)$. Hence (i, j) is not smaller than $\tilde{w}$ in $W/W_{P_{\hat{j}}}$, so the Plücker coordinate $p^{(j)}_{(i,j)}$ vanishes on $X_{\tilde{P}}(\tilde{w})$. Therefore for such (i, j), $x_{ij} \equiv 0$ on $Y_{\tilde{P}}(\tilde{w})$, by (2.2.3).

On the other hand, note that the reflections (i, j) with $j \le n$ and $i \ge \max\{a_j+1, s+1\}$ are precisely the reflections s_α with $\alpha \in R^- \setminus R^-_{\tilde{P}}$ and $\tilde{w} \not\ge s_\alpha$ in $W/W_{\tilde{P}}$. Since $X_{\tilde{P}}(\tilde{w})$ is smooth, this implies (see Proposition 3.1) that the codimension of $Y_{\tilde{P}}(\tilde{w})$ in $O^-_{\mathrm{GL}_N/\tilde{P}}$ equals

$$\#\big\{(i, j) \mid j \le n \ \text{ and } \ i \ge \max\{a_j+1, s+1\}\big\}$$

so $Y_{\tilde{P}}(\tilde{w})$ is the linear subspace of $O^-_{\mathrm{GL}_N/\tilde{P}}$ defined by the vanishing of

$$\big\{x_{ij} \mid j \le n \ \text{ and } \ i \ge \max\{a_j+1, s+1\}\big\}.$$

This gives the asserted identification of $Y_{\tilde{P}}(\tilde{w})$.

Now the second case: $w \in \mathcal{W}_r$ for some $1 \le r \le n-1$ and $s = r$. Note that $X_{Q_s}(w') = \mathrm{GL}_n/B_n$, because of the choice of w and s. Therefore, an argument similar to the one above, along with counting dimensions, shows that $Y_{\tilde{P}}(\tilde{w})$ is defined inside $O^-_{G/\tilde{P}}$ by

$$x_{ij} = 0 \quad \text{if} \quad \begin{cases} 1 \le j \le r, \quad \text{or} \\ r+1 \le j \le n-1 \ \text{ and } \ a_j - n < i \le m. \end{cases}$$

This gives the asserted identification of $Y_{\tilde{P}}(\tilde{w})$. □

Let $Z_{\tilde{P}}(\tilde{w}) := Y_P(w) \times_{X_P(w)} X_{\tilde{P}}(\tilde{w}) = (O^-_{\mathrm{GL}_N/P} \times P/\tilde{P}) \cap X_{\tilde{P}}(\tilde{w})$. Write p for the composite map

$$Z_{\tilde{P}}(\tilde{w}) \to O^-_{\mathrm{GL}_N/P} \times P/\tilde{P} \to P/\tilde{P},$$

where the first map is the inclusion (as a closed subvariety) and the second map is projection. Using Proposition 2.2.1(c) and (d), we see that

$$p\left(\begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \bmod \tilde{P}\right) = A \mod Q_s.$$

(A is invertible by Proposition 2.2.1(b).) Using the injective map

$$B_n \longrightarrow B_N, \quad A \mapsto \begin{bmatrix} A & 0_{n\times m} \\ 0_{m\times n} & I_m \end{bmatrix},$$

B_n can be thought of as a subgroup of B_N. With this identification, we have the following Proposition:

Proposition 3.6. *$Z_{\tilde{P}}(\tilde{w})$ is B_n-stable (for the action on the left by multiplication). Further, p is B_n-equivariant.*

Proof. Let

$$z := \begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \in \mathrm{GL}_N$$

be such that $z\tilde{P} \in Z_{\tilde{P}}(\tilde{w})$. Since $X_{B_N}(\tilde{w}) \to X_{\tilde{P}}(\tilde{w})$ is surjective, we may assume that $z \pmod{B_N} \in X_{B_N}(\tilde{w})$, i.e., $z \in \overline{B_N \tilde{w} B_N}$. Then for every $A' \in B_n$,

$$\begin{bmatrix} A' & 0_{n\times m} \\ 0_{m\times n} & I_m \end{bmatrix} z = \begin{bmatrix} A'A & 0 \\ D & I_m \end{bmatrix} =: z'.$$

Then $z' \in \overline{B_N \tilde{w} B_N}$, so $z' \pmod{\tilde{P}} \in X_{\tilde{P}}(\tilde{w})$. By Proposition 2.2.1(b), we have that A is invertible, and hence AA' is invertible; this implies (again by Proposition 2.2.1(b)) that $z' \pmod{\tilde{P}} \in Z_{\tilde{P}}(\tilde{w})$. Thus $Z_{\tilde{P}}(\tilde{w})$ is B_n-stable. Also, $p(A'z) = p(z') = A'A = A'p(z)$. Hence p is B_n-equivariant. □

Theorem 3.7. *With notation as above,*

(a) *The natural map $X_{\tilde{P}}(\tilde{w}) \to X_P(w)$ is proper and birational. In particular, the map $Z_{\tilde{P}}(\tilde{w}) \to Y_P(w)$ is proper and birational.*

(b) *$X_{Q_s}(w')$ is the fibre of the natural map $Z_{\tilde{P}}(\tilde{w}) \to Y_P(w)$ at $e_{\mathrm{id}} \in Y_P(w)$ (with w' as in Notation 3.2).*

(c) *Suppose that w and s satisfy the conditions of Proposition 3.4. Then $X_{Q_s}(w')$ is the image of p. Further, p is a fibration with fibre isomorphic to $\mathcal{V}_w$.*

(d) *Suppose that w and s satisfy the conditions of Proposition 3.4. Then p identifies $Z_{\tilde{P}}(\tilde{w})$ as a subbundle of the trivial bundle $O^{-}_{\mathrm{GL}_N/P} \times X_{Q_s}(w')$, which arises as the restriction of the vector bundle on GL_n/Q_s associated to the Q_s-module $\mathcal{V}_w$ (which, in turn, is a Q_s-submodule of $O^{-}_{\mathrm{GL}_N/P}$).*

We believe that all the assertions above hold without the hypothesis that $X_{\tilde{P}}(\tilde{w})$ is smooth.

Proof. (a) The map $X_{\tilde{P}}(\tilde{w}) \hookrightarrow \mathrm{GL}_N/\tilde{P} \to \mathrm{GL}_N/P$ is proper and its (scheme-theoretic) image is $X_P(w)$; hence $X_{\tilde{P}}(\tilde{w}) \to X_P(w)$ is proper. Birationality follows from the fact that $\tilde{w}$ is the minimal representative of the coset $w\tilde{P}$ (see Remark 2.2.5).

(b) The fibre at $e_{\mathrm{id}} \in Y_P(w)$ of the map $Y_{\tilde{P}}(\tilde{w}) \to Y_P(w)$ is $\{0\} \times \mathcal{V}'_w$ (contained in $\mathcal{V}_w \times \mathcal{V}'_w = Y_{\tilde{P}}(\tilde{w})$). Since $Z_{\tilde{P}}(\tilde{w})$ is the closure of $Y_{\tilde{P}}(\tilde{w})$ inside $O^{-}_{\mathrm{GL}_N/P} \times P/\tilde{P}$ and $X_{Q_s}(w')$ is the closure of $\mathcal{V}'_w$ inside $P/\tilde{P}$ (note that, as a subvariety of $O^{-}_{P/\tilde{P}}$, $Y_{Q_s}(w')$ is identified with $\mathcal{V}'_w$), we see that fibre of $Z_{\tilde{P}}(\tilde{w}) \to Y_P(w)$ at $e_{\mathrm{id}} \in Y_P(w)$ is $X_{Q_s}(w')$.

(c) From Theorem 3.5 it follows that

$$Y_{\tilde{P}}(\tilde{w}) = \left\{ \begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ D_{m\times n} & I_m \end{bmatrix} \bmod \tilde{P} \;\middle|\; A \in \mathcal{V}'_w \text{ and } D \in \mathcal{V}_w \right\}.$$

Hence $p(Y_{\tilde{P}}(\tilde{w})) = \mathcal{V}'_w \subseteq X_{Q_s}(w')$. Since $Y_{\tilde{P}}(\tilde{w})$ is dense inside $Z_{\tilde{P}}(\tilde{w})$ and $X_{Q_s}(w')$ is closed in GL_n/Q_s, we see that $p(Z_{\tilde{P}_r}(\tilde{w})) \subseteq X_{Q_s}(w')$. The other inclusion $X_{Q_s}(w') \subseteq p(Z_{\tilde{P}_r}(\tilde{w}))$ follows from (b). Hence, $p(Z_{\tilde{P}_r}(\tilde{w}))$ equals $X_{Q_s}(w')$.

Next, to prove the second assertion in (c), we shall show that for every $A \in \mathrm{GL}_n$ with $A \bmod Q_s \in X_{Q_s}(w')$,

$$p^{-1}(A \bmod Q_s) = \left\{ \begin{bmatrix} A & 0_{n\times m} \\ D & I_m \end{bmatrix} \bmod \tilde{P} \;\middle|\; D \in \mathcal{V}_w \right\}. \tag{3.8}$$

Towards proving this, we first observe that $p^{-1}(e_{\mathrm{id}})$ equals $\mathcal{V}_w$ (in view of Theorem 3.5). Next, we observe that every B_n-orbit inside $X_{Q_s}(w')$ meets $\mathcal{V}'_w$ (which equals $Y_{Q_s}(w')$); further, p is B_n-equivariant (see Proposition 3.6). The assertion (3.8) now follows.

(d) First observe that for the action of right multiplication by GL_n on $O^{-}_{G/P}$ (being identified with $M_{m,n}$, the space of $m \times n$ matrices), $\mathcal{V}_w$ is stable; we thus get the homogeneous bundle $\mathrm{GL}_n \times^{Q_s} \mathcal{V}_w \to \mathrm{GL}_n/Q_s$ (Definition 2.3.1). Now to prove the assertion about $Z_{\tilde{P}_s}(\tilde{w}))$ being a vector bundle over $X_{Q_s}(w')$, we will show that there is a commutative diagram given as below, with ψ an isomorphism:

$$\begin{tikzcd} Z_{\tilde{P}_s}(\tilde{w}) \arrow[dr, "\psi"] \arrow[drr, "\phi"] \arrow[ddr, "p"'] & & \\ & (\mathrm{GL}_n \times^{Q_s} \mathcal{V}_w)|_{X_{Q_s}(w')} \arrow[r] \arrow[d] & \mathrm{GL}_n \times^{Q_s} \mathcal{V}_w \arrow[d, "\alpha"] \\ & X_{Q_s}(w') \arrow[r, "\beta"] & \mathrm{GL}_n/Q_s \end{tikzcd}$$

The map α is the homogeneous bundle map and β is the inclusion. Define ϕ by

$$\phi : \begin{bmatrix} A & 0_{n\times m} \\ D & I_m \end{bmatrix} \bmod \tilde{P} \longmapsto (A, D)/\sim .$$

Using Proposition 2.2.1(c) and (3.8), we conclude the following: ϕ is well defined and injective; $\beta \cdot p = \alpha \cdot \phi$; hence, by the universal property of products, the map ψ exists; and, finally, the injective map ψ is in fact an isomorphism (by dimension considerations). □

Corollary 3.9. *If $X_{\tilde{P}}(\tilde{w})$ is smooth, then we have the following realisation of the diagram in (1-2):*

$$\begin{tikzcd} Z_{\tilde{P}}(\tilde{w}) \arrow[r, hook] \arrow[d, "q'"] & O^-_{\mathrm{GL}_N/P} \times X_{Q_s}(w') \arrow[r] \arrow[d, "q"] & X_{Q_s}(w') \\ Y_P(w) \arrow[r, hook] & O^-_{\mathrm{GL}_N/P} & \end{tikzcd} .$$

Example 3.10. This example shows that even with $r = s$, $X_{Q_s}(w')$ need not be smooth for arbitrary $w \in W^P$. Let $n = m = 4$ and $w = (2, 4, 7, 8)$. Then $r = 2$; take $s = 2$. Then we obtain $w_{\max} = (4, 2, 7, 8, 5, 6, 3, 1)$, which has a 4231 pattern.

4. Free resolutions

The Kempf–Lascoux–Weyman geometric technique. We now summarise the geometric technique of computing free resolutions, following [Weyman 2003, Chapter 5]. Consider (1-1). There is a natural map $f : V \to \mathrm{Gr}_{r,d}$ (where $r = \mathrm{rk}_V Z$ and $d = \dim \mathbb{A}$) such that the inclusion $Z \subseteq \mathbb{A} \times V$ is the pullback of the tautological sequence (2.3.4); here $\mathrm{rk}_V Z$ denotes the rank of Z as a vector bundle over V, i.e., $\mathrm{rk}_V Z = \dim Z - \dim V$. Let $\xi = (f^*\mathcal{Q})^*$. Write R for the polynomial ring $\mathbb{C}[\mathbb{A}]$ and $\mathfrak{m}$ for its homogeneous maximal ideal. (The grading on R arises as follows. In (1-1), $\mathbb{A}$ is thought of as the fibre of a trivial vector bundle, so it has a distinguished point, its origin. Now, being a subbundle, Z is defined by linear equations in each fibre; i.e., for each $v \in V$, there exist $s := (\dim \mathbb{A} - \mathrm{rk}_V Z)$ linearly independent linear polynomials $\ell_{v,1}, \dots, \ell_{v,s}$ that vanish along Z and define it. Now $Y = \{y \in \mathbb{A} \mid \text{there exists } v \in V \text{ such that } \ell_{v,1}(y) = \cdots = \ell_{v,s}(y) = 0\}$. Hence Y is defined by homogeneous polynomials. This explains why the resolution obtained below is graded.) Let $\mathfrak{m}$ be the homogeneous maximal ideal, i.e., the ideal defining the origin in $\mathbb{A}$.

Theorem 4.1 [Weyman 2003, basic theorem 5.1.2]. *With notation as above, there is a finite complex $(F_\bullet, \partial_\bullet)$ of finitely generated graded free R-modules that is quasi-isomorphic to $\boldsymbol{R}q'_*\mathcal{O}_Z$, with*

$$F_i = \bigoplus_{j \geq 0} \mathrm{H}^j(V, \textstyle\bigwedge^{i+j} \xi) \otimes_{\mathbb{C}} R(-i-j),$$

and $\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}$. Furthermore, the following are equivalent:

(a) *Y has rational singularities; i.e., $\boldsymbol{R}q'_*\mathcal{O}_Z$ is quasi-isomorphic to $\mathcal{O}_Y$;*

(b) *$F_\bullet$ is a minimal R-free resolution of $\mathbb{C}[Y]$, i.e., $F_0 \simeq R$ and $F_{-i} = 0$ for all $i > 0$.*

We give a sketch of the proof because one direction of the equivalence is only implicit in the proof of [op. cit., 5.1.3].

Sketch of the proof. One constructs a suitable q_*-acyclic resolution $\mathcal{I}^\bullet$ of the Koszul complex that resolves $\mathcal{O}_Z$ as an $\mathcal{O}_{\mathbb{A}\times V}$-module so that the terms in $q_*\mathcal{I}^\bullet$ are *finitely generated free* graded R-modules. One places the Koszul complex on the negative horizontal axis and thinks of $\mathcal{I}^\bullet$ as a second-quadrant double complex, thus to obtain a complex $G_\bullet$ of finitely generated free R-modules whose homology at the i-th position is $R^{-i}q_*\mathcal{O}_Z$. Then, using standard homological considerations, one constructs a subcomplex $(F_\bullet, \partial_\bullet)$ of $G_\bullet$ that is quasi-isomorphic to $G_\bullet$ with $\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ (we say that $F_\bullet$ is *minimal* if this happens), and since $\mathrm{H}_i(G_\bullet) = 0$ for every $|i| \gg 0$, $F_i = 0$ for every $|i| \gg 0$. Now using the minimality of $F_\bullet$, we see that $R^i q_*\mathcal{O}_Z = 0$ for every $i \geq 1$ if and only if $F_{-i} = 0$ for every $i \geq 1$. When one of these conditions holds, then $F_\bullet$ becomes a minimal free resolution of $q_*\mathcal{O}_Z$ which is a finitely generated $\mathcal{O}_Y$-module, and therefore $q_*\mathcal{O}_Z = \mathcal{O}_Y$ if and only if $q_*\mathcal{O}_Z$ is generated by one element as an $\mathcal{O}_Y$-module if and only if $q_*\mathcal{O}_Z$ is a generated by one element as an R-module if and only if F_0 is a free R-module of rank one if and only if $F_0 = R(0)$ since $\mathrm{H}^0(V, \bigwedge^0 \xi) \otimes R$ is a summand of F_0. □

Our situation. We now apply Theorem 4.1 to our situation. We keep the notation of Theorem 3.7. Theorem 4.1 and Corollary 3.9 yield the following result:

Theorem 4.2. *Suppose that $X_{\tilde{P}_s}(\tilde{w})$ is smooth. Write $\mathcal{U}_w$ for the restriction to $X_{Q_s}(w')$ of the vector bundle on GL_n/Q_s associated to the Q_s-module $(O^-_{\mathrm{GL}_N/P}/\mathcal{V}_w)^*$. (This is the dual of the quotient of $O^-_{\mathrm{GL}_N/P} \times X_{Q_s}(w')$ by $Z_{\tilde{P}_s}(\tilde{w})$.) Then we have a* *minimal R-free resolution* $(F_\bullet, \partial_\bullet)$ *of* $\mathbb{C}[Y_P(w)]$ *with*

$$F_i = \bigoplus_{j\geq 0} \mathrm{H}^j(X_{Q_s}(w'), \textstyle\bigwedge^{i+j}\mathcal{U}_w) \otimes_{\mathbb{C}} R(-i-j).$$

In the first case, $Q_s = B_n$, so p makes $Z_{\tilde{P}_1}(\tilde{w})$ a vector bundle on a smooth Schubert subvariety $X_{B_1}(w')$ of GL_n/B_n. In the second case, w' is the maximal word in S_n, so $X_{Q_r}(w') = \mathrm{GL}_n/Q_r$; see Discussion 4.3 for further details.

Computing the cohomology groups required in Theorem 4.2 in the general situation of Kempf's desingularization (Proposition 3.4(a)) is a difficult problem, even though the relevant Schubert variety $X_{B_n}(w')$ is smooth. Hence we are forced to restrict our attention to the subset of W^P considered in Proposition 3.4(b).

The stipulation that, for $w \in \mathcal{W}_r$, w sends n to N is not very restrictive. This can be seen in two (related) ways. Suppose that w does not send n to N. Then, firstly, $X_P(w)$ can be thought of as a Schubert subvariety of a smaller Grassmannian. Or, secondly, $\mathcal{U}_w$ will contain the trivial bundle $\mathcal{U}_n$ as a summand, so $\mathrm{H}^0(\mathrm{GL}_n/Q_r, \xi) \neq 0$, i.e., $R(-1)$ is a summand of F_1. In other words, the defining ideal of $Y_P(w)$ contains a linear form.

Discussion 4.3. We give some more details of the situation in Proposition 3.4(b) that will be used in the next section. Let

$$w = (n-r+1, n-r+2, \ldots, n, a_{r+1}, \ldots, a_{n-1}, N) \in \mathcal{W}_r.$$

The space of $(m \times n)$ matrices is a GL_n-module with a right action; the subspace $\mathcal{V}_w$ is Q_r-stable under this action. Thus $\mathcal{V}_w$ is a Q_r-module, and gives an associated vector bundle $(\mathrm{GL}_n \times^{Q_r} \mathcal{V}_w)$ on GL_n/Q_r. The action on the right of GL_n on the space of $(m \times n)$ matrices breaks by rows; each row is a natural n-dimensional representation of GL_n. For each $1 \leq j \leq m$, there is a unique $r \leq i_j \leq n-1$ such that $a_{i_j} < j+n \leq a_{i_j+1}$. (Note that $a_r = n$ and $a_n = N$.) In row j, $\mathcal{V}_w$ has rank $n - i_j$, and is a subbundle of the natural representation. Hence the vector bundle associated to the j-th row of $\mathcal{V}_w$ is the pullback of the tautological subbundle (of rank $(n-i_j)$) on $\mathrm{Gr}_{n-i_j,n}$. We denote this by $\mathcal{R}_{n-i_j}$. Therefore $(\mathrm{GL}_n \times^{Q_r} \mathcal{V}_w)$ is the vector bundle $\mathcal{R}_w := \bigoplus_{j=1}^m \mathcal{R}_{n-i_j}$. Let $\mathcal{Q}_w := \bigoplus_{j=1}^m \mathcal{Q}_{i_j}$ where $\mathcal{Q}_{i_j}$ is the tautological quotient bundle corresponding to $\mathcal{R}_{n-i_j}$. Then the vector bundle $\mathcal{U}_w$ on GL_n/Q_r that was defined in Theorem 4.2 is $\mathcal{Q}_w^*$.

5. Cohomology of homogeneous vector bundles

It is, in general, difficult to compute the cohomology groups $\mathrm{H}^j(\mathrm{GL}_n/Q_r, \bigwedge^t \mathcal{U}_w)$ in Theorem 4.2 for arbitrary $w \in \mathcal{W}_r$. In this section, we will discuss some approaches. We believe that this is a problem of independent interest. Our method involves replacing Q_r inductively by increasingly bigger parabolic subgroups, so we give the general setup below.

Setup 5.1. Let $1 \le r \le n-1$. Let $m_r, \dots, m_{n-1}$ be nonnegative integers such that $m_r + \cdots + m_{n-1} = m$. Let Q be a parabolic subgroup of GL_n such that $Q \subseteq P_{\hat{\imath}}$ for every $r \le i \le n-1$ such that $m_i > 0$. We consider the homogeneous vector bundle $\xi = \bigoplus_{i=r}^{n-1} \mathcal{U}_i^{m_i}$ on GL_n/Q, We want to compute the vector-spaces $\mathrm{H}^j(\mathrm{GL}_n/Q_r, \bigwedge^t \xi)$.

Lemma 5.2. *Let $f : X' \to X$ be a fibration with fibre some Schubert subvariety Y of some (partial) flag variety. Then $f_* \mathbb{O}_{X'} = \mathbb{O}_X$ and $R^i f_* \mathbb{O}_{X'} = 0$ for every $i \ge 1$. In particular, for every locally free coherent sheaf L on X, $\mathrm{H}^i(X', f^* L) = \mathrm{H}^i(X, L)$ for every $i \ge 0$.*

Proof. The first assertion is a consequence of Grauert's theorem [Hartshorne 1977, III.12.9] and the fact (see, for example, [Seshadri 2007, Theorem 3.2.1]) that

$$\mathrm{H}^i(Y, \mathbb{O}_Y) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The second assertion follows from the projection formula and the Leray spectral sequence. □

Proposition 5.3. *Let $m_i, r \le i \le n-1$ be as in Setup 5.1. Let*

$$Q' = \bigcap_{\substack{r \le i \le n-1 \\ m_i > 0}} P_{\hat{\imath}}.$$

Then $\mathrm{H}^*(\mathrm{GL}_n/Q, \bigwedge^t \xi) = \mathrm{H}^*(\mathrm{GL}_n/Q', \bigwedge^t \xi)$ for every t.

Proof. The assertion follows from Lemma 5.2, noting that $\bigwedge^t \xi$ on GL_n/Q is the pullback of $\bigwedge^t \xi$ on GL_n/Q', under the natural morphism $\mathrm{GL}_n/Q \to \mathrm{GL}_n/Q'$. □

Proposition 5.4. *For all j, $\mathrm{H}^j(\mathrm{GL}_n/Q, \xi) = 0$.*

Proof. We want to show that $\mathrm{H}^j(\mathrm{GL}_n/Q, \mathcal{U}_i) = 0$ for every $r \le i \le n-1$ and for every j. By Lemma 5.2 (and keeping Discussion 2.3.3 in mind), it suffices to show that $\mathrm{H}^j(\mathrm{Gr}_{n-i,n}, \mathcal{U}_i) = 0$ for every $r \le i \le n-1$ and for every j. To this end, we apply the Bott's algorithm [Weyman 2003, (4.1.5)] to the weight

$$\alpha := (\underbrace{0, \dots, 0}_{n-i}, 1, \underbrace{0, \dots, 0}_{i-1}).$$

Note that there is a permutation σ such that $\sigma \cdot \alpha = \alpha$, yielding the proposition. □

An inductive approach. We are looking for a way to compute $\mathrm{H}^*(\mathrm{GL}_n/Q, \bigwedge^t \xi)$ for a homogeneous bundle

$$\xi = \bigoplus_{i \in A} \mathcal{U}_i^{\oplus m_i}$$

where $A \subseteq \{r, \dots, n-1\}$ and $m_i > 0$ for every $i \in A$. Using Proposition 5.3, we assume that $Q = P_{\hat{A}}$. (Using Proposition 5.8 below, we may further assume that $m_i \geq 2$, but this is not necessary for the inductive argument to work.)

Let j be such that $Q \subseteq P_{\hat{j}}$ and $\mathcal{Q}_j$ (equivalently $\mathcal{U}_j$) be of least dimension; in other words, j is the smallest element of A. If $Q = P_{\hat{j}}$ (i.e., $|A| = 1$), then the $\bigwedge^t \xi$ is completely reducible, and we may use the Borel–Weil–Bott theorem to compute the cohomology groups. Hence suppose that $Q \neq P_{\hat{j}}$; write $Q = Q' \cap P_{\hat{j}}$ nontrivially, with Q' being a parabolic subgroup. Consider the diagram

$$\begin{tikzcd} \mathrm{GL}_n/Q \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathrm{GL}_n/P_{\hat{j}} \\ \mathrm{GL}_n/Q' \end{tikzcd}$$

Note that $\bigwedge^t \xi$ decomposes as a direct sum of bundles of the form $(p_1)^*\eta \otimes (p_2)^*(\bigwedge^{t_1} \mathcal{U}_j^{\oplus m_j})$ where η is a homogeneous bundle on GL_n/Q'. We must compute

$$\mathrm{H}^*\big(\mathrm{GL}_n/Q,\ (p_1)^*\eta \otimes (p_2)^*(\textstyle\bigwedge^{t_1} \mathcal{U}_j^{\oplus m_j})\big).$$

Using the Leray spectral sequence and the projection formula, we can compute this from

$$\mathrm{H}^*\big(\mathrm{GL}_n/Q',\ \eta \otimes R^*(p_1)_*(p_2)^*(\textstyle\bigwedge^{t_1} \mathcal{U}_j^{\oplus m_j})\big).$$

Now $\bigwedge^{t_1} \mathcal{U}_j^{\oplus m_j}$, in turn, decomposes as a direct sum of $\mathrm{S}_\mu \mathcal{U}_j$, so we must compute

$$\mathrm{H}^*\big(\mathrm{GL}_n/Q',\ \eta \otimes R^*(p_1)_*(p_2)^* \,\mathrm{S}_\mu \mathcal{U}_j\big).$$

The Leray spectral sequence and the projection formula respect the various direct-sum decompositions mentioned above. It would follow from Proposition 5.5 below that for each μ, at most one of the $R^p(p_1)_*(p_2)^* \,\mathrm{S}_\mu \mathcal{U}_j$ is nonzero, so the abutment of the spectral sequence is, in fact, an equality.

Proposition 5.5. *With notation as above, let θ be a homogeneous bundle on $\mathrm{GL}_n/P_{\hat{j}}$. Then $R^i {p_1}_* {p_2}^*\theta$ is the locally free sheaf associated to the vector bundle $\mathrm{GL}_n \times^{Q'} \mathrm{H}^i(Q'/Q, {p_2}^*\theta|_{Q'/Q})$ over GL_n/Q'.*

Proof. This proposition follows from Proposition 2.3.5. □

We hence want to determine the cohomology of the restriction of $\mathrm{S}_\mu \mathcal{U}_j$ on Q'/Q. It follows from the definition of j that Q'/Q is a Grassmannian whose tautological quotient bundle and its dual are, respectively, $\mathcal{Q}_j|_{Q'/Q}$ and $\mathcal{U}_j|_{Q'/Q}$. We can therefore compute $\mathrm{H}^i(Q'/Q, \mathrm{S}_\mu \mathcal{U}_j|_{Q'/Q})$ using the Borel–Weil–Bott theorem.

Example 5.6. Suppose that $n = 6$ and that $Q = P_{\widehat{\{2,4\}}}$. Then we have the diagram

$$\begin{tikzcd} \mathrm{GL}_6 / Q \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathrm{GL}_6 / P_{\hat{2}} \\ \mathrm{GL}_6 / P_{\hat{4}} & \end{tikzcd}$$

The fibre of p_1 is isomorphic to $P_{\hat{4}}/Q$ which is a Grassmannian of two-dimensional subspaces of a four-dimensional vector-space. Let $\mu = (\mu_1, \mu_2)$ be a weight. Then we can compute the cohomology groups $\mathrm{H}^*(P_{\hat{4}}/Q,\ \mathrm{S}_\mu \mathcal{U}_2|_{P_{\hat{4}}/Q})$ applying the Borel–Weil–Bott theorem [Weyman 2003, (4.1.5)] to the sequence $(0, 0, \mu_1, \mu_2)$. Note that $\mathrm{H}^*(P_{\hat{4}}/Q,\ \mathrm{S}_\mu \mathcal{U}_2|_{P_{\hat{4}}/Q})$ is, if it is nonzero, $\mathrm{S}_\lambda W$ where W is a four-dimensional vector-space that is the fibre of the dual of the tautological quotient bundle of $\mathrm{GL}_4 / P_{\hat{4}}$ and λ is a partition with at most four parts. Hence, by Proposition 5.5, we see that $R^i(p_1)_*(p_2)^* \mathrm{S}_\mu \mathcal{U}_2$ is, if it is nonzero, $\mathrm{S}_\lambda \mathcal{U}_4$ on $\mathrm{GL}_6 / P_{\hat{4}}$.

We summarise the above discussion as a theorem:

Theorem 5.7. *For $w \in \mathcal{W}_r$ the modules in the free resolution of $\mathbb{C}[Y_P(w)]$ given in Theorem 4.2 can be computed.*

We end this section with some observations.

Proposition 5.8. *Suppose that there exists i such that $r+1 \le i \le n-1$ and such that ξ contains exactly one copy of $\mathcal{U}_i$ as a direct summand. Let*

$$\xi' = \mathcal{U}_{i-1} \oplus \bigoplus_{\substack{j=1 \\ i_j \neq i}}^{m} \mathcal{U}_{i_j}.$$

Then $\mathrm{H}^*(\mathrm{GL}_n / Q, \bigwedge^t \xi) = \mathrm{H}^*(\mathrm{GL}_n / Q, \bigwedge^t \xi')$ for every t.

Proof. Note that ξ' is a subbundle of ξ with quotient $\mathcal{U}_i/\mathcal{U}_{i-1}$. We claim that $\mathcal{U}_i/\mathcal{U}_{i-1} \simeq L_{\omega_{i-1}-\omega_i}$, where for $1 \le j \le n$, ω_j is the j-th fundamental weight. Assume the claim. Then we have an exact sequence

$$0 \longrightarrow \textstyle\bigwedge^t \xi' \longrightarrow \bigwedge^t \xi \longrightarrow \bigwedge^{t-1} \xi' \otimes L_{\omega_{i-1}-\omega_i} \longrightarrow 0$$

Let

$$Q' = \bigcap_{\substack{r \le l \le n-1 \\ l \neq i}} P_{\hat{l}};$$

then $Q = Q' \cap P_{\hat{\imath}}$. Let $p : \mathrm{GL}_n / Q \to \mathrm{GL}_n / Q'$ be the natural projection; its fibres are isomorphic to $Q'/Q \simeq \mathrm{GL}_2 / B_N \simeq \mathbb{P}^1$. Note that $\bigwedge^{t-1} \xi' \otimes L_{\omega_{i-1}}$ is the pullback along p of some vector bundle on GL_n / Q'; hence it is constant on the fibres of p.

On the other hand, L_{ω_i} is the ample line bundle on $\mathrm{GL}_n / P_{\hat{\imath}}$ that generates its Picard group, so $L_{-\omega_i}$ restricted to any fibre of p is $\mathbb{O}(-1)$. Hence the bundle $\bigwedge^{t-1}\xi' \otimes L_{\omega_{i-1}-\omega_i}$ on any fibre of p is a direct sum of copies of $\mathbb{O}(-1)$ and hence it has no cohomology. By Grauert's theorem [Hartshorne 1977, III.12.9], $R^i p_*(\bigwedge^{t-1}\xi' \otimes L_{\omega_{i-1}-\omega_i}) = 0$ for every i, so, using the Leray spectral sequence, we conclude that $\mathrm{H}^*(\mathrm{GL}_n/Q, \bigwedge^{t-1}\xi' \otimes L_{\omega_{i-1}-\omega_i}) = 0$. This gives the proposition.

Now to prove the claim, note that $\mathcal{U}_i/\mathcal{U}_{i-1} \simeq (\mathcal{R}_{n-i+1}/\mathcal{R}_{n-i})^*$. Let $e_1, \dots, e_n$ be a basis for $\mathbb{C}^n$ such that the subspace spanned by $e_i, \dots, e_n$ is B_N-stable for every $1 \le i \le n$. (Recall that we take the right action of B_N on $\mathbb{C}^n$.) Hence $\mathcal{R}_{n-i+1}/\mathcal{R}_{n-i}$ is the invertible sheaf on which B_N acts through the character $\omega_i - \omega_{i-1}$, which implies the claim. □

Remark 5.9 (determinantal case). Recall (see the paragraph after Definition 3.3) that $Y_P(w) = D_k$ if $w = (k+1, \dots, n, N-k+1, \dots N) \in \mathcal{W}_{n-k}$. In this case,

$$\mathcal{U}_w = \mathcal{U}_{n-k}^{\oplus(m-k+1)} \oplus \bigoplus_{i=n-k+1}^{n-1} \mathcal{U}_i.$$

Therefore

$$\mathrm{H}^*(\mathrm{GL}_n/Q_{n-k}, \textstyle\bigwedge^*\xi) = \mathrm{H}^*(\mathrm{GL}_n/Q_{n-k}, \bigwedge^*\mathcal{U}_{n-k}^{\oplus m}) = \mathrm{H}^*(\mathrm{GL}_n / P_{\widehat{n-k}}, \bigwedge^*\mathcal{U}_{n-k}^{\oplus m})$$

where the first equality comes from a repeated application of Proposition 5.8 and the second one follows by Lemma 5.2, applied to the natural map $f : \mathrm{GL}_n/Q \to \mathrm{GL}_n / P_{\widehat{n-k}}$. Hence our approach recovers Lascoux's resolution of the determinantal ideal [Lascoux 1978]; see also [Weyman 2003, Chapter 6].

6. Examples

We illustrate our approach with two examples. Firstly, we compute the resolution of a determinantal variety using the inductive method from the last section.

Example 6.1 ($n \times m$ matrices of rank $\le k$). If $k = 1$, then $w = (2, \dots, n, n+m)$, and, hence, $\xi = \mathcal{U}_{n-1}^{\oplus m}$. Since this would not illustrate the inductive argument, let us take $k = 2$.

Consider the ideal generated by the 3×3 minors of a 4×3 matrix of indeterminates. It is generated by four cubics, which have a linear relation. Hence minimal free resolution of the quotient ring looks like

$$0 \longrightarrow R(-4)^{\oplus 3} \longrightarrow R(-3)^{\oplus 4} \longrightarrow R \longrightarrow 0. \tag{6.2}$$

Note that $w = (3, 4, 6, 7)$ and $\xi = \mathcal{U}_2^{\oplus 2} \bigoplus \mathcal{U}_3$. Write $G = \mathrm{GL}_4$ and $Q = P_{\widehat{\{2,3\}}}$. Then $j = 2$, $Q' = P_{\hat{3}}$ and $Q'/Q \simeq \mathrm{GL}_3 / P_{\hat{2}} \simeq \mathbb{P}^2$. Now there is a decomposition

$$\textstyle\bigwedge^t\xi = \bigoplus_{|\mu|\le t} \mathrm{S}_{\mu'}\,\mathbb{C}^2 \otimes \mathrm{S}_\mu\,\mathcal{U}_2 \otimes \bigwedge^{t-|\mu|}\mathcal{U}_3$$

Hence we need to consider only $\mu = (\mu_1, \mu_2) \le (2, 2)$. On $Q'/Q \simeq \mathrm{GL}_3 / P_{\hat{2}}$, we would apply the Borel–Weil–Bott theorem [op. cit., (4.1.5)] to the weight $(0, \mu_1, \mu_2)$ to compute the cohomology of $\mathrm{S}_\mu \mathcal{U}_j$. Thus we see that we need to consider only $\mu = (0, 0)$, $\mu = (2, 0)$ and $\mu = (2, 1)$. From this, we conclude that

$$R^i (p_1)_*(p_2)^*(\mathrm{S}_{\mu'} \mathbb{C}^2 \otimes \mathrm{S}_\mu \mathcal{U}_2) = \begin{cases} \mathcal{O}_{G/P_{\hat{3}}} & \text{if } i = 0 \text{ and } \mu = (0, 0), \\ \bigwedge^2 \mathcal{U}_3 & \text{if } i = 1 \text{ and } \mu = (2, 0), \\ (\bigwedge^3 \mathcal{U}_3)^{\oplus 2} & \text{if } i = 1 \text{ and } \mu = (2, 1), \\ 0 & \text{otherwise.} \end{cases}$$

We have to compute the cohomology groups of $(R^i (p_1)_*(p_2)^*(\mathrm{S}_{\mu'} \mathbb{C}^2 \otimes \mathrm{S}_\mu \mathcal{U}_2)) \otimes \bigwedge^{t-|\mu|} \mathcal{U}_3$ on $G/P_{\hat{3}}$. Now, $\mathrm{H}^*(G/P_{\hat{3}}, \bigwedge^i \mathcal{U}_3) = 0$ for every $i > 0$. Further

$$\begin{aligned} \textstyle\bigwedge^2 \mathcal{U}_3 \otimes \mathcal{U}_3 &\simeq \textstyle\bigwedge^3 \mathcal{U}_3 \oplus \mathrm{S}_{2,1} \mathcal{U}_3 && \text{for } \mu = (2, 0) \text{ and } t = 3, \\ \textstyle\bigwedge^2 \mathcal{U}_3 \otimes \bigwedge^2 \mathcal{U}_3 &\simeq \mathrm{S}_{2,1,1} \mathcal{U}_3 \oplus \mathrm{S}_{2,2} \mathcal{U}_3 && \text{for } \mu = (2, 0) \text{ and } t = 4, \\ \textstyle\bigwedge^2 \mathcal{U}_3 \otimes \bigwedge^3 \mathcal{U}_3 &\simeq \mathrm{S}_{2,2,1} \mathcal{U}_3 && \text{for } (\mu = (2, 0) \text{ or } \mu = (2, 1)) \text{ and } t = 5, \\ \textstyle\bigwedge^3 \mathcal{U}_3 \otimes \mathcal{U}_3 &\simeq \mathrm{S}_{2,1,1} \mathcal{U}_3 && \text{for } \mu = (2, 1) \text{ and } t = 4, \\ \textstyle\bigwedge^3 \mathcal{U}_3 \otimes \bigwedge^3 \mathcal{U}_3 &\simeq \mathrm{S}_{2,2,2} \mathcal{U}_3 && \text{for } \mu = (2, 1) \text{ and } t = 6. \end{aligned}$$

Again, by applying the Borel–Weil–Bott theorem [loc. cit.] for $G/P_{\hat{3}}$, we see that $\mathrm{S}_{2,2} \mathcal{U}_3$, $\mathrm{S}_{2,2,1} \mathcal{U}_3$ and $\mathrm{S}_{2,2,2} \mathcal{U}_3$ have no cohomology. Therefore we conclude that

$$\mathrm{H}^j (G/Q, \textstyle\bigwedge^t \xi) = \begin{cases} \bigwedge^0 \mathbb{C}^{\oplus 4} & \text{if } t = 0 \text{ and } j = 0, \\ \bigwedge^3 \mathbb{C}^{\oplus 4} & \text{if } t = 3 \text{ and } j = 2, \\ (\bigwedge^4 \mathbb{C}^{\oplus 4})^{\oplus 3} & \text{if } t = 4 \text{ and } j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

These ranks agree with the expected ranks from (6.2).

Example 6.3. Let $n = 6$, $m = 6$, $k = 4$ and $w = (5, 6, 8, 9, 11, 12)$. For this, $Q = P_{\widehat{\{2,\dots,5\}}}$ and $\mathcal{U}_w = \mathcal{U}_2^{\oplus 2} \oplus \mathcal{U}_3 \oplus \mathcal{U}_4^{\oplus 2} \oplus \mathcal{U}_5$. After applying Propositions 5.3 and 5.8, we reduce to the situation $Q = P_{\widehat{\{2,4\}}}$ and $\xi = \mathcal{U}_2^{\oplus 3} \oplus \mathcal{U}_4^{\oplus 3}$. Write $\xi = (\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2) \oplus (\mathbb{C}^3 \oplus \mathcal{U}_4)$. Now we project away from $\mathrm{GL}_6 / P_{\hat{2}}$.

$$\begin{tikzcd} \mathrm{GL}_6 / Q \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathrm{GL}_6 / P_{\hat{2}} \\ \mathrm{GL}_6 / P_{\hat{4}} & \end{tikzcd}$$

The fibre of p_1 is isomorphic to $P_{\hat{4}}/Q$ which is a Grassmannian of two-dimensional subspaces of a four-dimensional vector-space. We use the spectral sequence

$$\mathrm{H}^j (G/P_{\hat{4}}, R^i p_{1*} \textstyle\bigwedge^t \xi) \implies \mathrm{H}^{i+j} (G/Q, \textstyle\bigwedge^t \xi). \tag{6.4}$$

Observe that $\bigwedge^t \xi = \bigoplus_{t_1} \bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2) \otimes \bigwedge^{t-t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_4)$; the above spectral sequence respects this decomposition. Further, using the projection formula, we see that we need to compute

$$\mathrm{H}^j\big(G/P_{\hat{4}},\ (R^i p_{1*}\textstyle\bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2)) \otimes \bigwedge^{t-t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_4)\big).$$

Now, $R^i p_{1*}\bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2)$ is the vector bundle associated to the $P_{\hat{4}}$-module

$$\mathrm{H}^i(P_{\hat{4}}/Q,\ \textstyle\bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2)|_{P_{\hat{4}}/Q}) = \mathrm{H}^i(P_{\hat{4}}/Q,\ \bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2|_{P_{\hat{4}}/Q})).$$

Note that $\mathcal{U}_2|_{P_{\hat{4}}/Q}$ is the dual of the tautological quotient bundle of $P_{\hat{4}}/Q \simeq \mathrm{GL}_4/P_{\hat{2}}$; we denote this also, by abuse of notation, by $\mathcal{U}_2$. Note, further, that $\bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2) = \bigoplus_{\mu \vdash t_1} \mathrm{S}_{\mu'}\,\mathbb{C}^3 \otimes \mathrm{S}_\mu\,\mathcal{U}_2$. We need only consider $\mu \leq (3,3)$. From the Borel–Weil–Bott theorem [Weyman 2003, (4.1.5)], it follows that

$$\mathrm{H}^i(P_{\hat{4}}/Q,\ \mathrm{S}_\mu\,\mathcal{U}_2) = \begin{cases} \bigwedge^0(\mathbb{C}^{\oplus 4}) & \text{if } i=0 \text{ and } \mu=(0,0),\\ \bigwedge^3(\mathbb{C}^{\oplus 4}) & \text{if } i=2 \text{ and } \mu=(3,0),\\ \bigwedge^4(\mathbb{C}^{\oplus 4}) & \text{if } i=2 \text{ and } \mu=(3,1),\\ 0, & \text{otherwise.} \end{cases}$$

Therefore we conclude that

$$R^i p_{1*}\textstyle\bigwedge^{t_1}(\mathbb{C}^3 \otimes_{\mathbb{C}} \mathcal{U}_2) = \begin{cases} \mathbb{O}_{\mathrm{GL}_4/P_{\hat{2}}} & \text{if } i=0 \text{ and } t_1=0,\\ \bigwedge^3\mathcal{U}_4 & \text{if } i=2 \text{ and } t_1=3,\\ (\bigwedge^4\mathcal{U}_4)^{\oplus 3} & \text{if } i=2 \text{ and } t_1=4,\\ 0 & \text{otherwise.} \end{cases}$$

Therefore for each pair (t, t_1) at most one column of the summand of the spectral sequence (6.4) is nonzero; hence the abutment in (6.4) is in fact an equality.

Fix a pair (t, t_1) and an integer l. Then we have

$$\begin{aligned}\mathrm{H}^l(G/Q,\ \textstyle\bigwedge^t\xi) = \mathrm{H}^l(G/P_{\hat{4}},\ \bigwedge^t(\mathbb{C}^3 \otimes \mathcal{U}_4))&\\ \oplus \mathrm{H}^{l-2}(G/P_{\hat{4}},\ \textstyle\bigwedge^3\mathcal{U}_4 \otimes \bigwedge^{t-3}(\mathbb{C}^3 \otimes \mathcal{U}_4))&\\ \oplus \mathrm{H}^{l-2}(G/P_{\hat{4}},\ (\textstyle\bigwedge^4\mathcal{U}_4)^{\oplus 3} \otimes \bigwedge^{t-4}(\mathbb{C}^3 \otimes \mathcal{U}_4)).&\end{aligned}$$

Write $h^i(-) = \dim_{\mathbb{C}} \mathrm{H}^i(-)$. Note that $\bigwedge^t(\mathbb{C}^3 \otimes \mathcal{U}_4) \simeq \bigoplus_{\lambda \vdash t} \mathrm{S}_{\lambda'}\,\mathbb{C}^3 \otimes \mathrm{S}_\lambda\,\mathcal{U}_4$, by the Cauchy formula. Write $d_{\mu'} = \dim_{\mathbb{C}} \mathrm{S}_{\mu'}\,\mathbb{C}^{\oplus 3}$. Thus, from the above equation, we see, that for every l and for every t,

$$\begin{aligned}(6.5)\quad h^l(\textstyle\bigwedge^t\xi) = \displaystyle\sum_{\mu \vdash t} d_{\mu'} h^l(\mathrm{S}_\mu\,\mathcal{U}_4)&\\ + \sum_{\mu \vdash t-3} d_{\mu'} h^{l-2}(\textstyle\bigwedge^3\mathcal{U}_4 \otimes \mathrm{S}_\mu\,\mathcal{U}_4) + 3\displaystyle\sum_{\mu \vdash t-4} d_{\mu'} h^{l-2}(\textstyle\bigwedge^4\mathcal{U}_4 \otimes \mathrm{S}_\mu\,\mathcal{U}_4)&\end{aligned}$$

(Here the cohomology is calculated over GL_6/Q on the left-hand-side and over $\mathrm{GL}_6/P_{\hat{4}}$ on the right-hand-side.) For any μ, if $d_{\mu'} \neq 0$, then $\mu_1 \leq 3$. Any μ that contributes a nonzero integer to the right-hand-side of (6.5) has at most four parts and $m_1 \leq 3$. Further, if $\mathrm{S}_\lambda \mathcal{U}_4$ is an irreducible summand of a representation on the right-hand-side of (6.5) with nonzero cohomology, then λ has at most four parts and is such that $\lambda_1 \leq 4$. Therefore for $\lambda \leq (4, 4, 4, 4)$, we compute the cohomology using the Borel–Weil–Borel theorem:

$$\mathrm{H}^i(G/P_{\hat{4}}, \mathrm{S}_\lambda \mathcal{U}_4) = \begin{cases} \bigwedge^0(\mathbb{C}^{\oplus 6}) & \text{if } i = 0 \text{ and } \lambda = 0, \\ \mathrm{S}_{(\lambda_1-2,1,1,\lambda_2,\lambda_3,\lambda_4)}(\mathbb{C}^{\oplus 6}) & \text{if } i = 2,\ \lambda_1 \in \{3, 4\}, \\ & \quad \text{and } (\lambda_2, \lambda_3, \lambda_4) \leq (1, 1, 1), \\ \mathrm{S}_{(2,2,2,2,\lambda_3,\lambda_4)}(\mathbb{C}^{\oplus 6}) & \text{if } i = 4,\ \lambda_1 = \lambda_2 = 4, \\ & \quad \text{and } (\lambda_3, \lambda_4) \leq (2, 2), \\ 0 & \text{otherwise.} \end{cases}$$

We put these together to compute $h^l(\bigwedge^t \xi)$; the result is listed in Table 1. From this we get the following resolution:

$$0 \to R(-12)^{26} \to R(-11)^{108} \to \begin{matrix} R(-6)^{10} \\ \oplus \\ R(-10)^{153} \end{matrix} \to \begin{matrix} R(-5)^{36} \\ \oplus \\ R(-7)^{36} \\ \oplus \\ R(-9)^{70} \end{matrix} \to \begin{matrix} R(-3)^{45} \\ \oplus \\ R(-5)^{53} \end{matrix} \to \begin{matrix} R(-2)^{20} \\ \oplus \\ R(-4)^{18} \end{matrix} \to R \to 0.$$

Note, indeed, that $\dim Y_Q(w) = \dim X_Q(w) = 4+4+5+5+6+6 = 30$ and that $\dim O^-_{\mathrm{GL}_N/P} = 6 \cdot 6 = 36$, so the codimension is 6. Since the variety is Cohen–Macaulay, the length of a minimal free resolution is 6.

7. Further remarks

A realisation of Lascoux's resolution for determinantal varieties. We already saw in Remark 5.9 that when $Y_P(w) = D_k$, computing $\mathrm{H}^*(\mathrm{GL}_n/Q_{n-k}, \bigwedge^* \xi)$ is reduced, by a repeated application of Proposition 5.8 to computing the cohomology groups of (completely reducible) vector bundles on the Grassmannian $\mathrm{GL}_n/P_{\widehat{n-k}}$. We thus realise Lascoux's resolution of the determinantal variety using our approach.

In this section, we give yet another desingularization of D_k (for a suitable choice of the parabolic subgroup) so that the variety V of (1-2) is in fact a Grassmannian. Recall (the paragraph after Definition 3.3 or Remark 5.9) that $Y_P(w) = D_k$ if $w = (k+1, \ldots, n, N-k+1, \ldots N) \in \mathcal{W}_{n-k}$. Let $\tilde{P} = P_{\widehat{\{n-k,n\}}} \subseteq \mathrm{GL}_N$. Let $\tilde{w}$ be the representative of the coset $w\tilde{P}$ in $W^{\tilde{P}}$.

Proposition 7.1. *$X_{\tilde{P}}(\tilde{w})$ is smooth and the natural map $X_{\tilde{P}}(\tilde{w}) \to X_P(w)$ is proper and birational, i.e., $X_{\tilde{P}}(\tilde{w})$ is a desingularization of $X_P(w)$.*

t	$h^0(\bigwedge^t \xi)$	$h^1(\bigwedge^t \xi)$	$h^2(\bigwedge^t \xi)$	$h^3(\bigwedge^t \xi)$	$h^4(\bigwedge^t \xi)$	$h^5(\bigwedge^t \xi)$	$h^6(\bigwedge^t \xi)$
0	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	20	0	0	0	0
4	0	0	45	0	0	0	0
5	0	0	36	0	18	0	0
6	0	0	10	0	53	0	0
7	0	0	0	0	36	0	0
8	0	0	0	0	0	0	0
9	0	0	0	0	0	0	70
10	0	0	0	0	0	0	153
11	0	0	0	0	0	0	90
12	0	0	0	0	0	0	26

Table 1. Ranks of the relevant cohomology groups.

Proof. The proof is similar to that of Proposition 3.4. Let

$$w_{\max} = (k+1, \ldots, n, N-k+1, \ldots N, N-k, \ldots, n+1, k, \ldots, 1) \in W.$$

Then $X_{B_N}(w_{\max})$ is the inverse image of $X_{\tilde{P}}(\tilde{w})$ under the natural morphism $\mathrm{GL}_N/B_N \to \mathrm{GL}_N/\tilde{P}$, and that $w_{\max}$ is a 4231 and 3412-avoiding element of $W = S_N$. $\square$

We have $P/\tilde{P} \cong \mathrm{GL}_n/P_{\widehat{n-k}}$. As in Section 3, we have the following. Denoting by Z the preimage inside $X_{\tilde{P}}(\tilde{w})$ of $Y_P(w)$ (under the restriction to $X_{\tilde{P}}(\tilde{w})$ of the natural projection $G/\tilde{P} \to G/P$), we have $Z \subset O^- \times P/\tilde{P}$, and the image of Z under the second projection is $V := P/\tilde{P}$ ($\cong \mathrm{GL}_n/P_{\widehat{n-k}}$). The inclusion $Z \hookrightarrow O^- \times V$ is a subbundle (over V) of the trivial bundle $O^- \times V$. Denoting by ξ the dual of the quotient bundle on V corresponding to Z, we have that the homogeneous bundles $\bigwedge^{i+j} \xi$ on $\mathrm{GL}_n/P_{\widehat{n-k}}$ are completely reducible, and hence may be computed using Bott's algorithm.

Multiplicity. We describe how the free resolution obtained in Theorem 4.2 can be used to get an expression for the multiplicity $\mathrm{mult}_{\mathrm{id}}(w)$ of the local ring of the Schubert variety $X_P(w) \subseteq \mathrm{GL}_N/P$ at the point e_{id}. Notice that $Y_P(w)$ is an affine neighbourhood of e_{id}. We noticed in Section 4 that $Y_P(w)$ is a closed subvariety of $O^-_{\mathrm{GL}_N/P}$ defined by homogeneous equations. In $O^-_{\mathrm{GL}_N/P}$, e_{id} is the origin; hence in $Y_P(w)$ it is defined by the unique homogeneous maximal ideal of $\mathbb{C}[Y_P(w)]$. Therefore $\mathbb{C}[Y_P(w)]$ is the associated graded ring of the local ring of $\mathbb{C}[Y_P(w)]$ at

e_{id} (which is also the local ring of $X_P(w)$ at e_{id}). Hence $\mathrm{mult}_{\mathrm{id}}(w)$ is the normalised leading coefficient of the Hilbert series of $\mathbb{C}[Y_P(w)]$.

Observe that the Hilbert series of $\mathbb{C}[Y_P(w)]$ can be obtained as an alternating sum of the Hilbert series of the modules F_i in Theorem 4.2. Write $h^j(-) = \dim_{\mathbb{C}} \mathrm{H}^j(X_{Q_s}(w'), -)$ for coherent sheaves on $X_{Q_s}(w')$. Then the Hilbert series of $\mathbb{C}[Y_P(w)]$ is

$$\frac{1}{(1-t)^{mn}} \sum_{i=0}^{mn} \sum_{j=0}^{\dim X_{Q_s}(w')} (-1)^i h^j\big(\textstyle\bigwedge^{i+j} \mathcal{U}_w\big) t^{i+j}. \tag{7.2}$$

We may harmlessly change the range of summation in (7.2) to $-\infty < i, j < \infty$; this is immediate for j, while for i, we note that the proof of Theorem 4.1 implies that $h^j(\bigwedge^{i+j}\mathcal{U}_w) = 0$ for every $i < 0$ and for every j. Hence we may write the summation in (7.2) as (with $k = i + j$)

$$\sum_{k=0}^{\infty} (-1)^k t^k \sum_{j=0}^{\infty} (-1)^j h^j\big(\textstyle\bigwedge^k \mathcal{U}_w\big) = \sum_{k=0}^{\mathrm{rk}\,\mathcal{U}_w} (-1)^k \chi\big(\textstyle\bigwedge^k \mathcal{U}_w\big) t^k. \tag{7.3}$$

Since $\bigwedge^k \mathcal{U}_w$ is also a T_n-module, where T_n is the subgroup of diagonal matrices in GL_n, one may decompose $\bigwedge^k \mathcal{U}_w$ as a sum of rank-one T_n-modules and use the Demazure character formula to compute the Euler characteristics above.

It follows from generalities on Hilbert series (see, e.g., [Bruns and Herzog 1993, Section 4.1]) that the polynomial in (7.3) is divisible by $(1-t)^c$ where c is the codimension of $Y_P(w)$ in $O^-_{\mathrm{GL}_N/P}$, and that after we divide it and substitute $t = 1$ in the quotient, we get $\mathrm{mult}_{\mathrm{id}}(w)$. This gives an expression for $e_{\mathrm{id}}(w)$ apart from those of [Lakshmibai and Weyman 1990; Kreiman and Lakshmibai 2004].

Castelnuovo–Mumford regularity. Since $\mathbb{C}[Y_P(w)]$ is a graded quotient ring of $\mathbb{C}[O^-_{\mathrm{GL}_N/P}]$, it defines a coherent sheaf over the corresponding projective space $\mathbb{P}^{mn-1}$.

Let F be a coherent sheaf on $\mathbb{P}^n$. The *Castelnuovo–Mumford regularity* of F (with respect to $\mathcal{O}_{\mathbb{P}^n}(1)$) is the smallest integer r such that $\mathrm{H}^i(\mathbb{P}^n, F \otimes \mathcal{O}_{\mathbb{P}^n}(r-i)) = 0$ for every $1 \le i \le n$; we denote it by $\mathrm{reg}\, F$. Similarly, if $R = \Bbbk[x_0, \dots, x_n]$ is a polynomial ring over a field $\Bbbk$ with $\deg x_i = 1$ for every i and M is a finitely generated graded R-module, the *Castelnuovo–Mumford regularity* of M is the smallest integer r such that $(\mathrm{H}^i_{(x_0,\dots,x_n)}(M))_{r+1-i} = 0$ for every $0 \le i \le n+1$; we denote it by $\mathrm{reg}\, M$. (Here $\mathrm{H}^i_{(x_0,\dots,x_n)}(M)$ is the i-th *local cohomology* module of M, and is a graded R-module.) It is known that

$$\mathrm{reg}\, F = \mathrm{reg}\Big(\bigoplus_{i \in \mathbb{Z}} \mathrm{H}^0(\mathbb{P}^n, F \otimes \mathcal{O}_{\mathbb{P}^n}(i))\Big)$$

for every coherent sheaf F and that if depth $M \geq 2$, then $\operatorname{reg} M = \operatorname{reg} \widetilde{M}$. See [Eisenbud 2005, Chapter 4] for details.

Proposition 7.4. *In the notation of* (1-1), $\operatorname{reg} \mathbb{C}[Y] = \max\{j : \mathrm{H}^j(V, \bigwedge^* \xi) \neq 0\}$.

Proof. Let $R = \mathbb{C}[\mathbb{A}]$. It is known that

$$\operatorname{reg} M = \max\{j : \operatorname{Tor}_i^R(\Bbbk, M)_{i+j} \neq 0 \text{ for some } i\};$$

see [loc. cit.] for a proof. The proposition now follows from noting that

$$\operatorname{Tor}_i^R(\mathbb{C}, \mathbb{C}[Y])_{i+j} \simeq \mathrm{H}^j(V, \textstyle\bigwedge^{i+j} \xi)$$

by Theorem 4.2. □

Now let $w = (n-r+1, n-r+2, \ldots, n, a_{r+1}, \ldots, a_{n-1}, N) \in \mathcal{W}_r$. We would like to determine $\operatorname{reg} \mathbb{C}[Y_P(w)] = \max\{j : \mathrm{H}^j(\mathrm{GL}_n/Q_r, \bigwedge^* \mathcal{U}_w) \neq 0\}$. Let $a_r = n$ and $a_n = N$. For $r \leq i \leq n-1$, define $m_i = a_{i+1} - a_i$. Note that $\mathcal{U}_i$ appears in $\mathcal{U}_w$ with multiplicity m_i and that $m_i > 0$. Based on the examples that we have calculated, we have the following conjecture.

Conjecture 7.5. With notation as above,

$$\operatorname{reg} \mathbb{C}[Y_P(w)] = \sum_{i=r}^{n-1} (m_i - 1)i.$$

(Note that since $Y_P(w)$ is Cohen–Macaulay, $\operatorname{reg} \mathbb{C}[Y_P(w)] = \operatorname{reg} \mathcal{O}_{Y_P(w)}$.) Consider the examples in Section 6. In Example 6.1, $m_2 = 2$, $m_3 = 1$, and $\operatorname{reg} \mathbb{C}[Y_P(w)] = (2-1)2+0 = 2$. In Example 6.3, $m_2 = m_4 = 2$ and $m_3 = m_5 = 1$, so $\operatorname{reg} \mathbb{C}[Y_P(w)] = (2-1)2+0+(2-1)4+0 = 6$, which in deed is the case, as we see from Table 1.

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FREE RESOLUTIONS OF SOME SCHUBERT SINGULARITIES IN THE LAGRANGIAN GRASSMANNIAN

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Dedicated to the memory of Professor Robert Steinberg

In this paper we construct free resolutions of a certain class of closed subvarieties of affine space of symmetric matrices (of a given size). Our class covers the symmetric determinantal varieties (i.e., determinantal varieties in the space of symmetric matrices), whose resolutions were first constructed by Józefiak, Pragacz and Weyman (1981). Our approach follows the techniques developed by Kummini, Lakshmibai, Pramathanath and Seshadri (2015), and uses the geometry of Schubert varieties.

1. Introduction

This paper is a sequel to [Kummini et al. 2015]. Lascoux [1978] constructed a minimal free resolution of the coordinate ring of the determinantal varieties (consisting of $m \times n$ matrices (over $\mathbb{C}$) of rank at most k, considered as a closed subvariety of the mn-dimensional affine space of all $m \times n$ matrices), as a module over the mn-dimensional polynomial ring (the coordinate ring of the mn-dimensional affine space).

In [Kummini et al. 2015], the authors construct free resolutions for a larger class of singularities, *viz.*, Schubert singularities, i.e., the intersection of a singular Schubert variety and the "opposite big cell" inside a Grassmannian.

Józefiak, Pragacz and Weyman [1981] constructed a minimal free resolution of the coordinate ring of the determinantal varieties (in the space of symmetric matrices) as a module over the coordinate ring of the space of symmetric matrices. In this paper we construct free resolutions for a certain class of closed subvarieties of the affine space of symmetric matrices, which includes the symmetric determinantal varieties. The technique adopted in [Kummini et al. 2015] is algebraic group-theoretic, and we follow this approach.

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We now describe the results of this paper. Let n be a positive integer. Let $V = \mathbb{C}^{2n}$ and let $(\cdot\,,\cdot)$ be a nondegenerate skew-symmetric bilinear form on V. Let $H = \mathrm{SL}(V)$ and $G = \mathrm{SP}(V) = \{Z \in \mathrm{SL}(V) \mid Z \text{ leaves the form } (\cdot\,,\cdot) \text{ invariant}\}$. We take the matrix of the form, with respect to the standard basis of V, to be

$$F = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$$

where J is the antidiagonal $(1, \dots, 1)$, in this case of size n. To simplify our notation we will normally omit specifying the size of J as it will be obvious from the context. We may realize $\mathrm{SP}(V)$ as the fixed point set of the involution $\sigma : H \to H$ given by $\sigma(Z) = F(Z^T)^{-1}F^{-1}$ (cf. [Steinberg 1968]).

Denoting by T_H and B_H the maximal torus in H consisting of diagonal matrices and the Borel subgroup in H consisting of upper triangular matrices, respectively, we have that T_H and B_H are stable under σ and we set $T_G = T_H^\sigma$, $B_G = B_H^\sigma$. It is easily checked that T_G is a maximal torus in G and B_G is a Borel subgroup in G.

Thus we obtain

$$W_G \hookrightarrow W_H$$

where W_G, W_H denote the Weyl groups of G, H respectively (with respect to T_G, T_H respectively). Further, σ induces an involution on W_H:

$$w = (a_1, \cdots, a_{2n}) \in W_H, \quad \sigma(w) = (c_1, \cdots, c_{2n}), \quad c_i = 2n + 1 - a_{2n+1-i}$$

and

$$W_G = W_H^\sigma.$$

Thus we obtain

$$W_G = \{(a_1 \cdots a_{2n}) \in S_{2n} \mid a_i = 2n + 1 - a_{2n+1-i},\ 1 \le i \le 2n\}.$$

(here, S_{2n} is the symmetric group on $2n$ letters). Thus $w = (a_1 \cdots a_{2n}) \in W_G$ is known once $(a_1 \cdots a_n)$ is known. We shall denote an element $(a_1 \cdots a_{2n})$ in W_G by just $(a_1 \cdots a_n)$. Further, for $w \in W_G$, denoting by $X_G(w)$ (resp. $X_H(w)$), the associated Schubert variety in G/B_G (resp. H/B_H), we have that under the canonical inclusion $G/B_G \hookrightarrow H/B_H$, $X_G(w) = X_H(w) \cap G/B_G$, scheme-theoretically.

Let $P = P_{\hat{n}}$, the maximal parabolic subgroup of G corresponding to omitting the simple root α_n, the set of simple roots of G being indexed as in [Bourbaki 1968]. Let $1 \le k < r \le n$ be positive integers, and let $w \in \mathcal{W}_{k,r}$ (cf. Notation 3.2). Our main result (cf. Theorem 3.22) is a description of the minimal free resolution of the coordinate ring of $Y_P(w) := X_P(w) \cap O^-_{G/P}$, the *opposite cell* of $X_P(w)$, as a module over the coordinate ring of $O^-_{G/P}$. For this, as in [Kummini et al. 2015], we use the Kempf–Lascoux–Weyman "geometric technique" of constructing minimal free resolutions; in fact we use the same notation and description of this technique as in [Kummini et al. 2015].

Suppose that we have a commutative diagram of varieties

$$
\begin{tikzcd}
Z \arrow[r, hook] \arrow[d, "q'"] & \mathbb{A} \times V \arrow[r] \arrow[d, "q"] & V \\
Y \arrow[r, hook] & \mathbb{A} &
\end{tikzcd}
\tag{1.1}
$$

where $\mathbb{A}$ is an affine space, Y a closed subvariety of $\mathbb{A}$ and V a projective variety. The map q is first projection, q' is proper and birational, and the inclusion $Z \hookrightarrow \mathbb{A} \times V$ is a subbundle (over V) of the trivial bundle $\mathbb{A} \times V$. Let ξ be the dual of the quotient bundle on V corresponding to Z. Then the derived direct image $\mathbf{R}q'_*\mathcal{O}_Z$ is quasi-isomorphic to a minimal complex $F_\bullet$ with

$$F_i = \bigoplus_{j \geq 0} H^j\big(V, \textstyle\bigwedge^{i+j}\xi\big) \otimes_{\mathbb{C}} R(-i-j).$$

Here R is the coordinate ring of $\mathbb{A}$; it is a polynomial ring and $R(k)$ refers to twisting with respect to its natural grading. If q' is such that the natural map $\mathcal{O}_Y \longrightarrow \mathbf{R}q'_*\mathcal{O}_Z$ is a quasi-isomorphism (for example, if q' is a desingularization of Y and Y has rational singularities) then $F_\bullet$ is a minimal free resolution of $\mathbb{C}[Y]$ over the polynomial ring R.

In applying this technique in any given situation, there are two main steps involved: one must find a suitable Z and a suitable morphism $q' : Z \longrightarrow Y$ such that the map $\mathcal{O}_Y \longrightarrow \mathbf{R}q'_*\mathcal{O}_Z$ is a quasi-isomorphism and such that Z is a vector bundle over a projective variety V; and, one must be able to compute the necessary cohomology groups. We carry this out for opposite cells $Y_P(w)$, $w \in \mathcal{W}_{k,r}$.

As the first step, we establish the existence of a diagram as above, using the geometry of Schubert varieties. We now describe this briefly.

We take $\mathbb{A} = O^-_{G/P}$ and $Y = Y_P(w)$. Let $\widetilde{P}$ be the two-step parabolic subgroup $P_{\widehat{r-k},\hat{n}}$ of G, and let $\tilde{w}$ be the minimal representative of $w\widetilde{P}$ in $W^{\widetilde{P}}$ (that is, the set of minimal coset representatives in W, under the Bruhat order, of $W/W_{\widetilde{P}}$, where $W_{\widetilde{P}}$ is the Weyl group of $\widetilde{P}$). Let $w' := (k+1, \dots, r, n, \dots, r+1, k, \dots, 1) \in S_n$, the Weyl group of GL_n. Let $Z_{\widetilde{P}}(\tilde{w}) := Y_P(w) \times_{X_P(w)} X_{\widetilde{P}}(w)$ $\big(= (O^-_{G/P} \times P/\widetilde{P}) \cap X_{\widetilde{P}}(w)\big)$. Then it turns out that $Z_{\widetilde{P}}(\tilde{w})$ is smooth (cf. Definition 3.20), and is a desingularization of $Y_P(w)$. Write p for the composite map $Z_{\widetilde{P}}(\tilde{w}) \hookrightarrow O^-_{G/P} \times P/\widetilde{P} \to P/\widetilde{P}$ where the first map is the inclusion and the second map is the projection. We have (cf. Theorem 3.22) that p identifies $Z_{\widetilde{P}}(\tilde{w})$ as a subbundle of the trivial bundle $O^-_{G/P} \times X_{P'_{\widehat{r-k}}}(w')$ over $X_{P'_{\widehat{r-k}}}(w')$, which arises as the restriction (to $X_{P'_{\widehat{r-k}}}(w')$) of a certain homogeneous vector bundle on $\mathrm{GL}_n / P'_{\widehat{r-k}}$. With $V := X_{P'_{\widehat{r-k}}}(w')$, we get:

$$
\begin{tikzcd}
Z_{\widetilde{P}}(\tilde{w}) \arrow[r, hook] \arrow[d, "q'"] & O^-_{G/P} \times V \arrow[r] \arrow[d, "q"] & V \\
Y_P(w) \arrow[r, hook] & O^-_{G/P} &
\end{tikzcd}
\tag{1.2}
$$

In this diagram, q' is a desingularization of $Y_P(w)$. Since it is known that Schubert varieties have rational singularities, we have that the map $\mathbb{O}_Y \longrightarrow \mathbf{R}q'_*\mathbb{O}_Z$ is a quasi-isomorphism, so $F_\bullet$ is a minimal resolution.

At the second step, we need to determine the cohomology of the bundles $\bigwedge^t \xi$ over V. In the above situation, $V = X_{P'_{\widehat{r-k}}}(w') \hookrightarrow \mathrm{GL}_n / P'_{\widehat{r-k}}$. As can be easily seen, $X_{P'_{\widehat{r-k}}}(w')$ is a Grassmannian, namely, $\mathrm{GL}_r / P''_{\widehat{r-k}}$; the bundles $\bigwedge^t \xi$ (on $\mathrm{GL}_r / P''_{\widehat{r-k}}$) are also homogeneous, but are not of Bott type: they are not completely reducible (so one can not apply the Bott algorithm for computing the cohomology). This can be resolved in two ways. In [Ottaviani and Rubei 2006] the authors determine the cohomology of general homogeneous bundles on Hermitian symmetric spaces, and thus their results can be used to determine $H^\bullet(V, \bigwedge^t \xi)$. Alternatively, using a technique from [Weyman 2003], we may compute the resolution of a related space (whose associated homogeneous vector bundle is of Bott type) from which we retrieve the resolution of the coordinate ring of $Y_P(w)$ as a subcomplex.

We hope to extend the results of this paper to Schubert varieties in the orthogonal Grassmannian. Details will appear in a subsequent paper.

The paper is organized as follows. Section 2 contains notations and conventions and the necessary background material on Schubert varieties in the flag variety (Section 2.1) and Schubert varieties in the symplectic flag variety (Sections 2.2 and 2.3) and homogeneous bundles (Section 2.4). In Section 3, we discuss properties of Schubert desingularization, including the construction of Diagram 1.2. Section 4 is devoted to a review of the Kempf–Lascoux–Weyman technique and completes step one of the two part process of the geometric technique. Section 5 explains how the cohomology groups of the homogeneous bundles constructed in step one may be calculated.

2. Preliminaries

In this section we collect various results about Schubert varieties in the flag variety and symplectic flag variety, homogeneous vector bundles, and the Bott algorithm.

2.1. ***Notation and conventions in type A.*** We collect the symbols used and the conventions adopted in the rest of the paper here. For details on algebraic groups and Schubert varieties, the reader may refer to [Borel 1991; Jantzen 2003; Billey and Lakshmibai 2000; Seshadri 2007].

Let N be positive integer. We denote by GL_N (respectively, B_N, B_N^-) the group of all (respectively, upper triangular, lower triangular) invertible $N \times N$ matrices over $\mathbb{C}$. The Weyl group W of GL_N is isomorphic to the group S_N of permutations of N symbols and is generated by the *simple reflections* s_i, which correspond to the transpositions $(i, i+1)$, for $1 \le i \le N-1$. For $w \in W$, its *length* is the smallest integer l such that $w = s_{i_1} \cdots s_{i_l}$ as a product of simple reflections. For every

$1 \le i \le N-1$, there is a *minimal parabolic subgroup* P_i containing s_i (thought of as an element of GL_N) and a *maximal parabolic subgroup* $P_{\hat{i}}$ *not containing* s_i. Any parabolic subgroup can be written as $P_{\widehat{A}} := \bigcap_{i \in A} P_{\hat{i}}$ for some $A \subset \{1, \ldots, N-1\}$. On the other hand, for $A \subseteq \{1, \ldots, N-1\}$ write P_A for the subgroup of GL_N generated by P_i for $i \in A$. Then P_A is a parabolic subgroup and $P_{\{1,\ldots,N-1\}\setminus A} = P_{\widehat{A}}$.

We write the elements of W in *one-line* notation: $(a_1, \ldots, a_N)$ is the permutation $i \mapsto a_i$. For any $A \subseteq \{1, \ldots, N-1\}$, define W_{P_A} to be the subgroup of W generated by $\{s_i : i \in A\}$. By W^{P_A} we mean the subset of W consisting of the minimal representatives (under the Bruhat order) in W of the elements of W/W_{P_A}. For $1 \le i \le N$, we represent the elements of $W^{P_{\hat{i}}}$ by sequences $(a_1, \ldots, a_i)$ with $1 \le a_1 < \cdots < a_i \le N$ since under the action of the group $W_{P_{\hat{i}}}$, every element of W can be represented minimally by such a sequence.

We identify $\mathrm{GL}_N = \mathrm{GL}(V)$ for some N-dimensional vector-space V. Let $A := \{i_1 < i_2 < \cdots < i_r\} \subseteq \{1, \ldots, N-1\}$. Then $\mathrm{GL}_N / P_{\widehat{A}}$ is the set of all flags $0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_r \subsetneq V$ of subspaces V_j of dimension i_j inside V. We call $\mathrm{GL}_N / P_{\widehat{A}}$ a *flag variety*. If $A = \{1, \ldots, N-1\}$ (i.e., $P_{\widehat{A}} = B_N$), then we call the flag variety a *full flag variety*; otherwise, a *partial flag variety*. The *Grassmannian* $\mathrm{Grass}_{i,N}$ of i-dimensional subspaces of V is $\mathrm{GL}_N / P_{\hat{i}}$.

Let $\widetilde{P}$ be any parabolic subgroup containing B_N and $\tau \in W$. The *Schubert variety* $X_{\widetilde{P}}(\tau)$ is the closure inside $\mathrm{GL}_N / \widetilde{P}$ of $B_N \cdot e_w$ where e_w is the coset $\tau \widetilde{P}$, endowed with the canonical reduced scheme structure. Hereafter, when we write $X_{\widetilde{P}}(\tau)$, we mean that τ is the representative in $W^{\widetilde{P}}$ of its coset. The *opposite big cell* $O^-_{\mathrm{GL}_N/\widetilde{P}}$ in $\mathrm{GL}_N / \widetilde{P}$ is the B_N^--orbit of the coset $(\mathrm{id} \cdot \widetilde{P})$ in $\mathrm{GL}_N / \widetilde{P}$. Let $Y_{\widetilde{P}}(\tau) := X_{\widetilde{P}}(\tau) \cap O^-_{\mathrm{GL}_N/\widetilde{P}}$; we refer to $Y_{\widetilde{P}}(\tau)$ as the *opposite cell* of $X_{\widetilde{P}}(\tau)$.

We will write R^+, R^-, $R^+_{\widetilde{P}}$, $R^-_{\widetilde{P}}$, to denote respectively, positive and negative roots for GL_N and for $\widetilde{P}$. We denote by ϵ_i the character that sends the invertible diagonal matrix with $t_1, \ldots, t_n$ on the diagonal to t_i.

2.2. ***Notation and conventions in type C.*** Below we review the properties of symplectic Schubert varieties relevant to this paper. For a more in-depth introduction the reader may refer to [Lakshmibai and Raghavan 2008, Chapter 6].

Let n be a positive integer. Let $V = \mathbb{C}^{2n}$ and let $(\cdot\,,\cdot)$ be a nondegenerate skew-symmetric bilinear form on V. Let $H = \mathrm{SL}(V)$ and $G = \mathrm{SP}(V) = \{Z \in \mathrm{SL}(V) \mid Z \text{ leaves the form } (\cdot\,,\cdot) \text{ invariant}\}$. We take the matrix of the form, with respect to the standard basis of V, to be

$$F = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$$

where J is the antidiagonal $(1, \ldots, 1)$, in this case of size n. To simplify our notation we will normally omit specifying the size of J as it will be obvious from the context.

We may realize $\mathrm{SP}(V)$ as the fixed point set of the involution $\sigma : H \to H$ given by $\sigma(Z) = F(Z^T)^{-1}F^{-1}$ (cf. [Steinberg 1968]). That is,

$$\begin{aligned} G &= \{Z \in \mathrm{SL}(V) \mid Z^T F Z = F\} \\ &= \{Z \in \mathrm{SL}(V) \mid F^{-1}(Z^T)^{-1} F = Z\} \\ &= \{Z \in \mathrm{SL}(V) \mid F(Z^T)^{-1} F^{-1} = Z\} \\ &= H^\sigma. \end{aligned}$$

Denote by T_H and B_H the maximal torus in H consisting of diagonal matrices and the Borel subgroup in H consisting of upper triangular matrices, respectively. It is easily seen that T_H and B_H are stable under σ and we set $T_G = T_H^\sigma$, $B_G = B_H^\sigma$. It is easily checked that T_G is a maximal torus in G and B_G is a Borel subgroup in G.

Thus we obtain

$$W_G \hookrightarrow W_H$$

where W_G, W_H denote the Weyl groups of G, H, respectively (with respect to T_G, T_H, respectively). Further, σ induces an involution on W_H:

$$w = (a_1, \cdots, a_{2n}) \in W_H, \quad \sigma(w) = (c_1, \cdots, c_{2n}), \quad c_i = 2n + 1 - a_{2n+1-i}$$

and

$$W_G = W_H^\sigma.$$

Thus we obtain

$$W_G = \{(a_1 \cdots a_{2n}) \in S_{2n} \mid a_i = 2n + 1 - a_{2n+1-i},\ 1 \le i \le 2n\}.$$

(here, S_{2n} is the symmetric group on $2n$ letters). Thus $w = (a_1 \cdots a_{2n}) \in W_G$ is known once $(a_1 \cdots a_n)$ is known. We shall denote an element $(a_1 \cdots a_{2n})$ in W_G by just $(a_1 \cdots a_n)$. For example, $(4231) \in S_4$ represents $(42) \in W_G$, $G = \mathrm{SP}(4)$.

The involution σ induces an involution on $X(T_H)$, the character group of T_H:

$$\chi \in X(T_H), \quad \sigma(\chi)(D) = \chi(\sigma(D)), \quad D \in T_H.$$

Let ϵ_i, for $1 \le i \le 2n$, be the character in $X(T_H)$, $\epsilon_i(D) = d_i$, the i-th entry in $D \in T_H$. We have

$$\sigma(\epsilon_i) = -\epsilon_{2n+1-i}$$

Now it is easily seen that the under the canonical surjective map

$$\varphi : X(T_H) \to X(T_G)$$

we have

$$\varphi(\epsilon_i) = -\varphi(\epsilon_{2n+1-i}), \quad 1 \le i \le 2n.$$

Let $R_H := \{\epsilon_i - \epsilon_j, 1 \le i, j \le 2n\}$ be the root system of H (relative to T_H), and $R_H^+ := \{\epsilon_i - \epsilon_j, 1 \le i < j \le 2n\}$ the set of positive roots (relative to B_H). We have the following:

(a) σ leaves R_H (resp. R_H^+) stable.

(b) For $\alpha, \beta \in R_H, \varphi(\alpha) = \varphi(\beta) \Leftrightarrow \alpha = \sigma(\beta)$.

(c) φ is equivariant for the canonical action of W_G on $X(T_H), X(T_G)$.

(d) $R_H^\sigma = \{\pm(\epsilon_i - \epsilon_{2n+1-i}), 1 \le i \le n\}$.

Let R_G (resp. R_G^+) be the set of roots of G with respect to T_G (resp. the set of positive roots with respect to B_G). Using the above facts and the explicit nature of the adjoint representation of G on Lie G, we deduce that

$$R_G = \varphi(R_H), \quad R_G^+ = \varphi(R_H^+).$$

In particular, R_G (resp. R_G^+) gets identified with the orbit space of R_H (resp. R_H^+) modulo the action of σ. Thus we obtain the following identification:

$$R_G = \{\pm(\epsilon_i \pm \epsilon_j),\ 1 \le i < j \le n\} \cup \{\pm 2\epsilon_i,\ i = 1, \dots, n\},$$
$$R_G^+ = \{(\epsilon_i \pm \epsilon_j),\ 1 \le i < j \le n\} \cup \{2\epsilon_i,\ i = 1, \dots, n\}.$$

The set S_G of simple roots in R_G^+ is given by

$$S_G := \{\alpha_i = \epsilon_i - \epsilon_{i+1},\ 1 \le i \le n-1\} \cup \{\alpha_n = 2\epsilon_n\}.$$

Let us denote the simple reflections in W_G by $\{s_i, 1 \le i \le n\}$, namely, $s_i =$ reflection with respect to $\epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n-1$, and $s_n =$ reflection with respect to $2\epsilon_n$. Then we have

$$s_i = \begin{cases} r_i r_{2n-i}, & \text{if } 1 \le i \le n-1, \\ r_n, & \text{if } i = n, \end{cases} \tag{2.2.1}$$

where r_i denotes the transposition $(i, i+1)$ in S_{2n} for $1 \le i \le 2n-1$.

For $w \in W_G$, let us denote by $l(w, W_H)$ (resp. $l(w, W_G)$) the length of w as an element of W_H (resp. W_G). For $w = (a_1, \cdots, a_{2n}) \in W_H$, denote

$$m(w) := \#\{i \le n \mid a_i > n\}. \tag{2.2.2}$$

Then for $w = (a_1, \cdots, a_{2n}) \in W_G$, we have $l(w, W_G) = \frac{1}{2}\big(l(w, W_H) + m(w)\big)$.

Proposition 2.2.3 [Lakshmibai and Raghavan 2008, Proposition 6.2.5.1]. *Let $w \in W_G$; let $X_G(w)$ (resp. $X_H(w)$) be the associated Schubert variety in G/B_G (resp. H/B_H). Under the canonical inclusion $G/B_G \hookrightarrow H/B_H$, we have $X_G(w) = X_H(w) \cap G/B_G$. Further, the intersection is scheme-theoretic.*

Notation 2.2.4. For the remainder of the paper we fix the following notation. Let $1 \le k < r \le n$ be positive integers. Let $Q = Q_{\hat{n}}$ to be the parabolic subgroup of H corresponding to omitting the root α_n and $P = P_{\hat{n}}$ to be the parabolic subgroup of G corresponding to omitting the root α_n. Let $\widetilde{P}$ be the two-step parabolic subgroup $P_{\widehat{r-k},\hat{n}}$ of G. Let $\widetilde{Q}$ be the three step parabolic subgroup $Q_{\widehat{r-k},\hat{n},2n-\widehat{(r-k)}}$ in H. Note that $P = Q^{\sigma}$ and $\widetilde{P} = \widetilde{Q}^{\sigma}$. Finally, we identify $P/\widetilde{P}$ with $\mathrm{GL}_n/P'_{\widehat{r-k}}$ where $P'_{\widehat{r-k}}$ is the parabolic subgroup of GL_n corresponding to omitting the root α_{r-k}.

Definition 2.2.5. A square $m \times m$ matrix X is *persymmetric* if $JX = X^T J$. Or, equivalently, if JX is symmetric.

Remark 2.2.6. We denote by Mat_n the space of $n \times n$ matrices. Let K be the subgroup of H consisting of matrices of the form

$$\begin{bmatrix} \mathrm{Id}_n & 0 \\ Y & \mathrm{Id}_n \end{bmatrix}, \qquad Y \in \mathrm{Mat}_n .$$

The canonical morphism $H \to H/Q$ induces a morphism $\psi_H : K \to H/Q$. We have that ψ_H is an open immersion, and $\psi_H(K)$ gets identified with the *opposite big cell* $O^-_{H/Q}$ in H/Q.

The cell $O^-_{H/Q}$ is σ-stable and by [Lakshmibai and Raghavan 2008, Corollary 6.2.4.3], we can identify the *opposite big cell* $O^-_{G/P}$ as

$$O^-_{G/P} = (O^-_{H/Q})^{\sigma} = \{z \in K \mid JY^T J = Y\}.$$

So $O^-_{G/P}$ is the subspace of K with Y persymmetric. Thus we can identify $O^-_{G/P}$ with the space of symmetric $n \times n$ matrices, Sym_n, under the map $O^-_{G/P} \longrightarrow \mathrm{Sym}_n$ given by

$$\begin{bmatrix} \mathrm{Id}_n & 0 \\ Y & \mathrm{Id}_n \end{bmatrix} \mapsto JY.$$

2.3. ***Opposite cells in Schubert varieties in the symplectic flag variety.*** A matrix $z \in \mathrm{SL}(V)$ with $n \times n$ block form

$$\begin{bmatrix} A_{n\times n} & C_{n\times n} \\ D_{n\times n} & E_{n\times n} \end{bmatrix}$$

is an element of G if and only if $z^T F z = F$, i.e., if and only if the following conditions hold on the $n \times n$ blocks:

$$A^T J D = D^T J A, \tag{2.3.1}$$

$$C^T J E = E^T J C, \tag{2.3.2}$$

$$J = (A^T J E - D^T J C) = (E^T J A - C^T J D). \tag{2.3.3}$$

The following proposition will prove useful throughout the rest of the paper.

Proposition 2.3.4. *Write U_P^- for the negative unipotent radical of P.*

(a) *$O_{G/P}^-$ can be naturally identified with $U_P^- P/P$*

(b) *For*

$$z = \begin{bmatrix} A_{n\times n} & C_{n\times n} \\ D_{n\times n} & E_{n\times n} \end{bmatrix} \in G,$$

$zP \in O_{G/P}^-$ if and only if A is invertible.

(c) *The inverse image of $O_{G/P}^-$ under the natural map $G/\widetilde{P} \to G/P$ is isomorphic to $O_{G/P}^- \times P/\widetilde{P}$ as schemes. Every element of $O_{G/P}^- \times P/\widetilde{P}$ is of the form*

$$\begin{bmatrix} A_{n\times n} & 0 \\ D_{n\times n} & J(A^T)^{-1}J \end{bmatrix} \bmod \widetilde{P} \in G/\widetilde{P}.$$

Moreover, two matrices

$$\begin{bmatrix} A_{n\times n} & 0_{n\times n} \\ D_{n\times n} & J(A^T)^{-1}J \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A'_{n\times n} & 0_{n\times n} \\ D'_{n\times n} & J(A'^T)^{-1}J \end{bmatrix}$$

in G represent the same element modulo $\widetilde{P}$ if and only if there exists a matrix $q \in P'_{\widehat{r-k}}$ (as defined in Notation 2.2.4) such that $A' = Aq$ and $D' = Dq$.

(d) *$P/\widetilde{P}$ is isomorphic to $\mathrm{GL}_n / P'_{\widehat{r-k}}$. In particular, the projection map $O_{G/P}^- \times P/\widetilde{P} \to P/\widetilde{P}$ is given by*

$$\begin{bmatrix} A_{n\times n} & 0 \\ D_{n\times n} & J(A^T)^{-1}J \end{bmatrix} \bmod \widetilde{P} \longmapsto A \bmod P'_{\widehat{r-k}} \in \mathrm{GL}_n / P'_{\widehat{r-k}} \cong P/\widetilde{P}.$$

Proof. (a): Note that U_P^- is the subgroup of G generated by the root subgroups $U_{-\alpha}$ for $\alpha \in R^+ \setminus R_P^+$. Under the canonical projection $G \to G/P$, $g \mapsto gP$, U_P^- is mapped isomorphically onto its image $O_{G/P}^-$ (cf. [Billey and Lakshmibai 2000, Section 4.4.4]). Thus we obtain the identification of $O_{G/P}^-$ with $U_P^- P/P$.

(b): Suppose that $zP \in O_{G/P}^-$. By (a) this means that $\exists\, n\times n$ matrices A', C', D', E' such that

$$z_1 = \begin{bmatrix} \mathrm{Id}_n & 0 \\ D' & \mathrm{Id}_n \end{bmatrix} \in U_P^- \quad \text{and} \quad z_2 = \begin{bmatrix} A' & C' \\ 0 & E' \end{bmatrix} \in P \quad \text{with} \quad z = \begin{bmatrix} A & C \\ D & E \end{bmatrix} = z_1 z_2.$$

Hence $A = A'$, and A' invertible implies A invertible.

Conversely, suppose A is invertible. Let

$$z = \begin{bmatrix} A & C \\ D & E \end{bmatrix} \in G.$$

Then A, C, D, E satisfy properties (2.3.1)–(2.3.2). Since A is invertible we may write

$$z = z_1 z_2 \quad \text{where} \quad z_1 = \begin{bmatrix} \mathrm{Id}_n & 0 \\ DA^{-1} & \mathrm{Id}_n \end{bmatrix}, z_2 = \begin{bmatrix} A & C \\ 0 & E - DA^{-1}C \end{bmatrix}$$

We shall now show that $z_1, z_2 \in G$. First, we note that (2.3.1) implies that

$$J(DA^{-1}) = (DA^{-1})^T J. \tag{2.3.5}$$

Then (2.3.5) shows that $z_1 \in U_P^-$, and hence $z_1 \in G$.

Now $z_1 \in G$ implies $z_1^{-1} \in G$, and $z \in G$ by assumption. Hence $z_2 = zz_1^{-1} \in G$. Further, since A is invertible, $z_2 \in P$. Hence the coset $zP = z_1 P$, which in view of the fact that $z_1 \in U_P^-$, implies by part (a) that $zP \in O_{G/P}^-$.

(c): Let $z \in U_P^- P \subset G$. Then we can write $z = z_1 z_2$ uniquely with $z_1 \in U_P^-$, $z_2 \in P$. Suppose that

$$\begin{bmatrix} \mathrm{Id}_n & 0 \\ D_{n\times n} & \mathrm{Id}_n \end{bmatrix}\begin{bmatrix} A_{n\times n} & C_{n\times n} \\ 0_{n\times n} & E_{n\times n} \end{bmatrix} = \begin{bmatrix} \mathrm{Id}_n & 0 \\ D'_{n\times n} & \mathrm{Id}_n \end{bmatrix}\begin{bmatrix} A'_{n\times n} & C'_{n\times n} \\ 0_{n\times n} & E'_{n\times n} \end{bmatrix}$$

then $A = A'$, $C = C'$, $DA = D'A'$ and $DC + E = D'C' + E'$, which yields that $D' = D$ (since $A = A'$ is invertible), and then $E = E'$. Hence $U_P^- \times_{\mathbb{C}} P = U_P^- P$. Thus for any parabolic subgroup $P' \subseteq P$, $U_P^- \times_{\mathbb{C}} P/P' = U_P^- P/P'$. The asserted isomorphism follows by part (a) from taking $P' = \widetilde{P}$.

To see the second assertion consider

$$z = \begin{bmatrix} A_{n\times n} & C_{n\times n} \\ D_{n\times n} & E_{n\times n} \end{bmatrix} \in G$$

with $zP \in O_{G/P}^-$. Note that the $n \times n$ block matrices satisfy properties (2.3.1)–(2.3.3) and by (b), A is invertible.

We have by the first part of (c) that the coset zP is an element of $O_{G/P}^- \times P/\widetilde{P}$, since $zP \in O_{G/P}^-$.

Claim. *We have a decomposition of z in G,*

$$\begin{bmatrix} A & C \\ D & E \end{bmatrix} = y_1 y_2 \text{ where } y_1 = \begin{bmatrix} A & 0 \\ D & J(A^T)^{-1}J \end{bmatrix} \in G,\ y_2 = \begin{bmatrix} \mathrm{Id}_n & A^{-1}C \\ 0 & \mathrm{Id}_n \end{bmatrix} \in \widetilde{P}.$$

We first check that $z = y_1 y_2$. We need the following identity

$$JA^T J(E - DA^{-1}C) = \mathrm{Id}_n, \tag{2.3.6}$$

which follows from

$$\begin{aligned} JA^T J(E - DA^{-1}C) &= J(A^T JE - A^T JDA^{-1}C) \\ &= J(A^T JE - D^T JAA^{-1}C) && (2.3.1) \\ &= JJ && (2.3.3) \\ &= \mathrm{Id}_n\,. \end{aligned}$$

So that

$$\begin{aligned} DA^{-1}C + J(A^T)^{-1}J &= DA^{-1}C + J(A^T)^{-1}JJA^TJ(E - DA^{-1}C) && (2.3.6) \\ &= DA^{-1}C + E - DA^{-1}C \\ &= E. \end{aligned}$$

With this it is easily verified that $z = y_1 y_2$.

It is clear that $y_1 \in G$. To show $y_2 \in G$ we need to check that $J(A^{-1}C)^T J = A^{-1}C$.

$$\begin{aligned} (A^{-1}C)^T J &= (A^{-1}C)^T JJA^TJ(E - DA^{-1}C) && (2.3.6) \\ &= C^T J(E - DA^{-1}C) \\ &= E^T JC - C^T JDA^{-1}C && (2.3.2) \\ &= (E - DA^{-1}C)^T JC && (2.3.5) \\ &= (E - DA^{-1}C)^T JAJJA^{-1}C \\ &= (JA^TJ(E - DA^{-1}C))^T J(A^{-1}C) \\ &= J(A^{-1}C) && (2.3.6) \end{aligned}$$

Thus $y_2 \in G$. It is clear additionally that $y_2 \in \widetilde{P}$ (in fact $y_2 \in B_G$).

Hence our claim follows and we have

$$\begin{bmatrix} A & C \\ D & E \end{bmatrix} = \begin{bmatrix} A & 0 \\ D & J(A^T)^{-1}J \end{bmatrix} \bmod \widetilde{P}.$$

Finally,

$$\begin{bmatrix} A_{n\times n} & 0_{n\times n} \\ D_{n\times n} & J(A^T)^{-1}J \end{bmatrix} = \begin{bmatrix} A'_{n\times n} & 0_{n\times n} \\ D'_{n\times n} & J(A'^T)^{-1}J \end{bmatrix} \mod \widetilde{P}$$

if and only if there exist matrices $q \in P'_{\widehat{r-k}}$, and $q' \in \mathrm{Mat}_n$ such that

$$\begin{bmatrix} A' & 0_{n\times n} \\ D' & J(A^T)^{-1}J \end{bmatrix} = \begin{bmatrix} A & 0_{n\times n} \\ D & J(A'^T)^{-1}J \end{bmatrix} \begin{bmatrix} q & q' \\ 0_{n\times n} & J(q^T)^{-1}J \end{bmatrix},$$

which holds if and only if $q' = 0$, $A' = Aq$ and $D' = Dq$ (since A and A' are invertible).

(d): There is a surjective morphism of $\mathbb{C}$-group schemes $P \to \mathrm{GL}_n$:

$$\begin{bmatrix} A & C \\ 0 & E \end{bmatrix} \to A.$$

This induces the required isomorphism. The element

$$\begin{bmatrix} A & C \\ D & E \end{bmatrix} \bmod \widetilde{P} \in O^-_{G/P} \times P/\widetilde{P}$$

decomposes uniquely as

$$\begin{bmatrix} \mathrm{Id}_n & 0 \\ DA^{-1} & \mathrm{Id}_n \end{bmatrix} \left(\begin{bmatrix} A & C \\ 0 & E - DA^{-1}C \end{bmatrix} \bmod \widetilde{P} \right)$$

and hence it is mapped to $A \bmod P'_{\widehat{r-k}}$. □

2.4. ***Homogeneous bundles and representations.*** Let Q be a parabolic subgroup of GL_n. We collect here some results about homogeneous vector bundles on GL_n/Q. Most of these results are well-known, but for some of them, we could not find a reference, so we give a proof here for the sake of completeness. Online notes of G. Ottaviani [1995] and of D. Snow [1994] discuss the details of many of these results.

Let L_Q and U_Q be respectively the Levi subgroup and the unipotent radical of Q. Let E be a finite-dimensional vector-space on which Q acts on the right.

Definition 2.4.1. Define $\mathrm{GL}_n \times^Q E := (\mathrm{GL}_n \times E)/\sim$ where $\sim$ is the equivalence relation $(g, e) \sim (gq, eq)$ for every $g \in \mathrm{GL}_n$, $q \in Q$ and $e \in E$. Then $\pi_E : \mathrm{GL}_n \times^Q E \longrightarrow \mathrm{GL}_n/Q$, $(g, e) \mapsto gQ$, is a vector bundle called the *vector bundle associated to* E (and the principal Q-bundle $\mathrm{GL}_n \longrightarrow \mathrm{GL}_n/Q$). For $g \in \mathrm{GL}_n$, $e \in E$, we write $[g, e] \in \mathrm{GL}_n \times^Q E$ for the equivalence class of $(g, e) \in \mathrm{GL}_n \times E$ under $\sim$. We say that a vector bundle $\pi : \mathbf{E} \longrightarrow \mathrm{GL}_n/Q$ is *homogeneous* if $\mathbf{E}$ has a GL_n-action and π is GL_n-equivariant, i.e, for every $y \in \mathbf{E}$, $\pi(g \cdot y) = g \cdot \pi(y)$.

Remark 2.4.2. There is a similar construction in the case when E is a left Q-module.

In this section, we abbreviate $\mathrm{GL}_n \times^Q E$ as $\widetilde{E}$. It is known that $\mathbf{E}$ is homogeneous if and only if $\mathbf{E} \simeq \widetilde{E}$ for some Q-module E. (If this is the case, then E is the fiber of $\mathbf{E}$ over the coset Q.) A homogeneous bundle $\widetilde{E}$ is said to be *irreducible* (respectively *indecomposable*, *completely reducible*) if E is an irreducible (respectively indecomposable, completely reducible) Q-module. It is known that E is completely reducible if and only if U_Q acts trivially and that E is irreducible if and only if additionally it is irreducible as a representation of L_Q. See [Snow 1994, Section 5] or [Ottaviani 1995, Section 10] for the details.

Discussion 2.4.3. For the cohomology group computations in this paper, we will primarily be interested in the case when GL_n/Q is a Grassmannian. Thus let $Q = P_{\hat{m}}$, with $1 \le m \le n-1$. A weight λ is said to be *Q-dominant* if and only if when we express λ as $\sum_{i=1}^n \lambda_i \epsilon_i$ (where ϵ_i, for $1 \le i \le n$, is the character that sends a diagonal matrix in T to its i-th entry), then $\lambda_1 \ge \ldots \ge \lambda_m$ and $\lambda_{m+1} \ge \ldots \ge \lambda_n$. We will write $\lambda = (\lambda_1, \ldots, \lambda_n)$ to mean that $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$. Every finite-dimensional irreducible Q-module is of the form $H^0(Q/B_n, L_\lambda)$ for a Q-dominant weight λ. Hence the irreducible homogeneous vector bundles on GL_n/Q are in correspondence with Q-dominant weights. We describe them now. If $Q = P_{\widehat{n-i}}$, then $\mathrm{GL}_n/Q = \mathrm{Grass}_{i,n}$. (Recall that, for us, the GL_n-action on $\mathbb{C}^n$ is on the right.) On $\mathrm{Grass}_{i,n}$, we have the *tautological sequence*

$$0 \longrightarrow \mathcal{R}_i \longrightarrow \mathbb{C}^n \otimes \mathbb{O}_{\mathrm{Grass}_{i,n}} \longrightarrow \mathcal{Q}_{n-i} \longrightarrow 0 \tag{2.4.4}$$

of homogeneous vector bundles. The bundle $\mathcal{R}_i$ is called the *tautological subbundle* (of the trivial bundle $\mathbb{C}^n$) and $\mathcal{Q}_{n-i}$ is called the *tautological quotient bundle*. Every irreducible homogeneous bundle on $\mathrm{Grass}_{i,n}$ is of the form $S_{(\lambda_1,\cdots,\lambda_{n-i})}\mathcal{Q}^*_{n-i} \otimes S_{(\lambda_{n-i+1},\cdots,\lambda_n)}\mathcal{R}^*_i$ for some $P_{\widehat{n-i}}$-dominant weight λ. Here S_μ denotes the *Schur functor* associated to the partition μ (cf. [Fulton and Harris 1991, §6.1]).

A Q-dominant weight is called (m)-*dominant* in [Weyman 2003, p. 114]. Although our definition looks like Weyman's definition, we should keep in mind that our action is on the right. We only have to be careful when we apply the Borel–Weil–Bott theorem (more specifically, the Bott algorithm). In this paper, our computations are done only on Grassmannians. If μ and ν are partitions, then (μ, ν) will be Q-dominant (for a suitable Q), and will give us the vector bundle $S_\mu\mathcal{Q}^* \otimes S_\nu\mathcal{R}^*$ (this is where the right-action of Q becomes relevant) and to compute its cohomology, we will have to apply the Bott algorithm to the Q-dominant weight (ν, μ). (In [Weyman 2003], one would get $S_\mu\mathcal{R}^* \otimes S_\nu\mathcal{Q}^*$ and would apply the Bott algorithm to (μ, ν).) $\square$

We now give a brief description of the Bott algorithm for computing the cohomology of irreducible homogeneous vector bundles on GL_n/Q [Weyman 2003, Remark 4.1.5].

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a weight. As in [Weyman 2003, Remark 4.1.5] we define an action of the permutation $\nu_i = (i, i+1)$ on the set of weights in the following way:

$$\nu_i\alpha = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1} - 1, \alpha_i + 1, \alpha_{i+2}, \dots, \alpha_n). \tag{2.4.5}$$

The Bott algorithm may be applied to our case as follows. Let $Q = P_{\hat{m}}$, with $1 \le m \le n-1$ and let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a Q-dominant weight with associated homogeneous vector bundle $V(\lambda) := S_{(\lambda_1,\dots,\lambda_m)}\mathcal{Q}^* \otimes S_{(\lambda_{m+1},\dots,\lambda_n)}\mathcal{R}^*$. We will apply the Bott algorithm to $\lambda' = (\lambda_{m+1}, \dots, \lambda_n, \lambda_1, \dots, \lambda_m)$ in keeping with the last paragraph of Discussion 2.4.3.

If λ' is nonincreasing, then $H^0(\mathrm{GL}_n/Q, V(\lambda)) = S_{\lambda'}\mathbb{C}^n$ and $H^i(\mathrm{GL}_n/Q, V(\lambda)) = 0$ for $i > 0$. Otherwise we start to apply the exchanges of type (2.4.5) to λ', trying to move smaller numbers on the left to the right. Two possibilities can occur:

(1) We apply an exchange of type (2.4.5) and it leaves the sequence unchanged. In this case $H^i(\mathrm{GL}_n/Q, V(\lambda)) = 0$ for $i \ge 0$.

(2) After applying j exchanges, we transform λ' into a nonincreasing sequence β. Then we have $H^i(\mathrm{GL}_n/Q, V(\lambda)) = 0$ for $i \ne j$ and $H^j(\mathrm{GL}_n/Q, V(\lambda)) = S_\beta\mathbb{C}^n$.

3. Properties of Schubert desingularization in type C

Recall the following result about the tangent space of a Schubert variety, see [Billey and Lakshmibai 2000, Chapter 4] for details.

Proposition 3.1. *Let Q be a parabolic subgroup of* SL_{2n}. *Let $\tau \in W^Q$. Then the dimension of the tangent space of $X_Q(\tau)$ at e_{id} is*

$$\#\{s_\alpha \mid \alpha \in R^- \backslash R_Q^- \quad \textit{and} \quad \tau \geq s_\alpha \textit{ in } W/W_Q\}.$$

In particular, *$X_Q(\tau)$ is smooth if and only if*

$$\dim X_Q(\tau) = \#\{s_\alpha \mid \alpha \in R^- \backslash R_Q^- \quad \textit{and} \quad \tau \geq s_\alpha \textit{ in } W/W_Q\}.$$

Notation 3.2. For an integer i with $1 \leq i \leq n$ we define $i' = 2n + 1 - i$. Let $1 \leq k < r \leq n$. Then

$$\mathcal{W}_{k,r} = \begin{cases} (k+1, \dots, r, n', \dots, (r+1)', k', \dots, 1') \in W^P, & \text{if } r < n, \\ (k+1, \dots, r, k', \dots, 1') \in W^P, & \text{if } r = n. \end{cases}$$

Let $1 \leq k < r \leq n$ be integers. Let $w \in \mathcal{W}_{k,r}$ with $\tilde{w}$ its minimal representative in $W^{\widetilde{P}}$.

Proposition 3.3. *The Schubert variety $X_{\widetilde{Q}}(\tilde{w})$ in $H/\widetilde{Q}$ is smooth.*

Proof. Let $w_{\max} \in W_H (= S_{2n})$ be the maximal representative of $\tilde{w}$. Then

$$w_{\max} = \begin{cases} \big([r, k+1][1', k'][(r+1)', n'][n, (r+1)][k, 1][(k+1)', r']\big), & \text{if } r < n, \\ \big([r, k+1][1', k'][k, 1][(k+1)', r']\big), & \text{if } r = n. \end{cases}$$

To see this we need to show that $X_{P_{\hat{i}}}(w_{\max}) = X_{P_{\hat{i}}}(\tilde{w})$ for $i = r-k, n, 2n-(r-k)$ and that $w_{\max}$ is the maximal element of W_H with this property. But this follows from the fact that for $\tau = (c_1, \dots, c_{2n}) \in W_H$ and $1 \leq i \leq 2n$ we have that $X_{P_{\hat{i}}}(\tau) = X_{P_{\hat{i}}}(\tau')$ where $\tau' \in W^{P_{\hat{i}}}$ is the element with $c_1, \dots, c_i$ written in increasing order.

Thus $X_{B_H}(w_{\max})$ is the inverse image of $X_{\widetilde{Q}}(\tilde{w})$ under the natural morphism $H/B_H \to H/\widetilde{Q}$. As $w_{\max}$ is a 4231 and 3142 avoiding element of W_H we have that $X_{B_H}(w_{\max})$ is nonsingular (see [Billey and Lakshmibai 2000, 8.1.1]). Since the morphism $H/B_H \to H/\widetilde{Q}$ has nonsingular fibers (namely $\widetilde{Q}/B_H$), $X_{\widetilde{Q}}(\tilde{w})$ must be smooth. □

Proposition 3.4. *The Schubert variety $X_{\widetilde{P}}(\tilde{w})$ in $G/\widetilde{P}$ is smooth .*

Proof. Let $w_{\max}$ be as in the proof of Proposition 3.3. Then clearly $w_{\max}$ is in W_G and $X_{B_G}(w_{\max})$ is the inverse image of $X_{\widetilde{P}}(\tilde{w})$ under the natural morphism $G/B_G \to G/\widetilde{P}$.

Claim. *$X_{B_G}(w_{\max})$ is smooth.*

Note that the claim implies the required result (since the canonical morphism $G/B_G \to G/\widetilde{P}$ is a fibration with nonsingular fibers (namely, $\widetilde{P}/B_G$)). To prove the claim, as seen in the proof of Proposition 3.3, we have that $X_{B_H}(w_{\max})$ is smooth.

We conclude the smoothness of $X_{B_G}(w_{\max})$ using the following two formulas [Lakshmibai 1987, §3(VI), Remark 5.8]:

$$l_G(\theta) = \tfrac{1}{2}[l_H(\theta) + m(\theta)], \tag{1}$$

where we let $\theta \in W_G$, say, $\theta = (a_1, \cdots a_n)$. With $m(\theta) = \#\{i,\ 1 \le i \le m \mid a_i > m\}$ (cf. (2.2.2)), we have

$$\dim T_{\mathrm{id}}(\theta, G) = \tfrac{1}{2}[\dim T_{\mathrm{id}}(\theta, H) + c(\theta)], \tag{2}$$

where $c(\theta) = \#\{1 \le i \le m \mid \theta \ge s_{\epsilon_{2i}}\}$, and $T_{\mathrm{id}}(\theta, G)$ (resp $T_{\mathrm{id}}(\theta, H)$) denotes the Zariski tangent space of $X_{B_G}(\theta)$ (resp $X_{B_H}(\theta)$) at e_{id}. Note that $s_{\epsilon_{2i}}$ is just the transposition (i, i') (cf. (2.2.1)). Now taking $\theta = w_{\max}$, we have, $c(w_{\max}) = m(w_{\max})$. Hence we obtain from (1), (2) that $\dim T_{\mathrm{id}}(w_{\max}, G) = l_G(w_{\max})$, proving that $X_{B_G}(w_{\max})$ is smooth at e_{id}, and hence is nonsingular (note that for a Schubert variety X, the singular locus of X, $\mathrm{Sing}(X)$, is B-stable implying $e_{\mathrm{id}} \in \mathrm{Sing}(X)$ if $\mathrm{Sing}(X) \neq \varnothing$). Thus the claim (and hence the required result) follows. □

Remark 3.5. We have that $X_{\tilde{P}}(\tilde{w})$ is the fixed point set under an automorphism of order two of the Schubert variety $X_{\tilde{Q}}(\tilde{w})$ and thus is smooth, provided $\mathrm{char}\, K \neq 2$ ([Edixhoven 1992, Proposition 3.4]).

Discussion 3.6. To give a characterization of $Y_{\tilde{Q}}(\tilde{w})$ we first need a review of the structure of $O^-_{H/\tilde{Q}}$ and its Plücker coordinates.

Recall that for the Plücker embedding of the Grassmannian $\mathrm{Grass}_{d,n}$, the *Plücker coordinate* $p_{\underline{i}}(U)$, $U \in \mathrm{Grass}_{d,n}$ and $\underline{i} = (i_1, \ldots, i_d)$ with $1 \le i_1 < \ldots < i_d < n$, is just the $d \times d$ minor of the matrix $A_{n\times d}$ with row indices $(i_1, \ldots, i_d)$ (here the matrix $A_{n\times d}$ represents the d-dimensional subspace U with respect to the standard basis).

The cell $O^-_{H/\tilde{Q}}$ can be identified with the affine space of lower-triangular matrices with possible nonzero entries x_{ij} at row i and column j where (i, j) is such that there exists an $l \in \{r-k, n, 2n-(r-k)\}$ such that $j \le l < i \le N$. To see this, note that we are interested in those (i, j) such that the root $\epsilon_i - \epsilon_j$ belongs to $R^- \setminus R^-_{\tilde{Q}}$. Since $R^-_{\tilde{Q}} = R^-_{Q_{\widehat{r-k}}} \cap R_{Q_{\hat{n}}} \cap R_{Q_{\widehat{2n-(r-k)}}}$, we see that we are looking for (i, j) such that $\epsilon_i - \epsilon_j \in R^- \setminus R^-_{Q_{\hat{l}}}$, for some $l \in \{r-k, n, 2n-(r-k)\}$. For the maximal parabolic subgroup $P_{\hat{l}}$, we have, $R^- \setminus R^-_{Q_{\hat{l}}} = \{\epsilon_i - \epsilon_j \mid 1 \le j \le l < i \le N\}$. We have $\dim O^-_{H/\tilde{Q}} = |R^- \setminus R^-_{\tilde{Q}}|$.

Thus we have the following identification

$$O^-_{H/\tilde{Q}} = \begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ \mathcal{D}_1 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ \mathcal{D}_3 & \mathcal{D}_4 & E' & \mathrm{Id}_{r-k} \end{bmatrix} \tag{3.7}$$

where the block matrices have possible nonzero entries x_{ij} given by

$$A' = \begin{bmatrix} x_{(r-k)+1\ 1} & \cdots & x_{(r-k)+1\ r-k} \\ \vdots & & \vdots \\ x_{n\ 1} & \cdots & x_{n\ r-k} \end{bmatrix}, \qquad E' = \begin{bmatrix} x_{2n-(r-k)+1\ n+1} & \cdots & x_{2n-(r-k)+1\ 2n-(r-k)} \\ \vdots & & \vdots \\ x_{2n\ n+1} & \cdots & x_{2n\ 2n-(r-k)} \end{bmatrix},$$

$$\mathcal{D}_1 = \begin{bmatrix} x_{n+1\ 1} & \cdots & x_{n+1\ r-k} \\ \vdots & & \vdots \\ x_{2n-(r-k)\ 1} & \cdots & x_{2n-(r-k)\ r-k} \end{bmatrix}, \qquad \mathcal{D}_2 = \begin{bmatrix} x_{n+1\ (r-k)+1} & \cdots & x_{n+1\ n} \\ \vdots & & \vdots \\ x_{2n-(r-k)\ (r-k)+1} & \cdots & x_{2n-(r-k)\ n} \end{bmatrix},$$

$$\mathcal{D}_3 = \begin{bmatrix} x_{2n-(r-k)+1\ 1} & \cdots & x_{2n-(r-k)+1\ r-k} \\ \vdots & & \vdots \\ x_{2n\ 1} & \cdots & x_{2n\ r-k} \end{bmatrix}, \quad \mathcal{D}_4 = \begin{bmatrix} x_{2n-(r-k)+1\ (r-k)+1} & \cdots & x_{2n-(r-k)+1\ n} \\ \vdots & & \vdots \\ x_{2n\ (r-k)+1} & \cdots & x_{2n\ n} \end{bmatrix}.$$

We may break the Plücker coordinates we want to understand into several cases.

Case 1: For $i > r$, $j \le r-k$ the Plücker coordinate $p^{(r-k)}_{(i,j)}$ on the Grassmannian $H/Q_{\widehat{r-k}}$ lifts to a regular function on $H/\widetilde{Q}$. Its restriction to $O^-_{H/\widetilde{Q}}$ is the $r-k \times r-k$ minor of (3.7) with column indices $\{1, 2, \dots, r-k\}$ and row indices $\{1, \dots, j-1, j+1, \dots, r-k, i\}$. This minor is the determinant of an $r-k \times r-k$ matrix with the top $(r-k)-1$ rows equal to Id_{r-k} omitting the j-th row, and the bottom row equal to the first $r-k$ entries of the i-th row of (3.7). The determinant of this matrix is thus $(-1)^{(r-k)-j} x_{ij}$. Thus for $i > r$, $j \le r-k$:

$$p^{(r-k)}_{(i,j)}\big|_{O^-_{H/\widetilde{Q}}} = (-1)^{(r-k)-j} x_{ij}. \tag{3.8}$$

Case 2: For $i > 2n-(r-k)$, $n < j \le 2n-(r-k)$ the Plücker coordinate $p^{(2n-(r-k))}_{(i,j)}$ on the Grassmannian $H/Q_{\widehat{2n-(r-k)}}$ lifts to a regular function on $H/\widetilde{Q}$. Its restriction to $O^-_{H/\widetilde{Q}}$ is the $2n-(r-k) \times 2n-(r-k)$ minor of (3.7) with column indices $\{1, 2, \dots, 2n-(r-k)\}$ and row indices $\{1, \dots, j-1, j+1, \dots, 2n-(r-k), i\}$. This minor is the determinant of

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 \\ \widehat{\mathcal{D}}_1 & \widehat{\mathcal{D}}_2 & \widehat{I}_1 \\ [x_{i\ 1} \dots x_{i\ r-k}] & [x_{i\ (r-k)+1} \dots x_{i\ n}] & [x_{i\ n+1} \dots x_{i\ 2n-(r-k)}] \end{bmatrix} \tag{3.9}$$

where $\widehat{\mathcal{D}}_1$, $\widehat{\mathcal{D}}_2$, and $\widehat{I}_1$ are equal to, respectively, $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathrm{Id}_{n-(r-k)}$ with their $(j-n)$-th rows omitted. The determinant of (3.9) is equal to the determinant of

$$\begin{bmatrix} \widehat{I}_1 \\ [x_{i\ n+1} \dots x_{i\ 2n-(r-k)}] \end{bmatrix}.$$

As above this is just an identity matrix with a single row replaced and so its determinant is just $(-1)^{2n-(r-k)-j} x_{ij}$. Thus for $i > 2n-(r-k)$, $n < j \le 2n-(r-k)$:

$$p^{(2n-(r-k))}_{(i,j)}\big|_{O^-_{H/\widetilde{Q}}} = (-1)^{2n-(r-k)-j} x_{ij}. \tag{3.10}$$

Case 3: For $i > 2n-(r-k)$, $r-k < j \le n$ the Plücker coordinate $p^{(2n-(r-k))}_{(i,j)}$ on the Grassmannian $H/Q_{\widehat{2n-(r-k)}}$ lifts to a regular function on $H/\widetilde{Q}$. Its restriction to $O^-_{H/\widetilde{Q}}$ is the $2n-(r-k)\times 2n-(r-k)$ minor of (3.7) with column indices $\{1,2,\dots,2n-(r-k)\}$ and row indices $\{1,\dots,j-1,j+1,\dots,2n-(r-k),i\}$. This minor is the determinant of

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 \\ \widehat{A'} & \widehat{I_2} & 0 \\ \mathcal{D}_1 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} \\ [x_{i\,1}\dots x_{i\,r-k}] & [x_{i\,(r-k)+1}\dots x_{i\,n}] & [x_{i\,n+1}\dots x_{i\,2n-(r-k)}] \end{bmatrix} \tag{3.11}$$

where $\widehat{A'}$ and $\widehat{I_2}$ are equal to, respectively, A' and $\mathrm{Id}_{n-(r-k)}$ with their $j-(r-k)$-th rows omitted. The determinant of (3.11) is equal to the determinant of

$$\begin{bmatrix} \widehat{I_2} & 0 \\ \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} \\ [x_{i\,(r-k)+1}\dots x_{i\,n}] & [x_{i\,n+1}\dots x_{i\,2n-(r-k)}] \end{bmatrix}. \tag{3.12}$$

To calculate this, shift the bottom row so that it becomes the $j-(r-k)$-th row of $\widehat{I_2}$. Let $M = 2n-(r-k)-j$. Then the determinant of (3.12) will be $(-1)^M$ times the determinant of

$$\begin{bmatrix} I_3 & Z \\ \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} \end{bmatrix}, \tag{3.13}$$

where I_3 is $\mathrm{Id}_{n-(r-k)}$ with the $j-(r-k)$-th row replaced by $[x_{i\,(r-k)+1}\dots x_{i\,n}]$ and Z is the zero matrix with the $j-(r-k)$-th row replaced by $[x_{i\,n+1}\dots x_{i\,2n-(r-k)}]$. Since the lower right block matrix of (3.13) commutes with its lower left block matrix we have that the determinant of (3.13) is equal to the determinant of $I_3 - Z\mathcal{D}_2$. We have that $Z\mathcal{D}_2$ is equal to the zero matrix with its $j-(r-k)$-th row replaced by

$$[x_{i\,(r-k)+1}\dots x_{i\,n}]\mathcal{D}_2.$$

And thus $I_3 - Z\mathcal{D}_2$ is equal to $\mathrm{Id}_{n-(r-k)}$ with the $j-(r-k)$-th row replaced by

$$[x_{i\,(r-k)+1}\dots x_{i\,n}] - [x_{i\,(r-k)+1}\dots x_{i\,n}]\mathcal{D}_2.$$

And so the determinant of $I_3 - Z\mathcal{D}_2$ is merely equal to the $j-(r-k)$-th entry of $I_3 - Z\mathcal{D}_2$ which is

$$x_{ij} - [x_{i\,(r-k)+1}\dots x_{i\,n}][x_{n+1\,j}\dots x_{2n-(r-k)\,j}]^T.$$

Combining all our steps, we finally have that for $i > 2n-(r-k)$, $r-k < j \le n$:

$$p^{(2n-(r-k))}_{(i,j)}\big|_{O^-_{H/\widetilde{Q}}} = (-1)^M\left(x_{ij} - [x_{i\,(r-k)+1}\dots x_{in}][x_{n+1\,j}\dots x_{2n-(r-k)\,j}]^T\right). \tag{3.14}$$ $\square$

Theorem 3.15. *The opposite cell $Y_{\widetilde{Q}}(\tilde{w})$ can be identified with the subspace of $O^-_{H/\widetilde{Q}}$ given by matrices of the form*

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ 0 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ 0 & E'\mathcal{D}_2 & E' & \mathrm{Id}_{r-k} \end{bmatrix}$$

with $\mathcal{D}_2 \in \mathrm{Mat}_{n-(r-k)}$, $A' \in \mathrm{Mat}_{n-(r-k)\times r-k}$ with the bottom $n-r$ rows of A' all zero, and $E' \in \mathrm{Mat}_{r-k\times n-(r-k)}$ with the left $n-r$ columns of E' all zero.

Proof. For $j \le r-k < i$ the reflection (i, j) equals $(1, 2, \dots, j-1, j+1, \dots, r-k, i)$ and $\tilde{w}$ equals $(k+1, \dots, r)$ in $W/W_{Q_{\widehat{r-k}}}$. Thus for $i > r$ and $j \le r-k$, the reflection (i, j) is not smaller than $\tilde{w}$ in $W/W_{Q_{\widehat{r-k}}}$ so the Plücker coordinate $p^{(r-k)}_{(i,j)}$ vanishes on $X_{\widetilde{Q}}(\tilde{w})$. We saw in (3.8) that for such (i, j) we have $p^{(r-k)}_{(i,j)} = (-1)^{(r-k)-j} x_{ij}$ and thus $x_{ij} \equiv 0$ on $Y_{\widetilde{Q}}(\tilde{w})$.

For $j \le n < i$ the reflection (i, j) equals $(1, 2, \dots, j-1, j+1, \dots, n, i)$ and $\tilde{w}$ is equal to $(k+1, \dots, r, n', \dots, (r+1)', k', \dots, 1')$ in $W/W_{Q_{\hat{n}}}$. Thus there is no choice of (i, j) such that (i, j) is not smaller than $\tilde{w}$ in $W/W_{Q_{\hat{n}}}$.

For $j \le 2n-(r-k) < i$ the reflection (i, j) equals $(1, 2, \dots, j-1, j+1, \dots, 2n-(r-k), i)$ and $\tilde{w}$ equals $(1, \dots, n, n', \dots, (r+1)', k', \dots, 1')$ in $W/W_{Q_{\widehat{2n-(r-k)}}}$. Thus for $i > 2n-(r-k)$, and $j \le 2n-r$ the reflection (i, j) is not smaller than $\tilde{w}$ in $W/W_{Q_{\widehat{2n-(r-k)}}}$. We break these into two cases, ignoring those $j \le r-k$ as we have already shown above that for $j \le r-k$ and $i > 2n-(r-k)$ we have $x_{ij} \equiv 0$ on $Y_{\widetilde{Q}}(\tilde{w})$.

The first case is for (i, j) with $i > 2n-(r-k)$, and $n < j \le 2n-r$. The fact that (i, j) is not smaller than $\tilde{w}$ in $W/W_{Q_{\widehat{2n-(r-k)}}}$ implies that the Plücker coordinate $p^{(2n-(r-k))}_{(i,j)}$ vanishes on $X_{\widetilde{Q}}(\tilde{w})$. We saw in (3.10) that for such (i, j) we have $p^{(2n-(r-k))}_{(i,j)} = (-1)^{2n-(r-k)-j} x_{ij}$ and thus $x_{ij} \equiv 0$ on $Y_{\widetilde{Q}}(\tilde{w})$.

The second case is for (i, j) with $i > 2n-(r-k)$ and $r-k < j \le n$. The reflection (i, j) is not smaller than $\tilde{w}$ in $W/W_{Q_{\widehat{2n-(r-k)}}}$ implies that the Plücker coordinate $p^{(2n-(r-k))}_{(i,j)}$ vanishes on $X_{\widetilde{Q}}(\tilde{w})$. We saw in (3.14) that for such (i, j) we have $p^{(2n-(r-k))}_{(i,j)} = (-1)^M \big(x_{ij} - [x_{i\,(r-k)+1} \dots x_{i\,n}][x_{n+1\,j} \dots x_{2n-(r-k)\,j}]^T\big)$. Combining these two facts we get $x_{ij} = [x_{i\,(r-k)+1} \dots x_{i\,n}][x_{n+1\,j} \dots x_{2n-(r-k)\,j}]^T$.

As $[x_{i\,(r-k)+1} \dots x_{i\,n}]$ is the $(2n-(r-k)-i)$-th row of E' and $[x_{n+1\,j} \dots x_{2n-(r-k)\,j}]^T$ is the $(2n-(r-k)-j)$-th column of $\mathcal{D}_2$ it is clear that on $Y_{\widetilde{Q}}(\tilde{w})$ we have $x_{ij} = (E'X)_{(2n-(r-k)-i)\,(2n-(r-k)-j)}$.

On the other hand note that the reflections (i, j) with $i > r$ and $j \le r-k$, and $i > 2n-(r-k)$ and $r-k < j \le 2n-r$ are precisely the reflections s_α with $\alpha \in R^- \setminus R^-_{\widetilde{Q}}$ and $\tilde{w} \not\geq s_\alpha$ in $W/W_{\widetilde{Q}}$. Since $X_{\widetilde{Q}}(\tilde{w})$ is smooth this implies by Proposition 3.1 that the codimension of $Y_{\widetilde{Q}}(\tilde{w})$ in $O^-_{H/\widetilde{Q}}$ is equal to

$\#\{(i,j) \mid i > r \text{ and } j \le r-k, \text{ or } i > 2n-(r-k) \text{ and } r-k < j \le 2n-r\}$. Above we have shown that for each such (i,j), x_{ij} either vanishes, or is completely dependent on the entries of $E'X$. Thus $Y_{\widetilde{Q}}(\tilde{w})$ is the subspace of $O^-_{H/\widetilde{Q}}$ defined by the vanishing of $\{x_{ij} \mid i > r \text{ and } j \le r-k, \text{ or } i > 2n-(r-k) \text{ and } n < j \le 2n-r\}$ and $x_{ij} = (E'X)_{(2n-(r-k)-i)\,(2n-(r-k)-j)}$ for $i > 2n-(r-k)$ and $r-k < j \le n$. $\square$

Example 3.16. Let $k=2$, $r=4$, and $n=5$. Then $\widetilde{Q} = Q_{\hat{2},\hat{5},\hat{8}}$, $w = (3,4,6,9,10)$, and $\tilde{w} = (3,4,6,9,10,1,2,5)$. Then

$$O^-_{H/\widetilde{Q}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{41} & x_{42} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{51} & x_{52} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & 1 & 0 & 0 & 0 & 0 \\ x_{71} & x_{72} & x_{73} & x_{74} & x_{75} & 0 & 1 & 0 & 0 & 0 \\ x_{81} & x_{82} & x_{83} & x_{84} & x_{85} & 0 & 0 & 1 & 0 & 0 \\ x_{91} & x_{92} & x_{93} & x_{94} & x_{95} & x_{96} & x_{97} & x_{98} & 1 & 0 \\ x_{101} & x_{102} & x_{103} & x_{104} & x_{105} & x_{106} & x_{107} & x_{108} & 0 & 1 \end{bmatrix}.$$

And $Y_{\widetilde{P}}(\tilde{w})$ will be the subspace of $O^-_{H/\widetilde{Q}}$ given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{41} & x_{42} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{63} & x_{64} & x_{65} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{73} & x_{74} & x_{75} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & x_{83} & x_{84} & x_{85} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x_{97}x_{73}+x_{98}x_{83} & x_{97}x_{74}+x_{98}x_{84} & x_{97}x_{75}+x_{98}x_{85} & 0 & x_{97} & x_{98} & 1 & 0 \\ 0 & 0 & x_{107}x_{73}+x_{108}x_{83} & x_{107}x_{74}+x_{108}x_{84} & x_{107}x_{75}+x_{108} & 0 & x_{107} & x_{108} & 0 & 1 \end{bmatrix}.$$

Corollary 3.17. *The opposite cell $Y_{\widetilde{P}}(\tilde{w})$ can be identified with the subspace of $O^-_{G/\widetilde{P}}$ given by matrices of the form*

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ 0 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ 0 & -J(A')^T J\mathcal{D}_2 & -J(A')^T J & \mathrm{Id}_{r-k} \end{bmatrix}$$

with $J\mathcal{D}_2 \in \mathrm{Sym}_{n-(r-k)}$ and $A' \in \mathrm{Mat}_{n-(r-k)\times r-k}$ with the bottom $n-r$ rows of A' all zero.

Proof. Let $y \in Y_{\widetilde{P}}(\tilde{w}) = (Y_{\widetilde{Q}}(\tilde{w}))^\sigma \subset Y_{\widetilde{Q}}(\tilde{w})$. So y is just an element of $Y_{\widetilde{Q}}(\tilde{w})$ that is fixed under the involution σ. That is, an element which satisfies (2.3.1)–(2.3.3). Theorem 3.15 gives us that y is of the form

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ 0 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ 0 & E'\mathcal{D}_2 & E' & \mathrm{Id}_{r-k} \end{bmatrix}$$

with $\mathcal{D}_2 \in \mathrm{Mat}_{n-(r-k)}$, $A' \in \mathrm{Mat}_{n-(r-k)\times r-k}$ with the bottom $n-r$ rows of A' all zero, and $E' \in \mathrm{Mat}_{r-k\times n-(r-k)}$ with the left $n-r$ columns of E' all zero. We must now check what restrictions on y are required for it to satisfy (2.3.1)–(2.3.3). For y to satisfy (2.3.3) we know that

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ A' & \mathrm{Id}_{n-(r-k)} \end{bmatrix}^T \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix} \begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ E' & \mathrm{Id}_{n-(r-k)} \end{bmatrix} \left(= \begin{bmatrix} (A')^T J + J E' & J \\ J & 0 \end{bmatrix} \right)$$

must equal

$$\begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}$$

which implies that $E' = -J(A')^T J$.

Any y clearly satisfies (2.3.2). And finally for y to satisfy (2.3.1),

$$\begin{bmatrix} 0 & \mathcal{D}_2 \\ 0 & -J(A')^T J\mathcal{D}_2 \end{bmatrix}^T \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix} \begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ A' & \mathrm{Id}_{n-(r-k)} \end{bmatrix} \left(= \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{D}_2^T J \end{bmatrix} \right)$$

must equal

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ A' & \mathrm{Id}_{n-(r-k)} \end{bmatrix}^T \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathcal{D}_2 \\ 0 & -J(A')^T J\mathcal{D}_2 \end{bmatrix} \left(= \begin{bmatrix} 0 & 0 \\ 0 & J\mathcal{D}_2 \end{bmatrix} \right)$$

which implies that $J\mathcal{D}_2 = \mathcal{D}_2^T J$, or equivalently $J\mathcal{D}_2 \in \mathrm{Sym}_{n-(r-k)}$. □

Remark 3.18. We may identify $O^-_{P/\widetilde{P}}$ with $O^-_{\mathrm{GL}_n/P'_{\widehat{r-k}}}$ under the map

$$\begin{bmatrix} \mathcal{A} & 0 \\ 0 & J(\mathcal{A}^T)^{-1}J \end{bmatrix} \mapsto \mathcal{A}.$$

Remark 3.19. Let V_w be the linear subspace of Sym_n given by $x_{ij} = 0$ if $j \le r-k$ or $i < n-(r-k)$. And let V'_w be the linear subspace of $O^-_{\mathrm{GL}_n/P'_{\widehat{r-k}}}$ given by $x_{ij} = 0$ if $i > r$ and $j \le r-k$.

Consider the map $\delta : Y_{\widetilde{P}}(\tilde{w}) \hookrightarrow O^-_{G/\widetilde{P}} = O^-_{G/P} \times O^-_{P/\widetilde{P}} \cong O^-_{G/P} \times O^-_{\mathrm{GL}_n/P'_{\widehat{r-k}}}$, where the first map is inclusion, the second is simply the product decomposition, and the final map is from Remark 3.18. This map is given explicitly by

$$\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ A' & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ 0 & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ 0 & -J(A')^T J\mathcal{D}_2 & -J(A')^T J & \mathrm{Id}_{r-k} \end{bmatrix} \mapsto$$

$$\left(\begin{bmatrix} \mathrm{Id}_{r-k} & 0 & 0 & 0 \\ 0 & \mathrm{Id}_{n-(r-k)} & 0 & 0 \\ -\mathcal{D}_2 A' & \mathcal{D}_2 & \mathrm{Id}_{n-(r-k)} & 0 \\ J(A')^T J\mathcal{D}_2 A' & -J(A')^T J\mathcal{D}_2 & 0 & \mathrm{Id}_{r-k} \end{bmatrix}, \begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ A' & \mathrm{Id}_{n-(r-k)} \end{bmatrix} \right).$$

Consider the isomorphism $\gamma : O^-_{G/P} \times O^-_{\mathrm{GL}_n/P'_{\widehat{r-k}}} \to \mathrm{Sym}_n \times O^-_{\mathrm{GL}_n/P'_{\widehat{r-k}}}$ (cf. Remark 2.2.6) given by

$$\left(\begin{bmatrix} \mathrm{Id}_n & 0 \\ L & \mathrm{Id}_n \end{bmatrix}, \begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ N & \mathrm{Id}_{n-(r-k)} \end{bmatrix} \right) \mapsto \left((LN)^T JN, \begin{bmatrix} \mathrm{Id}_{r-k} & 0 \\ N & \mathrm{Id}_{n-(r-k)} \end{bmatrix} \right).$$

We have that under the map $\gamma \circ \delta$, $Y_{\widetilde{P}}(\tilde{w})$ gets identified with $V_w \times V'_w$. This follows by a simple computation and Corollary 3.17.

Definition 3.20. Now let $Z_{\widetilde{P}}(\tilde{w}) := Y_P(w) \times_{X_P(w)} X_{\widetilde{P}}(\tilde{w})$. Then $Z_{\widetilde{P}}(\tilde{w}) = (O^-_{G/P} \times P/\widetilde{P}) \cap X_{\widetilde{P}}(\tilde{w})$. Hence $Z_{\widetilde{P}}(\tilde{w})$ is smooth, being open in the smooth $X_{\widetilde{P}}(\tilde{w})$ (cf. Proposition 3.3).

Write p for the composite map $Z_{\widetilde{P}}(\tilde{w}) \to O^-_{G/P} \times P/\widetilde{P} \to P/\widetilde{P}$ ($\cong \mathrm{GL}_n / P'_{\widehat{r-k}}$) where the first map is the inclusion and the second map is the projection. Using Proposition 2.3.4(c) and (d) we see that

$$p\left(\begin{bmatrix} A & 0 \\ D & J(A^T)^{-1}J \end{bmatrix} \pmod{\widetilde{P}} \right) = A \pmod{P'_{\widehat{r-k}}}.$$

Note that A is invertible by 2.3.4(b).

Using the injective map

$$A \in B_n \longmapsto \begin{bmatrix} A & 0_{n\times n} \\ 0_{n\times n} & J(A^T)^{-1}J \end{bmatrix} \in B_G,$$

B_n can be thought of as a subgroup of B_G. With this identification we have the following proposition.

Proposition 3.21. *$Z_{\widetilde{P}}(\tilde{w})$ is B_n-stable for the action on the left by multiplication. Further p is B_n equivariant.*

Proof. Let $z \in \mathrm{SP}_{2n}$ such that $z\widetilde{P} \in Z_{\widetilde{P}}(\tilde{w})$. Then by Proposition 2.3.4(c) we may write

$$z = \begin{bmatrix} A & 0 \\ D & J(A^T)^{-1}J \end{bmatrix} \bmod \widetilde{P},$$

such that $z\widetilde{P} \in Z_{\widetilde{P}}(\tilde{w})$. Since $X_{B_G}(\tilde{w}) \to X_{\widetilde{P}}(\tilde{w})$ is surjective, we may assume that $z \pmod{B_G} \in X_{B_G}(\tilde{w})$, i.e., $z \in \overline{B_G \tilde{w} B_G}$. Then for every $A' \in B_n$:

$$\begin{bmatrix} A' & 0_{n\times n} \\ 0_{n\times n} & J(A'^T)^{-1}J \end{bmatrix} z = \begin{bmatrix} A'A & 0 \\ J(A'^T)^{-1}JD & J(A'^T)^{-1}(A'^T)^{-1}J \end{bmatrix} =: z'.$$

Then $z' \in \overline{B_G \tilde{w} B_G}$, so $z' \pmod{\widetilde{P}} \in X_{\widetilde{P}}(\tilde{w})$. By Proposition 2.3.4(b), we have that A is invertible, and hence AA'. This implies again by Proposition 2.3.4(b) that $z' \pmod{\widetilde{P}} \in Z_{\widetilde{P}}(\tilde{w})$. Thus $Z_{\widetilde{P}}(\tilde{w})$ is B_n stable. Also $p(A'z) = p(z') = A'A = A'p(z)$. Hence p is B_n-equivariant. □

Theorem 3.22. *With notation as above, let* $w' := (k+1, \dots, r, n, \dots, r+1, k, \dots, 1)$ *be an element of* S_n, *the Weyl group of* GL_n. *Then*:

(a) *The natural map* $X_{\widetilde{P}}(\tilde{w}) \longrightarrow X_P(w)$ *is proper and birational. In particular, the map* $Z_{\widetilde{P}}(\tilde{w}) \longrightarrow Y_P(w)$ *is proper and birational. And therefore,* $Z_{\widetilde{P}}(\tilde{w})$ *is a desingularization of* $Y_P(w)$.

(b) $X_{P'_{\widehat{r-k}}}(w')$ *is the fiber of the natural map* $Z_{\widetilde{P}}(\tilde{w}) \longrightarrow Y_P(w)$ *at* $e_{\mathrm{id}} \in Y_P(w)$.

(c) $X_{P'_{\widehat{r-k}}}(w')$ *is the image of* p. *Further,* p *is a fibration with fiber isomorphic to* V_w.

(d) p *identifies* $Z_{\widetilde{P}}(\tilde{w})$ *as a subbundle of the trivial bundle* $O^-_{G/P} \times X_{P'_{\widehat{r-k}}}(w')$, *which arises as the restriction of the vector bundle on* $\mathrm{GL}_n / P'_{\widehat{r-k}}$ *associated to the* $P'_{\widehat{r-k}}$*-module* V_w (*which, in turn, is a* $P'_{\widehat{r-k}}$*-submodule of* $O^-_{G/P}$).

Proof. (a): The map $X_{\widetilde{P}}(\tilde{w}) \hookrightarrow G/\widetilde{P} \to G/P$ is proper and its (scheme-theoretic) image is $X_P(w)$, hence $X_{\widetilde{P}}(\tilde{w}) \to X_P(w)$ is proper. Birationality follows from the fact that $\tilde{w}$ is the minimal representative of the coset $w\widetilde{P}$.

(b): The fiber at $e_{\mathrm{id}} \in Y_P(w)$ of the map $Y_{\widetilde{P}}(\tilde{w}) \longrightarrow Y_P(w)$ is $0 \times V'_w$, inside $V_w \times V'_w = Y_{\widetilde{P}}(\tilde{w})$. Since $Z_{\tilde{P}}(\tilde{w})$ is the closure of $Y_{\widetilde{P}}(\tilde{w})$ inside $O^-_{G/P} \times P/\widetilde{P}$ and $X_{P'_{\widehat{r-k}}}(w')$ is the closure of V'_w inside $P/\widetilde{P}$ (note that as a subvariety of $O^-_{P/\widetilde{P}}$, $Y_{P'_{\widehat{r-k}}}(w')$ is identified with V'_w), we see that the fiber at e_{id} (belonging to $Y_P(w)$) of $Z_{\widetilde{P}}(\tilde{w}) \longrightarrow Y_P(w)$ is $X_{P'_{\widehat{r-k}}}(w')$.

(c): From Remark 3.19 we have $p(Y_{\widetilde{P}}(\tilde{w})) = V'_w \subseteq X_{P'_{\widehat{r-k}}}(w')$. Since $Y_{\tilde{P}}(\tilde{w})$ is dense inside $Z_{\widetilde{P}}(\tilde{w})$ and $X_{P'_{\widehat{r-k}}}(w')$ is closed in $\mathrm{GL}_n / P'_{\widehat{r-k}}$ we see that $p(Z_{\tilde{P}}(\tilde{w})) \subseteq X_{P'_{\widehat{r-k}}}(w')$. The other inclusion $X_{P'_{\widehat{r-k}}}(w') \subseteq p(Z_{\tilde{P}}(\tilde{w}))$ follows from (b). Hence, $p(Z_{\widetilde{P}}(\tilde{w})) = X_{P'_{\widehat{r-k}}}(w')$. To prove the second assertion of (c) we shall show that for every $A \in \mathrm{GL}_n$ with A mod $P'_{\widehat{r-k}} \in X_{P'_{\widehat{r-k}}}(w')$, we have that $p^{-1}(A \bmod P'_{\widehat{r-k}})$ is isomorphic to V_w.

To prove this we first observe that $p^{-1}(e_{\mathrm{id}})$ is isomorphic to V_w in view of Remark 3.19. Next observe that every B_n-orbit inside $X_{P'_{\widehat{r-k}}}(w')$ meets V'_w (which equals $Y_{P'_{\widehat{r-k}}}(w')$); further p is B_n-equivariant by Proposition 3.21 and hence every fiber is isomorphic to the fiber at e_{id}, i.e., isomorphic to V_w.

(d): Define a right action of GL_n on $O^-_{G/P}$(identified with Sym_n as in Remark 2.2.6) as $g \circ v = g^T v g$ for $g \in \mathrm{GL}_n$, $v \in \mathrm{Sym}_n$. This induces an action of $P'_{\widehat{r-k}}$ on $O^-_{G/P}$ under which V_w is stable. Thus we get the homogeneous bundle

$$\mathrm{GL}_n \times^{P'_{\widehat{r-k}}} V_w \longrightarrow \mathrm{GL}_n / P'_{\widehat{r-k}}.$$

Now to prove the assertion about $Z_{\widetilde{P}}(\tilde{w}))$ being a vector bundle over $X_{P'_{\widehat{r-k}}}(w')$, we will show that there is a commutative diagram given as below, with ψ an isomorphism:

$$\begin{tikzcd}
Z_{\widetilde{P}}(\tilde{w}) \arrow[dr, "\psi"] \arrow[drr, "\phi"] \arrow[ddr, "p"'] & & \\
& (\mathrm{GL}_n \times^{P'_{\widehat{r-k}}} V_w)|_{X_{P'_{\widehat{r-k}}}(w')} \arrow[r] \arrow[d] & \mathrm{GL}_n \times^{P'_{\widehat{r-k}}} V_w \arrow[d, "\alpha"] \\
& X_{P'_{\widehat{r-k}}}(w') \arrow[r, "\beta"] & \mathrm{GL}_n / P'_{\widehat{r-k}}
\end{tikzcd}$$

The map α is the homogeneous bundle map and β is the inclusion map. Define ϕ by

$$\phi : \begin{bmatrix} A & 0_{n\times n} \\ D & J(A^T)^{-1}J \end{bmatrix} \bmod \widetilde{P} \longmapsto (A, D^T J A)/\sim .$$

Using Proposition 2.3.4(c) and Remark 3.19 we conclude the following: ϕ is well-defined and injective; $\beta \cdot p = \alpha \cdot \phi$; hence, by the universal property of products, the map ψ exists; and, finally, the injective map ψ is in fact an isomorphism (by dimension considerations). □

As an immediate consequence of Theorem 3.22 we have

Corollary 3.23. *We have the following realization of Diagram 1.2*:

$$\begin{tikzcd}
Z_{\widetilde{P}}(\tilde{w}) \arrow[r, hook] \arrow[d, "q'"] & O^-_{G/P} \times X_{P'_{\widehat{r-k}}}(w') \arrow[r] \arrow[d, "q"] & X_{P'_{\widehat{r-k}}}(w') \\
Y_P(w) \arrow[r, hook] & O^-_{G/P} &
\end{tikzcd}$$

Proposition 3.24. (1) *The Schubert variety $X_{P'_{\widehat{r-k}}}(w')$ is isomorphic to the Grassmannian $\mathrm{GL}_r / P''_{\widehat{r-k}}$, where $P''_{\widehat{r-k}}$ is the parabolic subgroup in GL_r obtained by omitting α_{r-k}.*

(2) $(\mathrm{GL}_n \times^{P'_{\widehat{r-k}}} V_w)|_{X_{P'_{\widehat{r-k}}}(w')} \cong (\mathrm{GL}_n \times^{P'_{\widehat{r-k}}} V_w)|_{\mathrm{GL}_r / P''_{\widehat{r-k}}} \cong \mathrm{GL}_r \times^{P''_{\widehat{r-k}}} V_w$ *as homogeneous vector bundles.*

Proof. (1): This is clear.

(2): Consider the embedding $i : \mathrm{GL}_r \hookrightarrow \mathrm{GL}_n$ given by

$$R \mapsto \begin{bmatrix} R & 0 \\ 0 & \mathrm{Id}_{n-r} \end{bmatrix}.$$

Define the action of GL_r on Sym_n as the action induced by this embedding. This induces an action of $P''_{\widehat{r-k}}$ on Sym_n. As $i(P''_{\widehat{r-k}}) \subset P'_{\widehat{r-k}}$, the $P'_{\widehat{r-k}}$ stability of V_w implies the $P''_{\widehat{r-k}}$ stability of V_w. Hence our result follows. $\square$

Corollary 3.25. *We have the following realization of Diagram 1.2*:

$$\begin{tikzcd} Z_{\widetilde{P}}(\tilde{w}) \arrow[r, hook] \arrow[d, "q'"] & O^-_{G/P} \times \mathrm{GL}_r / P''_{\widehat{r-k}} \arrow[r] \arrow[d, "q"] & \mathrm{GL}_r / P''_{\widehat{r-k}} \\ Y_P(w) \arrow[r, hook] & O^-_{G/P} & \end{tikzcd}$$

4. Free resolutions

Kempf–Lascoux–Weyman geometric technique. We summarize the geometric technique of computing free resolutions, following [Weyman 2003, Chapter 5].

Consider Diagram 1.1. There is a natural map $f : V \longrightarrow \mathrm{Grass}_{r,d}$ (where $r = \mathrm{rk}_V Z$ and $d = \dim \mathbb{A}$) such that the inclusion $Z \subseteq \mathbb{A} \times V$ is the pull-back of the tautological sequence (2.4.4); here $\mathrm{rk}_V\, Z$ denotes the rank of Z as a vector bundle over V, i.e., $\mathrm{rk}_V\, Z = \dim Z - \dim V$. Let $\xi = (f^*\mathcal{Q})^*$. Write R for the polynomial ring $\mathbb{C}[\mathbb{A}]$ and $\mathfrak{m}$ for its homogeneous maximal ideal. (The grading on R arises as follows. In Diagram 1.1, $\mathbb{A}$ is thought of as the fiber of a trivial vector bundle, so it has a distinguished point, its origin. Now, being a subbundle, Z is defined by linear equations in each fiber; i.e., for each $v \in V$, there exist $s := (\dim \mathbb{A} - \mathrm{rk}_V\, Z)$ linearly independent linear polynomials $\ell_{v,1}, \dots, \ell_{v,s}$ that vanish along Z and define it. Now $Y = \{y \in \mathbb{A} : \text{there exists } v \in V \text{ such that } \ell_{v,1}(y) = \cdots = \ell_{v,s}(y) = 0\}$. Hence Y is defined by homogeneous polynomials. This explains why the resolution obtained below is graded.) Let $\mathfrak{m}$ be the homogeneous maximal ideal, i.e., the ideal defining the origin in $\mathbb{A}$. Then:

Theorem 4.1 [Weyman 2003, Basic Theorem 5.1.2]. *With notation as above, there is a finite complex $(F_\bullet, \partial_\bullet)$ of finitely generated graded free R-modules that is quasi-isomorphic to $\mathbf{R}q'_*\mathcal{O}_Z$, with*

$$F_i = \bigoplus_{j \geq 0} H^j\big(V, \textstyle\bigwedge^{i+j} \xi\big) \otimes_{\mathbb{C}} R(-i-j),$$

and $\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}$. Furthermore, the following are equivalent:

(a) *Y has rational singularities i.e., $\mathbf{R}q'_*\mathcal{O}_Z$ is quasi-isomorphic to $\mathcal{O}_Y$*;

(b) *$F_\bullet$ is a minimal R-free resolution of $\mathbb{C}[Y]$, i.e., $F_0 \simeq R$ and $F_{-i} = 0$ for every $i > 0$.*

A sketch of the proof is given in [Kummini et al. 2015, Section 4], and [Weyman 2003, 5.1.3] may be consulted for a more comprehensive account.

Our situation. We now apply Theorem 4.1 to our situation. We keep the notation of Theorem 3.22. Theorem 4.1 and Corollary 3.25 yield the following result:

Theorem 4.2. *Write ξ for the homogeneous vector bundle on $\mathrm{GL}_r / P''_{\widehat{r-k}}$ associated to the $P''_{\widehat{r-k}}$-module $(O^-_{G/P}/V_w)^*$ (this is the dual of the quotient of $O^-_{G/P} \times \mathrm{GL}_r / P''_{\widehat{r-k}}$ by $Z_{\tilde{P}}(\tilde{w})$). Then we have a minimal R-free resolution $(F_\bullet, \partial_\bullet)$ of $\mathbb{C}[Y_P(w)]$ with*

$$F_i = \bigoplus_{j \geq 0} H^j(\mathrm{GL}_r / P''_{\widehat{r-k}}, \bigwedge^{i+j} \xi) \otimes_{\mathbb{C}} R(-i-j).$$

Computing the cohomology groups required in Theorem 4.2 in the general situation is a difficult problem. Techniques for computing them in our specific case are discussed in the following section.

5. Cohomology of homogeneous vector bundles

We have shown in Theorem 4.2 that the calculation of a minimal R-free resolution of $\mathbb{C}[Y_P(w)]$ comes down to the computation of the cohomology of certain homogeneous bundles over $\mathrm{GL}_r / P''_{\widehat{r-k}}$. In particular we need to calculate

$$H^\bullet\big(\mathrm{GL}_r / P''_{\widehat{r-k}}, \textstyle\bigwedge^t \xi\big) \tag{5.1}$$

for arbitrary t.

The $P''_{\widehat{r-k}}$-module $(O^-_{G/P}/V_w)^*$ is not completely reducible (the unipotent radical of $P''_{\widehat{r-k}}$ does not act trivially), and thus we can not use the Bott algorithm to compute its cohomology. In [Ottaviani and Rubei 2006] the authors determine the cohomology of general homogeneous bundles on Hermitian symmetric spaces. As $\mathrm{GL}_r / P''_{\widehat{r-k}}$ is such a space their results could be used to determine (5.1). In practice, proceeding along these lines is possible though extremely complicated.

Another approach to the calculation of these cohomologies comes from using a technique employed in [Weyman 2003, Chapter 6.3]. There the minimal R-free resolution of a related space is computed and the minimal R-free resolution of $\mathbb{C}[Y_P(w)]$ can be seen as a subresolution. In [Weyman 2003] this method is used for the case when $n = r$. That is, the case where $Y_P(w)$ is the symmetric determinental variety. In this case the authors assume that $k = 2u$ (the odd case can be reduced to this even case). They look at the subspace T_w of Sym_n given by symmetric matrices of block form

$$\begin{bmatrix} 0_{n-u \times n-u} & R \\ R^T & S_{u \times u} \end{bmatrix}.$$

Let $P'_{\widehat{n-u}}$ be the parabolic subgroup of GL_n omitting the root α_{n-u}, then T_w is a $P'_{\widehat{n-u}}$-module under the same action. If Z_w is the homogeneous vector bundle associated with T_w we have the following diagram

$$\begin{tikzcd} Z_w \arrow[r, hook] \arrow[d, "q'"] & \mathrm{Sym}_n \times \mathrm{GL}_n / P'_{\widehat{n-u}} \arrow[r] \arrow[d, "q"] & \mathrm{GL}_n / P'_{\widehat{n-u}} \\ Y \arrow[r, hook] & \mathrm{Sym}_n & \end{tikzcd}$$

They show that the resolution of $\mathbb{C}[Y_P(w)]$ can be realized as a subresolution of the resolution of $\mathbb{C}[Y]$. In this case, the $P'_{\widehat{n-u}}$-module $(\mathrm{Sym}_n / T_w)^*$ (this is the dual of the quotient of $\mathrm{Sym}_n \times \mathrm{GL}_n / P'_{\widehat{n-u}}$ by Z_w) is completely reducible and thus the cohomology of the corresponding homogeneous vector bundles $\bigwedge^t \xi$ may be computed using the Bott algorithm, leading to this:

Theorem 5.2 [Weyman 2003, Theorem 6.3.1(c)]. *The i-th term G_i of the minimal free resolution of $\mathbb{C}[Y_P(w)]$ as an R module is given by the formula*

$$G_i = \bigoplus_{\substack{\lambda \in Q_{k-1}(2t) \\ \mathrm{rank}\,\lambda \text{ even} \\ i = t - k\frac{1}{2}\mathrm{rank}\,\lambda}} S_{\lambda^\vee}\mathbb{C}^n \otimes_{\mathbb{C}} R.$$

Here $Q_{k-1}(2t)$ is the set of partitions λ of $2t$ which in hook notation can be written as $\lambda = (a_1, \dots, a_s | b_1, \dots, b_s)$, where s is a positive integer, and for each j we have $a_j = b_j + (k-1)$. And $\lambda^\vee$ is the conjugate (or dual) partition of λ. And finally, rank λ is defined as being equal to l, where the largest square fitting inside λ is of size $l \times l$.

Similar methods may be used to compute a closed form formula for the minimal free resolution of $\mathbb{C}[Y_P(w)]$ as an R module in the case $r \neq n$.

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DISTINGUISHED UNIPOTENT ELEMENTS AND MULTIPLICITY-FREE SUBGROUPS OF SIMPLE ALGEBRAIC GROUPS

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In memory of Robert Steinberg, whose elegant mathematics continues to inspire us.

For G a simple algebraic group over an algebraically closed field of characteristic 0, we determine the irreducible representations $\rho : G \to I(V)$, where $I(V)$ denotes one of the classical groups $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$, $\mathrm{SO}(V)$, such that ρ sends some distinguished unipotent element of G to a distinguished element of $I(V)$. We also settle a base case of the general problem of determining when the restriction of ρ to a simple subgroup of G is multiplicity-free.

1. Introduction

Let G be a simple algebraic group of rank at least 2 defined over an algebraically closed field of characteristic 0 and let $\rho : G \to I(V)$ be an irreducible representation, where $I(V)$ denotes one of the classical groups $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$, or $\mathrm{SO}(V)$. In this paper we consider two closely related problems. We determine those representations for which some distinguished unipotent element of G is sent to a distinguished element of $I(V)$. Also we settle a base case of the general problem of determining when the restriction of ρ to a simple subgroup of G is multiplicity-free.

A unipotent element of a simple algebraic group is said to be *distinguished* if it is not centralized by a nontrivial torus. Let $u \in G$ be a unipotent element. If $\rho(u)$ is distinguished in $I(V)$ then u must be distinguished in G. The distinguished unipotent elements of $I(V)$ can be decomposed into Jordan blocks of distinct sizes. Indeed they are a single Jordan block, the sum of blocks of distinct even sizes, or the sum of blocks of distinct odd sizes, according to whether $I(V)$ is $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$, or $\mathrm{SO}(V)$, respectively; see [Liebeck and Seitz 2012, Proposition 3.5].

Now u can be embedded in a subgroup A of G of type A_1 by the Jacobson–Morozov theorem; given u, the subgroup A is unique up to conjugacy in G. If $\rho(u)$ is distinguished, then $\rho(A)$ acts on V with irreducible summands of the same dimensions as the Jordan blocks of u, and hence the restriction $V \downarrow \rho(A)$ is

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multiplicity-free — that is, each irreducible summand appears with multiplicity 1. Indeed, $V \downarrow \rho(A)$ is either irreducible, or the sum of irreducibles of distinct even dimensions or of distinct odd dimensions.

Our main result determines those situations where $V \downarrow \rho(A)$ is multiplicity-free. In order to state it, we recall that a subgroup of G is said to be *G-irreducible* if it is contained in no proper parabolic subgroup of G. It follows directly from the definition that an A_1 subgroup of G is G-irreducible if and only if its nonidentity unipotent elements are distinguished in G. If these unipotent elements are regular in G, we call the subgroup a *regular A_1* in G.

Theorem 1. *Let G be a simple algebraic group of rank at least 2 over an algebraically closed field K of characteristic zero, let $A \cong A_1$ be a G-irreducible subgroup of G, let $u \in A$ be a nonidentity unipotent element, and let V be an irreducible KG-module of highest weight λ. Then $V \downarrow A$ is multiplicity-free if and only if λ and u are as in Tables 1 or 2, where λ is given up to graph automorphisms of G. Table 1 lists the examples where u is regular in G, and Table 2 lists those where u is nonregular.*

Theorem 1 is the base case of a general project in progress, which aims to determine all irreducible KG-modules V and G-irreducible subgroups X of G for which $V \downarrow X$ is multiplicity-free.

The answer to the original question on distinguished unipotent elements is as follows.

Corollary 2. *Let G be as in the theorem, and let $\rho : G \to I(V)$ be an irreducible representation with highest weight λ, where $I(V)$ is* $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$, *or* $\mathrm{SO}(V)$. *Let $u \in G$ be a nonidentity unipotent element, and suppose that $\rho(u)$ is a distinguished element of $I(V)$.*

(i) *If $I(V) = \mathrm{SL}(V)$, then $G = A_n$, B_n, C_n, or G_2, and $\lambda = \omega_1$ (or ω_n if $G = A_n$); moreover, u is regular in G.*

(ii) *If $I(V) = \mathrm{Sp}(V)$ or $I(V) = \mathrm{SO}(V)$, then λ and u are as in one of the cases in Tables 1 or 2, for which $V = V_G(\lambda)$ is a self-dual module (equivalently, $\lambda = -w_0(\lambda)$, where w_0 is the longest element of the Weyl group of G). Conversely, for each such case in the tables, $\rho(u)$ is distinguished in $I(V)$.*

The layout of the paper is as follows. Section 2 consists of notation and preliminary lemmas. This is followed by Sections 3, 4, 5, where we prove Theorem 1 in the special case where A is a regular A_1 subgroup of G. Then in Section 6 we consider the remaining cases where A is nonregular. There are far fewer examples in that situation. Finally, Section 7 contains the proof of the corollary.

G	λ
A_n	$\omega_1,\ \omega_2,\ 2\omega_1,\ \omega_1+\omega_n,$ $\omega_3\ (5\le n\le 7),$ $3\omega_1\ (n\le 5),\ 4\omega_1\ (n\le 3),\ 5\omega_1\ (n\le 3)$
A_3	110
A_2	$c1,\ c0$
B_n	$\omega_1,\ \omega_2,\ 2\omega_1,$ $\omega_n\ (n\le 8)$
B_3	$101,\ 002,\ 300$
B_2	$b0,\ 0b\,(1\le b\le 5),\ 11,\ 12,\ 21$
C_n	$\omega_1,\ \omega_2,\ 2\omega_1,$ $\omega_3\ (3\le n\le 5),$ $\omega_n\ (n=4,5)$
C_3	300
C_2	$b0,\ 0b\,(1\le b\le 5),\ 11,\ 12,\ 21$
$D_n\ (n\ge 4)$	$\omega_1,\ \omega_2\ (n=2k+1),\ 2\omega_1\ (n=2k),$ $\omega_n\ (n\le 9)$
E_6	$\omega_1,\ \omega_2$
E_7	$\omega_1,\ \omega_7$
E_8	ω_8
F_4	$\omega_1,\ \omega_4$
G_2	$10,\ 01,\ 11,\ 20,\ 02,\ 30$

Table 1. $V\downarrow A$ multiplicity-free, $u\in G$ regular in G.

G	λ	class of u in G
$B_n,\ C_n,\ D_n$	ω_1	any
$D_n\ (5\le n\le 7)$	ω_n	regular in $B_{n-2}B_1$
F_4	ω_4	$F_4(a_1)$
E_6	ω_1	$E_6(a_1)$
E_7	ω_7	$E_7(a_1)$ or $E_7(a_2)$
E_8	ω_8	$E_8(a_1)$

Table 2. $V\downarrow A$ multiplicity-free, $u\in G$ distinguished but not regular.

For many of the proofs we need to calculate dimensions of weight spaces in various G-modules. When the rank of G is small, such dimensions can be computed using Magma [Bosma et al. 1997], and we make occasional use of this facility.

2. Preliminary lemmas

Continue to let G be a simple algebraic group over an algebraically closed field K of characteristic zero. Let $A \cong A_1$ be a G-irreducible subgroup of G, let u be a nonidentity unipotent element of A, and let $T < A$ be a 1-dimensional torus such that the conjugates of u under T form the nonidentity elements of a maximal unipotent group of A.

We fix some notation that will be used throughout the paper. Let $T \leq T_G$, where T_G is a maximal torus of G and let $\Pi_G = \{\alpha_1, \ldots, \alpha_n\}$ denote a fundamental system of roots. We label the nodes of the Dynkin diagram of G with these roots as in [Bourbaki 1968, p. 250]. Write s_i for the reflection in α_i, an element of the Weyl group $W(G)$. When $G = D_n$ we assume that $n \geq 4$ (and regard D_3 as the group A_3).

The torus T determines a labelling of the Dynkin diagram by 0s and 2s (see [Liebeck and Seitz 2012, Theorem 3.18 and Table 13.2]), which gives the weights of T on fundamental roots. When u is regular in G these labels are all 2s.

Denote by $\omega_1, \ldots, \omega_n$ the fundamental dominant weights of G. For a dominant weight $\lambda = \sum c_i\omega_i$, let $V_G(\lambda)$ be the irreducible KG-module of highest weight λ. For $A \cong A_1$ and a nonnegative integer r, we abbreviate the irreducible module $V_A(r)$ by V_r or just r. More generally we frequently denote the module $V_G(\lambda)$ by just the weight λ, or the string $c_1 \cdots c_l$ (where l is the rank).

Let $V = V_G(\lambda)$ and let λ afford weight r when restricted to T. Since all weights of V can be obtained by subtracting roots from the highest weight, the restriction of each weight to T has the form $r - 2k$ for some nonnegative integer k. If $V \downarrow A$ is multiplicity-free, then $V \downarrow A = V_{r_1} + V_{r_2} + V_{r_3} + \cdots$, where $r = r_1 > r_2 > r_3 > \cdots$. Then the T-weights on V are

$$(r_1, r_1 - 2, \ldots, -r_1), \quad (r_2, r_2 - 2, \ldots, -r_2), \quad (r_3, r_3 - 2, \ldots, -r_3), \quad \ldots.$$

Note that weight r, respectively $r - 2$, arises as the restriction of $\lambda - \alpha_i$ for those i having label 0, resp. 2, and with $c_i > 0$. Therefore, if $c_i > 0$ then α_i has label 2, and there can be at most two values of i with $c_i > 0$.

We often use the following short hand notation. We simply write $\lambda - i^x j^y k^z \cdots$ rather than $\lambda - x\alpha_i - y\alpha_j - z\alpha_k - \cdots$.

Lemma 2.1. *If $V \downarrow A$ is multiplicity-free, then* $\dim V \leq \left(\frac{r}{2} + 1\right)^2$ *or* $\left(\frac{r+1}{2}\right)\left(\frac{r+3}{2}\right)$, *according as r is even or odd, respectively.*

Proof. If $V \downarrow A$ is multiplicity-free, then $V \downarrow A$ is a direct summand of the module $r + (r-2) + (r-4) + \cdots$. The assertion follows by taking dimensions. □

Lemma 2.2. *Assume $V \downarrow A$ is multiplicity-free.*

(i) *If $c \geq 1$ then the T-weight $r - 2c$ occurs with multiplicity at most one more than the multiplicity of the T-weight $r - 2(c-1)$.*

(ii) *For* $c \geq 1$, *the* T*-weight* $r-2c$ *occurs with multiplicity at most* $c+1$.

(iii) *If the* T*-weight* $r-2$ *occurs with multiplicity* 1, *e.g., if all labels are* 2 *and* $\lambda = b\omega_i$, *and if* $c \geq 1$, *then the* T*-weight* $r-2c$ *occurs with multiplicity at most* c.

Proof. Suppose i is maximal with $r-2c$ in the weight string $r_i, \dots, -r_i$. Then T-weight $r-2c$ occurs with the same multiplicity as does T-weight r_i. And weight r_i occurs with multiplicity at most one more than weight r_{i-1} as otherwise there would be two direct summands of highest weight r_i. Now (i) follows as does (ii). Part (iii) also follows, since the assumption rules out a summand of highest weight $r-2$. □

Lemma 2.3. *Assume* $V \downarrow A$ *is multiplicity-free and that* $\lambda = b\omega_i$ *with* $b > 1$.

(i) *Then* α_i *is an end-node of the Dynkin diagram.*

(ii) *If* G *has rank at least* 3, *then the node adjacent to* α_i *has label* 2.

Proof. (i) Suppose that $\alpha_j \neq \alpha_k$ both adjoin α_i in the Dynkin diagram. If both these roots have label 0, then T-weight $r-2$ is afforded by each of $\lambda - i$, $\lambda - ij$, $\lambda - ik$, $\lambda - ijk$, contradicting Lemma 2.2(ii). Next assume α_j has label 2 and α_k has label 0. Here we consider $r-4$ which is afforded by $\lambda - i^2$, $\lambda - i^2k$, $\lambda - i^2k^2$, $\lambda - ij$, again contradicting Lemma 2.2(ii). If both labels are 2, then $r-4$ is afforded by $\lambda - i^2$, $\lambda - ij$, $\lambda - ik$. But here $r-2$ only occurs from $\lambda - \alpha_i$, so this contradicts Lemma 2.2(iii).

(ii) Assume G has rank at least 3. By (i) α_i is an end-node. Let α_j be the adjoining node. We must show α_j has label 2. Suppose the label is 0 and let α_k be another node adjoining α_j. If α_k has label 0, then $r-2$ is afforded by each of $\lambda - i$, $\lambda - ij$, $\lambda - ijk$, a contradiction. Therefore α_k has label 2. But then $r-4$ is afforded by each of $\lambda - i^2$, $\lambda - i^2j$, $\lambda - i^2j^2$, $\lambda - ijk$, a contradiction. □

The next lemma will be frequently used, often implicitly, in what follows.

Lemma 2.4. *If* $c \geq d \geq 0$ *are integers, then the tensor product* $c \otimes d$ *of* A_1*-modules decomposes as* $c \otimes d = (c+d) \oplus (c+d-2) \oplus \cdots \oplus (c-d)$.

Proof. This follows from a consideration of weights in the tensor product. □

Lemma 2.5. *Suppose that* $\lambda = \omega_i + \omega_j$ *with* $j > i$ *and that the subdiagram with base* $\{\alpha_i, \dots, \alpha_j\}$ *is of type* A, *or is of rank at most* 3, *or is of type* F_4. *Then the* T_G*-weight* $\lambda - i(i+1)\cdots j$ *occurs with multiplicity* $j-i+1$.

Proof. Since the weight space lies entirely within the corresponding irreducible for the Levi factor with base $\{\alpha_i, \dots, \alpha_j\}$, we may assume that G is equal to this Levi factor; that is, $i = 1$ and $j = n$. Then the hypothesis of the lemma implies that G is A_n, B_2, B_3, C_2, C_3, G_2 or F_4. For all but the first case the conclusion follows by computation using Magma.

Now suppose $G = A_n$. Then $\omega_1 \otimes \omega_n = \lambda \oplus 0$. In the tensor product we see precisely $n+1$ times the weight $\lambda - \alpha_1 - \cdots - \alpha_n$ by taking weights of the form

$(\omega_1 - 1 \cdots j) \otimes (\omega_n - (j+1) \cdots n)$ for $1 \leq j \leq n-1$, together with the weights $\omega_1 \otimes (\omega_n - 1 \cdots n)$ and $(\omega_1 - 1 \cdots n) \otimes \omega_n$. Each occurs with multiplicity 1, so the conclusion follows, as $\lambda - \alpha_1 - \cdots - \alpha_n = 0$. □

Lemma 2.6. *Assume that there exist $i < j$ with $c_i \neq 0 \neq c_j$ and that $V \downarrow A$ is multiplicity-free.*

(i) *Then $c_k = 0$ for $k \neq i, j$.*

(ii) *Nodes adjoining α_i and α_j have label* 2.

(iii) *Either $c_i = 1$ or $c_j = 1$. Moreover, $c_i = c_j = 1$ unless α_i and α_j are adjacent.*

(iv) *Either α_i or α_j is an end-node.*

(v) *If either $c_i > 1$ or $c_j > 1$, then G has rank* 2.

(vi) *If α_i, α_j are nonadjacent and if all nodes have label* 2, *then both α_i and α_j are end-nodes.*

Proof. (i) This is immediate, as otherwise $\lambda - i$, $\lambda - j$, $\lambda - k$ all afford T-weight $r-2$, contradicting Lemma 2.2(ii).

(ii) Suppose (ii) is false. By symmetry we can assume α_k adjoins α_i and has label 0. Then $\lambda - i$, $\lambda - j$, $\lambda - ik$ all afford $r-2$, a contradiction.

(iii) By (ii), nodes adjacent to α_i and α_j have label 2. Consider T-weight $r-4$ which has multiplicity at most 3 by Lemma 2.2. Suppose $c_k > 1$ for $k = i$ or j. Then $\lambda - k^2$ and $\lambda - ij$ both afford weight $r-4$. Assume α_i and α_j are not adjacent. We give the argument when the diagram has no triality node. The other cases require only a slight change of notation. With this assumption we also get $r-4$ from $\lambda - i(i+1)$ and $\lambda - (j-1)j$, a contradiction. So $c_k > 1$ implies that α_i, α_j are adjacent. If both $c_i > 1$ and $c_j > 1$, then we again have a contradiction, since $r-4$ is afforded by $\lambda - i^2$, $\lambda - j^2$, and $\lambda - ij$, and the latter appears with multiplicity 2 by [Testerman 1988, §1.35].

(iv) Suppose neither α_i nor α_j is an end-node. We give details assuming there is no triality node. The remaining cases just require a slight change of notation. Consider weight $r-4$. This is afforded by $\lambda - ij$, $\lambda - (i-1)i$, and $\lambda - j(j+1)$. If $c_i > 1$ then $\lambda - i^2$ also affords $r-4$. This forces $c_i = 1$, and similarly $c_j = 1$. If $j = i+1$, then $\lambda - ij$ has multiplicity 2 by Lemma 2.5, again a contradiction. And if $j > i+1$, then $\lambda - i(i+1)$ and $\lambda - (j-1)j$ afford weight $r-4$. In either case $r-4$ appears with multiplicity at least 4, contradicting Lemma 2.2.

(v) Suppose $c_k > 1$ for $k = i$ or j. By (iv) we can assume α_i is an end-node. If G has rank at least 3, let α_l adjoin α_j, where $l \neq i$. Then (ii) implies that $r-4$ is afforded by $\lambda - ij$, $\lambda - k^2$, $\lambda - jl$. If α_j is adjacent to α_i then the first weight occurs with multiplicity 2 by [loc. cit.]. Otherwise there is another node α_m adjacent to α_i and $\lambda - im$ affords $r-4$. In either case we contradict Lemma 2.2.

(vi) As above we treat the case where the Dynkin diagram has no triality node. By (iv) and symmetry we can assume α_i is an end-node. Suppose $j < n$. Then $r-4$ is afforded by each of $\lambda - i(i+1)$, $\lambda - (j-1)j$, $\lambda - j(j+1)$, $\lambda - ij$, contradicting Lemma 2.2. Therefore, $j = n$. □

Lemma 2.7. *Suppose $\lambda = \omega_i$ and the Dynkin diagram has a string $\alpha_{i-3}, \dots, \alpha_{i+3}$ for which each node has T-label 2. Then $r-8$ occurs with multiplicity at least 5. In particular $V \downarrow A$ is not multiplicity-free.*

Proof. The T-weight $r-8$ arises from each of the following weights:

$$\lambda - i(i+1)(i+2)(i+3), \quad \lambda-(i-1)i(i+1)(i+2), \quad \lambda-(i-2)(i-1)i(i+1),$$
$$\lambda-(i-3)(i-2)(i-1)i, \quad \lambda-(i-1)i^2(i+1);$$

the last is a weight as it is equal to $(\lambda - (i-1)i(i+1))^{s_i}$. This proves the first assertion and the second assertion follows from Lemma 2.2(iii). □

The final lemma is an inductive tool. Let L be a Levi subgroup of G in our fixed system of roots, and let μ be the corresponding highest weight of L', namely, $\mu = \sum c_j \omega_j$, where the sum runs just over those fundamental weights corresponding to simple roots in the subsystem determined by L.

Lemma 2.8. *Fix $c \geq 1$ and let s denote the sum of the dimensions of all weight spaces of $V_{L'}(\mu)$ for all weights of form $\mu - \sum d_j \alpha_j$ such that $\sum d_j = c$ and each α_j such that $d_j \neq 0$ has label 2.*

(i) *If $s > c+1$, then $V \downarrow A$ is not multiplicity-free.*

(ii) *If T-weight $r-2$ occurs with multiplicity 1 (e.g., if all labels are 2 and $\lambda = b\omega_i$) and $s > c$, then $V \downarrow A$ is not multiplicity-free.*

Proof. This is immediate from Lemma 2.2, since $T \leq L$ and the weight $\mu - \sum d_j \alpha_j$ corresponds to a weight $\lambda - \sum d_j \alpha_j$ which affords T-weight $r - 2c$. □

3. The case where A is regular and $\lambda \neq c\omega_i$

As in the hypothesis of Theorem 1, let G be a simple algebraic group of rank at least 2, let $A \cong A_1$ be a G-irreducible subgroup, and let $V = V_G(\lambda)$, where $\lambda = \sum c_i \lambda_i$. This section and the next two concern the case of Theorem 1 where A is a regular A_1 of G (recall that this means that unipotent elements of A are regular in G). In this case all the T-labels of the Dynkin diagram of G are equal to 2. In this section we handle situations where $c_i > 0$ for at least two values of i.

If $V \downarrow A$ is multiplicity-free, $\lambda \neq c\omega_i$, and G has rank at least 3, then Lemma 2.6 implies that $\lambda = \omega_i + \omega_j$, where either α_i, α_j are both end-nodes, or one is an end-node and the other is adjacent to it.

Proposition 3.1. *Assume $V \downarrow A$ is multiplicity-free. Then there exist at least two values of i for which $c_i > 0$ if and only if G and λ are in the following table, up to graph automorphisms.*

G	λ
A_2	$c1$
A_3	110
$B_2,\ C_2$	$11,\ 12,\ 21$
G_2	11
B_3	101
A_n	$10\cdots01$

The proof will be in a series of lemmas.

Lemma 3.2. *Suppose $G = A_2$ and $\lambda = c1$ for $c \geq 1$. Then $V \downarrow A$ is multiplicity-free.*

Proof. Assume $G = A_2$. The weight $c1-\alpha_1-\alpha_2 = (c-1)0$ occurs with multiplicity 2 in the module $c1$ and multiplicity 3 in $c0 \otimes 01$. A dimension comparison shows that $c0 \otimes 01 = c1 + (c-1)0$.

Now $c0 = S^c(10)$, so weight considerations show that for c even, $S^c(10) \downarrow A = 2c \oplus (2c-4) \oplus (2c-8) \oplus \cdots \oplus 0$ and $S^{c-1}(10) = (2c-2) \oplus (2c-6) \oplus \cdots \oplus 2$. Therefore, Lemma 2.4 implies that

$$(c0 \otimes 01) \downarrow A = ((2c+2) + 2c + (2c-2)) + ((2c-2) + (2c-4) + (2c-6)) + \cdots + (6+4+2) + 2,$$

and it follows from the first paragraph that $V \downarrow A$ is multiplicity free. A similar argument applies for c odd. □

Lemma 3.3. *(i) If $G = C_2$ and $V = V_G(\lambda)$ with $\lambda = c1$ or $1c$ for $c \geq 1$, then $V \downarrow A$ is multiplicity-free if and only if $\lambda = 11, 21,$ or 12.*

(ii) If $G = G_2$ and $V = V_G(\lambda)$ with $\lambda = c1$ or $1c$ for $c \geq 1$, then $V \downarrow A$ is multiplicity-free if and only if $\lambda = 11$.

Proof. (i) Let $G = C_2$. We first settle the cases which are multiplicity-free. A Magma computation shows that $10 \otimes 01 = 11 + 10$, and hence $11 \downarrow A = 7 + 5 + 1$, which is multiplicity-free. Next consider $\lambda = 12$. First note that $10 \otimes 02 = 12 + 11$ and $02 = S^2(01) - 00$. It follows that

$$\begin{aligned} 12 \downarrow A &= 3 \otimes (S^2(4) - 0) - (7+5+1) = 3 \otimes (8+4) - (7+5+1) \\ &= (11+9+7+5) + (7+5+3+1) - (7+5+1) = 11+9+7+5+3 \end{aligned}$$

and $V \downarrow A$ is multiplicity-free. Finally, consider $\lambda = 21$. In this case $20 \otimes 01 = 21 + 20 + 01$. Now $20 \downarrow A = S^2(3) = 6 + 2$, so that $(20 \otimes 01) \downarrow A = (6+2) \otimes 4 =$

$(10+8+6+4+2)+(6+4+2)$. It follows that $21\downarrow A = 10+8+6+4+2$ and $V\downarrow A$ is multiplicity-free.

If $\lambda = 1b$ for $b \geq 3$, then $r = 3+4b$ and $\dim V = \frac{1}{3}(b+1)(b+3)(2b+4)$. Similarly, if $\lambda = b1$ for $b \geq 3$, then $r = 3b+4$ and $\dim V = \frac{1}{3}(b+1)(b+3)(b+5)$, Now Lemma 2.1 shows that $V\downarrow A$ cannot be multiplicity-free.

(ii) Let $G = G_2$. First consider $\lambda = 11$. A Magma computation yields $10\otimes 01 = 11+20+10$. Also, $10\downarrow A = 6$ and $01\downarrow A = 10+2$. Using the fact that $S^2(10) = 20+00$, we find that $V\downarrow A = 16+14+10+8+6+4$, which is multiplicity-free.

If $\lambda = c1$ with $c > 1$, then $\dim V = \frac{1}{60}(c+1)(c+3)(c+5)(c+7)(2c+8)$ and $r = 6c+10$. Similarly, if $\lambda = 1c$ with $c > 1$, then $r = 10c+6$ and $\dim V = \frac{1}{60}(c+1)(c+3)(2c+4)(3c+5)(3c+7)$. In either case, Lemma 2.1 shows that $V\downarrow A$ is not multiplicity-free. $\square$

Lemma 3.4. *Suppose G has rank at least 3 and $\lambda = \omega_i + \omega_j$, where α_i, α_j are adjacent and one of them is an end-node. Then $V\downarrow A$ is multiplicity-free if and only if $G = A_3$.*

Proof. First assume that $G = A_n$, B_n, C_n or D_n and $\lambda = \omega_1+\omega_2$. If $n \geq 4$, then the weights $\lambda-123 = (\lambda-12)^{s_3}$, $\lambda-234$, $\lambda-1^22 = (\lambda-2)^{s_1}$, $\lambda-12^2 = (\lambda-1)^{s_2}$ occur with multiplicities 2, 1, 1, 1 and all afford T weight $r-6$. Hence this weight occurs with multiplicity at least 5, and Lemma 2.2 shows that $V\downarrow A$ is not multiplicity-free. If $G = B_3$ or C_3, then of the above weights only $\lambda-234$ does not occur; however the weight $\lambda-23^2 = (\lambda-2)^{s_3}$ or $\lambda-2^23 = (\lambda-23)^{s_2}$ occurs, respectively, affording T weight $r-6$, which again gives the conclusion by Lemma 2.2. And if $G = A_3$, then $100\otimes 010 = 110+001$, and restricting to A we have $3\otimes(4+0) = (7+5+3+1)+3$. Therefore, $110\downarrow A = 7+5+3+1$ which is multiplicity-free, as in the conclusion.

Next consider $G = B_n$ or C_n with $\lambda = \omega_{n-1}+\omega_n$. For B_n, the weight $r-6$ is afforded by $\lambda-(n-2)(n-1)n$, $\lambda-(n-1)n^2 = (\lambda-(n-1)n)^{s_n}$, and $(\lambda-(n-1)^2n) = (\lambda-n)^{s_{n-1}}$. Moreover the first two weights occur with multiplicity 2, and so $r-6$ appears with multiplicity 5, so that $V\downarrow A$ is not multiplicity-free. A similar argument applies for C_n.

For $G = F_4$, the conclusion follows by using Lemma 2.8, applied to a Levi subgroup B_3 or C_3. Likewise, for D_n ($n \geq 5$) with $\lambda = \omega_n+\omega_{n-2}$ or $\omega_{n-1}+\omega_{n-2}$, or for $G = E_n$, we use a Levi subgroup A_r with $r \geq 4$. Finally, for D_4 the result follows from the first paragraph using a triality automorphism. $\square$

Lemma 3.5. *Assume $n \geq 3$ and $G = A_n$, B_n, C_n, or D_n and $\lambda = \omega_i+\omega_j$, where α_i, α_j are end-nodes. Then $V\downarrow A$ is multiplicity-free if and only if $\lambda = \omega_1+\omega_n$ and $G = A_n$ or B_3.*

Proof. First consider $G = A_n$, B_n, C_n. By Lemma 2.6(vi) we have $\lambda = \omega_1+\omega_n$. If $G = B_n$ with $n \geq 4$, then $\lambda-123$, $\lambda-(n-2)(n-1)n$, $\lambda-1(n-1)n$, $\lambda-12n$, and $\lambda-(n-1)n^2=(\lambda-(n-1)n)^{s_n}$ all restrict to $r-6$ on T, so $V\downarrow A$ is not multiplicity-free by Lemma 2.2. We argue similarly for $G=C_n$ with $n\geq 4$, replacing the last weight by $\lambda-(n-1)^2n=(\lambda-(n-1)n)^{s_{n-1}}$. And if $G=A_n$, then $V\downarrow A$ is just $(n\otimes n)-0$ and hence is multiplicity-free.

Now suppose $n=3$ and $\lambda=101$. If $G=B_3$, then Magma gives $100\otimes 001=101+001$. Restricting to A, the left side is $6\otimes(6+0)$ and we find that $101\downarrow A=12+10+8+6+4+2$, which is multiplicity-free. For $G=C_3$, Magma yields $100\otimes 001=101+010$, $\bigwedge^2(100)=010+000$, and $\bigwedge^3(100)=001+100$. Restricting to A and considering weights we have $101\downarrow A=14+12+10+8+6^2+4+2$, which is not multiplicity-free.

Finally, consider $G=D_n$ with $n\geq 4$. First consider $\lambda=\omega_1+\omega_{n-1}$. The T-weight $r-2(n-1)$ is afforded by $\lambda-1\cdots(n-1)$, $\lambda-2\cdots n$, $\lambda-1\cdots(n-2)n$, which, using Lemma 2.5, occur with multiplicities $n-1$, 1, 1 respectively, giving the conclusion by Lemma 2.2. A similar argument applies if $\lambda=\omega_1+\omega_n$. Finally assume $\lambda=\omega_{n-1}+\omega_n$. Here, T-weight $r-6$ is afforded by $\lambda-(n-2)(n-1)n$, $\lambda-(n-3)(n-2)(n-1)$, $\lambda-(n-3)(n-2)n$, with multiplicities 3, 1, 1, so again Lemma 2.2 applies. □

Lemma 3.6. *Assume $G=E_6$, E_7, E_8, or F_4 and $\lambda=\omega_i+\omega_j$, where α_i,α_j are end-nodes. Then $V\downarrow A$ is not multiplicity-free.*

Proof. First assume $G=F_4$. Then $\lambda=1001$ and we consider T-weight $r-8$ which is afforded by weights $\lambda-1234$, $\lambda-123^2=(\lambda-12)^{s_3}$, $\lambda-23^24=(\lambda-234)^{s_3}$, occurring with multiplicities 4, 1, 1, respectively, giving the result by Lemma 2.2.

So now assume $G=E_n$. If $\lambda=\omega_1+\omega_n$ then the weights $\lambda-134\cdots n$, $\lambda-1234\cdots(n-1)$, $\lambda-23\cdots n$ all afford T-weight $r-2(n-1)$ and, by Lemma 2.5, occur with multiplicities $n-1$, 1, 1 respectively, and now we apply Lemma 2.2. If $\lambda=\omega_1+\omega_2$, we argue similarly using weights $\lambda-1234$, $\lambda-1345$, $\lambda-2345$. And if $\lambda=\omega_2+\omega_n$, we use weights $\lambda-245\cdots n$, $\lambda-345\cdots n$, $\lambda-23\cdots(n-1)$. □

This completes the proof of Proposition 3.1.

4. The case where A is regular and $\lambda=b\omega_i$, $b\geq 2$

Continue to assume that G is a simple algebraic group, A is a regular A_1 in G, and $V=V_G(\lambda)$. In this section we prove Theorem 1 in the case where $\lambda=b\omega_i$ for some i and some $b\geq 2$. In this case, the T-weight $r-2$ appears in V with multiplicity 1 and Lemma 2.2(iii) applies. Also Lemma 2.3 implies that if $V\downarrow A$ is multiplicity-free then α_i is an end-node.

Proposition 4.1. *Assume $\lambda=b\omega_i$ with $b>1$. Then $V\downarrow A$ is multiplicity-free if and only if G and λ are as in the following table, up to graph automorphisms of A_n or D_4.*

λ	G
$2\omega_1$	$A_n,\ B_n,\ C_n,\ D_n\ (n=2k),\ G_2$
$3\omega_1$	$A_n\ (n\le 5),\ B_n\ (n=2,3),\ C_n(n=2,3),\ G_2$
$4\omega_1,\ 5\omega_1$	$A_n\ (n=2,3),\ B_2,\ C_2$
$b\omega_1\ (b\ge 6)$	A_2
$b\omega_1\ (b\le 5)$	C_2
$2\omega_3$	B_3
$2\omega_2$	G_2

The proof is carried out in a series of lemmas.

Lemma 4.2. *Assume that $\lambda=2\omega_1$. If $G=A_n$, B_n, or C_n, then $V\downarrow A$ is multiplicity-free. If $G=D_n$, then $V\downarrow A$ is multiplicity-free if and only if n is even.*

Proof. If $G=A_n$, then $V\downarrow A$ is just $S^2(n)$ and a consideration of weights shows that this is $2n+(2n-4)+(2n-8)+\cdots$, hence is multiplicity-free. If $G=B_n$ or C_n we can embed G in A_{2n} or A_{2n-1}, respectively. In each case A acts irreducibly on the natural module with highest weight $2n$ or $2n-1$, respectively, and the conclusion follows from the first sentence.

Now consider $G=D_n$. In this case A acts on the natural module ω_1 for G, as $(2n-2)+0$. Now $S^2(\omega_1)=V+0$ and hence $V\downarrow A=S^2(2n-2)+(2n-2)=((4n-4)+(4n-8)+\cdots)+(2n-2)$. If n is odd, we find that $2n-2$ appears with multiplicity 2, while if n is even, $V\downarrow A$ is multiplicity-free. □

Lemma 4.3. *Assume that $G=B_n$ $(n\ge 3)$, C_n $(n\ge 3)$, or D_n $(n\ge 4)$ and that $\lambda=b\omega_i$ with $b>1$ and $i>1$. Then $V\downarrow A$ is multiplicity-free if and only if $G=B_3$ and $\lambda=2\omega_3$ or $G=D_4$ and $\lambda=2\omega_i$ for $i=3$ or 4.*

Proof. By Lemma 2.3 we can assume that α_i is an end-node, so we may take $i=n$. First consider C_n. If $b\ge 3$, then the weight $r-6$ occurs with multiplicity at least 4 (from $\lambda-(n-2)(n-1)n$, $\lambda-(n-1)n^2$, $\lambda-n^3$, $\lambda-(n-1)^2n=(\lambda-n)^{s_{n-1}}$) and so $V\downarrow A$ is not multiplicity-free. For $b=2$ first consider $G=C_3$. We have $S^2(001)=V+200$. As $001\downarrow A=9+3$, it follows that $V\downarrow A$ contains 6^2. Next suppose that $G=C_n$ with $n\ge 4$ and $b=2$. This case essentially follows from the C_3 result. We need only show that there are at least two more weights $r-12$ than weights $r-10$. For $n=4$ the only weights $r-10$ that do not arise from the C_3 Levi are $\lambda-123^24$, $\lambda-1234^2$. Correspondingly, there are new $r-12$ weights, $\lambda-12^23^24$, $\lambda-123^24^2$. Similar reasoning applies for C_5, where $\lambda-12345$ is the only weight $r-10$ not appearing for C_4 and we conjugate by s_4 to get a new weight $r-12$. And for $n\ge 6$ there are no $r-10$ weights that were not present in a C_5 Levi factor.

Now let $G=B_n$. If $b\ge 3$ we find that T weight $r-6$ appears with multiplicity at least 4. Indeed, for the B_2 Levi the module $0b=S^b(01)$ and this yields weights

$\lambda - n^3$, $\lambda - (n-1)n^2$, the latter with multiplicity 2. Also $\lambda - (n-2)(n-1)n$ affords T-weight $r-6$, which yields the assertion.

Now assume $b = 2$. First consider $G = B_3$, so that $\lambda = 002$. The module 001 for B_3 is the spin module where A acts as $6+0$. We have $S^2(001) = 002 + 000$, and it follows that $V \downarrow A = 12+8+6+4+0$, which is multiplicity-free. Now assume $n > 3$. Here we show that T-weight $r-8$ occurs with multiplicity 5. The above shows that $r-8$ occurs with multiplicity 4 just working in the B_3 Levi. As $\lambda - (n-3)(n-2)(n-1)n$ affords $r-8$ the assertion follows.

Finally, consider $G = D_n$. If $b \geq 3$ then T-weight $r-6$ occurs with multiplicity 4 (from $\lambda - n^3$, $\lambda - (n-2)n^2$, $\lambda - (n-1)(n-2)n$, $\lambda - (n-3)(n-2)(n)$), and so $V \downarrow A$ is not multiplicity-free by Lemma 2.2(iii). Now assume $b = 2$. Applying a graph automorphism if necessary, we can assume $n \geq 5$ (the conclusion allows for D_4 using Lemma 4.2). Then T-weight $r-8$ occurs with multiplicity at least 5 (from $\lambda - (n-4)(n-3)(n-2)n$, $\lambda - (n-3)(n-2)(n-1)n$, $\lambda - (n-3)(n-2)n^2$, $\lambda - (n-1)(n-2)n^2$, $\lambda - (n-2)^2n^2$). Therefore, $V \downarrow A$ is not multiplicity-free. □

Lemma 4.4. *Assume that $G = A_n$, B_n $(n \geq 3)$, C_n $(n \geq 3)$ or D_n $(n \geq 4)$, and that $\lambda = b\omega_1$ with $b \geq 3$. Then $V \downarrow A$ is multiplicity-free only for the cases listed in rows 2 to 4 of the table in Proposition 4.1.*

Proof. First let $G = A_n$, so $V = V_G(b\omega_1) = S^b(\omega_1)$. First consider $b = 3$, so that $r = 3n$. If $n \geq 6$, then T-weight $3n - 12$ occurs with multiplicity at least 7 and $V \downarrow A$ cannot be multiplicity-free. Indeed, independent vectors of weight $3n-12$ occur as tensor symmetric powers of vectors of weights (i, j, k), where (i, j, k) is one of $(n, n, n-12)$, $(n, n-2, n-10)$, $(n, n-4, n-8)$, $(n, n-6, n-6)$, $(n-2, n-2, n-8)$, $(n-2, n-4, n-6)$, or $(n-4, n-4, n-4)$. On the other hand, for $n \leq 5$ the restriction is multiplicity-free.

Next consider $b = 4$, so that $r = 4n$. If $n \geq 4$, then $4n-8$ appears with multiplicity at least 5 and hence $V \downarrow A$ is not multiplicity-free. Indeed, independent vectors arise from symmetric powers of vectors of weights $(n, n, n, n-8)$, $(n, n, n-2, n-6)$, $(n, n, n-4, n-4)$, $(n, n-2, n-2, n-4)$, $(n-2, n-2, n-2, n-2)$. And for $n \leq 3$ a direct check shows that $S^b(\omega_1) \downarrow A$ is multiplicity-free. If $b \geq 5$, $n \geq 3$, and $(b, n) \neq (5, 3)$ then a similar argument shows that the weight $bn - 12$ occurs with multiplicity at least two more than does $bn - 10$; hence $V \downarrow A$ is not multiplicity-free in these cases. And if $(b, n) = (5, 3)$ one checks that $V \downarrow A = S^5(3) = 15 + 11 + 9 + 7 + 5 + 3$, which is multiplicity-free.

The final case for $G = A_n$ is when $n = 2$. We first note that the multiplicity of weight $2j$ in $S^b(2)$ is precisely the multiplicity of weight 0 in $S^{b-j}(2)$. Indeed, if we write $2^c0^d(-2)^e$ to denote a symmetric tensor of c vectors of weight 2, d vectors of weight 0 and e vectors of weight -2, then a basis for the $2j$-weight space is given by vectors $2^j0^{b-j}(-2)^0$, $2^{j+1}0^{b-j-2}(-2)^1$, $2^{j+2}0^{b-j-4}(-2)^2, \ldots$ and ignoring the first j terms in each tensor we obtain the assertion. The multiplicity of weight 0 in $S^{b-j}(2)$ is easily seen to be $(b-j+1)/2$ if $b-j$ is odd and $(b-j+2)/2$ if $b-j$ is even. From this information we see that $S^b(2) = 2b + (2b-4) + (2b-8) + \cdots$ and hence $V \downarrow A$ is multiplicity-free.

Now consider $G = B_n$, C_n, or D_n. The C_n case follows from the A_{2n-1} case since $V = S^b(\omega_1)$; see [Seitz 1987]. If $G = D_n$ with $n \geq 4$, then $A \leq B_{n-1} < G$. If the corresponding module for this subgroup is not multiplicity-free, then the same holds for G since it appears as a direct summand of V.

So assume $G = B_n$. If $b \geq 4$, then T-weight $r-8$ occurs with multiplicity at least 5. Indeed, if $n \geq 4$ this weight arises from $\lambda - 1234$, $\lambda - 1^2 23$, $\lambda - 1^2 2^2$, $\lambda - 1^3 2$, $\lambda - 1^4$; whereas, if $n = 3$ replace the first of these weights by $\lambda - 123^2 = (\lambda - 12)^{s_3}$. Now consider $b = 3$. If $n = 4$, then $S^3(\lambda_1) = 3000 + 1000$ and one checks that T-weight $r - 12 = 12$ occurs with multiplicity 7, and so $V \downarrow A$ is not multiplicity-free. And for $n > 4$ we apply Lemma 2.8 to get the same conclusion. Finally, if $n = 3$ then $S^3(\lambda_1) = V + 100$, and a direct check of weights shows that $S^3(\lambda_1) \downarrow A = 18 + 14 + 12 + 10 + 8 + 6^2 + 2$, which implies that $V \downarrow A$ is multiplicity-free.

The only remaining case is when $G = D_4$ and $b = 3$, since here the module $300 \downarrow A$ for B_3 is multiplicity-free. As a module for G we have $S^3(\omega_1) = 3\omega_1 \oplus \omega_1$, so that $V \downarrow A = S^3(6+0) - (6+0)$, which one easily checks is not multiplicity-free. □

Lemma 4.5. *Assume that $G = B_2$, C_2, or G_2 and $\lambda = b\omega_i$ (with $b \geq 2$). Then $V \downarrow A$ is multiplicity-free if and only if one of the following holds*:

(i) *$G = B_2$ or C_2 and $\lambda = b0, 0b$ ($b \leq 5$).*

(ii) *$G = G_2$ and $\lambda = 20$, 30, or 02.*

Proof. (i) Let $G = B_2$. Then $0b = S^b(01)$, which restricts to A as $S^b(3)$. Therefore, the assertion follows from the A_3 result which has already been established.

Now assume $\lambda = b0$. Here $\dim(b0) = (b+1)(b+2)(2b+3)/6$ and the highest weight of $V \downarrow A$ is $4b$. If the restriction were multiplicity-free, then weight $4b-2$ would only occur with multiplicity 1, and the restriction with largest possible dimension would have composition factors $4b + (4b-4) + (4b-6) + \cdots + 2 + 0$ which totals $4b^2 + 2$. For $b \geq 7$, this is less than the above dimension of $b0$ and so the restriction cannot be multiplicity-free. And for $b \leq 3$, V is a summand of $S^b(4)$ which we have already seen to be multiplicity-free. This leaves the cases $b = 4, 5, 6$.

A computation gives the following decompositions of symmetric powers of the G-module 10:

$$\begin{aligned}
S^6(10) &= 60+40+20+00,\\
S^5(10) &= 50+30+10,\\
S^4(10) &= 40+20+00,\\
S^3(10) &= 30+10,\\
S^2(10) &= 20+00.
\end{aligned}$$

It follows that $40 \downarrow A = 16+12+10+8+4$ and $50 \downarrow A = 20+16+14+12+10+8+4$, so these are both multiplicity-free. Also $S^6(4) = 24+20+18+16^2+14+12^3+\cdots$. This and the above imply that $60 \downarrow A$ is not multiplicity-free. This completes the proof of (i).

(ii) It follows from [Seitz 1987] that $V_{B_3}(b00)$ is irreducible upon restriction to G_2, with highest weight $b0$, and also a regular A in B_3 lies in a subgroup G_2. So for $i = 1$ the assertion follows from our results for B_3. Now assume $i = 2$. Then

$$\dim(0b) = \frac{1}{120}(b+1)(b+2)(2b+3)(3b+4)(3b+5),$$

and the highest T-weight is $10b$. First let $b = 2$. Then $V \downarrow A$ is a direct summand of $S^2(01) \downarrow A = 20+16+12^2+10+8^2+4^2+0^2$. We have $S^2(01) = V \oplus 20 \oplus 00$ and hence $V \downarrow A = 20+16+12+10+8+4+0$, which is multiplicity-free. On the other hand if $b \geq 3$, then Lemma 2.1 implies that $V \downarrow A$ is not multiplicity-free. □

Lemma 4.6. *If $G = E_n$ and $\lambda = b\omega_i$ with $b > 1$, then $V \downarrow A$ is not multiplicity-free.*

Proof. By Lemma 2.3, we can take α_i to be an end-node. First assume $i = 1$. If $b = 2$ one checks that $r-6$ is only afforded by $\lambda-134$, $\lambda-1^23$, while $r-8$ is afforded by $\lambda-1234$, $\lambda-1345$, $\lambda-1^234$, $\lambda-1^23^2$, so that $V \downarrow A$ is not multiplicity-free by Lemma 2.2(ii). Similarly for $b \geq 3$ as T-weight $r-6$ appears with multiplicity 3 (from $\lambda-134$, $\lambda-1^23$, $\lambda-1^3$), but $r-8$ appears with multiplicity at least 5 (from $\lambda-1345$, $\lambda-1234$, $\lambda-1^234$, $\lambda-1^22^2$, $\lambda-1^33$).

If $i = 2$, we see that weight $r-8$ appears with multiplicity at least 5, since it is afforded by each of $\lambda-2345$, $\lambda-1234$, $\lambda-2456$, $\lambda-2^234$, $\lambda-2^245$. So $V \downarrow A$ is not multiplicity-free by Lemma 2.2(iii).

Finally, assume that $i = n$. For $n = 6$, V is just the dual of $V_G(\lambda_1)$, so suppose $G = E_7$ or E_8. If $b \geq 4$ it is easy to list weights and verify that T-weight $r-8$ appears with multiplicity at least 5, so Lemma 2.2(iii) shows that $V \downarrow A$ is not multiplicity-free. And if $b = 2$ or 3, we see that T-weight $r-12$ appears with multiplicity at least 2 more than T-weight $r-10$. □

Lemma 4.7. *If $G = F_4$ and $\lambda = b\omega_i$ with $b > 1$, then $V \downarrow A$ is not multiplicity-free.*

Proof. As usual we can take α_i to be an end-node. First assume $i = 1$. If $b = 2$, then T weight $r-6$ occurs with multiplicity 2 (from $\lambda-123$, $\lambda-1^22$); whereas, $r-8$ occurs with multiplicity 4 (from $\lambda-1234$, $\lambda-123^2 = (\lambda-12)^{s_3}$, $\lambda-1^223$, $\lambda-1^22^2$). If $b \geq 3$, then the weight $r-6$ appears with multiplicity 3 due to the additional weight $\lambda-1^3$. But we also get an additional weight $r-8$ from $\lambda-1^32$. In either case, Lemma 2.2 implies that $V \downarrow A$ is not multiplicity-free.

Now assume $i = 4$. First assume $b = 2$. Then $S^2(0001) = V + 0001 + 0000$. Moreover, a consideration of weights shows that $0001 \downarrow A = 16 + 8$, and we conclude that $V \downarrow A$ is not multiplicity-free as there is a summand 20^2.

Finally, assume $b \geq 3$. The T-weight $r-6$ occurs with multiplicity 3 (from $\lambda-234$, $\lambda-34^2$, $\lambda-4^3$), whereas T-weight $r-8$ occurs with multiplicity at least 5 (from $\lambda-1234$, $\lambda-23^24 = (l-234)^{s_3}$, $\lambda-234^2$, $\lambda-3^24^2$, $\lambda-34^3$). □

This completes the proof of Proposition 4.1.

5. The case where A is regular and $\lambda = \omega_i$

Continue to assume that G is a simple algebraic group, A is a regular A_1 in G, and $V = V_G(\lambda)$. In this section we prove Theorem 1 in the case where $\lambda = \omega_i$ for some i.

Proposition 5.1. *Assume that $\lambda = \omega_i$ for some i. Then $V \downarrow A$ is multiplicity-free if and only if G and λ are as in the following table, up to graph automorphisms.*

λ	G
$\omega_1,\ \omega_2$	$A_n,\ B_n,\ C_n,\ D_n\ (n = 2k+1),\ G_2$
ω_3	$A_n\ (n \leq 7),\ C_n\ (n \leq 5)$
ω_n	$C_4,\ C_5$
ω_n	$B_n\ (n \leq 8),\ D_n\ (n \leq 9)$
$\omega_1,\ \omega_2$	E_6
$\omega_1,\ \omega_7$	E_7
ω_8	E_8
$\omega_1,\ \omega_4$	F_4

The proof is carried out in a series of lemmas.

Lemma 5.2. *Assume that $\lambda = \omega_i$.*

(i) *Then $V \downarrow A$ is not multiplicity-free if $G = A_n$, B_n, C_n or D_n and $4 \leq i \leq n-3$.*

(ii) *If $G = A_n$, $i = 3$, and $n \geq 5$, then $V \downarrow A$ is multiplicity-free if and only if $n \leq 7$.*

(iii) *If $G = A_n$, B_n, C_n, D_n, or G_2 and $i = 1$ or 2, then $V \downarrow A$ is multiplicity-free except when $G = D_n$, $i = 2$, and n even.*

Proof. (i) This follows from Lemma 2.7.

(ii) Assume $G = A_n$ and $i = 3$ with $n \geq 5$. Then $V = \bigwedge^3(\omega_1)$ and a computation using Magma shows that $V \downarrow A$ is multiplicity-free for $n = 5, 6, 7$. If $n \geq 8$ one checks that T-weight $r - 12$ occurs with multiplicity at least 7. Indeed, here $r = 3n - 6$, and $r - 12 = 3n - 18$ is afforded by the wedge of tensors of weight vectors for each of the following weights:

$$n(n-2)(n-16), \quad n(n-4)(n-14),$$
$$n(n-6)(n-12), \quad n(n-8)(n-10), \quad (n-2)(n-4)(n-12),$$
$$(n-2)(n-6)(n-10), \quad (n-4)(n-6)(n-8).$$

Hence $V \downarrow A$ is not multiplicity-free for $n \geq 8$ by Lemma 2.2(iii).

(iii) If $G = A_n$ then A is irreducible on the natural module (i.e., ω_1) for G with highest weight n. And if $i = 2$, then $V \downarrow A = \bigwedge^2(n)$ is a direct summand of $n \otimes n = 2n + (2n-2) + (2n-4) + \cdots + 0$, and hence $V \downarrow A$ is multiplicity-free. Now consider $G = B_n, C_n, D_n$ embedded in $X = A_{2n}, A_{2n-1}, A_{2n-1}$. In the first two cases A acts irreducibly on the natural module, $V_X(\omega_1)$, and in the third case A acts as $(2n-2) + 0$. So $V \downarrow A$ is obviously multiplicity-free for $i = 1$. Now consider $i = 2$. Then $V_X(\omega_2) \downarrow G = V$ if $G = B_n$ or D_n [Seitz 1987] and equals $V + 0$ if $G = C_n$ (the fixed space corresponds to a fixed alternating form). Therefore, $V \downarrow A = \bigwedge^2(2n)$, $\bigwedge^2((2n-2)+0)$, or $\bigwedge^2(2n-1) - 0$, respectively. So $V \downarrow A$ is multiplicity-free if $G = B_n$ or C_n. But if $G = D_n$, then

$$V \downarrow A = \bigwedge^2((2n-2)+0) = (2n-2) + (4n-6) + (4n-10) + \cdots$$

and this is multiplicity-free only if n is odd. Finally consider $G = G_2$ viewed as a subgroup of A_6. Then A is irreducible on the natural 7-dimensional module $V_G(\omega_1)$. Also $V_G(\omega_2)$ is a direct summand of $\bigwedge^2(V_G(\omega_1))$. So $V \downarrow A$ is multiplicity-free in both cases. □

Lemma 5.3. *Suppose that $G = B_n$, C_n or D_n, that $\lambda = \omega_i$ for $i \geq 3$ and that V is not a spin module for B_n or D_n. Then $V \downarrow A$ is multiplicity-free if and only if one of the following holds*:

(i) *$i = n$ and $G = C_4$ or C_5.*

(ii) *$i = 3$ and $G = C_n$ for $n = 3, 4, 5$.*

Proof. If $G = B_n$ or D_n, then $V = \bigwedge^i(\omega_1)$ and the result follows from the A_{2n} or A_{2n-1} part of Lemma 5.2. Indeed, if $G = B_n$, then A is regular in A_{2n} while if $G = D_n$, $A < B_{n-1} < D_n$. Therefore, we may assume that $G = C_n$. If $4 \leq i \leq n-3$ then $V \downarrow A$ is not multiplicity-free by Lemma 5.2.

Suppose $i \geq 4$. By the previous paragraph we can assume that $i > n-3$. If $i = n-2$, then T-weight $r-8$ occurs with multiplicity at least 5 as it is afforded by

$$\lambda-(i-3)(i-2)(i-1)i, \quad \lambda-(i-2)(i-1)i(i+1),$$
$$\lambda-(i-1)i(i+1)(i+2), \quad \lambda-(i-1)i^2(i+1),$$
$$\lambda-i(i+1)^2(i+2) = (\lambda-i(i+1)(i+2))^{s_{i+1}},$$

so $V \downarrow A$ is not multiplicity-free by Lemma 2.2(iii).

Next assume $i = n-1$. First consider $n = 5$, where $\bigwedge^4(\omega_1) = \omega_4 + \omega_2 + 0$. Here $r = 24$ and a computation shows that $r - 12 = 12$ occurs with multiplicity 9 in $\bigwedge^4(\omega_1)$ but it only occurs twice in $\bigwedge^2(\omega_1) = \omega_2 + 0$. Therefore, this weight occurs with multiplicity 7 in V and hence $V \downarrow A$ is not multiplicity-free by Lemma 2.2(iii). Now return to the general case with $i = n-1$. Then an application of Lemma 2.8(ii) to a C_5 Levi subgroup shows that T-weight $r - 12$ appears with multiplicity at least 7, against Lemma 2.2.

A similar argument settles the case where $n = i$. If $n = 4$ or 5, then a Magma computation shows that $V \downarrow A$ is multiplicity-free. If $n = 6$, weights $24 = r - 12$ and $26 = r - 10$ occur with multiplicities 6 and 4 respectively, and so Lemma 2.2(i) implies that $V \downarrow A$ is not multiplicity-free. For $n > 6$ we also compare weights $r - 10$ and $r - 12$. These must already be weights of the C_6 Levi subgroups, so again this contradicts Lemma 2.2(i).

Now assume $i = 3$ with $G = C_n$. Then $\bigwedge^3(\omega_1) = V + \omega_1$. Also A is irreducible on the natural module for A_{2n-1}. In the proof of Lemma 5.2(ii) we saw that for $n \geq 5$ the weight $r - 12 = 6n - 21$ occurs in $\bigwedge^3(\omega_1)$ with multiplicity at least 7. If $n \geq 6$, then all these weights occur within V, so $V \downarrow A$ is not multiplicity-free. This leaves $n = 3, 4, 5$. In these cases, a simple check of weights shows that $V \downarrow A$ is multiplicity-free. □

Lemma 5.4. *Assume V is a spin module for B_n or D_n. Then $V \downarrow A$ is multiplicity-free if and only if $n \leq 8$ for B_n and $n \leq 9$ for D_n.*

Proof. If $G = D_n$, then $A \leq B_{n-1} < G$ and B_{n-1} is irreducible on V, so it will suffice to settle the $G = B_n$ case. In terms of roots, $\omega_n = \sum(i\alpha_i)/2$, so that $r = n(n+1)/2$. As $\dim V = 2^n$, Lemma 2.1 shows that $V \downarrow A$ is not multiplicity-free if $n \geq 10$. If $n = 9$ then $\dim V = 2^9 = 512$, while the sum in Lemma 2.1 is 552. However, $V \downarrow A$ does not contain a summand of highest weight $r-2 = 43$, so $\dim V \leq 552 - 44 = 508$. So here too, $V \downarrow A$ fails to be multiplicity-free. This leaves the case $n \leq 8$.

Consider the restriction $V \downarrow L$, where $L = \mathrm{GL}_n$ is a Levi subgroup. One checks (see [Liebeck and Seitz 2012, Lemma 11.15]) that the restriction to SL_n consists of the natural module and all its wedge powers together with two trivial modules. For example, when $n = 8$ the restriction to A of the weights λ, $\lambda-8$, $\lambda-78^2 = (\lambda-8)^{s_7s_8}$, $\lambda-67^28^3 = (\lambda-78^2)^{s_6s_7s_8}, \ldots$ afford the modules $0, \omega_7, \omega_6, \omega_5, \ldots$ for the A_7

factor. However, the T-weights are shifted in accordance with the number of fundamental roots subtracted. In the above example, the T-weight of 0 is just that of λ, namely 36 and the T-weights of ω_7 are 34, 32, ..., 20, etc.

Here we indicate some of the decompositions for $V \downarrow A$ for later use.

n	decomposition
8	$36+30+26+24+22+20+18+16+14+12+10+8+6+0$
7	$28+22+18+16+14+10+8+4$
6	$21+15+11+9+3$
5	$15+9+5$
4	$10+4$
3	$6+0$

Carrying out the above we obtain the conclusion. □

Lemma 5.5. *Assume that $G = E_n$ or F_4. Then $V \downarrow A$ is multiplicity-free if and only if λ is as in the following table.*

G	λ
E_6	$\omega_1,\ \omega_2,\ \omega_6$
E_7	$\omega_1,\ \omega_7$
E_8	ω_8
F_4	$\omega_1,\ \omega_4$

Proof. First assume $G = F_4$ and $\lambda = \omega_4$. It is straightforward to list the first few weights and see that $V \downarrow A = 16+8$. [Liebeck and Seitz 1996, Propositions 2.4 and 2.5] show that $V \downarrow A$ is multiplicity-free for each of the remaining cases listed in the table.

It remains to show that all other possibilities fail to be multiplicity-free. To do this, we use Lemma 2.1 along with the dimensions of $V = V(\omega_i)$, which can be found using Magma; the values of r can be calculated using the expressions for ω_i in terms of roots, given in [Bourbaki 1968, p. 250]. □

This completes the proof of Proposition 5.1

6. The case where A is nonregular

Assume that G is a simple algebraic group, and $A \cong A_1$ is a G-irreducible subgroup of G. Recall from the introduction that this means that a nonidentity unipotent element u of A is distinguished in G. In this section we prove Theorem 1, classifying G-modules $V = V_G(\lambda)$ such that $V \downarrow A$ is multiplicity-free, in the case where u is distinguished, but not a regular element of G. Such elements exist for G of type

B_n, $(n \geq 4)$, C_n, $(n \geq 3)$, D_n, $(n \geq 4)$, E_6, E_7, E_8, F_4 or G_2. We shall see that there are relatively few examples; they are listed in Table 2 of Theorem 1.

We begin with the analysis of the classical groups.

Proposition 6.1. *Assume that $G = B_n$, C_n or D_n and u is distinguished but not regular. Then up to graph automorphisms of D_n, $V_G(\lambda) \downarrow A$ is multiplicity-free if and only if one of the following holds*:

(i) $\lambda = \omega_1$.

(ii) *$G = D_n$ with $5 \leq n \leq 7$, $\lambda = \omega_n$, and $A < B_{n-2}B_1$, projecting to a regular A_1 in each factor.*

For the next four lemmas assume the hypotheses of Proposition 6.1. The natural G-module, when restricted to A, is a direct sum of irreducible modules of distinct highest weights, and we first discuss the corresponding T-labelling of the Dynkin diagram of G. A full description can be found in [Liebeck and Seitz 2012, Theorem 3.18]. As an example, consider $G = C_{15}$ with A acting as $15 + 9 + 3$. The T-weights are 15, 13, 11, 9^2, 7^2, 5^2, 3^3, 1^3 plus negatives. The corresponding labelling of the Dynkin diagram is 222020202002002. So the labelling begins with an *initial string* of 2s, then a number of terms 20, several of type 200, and so on. For C_n, the end-node α_n has label 2, and for B_n it has label 0. For D_n both of α_{n-1}, α_n have the same label; it is 2 or 0, according to whether there are just two summands for A or more than two, respectively.

As in previous sections, let $V = V_G(\lambda)$, of highest weight $\lambda = \sum c_i\omega_i$ affording T-weight r.

Lemma 6.2. *Assume $V \downarrow A$ is multiplicity-free. Then the following hold*:

(i) *$c_i = 0$ if α_i has label 0.*

(ii) *$c_i = 0$ if α_i has label 2 and α_i is adjacent to two nodes having label 0.*

(iii) *$\lambda = b\omega_i$ for some i.*

(iv) *If $\lambda = b\omega_i$ with $b > 1$, then $i = 1$.*

(v) *$\lambda \neq \omega_n$ if $G = B_n$ or C_n.*

Proof. (i) Assume α_i has label 0 but $c_i \neq 0$. Then $\lambda - \alpha_i$ is a weight affording T-weight r, which implies that r^2 is a summand of $V \downarrow A$, a contradiction.

(ii) Next suppose that α_i has label 2 but nodes on either side have label 0. If we label these nodes $\alpha_i, \alpha_j, \alpha_k$, then $\lambda - i$, $\lambda - ij$, $\lambda - ik$ all afford T-weight $r - 2$, contradicting Lemma 2.2.

(iii) Assume $c_i \neq 0 \neq c_j$. Then $\lambda - i$ and $\lambda - j$ afford the only T-weights $r - 2$. This implies that neither α_i nor α_j can be adjacent to a node with 0 label, as otherwise $r - 2$ would occur with multiplicity at least 3. Therefore, both occur in the initial string of 2s, and within this string we can argue exactly as in the regular case. Indeed, the argument of parts (iv), (v), and (vi) of Lemma 2.6 implies that $i = 1$, $j = 2$, and $c_i = c_j = 1$. Then the first paragraph of the proof of Lemma 3.4 implies that the initial string of 2s has length 3. But then T-weight $r-4$ is afforded by $\lambda - 12$ (multiplicity 2), $\lambda - 23$, and $\lambda - 234$, contradicting Lemma 2.2.

(iv) Assume $\lambda = b\omega_i$ with $b > 1$. By Lemma 2.3(i), α_i is an end-node. Suppose $i = n$. Then $G \neq B_n$, as otherwise α_n has label 0, against (i). If $G = C_n$, then $\lambda - n$, $\lambda - n(n-1)$, $\lambda - n(n-1)^2 = (\lambda - n(n-1))^{s_{n-1}}$ all afford $r-2$. And for D_n, $r-4$ is afforded by $\lambda - n^2$, $\lambda - n^2(n-2)$, $\lambda - n^2(n-2)^2$, $\lambda - n(n-2)(n-1)$. This is a contradiction. A similar argument applies if $G = D_n$ and $i = n-1$.

(v) Suppose $\lambda = \omega_n$. The last argument of the previous paragraph also shows that $V \downarrow A$ is not multiplicity-free if $G = C_n$. And if $G = B_n$ then α_n has label 0, contradicting (i). $\square$

Lemma 6.3. *Suppose $G = D_n$ with $n \geq 5$, and $\lambda = \omega_n$. Then $V \downarrow A$ is multiplicity-free if and only if $n \leq 7$ and $A < B_{n-2}B_1$, projecting to a regular A_1 in each factor.*

Proof. Assume $G = D_n$ and $\lambda = \omega_n$. Then the labels of α_{n-1} and α_n are both 2, and A has two irreducible summands on the natural G-module. The label of α_{n-2} is 0.

Suppose that $V \downarrow A$ is multiplicity-free. If α_{n-3} also has label 0, then $\lambda - n$, $\lambda - (n-2)n$, $\lambda - (n-3)(n-2)n$ all afford $r-2$, a contradiction. Therefore, α_{n-3} has label 2. Next consider α_{n-4}. If α_{n-4} has label 0, then $n \geq 6$ and α_{n-5} must have label 2. Hence $r-6$ is afforded by each of

$$\lambda-(n-3)(n-2)(n-1)n, \quad \lambda-(n-4)(n-3)(n-2)(n-1)n,$$
$$\lambda-(n-3)(n-2)^2(n-1)n, \quad \lambda-(n-4)(n-3)(n-2)^2(n-1)n,$$
$$\lambda-(n-5)(n-4)(n-3)(n-2)n,$$

again a contradiction. Therefore, α_{n-4} has label 2. This forces the full labelling to be $22\cdots22022$.

Hence A acts on the natural G-module as $(2n-4)+2$ and so lies in a subgroup $B_{n-2}B_1$, which acts on V as the tensor product of spin modules for the factors. That is, $V \downarrow A = X \otimes 1$ where X is the restriction of the spin module of B_{n-2} to a regular A_1. As we are assuming $V \downarrow A$ to be multiplicity-free, this forces X to be multiplicity-free. Applying Lemma 5.4 we see that this implies $n-2 \leq 8$. Moreover, at the end of the proof of Lemma 5.4 we listed the decompositions of X when this occurs. Tensoring these with 1 it is immediate from Lemma 2.4 that the V is multiplicity-free if and only if $n \leq 7$. $\square$

Lemma 6.4. (i) *Assume $\lambda = b\omega_1$ with $b > 1$. Then $V \downarrow A$ is not multiplicity-free.*

(ii) *Assume $\lambda = \omega_2$. Then $V \downarrow A$ is not multiplicity-free.*

Proof. (i) First suppose $b=2$. Note that $S^2(\omega_1)=V$ if $G=C_n$, while $S^2(\omega_1)=V+0$ if $G=B_n$ or D_n. Let A act on the natural module for G as $c+d+\cdots$, where $c>d>\cdots$. Note that if $d=0$, then u is a regular element of B_{n-1} and is hence regular in $G=D_n$, which we are assuming is not the case. Hence $d>0$.

Now $S^2(\omega_1)\downarrow A$ contains direct summands $S^2(c)=2c+(2c-4)+\cdots$ and $c\otimes d=(c+d)+(c+d-2)+\cdots$. If $c-d=4k$, then $2c-4k=c+d$ is common to both summands. And if $c-d=4k-2$, then $2c-4k=c+d-2$ is common to both summands. In either case we see that $V\downarrow A$ is not multiplicity-free.

Now assume that $b\geq 3$ and that $V\downarrow A$ is multiplicity-free. We first settle some special cases. If the T- labelling is $202\ldots$, then $r-4$ is afforded by $\lambda-1^2$, $\lambda-1^22$, $\lambda-1^22^2$, $\lambda-123$, a contradiction. Similarly, if the labelling is $2202\ldots$, then $r-4$ is afforded by $\lambda-12$, $\lambda-123$, $\lambda-1^2$, which contradicts Lemma 2.2(iii). And if the labelling is $22202\ldots$, then $r-8$ is afforded by $\lambda-12345$, $\lambda-1^223$, $\lambda-1^2234$, $\lambda-1^22^2$, $\lambda-1^32$, again contradicting Lemma 2.2(iii).

Now suppose that the initial string of 2s has length at least 4. If $b\geq 4$, the weights $\lambda-1234$, $\lambda-1^223$, $\lambda-1^22^2$, $\lambda-1^32$, $\lambda-1^4$ all afford $r-8$, against Lemma 2.2(iii). So assume $b=3$. Then $S^3(\omega_1)=V$ or $V+\omega_1$ according to whether or not $G=C_n$. One checks $S^3(\omega_1)$ to see that $r-12$ occurs with multiplicity at least 7 in $V\downarrow A$, and hence $V\downarrow A$ is not multiplicity-free.

(ii) The argument is similar to the $b=2$ case in (i). Assume A acts on the natural module as $c+d+\cdots$, where $c>d>\cdots$. Note that $d>0$, as otherwise u would be a regular element of $G=D_n$. Then $\bigwedge^2(\omega_1)=V$ or $V+0$ according to whether or not G is an orthogonal group. So $\bigwedge^2(\omega_1)\downarrow A$ contains $\bigwedge^2(c)=(2c-2)+(2c-6)+\cdots$, as well as $c\otimes d=(c+d)+(c+d-2)+\cdots$, as direct summands. If $c-d=4k+2$, then $2c-2-4k=c+d$ and if $c-d=4k$, then $2c-2-4k=c+d-2$. In either case $V\downarrow A$ is not multiplicity-free. □

Lemma 6.5. *Assume $\lambda=\omega_i$ for $3\leq i<n$ and V is not a spin module for D_n. Then $V\downarrow A$ is not multiplicity-free.*

Proof. Assume $V\downarrow A$ is multiplicity-free. By Lemma 6.2(ii) we know that α_i is in the initial string of 2s. Suppose the end of this string is at α_j. First assume $i\geq 4$. If in addition, $i\leq j-3$, then the result follows from Lemma 2.7. So we now consider situations where $i>j-3$ (still with $i\geq 4$).

Suppose $i=j$. Then α_{i+1} has label 0. If $n=i+1$, then $G=B_n$ and each of $\lambda-i$, $\lambda-i(i+1)$, $\lambda-i(i+1)^2=(\lambda-i(i+1))^{s_{i+1}}$ afford $r-2$, a contradiction. Therefore $n>i+1$. If α_{i+2} has label 0 we obtain the same contradiction from $\lambda-i$, $\lambda-i(i+1)$, $\lambda-i(i+1)(i+2)$. So suppose α_{i+2} has label 2. Then $r-4$ is afforded by each of $\lambda-(i-1)i$, $\lambda-(i-1)i(i+1)$, $\lambda-i(i+1)(i+2)$, which is not yet a contradiction. If $n=i+2$, then $G=C_n$ and we also get $r-4$ from $\lambda-i(i+1)^2(i+2)=(\lambda-i(i+1)(i+2))^{s_{i+2}}$. And if $n>i+2$, either α_{i+3} has label 0 or else $G = D_{i+3}$. In either case we get an extra weight affording $r-4$, which does contradict Lemma 2.2.

Therefore $i < j$. Then $r-2$ appears with multiplicity 1 and Lemma 2.2(iii) applies. By assumption, α_{j+1} has label 0. Suppose $i = j-1$. Then $r-4$ is afforded by each of $\lambda-(i-1)i$, $\lambda-ij$, $\lambda-ij(j+1)$ a contradiction. And if $i = j-2$, then $r-8$ is afforded by each of

$$\lambda-(i-3)(i-2)(i-1)i, \quad \lambda-(i-2)(i-1)i(i+1),$$
$$\lambda-(i-1)i(i+1)(i+2), \quad \lambda-(i-1)i(i+1)(i+2)(i+3),$$
$$\lambda-(i-1)i^2(i+1),$$

contradicting Lemma 2.2(iii).

Now assume $i = 3$. Then $\bigwedge^3(\omega_1)$ equals V or $V+\omega_1$ depending on whether or not G is an orthogonal group. Write $\omega_1 \downarrow A = a+b+\cdots$ with $a > b > \cdots$. We know that α_3 is in the initial string of 2s, and this forces $a-b \geq 6$ so that $r = 3a-6$. If G is an orthogonal group, then $a, b, \ldots$ are even and so $a \geq 8$ (note that $b > 0$ as A is not regular). Then $V \downarrow A$ contains $\bigwedge^3(a)$ as a direct summand which is not multiplicity-free by Lemma 5.2(ii). Indeed, there is a direct summand of highest weight $r-12 = 3a-18$ appearing with multiplicity 2. Now consider $G = C_n$. The same argument applies provided $3a-18 > a$. So it remains to consider $a \leq 9$. The cases are $(a, b) = (7, 1), (9, 3), (9, 1)$. Then $\bigwedge^3(\omega_1) \downarrow A$ contains $\bigwedge^3(a)$ and $\bigwedge^2(a) \otimes b$ as direct summands. As $\bigwedge^3(a) = (3a-6)+(3a-10)+\cdots$ and $\bigwedge^2(a) \otimes b = (2a-2+b)+(2a-4+b)+\cdots$, it follows that in each case, $3a-10$ occurs with multiplicity at least 2 and is not present in ω_1. $\square$

This completes the proof of Proposition 6.1.

It remains to consider the exceptional groups. Here we label the distinguished nonregular classes as in [Liebeck and Seitz 2012]. For convenience we reproduce the list in Table 3.

Proposition 6.6. *Assume G is an exceptional group and u is distinguished but not regular. Then up to graph automorphisms of E_6, $V_G(\lambda) \downarrow A$ is multiplicity-free if and only if λ and u are as in the following table.*

G	u	λ
F_4	$F_4(a_1)$	ω_4
E_6	$E_6(a_1)$	ω_1
E_7	$E_7(a_1)$ or $E_7(a_2)$	ω_7
E_8	$E_8(a_1)$	ω_8

G	classes	labellings
G_2	$G_2(a_1)$	02
F_4	$F_4(a_1),\ F_4(a_2),\ F_4(a_3)$	2202, 0202, 0200
E_6	$E_6(a_1),\ E_6(a_3)$	222022, 200202
E_7	$E_7(a_1),\ E_7(a_2),\ E_7(a_3),$ $E_7(a_4),\ E_7(a_5)$	2220222, 2220202, 2002022, 2002002, 0002002
E_8	$E_8(a_1),\ E_8(a_2),\ E_8(a_3),$ $E_8(a_4),\ E_8(a_5),\ E_8(a_6),$ $E_8(a_7),\ E_8(b_4),\ E_8(b_5),$ $E_8(b_6)$	22202222, 22202022, 20020222, 20020202, 20020020, 00020020, 00002000, 20020022, 00020022, 00020002

Table 3. Distinguished nonregular classes in exceptional groups.

Lemma 6.7. *Proposition 6.6 holds if $G = G_2$ or F_4.*

Proof. First consider $G = F_4$. Suppose $V \downarrow A$ is multiplicity-free. If there exist $i \neq j$ with $c_i \neq 0 \neq c_j$, then either α_i or α_j is adjacent to a node with label 0, contradicting Lemma 2.6(ii). Therefore $\lambda = b\omega_i$ for some i. From the diagrams in Table 3, and considering the multiplicity of $r-2$ using Lemma 6.2(ii), we see that u cannot be in the class $F_4(a_3)$, and that if $u = F_4(a_2)$ then $i = 4$. But then $\lambda - 234$, $\lambda - 1234$, $\lambda - 23^24$, $\lambda - 123^24$ all afford $r-4$, contradicting Lemma 2.2.

Now consider u in class $F_4(a_1)$. If $i = 2$, then $\lambda - 2$, $\lambda - 23$, $\lambda - 23^2$ all afford $r-2$, a contradiction. If $i = 1$, then $r-2$ appears with multiplicity 1, but $\lambda - 12$, $\lambda - 123$, $\lambda - 123^2$ all afford $r-4$, contradicting Lemma 2.2(i). Therefore $i = 4$. If $b > 1$, $r-4$ appears with multiplicity 4, which is impossible. And if $\lambda = \omega_4$ it follows from [Seitz 1991, Table A, p. 65] and the tables at the end of [Liebeck and Seitz 1996] that $A < B_4$, and $\omega_4 \downarrow B_4 = 1000 + 0001 + 0000$. Using the information at the end of the proof of Lemma 5.4, we find that $V \downarrow A = 8 + (10+4) + 0$ and hence $V \downarrow A$ is multiplicity-free.

Finally consider G_2 where the only labelling is 02. Hence $\lambda = b\omega_2$. Then $\lambda - 2$, $\lambda - 12$, $\lambda - 1^32$ all afford $r-2$, a contradiction. □

Lemma 6.8. *Proposition 6.6 holds if $G = E_n$.*

Proof. Assume $G = E_n$ and $V \downarrow A$ is multiplicity-free. First suppose that there exist $i > j$ with $c_i \neq 0 \neq c_j$. Lemma 2.6 shows these are the only two such nodes, that neither can adjoin a node with label 0, that at least one must be an end-node, and that $c_i = c_j = 1$. Suppose $j = 1$. Then α_3 must be labelled 2 and from the list of possible labellings in Table 3 we see that α_4 has label 0. This forces $i \geq 6$. But then $r-4$ is afforded by $\lambda - 13$, $\lambda - 134$, $\lambda - 1i$, $\lambda - (i-1)i$, a contradiction.

Therefore, $j \neq 1$ and hence $i = n$. If $j \neq n-1$, then we must have $G = E_8$, $j = 6$, and $u = E_8(a_1)$. But here we see that $r-4$ occurs with multiplicity at least 5, a contradiction.

Suppose $i = n$, $j = n-1$. If α_{n-3} has label 2, then $r-6$ occurs with multiplicity at least 5 from $\lambda-(n-2)(n-1)n$ (multiplicity 2), $\lambda-(n-1)^2n = (\lambda-n)^{s_{n-1}}$, $\lambda-(n-1)n^2 = (\lambda-(n-1))^{s_n}$, $\lambda-(n-3)(n-2)(n-1)$. We get the same contradiction if α_{n-3} has label 0, by replacing the last weight with $\lambda-(n-3)(n-2)(n-1)n$, (it even appears with multiplicity 2).

Hence $\lambda = b\omega_i$ for some i. Suppose $b > 1$. Then Lemma 2.3 implies that α_i is an end-node with label 2 and that the adjacent node has label 2. Therefore $i = 1$ or $i = n$. If $i = 1$, then $r-6$ is afforded by $\lambda-1234$, $\lambda-1345$, $\lambda-1^23$, $\lambda-1^234$, contradicting Lemma 2.2(iii).

Next consider $i = n$ where we can assume $n = 7$ or 8 since the E_6 case follows from the above via a graph automorphism. If α_{n-2} has label 0, then $r-4$ is afforded by $\lambda-(n-1)n$, $\lambda-(n-2)(n-1)n$, $\lambda-n^2$, contradicting Lemma 2.2(iii). Therefore, α_{n-2} has label 2. The only possibilities satisfying these conditions are $u = E_7(a_1)$, $E_8(a_1)$, $E_8(a_3)$. If $u = E_8(a_1)$, then $r-12$ arises from

$$\begin{aligned}
&\lambda-1345678,\\
&\lambda-2345678,\\
&\lambda-234^25678,\\
&\lambda-345678^2,\\
&\lambda-245678^2,\\
&\lambda-567^28^2,\\
&\lambda-6^27^28^2,
\end{aligned}$$

a contradiction. A similar argument applies to $E_7(a_1)$ and $E_8(a_3)$, using the weight $r-8$.

At this point we have $\lambda = \omega_i$. As in the proof of Lemma 5.5, we use Lemma 2.1 to reduce to the cases $(G; i) = (E_6; 1, 2, 6)$, $(E_7; 1, 7)$, and $(E_8; 8)$. The action of A on $L(G)$ is given in [Seitz 1991, Table A, p. 65 and Table 1, p. 193]. This settles all but the 27 dimensional modules ω_1, ω_6 for E_6 and the 56 dimensional module ω_7 for E_7.

Suppose $G = E_6$. From p. 65 of that reference we see that u is a regular element in C_4 or A_1A_5 according to whether $u = E_6(a_1)$ or $E_6(a_3)$. Then [Liebeck and Seitz 1996, Propositions 2.3 and 2.5] show that only the first case is multiplicity-free.

Finally assume that $G = E_7$ and $\lambda = \omega_7$. [loc. cit., Proposition 2.5] shows that $V \downarrow A$ is multiplicity-free if $u = E_7(a_1)$. But if $u = E_7(a_2)$, then $A \leq A_1F_4$ by [Seitz 1991, p. 65], and [Liebeck and Seitz 1996, Proposition 2.5] shows that $V \downarrow A = (1 \otimes (16+8)) + 3$, which is multiplicity-free. If $u = E_7(a_4)$ or $E_7(a_5)$, then both α_5 and α_6 have label 0 so that $r-2$ occurs with multiplicity 3, a contradiction. This leaves $u = E_7(a_3)$, in which case [Seitz 1991, p. 65] shows that $A < A_1B_5 < A_1D_6$. Then [Liebeck and Seitz 1996, Proposition 2.3] shows that $V \downarrow A_1D_6 = 1\otimes\omega_1 + 0\otimes\omega_5$. Applying the decomposition at the end of the proof of Lemma 5.4, we see that this is not multiplicity-free. □

This completes the proof of Theorem 1.

7. Proof of Corollary 2

Now we prove Corollary 2. Let G be a simple algebraic group of rank at least 2, let $u \in G$ be a distinguished unipotent element, and let A be an A_1 subgroup of G containing u. Let $\rho : G \to I(V)$ be an irreducible representation with highest weight λ.

If $I(V) = \mathrm{SL}(V)$, then $\rho(u)$ is distinguished in $I(V)$ if and only if $V \downarrow \rho(A)$ is irreducible, so the conclusion goes back to Dynkin [1957], but see also [Seitz 1987, Theorem 7.1] where the result is given explicitly. Alternatively it is easy to check in Tables 1 and 2 of Theorem 1, that except for ω_1 for A_n, B_n, C_n, and 10 for G_2, the subgroup acts reducibly on $V_G(\lambda)$.

Now suppose $I(V) = \mathrm{Sp}(V)$ or $\mathrm{SO}(V)$. If $\rho(u)$ is distinguished in $I(V)$, then $V \downarrow \rho(A)$ is multiplicity-free, and so λ is as in Tables 1 or 2 of Theorem 1. Moreover V is self-dual, so that $\lambda = -w_0(\lambda)$. Conversely, for all such λ in the tables, $V \downarrow \rho(A)$ is multiplicity-free, and so $\rho(u)$ has Jordan blocks on V of distinct sizes, hence is distinguished. This completes the proof.

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ACTION OF LONGEST ELEMENT ON A HECKE ALGEBRA CELL MODULE

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Dedicated to the memory of Robert Steinberg

By a result of Mathas, the basis element T_{w_0} of the Hecke algebra of a finite Coxeter group acts in the canonical basis of a cell module as a permutation matrix times plus or minus a power of v. We generalize this result to the unequal parameter case. We also show that the image of T_{w_0} in the corresponding asymptotic Hecke algebra is given by a simple formula.

Introduction

0.1. The Hecke algebra $\mathcal{H}$ (over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, v an indeterminate) of a finite Coxeter group W has two bases as an $\mathcal{A}$-module: the standard basis $\{T_x; x \in W\}$ and the basis $\{C_x; x \in W\}$ introduced in [Kazhdan and Lusztig 1979]. The second basis determines a decomposition of W into two-sided cells and a partial order for the set of two-sided cells, see [Kazhdan and Lusztig 1979]. Let $l \to \mathbb{N}$ be the length function, let w_0 be the longest element of W and let $\boldsymbol{c}$ be a two-sided cell. Let a (resp. a') be the value of the $\boldsymbol{a}$-function [Lusztig 2003, 13.4] on $\boldsymbol{c}$ (resp. on $w_0\boldsymbol{c}$). The following result was proved by Mathas [1996].

(a) *There exists a unique permutation $u \mapsto u^*$ of $\boldsymbol{c}$ such that for any $u \in \boldsymbol{c}$ we have $T_{w_0}(-1)^{l(u)}C_u = (-1)^{l(w_0)+a'}v^{-a+a'}(-1)^{l(u^*)}C_{u^*}$ plus an $\mathcal{A}$-linear combination of elements $C_{u'}$ with u' in a two-sided cell strictly smaller than $\boldsymbol{c}$. Moreover, for any $u \in \boldsymbol{c}$ we have $(u^*)^* = u$.*

A related (but weaker) result appears in [Lusztig 1984, (5.12.2)]. A result similar to (a) which concerns canonical bases in representations of quantum groups appears in [Lusztig 1990, Corollary 5.9]; now, in the case where W is of type A, (a) can be deduced from [loc. cit.] using the fact that irreducible representations of the Hecke algebra of type A (with their canonical bases) can be realized as 0-weight spaces of certain irreducible representations of a quantum group with their canonical bases.

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As R. Bezrukavnikov pointed out to the author, (a) specialized for $v=1$ (in the group algebra of W instead of $\mathcal{H}$) and assuming that W is crystallographic can be deduced from [Bezrukavnikov et al. 2012, Proposition 4.1] (a statement about Harish-Chandra modules), although it is not explicitly stated there.

In this paper we shall prove a generalization of (a) which applies to the Hecke algebra associated to W and any weight function assumed to satisfy the properties P1–P15 in [Lusztig 2003, §14], see Theorem 2.3; (a) corresponds to the special case where the weight function is equal to the length function. As an application we show that the image of T_{w_0} in the asymptotic Hecke algebra is given by a simple formula (see Corollary 2.8).

0.2. ***Notation.*** W is a finite Coxeter group; the set of simple reflections is denoted by S. We shall adopt many notations of [Lusztig 2003]. Let $\leq$ be the standard partial order on W. Let $l \to \mathbb{N}$ be the length function of W and let $L \to \mathbb{N}$ be a weight function (see [Lusztig 2003, 3.1]), that is, a function such that $L(ww') = L(w)+L(w')$ for any w, w' in W such that $l(ww') = l(w)+l(w')$; we assume that $L(s)>0$ for any $s \in S$. Let $w_0, \mathcal{A}$ be as in Section 0.1 and let $\mathcal{H}$ be the Hecke algebra over $\mathcal{A}$ associated to W, L as in [Lusztig 2003, 3.2]; we shall assume that properties P1–P15 in [Lusztig 2003, §14] are satisfied. (This holds automatically if $L=l$ by [Lusztig 2003, §15] using the results of [Elias and Williamson 2014]. This also holds in the quasisplit case, see [Lusztig 2003, §16].) We have $\mathcal{A} \subset \mathcal{A}' \subset K$ where $\mathcal{A}' = \mathbb{C}[v, v^{-1}]$, $K = \mathbb{C}(v)$. Let $\mathcal{H}_K = K \otimes_{\mathcal{A}} \mathcal{H}$ (a K-algebra). Recall that $\mathcal{H}$ has an $\mathcal{A}$-basis $\{T_x; x \in W\}$, see [Lusztig 2003, 3.2] and an $\mathcal{A}$-basis $\{c_x; x \in W\}$, see [Lusztig 2003, 5.2]. For $x \in W$ we have $c_x = \sum_{y \in W} p_{y,x} T_y$ and $T_x = \sum_{y \in W} (-1)^{l(xy)} p_{w_0x,w_0y} c_y$ (see [Lusztig 2003, 11.4]) where $p_{x,x}=1$ and $p_{y,x} \in v^{-1}\mathbb{Z}[v^{-1}]$ for $y \neq x$. We define preorders $\leq_{\mathcal{L}}, \leq_{\mathcal{R}}, \leq_{\mathcal{LR}}$ on W in terms of $\{c_x; x \in W\}$ as in [Lusztig 2003, 8.1]. Let $\sim_{\mathcal{L}}, \sim_{\mathcal{R}}, \sim_{\mathcal{LR}}$ be the corresponding equivalence relations on W, see [Lusztig 2003, 8.1] (the equivalence classes are called left cells, right cells, two-sided cells). Let $\bar{\ }: \mathcal{A} \to \mathcal{A}$ be the ring involution such that $\overline{v^n} = v^{-n}$ for $n \in \mathbb{Z}$. Let $\bar{\ }: \mathcal{H} \to \mathcal{H}$ be the ring involution such that $\overline{fT_x} = \bar{f} T_{x^{-1}}^{-1}$ for $x \in W$, $f \in \mathcal{A}$. For $x \in W$ we have $\overline{c_x} = c_x$. Let $h \mapsto h^\dagger$ be the algebra automorphism of $\mathcal{H}$ or of $\mathcal{H}_K$ given by $T_x \mapsto (-1)^{l(x)} T_{x^{-1}}^{-1}$ for all $x \in W$, see [Lusztig 2003, 3.5]. Then the basis $\{c_x^\dagger; x \in W\}$ of $\mathcal{H}$ is defined. (In the case where $L=l$, for any x we have $c_x^\dagger = (-1)^{l(x)} C_x$ where C_x is as in Section 0.1.) Let $h \mapsto h^\flat$ be the algebra antiautomorphism of $\mathcal{H}$ given by $T_x \mapsto T_{x^{-1}}$ for all $x \in W$, see [Lusztig 2003, 3.5]; for $x \in W$ we have $c_x^\flat = c_{x^{-1}}$, see [Lusztig 2003, 5.8]. For $x, y \in W$ we have $c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$, $c_x^\dagger c_y^\dagger = \sum_{z \in W} h_{x,y,z} c_z^\dagger$, where $h_{x,y,z} \in \mathcal{A}$. For any $z \in W$ there is a unique number $\boldsymbol{a}(z) \in \mathbb{N}$ such that for any x, y in W we have

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{\boldsymbol{a}(z)} + \text{strictly smaller powers of } v,$$

where $g_{x,y,z^{-1}} \in \mathbb{Z}$ and $g_{x,y,z^{-1}} \neq 0$ for some x, y in W. We have also

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{-\boldsymbol{a}(z)} + \text{strictly larger powers of } v.$$

Moreover $z \mapsto \boldsymbol{a}(z)$ is constant on any two-sided cell. The free abelian group J with basis $\{t_w;\, w \in W\}$ has an associative ring structure given by $t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z$; it has a unit element of the form $\sum_{d \in \mathcal{D}} n_d t_d$ where $\mathcal{D}$ is a subset of W consisting of certain elements with square 1 and $n_d = \pm 1$. Moreover for $d \in \mathcal{D}$ we have $n_d = \gamma_{d,d,d}$.

For any $x \in W$ there is a unique element $d_x \in \mathcal{D}$ such that $x \sim_{\mathcal{L}} d_x$. For a commutative ring R with 1 we set $J_R = R \otimes J$ (an R-algebra).

There is a unique $\mathcal{A}$-algebra homomorphism $\phi : \mathcal{H} \to J_{\mathcal{A}}$ such that $\phi(c_x^\dagger) = \sum_{d \in \mathcal{D}, z \in W; d_z = d} h_{x,d,z} n_d t_z$ for any $x \in W$. After applying $\mathbb{C} \otimes_{\mathcal{A}}$ to ϕ (we regard $\mathbb{C}$ as an $\mathcal{A}$-algebra via $v \mapsto 1$), ϕ becomes a $\mathbb{C}$-algebra isomorphism $\phi_{\mathbb{C}} : \mathbb{C}[W] \xrightarrow{\sim} J_{\mathbb{C}}$ (see [Lusztig 2003, 20.1(e)]). After applying $K \otimes_{\mathcal{A}}$ to ϕ, ϕ becomes a K-algebra isomorphism $\phi_K : \mathcal{H}_K \xrightarrow{\sim} J_K$ (see [Lusztig 2003, 20.1(d)]).

For any two-sided cell $\boldsymbol{c}$ let $\mathcal{H}^{\leq \boldsymbol{c}}$ (resp. $\mathcal{H}^{< \boldsymbol{c}}$) be the $\mathcal{A}$-submodule of $\mathcal{H}$ spanned by $\{c_x^\dagger, x \in W, x \leq_{\mathcal{LR}} x' \text{ for some } x' \in \boldsymbol{c}\}$ (resp. $\{c_x^\dagger, x \in W, x <_{\mathcal{LR}} x' \text{ for some } x' \in \boldsymbol{c}\}$). Note that $\mathcal{H}^{\leq \boldsymbol{c}}$, $\mathcal{H}^{< \boldsymbol{c}}$ are two-sided ideals in $\mathcal{H}$. Hence $\mathcal{H}^{\boldsymbol{c}} := \mathcal{H}^{\leq \boldsymbol{c}} / \mathcal{H}^{< \boldsymbol{c}}$ is an $(\mathcal{H}, \mathcal{H})$-bimodule. It has an $\mathcal{A}$-basis $\{c_x^\dagger, x \in \boldsymbol{c}\}$. Let $J^{\boldsymbol{c}}$ be the subgroup of J spanned by $\{t_x;\, x \in \boldsymbol{c}\}$. This is a two-sided ideal of J. Similarly, $J_{\mathbb{C}}^{\boldsymbol{c}} := \mathbb{C} \otimes J^{\boldsymbol{c}}$ is a two-sided ideal of $J_{\mathbb{C}}$ and $J_K^{\boldsymbol{c}} := K \otimes J^{\boldsymbol{c}}$ is a two-sided ideal of J_K.

We write $E \in \operatorname{Irr} W$ whenever E is a simple $\mathbb{C}[W]$-module. We can view E as a (simple) $J_{\mathbb{C}}$-module $E_\spadesuit$ via the isomorphism $\phi_{\mathbb{C}}^{-1}$. Then the (simple) J_K-module $K \otimes_{\mathbb{C}} E_\spadesuit$ can be viewed as a (simple) $\mathcal{H}_K$-module E_v via the isomorphism ϕ_K. Let $E^\dagger$ be the simple $\mathbb{C}[W]$-module which coincides with E as a $\mathbb{C}$-vector space but with the w action on $E^\dagger$ (for $w \in W$) being $(-1)^{l(w)}$ times the w-action on E. Let $\boldsymbol{a}_E \in \mathbb{N}$ be as in [Lusztig 2003, 20.6(a)].

1. Preliminaries

1.1. Let $\sigma : W \to W$ be the automorphism given by $w \mapsto w_0 w w_0$; it satisfies $\sigma(S) = S$ and it extends to a $\mathbb{C}$-algebra isomorphism $\sigma : \mathbb{C}[W] \to \mathbb{C}[W]$. For $s \in S$ we have $l(w_0) = l(w_0 s) + l(s) = l(\sigma(s)) + l(\sigma(s) w_0)$ hence $L(w_0) = L(w_0 s) + L(s) = L(\sigma(s)) + L(\sigma(s) w_0) = L(\sigma(s)) + L(w_0 s)$ so that $L(\sigma(s)) = L(s)$. It follows that $L(\sigma(w)) = L(w)$ for all $w \in W$ and that we have an $\mathcal{A}$-algebra automorphism $\sigma : \mathcal{H} \to \mathcal{H}$ where $\sigma(T_w) = T_{\sigma(w)}$ for any $w \in W$. This extends to a K-algebra isomorphism $\sigma : \mathcal{H}_K \to \mathcal{H}_K$. We have $\sigma(c_w) = c_{\sigma(w)}$ for any $w \in W$. For any $h \in \mathcal{H}$ we have $\sigma(h^\dagger) = (\sigma(h))^\dagger$. Hence we have $\sigma(c_w^\dagger) = c_{\sigma(w)}^\dagger$ for any $w \in W$. We have $h_{\sigma(x),\sigma(y),\sigma(z)} = h_{x,y,z}$ for all $x, y, z \in W$. It follows that $\boldsymbol{a}(\sigma(w)) = \boldsymbol{a}(w)$ for all $w \in W$ and $\gamma_{\sigma(x),\sigma(y),\sigma(z)} = \gamma_{x,y,z}$ for all $x, y, z \in W$ so that we have a ring isomorphism $\sigma : J \to J$ where $\sigma(t_w) = t_{\sigma(w)}$ for any $w \in W$. This extends to an $\mathcal{A}$-algebra isomorphism $\sigma : J_{\mathcal{A}} \to J_{\mathcal{A}}$, to a $\mathbb{C}$-algebra isomorphism $\sigma : J_{\mathbb{C}} \to J_{\mathbb{C}}$ and to a K-algebra isomorphism $\sigma : J_K \to J_K$. From the definitions we see that $\phi : \mathcal{H} \to J_{\mathcal{A}}$ (see Section 0.2) satisfies $\phi\sigma = \sigma\phi$. Hence $\phi_{\mathbb{C}}$ satisfies $\phi_{\mathbb{C}}\sigma = \sigma\phi_{\mathbb{C}}$ and ϕ_K satisfies $\phi_K\sigma = \sigma\phi_K$. We show:

(a) *For $h \in \mathcal{H}$ we have $\sigma(h) = T_{w_0} h T_{w_0}^{-1}$.*

It is enough to show this for h running through a set of algebra generators of $\mathcal{H}$. Thus we can assume that $h = T_s^{-1}$ with $s \in S$. We must show that $T_{\sigma(s)}^{-1} T_{w_0} = T_{w_0} T_s^{-1}$: both sides are equal to $T_{\sigma(s)w_0} = T_{w_0 s}$.

Lemma 1.2. *For any $x \in W$ we have $\sigma(x) \sim_{\mathcal{LR}} x$.*

From 1.1(a) we deduce that $T_{w_0} c_x T_{w_0}^{-1} = c_{\sigma(x)}$. In particular, $\sigma(x) \le_{\mathcal{LR}} x$. Replacing x by $\sigma(x)$ we obtain $x \le_{\mathcal{LR}} \sigma(x)$. The lemma follows.

1.3. Let $E \in \mathrm{Irr} W$. We define $\sigma_E : E \to E$ by $\sigma_E(e) = w_0 e$ for $e \in E$. We have $\sigma_E^2 = 1$. For $e \in E$, $w \in W$, we have $\sigma_E(we) = \sigma(w)\sigma_E(e)$. We can view σ_E as a vector space isomorphism $E_\spadesuit \xrightarrow{\sim} E_\spadesuit$. For $e \in E_\spadesuit$, $w \in W$ we have $\sigma_E(t_w e) = t_{\sigma(w)}\sigma_E(e)$. Now $\sigma_E : E_\spadesuit \to E_\spadesuit$ defines by extension of scalars a vector space isomorphism $E_v \to E_v$ denoted again by σ_E. It satisfies $\sigma_E^2 = 1$. For $e \in E_v$, $w \in W$ we have $\sigma_E(T_w e) = T_{\sigma(w)}\sigma_E(e)$.

Lemma 1.4. *Let $E \in \mathrm{Irr} W$. There is a unique (up to multiplication by a scalar in $K - \{0\}$) vector space isomorphism $g : E_v \to E_v$ such that $g(T_w e) = T_{\sigma(w)} g(e)$ for all $w \in W$, $e \in E_v$. We can take for example $g = T_{w_0} : E_v \to E_v$ or $g = \sigma_E : E_v \to E_v$. Hence $T_{w_0} = \lambda_E \sigma_E : E_v \to E_v$ where $\lambda_E \in K - \{0\}$.*

The existence of g is clear from the second sentence of the lemma. If g' is another isomorphism $g' : E_v \to E_v$ such that $g'(T_w e) = T_{\sigma(w)} g'(e)$ for all $w \in W$, $e \in E_v$, then for any $e \in E_v$ we have $g^{-1}g'(T_w e) = g^{-1}T_{\sigma(w)} g'(e) = T_w g^{-1} g'(e)$ and using Schur's lemma we see that $g^{-1}g'$ is a scalar. This proves the first sentence of the lemma hence the third sentence of the lemma.

1.5. Let $E \in \mathrm{Irr} W$. We have

(a) $$\sum_{x \in W} \mathrm{tr}(T_x, E_v)\,\mathrm{tr}(T_{x^{-1}}, E_v) = f_{E_v} \dim(E)$$

where $f_{E_v} \in \mathcal{A}'$ is of the form

(b) $$f_{E_v} = f_0 v^{-2\boldsymbol{a}_E} + \text{strictly higher powers of } v$$

and $f_0 \in \mathbb{C} - \{0\}$. (See [Lusztig 2003, 19.1(e), 20.1(c), 20.7].)

From Lemma 1.4 we see that $\lambda_E^{-1}T_{w_0}$ acts on E_v as σ_E. Using [Lusztig 2005, 34.14(e)] with $c=\lambda_E^{-1}T_{w_0}$ (an invertible element of $\mathcal{H}_K$) we see that

(c) $$\sum_{x\in W}\mathrm{tr}(T_x\sigma_E,E_v)\,\mathrm{tr}(\sigma_E^{-1}T_{x^{-1}},E_v)=f_{E_v}\dim(E).$$

Lemma 1.6. *Let $E\in\mathrm{Irr}W$. We have $\lambda_E=v^{n_E}$ for some $n_E\in\mathbb{Z}$.*

For any $x\in W$ we have

$$\mathrm{tr}(\sigma_E c_x^\dagger,E_v)=\sum_{d\in\mathcal{D},z\in W;d=d_z}h_{x,d,z}n_d\,\mathrm{tr}(\sigma_E t_z,E_\spadesuit)\in\mathcal{A}'$$

since $\mathrm{tr}(\sigma_E t_z,E_\spadesuit)\in\mathbb{C}$. It follows that $\mathrm{tr}(\sigma_E h,E_v)\in\mathcal{A}'$ for any $h\in\mathcal{H}$. In particular, both $\mathrm{tr}(\sigma_E T_{w_0},E_v)$ and $\mathrm{tr}(T_{w_0}^{-1}\sigma_E,E_v)$ belong to $\mathcal{A}'$. Thus $\lambda_E\dim E$ and $\lambda_E^{-1}\dim E$ belong to $\mathcal{A}'$ so that $\lambda_E=bv^n$ for some $b\in\mathbb{C}-\{0\}$ and $n\in\mathbb{Z}$. From the definitions we have $\lambda_E|_{v=1}=1$ (for $v=1$, T_{w_0} becomes w_0) hence $b=1$. The lemma is proved.

Lemma 1.7. *Let $E\in\mathrm{Irr}W$. There exists $\epsilon_E\in\{1,-1\}$ such that for any $x\in W$ we have*

(a) $$\mathrm{tr}(\sigma_{E^\dagger}T_x,(E^\dagger)_v)=\epsilon_E(-1)^{l(x)}\,\mathrm{tr}(\sigma_E T_{x^{-1}}^{-1},E_v).$$

Let $(E_v)^\dagger$ be the $\mathcal{H}_K$-module with underlying vector space E_v such that the action of $h\in\mathcal{H}_K$ on $(E_v)^\dagger$ is the same as the action of $h^\dagger$ on E_v. From the proof in [Lusztig 2003, 20.9] we see that there exists an isomorphism of $\mathcal{H}_K$-modules $b:(E_v)^\dagger\xrightarrow{\sim}(E^\dagger)_v$. Let $\iota:(E_v)^\dagger\to(E_v)^\dagger$ be the vector space isomorphism which corresponds under b to $\sigma_{E^\dagger}:(E^\dagger)_v\to(E^\dagger)_v$. Then we have $\mathrm{tr}(\sigma_{E^\dagger}T_x,(E^\dagger)_v)=\mathrm{tr}(\iota T_x,(E_v)^\dagger)$. It is enough to prove that $\iota=\pm\sigma_E$ as a K-linear map of the vector space $E_v=(E_v)^\dagger$ into itself. From the definition we have $\iota(T_we)=T_{\sigma(w)}\iota(e)$ for all $w\in W,e\in(E_v)^\dagger$. Hence $(-1)^{l(w)}\iota(T_{w^{-1}}^{-1}e)=(-1)^{l(w)}T_{\sigma(w^{-1})}^{-1}\iota(e)$ for all $w\in W,e\in E_v$. It follows that $\iota(he)=(-1)^{l(w)}T_{\sigma(h)}\iota(e)$ for all $h\in\mathcal{H},e\in E_v$. Hence $\iota(T_we)=T_{\sigma(w)}\iota(e)$ for all $w\in W,e\in E_v$. By the uniqueness in Lemma 1.4 we see that $\iota=\epsilon_E\sigma_E:E_v\to E_v$ where $\epsilon_E\in K-\{0\}$. Since $\iota^2=1$, $\sigma_E^2=1$, we see that $\epsilon_E=\pm1$. The lemma is proved.

Lemma 1.8. *Let $E\in\mathrm{Irr}W$. We have $n_E=-\boldsymbol{a}_E+\boldsymbol{a}_{E^\dagger}$.*

For $x\in W$ we have (using Lemmas 1.4 and 1.6)

(a) $$\mathrm{tr}(T_{w_0x},E_v)=\mathrm{tr}(T_{w_0}T_{x^{-1}}^{-1},E_v)=v^{n_E}\,\mathrm{tr}(\sigma_E T_{x^{-1}}^{-1},E_v).$$

Making a change of variable $x \mapsto w_0 x$ in 1.5(a) and using that $T_{x^{-1}w_0} = T_{w_0\sigma(x)^{-1}}$ we obtain

$$f_{E_v}\dim(E) = \sum_{x\in W} \operatorname{tr}(T_{w_0 x}, E_v)\operatorname{tr}(T_{w_0\sigma(x)^{-1}}, E_v)$$
$$= v^{2n_E}\sum_{x\in W} \operatorname{tr}(\sigma_E T_{x^{-1}}^{-1}, E_v)\operatorname{tr}(\sigma_E T_{\sigma(x)}^{-1}, E_v).$$

Using now Lemma 1.7 and the equality $l(x) = l(\sigma(x^{-1}))$ we obtain

$$f_{E_v}\dim(E) = v^{2n_E}\sum_{x\in W} \operatorname{tr}(\sigma_{E^\dagger}T_x, (E^\dagger)_v)\operatorname{tr}(\sigma_{E^\dagger}T_{\sigma(x^{-1})}, (E^\dagger)_v)$$
$$= v^{2n_E}\sum_{x\in W} \operatorname{tr}(\sigma_{E^\dagger}T_x, (E^\dagger)_v)\operatorname{tr}(T_{\xi^{-1}}\sigma_{E^\dagger}, (E^\dagger)_v)$$
$$= v^{2n_E} f_{(E^\dagger)_v}\dim(E^\dagger).$$

(The last step uses 1.5(c) for $E^\dagger$ instead of E.) Thus we have $f_{E_v} = v^{2n_E} f_{(E^\dagger)_v}$. The left-hand side is as in 1.5(b) and similarly the right-hand side of the form

$$f_0' v^{2n_E - 2\boldsymbol{a}_{E^\dagger}} + \text{strictly higher powers of } v$$

where $f_0, f_0' \in \mathbb{C} - \{0\}$. It follows that $-2\boldsymbol{a}_E = 2n_E - 2\boldsymbol{a}_{E^\dagger}$. The lemma is proved.

Lemma 1.9. *Let $E \in \operatorname{Irr} W$ and let $x \in W$. We have*

(a) $$\operatorname{tr}(T_x, E_v) = (-1)^{l(x)} v^{-\boldsymbol{a}_E}\operatorname{tr}(t_x, E_\spadesuit) \mod v^{-\boldsymbol{a}_E+1}\mathbb{C}[v],$$

(b) $$\operatorname{tr}(\sigma_E T_x, E_v) = (-1)^{l(x)} v^{-\boldsymbol{a}_E}\operatorname{tr}(\sigma_E t_x, E_\spadesuit) \mod v^{-\boldsymbol{a}_E+1}\mathbb{C}[v].$$

For a proof of (a), see [Lusztig 2003, 20.6(b)]. We now give a proof of (b) along the same lines as that of (a). There is a unique two sided cell $\boldsymbol{c}$ such that $t_z|_{E_\spadesuit} = 0$ for $z \in W - \boldsymbol{c}$. Let $a = \boldsymbol{a}(z)$ for all $z \in \boldsymbol{c}$. By [Lusztig 2003, 20.6(c)] we have $a = \boldsymbol{a}_E$. From the definition of c_x we see that $T_x = \sum_{y\in W} f_y c_y$, where $f_x = 1$ and $f_y \in v^{-1}\mathbb{Z}[v^{-1}]$ for $y \neq x$. Applying $\dagger$ we obtain $(-1)^{l(x)}T_{x^{-1}}^{-1} = \sum_{y\in W} f_y c_y^\dagger$; applying $\bar{\ }$ we obtain $(-1)^{l(x)}T_x = \sum_{y\in W} \bar{f}_y c_y^\dagger$. Thus we have

$$(-1)^{l(x)}\operatorname{tr}(\sigma_E T_x, E_v) = \sum_{y\in W} \bar{f}_y \operatorname{tr}(\sigma_E c_y^\dagger, E_v) = \sum_{\substack{y,z\in W\\ d\in\mathcal{D};\, d=d_z}} \bar{f}_y h_{y,d,z} n_d \operatorname{tr}(\sigma_E t_z, E_\spadesuit).$$

In the last sum we can assume that $z \in \boldsymbol{c}$ and $d \in \boldsymbol{c}$ so that $h_{y,d,z} = \gamma_{y,d,z^{-1}} v^{-a}$ mod $v^{-a+1}\mathbb{Z}[v]$. Since $\bar{f}_x = 1$ and $\bar{f}_y \in v\mathbb{Z}[v]$ for all $y \neq x$ we see that

$$(-1)^{l(x)}\operatorname{tr}(\sigma_E T_x, E_v) = \sum_{\substack{z\in\boldsymbol{c}\\ d\in\mathcal{D}\cap\boldsymbol{c}}} \gamma_{x,d,z^{-1}} n_d v^{-a}\operatorname{tr}(\sigma_E t_z, E_\spadesuit) \mod v^{-a+1}\mathbb{C}[v].$$

If $x \notin \boldsymbol{c}$ then $\gamma_{x,d,z^{-1}} = 0$ for all d, z in the sum so that $\operatorname{tr}(\sigma_E T_x, E_v) = 0$; we have also $\operatorname{tr}(\sigma_E t_x, E_\spadesuit) = 0$ and the desired formula follows. We now assume that $x \in \boldsymbol{c}$. Then for d, z as above we have $\gamma_{x,d,z^{-1}} = 0$ unless $x = z$ and $d = d_x$ in which case $\gamma_{x,d,z^{-1}} n_d = 1$. Thus (b) holds again. The lemma is proved.

Lemma 1.10. *Let $E \in \operatorname{Irr} W$. Let $\boldsymbol{c}$ be the unique two sided cell such that $t_z|_{E_\spadesuit} = 0$ for $z \in W - \boldsymbol{c}$. Let $\boldsymbol{c}'$ be the unique two sided cell such that $t_z|_{(E^\dagger)_\spadesuit} = 0$ for $z \in W - \boldsymbol{c}'$. We have $\boldsymbol{c}' = w_0 \boldsymbol{c}$.*

Using 1.8(a) and 1.7(a) we have

(a) $$\operatorname{tr}(T_{w_0 x}, E_v) = v^{n_E} \operatorname{tr}(\sigma_E T^{-1}_{x^{-1}}, E_v) = v^{n_E} \epsilon_E (-1)^{l(x)} \operatorname{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v).$$

Using 1.9(a) for E and 1.9(b) for $E^\dagger$ we obtain

$$\operatorname{tr}(T_{w_0 x}, E_v) = (-1)^{l(w_0 x)} v^{-\boldsymbol{a}_E} \operatorname{tr}(t_{w_0 x}, E_\spadesuit) \mod v^{-\boldsymbol{a}_E + 1}\mathbb{C}[v],$$
$$\operatorname{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v) = (-1)^{l(x)} v^{-\boldsymbol{a}_{E^\dagger}} \operatorname{tr}(\sigma_{E^\dagger} t_x, E^\dagger_\spadesuit) \mod v^{-\boldsymbol{a}_{E^\dagger} + 1}\mathbb{C}[v].$$

Combining with (a) we obtain

$$\begin{aligned}&(-1)^{l(w_0 x)} v^{-\boldsymbol{a}_E} \operatorname{tr}(t_{w_0 x}, E_\spadesuit) + \text{strictly higher powers of } v\\ &= v^{n_E} \epsilon_E v^{-\boldsymbol{a}_{E^\dagger}} \operatorname{tr}(\sigma_{E^\dagger} t_x, E^\dagger_\spadesuit) + \text{strictly higher powers of } v.\end{aligned}$$

Using the equality $n_E = -\boldsymbol{a}_E + \boldsymbol{a}_{E^\dagger}$ (see Lemma 1.8) we deduce

$$(-1)^{l(w_0 x)} \operatorname{tr}(t_{w_0 x}, E_\spadesuit) = \epsilon_E \operatorname{tr}(\sigma_{E^\dagger} t_x, E^\dagger_\spadesuit).$$

Now we can find $x \in W$ such that $\operatorname{tr}(t_{w_0 x}, E_\spadesuit) \neq 0$ and the previous equality shows that $t_x|_{(E^\dagger)_\spadesuit} \neq 0$. Moreover from the definition we have $w_0 x \in \boldsymbol{c}$ and $x \in \boldsymbol{c}'$ so that $w_0 \boldsymbol{c} \cap \boldsymbol{c}' \neq \varnothing$. Since $w_0 \boldsymbol{c}$ is a two-sided cell (see [Lusztig 2003, 11.7(d)]) it follows that $w_0 \boldsymbol{c} = \boldsymbol{c}'$. The lemma is proved.

Lemma 1.11. *Let $\boldsymbol{c}$ be a two-sided cell of W. Let $\boldsymbol{c}'$ be the two-sided cell $w_0 \boldsymbol{c} = \boldsymbol{c} w_0$ (see Lemma 1.2). Let $a = \boldsymbol{a}(x)$ for any $x \in \boldsymbol{c}$; let $a' = \boldsymbol{a}(x')$ for any $x' \in \boldsymbol{c}'$. The K-linear map $J_K^{\boldsymbol{c}} \to J_K^{\boldsymbol{c}}$ given by $\xi \mapsto \phi(v^{a-a'} T_{w_0})\xi$ (left multiplication in J_K) is obtained from a $\mathbb{C}$-linear map $J_{\mathbb{C}}^{\boldsymbol{c}} \to J_{\mathbb{C}}^{\boldsymbol{c}}$ (with square 1) by extension of scalars from $\mathbb{C}$ to K.*

We can find a direct sum decomposition $J_{\mathbb{C}}^{\boldsymbol{c}} = \oplus_{i=1}^m E^i$ where E^i are simple left ideals of $J_{\mathbb{C}}$ contained in $J_{\mathbb{C}}^{\boldsymbol{c}}$. We have $J_K^{\boldsymbol{c}} = \oplus_{i=1}^m K \otimes E^i$. It is enough to show that for any i, the K-linear map $K \otimes E^i \to K \otimes E^i$ given by the action of $\phi(v^{a-a'} T_{w_0})$ in the left J_K-module structure of $K \otimes E^i$ is obtained from a $\mathbb{C}$-linear map $E^i \to E^i$ (with square 1) by extension of scalars from $\mathbb{C}$ to K. We can find $E \in \operatorname{Irr} W$ such that E^i is isomorphic to $E_\spadesuit$ as a $J_{\mathbb{C}}$-module. It is then enough to show that the action of $v^{a-a'} T_{w_0}$ in the left $\mathcal{H}_K$-module structure of E_v is obtained from the map $\sigma_E : E \to E$ by extension of scalars from $\mathbb{C}$ to K. This follows from the equality

$v^{a-a'}T_{w_0} = \sigma_E : E_v \to E_v$ (since σ_E is obtained by extension of scalars from a $\mathbb{C}$-linear map $E \to E$ with square 1) provided that we show that $-n_E = a - a'$. Since $n_E = -\boldsymbol{a}_E + \boldsymbol{a}_{E^\dagger}$ (see Lemma 1.8) it is enough to show that $a = \boldsymbol{a}_E$ and $a' = \boldsymbol{a}_{E^\dagger}$. The equality $a = \boldsymbol{a}_E$ follows from [Lusztig 2003, 20.6(c)]. The equality $a' = \boldsymbol{a}_{E^\dagger}$ also follows from [Lusztig 2003, 20.6(c)] applied to $E^\dagger$, $\boldsymbol{c}' = w_0\boldsymbol{c}$ instead of $E, \boldsymbol{c}$ (see Lemma 1.10). The lemma is proved.

Lemma 1.12. *In the setup of Lemma 1.11 we have*

(a) $$\phi(v^{a-a'}T_{w_0})t_x = \sum_{x' \in \boldsymbol{c}} m_{x',x} t_{x'}$$

and

(b) $$\phi(v^{2a-2a'}T_{w_0}^2)t_x = t_x$$

for any $x \in \boldsymbol{c}$, where $m_{x',x} \in \mathbb{Z}$.

Now (b) and the fact that (a) holds with $m_{x',x} \in \mathbb{C}$ is just a restatement of Lemma 1.11. Since $\phi(v^{a-a'}T_{w_0}) \in J_{\mathcal{A}}$ we have also $m_{x',x} \in \mathcal{A}$. We now use that $\mathcal{A} \cap \mathbb{C} = \mathbb{Z}$ and the lemma follows.

Lemma 1.13. *In the setup of Lemma 1.11 we have for any $x \in \boldsymbol{c}$ the equalities*

(a) $$v^{a-a'}T_{w_0}c_x^\dagger = \sum_{x' \in \boldsymbol{c}} m_{x',x} c_{x'}^\dagger$$

and

(b) $$v^{2a-2a'}T_{w_0}^2 c_x^\dagger = c_x^\dagger$$

in $\mathcal{H}^{\boldsymbol{c}}$, where $m_{x',x} \in \mathbb{Z}$ are the same as in Lemma 1.12. Moreover, if $m_{x',x} \neq 0$ then $x' \sim_{\mathcal{L}} x$.

The first sentence follows from Lemma 1.12 using [Lusztig 2003, 18.10(a)]. Clearly, if $m_{x',x} \neq 0$ then $x' \leq_{\mathcal{L}} x$, which together with $x' \sim_{\mathcal{LR}} x$ implies $x' \sim_{\mathcal{L}} x$.

2. The main results

2.1. In this section we fix a two-sided cell $\boldsymbol{c}$ of W; a, a' are as in Lemma 1.11. We define an $\mathcal{A}$-linear map $\theta : \mathcal{H}^{\leq \boldsymbol{c}} \to \mathcal{A}$ by $\theta(c_x^\dagger) = 1$ if $x \in \mathcal{D} \cap \boldsymbol{c}$, $\theta(c_x^\dagger) = 0$ if $x \leq_{\mathcal{LR}} x'$ for some $x' \in \boldsymbol{c}$ and $x \notin \mathcal{D} \cap \boldsymbol{c}$. Note that θ is zero on $\mathcal{H}^{<\boldsymbol{c}}$ hence it can be viewed as an $\mathcal{A}$-linear map $\mathcal{H}^{\boldsymbol{c}} \to \mathcal{A}$.

Lemma 2.2. *Let $x, x' \in \boldsymbol{c}$. We have*

(a) $$\theta(c_{x^{-1}}^\dagger c_{x'}^\dagger) = n_{d_x}\delta_{x,x'}v^a + \textit{strictly lower powers of } v.$$

The left-hand side of (a) is

$$\sum_{d\in\mathcal{D}\cap\boldsymbol{c}} h_{x^{-1},x',d} = \sum_{d\in\mathcal{D}\cap\boldsymbol{c}} \gamma_{x^{-1},x',d}v^a + \text{strictly lower powers of } v$$
$$= n_{d_x}\delta_{x,x'}v^a + \text{strictly lower powers of } v.$$

The lemma is proved.

We now state one of the main results of this paper.

Theorem 2.3. *There exists a unique permutation $u \mapsto u^*$ of $\boldsymbol{c}$ (with square 1) such that for any $u \in \boldsymbol{c}$ we have*

(a) $$v^{a-a'}T_{w_0}c_u^\dagger = \epsilon_u c_{u^*}^\dagger \mod \mathcal{H}^{<\boldsymbol{c}}$$

where $\epsilon_u = \pm 1$. For any $u \in \boldsymbol{c}$ we have $\epsilon_{u^{-1}} = \epsilon_u = \epsilon_{\sigma(u)} = \epsilon_{u^*}$ and $\sigma(u^*) = (\sigma(u))^* = ((u^{-1})^*)^{-1}$.

Let $u \in \boldsymbol{c}$. We set $Z = \theta((v^{a-a'}T_{w_0}c_u^\dagger)^\flat v^{a-a'}T_{w_0}c_u^\dagger)$. We compute Z in two ways, using Lemma 2.2 and Lemma 1.13. We have

$$Z = \theta(c_{u^{-1}}^\dagger v^{2a-2a'}T_{w_0}^2 c_u^\dagger) = \theta(c_{u^{-1}}^\dagger c_u^\dagger) = n_{d_u}v^a + \text{strictly lower powers of } v,$$

$$\begin{aligned} Z &= \theta\left(\left(\sum_{y\in\boldsymbol{c}} m_{y,u}c_y^\dagger\right)^\flat \sum_{y'\in\boldsymbol{c}} m_{y',u}c_{y'}^\dagger\right) = \sum_{y,y'\in\boldsymbol{c}} m_{y,u}m_{y',u}\theta(c_{y^{-1}}^\dagger c_{y'}^\dagger) \\ &= \sum_{y,y'\in\boldsymbol{c}} m_{y,u}m_{y',u}n_{d_y}\delta_{y,y'}v^a + \text{strictly lower powers of } v \\ &= \sum_{y\in\boldsymbol{c}} n_{d_y}m_{y,u}^2 v^a + \text{strictly lower powers of } v \\ &= \sum_{y\in\boldsymbol{c}} n_{d_u}m_{y,u}^2 v^a + \text{strictly lower powers of } v \end{aligned},$$

where $m_{y,u} \in \mathbb{Z}$ is zero unless $y \sim_{\mathcal{L}} u$ (see Lemma 1.13), in which case we have $d_y = d_u$. We deduce that $\sum_{y\in\boldsymbol{c}} m_{y,u}^2 = 1$, so that we have $m_{y,u} = \pm 1$ for a unique $y \in \boldsymbol{c}$ (denoted by u^*) and $m_{y,u} = 0$ for all $y \in \boldsymbol{c} - \{u^*\}$. Then 2.3(a) holds. Using 2.3(a) and Lemma 1.13(b) we see that $u \mapsto u^*$ has square 1 and that $\epsilon_u\epsilon_{u^*} = 1$.

The automorphism $\sigma : \mathcal{H} \to \mathcal{H}$ (see Section 1.1) satisfies the equality $\sigma(c_u^\dagger) = c_{\sigma(u)}^\dagger$ for any $u \in W$; note also that $w \in \boldsymbol{c} \leftrightarrow \sigma(w) \in \boldsymbol{c}$ (see Lemma 1.2). Applying σ to 2.3(a) we obtain

$$v^{a-a'}T_{w_0}c_{\sigma(u)}^\dagger = \epsilon_u c_{\sigma(u^*)}^\dagger$$

in $\mathcal{H}^{\boldsymbol{c}}$. By 2.3(a) we have also $v^{a-a'}T_{w_0}c_{\sigma(u)}^\dagger = \epsilon_{\sigma(u)}c_{(\sigma(u))^*}^\dagger$ in $\mathcal{H}^{\boldsymbol{c}}$. It follows that $\epsilon_u c_{\sigma(u^*)}^\dagger = \epsilon_{\sigma(u)}c_{(\sigma(u))^*}^\dagger$ hence $\epsilon_u = \epsilon_{\sigma(u)}$ and $\sigma(u^*) = (\sigma(u))^*$.

Applying $h \mapsto h^\flat$ to 2.3(a) we obtain

$$v^{a-a'}c_{u^{-1}}^\dagger T_{w_0} = \epsilon_u c_{(u^*)^{-1}}^\dagger$$

in $\mathcal{H}^{\boldsymbol{c}}$. By 2.3(a) we have also

$$v^{a-a'} c^{\dagger}_{u^{-1}} T_{w_0} = v^{a-a'} T_{w_0} c^{\dagger}_{\sigma(u^{-1})} = \epsilon_{\sigma(u^{-1})} c^{\dagger}_{(\sigma(u^{-1}))^*}$$

in $\mathcal{H}^{\boldsymbol{c}}$. It follows that $\epsilon_u c^{\dagger}_{(u^*)^{-1}} = \epsilon_{\sigma(u^{-1})} c^{\dagger}_{(\sigma(u^{-1}))^*}$ hence $\epsilon_u = \epsilon_{\sigma(u^{-1})}$ and $(u^*)^{-1} = (\sigma(u^{-1}))^*$. Since $\epsilon_{\sigma(u^{-1})} = \epsilon_{u^{-1}}$, we see that $\epsilon_u = \epsilon_{u^{-1}}$. Replacing u by u^{-1} in $(u^*)^{-1} = (\sigma(u^{-1}))^*$ we obtain $((u^{-1})^*)^{-1} = (\sigma(u))^*$ as required. The theorem is proved.

2.4. For $u \in \boldsymbol{c}$ we have

(a) $$u \sim_{\mathcal{L}} u^*,$$

(b) $$\sigma(u) \sim_{\mathcal{R}} u^*.$$

Indeed, (a) follows from Lemma 1.13. To prove (b) it is enough to show that $\sigma(u)^{-1} \sim_{\mathcal{L}} (u^*)^{-1}$. Using (a) for $\sigma(u)^{-1}$ instead of u we see that it is enough to show that $(\sigma(u^{-1}))^* = (u^*)^{-1}$; this follows from Theorem 2.3.

If we assume that

(c) *any left cell in* $\boldsymbol{c}$ *intersects any right cell in* $\boldsymbol{c}$ *in exactly one element*

then by (a), (b), for any $u \in \boldsymbol{c}$,

(d) u^* *is the unique element of* $\boldsymbol{c}$ *in the intersection of the left cell of* u *with right cell of* $\sigma(u)$.

Note that condition (c) is satisfied for any $\boldsymbol{c}$ if W is of type A_n or if W is of type B_n ($n \geq 2$) with $L(s) = 2$ for all but one $s \in S$ and $L(s) = 1$ or 3 for the remaining $s \in S$. (In this last case we are in the quasisplit case and we have $\sigma = 1$ hence $u^* = u$ for all u.)

Theorem 2.5. *For any* $x \in W$ *we set* $\vartheta(x) = \gamma_{w_0 d_{w_0 x^{-1}}, x, (x^*)^{-1}}$.

(a) *If* $d \in \mathcal{D}$ *and* $x, y \in \boldsymbol{c}$ *satisfy* $\gamma_{w_0 d, x, y} \neq 0$ *then* $y = (x^*)^{-1}$.

(b) *If* $x \in \boldsymbol{c}$ *then there is a unique* $d \in \mathcal{D} \cap w_0 \boldsymbol{c}$ *such that* $\gamma_{w_0 d, x, (x^*)^{-1}} \neq 0$, *namely* $d = d_{w_0 x^{-1}}$. *Moreover we have* $\vartheta(x) = \pm 1$.

(c) *For* $u \in \boldsymbol{c}$ *we have* $\epsilon_u = (-1)^{l(w_0 d)} n_d \vartheta(u)$ *where* $d = d_{w_0 u^{-1}}$.

Applying $h \mapsto h^{\dagger}$ to 2.3(a) we obtain for any $u \in \boldsymbol{c}$:

(d) $$v^{a-a'} (-1)^{l(w_0)} \overline{T_{w_0}} c_u = \sum_{z \in \boldsymbol{c}} \delta_{z,u^*} \epsilon_u c_z \mod \sum_{z' \in W - \boldsymbol{c}} \mathcal{A} c_{z'}.$$

We have $T_{w_0} = \sum_{y \in W} (-1)^{l(w_0 y)} p_{1, w_0 y} c_y$ hence $\overline{T_{w_0}} = \sum_{y \in W} (-1)^{l(w_0 y)} \overline{p_{1, w_0 y}} c_y$. Introducing this in (d) we obtain

$$v^{a-a'} \sum_{y \in W} (-1)^{l(y)} \overline{p_{1, w_0 y}} c_y c_u = \sum_{z \in \boldsymbol{c}} \delta_{z,u^*} \epsilon_u c_z \mod \sum_{z' \in W - \boldsymbol{c}} \mathcal{A} c_{z'},$$

that is,

$$v^{a-a'} \sum_{y,z \in W} (-1)^{l(y)} \overline{p_{1,w_0 y}} h_{y,u,z} c_z = \sum_{z \in \boldsymbol{c}} \delta_{z,u^*} \epsilon_u c_z \mod \sum_{z' \in W - \boldsymbol{c}} \mathcal{A} c_{z'}.$$

Thus, for $z \in \boldsymbol{c}$ we have

(e) $$v^{a-a'} \sum_{y \in W} (-1)^{l(y)} \overline{p_{1,w_0 y}} h_{y,u,z} = \delta_{z,u^*} \epsilon_u.$$

Here we have $h_{y,u,z} = \gamma_{y,u,z^{-1}} v^{-a} \mod v^{-a+1}\mathbb{Z}[v]$ and we can assume than $z \leq_{\mathcal{R}} y$ so that $w_0 y \leq_{\mathcal{R}} w_0 z$ and $\boldsymbol{a}(w_0 y) \geq \boldsymbol{a}(w_0 z) = a'$.

For $w \in W$ we set $s_w = n_w$ if $w \in \mathcal{D}$ and $s_w = 0$ if $w \notin \mathcal{D}$. By [Lusztig 2003, 14.1] we have $p_{1,w} = s_w v^{-\boldsymbol{a}(w)} \mod v^{-\boldsymbol{a}(w)-1}\mathbb{Z}[v^{-1}]$ hence $\overline{p_{1,w}} = s_w v^{\boldsymbol{a}(w)} \mod v^{\boldsymbol{a}(w)+1}\mathbb{Z}[v]$. Hence for y in the sum above we have $\overline{p_{1,w_0 y}} = s_{w_0 y} v^{\boldsymbol{a}(w_0 y)} \mod v^{\boldsymbol{a}(w_0 y)+1}\mathbb{Z}[v]$. Thus (e) gives

$$v^{a-a'} \sum_{y \in \boldsymbol{c}} (-1)^{l(y)} s_{w_0 y} \gamma_{y,u,z^{-1}} v^{\boldsymbol{a}(w_0 y) - a} - \delta_{z,u^*} \epsilon_u \in v\mathbb{Z}[v]$$

and using $\boldsymbol{a}(w_0 y) = a'$ for $y \in \boldsymbol{c}$ we obtain

$$\sum_{y \in \boldsymbol{c}} (-1)^{l(y)} s_{w_0 y} \gamma_{y,u,z^{-1}} = \delta_{z,u^*} \epsilon_u.$$

Using the definition of $s_{w_0 y}$ we obtain

(f) $$\sum_{d \in \mathcal{D} \cap w_0 \boldsymbol{c}} (-1)^{l(w_0 d)} n_d \gamma_{w_0 d,u,z^{-1}} = \delta_{z,u^*} \epsilon_u.$$

Next we note that

(g) *if $d \in \mathcal{D}$ and $x, y \in \boldsymbol{c}$ satisfy $\gamma_{w_0 d,x,y} \neq 0$ then $d = d_{w_0 x^{-1}}$.*

Indeed from [Lusztig 2003, §14, P8] we deduce $w_0 d \sim_{\mathcal{L}} x^{-1}$. Using [Lusztig 2003, 11.7] we deduce $d \sim_{\mathcal{L}} w_0 x^{-1}$ so that $d = d_{w_0^{-1} x^{-1}}$. This proves (g).

Using (g) we can rewrite (f) as follows.

(h) $$(-1)^{l(w_0)} (-1)^{l(d)} n_d \gamma_{w_0 d,u,z^{-1}} = \delta_{z,u^*} \epsilon_u$$

where $d = d_{w_0 u^{-1}}$.

We prove (a). Assume that $d \in \mathcal{D}$ and $x, y \in \boldsymbol{c}$ satisfy $\gamma_{w_0 d,x,y} \neq 0$, $y \neq (x^*)^{-1}$. Using (g) we have $d = d_{w_0 x^{-1}}$. Using (h) with $u = x$, $z = y^{-1}$ we see that $\gamma_{w_0 d,x,y} = 0$, a contradiction. This proves (a).

We prove (b). Using (h) with $u = x$, $z = x^*$ we see that

(i) $$(-1)^{l(w_0 d)} n_d \gamma_{w_0 d,x,(x^*)^{-1}} = \epsilon_u$$

where $d = d_{w_0 x^{-1}}$. Hence the existence of d in (b) and the equality $\vartheta(x) = \pm 1$ follow; the uniqueness of d follows from (g).

Now (c) follows from (i). This completes the proof of the theorem.

2.6. In the case where $L = l$, $\vartheta(u)$ (in 2.5(c)) is ≥ 0 and ± 1 hence 1; moreover, $n_d = 1$, $(-1)^{l(d)} = (-1)^{a'}$ for any $d \in \mathcal{D} \cap w_0\boldsymbol{c}$ (by the definition of $\mathcal{D}$). Hence we have $\epsilon_u = (-1)^{l(w_0)+a'}$ for any $u \in \boldsymbol{c}$, a result of Mathas [1996].

Now Theorem 2.5 also gives a characterization of u^* for $u \in \boldsymbol{c}$; it is the unique element $u' \in \boldsymbol{c}$ such that $\gamma_{w_0 d,u,u'^{-1}} \neq 0$ for some $d \in \mathcal{D} \cap w_0\boldsymbol{c}$.

We will show:

(a) *The subsets* $X = \{d^*; d \in \mathcal{D} \cap \boldsymbol{c}\}$ *and* $X' = \{w_0 d'; d' \in \mathcal{D} \cap w_0\boldsymbol{c}\}$ *of* $\boldsymbol{c}$ *coincide.*

Let $d \in \mathcal{D} \cap \boldsymbol{c}$. By 2.5(b) we have $\gamma_{w_0 d',d,(d^*)^{-1}} = \pm 1$ for some $d' \in \mathcal{D} \cap w_0\boldsymbol{c}$. Hence $\gamma_{(d^*)^{-1},w_0 d',d} = \pm 1$. Using [Lusztig 2003, 14.2, P2] we deduce $d^* = w_0 d'$. Thus $X \subset X'$. Let Y (resp. Y') be the set of left cells contained in $\boldsymbol{c}$ (resp. $w_0\boldsymbol{c}$). We have $\sharp(X) = \sharp(Y)$ and $\sharp(X') = \sharp(Y')$. By [Lusztig 2003, 11.7(c)] we have $\sharp(Y) = \sharp(Y')$. It follows that $\sharp(X) = \sharp(X')$. Since $X \subset X'$, we must have $X = X'$. This proves (a).

Theorem 2.7. *We have*

$$\phi(v^{a-a'} T_{w_0}) = \sum_{d \in \mathcal{D} \cap \boldsymbol{c}} \vartheta(d) \epsilon_d t_{d^*} \mod \sum_{u \in W - \boldsymbol{c}} \mathcal{A} t_u.$$

We set $\phi(v^{a-a'} T_{w_0}) = \sum_{u \in W} p_u t_u$ where $p_u \in \mathcal{A}$. Combining 1.12a, 1.13a, 2.3(a) we see that for any $x \in \boldsymbol{c}$ we have

$$\phi(v^{a-a'} T_{w_0}) t_x = \epsilon_x t_{x^*},$$

hence

$$\epsilon_x t_{x^*} = \sum_{u \in \boldsymbol{c}} p_u t_u t_x = \sum_{u,y \in \boldsymbol{c}} p_u \gamma_{u,x,y^{-1}} t_y.$$

It follows that for any $x, y \in \boldsymbol{c}$ we have

$$\sum_{u \in \boldsymbol{c}} p_u \gamma_{u,x,y^{-1}} = \delta_{y,x^*} \epsilon_x.$$

Taking $x = w_0 d$ where $d = d_{w_0 y} \in \mathcal{D} \cap w_0\boldsymbol{c}$ we obtain

$$\sum_{u \in \boldsymbol{c}} p_u \gamma_{w_0 d_{w_0 y},y^{-1},u} = \delta_{y,(w_0 d_{w_0 y})^*} \epsilon_{w_0 d_{w_0 y}}$$

which, by Theorem 2.5, can be rewritten as

$$p_{((y^{-1})^*)^{-1}} \vartheta(y^{-1}) = \delta_{y,(w_0 d_{w_0 y})^*} \epsilon_{w_0 d_{w_0 y}}.$$

We see that for any $y \in \boldsymbol{c}$ we have

$$p_{\sigma(y^*)} = \delta_{y,(w_0 d_{w_0 y})^*} \vartheta(y^{-1}) \epsilon_{w_0 d_{w_0 y}}.$$

In particular we have $p_{\sigma(y^*)} = 0$ unless $y = (w_0 d_{w_0 y})^*$ in which case

$$p_{\sigma(y^*)} = p_{(\sigma(y))^*)} = \vartheta(y^{-1}) \epsilon_y.$$

(We use that $\epsilon_{y^*} = \epsilon_y$.) If $y = (w_0 d_{w_0 y})^*$ then $y^* \in X'$ hence by 2.6(a), $y^* = d^*$ that is $y = d$ for some $d \in \mathcal{D}$. Conversely, if $y \in \mathcal{D}$ then $w_0 y^* \in \mathcal{D}$ (by 2.6(a)) and $w_0 y^* \sim_{\mathcal{L}} w_0 y$ (since $y^* \sim_{\mathcal{L}} y$) hence $d_{w_0 y} = w_0 y^*$. We see that $y = (w_0 d_{w_0 y})^*$ if and only if $y \in \mathcal{D}$. We see that

$$\phi(v^{a-a'} T_{w_0}) = \sum_{d \in \mathcal{D} \cap \boldsymbol{c}} \vartheta(d^{-1}) \epsilon_d t_{(\sigma(d))^*} + \sum_{u \in W - \boldsymbol{c}} p_u t_u.$$

Now $d \mapsto \sigma(d)$ is a permutation of $\mathcal{D} \cap \boldsymbol{c}$ and $\vartheta(d^{-1}) = \vartheta(d) = \vartheta(\sigma(d))$, $\epsilon_{\sigma(d)} = \epsilon_d$. The theorem follows.

Corollary 2.8. $$\phi(T_{w_0}) = \sum_{d \in \mathcal{D}} \vartheta(d) \epsilon_d v^{-\boldsymbol{a}(d) + \boldsymbol{a}(w_0 d)} t_{d^*} \in J_{\mathcal{A}}.$$

2.9. We set $\mathfrak{T}_{\boldsymbol{c}} = \sum_{d \in \mathcal{D} \cap \boldsymbol{c}} \vartheta(d) \epsilon_d t_{d^*} \in J^{\boldsymbol{c}}$. We show:

(a) $\mathfrak{T}_{\boldsymbol{c}}^2 = \sum_{d \in \mathcal{D} \cap \boldsymbol{c}} n_d t_d$.

(b) $t_x \mathfrak{T}_{\boldsymbol{c}} = \mathfrak{T}_{\boldsymbol{c}} t_{\sigma(x)}$ for any $x \in W$.

By Theorem 2.7 we have $\phi(v^{a-a'} T_{w_0}) = \mathfrak{T}_{\boldsymbol{c}} + \xi$ where $\xi \in J_K^{W-\boldsymbol{c}} := \sum_{u \in W - \boldsymbol{c}} K t_u$. Since $J_K^{\boldsymbol{c}}$, $J_K^{W-\boldsymbol{c}}$ are two-sided ideals of J_K with intersection zero and $\phi_K : \mathcal{H}_K \to J_K$ is an algebra homomorphism, it follows that

$$\phi(v^{2a-2a'} T_{w_0}^2) = (\phi(v^{a-a'} T_{w_0}))^2 = (\mathfrak{T}_{\boldsymbol{c}} + \xi)^2 = \mathfrak{T}_{\boldsymbol{c}}^2 + \xi'$$

where $\xi' \in J_K^{W-\boldsymbol{c}}$. Hence, for any $x \in \boldsymbol{c}$ we have $\phi(v^{2a-2a'} T_{w_0}^2) t_x = \mathfrak{T}_{\boldsymbol{c}}^2 t_x$ so that (using 1.12b): $t_x = \mathfrak{T}_{\boldsymbol{c}}^2 t_x$. We see that $\mathfrak{T}_{\boldsymbol{c}}^2$ is the unit element of the ring $J_K^{\boldsymbol{c}}$. Thus (a) holds.

We prove (b). For any $y \in W$ we have $T_y T_{w_0} = T_{w_0} T_{\sigma(y)}$ hence, applying ϕ_K,

$$\phi(T_y) \phi(v^{a-a'} T_{w_0}) = \phi(v^{a-a'} T_{w_0}) \phi(T_{\sigma(y)}),$$

that is, $\phi(T_y)(\mathfrak{T}_{\boldsymbol{c}} + \xi) = (\mathfrak{T}_{\boldsymbol{c}} + \xi) \phi(T_{\sigma(y)})$. Thus, $\phi(T_y) \mathfrak{T}_{\boldsymbol{c}} = \mathfrak{T}_{\boldsymbol{c}} \phi(T_{\sigma(y)}) + \xi_1$ where $\xi_1 \in J_K^{W-\boldsymbol{c}}$. Since ϕ_K is an isomorphism, it follows that for any $x \in W$ we have $t_x \mathfrak{T}_{\boldsymbol{c}} = \mathfrak{T}_{\boldsymbol{c}} t_{\sigma(x)} \mod J_K^{W-\boldsymbol{c}}$. Thus (b) holds.

2.10. In this subsection we assume that $L = l$. In this case Corollary 2.8 becomes

$$\phi(T_{w_0}) = \sum_{d \in \mathcal{D}} (-1)^{l(w_0)+\boldsymbol{a}(w_0 d)} v^{-\boldsymbol{a}(d)+\boldsymbol{a}(w_0 d)} t_{d^*} \in J_{\mathcal{A}}.$$

(We use that $\vartheta(d) = 1$.)

For any left cell Γ contained in $\boldsymbol{c}$ let n_Γ be the number of fixed points of the permutation $u \mapsto u^*$ of Γ. Now Γ carries a representation $[\Gamma]$ of W and from Theorem 2.3 we see that $\mathrm{tr}(w_0, [\Gamma]) = \pm n_\Gamma$. Thus n_Γ is the absolute value of the integer $\mathrm{tr}(w_0, [\Gamma])$. From this the number n_Γ can be computed for any Γ. In this way we see for example that if W is of type E_7 or E_8 and $\boldsymbol{c}$ is not an exceptional two-sided cell, then $n_\Gamma > 0$.

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GENERIC STABILISERS FOR ACTIONS OF REDUCTIVE GROUPS

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In memory of Robert Steinberg

Let G be a reductive algebraic group over an algebraically closed field and let V be a quasiprojective G-variety. We prove that the set of points $v \in V$ such that $\dim(G_v)$ is minimal and G_v is reductive is open. We also prove some results on the existence of principal stabilisers in an appropriate sense.

1. Introduction

Let G be a reductive linear algebraic group over an algebraically closed field k and let V be a quasiprojective G-variety. For convenience, we assume throughout the paper that G permutes the irreducible components of V transitively (the extension of our results to the general case is straightforward). An important question in geometric invariant theory is the following: what can we say about generic stabilisers for the G-action? For instance, given $v \in V$, what does the stabiliser G_v tell us about the stabilisers G_w for w near v? Define V_0 to be the set of points $v \in V$ such that the stabiliser G_v has minimal dimension. The basic theory tells us that V_0 is open (Lemma 2.1). Here is a deeper result [Bardsley and Richardson 1985, Proposition 8.6]: if V is affine and there exists an étale slice through v for the G-action then there exists an open neighbourhood U of v such that for all $w \in U$, G_w is conjugate to a subgroup of G_v. In particular, if $\dim(G_v)$ is minimal in this case then G_w^0 is conjugate to G_v^0 for all $w \in U$. The existence of an étale slice requires, among other conditions, that V be affine and the orbit $G \cdot v$ be closed and separable. If V is affine and k has characteristic 0 then every $v \in V$ such that $G \cdot v$ is closed admits an étale slice, but if k has positive characteristic then it can happen that there are no étale slices at all, since, for example, orbits need not be separable.

In this paper we prove some results about properties of generic stabilisers. Most previous work in this area has dealt with affine varieties and/or fields of characteristic 0 only. Our results hold for quasiprojective varieties and in arbitrary

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characteristic, although in some cases we get stronger results in characteristic 0. We need no assumptions on the existence of or properties of closed orbits, and we allow G to be nonconnected.

Let $V_{\mathrm{red}} = \{v \in V_0 \mid G_v \text{ is reductive}\}$. It is possible for V_{red} to be empty (see Example 8.2). Our first main result implies that if V_{red} is nonempty then generic stabilisers are reductive.

Theorem 1.1. *V_{red} is an open subvariety of V.*

A key ingredient in the proof is the Projective Extension Theorem (see Lemma 3.1).

We mention two related results. First, it follows from [Richardson 1972a, Corollary 9.1.2] that if G is a complex linear algebraic group — not necessarily reductive — and V is a smooth algebraic transformation space for G then V_{red} is open. Second, V. Popov [1972] proved the following (cf. [Luna and Vust 1974]). Let G be a connected linear algebraic group — not necessarily reductive, and in arbitrary characteristic — such that G has no nontrivial characters, and let V be an irreducible normal algebraic variety on which G acts such that the divisor class group $\mathrm{Cl}(V)$ has no elements of infinite order. Then generic G-orbits on V are closed if generic G-orbits on V are affine, and the converse also holds if V is affine.

Richardson [1977, Theorem A] proved that if G is reductive and V is an affine G-variety then an orbit $G \cdot v$ is affine if and only if the stabiliser G_v is reductive. Suppose V is affine and there exists a closed orbit $G \cdot v$ of maximum dimension; then the union of the closed orbits of maximal dimension is open in V [Newstead 1978, Proposition 3.8]. It follows from Richardson's result that there is an open dense set of points $v \in V$ such that G_v is reductive. Theorem 1.1 extends this to the case when generic orbits are not closed, without the affineness assumption.

Richardson's result also gives an immediate corollary to Theorem 1.1 (note that G_v has minimal dimension if and only if the orbit $G \cdot v$ has maximum dimension).

Corollary 1.2. *Suppose V is affine. Then the set $v \in V$ such that $\dim(G \cdot v)$ is maximal and $G \cdot v$ is affine is open.*

We give an application of Theorem 1.1. Nisnevič [1973] proved the following result when $\mathrm{char}(k) = 0$ and $t = 1$.[1] He also proved that the subset A is nonempty in this special case.

Theorem 1.3. *Let $M, H_1, \dots, H_t$ be subgroups of a reductive group G such that M is reductive. Let*

$$A = \big\{(g_1, \dots, g_t) \in G^t \mid M \cap g_1 H_1 g_1^{-1} \cap \cdots \cap g_t H_t g_t^{-1} \\ \textit{is reductive and has minimal dimension}\big\}.$$

Then A is open.

[1] In a private communication, Wallach has also given a proof in this case.

We do not know in general whether A can be empty in positive characteristic, not even when $t = 1$ and $H_1 = M$.

If generic stabilisers are reductive, it is reasonable to try to pin down which reductive subgroups of G actually appear as stabilisers. We say that a subgroup H of G is a *principal stabiliser* for the G-variety V if there is a nonempty open subset O of V such that G_v is conjugate to H for all $v \in O$. We then say that V has a *principal orbit type*. Under our assumptions on G and V, a principal stabiliser is unique up to conjugacy if it exists. Richardson proved that if $\mathrm{char}(k) = 0$ and V is smooth and affine then a principal stabiliser exists [1972b, Proposition 5.3].

It turns out that in positive characteristic, the condition of conjugacy of the stabilisers is too strong: Example 8.3 below shows that even if generic stabilisers are connected and reductive, a principal stabiliser need not exist. To obtain a result, we need to weaken the notion of principal stabiliser. Let $M \leq G$ and let P be a minimal R-parabolic subgroup of G containing M (see Section 2 for the definition of R-parabolic subgroups), let L be an R-Levi subgroup of P and let $\pi_L \colon P \to L$ be the canonical projection. It can be shown that up to G-conjugacy, $\pi_L(M)$ does not depend on the choice of P and L (cf. [Bate et al. 2013, Proposition 5.14(i)]). We define $\mathfrak{D}(M)$ to be the conjugacy class $G \cdot \pi_L(M)$, and we call this the *G-completely reducible degeneration* of M (see Section 4 for the definition of G-complete reducibility). Our second main result says that the $\mathfrak{D}(G_v)$ are equal for generic v.

Theorem 1.4. *There exist a G-completely reducible subgroup H of G and a nonempty open subset O of V such that $\mathfrak{D}(G_v) = G \cdot H$ for all $v \in O$.*

If G is connected and every stabiliser is unipotent then $\mathfrak{D}(G_v) = 1$ for all $v \in V$, so we don't learn much about the structure of the stabilisers. Under some extra hypotheses, however, we can deduce the existence of a principal stabiliser.

Corollary 1.5. *Suppose there is a nonempty open subset O of V such that G_v is G-completely reducible for all $v \in O$. Then the subgroup H from Theorem 1.4 is a principal stabiliser for V.*

Corollary 1.6. *Suppose* $\mathrm{char}(k) = 0$ *and V_{red} is nonempty. Then the subgroup H from Theorem 1.4 is a principal stabiliser for V.*

If we restrict ourselves to the identity components of stabilisers then we get slightly stronger results.

Theorem 1.7. *Suppose V_{red} is nonempty. There exists a connected G-completely reducible subgroup H of G such that $\mathfrak{D}(G_v^0) = G \cdot H$ for all $v \in V_{\mathrm{red}}$.*

In fact, we prove a version of Theorem 1.7 which applies even when V_{red} is empty (see Theorem 7.6).

We briefly explain our approach to the proof of Theorems 1.4 and 1.7. We may regard the subgroups G_v as a family of subgroups of G parametrised by V. There is no obvious way to endow a set of subgroups of G with a geometric structure, so instead we follow the approach of R.W. Richardson [1967; 1988] and consider the set of tuples that generate these subgroups.

Definition 1.8. Let $N \in \mathbb{N}$. Define

$$C = C_N = \{(v, g_1, \ldots, g_N) \mid v \in V, g_1, \ldots, g_N \in G_v\}.$$

We call C the *stabiliser variety* of V.

Our results follow from a study of the geometry of C, using the theory of character varieties and the theory of G-complete reducibility. A major technical problem is that C can be reducible even when G is connected and V is irreducible, so the projection into V of a nonempty open subset of C need not be dense (see Remarks 7.9 and 7.13, for example). The situation is better if we consider only the identity components of stabilisers: we can work with a canonically defined subvariety $\widetilde{C}$ of C with nicer properties (see Lemma 7.1).

The paper is laid out as follows. Section 2 contains preliminary material. In Section 3 we prove Theorems 1.1 and 1.3. Section 4 reviews G-complete reducibility and Section 5 introduces a technical tool needed in Section 6, where we prove Theorem 1.4 and Corollaries 1.5 and 1.6. We study the irreducible components of C in Section 7 and prove Theorem 1.7. The final section contains some examples.

2. Preliminaries

Throughout the paper, N denotes a positive integer, G is a reductive linear algebraic group — possibly nonconnected — over an algebraically closed field k and V is a quasiprojective G-variety over k. The stabiliser variety C_N depends on the choice of N, but to ease notation we suppress the subscript and write just C. All subgroups of G are assumed to be closed. If H is a linear algebraic group then we write $\kappa(H)$ for the number of connected components of H, $R_u(H)$ for the unipotent radical of H and α_H for the canonical projection $H \to H/R_u(H)$. The irreducible components of H^N are the subsets of the form $H_1 \times \cdots \times H_N$, where each H_i is a connected component of H. If X' is a subset of a variety X then we denote the closure of X' in X by $\overline{X'}$. Below we will use the following results on fibres of morphisms (cf. [Borel 1991, AG.10.1 Theorem]): if $f : X \to Y$ is a dominant morphism of irreducible quasiprojective varieties then for all $y \in Y$, every irreducible component of $f^{-1}(y)$ has dimension at least $\dim(X) - \dim(Y)$, and there is a nonempty open subset U of Y such that if $y \in U$ then equality holds. More generally, if Z is a closed irreducible subset of Y and W is an irreducible component of $f^{-1}(Z)$ that dominates Z then $\dim(W) \geq \dim(Z) + \dim(X) - \dim(Y)$.

The next result is Lemma 3.7 of [Newstead 1978].

Lemma 2.1. *Let a linear algebraic group H act on a quasiprojective variety W. For any $t \in \mathbb{N} \cup \{0\}$, the set $\{w \in W \mid \dim(H_w) \geq t\}$ is closed.*

Our assumption that G permutes the irreducible components of V transitively implies that these components all have the same dimension, which we denote by n, and also that nonempty open G-stable subsets of V are dense. In particular, the open subset V_0 is dense; we denote the dimension of G_v for any $v \in V_0$ by r.

The group G acts on G^N by simultaneous conjugation: $g \cdot (g_1, \dots, g_N) = (gg_1g^{-1}, \dots, gg_Ng^{-1})$. We define $\phi\colon C \to G^N$ and $\eta\colon C \to V$ to be the canonical projections. We allow G to act on C in the obvious way, so that ϕ and η are G-equivariant.

We recall an approach to parabolic subgroups and Levi subgroups using cocharacters [Springer 1998, Section 8.4; Bate et al. 2005, Lemma 2.4 and Section 6]. We denote by $Y(G)$ the set of cocharacters of G. The subgroup

$$P_\lambda := \left\{ g \in G \mid \lim_{a \to 0} \lambda(a) g \lambda(a)^{-1} \text{ exists} \right\}$$

is called an *R-parabolic subgroup* of G, and the subset $L_\lambda := C_G(\lambda(k^*))$ is called an *R-Levi subgroup* of P_λ. An R-parabolic subgroup P is parabolic in the sense that G/P is complete, and P^0 is a parabolic subgroup of G^0. If G is connected then an R-parabolic (resp. R-Levi) subgroup is a parabolic (resp. Levi) subgroup, and every parabolic subgroup P and every Levi subgroup L of P arise in this way. The normaliser $N_G(P)$ of a parabolic subgroup P of G^0 is an R-parabolic subgroup. The subset $\{g \in G \mid \lim_{a \to 0} \lambda(a) g \lambda(a)^{-1} = 1\}$ is the unipotent radical $R_u(P_\lambda)$, and this coincides with $R_u(P_\lambda^0)$. We denote the canonical projection from P_λ to L_λ by c_λ. There are only finitely many conjugacy classes of R-parabolic subgroups [Martin 2003, Proposition 5.2(e)].

We finish with some results that are well known; we give proofs here as we could not find any in the literature. These results are not needed in the proofs of Theorems 1.1 and 1.3.

Lemma 2.2. *Let $\psi\colon X \to Y$ be a morphism of quasiprojective varieties over k. There exists $d \in \mathbb{N}$ such that any fibre of ψ has at most d irreducible components.*

Proof. By noetherian induction on closed subsets of X and Y, we are free to pass to open affine subvarieties of X and Y whenever this is convenient. So assume that X and Y are affine and let R and S be the coordinate rings of X and Y, respectively. Suppose first that X and Y are irreducible and that ψ is finite and dominant. By a simple induction argument, we can assume that $R = S[f]$ for some $f \in R$. Let $m(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_0$ be the minimal polynomial of f with respect to the quotient field of S. Passing to open subvarieties, we can assume that the a_i are defined on Y. Let $y \in Y$. If $x \in X$ with $\psi(x) = y$ then we have $f(x)^d + a_{d-1}(y)f(x)^{d-1} + \cdots + a_0(y) = 0$; it follows that there can be at most d such values of x. Thus the fibres of ψ have cardinality at most d.

Now consider the general case. Passing to open subvarieties, we can assume that X and Y are irreducible and affine and that ψ is dominant. We can write $R = S[f_1, \ldots, f_t]$ for some t and some $f_1, \ldots, f_t \in R$. After reordering the f_i if necessary, there exists s with $0 \leq s \leq t$ such that $f_1, \ldots, f_s$ are algebraically independent over S and $f_{s+1}, \ldots, f_t$ are algebraic over $S[f_1, \ldots, f_s]$. The inclusion $S \subseteq S[f_1, \ldots, f_s] \subseteq R$ corresponds to a factorisation of ψ as

$$\psi = X \xrightarrow{\psi'} Y' \xrightarrow{g} Y,$$

where Y' is the affine variety with coordinate ring $S[f_1, \ldots, f_s]$. Then we have $\dim(X) = \dim(Y')$ and ψ' is dominant. By passing to open affine subvarieties, we can assume that ψ' is finite and Y' is normal. By the special case above, the cardinality of the fibres of ψ' is bounded by some d.

Suppose that for some $y \in Y$, the fibre $F := \psi^{-1}(y)$ has $d+1$ distinct irreducible components, say $F_1, \ldots, F_{d+1}$. The fibre $F' := g^{-1}(y)$ is clearly isomorphic to k^s and we have $F = (\psi')^{-1}(F')$. Since ψ' is finite and Y' is normal, every irreducible component of F has dimension s and is mapped surjectively to F' [Humphreys 1975, 4.2 Proposition (b)]. But this means that for generic $y' \in F'$, $(\psi')^{-1}(y')$ has at least $d+1$ elements, a contradiction. We deduce that F has at most d irreducible components, as required. □

Definition 2.3. Applying Lemma 2.2 to the map $\eta\colon C \to V$, we see there is a uniform bound on $\kappa(G_v)$ as v ranges over the elements of V, since the number of irreducible components of G_v^N is $\kappa(G_v)^N$. We denote the least such bound by Θ.

Lemma 2.4. *Let Ω/k be a proper extension of algebraically closed fields. Let $t \in \mathbb{N}$ and let X be an Ω-defined constructible subset of Ω^t. Let $\{X_i \mid i \in I\}$ be a family of k-defined constructible subsets of Ω^t such that $X \subseteq \bigcup_{i\in I} X_i$. Then there exists $i \in I$ such that $X \cap X_i$ has nonempty interior in X. Moreover, there exists a finite subset F of I such that $X \subseteq \bigcup_{i\in F} X_i$.*

Proof. Clearly we can reduce to the case when X and each of the X_i is irreducible and locally closed in Ω^t. The second assertion follows from the first by Noetherian induction on closed subsets of X, so it is enough to prove the first assertion. Let $m = \dim(X)$. It suffices to show that $\dim(X \cap X_i) = m$ for some $i \in I$. We use induction on m. The result is trivial if $m = 0$. Choose polynomials $f_1, \ldots, f_m \in k[T_1, \ldots, T_t]$ such that the restrictions of the f_i to X form a subset of the coordinate ring $\Omega[X]$ that is algebraically independent over Ω. Define $f\colon \Omega^t \to \Omega^m$ by $f(x) = (f_1(x), \ldots, f_m(x))$; note that f is k-defined. Any proper closed subset of X is a union of irreducible components of dimension less than m.

By induction, we are therefore free to replace X with any nonempty open subset of X, so we can assume that $f|_X$ gives a finite map from X onto an open subset of Ω^m. Then $f(X) \subseteq \bigcup_{i\in I} f(X_i)$ and each $f(X_i)$ is k-constructible. It is enough to prove that $f(X) \cap f(X_i)$ has nonempty interior in $f(X)$. Hence we can assume without loss that $t = m$ and X is an open subset of Ω^m.

Let $\pi : \Omega^m \to \Omega$ be the projection onto the first coordinate. Since X is an open and dense subset of Ω^m, $\pi(X)$ is a dense constructible subset of Ω, so $\Omega \setminus \pi(X)$ is finite. Hence there exists $y \in \pi(X)$ such that $y \notin k$. Let $\widetilde{X} = X \cap \pi^{-1}(y)$. Then $\widetilde{X}$ is an Ω-defined locally closed subset of Ω^m, $\widetilde{X}$ is irreducible of dimension $m-1$ and $\widetilde{X} \subseteq \bigcup_{i\in I} X_i$. By induction, there exists $j \in I$ such that $\widetilde{X} \cap X_j$ has an irreducible component of dimension $m-1$. Hence $\pi^{-1}(y) \cap X_j$ has an irreducible component of dimension at least $m-1$. Note that we retain our assumption that the X_i are irreducible. Now X_j cannot be contained in $\pi^{-1}(y)$ because $\pi^{-1}(y)$ has no k-points, so $\pi^{-1}(y) \cap X_j$ is a proper closed subset of X_j. Hence $\dim(X_j) = m$, as required. □

Corollary 2.5. *Let Ω be an uncountable algebraically closed field. Let $t \in \mathbb{N}$ and let X be an Ω-defined constructible subset of Ω^t. Let $\{X_i \mid i \in I\}$ be a countable family of Ω-constructible subsets of X such that $X \subseteq \bigcup_{i\in I} X_i$. Then there exists $i \in I$ such that X_i has nonempty interior in X. Moreover, there exists a finite subset F of I such that $X \subseteq \bigcup_{i\in F} X_i$.*

Proof. Each of the X_i is defined over a subfield of Ω that is finitely generated over the algebraic closure of the prime field, so there exists a countable subfield k of Ω such that each of the X_i is defined over k. Since k is countable and Ω is not, Ω/k is a proper field extension. Now apply Lemma 2.4. □

Corollary 2.6. *If X is irreducible and the X_i are closed in Corollary 2.5 then there exists $i \in I$ such that $X \subseteq X_i$.*

Proof. This is immediate from Corollary 2.5. □

3. Proof of Theorem 1.1

We now prove our first main result.

Lemma 3.1. *Let X be a quasiprojective variety, let Y be a projective variety and let Z be a closed subvariety of $X \times Y$. Then the projection of Z onto X is a closed variety.*

Proof. Choose a covering of X by open affine subvarieties $X_1, \dots, X_m$. A subset S of X (resp. $X \times Y$) is closed if and only if its intersection with X_i (resp. $X_i \cap Y$) is closed for all i, so we can assume that X is affine. The result now follows from the Projective Extension Theorem [Cox et al. 2015, Chapter 8, Section 5, Theorem 6]. □

Lemma 3.2. *Let P be an R-parabolic subgroup of G and let W be a closed P-stable subset of V. Then $G \cdot W$ is closed in V.*

Proof. Set $D = \{(v, g) \in V \times G \mid g^{-1} \cdot v \in W\}$, a closed subvariety of $V \times G$. We let P act on $V \times G$ by $h \cdot (v, g) = (v, gh^{-1})$; then D is P-stable as W is. Let $\pi_P \colon G \to G/P$ be the canonical projection and define $\theta \colon V \times G \to V \times G/P$ by $\theta(v, g) = (v, \pi_P(g))$. Since π_P is smooth, π_P is flat, so $(\theta, V \times G/P)$ is a geometric quotient by [Bongartz 1998, Lemma 5.9(a)]. Then θ takes closed P-stable subvarieties of $V \times G$ to closed subvarieties of $V \times G/P$, so $\theta(D)$ is a closed subvariety of $V \times G/P$. Note that the projection of $\theta(D)$ onto V is $G \cdot W$. Lemma 3.1 implies that $G \cdot W$ is closed in V, so we are done. □

Remark 3.3. We record one corollary (cf. [Sikora 2012, Proposition 27]). Recall that G acts on G^N by simultaneous conjugation. Let P be an R-parabolic subgroup of G. Then $G \cdot P^N$ is closed in G^N. This follows immediately from Lemma 3.2, taking $V = G^N$ and $W = P^N$.

Proposition 3.4. *Let P be an R-parabolic subgroup of G with unipotent radical U. Define $V_P = \{v \in V_0 \mid \dim(P_v) = r\} = \{v \in V_0 \mid G_v^0 \leq P\}$ and, for each t, $V_{P,t} = \{v \in V_P \mid \dim(U_v) \geq t\}$. Then $G \cdot V_{P,t}$ is closed in V_0 for each t.*

Proof. This follows from Lemma 3.2 (applied to V_0), as each $V_{P,t}$ is P-stable and closed in V_0 (Lemma 2.1). □

Proof of Theorem 1.1. We show that G_v is nonreductive if and only if $v \in \bigcup_P G \cdot V_{P,1}$, where the union is over a set of representatives of the conjugacy classes of R-parabolic subgroups of G. Since there are only finitely many R-parabolic subgroups up to conjugacy and each subset $G \cdot V_{P,1}$ is closed in V_0 (Proposition 3.4), this suffices to prove the theorem.

If $v \in G \cdot V_{P,1}$ — say, $g \cdot v \in V_{P,1}$ — then $G_v^0 \leq g^{-1} P g$ and G_v^0 contains a positive-dimensional subgroup M of $g^{-1} U g = R_u(g^{-1} P g)$. Thus G_v^0 is not reductive, as G_v^0 normalises the connected unipotent subgroup of $g^{-1} U g$ generated by the G_v^0-conjugates of M. Hence G_v is not reductive, either. Conversely, if $v \in V_0$ and G_v has nontrivial unipotent radical H then we can pick a minimal R-parabolic subgroup P of G containing G_v; then $H \leq R_u(P)$ (see the paragraph following Lemma 4.1), so $v \in G \cdot V_{P,1}$. The result now follows. □

Remark 3.5. More generally, set $V(t) = \{v \in V_0 \mid \dim(R_u(G_v)) \geq t\}$. A similar argument to the one above shows that $V(t) = \bigcup_P G \cdot V_{P,t}$, where the union is over a set of representatives of the conjugacy classes of R-parabolic subgroups of G, so $V(t)$ is closed. In particular, define $V_{\min} = \{v \in V_0 \mid \dim(R_u(G_v)) \text{ is minimal}\}$; then $V_{\min}$ is a nonempty open subset of V_0. Note that $V_{\min} = V_{\mathrm{red}}$ if V_{red} is nonempty.

We finish the section with the proof of Theorem 1.3. Each coset space G/H_i is quasiprojective, and the reductive group M acts on G/H_i by left multiplication.

Let $V = G/H_1 \times \cdots \times G/H_t$, with M acting on V by the product action. For any $(g_1, \ldots, g_t) \in G^t$, the stabiliser $M_{(g_1 H_1, \ldots, g_t H_t)}$ equals $M \cap g_1 H_1 g_1^{-1} \cap \cdots \cap g_t H_t g_t^{-1}$. Hence the set A equals the preimage of V_{red} under the map from G^t to V that sends $(g_1, \ldots, g_t)$ to $(g_1 H_1, \ldots, g_t H_t)$. But V_{red} is open by Theorem 1.1, so A is open. This completes the proof.

Remark 3.6. In the setup in the proof of Theorem 1.3, we do not know whether the subgroups $M \cap g_1 H_1 g_1^{-1} \cap \cdots \cap g_t H_t g_t^{-1}$ are all conjugate for generic $(g_1, \ldots, g_t)$. This is the case, however, if these subgroups are G-completely reducible for generic $(g_1, \ldots, g_t)$ (cf. Example 8.4).

4. *G*-complete reducibility and orbits of tuples

Let H be a subgroup of G. We say that H is *G-completely reducible* (G-cr) if whenever H is contained in an R-parabolic subgroup P of G, there is an R-Levi subgroup L of P such that H is contained in L. This notion is due to Serre [2005]; see [Serre 1998; 1997] for more details. In particular, we say that H is *G-irreducible* (G-ir) if H is not contained in any proper R-parabolic subgroup of G at all; then H is G-cr. A G-cr subgroup of G is reductive (cf. [Bate et al. 2005, Section 2.5 and Theorem 3.1]), and the converse holds in characteristic 0. A linearly reductive subgroup is G-cr, while a nontrivial unipotent subgroup of G^0 is never G-cr. A normal subgroup of a G-cr subgroup is G-cr [Bate et al. 2005, Theorem 3.10]. We denote by $\mathscr{C}(G)_{\mathrm{cr}}$ the set of conjugacy classes of G-cr subgroups of G.

Lemma 4.1. *$\mathscr{C}(G)_{\mathrm{cr}}$ is countable.*

Proof. Let F be the algebraic closure of the prime field. Then G has an F-structure, by [Martin 2003, Proposition 3.2]. By [Martin 2003, Theorem 10.3] and [Bate et al. 2005, Theorem 3.1], any G-cr subgroup of G is G-conjugate to an F-defined subgroup. But $G(F)$ has only countably many $G(F)$-conjugacy classes of $G(F)$-cr subgroups since F is countable. The result follows. □

Let H be a subgroup of G. Let $P = P_\lambda$ be minimal amongst the R-parabolic subgroups of G that contain H. Then $c_\lambda(H)$ is an L_λ-ir subgroup of L_λ (see the proof of [Bate et al. 2013, Proposition 5.14(i)]), so $c_\lambda(H)$ is G-cr. As observed in Section 1, $c_\lambda(H)$ does not depend on the choice of λ up to conjugacy, and we set $\mathscr{D}(H) = G \cdot c_\lambda(H)$. We have $\mathscr{D}(H) = G \cdot H$ if and only if $c_\lambda(H)$ is conjugate to H if and only if H is G-cr [Bate et al. 2013, Proposition 5.14(i)]. For any $\mu \in Y(G)$ such that $H \leq P_\mu$, if H is G-cr then $c_\mu(H)$ is conjugate to H, and if $c_\mu(H)$ is G-ir then $L_\mu = G$, so $H = c_\mu(H)$ is G-ir. Since $c_\lambda(H)$ is reductive, $R_u(H) \leq R_u(P_\lambda)$ and H is reductive if and only if $H \cap R_u(P_\lambda)$ is finite if and only if $\dim(H) = \dim(c_\lambda(H))$. Moreover, $\dim(C_G(H)) \leq \dim(C_G(c_\lambda(H)))$, with equality if and only if H is G-cr

[Bate et al. 2013, Theorem 5.8(ii)], and $\dim(c_\lambda(H)) = \dim(H) - \dim(R_u(H))$. If $M \leq H$ and $\alpha_H(M) = H/R_u(H)$ then $\mathcal{D}(M) = \mathcal{D}(H)$.

If $\mathrm{char}(k) = 0$ then H has a Levi subgroup M by [Hochschild 1981, VIII, Theorem 4.3]; that is, H has a reductive subgroup M such that $H \cong M \ltimes R_u(H)$. Then $c_\lambda(H) = c_\lambda(M)$ is conjugate to M, since M is G-cr, so $\mathcal{D}(H) = G \cdot M$.

The paper [Bate et al. 2005] laid out an approach to the theory of G-complete reducibility using geometric invariant theory; we briefly review this now. As described in Section 1, the idea is to study subgroups of G indirectly by looking instead at generating tuples for subgroups. Given $s \in \mathbb{N}$ and $\boldsymbol{g} = (g_1, \dots, g_s) \in G^s$, we denote by $\mathcal{G}(\boldsymbol{g})$ or $\mathcal{G}(g_1, \dots, g_s)$ the closed subgroup generated by $g_1, \dots, g_s$. If H is of the form $\mathcal{G}(g_1, \dots, g_s)$ for some $g_1, \dots, g_s \in G$ then we say that H is *topologically finitely generated*, and we call $\boldsymbol{g}$ a *generating s-tuple* or *generating tuple* for H. The structure of the set of generating s-tuples is complicated; for instance, if $H = k^*$ and k is solid (Definition 4.2) then both $\{\boldsymbol{h} \in H^s \mid \mathcal{G}(\boldsymbol{h}) = H\}$ and $\{\boldsymbol{h} \in H^s \mid \mathcal{G}(\boldsymbol{h}) \neq H\}$ are dense in H^s, even when $s = 1$.

Recall that G acts on G^N by simultaneous conjugation. We call the quotient space G^N/G a *character variety* and we denote the canonical projection from G^N to G^N/G by π_G. If $\lambda \in Y(G)$ then we abuse notation and denote the map $c_\lambda \times \cdots \times c_\lambda \colon P_\lambda^N \to L_\lambda^N$ by c_λ. We have $\pi_G(\boldsymbol{g}) = \pi_G(c_\lambda(\boldsymbol{g}))$ and $\mathcal{G}(c_\lambda(\boldsymbol{g})) = c_\lambda(\mathcal{G}(\boldsymbol{g}))$ for all $\boldsymbol{g} \in P_\lambda^N$. If $\boldsymbol{g} \in P_\lambda^N$ and $\boldsymbol{g}' \in P_{\lambda'}^N$ such that $G \cdot c_\lambda(\boldsymbol{g})$ and $G \cdot c_{\lambda'}(\boldsymbol{g}')$ are closed then $\pi_G(\boldsymbol{g}) = \pi_G(\boldsymbol{g}')$ if and only if $c_\lambda(\boldsymbol{g})$ and $c_{\lambda'}(\boldsymbol{g}')$ are conjugate (see [Newstead 1978, Corollary 3.5.2]). In particular, if $G \cdot \boldsymbol{g}'$ is closed then we can take $\lambda' = 0$, so $\pi_G(\boldsymbol{g}) = \pi_G(\boldsymbol{g}')$ if and only if $c_\lambda(\boldsymbol{g})$ is conjugate to $\boldsymbol{g}'$.

We need a condition on the field to ensure that reductive groups are topologically finitely generated.

Definition 4.2. An algebraically closed field is *solid* if either it has characteristic 0 or it has characteristic $p > 0$ and is transcendental over $\mathbb{F}_p$.

The next result allows us to understand subgroups of G by studying generating tuples; several of the results stated above for subgroups have equivalent formulations given for tuples below.

Proposition 4.3 [Martin 2003, Lemma 9.2]. *Suppose k is solid. Let H be a reductive algebraic group and suppose that $N \geq \kappa(H) + 1$. Then there exists $\boldsymbol{h} \in H^N$ such that $\mathcal{G}(\boldsymbol{h}) = H$.*

Proposition 4.3 fails if $k = \overline{\mathbb{F}_p}$, for then any topologically finitely generated subgroup of G is finite. This is the reason for some of the technical complexity in what follows. We can, however, formulate the results of this section for arbitrary k, for example by using the notion of a "generic tuple" [Bate et al. 2013, Definition 5.4]. Even when k is solid, nonreductive subgroups need not be topologically finitely generated (for example, a topologically finitely generated subgroup of a unipotent group in positive characteristic is finite). This is why we need to work with $H/R_u(H)$ rather than just H in Definition 5.1.

The next result is [Bate et al. 2005, Corollary 3.7].

Theorem 4.4. *Let $\boldsymbol{g} \in G^N$. Then the orbit $G \cdot \boldsymbol{g}$ is closed if and only if $\mathcal{G}(\boldsymbol{g})$ is G-cr.*

Let H be a reductive subgroup of G. The inclusion of H^N in G^N gives rise to a morphism $\Psi_H^G\colon H^N/H \to G^N/G$, given by $\Psi_H^G(\pi_H(\boldsymbol{h})) = \pi_G(\boldsymbol{h})$ for $\boldsymbol{h} \in H^N$. The next result is Theorem 1.1 of [Martin 2003].

Theorem 4.5. *The morphism Ψ_H^G is finite. In particular, $\Psi_H^G(H^N/H)$ is closed in G^N/G.*

Remark 4.6. (i) The set $(G^N)_{\mathrm{ir}} := \{\boldsymbol{g} \in G^N \mid \mathcal{G}(\boldsymbol{g})$ is G-ir$\}$ is open; this was proved in [Martin 2003, Corollary 8.4] but it also follows from Remark 3.3.

(ii) Suppose V is irreducible, $N \geq 2$ and there exists $v \in V_0$ such that G_v^0 is G-ir. Then $\phi^{-1}((G^N)_{\mathrm{ir}})$ is a nonempty open G-stable subset of C by (i), and it follows from arguments in Section 7 that $\eta(\phi^{-1}((G^N)_{\mathrm{ir}}))$ is a dense subset of V (cf. Remark 7.9). This means that generic stabilisers are "large" in the sense of not being contained in any proper R-parabolic subgroup of G. On the other hand, we can interpret Lemma 2.1 as saying that generic stabilisers are "small". This special case illustrates the tension between largeness and smallness, from which several of our results spring.

5. The partial order $\preceq$

In this section we introduce a technical tool which we need for the proof of Theorem 1.4. For simplicity, we assume throughout the section that k is solid; see Remark 5.14 for a discussion of arbitrary k.

Definition 5.1. Let H, M be subgroups of G. We define $G \cdot H \preceq G \cdot M$ if there exist $s \in \mathbb{N}$, $\boldsymbol{h} \in H^s$ and $\boldsymbol{m} \in M^s$ such that $\alpha_H(\mathcal{G}(\boldsymbol{h})) = H/R_u(H)$ and $\pi_G(\boldsymbol{m}) = \pi_G(\boldsymbol{h})$. (It is clear that this does not depend on the choice of subgroup in the conjugacy classes $G \cdot H$ and $G \cdot M$.) We define $G \cdot H \prec G \cdot M$ if $G \cdot H \preceq G \cdot M$ and $G \cdot H \neq G \cdot M$.

Lemma 5.2. *Let $H, M \leq G$. Then $G \cdot H \preceq G \cdot M$ if and only if $\mathfrak{D}(H) \preceq \mathfrak{D}(M)$.*

Proof. Pick $\lambda, \mu \in Y(G)$ such that $H \leq P_\lambda$, $c_\lambda(H)$ is G-cr, $M \leq P_\mu$ and $c_\mu(M)$ is G-cr. Since $\mathfrak{D}(H) = G \cdot c_\lambda(H)$ and $\mathfrak{D}(M) = G \cdot c_\mu(M)$, it is enough to show that $G \cdot H \preceq G \cdot M$ if and only if $G \cdot c_\lambda(H) \preceq G \cdot c_\mu(M)$.

So suppose $G \cdot H \preceq G \cdot M$. There exist $s \in \mathbb{N}$, $\boldsymbol{m} = (m_1, \ldots, m_s) \in M^s$ and $\boldsymbol{h} = (h_1, \ldots, h_s) \in H^s$ such that $\alpha_H(\mathcal{G}(\boldsymbol{h})) = H/R_u(H)$ and $\pi_G(\boldsymbol{m}) = \pi_G(\boldsymbol{h})$. Then $c_\mu(\boldsymbol{m}) \in c_\mu(M)^s$ and $\pi_G(c_\mu(\boldsymbol{m})) = \pi_G(c_\lambda(\boldsymbol{h}))$. Now $c_\lambda(H)$ is reductive, so $c_\lambda(R_u(H)) = 1$. It follows that $\mathcal{G}(c_\lambda(\boldsymbol{h})) = c_\lambda(\mathcal{G}(\boldsymbol{h})) = c_\lambda(H)$. This shows that $G \cdot c_\lambda(H) \preceq G \cdot c_\mu(M)$.

Conversely, suppose that $G \cdot c_\lambda(H) \preceq G \cdot c_\mu(M)$. Then there exist $s \in \mathbb{N}$ and $\boldsymbol{y} = (y_1, \dots, y_s) \in c_\mu(M)^s$ and $\boldsymbol{x} = (x_1, \dots, x_s) \in c_\lambda(H)^s$ such that $\mathcal{G}(\boldsymbol{x}) = c_\lambda(H)$ and $\pi_G(\boldsymbol{y}) = \pi_G(\boldsymbol{x})$. The maps $c_\lambda \colon H^s \to c_\lambda(H)^s$ and $c_\mu \colon M^s \to c_\mu(M)^s$ are surjective, so there exist $\boldsymbol{h} = (h_1, \dots, h_s) \in H^s$ and $\boldsymbol{m} = (m_1, \dots, m_s) \in M^s$ such that $c_\lambda(\boldsymbol{h}) = \boldsymbol{x}$ and $c_\mu(\boldsymbol{m}) = \boldsymbol{y}$.

As $c_\lambda(H)$ is reductive, $R_u(H) \leq R_u(P_\lambda)$. As $(R_u(P_\lambda) \cap H)^0$ is a connected normal unipotent subgroup of H, we must have $(R_u(P_\lambda) \cap H)^0 \leq R_u(H)$, and it follows that $(R_u(P_\lambda) \cap H)^0 = R_u(H)$. Choose $h_{s+1}, \dots, h_{s+t} \in R_u(P_\lambda) \cap H$ such that the $\alpha_H(h_i)$ for $s+1 \leq i \leq s+t$ generate the finite group $(R_u(P_\lambda) \cap H)/R_u(H)$. Set

$$\begin{aligned}
\boldsymbol{h}' &= (h_1, \dots, h_s, h_{s+1}, \dots, h_{s+t}) \in H^{s+t}, \\
\boldsymbol{x}' &= (x_1, \dots, x_s, 1, \dots, 1) \in c_\lambda(H)^{s+t}, \\
\boldsymbol{m}' &= (m_1, \dots, m_s, 1, \dots, 1) \in M^{s+t}, \\
\boldsymbol{y}' &= (y_1, \dots, y_s, 1, \dots, 1) \in c_\mu(M)^{s+t}.
\end{aligned}$$

Then $c_\lambda(\boldsymbol{h}') = \boldsymbol{x}'$ and $c_\mu(\boldsymbol{m}') = \boldsymbol{y}'$; moreover, $\alpha_H(\mathcal{G}(\boldsymbol{h}')) = H/R_u(H)$ by construction.

To finish, it is enough to show that $\pi_G(\boldsymbol{x}') = \pi_G(\boldsymbol{y}')$. As $\pi_G(\boldsymbol{x}) = \pi_G(\boldsymbol{y})$ and $\mathcal{G}(\boldsymbol{x}) = c_\lambda(H)$ is G-cr, there exists $\nu \in Y(G)$ such that $\mathcal{G}(\boldsymbol{y}) \leq P_\nu$ and $c_\nu(\boldsymbol{y})$ is conjugate to $\boldsymbol{x}$. It is then immediate that $\mathcal{G}(\boldsymbol{y}') \leq P_\nu$ and $c_\nu(\boldsymbol{y}')$ is conjugate to $\boldsymbol{x}'$. Hence $\pi_G(\boldsymbol{x}') = \pi_G(\boldsymbol{y}')$, as required. $\square$

Lemma 5.3. *Let $H, M \leq G$. Suppose that H is G-cr. Then $G \cdot H \preceq \mathcal{D}(M)$ if and only if $G \cdot H \preceq G \cdot M$ if and only if there exist $\lambda \in Y(G)$ and $M_1 \leq P_\lambda \cap M$ such that $c_\lambda(M_1)$ is conjugate to H.*

Proof. The first equivalence follows from Lemma 5.2. We prove the second equivalence. As H is G-cr, H is reductive. Suppose $G \cdot H \preceq G \cdot M$. There exist $s \in \mathbb{N}$, $\boldsymbol{h} \in H^s$ and $\boldsymbol{m} \in M^s$ such that $\mathcal{G}(\boldsymbol{h}) = H$ and $\pi_G(\boldsymbol{m}) = \pi_G(\boldsymbol{h})$. Set $M_1 = \mathcal{G}(\boldsymbol{m})$. As $H = \mathcal{G}(\boldsymbol{h})$ is G-cr, there exist $\lambda \in Y(G)$ and $g \in G$ such that $M_1 \leq P_\lambda$ and $c_\lambda(\boldsymbol{m}) = g \cdot \boldsymbol{h}$. Then

$$c_\lambda(M_1) = c_\lambda(\mathcal{G}(\boldsymbol{m})) = \mathcal{G}(c_\lambda(\boldsymbol{m})) = \mathcal{G}(g \cdot \boldsymbol{h}) = g\mathcal{G}(\boldsymbol{h})g^{-1} = gHg^{-1},$$

as required.

Conversely, suppose there exist $\lambda \in Y(G)$ and $M_1 \leq P_\lambda \cap M$ such that $c_\lambda(M_1)$ is conjugate to H. Pick $s \geq \kappa(H) + 1$. By Proposition 4.3, there exists $\boldsymbol{h} \in H^s$ such that $\mathcal{G}(\boldsymbol{h}) = H$. We can pick $\boldsymbol{m} \in M_1^s$ such that $c_\lambda(\boldsymbol{m})$ is conjugate to $\boldsymbol{h}$. Then $\pi_G(\boldsymbol{m}) = \pi_G(c_\lambda(\boldsymbol{m})) = \pi_G(\boldsymbol{h})$, so $G \cdot H \preceq G \cdot M$, and we are done. $\square$

Lemma 5.4. *Let $H, M, K \leq G$. If $G \cdot H \preceq G \cdot M$ and $G \cdot M \preceq G \cdot K$ then $G \cdot H \preceq G \cdot K$.*

Proof. Suppose $G \cdot H \preceq G \cdot M$ and $G \cdot M \preceq G \cdot K$. By Lemma 5.2, we can assume H, M and K are G-cr. By Lemma 5.3, there exist $\lambda \in Y(G)$ and $K_1 \leq P_\lambda \cap K$ such that $c_\lambda(K_1)$ is conjugate to M. Replacing (K, λ) with a conjugate of (K, λ) if necessary, we can assume that $c_\lambda(K_1) = M$. Pick $s \in \mathbb{N}$, $\boldsymbol{h} \in H^s$ and $\boldsymbol{m} \in M^s$ such that $\mathcal{G}(\boldsymbol{h}) = H$ and $\pi_G(\boldsymbol{h}) = \pi_G(\boldsymbol{m})$. There exists $\boldsymbol{k} \in K_1^s$ such that $c_\lambda(\boldsymbol{k}) = \boldsymbol{m}$. Then $\pi_G(\boldsymbol{k}) = \pi_G(c_\lambda(\boldsymbol{k})) = \pi_G(\boldsymbol{m}) = \pi_G(\boldsymbol{h})$, so $G \cdot H \preceq G \cdot K$. □

If H and M are subgroups of G and H is conjugate to a subgroup of M then $G \cdot H \preceq G \cdot M$ (and so $\mathcal{D}(H) \preceq \mathcal{D}(M)$ by Lemma 5.2); in particular, $G \cdot H \preceq G \cdot H$. For without loss we can assume that $H \leq M$, and if we take $s \geq \kappa(H/R_u(H)) + 1$ then by Proposition 4.3 we can choose $\boldsymbol{m} = \boldsymbol{h} \in H^s$ such that $\alpha_H(\boldsymbol{h})$ generates the reductive group $H/R_u(H)$. The following example shows that the converse is false, even when H and M are G-cr.

Example 5.5. Let $\mathrm{char}(k) = 2$, let $G = \mathrm{SL}_8(k)$ and let M be $\mathrm{PGL}_3(k)$ embedded in G via the adjoint representation on $\mathrm{Lie}(M) \cong k^8$. Since $\mathrm{Lie}(M)$ is a simple M-module, M is G-cr (in fact, G-ir). It follows from elementary representation-theoretic arguments that M contains exactly two subgroups of type A_1 up to M-conjugacy: the derived group H_1 of a Levi subgroup of a rank 1 parabolic subgroup of M, and the image H_2 of $\mathrm{SL}_2(k)$ under the map $\mathrm{SL}_2(k) \to \mathrm{SL}_3(k) \to M$, where the first arrow is the adjoint representation of $\mathrm{SL}_2(k)$ and the second is the canonical projection. It is easily checked that H_1 is M-cr but H_2 is not; in fact, there exists $\lambda \in Y(M)$ such that $c_\lambda(H_2) = H_1$.

Now H_1 is not G-cr because $\mathrm{Lie}(H_1)$ is an H_1-stable submodule of $\mathrm{Lie}(M)$ and H_1 does not act completely reducibly on $\mathrm{Lie}(H_1)$. Choose $\mu \in Y(G)$ such that $H_1 \leq P_\mu$ and $H := c_\mu(H_1)$ is G-cr. We have $G \cdot H_1 \preceq G \cdot M$ as $H_1 \leq M$, so $G \cdot H \preceq G \cdot M$ by Lemma 5.2. We claim that H is not G-conjugate to a subgroup of M. First, H is not G-conjugate to H_1 because H is G-cr but H_1 is not. If H is G-conjugate to H_2 then H_2 is G-cr, so $H_1 = c_\lambda(H_2)$ is G-conjugate to H_2; but then H is G-conjugate to H_1, a contradiction. This proves the claim.

We do, however, have the following result.

Lemma 5.6. *Let $H, M \leq G$. If $G \cdot H \preceq G \cdot M$ and $G \cdot M \preceq G \cdot H$ then $\mathcal{D}(H) = \mathcal{D}(M)$. In particular, if H and M are G-cr then $G \cdot H = G \cdot M$.*

Proof. By Lemma 5.2, we can assume H and M are G-cr; in particular, H and M are reductive. By Lemma 5.3, there exist $\lambda \in Y(G)$ and $M_1 \leq P_\lambda \cap M$ such that $c_\lambda(M_1)$ is conjugate to H. Replacing (M, λ) with a conjugate of (M, λ) if necessary, we can assume that $c_\lambda(M_1) = H$. We have

$$\dim(H) = \dim(c_\lambda(M_1)) \leq \dim(M_1) \leq \dim(M).$$

By symmetry, $\dim(M) \leq \dim(H)$, so

$$\dim(H) = \dim(c_\lambda(M_1)) = \dim(M_1) = \dim(M).$$

It now follows that

$$\kappa(H) = \kappa(c_\lambda(M_1)) \leq \kappa(M_1) \leq \kappa(M).$$

By symmetry, $\kappa(M) \leq \kappa(H)$, so

$$\kappa(H) = \kappa(c_\lambda(M_1)) = \kappa(M_1) = \kappa(M).$$

This implies that $M_1 = M$ since $M_1 \leq M$, so $H = c_\lambda(M)$. But M is G-cr, so M is conjugate to H. This completes the proof. □

The next result follows immediately from Lemmas 5.4 and 5.6.

Corollary 5.7. *The relation $\preceq$ is a partial order on $\mathscr{C}(G)_{\mathrm{cr}}$.*

Remark 5.8. The proof of Lemma 5.6 shows that if H and M are G-cr subgroups of G and $G \cdot H \prec G \cdot M$ then either $\dim(H) < \dim(M)$, or $\dim(H) = \dim(M)$ and $\kappa(H) < \kappa(M)$. It follows that $\mathscr{C}(G)_{\mathrm{cr}}$ satisfies the descending chain condition with respect to $\preceq$.

Given a reductive subgroup H of G, set $S(H) = \{\boldsymbol{g} \in G^N \mid \pi_G(\boldsymbol{g}) \in \Psi_H^G(H^N/H)\}$. Theorem 4.5 implies that $S(H)$ is closed.

Lemma 5.9. *Let $\boldsymbol{g} \in G^N$ and let $H \leq G$ be reductive. Then $\boldsymbol{g} \in S(H)$ if and only if $G \cdot \mathscr{G}(\boldsymbol{g}) \preceq G \cdot H$ if and only if $\mathfrak{D}(\mathscr{G}(\boldsymbol{g})) \preceq \mathfrak{D}(H)$.*

Proof. We prove the first equivalence. If $\boldsymbol{g} \in S(H)$ then there exists $\boldsymbol{h} \in H^N$ such that $\pi_G(\boldsymbol{h}) = \pi_G(\boldsymbol{g})$, so $G \cdot \mathscr{G}(\boldsymbol{g}) \preceq G \cdot H$ as $\boldsymbol{g}$ generates $\mathscr{G}(\boldsymbol{g})$. Conversely, suppose $G \cdot \mathscr{G}(\boldsymbol{g}) \preceq G \cdot H$. Set $M = \mathscr{G}(\boldsymbol{g})$. Then $\mathfrak{D}(M) \preceq \mathfrak{D}(H)$ by Lemma 5.2. Choose $\mu \in Y(G)$ such that $H \leq P_\mu$ and $c_\mu(H)$ is G-cr. Choose $\nu \in Y(G)$ such that $M \leq P_\nu$ and $c_\nu(M)$ is G-cr. Then $\mathfrak{D}(H) = G \cdot c_\mu(H)$ and $\mathfrak{D}(M) = G \cdot c_\nu(M)$. By Lemma 5.3, there exist $K \leq c_\mu(H)$ and $\lambda \in Y(G)$ such that $G \cdot c_\lambda(K) = G \cdot c_\nu(M)$. There exists $\boldsymbol{k} \in K^N$ such that $G \cdot c_\lambda(\boldsymbol{k}) = G \cdot c_\nu(\boldsymbol{g})$. There exists $\boldsymbol{h} \in H^N$ such that $c_\mu(\boldsymbol{h}) = \boldsymbol{k}$. We have $\pi_G(\boldsymbol{h}) = \pi_G(c_\mu(\boldsymbol{h})) = \pi_G(\boldsymbol{k}) = \pi_G(c_\lambda(\boldsymbol{k})) = \pi_G(c_\nu(\boldsymbol{g})) = \pi_G(\boldsymbol{g})$, so $\boldsymbol{g} \in S(H)$, as required.

The second equivalence follows from Lemma 5.2. □

To prove our results in Section 6, we need to investigate the behaviour of the relation $\preceq$ under field extensions. We assume for the rest of the section that $N \geq \Theta + 1$. Fix a G-cr subgroup H of G such that $N \geq \kappa(H) + 1$.

Definition 5.10. Define $B_H = \{v \in V \mid \mathfrak{D}(G_v) = G \cdot H\}$.

Let $v \in V$. For all $\boldsymbol{g} \in G_v^N$, we have $\mathscr{G}(\boldsymbol{g}) \leq G_v$, and so $\mathfrak{D}(\mathscr{G}(\boldsymbol{g})) \preceq \mathfrak{D}(G_v)$. Moreover, since $N \geq \Theta+1 \geq \kappa(G_v)+1 \geq \kappa(G_v/R_u(G_v))+1$, there exists $\boldsymbol{g}' \in G^N$ such that $\alpha_{G_v}(\mathscr{G}(\boldsymbol{g}')) = G_v/R_u(G_v)$ by Proposition 4.3, so $\mathfrak{D}(\mathscr{G}(\boldsymbol{g}')) = \mathfrak{D}(G_v)$. Lemma 5.4 now implies that $\mathfrak{D}(G_v) \preceq G \cdot H$ if and only if $\mathfrak{D}(\mathscr{G}(\boldsymbol{g})) \preceq \mathfrak{D}(H)$ for all $\boldsymbol{g} \in G_v^N$ if and only if $\pi_G(\boldsymbol{g}) \in S(H)$ for all $\boldsymbol{g} \in G_v^N$, where the last equivalence follows from Lemma 5.9. This is the case if and only if the following formula holds:

$$(\forall \boldsymbol{g} \in G_v^N)\,(\exists \boldsymbol{h} \in H^N)\ \pi_G(\boldsymbol{h}) = \pi_G(\boldsymbol{g}). \tag{5.11}$$

Conversely, $G \cdot H \preceq \mathfrak{D}(G_v)$ if and only if there exist $M_1 \leq G_v$ and $\lambda \in Y(G)$ such that $M_1 \leq P_\lambda$ and $c_\lambda(M_1)$ is conjugate to H (Lemma 5.3). This is the case if and only if the following formula holds:

$$(\exists \boldsymbol{g} \in G_v^N)\,(\exists g \in G)\ \pi_G(\boldsymbol{g}) = g \cdot \boldsymbol{h}_0, \tag{5.12}$$

where $\boldsymbol{h}_0$ is a fixed element of H^N such that $\mathscr{G}(\boldsymbol{h}_0) = H$. For, given $\boldsymbol{g} \in G_v^N$ and $g \in G$ such that $\pi_G(\boldsymbol{g}) = g \cdot \boldsymbol{h}_0$, we set $M_1 = \mathscr{G}(\boldsymbol{g})$; conversely, given $M_1 \leq G_v$ and $\lambda \in Y(G)$ such that $M_1 \leq P_\lambda$ and $g \in G$ such that $c_\lambda(M_1) = gHg^{-1}$, we choose $\boldsymbol{g} \in M_1^N$ such that $c_\lambda(\boldsymbol{g}) = g \cdot \boldsymbol{h}_0$.

We summarise the above argument as follows.

Lemma 5.13. *Let H be a G-cr subgroup of G such that $N \geq \kappa(H)+1$. Then $B_H \subseteq V$ is the set of solutions to the formulas (5.11) and (5.12). In particular, B_H is constructible.*

Remark 5.14. It can be shown that Lemma 5.13 holds for arbitrary k, where we take $\boldsymbol{h}$ to be a generic tuple for H in the sense of [Bate et al. 2013, Definition 5.4]. To do this, one replaces generating tuples with generic tuples in the definition of $\preceq$ and makes the obvious modifications to the arguments of this section.

6. Proof of Theorem 1.4

We assume throughout this section that $N \geq \Theta + 1$.

Proof of Theorem 1.4. We will show that there is a G-cr subgroup H of G such that $N \geq \kappa(H)+1$ and B_H has nonempty interior. By Lemma 5.13 and Remark 5.14, it is enough to prove this after extending the ground field to an uncountable algebraically closed field Ω (recall from the proof of Lemma 4.1 that any $G(\Omega)$-cr subgroup of $G(\Omega)$ is $G(\Omega)$-conjugate to a k-defined G-cr subgroup). Thus we can assume without loss that k is uncountable (and hence solid).

Let $D_1, \ldots, D_t$ be the irreducible components of C such that $\overline{\eta(G \cdot D_j)} = V$ for $1 \leq j \leq t$ — it follows from Lemma 7.1(b) below that there is at least one such component — and let $D'_1, \dots, D'_{t'}$ be the other irreducible components of C. Let

$$V' = V \setminus \bigcup_{j=1}^{t'} \overline{\eta(G \cdot D'_j)}.$$

For $1 \le j \le t$, set $E_j = \{(v, \boldsymbol{g}) \in D_j \mid \alpha_{G_v}(\mathscr{G}(\boldsymbol{g})) = G_v/R_u(G_v)\}$; note that E_j is neither closed nor open in general, and if $(v, \boldsymbol{g}) \in E_j$ then $\mathscr{D}(\mathscr{G}(\boldsymbol{g})) = \mathscr{D}(G_v)$. For any $v \in V'$, $N \ge \Theta + 1 \ge \kappa(G_v) + 1 \ge \kappa(G_v/R_u(G_v)) + 1$, so by Proposition 4.3 there exists $\boldsymbol{g} \in G^N$ such that $\alpha_{G_v}(\mathscr{G}(\boldsymbol{g})) = G_v/R_u(G_v)$. Then $(v, \boldsymbol{g}) \in D_j$ for some $1 \le j \le t$, so $(v, \boldsymbol{g}) \in E_j$. Hence $\bigcup_{1 \le j \le t} \eta(G \cdot E_j) \supseteq V'$. As G permutes the irreducible components of V transitively, $\eta(G \cdot E_m)$ is dense in V for some $1 \le m \le t$.

Choose G-cr subgroups H_i such that $\mathscr{H} := \{H_i \mid i \in I\}$ is a set of representatives for the conjugacy classes in $\mathscr{C}(G)_{\mathrm{cr}}$; Lemma 4.1 implies that I is countable. Let $\Lambda = \{H_i \mid G \cdot D_m \subseteq \phi^{-1}(S(H_i))\}$. Then $G \in \Lambda$, so Λ is nonempty. By Remark 5.8, we can pick $H \in \Lambda$ such that H is minimal with respect to $\preceq$. We claim that $G \cdot D_j \subseteq \phi^{-1}(S(H))$ for all $1 \le j \le t$. To prove this, let $(v, \boldsymbol{g}) \in D_j$ such that $v \in \eta(E_m)$. There exists $\boldsymbol{g}' \in G_v^N$ such that $(v, \boldsymbol{g}') \in E_m$. Then $(v, \boldsymbol{g}') \in \phi^{-1}(S(H))$, so $\boldsymbol{g}' \in S(H)$. Now $\mathscr{G}(\boldsymbol{g}) \le G_v$, so $\mathscr{D}(\mathscr{G}(\boldsymbol{g})) \preceq \mathscr{D}(G_v) = \mathscr{D}(\mathscr{G}(\boldsymbol{g}')) \preceq G \cdot H$ by Lemma 5.9. Hence $(v, \boldsymbol{g}) \in \phi^{-1}(S(H))$ by Lemma 5.9. As $S(H)$ is G-stable, it now follows that if $(v, \boldsymbol{g}) \in D_j$ and $v \in \eta(G \cdot E_m)$ then $(v, \boldsymbol{g}) \in \phi^{-1}(S(H))$. But $\eta^{-1}(\eta(G \cdot E_m)) \cap D_j$ is dense in D_j as $\eta(G \cdot E_m)$ is dense in V, so $D_j \subseteq \phi^{-1}(S(H))$. As $S(H)$ is G-stable, $G \cdot D_j \subseteq \phi^{-1}(S(H))$, as claimed. It follows from Lemma 5.9 that $\mathscr{D}(\mathscr{G}(\boldsymbol{g})) \preceq G \cdot H$ for all $1 \le j \le t$ and all $(v, \boldsymbol{g}) \in G \cdot D_j$. In particular, for any $v \in V'$, there exist j and $\boldsymbol{g}' \in G^N$ such that $(v, \boldsymbol{g}') \in E_j$, so $\mathscr{D}(G_v) = \mathscr{D}(\boldsymbol{g}') \preceq G \cdot H$.

To finish, we show that B_H has nonempty interior in V. Suppose otherwise. As B_H is constructible (Lemma 5.13), $\overline{B_H}$ is a proper closed subset of V, so $V \setminus B_H$ is a G-stable subset with nonempty interior. Now $\eta\big(\phi^{-1}(S(H))\big)$ is dense in V as it contains $\eta(G \cdot D_m)$. Hence there is a nonempty open G-stable subset O of $\eta\big(\phi^{-1}(S(H))\big) \cap V'$ such that $B_H \cap O$ is empty. Let $v \in O$ and let $\boldsymbol{g} \in G^N$ such that $(v, \boldsymbol{g}) \in D_m$. Then $\mathscr{D}(\mathscr{G}(\boldsymbol{g})) \preceq \mathscr{D}(G_v) \preceq G \cdot H$; but $v \notin B_H$, so $\mathscr{D}(G_v) \neq G \cdot H$, and it follows from Corollary 5.7 that $\mathscr{D}(\mathscr{G}(\boldsymbol{g})) \prec G \cdot H$. Hence $\mathscr{D}(\mathscr{G}(\boldsymbol{g})) = G \cdot H_i$ for some $i \in I$ such that $G \cdot H_i \prec G \cdot H$. Lemma 5.9 now implies that $\eta^{-1}(O) \cap D_m \subseteq \bigcup_{i \in I'} \phi^{-1}(S(H_i))$, where $I' := \{i \in I \mid G \cdot H_i \prec G \cdot H\}$. By Corollary 2.6, there exists $i \in I'$ such that $\eta^{-1}(O) \cap D_m \subseteq \phi^{-1}(S(H_i))$. Since $\eta^{-1}(O) \cap D_m$ is a nonempty open subset of D_m and $\phi^{-1}(S(H_i))$ is closed and G-stable, $G \cdot D_m \subseteq \phi^{-1}(S(H_i))$. But $G \cdot H_i \prec G \cdot H$, which contradicts the minimality of H. We conclude that B_H has nonempty interior in V after all. Finally, since $G \cdot H = \mathscr{D}(G_v)$ for some $v \in V$, we have $\kappa(H) \le \kappa(G_v) \le \Theta$, so $N \ge \kappa(H) + 1$. This completes the proof. □

Proof of Corollaries 1.5 and 1.6. We can assume O is G-stable. By Theorem 1.4, there is a nonempty open G-stable subset O' of V and a G-cr subgroup H of G

such that $\mathscr{D}(G_v) = G \cdot H$ for all $v \in O'$. Now $O \cap O'$ is a nonempty open G-stable subset of V, and for all $v \in O \cap O'$, $\mathscr{D}(G_v) = G \cdot H$. Since G_v is G-cr for $v \in O \cap O'$, G_v is conjugate to H. It follows that V has a principal stabiliser.

In particular, the hypotheses of Corollary 1.5 are satisfied if $\mathrm{char}(k) = 0$ and V_{red} is nonempty, since then V_{red} is open by Theorem 1.1 and for all $v \in V_{\mathrm{red}}$, G_v — being reductive — is G-cr. This proves Corollary 1.6. □

Remark 6.1. Here is a generalisation of Corollary 1.6. If $\mathrm{char}(k) = 0$ and O is as in Theorem 1.4 then $G \cdot M_v = \mathscr{D}(G_v) = G \cdot H$ for all $v \in O$, where M_v is any Levi subgroup of G_v.

7. Irreducible components of the stabiliser variety

In this section we study the irreducible components of the stabiliser variety C. We use the information we obtain to prove results analogous to those in Section 6, but for the subgroups G_v^0 rather than the subgroups G_v. We assume throughout the section that $N \geq 3$.

Lemma 7.1. (*a*) *Let D be an irreducible component of C such that $\eta(G \cdot D)$ is dense in V. Then* $\dim(D) = n + Nr$ *and for all $v \in V_0$, the fibre $(\eta|_D)^{-1}(v)$ either is empty or has dimension Nr and is isomorphic* (*via ϕ*) *to a union of irreducible components of G_v^N.*

(*b*) *There is a unique closed subset $\widetilde{C}$ of C such that $\widetilde{C}$ contains $V \times \{\mathbf{1}\}$, $\widetilde{C}$ is a union of irreducible components of C and G permutes these irreducible components transitively. The variety $\widetilde{C}$ is the closure of the set $\{(v, \boldsymbol{g}) \mid v \in V_0, \boldsymbol{g} \in (G_v^0)^N\}$, and each irreducible component of $\widetilde{C}$ has dimension $n + Nr$.*

Proof. Clearly it is enough to prove the result when G is connected and V is irreducible, so we assume this.

(a) Define $f \colon V \times G^N \to V \times V^N$ by

$$f(v, \boldsymbol{g}) = (v, g_1 \cdot v, \ldots, g_N \cdot v).$$

Let Y be the closure of the image of f. Let Δ be the diagonal in $V \times V^N$; then $C = f^{-1}(\Delta)$. The variety Y is irreducible because G and V are irreducible. Let $v \in V$ and let $\boldsymbol{g} \in G^N$. Then $f^{-1}(v, g_1 \cdot v, \ldots, g_N \cdot v) = \{v\} \times g_1 G_v \times \cdots \times g_N G_v$. Hence irreducible components of generic fibres of f over Y have dimension Nr. It follows that

$$\dim(Y) = \dim(V \times G^N) - Nr = n + N\dim(G) - Nr = n + N(\dim(G) - r).$$

As $\eta(D)$ is dense in V, $f(D)$ is dense in Δ, so $\dim(D) \geq \dim(\Delta) + Nr = n + Nr$.

If $v \in \eta(D) \cap V_0$ and Z is an irreducible component of $(\eta|_D)^{-1}(v)$, then we have $\dim(Z) \geq \dim(D) - \dim(V) \geq n + Nr - n = Nr$. But $\phi(\eta^{-1}(v))$ is a subset of

G_v^N and the irreducible components of G_v^N all have dimension $\dim(G_v^N) = Nr$. This forces Z to be isomorphic (via ϕ) to an irreducible component of G_v^N. Hence irreducible components of generic fibres of $\eta|_D$ have dimension Nr, which implies that $\dim(D) = n + Nr$. Part (a) now follows.

(b) Since $V \times \{\mathbf{1}\}$ is irreducible, there is some irreducible component $\widetilde{C}$ of C such that $\widetilde{C}$ contains $V \times \{\mathbf{1}\}$. For any $v \in V_0$, let Z be an irreducible component of the fibre $(\eta|_{\widetilde{C}})^{-1}(v)$ such that $(v, \mathbf{1}) \in Z$. By part (a), $\dim(Z) = Nr$, so Z is isomorphic via ϕ to an irreducible component of G_v^N. But the only component of G_v^N that contains $\mathbf{1}$ is $(G_v^0)^N$, so $\{v\} \times (G_v^0)^N \subseteq Z$. Hence $\widetilde{C}$ contains the closure of $\{(v, \boldsymbol{g}) \mid v \in V_0, \boldsymbol{g} \in (G_v^0)^N\}$ — call this closure C'.

Let $A_1, \dots, A_m$ be the irreducible components of C' such that $\overline{\eta(A_j)} = V$ (there is at least one, since $\eta(C') = V$). Let $s_i = \dim(A_i)$ for $1 \leq i \leq m$ and let $\eta_i \colon A_i \to V$ be the restriction of η. There is a nonempty open subset U of V such that for all $v \in U$, $\eta^{-1}(v) \subseteq A_1 \cup \dots \cup A_m$ and every irreducible component of $\eta_i^{-1}(v)$ has dimension $s_i - n$. Since $\{v\} \times (G_v^0)^N \subseteq C'$ for all $v \in V_0$, if $v \in U \cap V_0$ then $\eta_j^{-1}(v)$ must contain $\{v\} \times (G_v^0)^N$ for some $1 \leq j \leq m$, which forces $s_j \geq n + Nr$. But $\dim(\widetilde{C}) = n + Nr$ by part (a), so A_j must be the whole of $\widetilde{C}$, so $C' = \widetilde{C}$. This completes the proof. □

Remark 7.2. The dimension inequality in Lemma 7.1(a) can fail if $\eta(G \cdot D)$ is not dense in V (Example 8.2). Moreover, $\widetilde{C}$ need not contain the whole of $\{(v, \boldsymbol{g}) \mid v \in V, \boldsymbol{g} \in (G_v^0)^N\}$: see Examples 8.1(a) and 8.2.

Remark 7.3. If G is connected and V is irreducible then $\widetilde{C}$ is irreducible and G-stable. More generally, any irreducible component of C is G-stable in this case.

Definition 7.4. We call $\widetilde{C}$ the *connected-stabiliser variety* of V.

Corollary 7.5. *If* $r = 0$ *then* $\widetilde{C} = V \times \{\mathbf{1}\}$.

Proof. The irreducible components of $V \times \{\mathbf{1}\}$ are isomorphic via η to the irreducible components of V, so they are permuted transitively by G and each has dimension n. It follows from the dimension formula in Lemma 7.1(a) that these irreducible components are irreducible components of C. The result now follows from Lemma 7.1(b). □

We denote by $\tilde{\phi} \colon \widetilde{C} \to G^N$ and $\tilde{\eta} \colon \widetilde{C} \to V$ the restrictions to $\widetilde{C}$ of ϕ and η, respectively, and if $v \in V$ then we denote $\tilde{\phi}\big(\tilde{\eta}^{-1}(v)\big)$ by F_v. If $v \in V_0$ then $(G_v^0)^N \subseteq F_v$; we do not know whether equality holds for all $v \in V_0$, or even for generic $v \in V_0$.

We now give a counterpart to Theorem 1.4. In the connected case, we obtain slightly more information: we can describe $\mathfrak{D}(G_v^0)$ for all $v \in V_{\min}$ (recall the definition of $V_{\min}$ from Remark 3.5).

Theorem 7.6. *There exists a connected G-completely reducible subgroup H of G such that*:

(a) *For all $v \in V_{\min}$, $\mathcal{D}(G_v^0) = G \cdot H$.*
(b) $\widetilde{C} \subseteq \tilde{\phi}^{-1}(S(H))$.

In particular, if V_{red} is nonempty then $\mathcal{D}(G_v^0) = G \cdot H$ for all $v \in V_{\mathrm{red}}$.

Proof. By Theorem 1.4, there exist a G-cr subgroup H' of G and a G-stable open subset O of V such that $\mathcal{D}(G_v) = G \cdot H'$ for all $v \in O$. Set $H = (H')^0$; then H is G-cr as $H \trianglelefteq H'$. Let t be the minimal dimension of $\dim(R_u(G_v))$ for $v \in V_0$. The G-stable open sets O and $V_{\min}$ have nonempty intersection, so there exists $v \in V_{\min} \cap O$ such that $\mathcal{D}(G_v) = G \cdot H'$ and $\dim(R_u(G_v)) = t$. This yields $\dim(H') = \dim(G_v) - \dim(R_u(G_v)) = r - t$.

By the proof of Theorem 1.4, $\widetilde{C} \subseteq \tilde{\phi}^{-1}(S(H'))$. Let $v \in V_{\min}$ and choose $\lambda \in Y(G)$ such that $\mathcal{D}(G_v) = c_\lambda(G_v)$. Then $c_\lambda(G_v)$ is G-cr and $c_\lambda(G_v^0) = c_\lambda(G_v)^0$ is a normal subgroup of $c_\lambda(G_v)$, so $c_\lambda(G_v^0)$ is G-cr. It follows that $\mathcal{D}(G_v^0) = G \cdot c_\lambda(G_v^0)$. We want to prove that $c_\lambda(G_v^0)$ is conjugate to H; that is, we want to prove that

$$(\exists m \in G)[(\forall g \in G_v^0)\, c_\lambda(g) \in mHm^{-1} \wedge (\forall h \in H)(\exists g \in G_v^0)\, c_\lambda(g) = mhm^{-1}].$$

Since this is a first-order formula, this is a constructible condition. Hence it is enough to prove that it holds after extending k to any larger algebraically closed field. So without loss of generality we assume k is solid.

By Proposition 4.3, we can choose $\boldsymbol{g}' \in (G_v^0)^N$ such that $\alpha_{G_v^0}(\mathcal{G}(\boldsymbol{g}')) = G_v^0/R_u(G_v^0)$. There exists $\boldsymbol{h} \in (H')^N$ such that $\pi_G(\boldsymbol{h}) = \pi_G(\boldsymbol{g}')$. Let $K = \mathcal{G}(\boldsymbol{h})$. Now $c_\lambda(\mathcal{G}(\boldsymbol{g}')) = c_\lambda(G_v^0)$ is G-cr, so there exists $\mu \in Y(G)$ such that $c_\mu(\boldsymbol{h})$ is conjugate to $c_\lambda(\boldsymbol{g}')$. Then $c_\lambda(\mathcal{G}(\boldsymbol{g}'))$ is conjugate to $c_\mu(\mathcal{G}(\boldsymbol{h}))$. But

$$\begin{aligned}(7.7)\quad \dim\big(c_\lambda(\mathcal{G}(\boldsymbol{g}'))\big) &= \dim\big(c_\lambda(G_v^0)\big) = \dim(H) \\ &\geq \dim(K) \geq \dim\big(c_\mu(K)\big) = \dim\big(c_\mu(\mathcal{G}(\boldsymbol{h}))\big),\end{aligned}$$

which forces $\dim(K)$ to equal $\dim(H)$. Hence $K \supseteq H$. Now $c_\mu(K)$ is conjugate to $c_\lambda(G_v^0)$, which is connected, so $c_\mu(K) = c_\mu(H)$. But $c_\mu(H)$ is conjugate to H since H is G-cr, so we deduce that $c_\lambda(G_v^0)$ is conjugate to H. Hence $\mathcal{D}(G_v^0) = G \cdot H$. This proves part (a). Moreover, if $\boldsymbol{g} \in (G_v^0)^N$ then $c_\lambda(\mathcal{G}(\boldsymbol{g})) = \mathcal{G}(c_\lambda(\boldsymbol{g}))$ is conjugate to a subgroup of H, so there exists $\boldsymbol{h} \in H^N$ such that $c_\lambda(\boldsymbol{g})$ is conjugate to $\boldsymbol{h}$; hence $(v, \boldsymbol{g}) \in \tilde{\phi}^{-1}(S(H))$. As $\{(v, \boldsymbol{g}) \mid v \in V_{\min}, \boldsymbol{g} \in (G_v^0)^N\}$ is dense in $\widetilde{C}$ by Lemma 7.1(b) and Remark 3.5, $\widetilde{C} \subseteq \tilde{\phi}^{-1}(S(H))$. This proves part (b). □

The next result is the counterpart to Corollaries 1.5 and 1.6. We omit the proof, which is similar.

Corollary 7.8. *Suppose there is a nonempty open subset O of V such that G_v^0 is G-cr for all $v \in O$ (in particular, this condition holds if* $\mathrm{char}(k) = 0$ *and V_{red} is* *nonempty). Let H be the connected G-cr subgroup from Theorem 7.6. Then G_v^0 is conjugate to H for all $v \in V_{\mathrm{red}}$.*

Remark 7.9. Suppose there exists $v \in V_0$ such that G_v^0 is G-ir. Then G_v is G-cr, so $v \in V_{\mathrm{red}}$, so V_{red} is nonempty. We have $\mathfrak{D}(G_v^0) = G \cdot H$ by Theorem 7.6(a). As G_v^0 is G-cr, $G \cdot G_v^0 = G \cdot H$. It follows that H is G-ir and G_w^0 is conjugate to H for all $w \in V_{\mathrm{red}}$; in particular, G_w^0 is G-ir for all $w \in V_{\mathrm{red}}$. The analogous result for the full stabiliser G_v is false (cf. Remarks 7.13 and 4.6(ii), and Examples 8.1(c) and 8.2). However, if O is as in Theorem 1.4 and there exists $v \in O$ such that G_v is G-ir then an argument like the one above shows that V has a G-ir principal stabiliser.

Theorem 7.6 gives rise to the following counterpart to Remark 6.1 for G_v^0; the proof is similar.

Corollary 7.10. *Suppose* $\mathrm{char}(k) = 0$. *Then H is conjugate to a Levi subgroup of G_v^0 for all $v \in V_{\min}$.*

We give a criterion to ensure that the fibres of $\tilde{\eta}$ are irreducible. Define $\widetilde{C}_{\min} = \tilde{\eta}^{-1}(V_{\min})$.

Proposition 7.11. *Suppose* $\mathrm{char}(k) = 0$ *and* $N \geq \Theta + 1$. *Then*

$$\widetilde{C}_{\min} = \{(v, \boldsymbol{g}) \mid v \in V_{\min}, \boldsymbol{g} \in (G_v^0)^N\}.$$

Proof. Let H be as in Theorem 7.6. Let $v \in V_{\min}$, and suppose F_v properly contains $(G_v^0)^N$. Then F_v contains an irreducible component $D \neq (G_v^0)^N$ of G_v^N by Lemma 7.1(a). Set $K = G_v$, set $M = K/R_u(K)$ and let $\alpha_K : K \to M$ be the canonical projection. Let K_1 be the subgroup of K generated by K^0 together with the components of each of the tuples in D; then K_1 properly contains K^0. As $R_u(G_v)$ is connected, $M_1 := \alpha_K(K_1)$ properly contains M^0. In particular, M_1 is reductive. By [Martin 2003, Lemma 9.2], there exists $\boldsymbol{g} \in D$ such that $\alpha_K(\boldsymbol{g})$ generates M_1. Hence $G \cdot K_1 \preceq G \cdot \mathcal{G}(\boldsymbol{g})$. Now $\boldsymbol{g} \in \widetilde{C}$, so $G \cdot \mathcal{G}(\boldsymbol{g}) \preceq G \cdot H$ (Theorem 7.6(b)). It follows from Lemma 5.4 that $G \cdot K_1 \preceq G \cdot H$.

We have $G \cdot H = \mathfrak{D}(K^0)$ by choice of v and Theorem 7.6, so $G \cdot H \preceq G \cdot K^0$ by Lemma 5.2. Now $G \cdot K^0 \preceq G \cdot K_1$ as $K^0 \leq K_1$, so $G \cdot H \preceq G \cdot K_1$ by Lemma 5.4. It follows from Lemmas 5.2 and 5.6 that $G \cdot H = \mathfrak{D}(K_1)$. Now $\mathfrak{D}(K_1) = G \cdot M_1$ as M_1 is reductive and $\mathrm{char}(k) = 0$, so $G \cdot H = G \cdot M_1$. But this is impossible as H is connected and M_1 is not. We conclude that $F_v = (G_v^0)^N$ after all. The result now follows. ∎

We have seen that we obtain stronger results if we know that generic stabilisers (or their identity components) are G-cr. Reductive subgroups are always G-cr in characteristic 0, but things are more complicated in positive characteristic. Our next result shows that if this G-complete reducibility condition fails for connected stabilisers then it fails badly: we prove that if there exists $v \in V_0$ such that G_v^0 is reductive but not G-cr then generic elements of V have the same property.

Proposition 7.12. *Let H be as in Theorem 7.6. Let*

$$\widetilde{B}'_H = \{v \in V_{\min} \mid G_v^0 \text{ is not } G\text{-cr}\}.$$

If $\widetilde{B}'_H$ is nonempty then $\widetilde{B}'_H$ has nonempty interior.

Proof. Note that $\mathcal{D}(G_v^0) = G \cdot H$ for all $v \in \widetilde{B}'_H$, by Theorem 7.6. The argument below shows that $\widetilde{B}'_H = V_{\min} \cap \tilde{\eta}(\tilde{\phi}^{-1}(U))$, where U is the open set defined below, so $\widetilde{B}'_H$ is constructible. It follows as in the proof of Theorem 7.6 that we can extend the ground field k; hence we can assume k is solid.

Suppose $\widetilde{B}'_H$ is nonempty. Let $v \in \widetilde{B}'_H$. By Proposition 4.3 we can choose $\boldsymbol{g} \in (G_v^0)^N$ such that $\alpha_{G_v^0}(\mathcal{G}(\boldsymbol{g})) = G_v^0/R_u(G_v^0)$ and such that $1 \neq g_N \in R_u(G_v^0)$ if G_v^0 is nonreductive. This ensures that $\mathcal{G}(\boldsymbol{g})$ is not G-cr. Since H is G-cr and $\mathcal{D}(G_v^0) = G \cdot H$, there exists $\lambda \in Y(G)$ such that $G_v^0 \leq P_\lambda$ and $c_\lambda(G_v^0)$ is conjugate to H. Then $c_\lambda(\mathcal{G}(\boldsymbol{g})) = c_\lambda(G_v^0)$ is conjugate to H, so

$$\dim(G_{\boldsymbol{g}}) = \dim\big(C_G(\mathcal{G}(\boldsymbol{g}))\big) < \dim(C_G(H)),$$

since $\mathcal{G}(\boldsymbol{g})$ is not conjugate to H (as $\mathcal{G}(\boldsymbol{g})$ is not G-cr). Consider G^N regarded as a G-variety. Let U be the set of all $\boldsymbol{m} \in G^N$ such that $\dim(G_{\boldsymbol{m}}) < \dim(C_G(H))$; then U is an open neighbourhood of $\boldsymbol{g}$, by Lemma 2.1.

Let $E = \{(w, \boldsymbol{g}) \in \widetilde{C} \cap \tilde{\phi}^{-1}(U) \cap \tilde{\eta}^{-1}(V_{\min}) \mid \boldsymbol{g} \in (G_w^0)^N\}$. By Lemma 7.1(b), E is dense in $\widetilde{C}$, so $\tilde{\eta}(E)$ is dense in V. To complete the proof, it is enough to show that $\tilde{\eta}(E) \subseteq \widetilde{B}'_H$, for then $\widetilde{B}'_H$, being constructible and dense, has nonempty interior. So let $w \in \tilde{\eta}(E)$. Pick $\boldsymbol{m}$ such that $(w, \boldsymbol{m}) \in E$. Then $\boldsymbol{m} \in (G_w^0)^N \cap U$, so $\dim(C_G(\boldsymbol{m})) < \dim(C_G(H))$, so $\dim(C_G(G_w^0)) < \dim(C_G(H))$ also. It follows by running the argument above for G_v^0 in reverse that G_w^0 is not G-cr. Hence $w \in \widetilde{B}'_H$, as required. □

Remark 7.13. A similar argument establishes the following. Let H be as in Theorem 1.4. If there exists $(v, \boldsymbol{g}) \in C$ such that $v \in V_0$, $\mathcal{D}(G_v) = G \cdot H$ and G_v is not G-cr then there is an open neighbourhood U of $(v, \boldsymbol{g}) \in C$ such that for all $(w, \boldsymbol{g}') \in U$, G_w is not G-cr. But this does not yield an analogue of Proposition 7.12 for G_v (see Example 8.1(b)) — the problem is that $\eta(U)$ need not be dense in V.

8. Examples

In this section we present some examples that show the limits of our results and illustrate some of the phenomena that can occur. We assume $N \geq \Theta + 1$.

Example 8.1. We consider a special case of the setup from the proof of Theorem 1.3. Let $G = \mathrm{PGL}_2(k)$, let $M \leq G$ and let V be the quasiprojective variety G/M with

G acting by left multiplication. We assume that $M \cap gMg^{-1} = 1$ for generic $g \in G$ (this will hold in all the cases we consider). Then $G_w = 1$ for generic $w \in V$, so the subset H from Theorem 1.4 is 1, and $\widetilde{C} = V \times \{\mathbf{1}\}$ by Corollary 7.5. In particular, $F_w = \{\mathbf{1}\}$ for all $w \in V$. Let $v = M \in G/M$; then $G_v = M$.

(a) Let M be a maximal torus of G. Then F_v is properly contained in M^N, so we see that F_v need not contain all of $(G_v^0)^N$ when $v \notin V_0$ (cf. Remark 7.2). The subset B_H is dense but not closed in V, as $\mathfrak{D}(G_v) = G \cdot M$.

(b) Let $M = \langle x \rangle$, where $x \in G$ is a nontrivial unipotent element. Then $V = V_0 = V_{\mathrm{red}}$ and G_w is unipotent for all $w \in V$, so $\mathfrak{D}(G_v) = \{1\}$ for all $w \in V$ (where 1 denotes the trivial subgroup). Now $G_w = 1$ is G-cr for generic $w \in V$ but G_v is not G-cr. Hence the set $\{w \in V_{\mathrm{red}} \mid \mathfrak{D}(G_w) = G \cdot H$ and G_v is not G-cr$\}$ is nonempty but not dense in V (cf. Remark 7.13). The irreducible components of C apart from $\widetilde{C}$ do not dominate V.

(c) Let $M = \mathrm{PGL}_2(q)$, where q is a power of the characteristic p. We have $V = V_0 = V_{\mathrm{red}}$. Now M is G-ir, so the set $\{w \in V_{\mathrm{red}} \mid G_w$ is G-ir$\}$ is nonempty but not dense in V. Moreover, the set O from Theorem 1.4 does not contain the whole of V_{red}.

Example 8.2. Suppose G is connected and not a torus. Let $m \in \mathbb{N}$ and let V be the variety of m-tuples of unipotent elements of G, with G acting on V by simultaneous conjugation. We claim that $\{(1, \dots, 1)\} \times G^N$ is an irreducible component of C. To see this, let D be an irreducible component of C such that $\{(1, \dots, 1)\} \times G^N \subseteq D$. Consider the element $(1, \dots, 1, \boldsymbol{g}) \in D$, where the components of $\boldsymbol{g} \in G^N$ are all regular semisimple elements of G. There is an open neighbourhood O of $\boldsymbol{g}$ in G^N consisting of tuples of regular semisimple elements. If $(v_1, \dots, v_m, \boldsymbol{g}') \in \phi^{-1}(O)$ then each component of $\boldsymbol{g}'$ is a regular semisimple element of g centralising the unipotent elements $v_1, \dots, v_m$ of G. But this forces $v_1, \dots, v_m$ to be 1. It follows that $D = \{(1, \dots, 1)\} \times G^N$, as claimed. Hence $\eta(D) = \{(1, \dots, 1)\}$ and $\eta(D) \cap V_0$ is empty (note also that if m is large enough then the dimension inequality from Lemma 7.1(a) is violated). We see that the set $\{w \in V \mid G_w^0$ is G-ir$\}$ is nonempty but not dense in V.

It is not hard to show that $F_{(1,\dots,1)} \subseteq \{g \cdot \boldsymbol{g} \mid \boldsymbol{g} \in U^N\}$, where U is a maximal unipotent subgroup of G; in particular, we see as in Example 8.1(a) that F_v need not contain all of $(G_v^0)^N$ when $v \notin V_0$. Moreover, since the centraliser of a nontrivial unipotent subgroup of a connected group can never be reductive, the only reductive stabiliser is $G_{(1,\dots,1)}$, so V_{red} is empty.

Example 8.3. Let X be an affine variety and M a reductive linear algebraic group. Suppose we have a morphism $f\colon X \times M \to X \times G$ of the form $f(x, m) = (x, f_x(m))$, and suppose further that each $f_x\colon M \to G$ is a homomorphism of algebraic groups.

Set $K_x = \mathrm{im}(f_x)$. Define actions of G and M on $X \times G$ by $g \cdot (x, g') = (x, gg')$ and $m \cdot (x, g') = (x, g' f_x(m)^{-1})$. These actions commute with each other, so we get an action of G on the quotient space $V := (X \times G)/M$.

Now suppose moreover that $\dim(K_x)$ is independent of x. Then the M-orbits on $X \times G$ all have the same dimension, so they are all closed. This means the canonical projection φ from $X \times G$ to V is a geometric quotient, so its fibres are precisely the M-orbits [Newstead 1978, Corollary 3.5.3]. A straightforward calculation shows that for any $(x, g) \in X \times G$, the stabiliser $G_{\varphi(x,g)}$ is precisely gK_xg^{-1}. It follows that if X is infinite and the subgroups K_x are pairwise nonconjugate as x runs over the elements of a dense subset of X then V has no principal stabiliser.

Here is a simple example. Let $G = \mathrm{SL}_2(k)$, let $X = k$ and let

$$M = C_p \times C_p = \langle \gamma_1, \gamma_2 \mid \gamma_1^p = \gamma_2^p = [\gamma_1, \gamma_2] = 1 \rangle.$$

Define $f\colon X \times M \to X \times G$ by $f(x, m) = (x, f_x(m))$, where

$$f_x(\gamma_1^{m_1}\gamma_2^{m_2}) := \begin{pmatrix} 1 & m_1x + m_2x^2 \\ 0 & 1 \end{pmatrix}.$$

It is easily checked that f has the desired properties, so $V := (X \times M)/G$ has no principal stabiliser. Note also that generic stabilisers are nontrivial finite unipotent groups, but the element $v = \varphi(0, 1)$ has trivial stabiliser.

Here is an example where the stabilisers are connected. Daniel Lond [Lond 2013, Section 6.5] produced a family, parametrised by $X := k$, of homomorphisms from $M := \mathrm{SL}_2(k)$ to $G := B_4$ in characteristic 2 with pairwise nonconjugate images. Using this one can construct a morphism $f\colon X \times M \to X \times G$ with the desired properties, giving rise to a G-variety $V := (X \times M)/G$ having no principal stabiliser and with all stabilisers connected and reductive. Results of David Stewart [2010, Section 5.4.3] give rise to a similar construction for $G = F_4$ in characteristic 2.

Example 8.4. We now give an example where there is a point with trivial stabiliser but generic stabilisers are finite and linearly reductive, using another special case of the setup from the proof of Theorem 1.3. We describe a recipe for producing such examples, given in [Burness et al. 2015, Corollary 3.10]. Take a simple algebraic group G of rank s in characteristic not 2 and set $M = C_G(\tau)$, where τ is an involution that inverts a maximal torus of G. Then the affine variety G/M, with M acting by left multiplication, has precisely one orbit that consists of points with trivial stabiliser. Let $V = G/M \times G/M$ with the product action of G. Then generic stabilisers of points in V are 2-groups of order 2^s, but V contains points with trivial stabiliser. Thus $V = V_0 = V_{\mathrm{red}}$ and $\widetilde{C} = V \times \{\mathbf{1}\}$. Since 2-groups are linearly reductive — and hence G-cr — in characteristic not 2, the G-cr subgroup H from Theorem 1.4 must be a 2-group of order 2^s, and moreover, H is a principal stabiliser for V by Corollary 1.5. The set $\{v \in V_{\mathrm{red}} \mid \mathfrak{D}(G_v) = G \cdot H\}$ does not contain the whole of V_{red} (cf. Theorem 7.6).

We claim that there is at least one irreducible component D of C such that $\overline{\eta(D)} = V$ but $\eta(D) \neq V$. Let $D_1, \ldots, D_t$ be the irreducible components of V apart from C. Then $\bigcup_{i=1}^{t} \overline{\eta(D_i)} = V$, so $\eta(D_j)$ is dense in V for some $1 \leq j \leq t$. There are only finitely many conjugacy classes of nontrivial elements of G of order dividing 2^s, and each such conjugacy class is closed because in characteristic not 2, elements of order a power of 2 are semisimple. Hence there are regular functions $f_1, \ldots, f_m \colon G \to k$ for some m such that for all $g \in G$, g is a nontrivial element of order dividing 2^s if and only if $f_1(g) = \cdots = f_m(g) = 0$. For $1 \leq l \leq N$, let Z_l be the closed subset $\{(v, g_1, \ldots, g_N) \in C \mid f_1(g_l) = \cdots = f_m(g_l) = 0\}$ of C and let $Z = Z_1 \cup \cdots \cup Z_N$. If $(v, \boldsymbol{g}) \in D_j \setminus (D_j \cap \widetilde{C})$ then $\boldsymbol{g} \neq 1$, so some component of $\boldsymbol{g}$ is a nontrivial element of G of order dividing 2^s, so $(v, \boldsymbol{g}) \in Z$. Hence the open dense subset $D_j \setminus (D_j \cap \widetilde{C})$ of D_j is contained in Z, and it follows that $D_j \subseteq Z$. This implies that if $v \in V$ and $G_v = 1$ then $v \notin \eta(D_j)$.

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ON THE EQUATIONS DEFINING AFFINE ALGEBRAIC GROUPS

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In memory of Robert Steinberg

For the coordinate algebras of connected affine algebraic groups, we explore the problem of finding a presentation by generators and relations canonically determined by the group structure.

1. Introduction

Connected algebraic groups constitute a remarkable class of irreducible quasiprojective algebraic varieties. It contains the subclasses of abelian varieties and affine algebraic groups. These subclasses are basic: by Chevalley's theorem, every connected algebraic group G has a unique connected normal affine algebraic subgroup L such that G/L is an abelian variety, whence the variety G is an L-torsor over the abelian variety G/L. The varieties from these subclasses can be embedded in many ways as closed subvarieties in, respectively, projective and affine spaces. A natural question then arises as to whether there are distinguished embeddings and equations of their images, which are canonically determined by the group structure. For abelian varieties, this is the existence problem for canonically defined bases in linear systems and that of presenting homogeneous coordinate rings of ample invertible sheafs by generators and relations. These problems were explored and solved by D. Mumford [1966]. For affine algebraic groups, it is the existence problem of the canonically defined presentations of the coordinate algebras of such groups by generators and relations. We explore this problem in the present paper.

We fix as the base field an algebraically closed field k of arbitrary characteristic. In this paper, as in [Borel 1991], "variety" means "algebraic variety" in the sense of Serre [1955, Subsection 34]; every variety is taken over k.

Let G be a connected affine algebraic group and let $R_u(G)$ be its unipotent radical. In view of [Grothendieck 1958, Propositions 1, 2] and [Rosenlicht 1956, Theorem 10], the underlying variety of G is isomorphic to the product of that of

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$G/R_u(G)$ and $R_u(G)$, and the latter is isomorphic to an affine space. Therefore, the problem under consideration is reduced to the case of reductive groups. Given this, henceforth G stands for a connected *reductive* algebraic group.

The simplest case of SL_2 is the guiding example. Take the polynomial k-algebra $k[x_1, x_2, x_3, x_4]$ in four variables x_i. The usual presentation of $k[\mathrm{SL}_2]$ is given by the surjective homomorphism

$$\mu\colon k[x_1, x_2, x_3, x_4] \to k[\mathrm{SL}_2], \quad \mu(x_i)\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = a_i, \tag{1}$$

whose kernel is the ideal $(x_1x_4 - x_2x_3 - 1)$. After rewriting, this presentation can be interpreted in terms of the group structure of SL_2 as follows.

We have $k[x_1, x_2, x_3, x_4] = k[x_1, x_3] \otimes_k k[x_2, x_4]$ and the restriction of μ to the subalgebra $k[x_1, x_3]$ (respectively, $k[x_2, x_4]$) is an isomorphism with the subalgebra $\mathcal{S}^+$ (respectively, $\mathcal{S}^-$) of $k[\mathrm{SL}_2]$ consisting of all regular functions invariant with respect to the subgroup U^+ (respectively, U^-) of all unipotent upper (respectively, lower) triangular matrices acting by right translations. Hence (1) yields the following presentation of $k[\mathrm{SL}_2]$ by generators and relations:

$$\begin{aligned} k[\mathrm{SL}_2] &\cong (\mathcal{S}^+ \otimes_k \mathcal{S}^-)/\mathcal{I}, \\ \mathcal{S}^+ &= k[\mu(x_1), \mu(x_3)] \cong k[x_1, x_3], \\ \mathcal{S}^- &= k[\mu(x_2), \mu(x_4)] \cong k[x_2, x_4], \\ \mathcal{I} &= \big(\mu(x_1) \otimes \mu(x_4) - \mu(x_2) \otimes \mu(x_3) - 1\big). \end{aligned} \tag{2}$$

The subgroups U^+, U^- are opposite maximal unipotent subgroups of SL_2. The subalgebras $\mathcal{S}^+$, $\mathcal{S}^-$ are stable with respect to SL_2 acting by left translations, and $f := \mu(x_1) \otimes \mu(x_4) - \mu(x_2) \otimes \mu(x_3) - 1$ is the unique element of $(\mathcal{S}^+ \otimes_k \mathcal{S}^-)^{\mathrm{SL}_2}$ determined by the conditions $f(e, e) = 1$, $k[f] = (\mathcal{S}^+ \otimes_k \mathcal{S}^-)^{\mathrm{SL}_2}$.

We show that there is an analogue of (2) for every connected reductive algebraic group G. Namely, we endow $k[G]$ with the G-module structure determined by left translations and fix in G a pair of opposite Borel subgroups B^+ and B^-. Let $U^\pm$ be the unipotent radical of $B^\pm$. Consider the G-stable subalgebras

$$\begin{aligned} \mathcal{S}^+ &:= \{f \in k[G] \mid f(gu) = f(g) \text{ for all } g \in G, u \in U^+\}, \\ \mathcal{S}^- &:= \{f \in k[G] \mid f(gu) = f(g) \text{ for all } g \in G, u \in U^-\} \end{aligned} \tag{3}$$

of $k[G]$ and the natural multiplication homomorphism of k-algebras

$$\mu\colon \mathcal{S}^+ \otimes_k \mathcal{S}^- \to k[G], \quad f_1 \otimes f_2 \mapsto f_1 f_2. \tag{4}$$

For $k = \mathbb{C}$, the following were put forward in [Flath and Towber 1992]:

Conjectures (D. E. Flath and J. Towber [1992]).

(S) *The homomorphism μ is surjective.*

(K) *The ideal* $\ker\mu$ *in $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ is generated by* $(\ker\mu)^G$.

If these conjectures are true, then the problem under consideration is reduced to the following:

(a) Find the canonically defined generators of the k-algebra $(\ker \mu)^G$.

(b) Find the canonically defined presentations of $\mathcal{S}^{\pm}$ by generators and relations.

In [Flath and Towber 1992], Conjectures (S) and (K) were proved for $k = \mathbb{C}$ and $G = \mathrm{SL}_n, \mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_n$ by means of lengthy direct computations of some Laplace decompositions, minors, and algebraic identities between them. In Theorems 3 and 9 below, we prove Conjectures (S) and (K) in full generality, with no restrictions on k and G.

In Theorems 11 and 20 below, we describe $\ker \mu$ as a vector space over k. In Theorem 21, we solve the above part (a) of the problem, finding the canonically defined generators of the k-algebra $(\ker \mu)^G$. We call them SL_2*-type relations* of the sought-for canonical presentation of $k[G]$ because for $G = \mathrm{SL}_2$, the element $\mu(x_1) \otimes \mu(x_4) - \mu(x_2) \otimes \mu(x_3) - 1$ is just such a generator of $\mathcal{I}$ (see (2)). All of them are inhomogeneous of degree 2. If G is semisimple, they are indexed by the elements of the Hilbert basis $\mathscr{H}$ of the monoid of dominant weights of G. Note that the cardinality $|\mathscr{H}|$ of $\mathscr{H}$ is at least $\operatorname{rank} G$ with equality for simply connected G, but in the general case it may be much bigger. For instance, if $G = \mathrm{PGL}_r$, then $|\mathscr{H}| \geqslant p(r) + \varphi(r) - 1$, where p and φ are, respectively, the classical partition function and the Euler function (see [Popov 2011, Example 3.15]). Note that the problem of determining a full set of generators of the ideal $\ker \mu$ was formulated in [Flath 1994, Section 4] and, for $k = \mathbb{C}$, $G = \mathrm{SL}_n, \mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_n$, solved in [Flath and Towber 1992] by lengthy direct computations.

For a semisimple group G whose monoid of dominant weights is freely generated (i.e., with $|\mathscr{H}| = \operatorname{rank} G$), a solution to the above part (b) of the problem in characteristic 0 was obtained (but not published) by B. Kostant; his proof appeared in [Lancaster and Towber 1979, Theorem 1.1]. In arbitrary characteristic, such a solution is given by Theorems 1, 2, 22 below, which are heavily based on the main results of [Ramanan and Ramanathan 1985] and [Kempf and Ramanathan 1987]. All relations in this case are homogeneous of degree 2. We call them *Plücker-type relations* of the sought-for canonical presentation of $k[G]$ because the k-algebra $\mathcal{S}^{\pm}$ for $G = \mathrm{SL}_n$ is the coordinate algebra of the affine multicone over the flag variety, and if $\operatorname{char} k = 0$, these relations are generated by the classical Plücker-type relations, obtained by Hodge [1942; 1943], that determine this multicone (see Section 6). The set of these relations is a union of finite-dimensional vector spaces canonically determined by the group structure of G; these spaces are indexed by the elements of $\mathscr{H} \times \mathscr{H}$ and different spaces have zero intersection (see Theorem 22). Thus in this case, we obtain a canonical presentation of $k[G]$, in which all relations are quadratic and divided into two families: homogeneous relations of Plücker type and inhomogeneous relations of SL_2-type. As a parallel, we recall that any abelian variety is canonically presented as an intersection of quadrics in a projective space given by the Riemann equations; see [Kempf 1989] and [Lange and Birkenhake 1992].

For an arbitrary reductive group G, let $\tau: \widehat{G} \to G$ be the universal covering. Then $\widehat{G} = Z \times C$, where Z is a torus, C is a simply connected semisimple group, $G = \widetilde{G}/\ker\tau$, and $\ker\tau$ is a finite central subgroup. The algebra $\mathcal{S}^{\pm}$ for $\widehat{G}$ is then the tensor product of $k[Z]$ and the algebra $\mathcal{S}^{\pm}$ for C. Since the presentation of $k[Z]$ is clear, and that of $\mathcal{S}^{\pm}$ for C are given by Theorems 1, 2, and 22, the above part (b) of the problem is reduced to finding a presentation for the invariant algebra of the finite abelian group $\ker\tau$.

As an illustration, in Section 6 we consider the example of $G = \mathrm{SL}_n$, $\operatorname{char} k = 0$, and describe explicitly how the ingredients of our construction and the canonical presentation of $k[G]$ look in this case.

The preprints [Popov 1995; 2000] of these results in characteristic 0 have been disseminated long ago. The validity of the results in arbitrary characteristic was announced in [Popov 2000]. The author is happy to finally present the complete proofs in the volume dedicated to the memory of Robert Steinberg who made a great contribution to the theory of algebraic groups.

Notation and conventions. Below we use freely the standard notation and conventions of [Borel 1991; Jantzen 1987; Popov and Vinberg 1994; Shafarevich 2013]. In particular, the algebra of functions regular on a variety X is denoted by $k[X]$, the field of rational functions on an irreducible X is denoted by $k(X)$, and the local ring of X at a point x is denoted by $\mathcal{O}_{x,X}$. For a morphism $\varphi: X \to Y$ of varieties, $\varphi^*: k[Y] \to k[X]$ denotes its comorphism.

All topological terms refer to the Zariski topology; the closure of Z in X is denoted by $\overline{Z}$ (each time it is clear from the context what is X).

The fixed point set of an action of a group P on a set S is denoted by S^P. Every action $\alpha: H \times X \to X$ of an algebraic group H on a variety X is always assumed to be regular (the latter means that α is a morphism). For every $h \in H$, $x \in X$, we write $g \cdot x$ in place of $\alpha(g, x)$. The H-orbit and the H-stabilizer of x are denoted respectively by $H \cdot x$ and H_x. Every homomorphism of algebraic groups is assumed to be algebraic.

The additively written group of characters (i.e., homomorphisms to the multiplicative group of k) of an algebraic group H is denoted by $\mathrm{X}(H)$. The value of a character $\lambda \in \mathrm{X}(H)$ at an element $h \in H$ is denoted by h^{λ}. Given a kH-module M, its weight space with weight $\lambda \in X(H)$ is denoted by M_{λ}.

We fix in G the maximal torus

$$T := B^+ \cap B^-$$

and identify $\mathrm{X}(B^{\pm})$ with $\mathrm{X}(T)$ by means of the restriction isomorphisms $\mathrm{X}(B^{\pm}) \to \mathrm{X}(T)$, $\lambda \mapsto \lambda|_T$.

By $\mathrm{X}(T)_+$ we denote the monoid of dominant weights of T determined by B^+. Below the highest weight of every simple G-module is assumed to be the highest weight with respect to T and B^+.

We denote by w_0 the longest element of the Weyl group of T and fix in the normalizer of T a representative $\dot{w}_0$ of w_0. We then have $\dot{w}_0 B^{\pm} \dot{w}_0^{-1} = B^{\mp}$ and $\dot{w}_0 U^{\pm} \dot{w}_0^{-1} = U^{\mp}$. For every $\lambda \in \mathrm{X}(T)_+$, we put $\lambda^* := -w_0(\lambda) \in \mathrm{X}(T)_+$.

The set of all nonnegative rational numbers is denoted by $\mathbb{Q}_{\geqslant 0}$ and we put $\mathbb{N} := \mathbb{Z} \cap \mathbb{Q}_{\geqslant 0}$.

If $m \in \mathbb{Z}$, $m > 0$, we put $[m] := \{a \in \mathbb{Z} \mid 1 \leqslant a \leqslant m\}$.

For $d \in \mathbb{N}$, we denote by $[m]_d$ the set of all increasing sequences of d elements of $[m]$ (if $d \notin [m]$, then $[m]_d = \varnothing$).

2. Proof of Conjecture (S)

For every $\lambda \in \mathrm{X}(T)$, the spaces

$$
\begin{aligned}
&\mathcal{S}^+(\lambda) := \{f \in \mathcal{S}^+ \mid f(gt) = t^{\lambda} f(g) \text{ for all } g \in G, t \in T\},\\
&\mathcal{S}^-(\lambda) := \{f \in \mathcal{S}^- \mid f(gt) = t^{w_0(\lambda)} f(g) \text{ for all } g \in G, t \in T\}
\end{aligned}
\tag{5}
$$

are the finite-dimensional (see, e.g., [Jantzen 1987, I.5.12.c)]) G-submodules of the G-modules $\mathcal{S}^+$ and $\mathcal{S}^-$ respectively. Since $\mathcal{S}^-(\lambda)$ is the right translation of $\mathcal{S}^+(\lambda)$ by $\dot{w}_0$, these G-submodules are isomorphic. In the notation of [Jantzen 1987, II.2.2], we have

$$
\mathcal{S}^-(\lambda) = H^0(\lambda^*),
\tag{6}
$$

so by (6) and [Jantzen 1987, II.2.6, 2.2, 2.3], the following properties hold:

$$
\left.
\begin{aligned}
&\text{(i) } \mathcal{S}^{\pm}(\lambda) \neq 0 \iff \lambda \in \mathrm{X}(T)_+.\\
&\text{(ii) } \operatorname{soc}_G \mathcal{S}^{\pm}(\lambda) \text{ is a simple } G\text{-module with the highest weight } \lambda^*.
\end{aligned}
\right\}
\tag{7}
$$

If $\operatorname{char} k = 0$, then the G-module $\mathcal{S}^+(\lambda)$ is semisimple and hence $\mathcal{S}^+(\lambda) = \operatorname{soc}_G \mathcal{S}^+(\lambda)$ by (7)(ii). If $\operatorname{char} k > 0$, then, in general, this equality does not hold. From (3), (5), and (7)(i) we infer that

$$
\begin{aligned}
\mathcal{S}^+ &= \bigoplus_{\lambda \in \mathrm{X}(T)_+} \mathcal{S}^+(\lambda), \quad \mathcal{S}^+(\lambda)\mathcal{S}^+(\mu) \subseteq \mathcal{S}^+(\lambda+\mu),\\
\mathcal{S}^- &= \bigoplus_{\lambda \in \mathrm{X}(T)_+} \mathcal{S}^-(\lambda), \quad \mathcal{S}^-(\lambda)\mathcal{S}^-(\mu) \subseteq \mathcal{S}^-(\lambda+\mu);
\end{aligned}
\tag{8}
$$

i.e., the decompositions (8) are the $\mathrm{X}(T)_+$-gradings of the algebras $\mathcal{S}^+$ and $\mathcal{S}^-$. They are obtained from each other by the right translation by $\dot{w}_0$.

Theorem 1. *The linear span of $\mathcal{S}^{\pm}(\lambda)\mathcal{S}^{\pm}(\mu)$ over k is $\mathcal{S}^{\pm}(\lambda+\mu)$.*

Proof. This statement is the main result of [Ramanan and Ramanathan 1985]. Note that the difficulty lies in the case of positive characteristic: since $\mathcal{S}^{\pm}$ is an integral domain, if $\operatorname{char} k = 0$, then the claim immediately follows from (7)(i) and the inclusions in (8) because then $\mathcal{S}^{\pm}(\lambda+\mu)$ is a simple G-module. $\square$

Theorem 2. (i) *If $\mathcal{G}$ is a generating set of the semigroup $\mathrm{X}(T)_+$, then the k-algebra $\mathcal{S}^{\pm}$ is generated by the subspace $\bigoplus_{\lambda\in\mathcal{G}}\mathcal{S}^{\pm}(\lambda)$.*

(ii) *The k-algebras S^{+} and $\mathcal{S}^{-}$ are finitely generated.*

Proof. Part (i) follows from (8) and Theorem 1. Being the intersection of the lattice $\mathrm{X}(T)$ with a convex cone in $\mathrm{X}(T)\otimes_{\mathbb{Z}}\mathbb{Q}$ generated by finitely many vectors, the semigroup $\mathrm{X}(T)_+$ is finitely generated. This, (i), and the inequality $\dim_k \mathcal{S}^{\pm}(\lambda)<\infty$ imply (ii). $\square$

Now we are ready to turn to the proof of Conjecture (S).

Theorem 3. *The homomorphism μ is surjective.*

Our proof of Theorem 3 is based on two general results. The first is the following well-known surjectivity criterion:

Lemma 4. *The following properties of a morphism $\varphi\colon X\to Y$ of affine algebraic varieties are equivalent*:

(a) *φ is a closed embedding.*

(b) *$\varphi^*\colon k[Y]\to k[X]$ is surjective.*

Proof. See, e.g., [Steinberg 1974, Section 1.5]. $\square$

The second is the closedness criterion for orbits of connected solvable affine algebraic groups that generalizes Rosenlicht's classical theorem [1961, Theorem 2] on the closedness of orbits of unipotent groups.

Theorem 5. *Let a connected solvable affine algebraic group S act on an affine algebraic variety Z. Let x be a point of Z. Consider the orbit morphism $\tau: S\to Z$, $s\mapsto s\cdot z$. Then the following properties are equivalent*:

(a) *The orbit $S\cdot z$ is closed in Z.*

(b) *The semigroup $\{\lambda\in\mathrm{X}(S)\mid$ the function $S\to k$, $s\mapsto s^{\lambda}$, lies in $\tau^*(k[Z])\}$ is a group.*

Proof. This is proved in [Popov 1989, Theorem 4] $\square$

Remark 6. Since $\mathrm{X}(S)$ in Theorem 5 is a finitely generated free abelian group, it can be naturally regarded as a lattice in $\mathrm{X}(S)\otimes_{\mathbb{Z}}\mathbb{Q}$. Hence the following general criterion is applicable for verifying condition (b).

Let M be a nonempty subset of a finite-dimensional vector space V over $\mathbb{Q}$. Let $\mathbb{Q}_{\geqslant 0}M$, $\operatorname{conv} M$, and $\mathbb{Q}M$ be, respectively, the convex cone generated by M, the convex hull of M, and the linear span of M in V. Then the following properties are equivalent (see [Popov 1989, p. 386]):

(i) 0 is an interior point of $\operatorname{conv} M$.

(ii) $\mathbb{Q}_{\geqslant 0}M = \mathbb{Q}M$.

If M is a subsemigroup of V, then (i) and (ii) are equivalent to

(iii) M is a group.

Proof of Theorem 3. 1. We consider the action of G on its underlying algebraic variety by left translations. By Theorem 2, there is an irreducible affine algebraic variety X endowed with an action of G and a G-equivariant dominant morphism

$$\alpha\colon G \to X \text{ such that } \alpha^* \text{ is an isomorphism } k[X] \xrightarrow{\cong} \mathcal{S}^+. \tag{9}$$

Let $x := \alpha(e)$. Since α is G-equivariant, we have

$$\alpha(g) = g \cdot x \text{ for every } g \in G, \tag{10}$$

and since α is dominant, the orbit $G \cdot x$ is open and dense in X. Consider the canonical projection $\pi\colon G \to G/U^+$. It is the geometric quotient for the action of U^+ on G by right translations. Therefore, (3) yields the isomorphism

$$\pi^*\colon k[G/U^+] \xrightarrow{\cong} \mathcal{S}^+, \tag{11}$$

and, since α is constant on the fibers π, there exists a G-equivariant morphism $\iota\colon G/U^+ \to X$ such that

$$\alpha = \iota \circ \pi. \tag{12}$$

From (12) we infer that the image of ι is $G \cdot x$. Since the group U^+ is unipotent, the algebraic variety G/U^+ is quasiaffine (see [Rosenlicht 1961, Theorem 3]). Therefore, $k(G/U^+)$ is the field of fractions of $k[G/U^+]$. On the other hand, $k(X)$ is the field of fractions of $k[X]$ inasmuch as X is affine. Using that (12) and isomorphisms (9), (11) yield the isomorphism $\iota^*\colon k[X] \xrightarrow{\cong} k[G/U^+]$, we conclude that ι is a birational isomorphism. Therefore, for a point z in general position in $G \cdot x$, the fiber $\iota^{-1}(z)$ is a single point. Being G-equivariant, ι is then injective. Finally, since G is smooth, $k[G]$ is integrally closed; therefore, $\mathcal{S}^{\pm}$ is integrally closed as well in view of (3) (see, e.g., [Popov and Vinberg 1994, Theorem 3.16]). Thus X is normal, and hence by Zariski's Main Theorem, $\iota\colon G/U^+ \to G \cdot x$ is an isomorphism. Using that π is separable (see, e.g., [Borel 1991, II.6.5]), from this we infer that the following properties hold:

(i$_1$) $G_x = U^+$.

(ii$_1$) $G \to G \cdot x$, $g \mapsto \alpha(g) = g \cdot x$, is a separable morphism.

2. Let $y := \dot{w}_0 \cdot x$. Consider the G-equivariant morphism

$$\beta\colon G \to X, \quad g \mapsto g \cdot y. \tag{13}$$

From (10), (13), (i_1), and (ii_1), we infer that the following properties hold:

(i_2) $G_y = U^-$.

(ii_2) $G \to G \cdot y$, $g \mapsto \beta(g) = g \cdot y$, is a separable morphism.

(iii_2) β^* is an isomorphism $k[X] \xrightarrow{\cong} \mathcal{S}^-$.

3. Now consider the G-equivariant morphism

$$\gamma := \alpha \times \beta\colon G \to X \times X, \quad g \mapsto g \cdot z, \text{ where } z := (x, y). \tag{14}$$

From (14), (i_1), and (i_2), we obtain

$$G_z = G_x \cap G_y = U^+ \cap U^- = \{e\}, \tag{15}$$

and hence γ is injective. We claim that γ is a closed embedding, i.e.,

(a) $G \to G \cdot z$, $g \mapsto g \cdot z$, is an isomorphism;

(b) $G \cdot z$ is closed in $X \times X$.

If this claim is proved, then the proof of Theorem 3 is completed as follows. Consider the isomorphism

$$k[X] \otimes_k k[X] \to k[X \times X], \quad f \otimes h \mapsto fh. \tag{16}$$

Then (4), (9), (iii_2), (14), (16) imply that μ is the composition of the homomorphisms

$$\mathcal{S}^+ \otimes_k \mathcal{S}^- \xrightarrow[\cong]{(\alpha^*)^{-1} \otimes (\beta^*)^{-1}} k[X] \otimes_k k[X] \xrightarrow[\cong]{(16)} k[X \times X] \xrightarrow{\gamma^*} k[G]. \tag{17}$$

Hence the surjectivity of μ is equivalent to the surjectivity of γ^*. By Lemma 4, the latter is equivalent to the property that γ is a closed embedding, i.e., that properties (a) and (b) hold.

Thus the proof of Theorem 3 is reduced to proving properties (a) and (b).

4. First, we shall prove property (a). Since γ is injective, this is reduced to proving the separability of γ. In turn, in view of (14), the latter is reduced to proving that $\ker d_e\gamma$ is contained in $\operatorname{Lie} G_z$, i.e., that $\ker d_e\gamma = \{0\}$ because of (15) (see [Borel 1991, II.6.7]). Using [loc. cit.], from (10), (13), (i_1), (ii_1), (i_2), (ii_2) we infer that $\ker d_e\alpha \subseteq \operatorname{Lie} U^+$, $\ker d_e\beta \subseteq \operatorname{Lie} U^-$. In view of (14), we then have $\ker d_e\gamma = \ker d_e\alpha \cap \ker d_e\beta \subseteq \operatorname{Lie} U^+ \cap \operatorname{Lie} U^- = \{0\}$. This proves property (a).

5. Now we shall prove property (b). Actually, we shall prove the stronger property that the orbit $B^+ \cdot z$ is closed in $X \times X$: since the algebraic variety G/B^+ is complete, this stronger property implies property (b) (see [Steinberg 1974, Section 2.13, Lemma 2]). Using that B^+ is connected solvable, to this end we shall apply Theorem 5.

Namely, consider the morphism $\tau\colon B^+ \to X \times X$, $b \mapsto b \cdot z$ and the following subsemigroup M in $\mathrm{X}(B^+)$:

$$M := \{\lambda \in \mathrm{X}(B^+) \mid \text{ the function } B^+ \to k,\, b \mapsto b^\lambda \text{ lies in } \tau^*(k[X \times X])\}.$$

We identify $\mathrm{X}(B^+)$ with the lattice in $L := \mathrm{X}(B^+) \otimes_{\mathbb{Z}} \mathbb{Q}$. In view of Theorem 5 and Remark 6, the orbit $B^+ \cdot z$ is closed if and only if

$$\mathbb{Q}_{\geqslant 0} M = \mathbb{Q} M. \tag{18}$$

Given this, the problem is reduced to proving that property (18) holds. This is done below.

6. Since $\tau = \gamma|_{B^+}$, the algebra $\tau^*(k[X \times X])$ is the image of the homomorphism $\gamma^*(k[X \times X]) \to k[B^+]$, $f \mapsto f|_{B^+}$. From (17) we see that $\gamma^*(k[X \times X])$ contains $\mathcal{S}^+$ and $\mathcal{S}^-$. Hence the restrictions of $\mathcal{S}^+$ and $\mathcal{S}^-$ to B^+ lie in $\tau^*(k[X \times X])$. We shall exhibit some characters of B^+ lying in these restrictions.

First consider the restriction of $\mathcal{S}^+(\lambda)$ to B^+ for $\lambda \in \mathrm{X}(T)_+$. Note that $\mathcal{S}^+(\lambda)$ contains a function f such that $f(e) \neq 0$. Indeed, in view of (7)(i) and Borel's fixed point theorem, $\mathcal{S}^+(\lambda)$ contains a B^--stable line ℓ. The group B^- acts on ℓ by means of a character $\nu \in \mathrm{X}(B^-)$. Take a nonzero function $f \in \ell$. For every $b \in B^-$, $u \in U^+$, we then have $f(b^{-1}u) = b^\nu f(u) \overset{(3)}{=} b^\nu f(e)$, whence $f(e) \neq 0$ because $B^- U^+$ is dense in G. This proves the existence of f. Multiplying f by $1/f(e)$, we may assume that $f(e) = 1$. Then for every $b \in B^+$, we deduce from (3), (5) that $f(b) = b^\lambda f(e) = b^\lambda$, i.e., $f|_{B^+}$ is the character $B^+ \to k$, $b \mapsto b^\lambda$. This proves the inclusion

$$\mathrm{X}(B^+)_+ \subseteq M. \tag{19}$$

Now consider the restriction of $\mathcal{S}^-(\lambda)$ to B^+ for $\lambda \in \mathrm{X}(T)_+$. In view of (7)(ii), there is a B^+-stable line ℓ in $\mathcal{S}^-(\lambda)$, on which B^+ acts by the character $\lambda^* \in \mathrm{X}(B^+)$. Take a nonzero function $f \in \ell$. We may assume that $f(e) = 1$: this is proved as above with $\nu = \lambda$, replacing B^- by B^+, and U^+ by U^-. For every $b \in B^+$, we then have $f(b^{-1}) = b^{\lambda^*}$, i.e., $f|_{B^+}$ is the character $B^+ \to k$, $b \mapsto b^{-\lambda^*} = b^{w_0(\lambda)}$. This proves the inclusion

$$-\mathrm{X}(B^+)_+ \subseteq M. \tag{20}$$

Since $\mathbb{Q}_{\geqslant 0}(\mathrm{X}(B^+)_+) - \mathbb{Q}_{\geqslant 0}(\mathrm{X}(B^+)_+) = L$, the inclusions (19), (20) imply the equality $\mathbb{Q}_{\geqslant 0} M = L$, whence a fortiori the equality (18) holds. This completes the proof of Theorem 3. $\square$

3. Proof of Conjecture (K)

We now intend to describe the ideal $\ker \mu$ in $\mathcal{S}^+ \otimes_k \mathcal{S}^-$. This is done in Sections 3 and 4 in several steps: first in Theorem 9 we prove that $\ker \mu$ is generated by

$(\ker\mu)^G$, then in Theorem 11 we describe $\ker\mu$ as a vector space, and finally in Theorem 21 we find a standard finite generating set of $\ker\mu$.

The first step is based on the following general statement:

Theorem 7. *Let Z be an affine algebraic variety endowed with an action of a reductive algebraic group H. Let $a \in Z$ be a point such that the orbit morphism*

$$\varphi\colon H \to Z, \quad h \mapsto h\cdot a,$$

is a closed embedding. Then the ideal $\ker\varphi^*$ in $k[Z]$ is generated by $(\ker\varphi^*)^H$.

For the proof of Theorem 7, we need the following:

Lemma 8. *Let $\psi\colon Y \to Z$ be a morphism of irreducible affine algebraic varieties and let $z \in \psi(Y)$ be a smooth point of Z. Assume that for each point $y \in \psi^{-1}(z)$, the following hold*:

(i) *y is a smooth point of Y.*

(ii) *The differential $d_y\psi$ is surjective.*

Then the ideal $\{f \in k[Y] \mid f|_{\psi^{-1}(z)} = 0\}$ of $k[Y]$ is generated by $\psi^*(\mathfrak{m})$, where $\mathfrak{m} := \{h \in k[Z] \mid h(z) = 0\}$.

Proof. Given a nonzero function $f \in k[Y]$, below we denote by Y_f the principal open subset $\{y \in Y \mid f(y) \neq 0\}$ of Y; it is affine and $k[Y_f] = k[Y]_f$.

1. Let $s_1, \ldots, s_d$ be a system of generators of the ideal $\mathfrak{m}$ of $k[Z]$. Put $t_i := \psi^*(s_i)$. Then we have

$$\{y \in Y \mid t_1(y) = \cdots = t_d(y) = 0\} = \psi^{-1}(z). \tag{21}$$

We claim that, for every point $a \in Y$, there is a function $h_a \in k[Y]$ such that the principal open subset $U = Y_{h_a}$ is a neighborhood of a and

$$I_U := \{f \in k[U] \mid f|_{\psi^{-1}(z)\cap U} = 0\}$$

is the ideal of $k[U]$ generated by $t_1|_U, \ldots, t_d|_U$.

Proving this, we consider two cases.

First, consider the case where $a \notin \psi^{-1}(z)$. Then any principal open neighborhood of a not intersecting $\psi^{-1}(z)$ may be taken as U because in this case $I_U = k[U]$ and, in view of (21) and Hilbert's Nullstellensatz, $k[U] = k[U]t_1|_U + \cdots + k[U]t_d|_U$.

Second, consider the case where $a \in \psi^{-1}(z)$. Let $n = \dim Y$, $m = \dim Z$. Since a and z are the smooth points, the assumption (ii) yields the equality

$$\dim\ker d_a\psi = n - m. \tag{22}$$

The functions $s_1, \ldots, s_d$ generate the maximal ideal of $\mathcal{O}_{z,Z}$. Therefore, renumbering them if necessary, we may (and shall) assume that $s_1, \ldots, s_m$ is a system of local parameters of Z at z, i.e., $\bigcap_{i=1}^m \ker d_z s_i = \{0\}$. Since $d_a t_i = d_a\psi \circ d_z s_i$, we then infer from (ii) that $\bigcap_{i=1}^{m} \ker d_a t_i = \ker d_a \psi$. In view of (22), the latter equality implies the existence of functions $f_1, \dots, f_{n-m} \in \mathcal{O}_{a,Y}$ such that $t_1, \dots, t_m, f_1, \dots, f_{n-m}$ is a system of local parameters of Y at a. Let

$$F := \{y \in Y \mid t_1(y) = \cdots = t_m(y) = 0\}. \tag{23}$$

By [Shafarevich 2013, Chapter II, Section 3.2, Theorem 2.13], there is a principal open neighborhood U of a such that $F \cap U$ is an irreducible smooth $(n-m)$-dimensional closed subvariety of U whose ideal in $k[U]$ is generated by $t_1|_U, \dots, t_m|_U$. On the other hand, (21) and (23) yield $\psi^{-1}(z) \subseteq F$ and, by the fiber dimension theorem, every irreducible component of $\psi^{-1}(z)$ has dimension $\geqslant n-m$. Hence $U \cap F = \psi^{-1}(z) \cap U$. This and (21) prove the claim.

2. Using this claim, the proof of Lemma 8 is completed as follows. Since $Y = \bigcup_{a \in Y} Y_{h_a}$ and Y is quasicompact, there exists a finite set of points $a_1, \dots, a_r \in Y$ such that

$$Y = \bigcup_{i=1}^{r} Y_{h_i}, \quad \text{where } h_i := h_{a_i}. \tag{24}$$

Now, let $f \in k[Y]$ be a function such that $f|_{\psi^{-1}(z)} = 0$. Then, in view of the definition of h_a, for every $i = 1, \dots, r$, we have

$$f h_i^{b_i} = c_{i,1} t_1 + \cdots + c_{i,d} t_d \quad \text{for some } c_{i,j} \in k[Y] \text{ and } b_i \in \mathbb{N}. \tag{25}$$

From (24) and Hilbert's Nullstellensatz, we infer that there are functions $q_1, \dots, q_r \in k[Y]$ such that

$$1 = q_1 h_1^{b_1} + \cdots + q_s h_s^{b_r}. \tag{26}$$

From (25) and (26), we then deduce that

$$f = \Bigl(\sum_{i=1}^{r} q_i c_{i,1}\Bigr) t_1 + \cdots + \Bigl(\sum_{i=1}^{r} q_i c_{i,d}\Bigr) t_d \in k[Y]t_1 + \cdots + k[Y]t_d. \qquad \square$$

Proof of Theorem 7. There is a closed equivariant embedding of Z in an affine space on which H operates linearly (see [Rosenlicht 1961, Lemma 2] and [Popov and Vinberg 1994, Theorem 1.5]). Hence we may (and shall) assume that Z is an irreducible smooth affine algebraic variety.

Since G is reductive, $k[Z]^G$ is a finitely generated k-algebra (see, e.g., [Mumford and Fogarty 1982, Theorem A.1.0] and the references therein). Denote by $Z /\!\!/ H$ the affine algebraic variety $\operatorname{Specm}(k[Z]^G)$ and by $\pi\colon Z \to Z /\!\!/ H$ the morphism corresponding to the inclusion homomorphism $k[Z]^G \hookrightarrow k[Z]$.

The condition on the point a implies that its H-stabilizer is trivial,

$$H_a = \{e\}. \tag{27}$$

Hence $H \cdot a$ is a closed H-orbit of maximal dimension. Taking into account that in every fiber of π there is a unique closed orbit lying in the closure of every orbit contained in this fiber (see [Mumford and Fogarty 1982, Corollaries 1.2, A.1.0]), from this we deduce the equality

$$\pi^{-1}(\pi(a)) = H \cdot a. \tag{28}$$

Since the group $\{e\}$ is linearly reductive, from (27) and the separability of φ, we infer by [Bardsley and Richardson 1985, Proposition 7.6] that there is a smooth affine subvariety S of the H-variety Z, which is an étale slice at $a \in S$. In view of (27), this means the following:

(i) The morphisms

$$\pi|_S : S \to Z /\!\!/ H \qquad \text{and} \qquad \psi : H \times S \to Z, \quad (h, s) \mapsto h \cdot s$$

are étale.

(ii) The diagram

$$\begin{array}{ccc} H \times S & \xrightarrow{\psi} & Z \\ {\scriptstyle \mathrm{pr}_2}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\pi|_S} & Z /\!\!/ H \end{array}$$

is a Cartesian square; i.e., it is commutative and the morphism

$$H \times S \to S \times_{Z /\!\!/ H} Z$$

determined by ψ and pr_2 is an isomorphism.

From (i) and (ii), we deduce that $\pi(a)$ is a smooth point of $Z /\!\!/ H$ and the differentials $d_{(e,a)}\psi$, $d_a(\pi|_S)$ are isomorphisms. Since $d_{(e,a)}\mathrm{pr}_2$ is clearly surjective, (ii) then implies that $d_a\pi$ is surjective, too.

Now, in view of (28) and transitivity of the action of H on $H \cdot a$, we conclude that $d_z\pi$ is surjective for every point $z \in \pi^{-1}(\pi(a))$. In view of Lemma 8, this implies the claim of Theorem 7. □

Theorem 9. *The ideal* $\ker \mu$ *in* $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ *is generated by* $(\ker \mu)^G$.

Proof. In the proof of Theorem 3, we have shown that

— the homomorphism μ is the composition of the homomorphisms (17);

— the morphism γ is a closed embedding.

In view of these facts, Theorem 9 is equivalent to the claim that the ideal $\ker \gamma^*$ in $k[X \times X]$ is generated by $(\ker \gamma^*)^G$. This claim follows from Theorem 7. □

4. Structure of $(\ker \mu)^G$

We shall use the following lemma for describing $(\ker \mu)^G$ as a vector space.

Lemma 10.

(29) $$\dim(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\nu))^G = \begin{cases} 1 & \text{if } \nu = \lambda^*, \\ 0 & \text{if } \nu \neq \lambda^* \end{cases} \quad \text{for every } \lambda, \nu \in \mathrm{X}(T)_+,$$

(30) $$(\mathcal{S}^+ \otimes_k \mathcal{S}^-)^G = \bigoplus_{\lambda \in \mathrm{X}(T)_+} (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G.$$

Proof. In view of (8), the equality (30) follows from (29). To prove (29), we note that

$$(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\nu))^G \cong \mathrm{Hom}_G(\mathcal{S}^+(\lambda)^*, \mathcal{S}^-(\nu))$$

and, in view of (6), the G-module $\mathcal{S}^+(\lambda)^*$ is the universal highest weight module of weight λ (the Weyl module); in particular, for each G-module M, there is an isomorphism

(31) $$\mathrm{Hom}_G(\mathcal{S}^+(\lambda)^*, M) \xrightarrow{\cong} (M^{U^+})_\lambda,$$

where the right-hand side of (31) is the weight space of T (see [Jantzen 1987, II.2.13, Lemma]). Since $\mathcal{S}^-(\nu)^{U^+}$ is a line on which B^+ acts by means of ν^* (see [Jantzen 1987, II.2.2, Proposition]), this proves (29). □

We identify $k[G] \otimes_k k[G]$ with $k[G \times G]$ by the isomorphism

(32) $$k[G] \otimes_k k[G] \to k[G \times G], \quad f_1 \otimes f_2 \mapsto ((a, b) \mapsto f_1(a) f_2(b)).$$

Thus $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ is regarded as a subalgebra of $k[G \times G]$, and (4), (32) yield the equality

(33) $$f(a, a) = \mu(f)(a) \text{ for every } f \in \mathcal{S}^+ \otimes_k \mathcal{S}^- \text{ and } a \in G.$$

Theorem 11. (i) *If* $f \in (\mathcal{S}^+ \otimes_k \mathcal{S}^-)^G$, *then* $f - f(e, e) \in (\ker \mu)^G$.

(ii) *Every* $h \in (\ker \mu)^G$ *can be uniquely written in the form*

(34) $$h = \sum (h_\lambda - h_\lambda(e, e)), \quad h_\lambda \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G,$$

where the sum is taken over a finite set of nonzero elements $\lambda \in \mathrm{X}(T)_+$.

Proof. (i) Since μ is G-equivariant, its restriction to $(\mathcal{S}^+ \otimes_k \mathcal{S}^-)^G$ is a homomorphism to $k[G]^G = k$. Hence $\mu(f)$ is a constant. In view of (33), this implies (i).

(ii) If (34) holds, then the decomposition (30) implies that h_λ is the natural projection of h to $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$ determined by this decomposition, whence the uniqueness of (34). To prove the existence, let h_λ be the aforementioned projection of h to $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$. Then $h = \sum_{\lambda \in F} h_\lambda$ for a finite set $F \subset \mathrm{X}(T)_+$.

Hence $0 = \mu(h) = \sum_{\lambda\in F} \mu(h_\lambda)$. As above, $\mu(h_\lambda) = h_\lambda(e, e)$; this implies equality (34), where the sum is taken over all $\lambda \in F$. Since h_0 is a constant, we may assume that F does not contain 0. This proves (ii). ☐

In the next lemma, for brevity, we put (cf. [Jantzen 1987])

$$V(\lambda) := \mathcal{S}^-(\lambda)^* \cong \mathcal{S}^+(\lambda)^*, \quad L(\lambda) := V(\lambda)/\operatorname{rad}_G V(\lambda),$$
$$\pi_\lambda\colon V(\lambda) \to L(\lambda) \text{ is the canonical projection.} \tag{35}$$

The G-module $V(\lambda)$ (hence $L(\lambda)$ as well) is generated by a B^+-stable line of weight λ (see [Jantzen 1987, II.2.13, Lemma]), whence $V(\lambda)$ is also generated by a B^--stable line of weight $-\lambda^*$.

Also, for the G-modules P and Q, we denote by $\mathcal{B}(P\times Q)$ the G-module of all bilinear maps $P \times Q \to k$; we then have the isomorphism of G-modules

$$P^* \otimes_k Q^* \xrightarrow{\cong} \mathcal{B}(P\times Q), \quad f\otimes h \mapsto fh. \tag{36}$$

Lemma 12. *For all elements $\lambda, \nu \in \mathrm{X}(T)_+$, the following hold*:

(a) $\dim \mathcal{B}(V(\lambda)\times V(\nu))^G = \begin{cases} 1 & \text{if } \nu = \lambda^*, \\ 0 & \text{if } \nu \neq \lambda^*. \end{cases}$

(b) $\dim \mathcal{B}(L(\lambda)\times L(\nu))^G = \begin{cases} 1 & \text{if } \nu = \lambda^*, \\ 0 & \text{if } \nu \neq \lambda^*. \end{cases}$

(c) *Every nonzero element $\theta \in \mathcal{B}(L(\lambda)\times L(\lambda^*))^G$ is a nondegenerate pairing $L(\lambda)\times L(\lambda^*) \to k$.*

(d) *If $l^+ \in L(\lambda)$, $l^- \in L(\lambda^*)$ are the nonzero semi-invariants of, respectively, B^+ and B^-, then $\theta(l^+, l^-) \neq 0$ for θ from* (c). *For every nonzero element $\epsilon \in k$, there exists a unique θ such that $\theta(l^+, l^-) = \epsilon$.*

(e) *Every element $\vartheta \in \mathcal{B}(V(\lambda)\times V(\lambda^*))^G$ vanishes on $\ker\pi_\lambda \times \ker\pi_{\lambda^*}$. If $\vartheta \neq 0$, then ϑ is a nondegenerate pairing $V(\lambda)\times V(\lambda^*) \to k$.*

(f) *Let $v^+ \in V(\lambda)$ and $v^- \in V(\lambda^*)$ be, respectively, the nonzero B^+- and B^--semi-invariants of weights λ and $-\lambda$ that generate the G-modules $V(\lambda)$ and $V(\lambda^*)$. Then $\vartheta(v^+, v^-) \neq 0$ for every nonzero element $\vartheta \in \mathcal{B}(V(\lambda)\times V(\lambda^*))^G$.*

Proof. Part (a) follows from (29), (36), (35). Part (b) is proved similarly, using that $L(\lambda)$ is a simple G-module with highest weight λ (see [Jantzen 1987, II.2.4]). The simplicity of $L(\lambda)$ implies (c) because the left and right kernels of θ are G-stable.

Proving (d), take a basis $\{p_1, \dots, p_s\}$ of $L(\lambda)$ such that $p_1 = l^+$ and every p_i is a weight vector of T. Let $\{p_1^*, \dots, p_s^*\}$ be the basis of $L(\lambda^*)$ dual to $\{p_1, \dots, p_s\}$ with respect to θ. Let $L(\lambda)'$ be the linear span over k of all p_i with $i > 1$. Then $L(\lambda)'$ is B^--stable, and, for every element $u \in U^-$, we have $u \cdot p_1 = p_1 + p'$, where $p' \in L(\lambda)'$ (see, e.g., [Steinberg 1974, Section 3.3, Proposition 2 and p. 84]). Then, for all elements $\alpha_1, \dots, \alpha_s \in k$, we have

$$\begin{aligned}(u \cdot p_1^*)\Big(\sum_{i=1}^{s} \alpha_i p_i\Big) &= p_1^*\Big(\sum_{i=1}^{s} \alpha_i (u^{-1} \cdot p_i)\Big)\\ &= p_1^*(\alpha_1 p_1 + \text{an element of } L(\lambda)')\\ &= \alpha_1 = p_1^*\Big(\sum_{i=1}^{s} \alpha_i p_i\Big),\end{aligned}$$

whence $u \cdot p_1^* = p_1^*$. Therefore, $l^- = \lambda p_1^*$ for a nonzero $\lambda \in k$, and hence $\theta(l^+, l^-) = \lambda \neq 0$. This and (b) prove (d).

It follows from (35), (a), and (b) that the embedding

$$\mathcal{B}(L(\lambda) \times L(\lambda^*))^G \to \mathcal{B}(V(\lambda) \times V(\lambda^*))^G, \quad \theta \mapsto \theta \circ (\pi_\lambda \times \pi_{\lambda^*}),$$

is an isomorphism. Part (e) follows from this and (c).

Part (f) follows from (d) and (e), because $\pi_\lambda(v^+)$ and $\pi_{\lambda^*}(v^-)$ are, in view of (35), the nonzero semi-invariants of, respectively, B^+ and B^-. □

Lemma 13. *Let an algebraic group H act on an algebraic variety Z and let V be a finite-dimensional submodule of the H-module $k[Z]$. Then the morphism*

(37) $$\varphi\colon Z \to V^*, \quad \varphi(a)(f) = f(a) \text{ for every } a \in Z, f \in V,$$

has the following properties:

(i) *φ is H-equivariant.*

(ii) *The restriction of φ^* to $(V^*)^*$ is an isomorphism $(V^*)^* \to V$.*

(iii) *φ^* exercises an isomorphism between $k[\overline{\varphi(Z)}]$ and the subalgebra of $k[Z]$ generated by V.*

Proof. Part (i) is proved by direct verification.

Every function $f \in V$ determines an element $l_f \in (V^*)^*$ by the formula $l_f(s) = s(f)$, $s \in V^*$. It is immediate that $V \to (V^*)^*$, $f \mapsto l_f$ is a vector space isomorphism and that (37) implies $\varphi^*(l_f) = f$. This proves (ii).

Let $\iota\colon (V^*)^* \to k[\overline{\varphi(Z)}]$ be the restriction homomorphism. The k-algebra $k[\overline{\varphi(Z)}]$ is generated by $\iota((V^*)^*)$. Part (iii) now follows from the fact that φ^* exercises an embedding of $k[\overline{\varphi(Z)}]$ in $k[Z]$ and, in view of (ii), the image of $\iota((V^*)^*)$ under this embedding is V. □

Corollary 14. *In the notation of Lemma 13, let $V \neq \{0\}$ and let the orbit $H \cdot a$ be dense in Z. Then $\varphi(a) \neq 0$.*

We call the morphism (37) the *covariant determined by the submodule V*.

Lemma 15. *Let λ be an element of* $\mathrm{X}(T)_+$ *and let*

$$\varphi^+\colon G \to \mathcal{S}^+(\lambda)^*, \quad \varphi^-\colon G \to \mathcal{S}^-(\lambda^*)^*$$

be the covariants determined by the submodules $\mathcal{S}^+(\lambda)$ and $\mathcal{S}^-(\lambda^*)$ of the G-module $k[G]$. Then $v^+ := \varphi^+(e)$ and $v^- := \varphi^-(e)$ are, respectively, the nonzero B^+- and B^--semi-invariants of weights λ and $-\lambda$.

Proof. First, we have $v^+ \neq 0$, $v^- \neq 0$ by Corollary 14. Next, for every $f \in \mathcal{S}^+(\lambda)$, $b \in B^+$, we have

$$\begin{aligned}(b \cdot v^+)(f) &= \varphi^+(e)(b^{-1} \cdot f) \overset{(37)}{=\!=} (b^{-1} \cdot f)(e) \\ &= f(b) \overset{(5)}{=} b^\lambda f(e) \overset{(37)}{=\!=} (b^\lambda v^+)(f),\end{aligned}$$

whence $b \cdot v^+ = b^\lambda v^+$; i.e., v^+ is a nonzero B^+-semi-invariant of weight λ, as claimed. For v^- the proof is similar. $\square$

Theorem 16. *The restriction of μ to $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda))^G$ for every $\lambda \in \mathrm{X}(T)_+$ is an isomorphism $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G \xrightarrow{\cong} k[G]^G = k$.*

Proof. In view of (33) and Lemma 10, the proof is reduced to showing that there is a function $f \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$ such that $f(e,e) \neq 0$.

Consider the covariants φ^+ and φ^- from Lemma 15 and the G-equivariant morphism

$$\varphi := \varphi^+ \times \varphi^-\colon G \times G \to \mathcal{S}^+(\lambda)^* \times \mathcal{S}^-(\lambda^*)^*.$$

Lemma 12(a) and (35) imply that $\mathcal{B}(\mathcal{S}^+(\lambda)^* \times \mathcal{S}^-(\lambda^*)^*)^G$ contains a nonzero element ϑ. By Lemma 13, the function $f := \vartheta \circ \varphi\colon G \times G \to k$ is contained in $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$. For this f, using Lemmas 15 and 12(f), we obtain

$$f(e,e) = \vartheta(\varphi(e,e)) = \vartheta(\varphi^+(e), \varphi^-(e)) \neq 0. \tag{38}$$

This completes the proof. $\square$

Corollary 17. *For every element $\lambda \in \mathrm{X}(T)_+$, there exists a unique element*

$$s_\lambda \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G \subseteq k[G \times G] \quad \textit{such that } s_\lambda(e,e) = 1. \tag{39}$$

If $\{f_1, \dots, f_d\}$ and $\{h_1, \dots, h_d\}$ are the bases of $\mathcal{S}^+(\lambda)$ and $\mathcal{S}^-(\lambda^*)$ dual with respect to a nondegenerate G-invariant pairing $\mathcal{S}^+(\lambda) \times \mathcal{S}^-(\lambda^*) \to k$ (the latter exists by (36) and Lemma 12), then $\varepsilon := \sum_{i=1}^d f_i(e)h_i(e) \neq 0$ and

$$s_\lambda = \varepsilon^{-1}\Bigl(\sum_{i=1}^d f_i \otimes h_i\Bigr).$$

Proof. First, note that if P, Q are the finite-dimensional kG-modules, $\theta \in \mathcal{B}(P,Q)^G$ is a nondegenerate pairing $P \times Q \to k$, and $\{p_1, \dots, p_m\}$ and $\{q_1, \dots, q_m\}$ are the bases of P and Q dual with respect to θ, then $\sum_{i=1}^m p_i \otimes q_i$ is a nonzero element of $(P \otimes_k Q)^G$ (not depending on the choice of these bases). Indeed, θ determines the isomorphism of G-modules

$$\begin{aligned} &\phi\colon P \otimes_k Q \to \operatorname{Hom}(P, P), \\ &(\phi(p \otimes q))(p') = \theta(p', q)p, \quad \text{where } p, p' \in P,\ q \in Q. \end{aligned} \tag{40}$$

From (40), we then obtain

$$\Big(\phi\Big(\sum_{i=1}^{m} p_i \otimes q_i\Big)\Big)(p_j) = \sum_{i=1}^{m} \theta(p_j, q_i)p_i = \sum_{i=1}^{m} \delta_{ij}\, p_i = p_j;$$

therefore, $\phi\big(\sum_{i=1}^{m} p_i \otimes q_i\big) = \mathrm{id}_P$, whence the claim.

For $P = \mathcal{S}^+(\lambda)$, $Q = \mathcal{S}^-(\lambda^*)$, it yields that $\sum_{i=1}^{d} f_i \otimes h_i$ is a nonzero element of $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$. Theorem 16 and (33) then complete the proof. □

Remark 18. For $\operatorname{char} k = 0$, there is another characterization of s_λ. Namely, let $\mathcal{U}$ be the universal enveloping algebra of $\operatorname{Lie} G$. Every $\mathcal{S}^{\pm}(\lambda)$ is endowed with the natural $\mathcal{U}$-module structure. Let $\{x_1, \dots, x_n\}$ and $\{x_1^*, \dots, x_n^*\}$ be the bases of $\operatorname{Lie} G$ dual with respect to the Killing form Φ. Identify $\operatorname{Lie} T$ with its dual space by means of Φ. Let σ be the sum of all positive roots. For every $\lambda \in \mathrm{X}(T)_+$, put

$$c_\lambda := \Phi(\lambda + \sigma, \lambda) + \Phi(\lambda^* + \sigma, \lambda^*) \tag{41}$$

and consider on the space $\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*)$ the linear operator

$$\Delta := \sum_{i=1}^{n} (x_i \otimes x_i^* + x_i^* \otimes x_i). \tag{42}$$

Proposition 19. *The following properties of an element $t \in \mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*)$ are equivalent*:

(i) $t = s_\lambda$.

(ii) $\Delta(t) = -c_\lambda t$ *and* $t(e, e) = 1$.

Proof. By [Bourbaki 1975, Chapitre VIII, §6.4, Corollaire], the Casimir element $\Omega := \sum_{i=1}^{n} x_i x_i^* \in \mathcal{U}$ acts on any simple $\mathcal{U}$-module with the highest weight γ as scalar multiplication by $\Phi(\gamma + \sigma, \gamma)$. Since $\Phi(\gamma + \sigma, \gamma) > 0$ if $\gamma \neq 0$, the kernel of Ω in any finite-dimensional $\mathcal{U}$-module V coincides with V^G. We apply this to $V = \mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*)$. For any elements $f \in \mathcal{S}^+(\lambda)$, $h \in \mathcal{S}^-(\lambda^*)$, we deduce from (41), (42) the following:

$$\begin{aligned} \Omega(f \otimes h) &= \sum_{i=1}^{n} \big(x_i x_i^*(f) \otimes h + x_i^*(f) \otimes x_i(h) + x_i(f) \otimes x_i^*(h) + f \otimes x_i x_i^*(h)\big) \\ &= \Omega(f) \otimes h + f \otimes \Omega(h) + \Delta(f \otimes h) = c_\lambda(f \otimes h) + \Delta(f \otimes h). \end{aligned}$$

Now Corollary 17 and the aforesaid about $\ker \Omega$ complete the proof. □

Theorem 20. *Let $\lambda_1, \dots, \lambda_m$ be a system of generators of the monoid* $\mathrm{X}(T)_+$. *Then* $(\ker \mu)^G$ *is the linear span over k of all monomials of the form*

$$(s_{\lambda_1} - 1)^{d_1} \cdots (s_{\lambda_m} - 1)^{d_m}, \quad \text{where } d_i \in \mathbb{N}, d_1 + \dots + d_m > 0,$$

where s_{λ_i} is defined in Corollary 17.

Proof. By Theorem 11(i), the linear span L referred to in Theorem 20 is contained in $(\ker \mu)^G$. In view of Theorem 11, to prove the converse inclusion $(\ker \mu)^G \subseteq L$, we have to show that, for every function

$$f \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G, \tag{43}$$

we have $f - f(e,e) \in L$. Since $\lambda_1, \dots, \lambda_m$ is a system of generators of $\mathrm{X}(T)_+$, there are integers $d_1, \dots, d_m \in \mathbb{N}$ such that $\lambda = \sum_{i=1}^m d_i \lambda_i$. From (39) and (8) we then infer that $h := \prod_{i=1}^m s_{\lambda_i}^{d_i} \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$ and $h(e,e) = 1$. This, (43), and (29) imply that $f = f(e,e)h$. Therefore,

$$f - f(e,e) = f(e,e)(h-1) = f(e,e)\biggl(\prod_{i=1}^m \bigl((s_{\lambda_i} - 1) + 1\bigr)^{d_i} - 1\biggr). \tag{44}$$

The right-hand side of (44) clearly lies in L. This completes the proof. □

Theorem 21. *Let $\lambda_1, \dots, \lambda_m$ be a system of generators of the monoid* $\mathrm{X}(T)_+$. *Then the ideal* $\ker \mu$ *in $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ is generated by* $s_{\lambda_1} - 1, \dots, s_{\lambda_m} - 1$, *where s_{λ_i} is defined in Corollary 17.*

Proof. This follows from Theorems 9 and 20. □

5. Presentation of $\mathcal{S}^\pm$

If the group G is semisimple, then the semigroup $\mathrm{X}(T)_+$ has no units other than 0. Hence the set $\mathcal{H}$ of all indecomposable elements of $\mathrm{X}(T)_+$ is finite,

$$\mathcal{H} = \{\lambda_1, \dots, \lambda_d\} \tag{45}$$

generates $\mathrm{X}(T)_+$, and every generating set of $\mathrm{X}(T)_+$ contains $\mathcal{H}$ (see, e.g., [Lorenz 2005, Lemma 3.4.3]). Note that $\mathcal{H}$, called the *Hilbert basis of* $\mathrm{X}(T)_+$, in general is not a free generating system of $\mathrm{X}(T)_+$ (i.e., it is not true that every element $\alpha \in \mathrm{X}(T)_+$ may be uniquely expressed in the form $\alpha = \sum_{i=1}^d c_i \lambda_i$, $c_i \in \mathbb{N}$). Namely, it is free if and only if $G = G_1 \times \dots \times G_s$, where every G_i is either a simply connected simple algebraic group or isomorphic to SO_{n_i} for an odd n_i (see [Steinberg 1975, §3], [Richardson 1979, Proposition 4.1], [Richardson 1982, Proposition 13.3] and [Popov 2011, Remark 3.16]). In particular, if G is simply connected, then $\mathcal{H}$ coincides with the set of all fundamental weights and generates $\mathrm{X}(T)_+$ freely. Note that $\lambda_i^* \in \mathcal{H}$ for every i.

To understand the presentation of $\mathcal{S}^{\pm}$, denote respectively by $\operatorname{Sym}\mathcal{S}^{\pm}(\lambda_i)$ and $\operatorname{Sym}^m \mathcal{S}^{\pm}(\lambda_i)$ the symmetric algebra and the m-th symmetric power of $\mathcal{S}^{\pm}(\lambda_i)$. The naturally $\mathbb{N}^d$-graded free commutative k-algebra

$$\mathcal{F}^{\pm} := \operatorname{Sym}\mathcal{S}^{\pm}(\lambda_1) \otimes_k \cdots \otimes_k \operatorname{Sym}\mathcal{S}^{\pm}(\lambda_d) \tag{46}$$

may be viewed as the algebra of regular functions $k[L^{\pm}]$ on the vector space

$$L^{\pm} := \mathcal{S}^{\pm}(\lambda_1)^* \oplus \cdots \oplus \mathcal{S}^{\pm}(\lambda_d)^*.$$

Let e_i be the i-th unit vector of $\mathbb{N}^d$ and let $\mathcal{F}^{\pm}_{p,q}$ be the homogeneous component of $\mathcal{F}^{\pm}$ of degree $e_p + e_q$. We have the natural isomorphisms of G-modules

$$\varphi^{\pm}_{p,q}\colon \mathcal{F}^{\pm}_{p,q} \xrightarrow{\cong} \mathcal{S}^{\pm}_{p,q} := \begin{cases} \mathcal{S}^{\pm}(\lambda_p) \otimes_k \mathcal{S}^{\pm}(\lambda_q) & \text{if } p \neq q, \\ \operatorname{Sym}^2 \mathcal{S}^{\pm}(\lambda_p) & \text{if } p = q. \end{cases} \tag{47}$$

By Theorems 1 and 2, the natural multiplication homomorphisms

$$\phi^{\pm}\colon \mathcal{F}^{\pm} \to \mathcal{S}^{\pm} \quad \text{and} \quad \psi^{\pm}_{p,q}\colon \mathcal{S}^{\pm}_{p,q} \to \mathcal{S}^{\pm}(\lambda_p + \lambda_q) \tag{48}$$

are surjective. Since $\mathcal{F}^{\pm}$ is a polynomial algebra, the surjectivity of $\phi^{\pm}$ reduces finding a presentation of $\mathcal{S}^{\pm}$ by generators and relations to describing $\ker\phi^{\pm}$. If $d = \dim T$, the following explicit description of $\ker\phi^{\pm}$ is available:

Theorem 22. *Let G be a connected semisimple group such that the Hilbert basis (45) freely generates the semigroup $\mathrm{X}(T)_+$. Then*

(i) *the ideal $\ker\phi^{\pm}$ of the $\mathbb{N}^d$-graded k-algebra $\mathcal{F}^{\pm}$ is homogeneous*;

(ii) *this ideal is generated by the union of all its homogeneous components of the total degree* 2;

(iii) *the set of these homogeneous components coincides with the set of all subspaces* $(\varphi^{\pm}_{p,q})^{-1}(\ker\psi^{\pm}_{p,q})$, $1 \leqslant p \leqslant q \leqslant d$.

Proof. This is the main result of [Kempf and Ramanathan 1987]. □

Remark 23. In characteristic 0, for the first time the proof of Theorem 22 was obtained (but not published) by B. Kostant; his proof appeared in [Lancaster and Towber 1979, Theorem 1.1]. In this case, (47) and the surjectivity of $\psi^{\pm}_{p,q}$ yield that $\psi^{\pm}_{p,q}$ is the projection of $\mathcal{S}^{\pm}_{p,q}$ to the Cartan component of $\mathcal{S}^{\pm}_{p,q}$, and $\ker\psi^{\pm}_{p,q}$ is the unique G-stable direct complement to this component. The subspace $\ker\psi^{\pm}_{p,q}$ admits the following description using the notation of Remark 18 [loc. cit.]. Let $\{x_1, \ldots, x_n\}$ and $\{x_1^*, \ldots, x_n^*\}$ be the dual bases of $\operatorname{Lie} G$ with respect to Φ. Then $\ker\psi^{+}_{p,q}$ is the image of the linear transformation $\bigl(\sum_{s=1}^{n}(x_s \otimes x_s^* + x_s^* \otimes x_s)\bigr) - 2\Phi(\lambda_p^*, \lambda_q^*)\mathrm{id}$ of the vector space $\mathcal{S}^{\pm}_{p,q}$.

Summing up, if G is a connected semisimple group such that the Hilbert basis (45) freely generates the semigroup $\mathrm{X}(T)_+$, then the sought-for canonical presentation of $k[G]$ is given by the surjective homomorphism

$$\phi := \phi^+ \otimes \phi^- \colon \mathcal{F} := \mathcal{F}^+ \otimes_k \mathcal{F}^- \to k[G] \tag{49}$$

of the polynomial k-algebra $\mathcal{F}$ and the following generating system $\mathcal{R}$ of the ideal $\ker\phi$. Identify $\mathcal{F}^+$ and $\mathcal{F}^-$ with subalgebras of $\mathcal{F}$ in the natural way. Then $\mathcal{R} = \mathcal{R}_1 \sqcup \mathcal{R}_2$, where

$$\mathcal{R}_1 = \bigcup_{p,q} \big((\varphi^+_{p,q})^{-1}(\ker\psi^+_{p,q}) \cup (\varphi^-_{p,q})^{-1}(\ker\psi^-_{p,q})\big) \tag{50}$$

(see the definition of $\varphi^\pm_{p,q}$, $\psi^\pm_{p,q}$ in (47), (48)) and

$$\mathcal{R}_2 = \{s_{\lambda_1} - 1, \dots, s_{\lambda_d} - 1\} \tag{51}$$

(see the definition of s_{λ_i} in Corollary 17). The elements of $\mathcal{R}_1$ (respectively, $\mathcal{R}_2$) are the Plücker-type (respectively, the SL_2-type) relations of the presentation.

The canonical presentation of $k[G]$ is redundant. To reduce the size of $\mathcal{R}_1$, we may replace every space $\ker\psi^\pm_{p,q}$ in (50) by a basis of this space. Finding such a basis falls within the framework of Standard Monomial Theory.

6. An example

As an illustration, here we explicitly describe the canonical presentation of $k[G]$ for $G = \mathrm{SL}_n$, $n \geqslant 2$, and $\operatorname{char} k = 0$.

Let T be the maximal torus of diagonal matrices in G, and let B^+ (respectively, B^-) be the Borel subgroup of lower (respectively, upper) triangular matrices in G. Then

$$\mathcal{H} = \{\varpi_1, \dots, \varpi_{n-1}\}, \quad \text{where}$$
$$\varpi_d \colon T \to k, \quad \operatorname{diag}(a_1, \dots, a_n) \mapsto a_{n-d+1} \cdots a_n.$$

Every pair $i_1, i_2 \in [n]$ determines the function

$$x_{i_1,i_2} \colon G \to k, \quad \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \dots & \dots & \dots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \mapsto a_{i_1,i_2}. \tag{52}$$

The k-algebra generated by all functions (52) is $k[G]$.

For every $d \in [n-1]$ and every sequence $i_1, \dots, i_d$ of d elements of $[n]$, put

$$f^-_{i_1,\dots,i_d} := \det \begin{pmatrix} x_{i_1,1} & \dots & x_{i_1,d} \\ \dots & \dots & \dots \\ x_{i_d,1} & \dots & x_{i_d,d} \end{pmatrix}, \quad f^+_{i_1,\dots,i_d} := \det \begin{pmatrix} x_{i_1,n-d+1} & \dots & x_{i_1,n} \\ \dots & \dots & \dots \\ x_{i_d,n-d+1} & \dots & x_{i_d,n} \end{pmatrix}.$$

For every fixed d, all functions $f^-_{i_1,\dots,i_d}$ (respectively, $f^+_{i_1,\dots,i_d}$) such that $i_1<\dots<i_d$ are linearly independent over k and their linear span over k is the simple G-module $\mathcal{S}^-(\varpi_d)$ (respectively, $\mathcal{S}^+(\varpi_d)$); see, e.g., [Flath and Towber 1992, Proposition 3.2]. Therefore, denoting by $x^\pm_{i_1,\dots,i_d}$ the element $f^\pm_{i_1,\dots,i_d}$ of the k-algebra $\mathcal{F}^\pm$ defined by (46), we identify $\mathcal{F}^\pm$ with the polynomial k-algebra in variables $x^\pm_{i_1,\dots,i_d}$, where d runs over $[n-1]$ and $i_1,\dots,i_d$ runs over $[n]_d$. Correspondingly, the k-algebra $\mathcal{F}$ is identified with the polynomial k-algebra in the variables $x^-_{i_1,\dots,i_d}$ and $x^+_{i_1,\dots,i_d}$, the homomorphism (49) takes the form

$$\phi\colon \mathcal{F}\to k[G],\qquad x^+_{i_1,\dots,i_d}\mapsto f^+_{i_1,\dots,i_d},\quad x^-_{i_1,\dots,i_d}\mapsto f^-_{i_1,\dots,i_d},$$

and $\phi^\pm=\phi|_{\mathcal{F}^\pm}$. Below the sets (50) and (51) are explicitly specified using this notation.

First, we will specify the Plücker-type relations. It is convenient to introduce the following elements of $\mathcal{F}^\pm$. Let $i_1,\dots,i_d$ be a sequence of $d\in[n-1]$ elements of $[n]$, and let $j_1,\dots,j_d$ be the nondecreasing sequence obtained from $i_1,\dots,i_d$ by permutation. Then we put

$$x^\pm_{i_1,\dots,i_d}=\begin{cases}\operatorname{sgn}(i_1,\dots,i_d)x^\pm_{j_1,\dots,j_d} & \text{if } i_p\neq i_q \text{ for all } p\neq q,\\ 0 & \text{otherwise.}\end{cases}$$

The k-algebra $\mathcal{S}^\pm$ is the coordinate algebra of the affine multicone over the flag variety; see [Towber 1979]. By the well-known classical Hodge's result [1942; 1943] (see also [Towber 1979, p. 434, Corollary 1]), the ideal $\ker\phi^\pm$ is generated by all elements of the form

$$\sum_{l=1}^{q+1}(-1)^l x^\pm_{i_1,\dots,i_{p-1},j_l}\,x^\pm_{j_1,\dots,\hat{j}_l,\dots,j_{q+1}},\tag{53}$$

where p and q run over $[n-1]$, $p\leqslant q$, and $i_1,\dots,i_{p-1}$ and $j_1,\dots,j_{q+1}$ run over $[n]_{p-1}$ and $[n]_{q+1}$ respectively. Since every element (53) is homogeneous of degree 2, this result together with Theorem 22 imply that, for every fixed $p,q\in[n-1]$, the set $(\varphi^\pm_{p,q})^{-1}(\ker\psi^\pm_{p,q})$ in (50) is the linear span of all elements (53), where $i_1,\dots,i_{p-1}$ and $j_1,\dots,j_{q+1}$ run over $[n]_{p-1}$ and $[n]_{q+1}$ respectively. This describes the Plücker-type relations (50).

Secondly, we will describe s_{ϖ_d}. If $\boldsymbol{i}\in[n]_{n-d}$ is a sequence $i_1,\dots,i_{n-d}$, we put $x^\pm_{\boldsymbol{i}}:=x^\pm_{i_1,\dots,i_{n-d}}$ and denote by $\boldsymbol{i}^*\in[n]_d$ the unique sequence $j_1,\dots,j_d$ whose intersection with $i_1,\dots,i_{n-d}$ is empty. Let $\operatorname{sgn}(\boldsymbol{i},\boldsymbol{i}^*)$ be the sign of the permutation $(i_1,\dots,i_{n-d},j_1,\dots,j_d)$. Then by [Flath and Towber 1992, Theorem 3.1(b)],

$$s_{\varpi_d}=\sum_{\boldsymbol{i}\in[n]_{n-d}}\operatorname{sgn}(\boldsymbol{i},\boldsymbol{i}^*)x^-_{\boldsymbol{i}}x^+_{\boldsymbol{i}^*}.$$

This describes the SL_2-type relations (51).

A similar description of the presentation of $k[G]$ may be given for the classical groups G of several other types: for them, the Plücker-type (respectively, the SL_2-type) relations are obtained using [Lancaster and Towber 1979; 1985] (respectively, [Flath and Towber 1992]).

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SMOOTH REPRESENTATIONS AND HECKE MODULES IN CHARACTERISTIC p

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Dedicated to the memory of Robert Steinberg.

Let G be a p-adic Lie group and $I \subseteq G$ be a compact open subgroup which is a torsionfree pro-p-group. Working over a coefficient field k of characteristic p we introduce a differential graded Hecke algebra for the pair (G, I) and show that the derived category of smooth representations of G in k-vector spaces is naturally equivalent to the derived category of differential graded modules over this Hecke DGA.

1. Background and motivation

Let G be a d-dimensional p-adic Lie group, and let k be any field. We denote by $\mathrm{Mod}_k(G)$ the category of smooth G-representations in k-vector spaces. It obviously has arbitrary direct sums.

Fix a compact open subgroup $I \subseteq G$. In $\mathrm{Mod}_k(G)$ we then have the representation

$$\mathrm{ind}_I^G(1) := \{k\text{-valued functions with finite support on } G/I\}$$

with G acting by left translations. Being generated by a single element, which is the characteristic function of the trivial coset, $\mathrm{ind}_I^G(1)$ is a compact object in $\mathrm{Mod}_k(G)$. It generates the full subcategory $\mathrm{Mod}_k^I(G)$ of all representations V in $\mathrm{Mod}_k(G)$ which are generated by their I-fixed vectors V^I. In general, $\mathrm{Mod}_k^I(G)$ is not an abelian category. The Hecke algebra of I by definition is the endomorphism ring

$$\mathcal{H}_I := \mathrm{End}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1))^{\mathrm{op}}.$$

We let $\mathrm{Mod}(\mathcal{H}_I)$ denote the category of left unital $\mathcal{H}_I$-modules. There is the pair of adjoint functors

$$\begin{aligned} H^0 : \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}(\mathcal{H}_I) \\ V &\longmapsto V^I = \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V), \end{aligned}$$

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and

$$T_0 : \mathrm{Mod}(\mathcal{H}_I) \longrightarrow \mathrm{Mod}_k^I(G) \subseteq \mathrm{Mod}_k(G)$$
$$M \longmapsto \mathrm{ind}_I^G(1) \otimes_{\mathcal{H}_I} M.$$

If the characteristic of k does not divide the pro-order of I then the functor H^0 is exact. Then $\mathrm{ind}_I^G(1)$ is a projective compact object in $\mathrm{Mod}_k(G)$. Since it does not generate the full category $\mathrm{Mod}_k(G)$, one cannot apply the Gabriel–Popescu theorem (compare [Kashiwara and Schapira 2006, Theorem 8.5.8]) to the functor H^0. Nevertheless, in this case, one might hope for a close relation between the categories $\mathrm{Mod}_k^I(G)$ and $\mathrm{Mod}(\mathcal{H}_I)$. This indeed happens, for example, for a connected reductive group G and its Iwahori subgroup I and the field $k = \mathbb{C}$; compare [Bernstein 1984, Corollary 3.9(ii)]. In addition, in this situation the algebra $\mathcal{H}_I$ turns out to be a generalized affine Hecke algebra so that its structure is explicitly known. Therefore, in characteristic zero, Hecke algebras have become one of the most important tools in the investigation of smooth G-representations.

In this light, it is a pressing question to better understand the relation between the two categories $\mathrm{Mod}_k(G)$ and $\mathrm{Mod}(\mathcal{H}_I)$ in the opposite situation where k has characteristic p. Since p always will divide the pro-order of I, the functor H^0 certainly is no longer exact. Both functors H^0 and T_0 now have a very complicated behavior and little is known [Koziol 2014; Ollivier 2009; Ollivier and Schneider 2015]. This suggests that one should work in a derived framework which takes into account the higher cohomology of I.

This paper will demonstrate that by doing this — not in a naive way but in an appropriate differential graded context — the situation does improve drastically. We will show the somewhat surprising result that the object $\mathrm{ind}_I^G(1)$ becomes a compact generator of the full derived category of G provided I is a torsionfree pro-p-group.

The main result of this paper was proved already in 2007 but remained unpublished. At the time, we gave a somewhat ad hoc proof. Although the main arguments remain unchanged we now, by appealing to a general theorem of Keller, have arranged them in a way which makes the reasoning more transparent. In the context of the search for a p-adic local Langlands program, there is increasing interest in studying derived situations; see [Harris 2015]. We also have now [Ollivier and Schneider 2015] the first examples of explicit computations of the cohomology groups $H^i(I, \mathrm{ind}_I^G(1))$. I hope that these are sufficient reasons to finally publish the paper.

2. The unbounded derived category of G

We assume from now on throughout the paper that the field k has characteristic p and that I is a torsionfree pro-p-group. Let us first of all collect a few properties of the abelian category $\mathrm{Mod}_k(G)$.

Lemma 1. (i) $\mathrm{Mod}_k(G)$ *is* (AB5), *i.e., it has arbitrary colimits and filtered colimits are exact.*

(ii) $\mathrm{Mod}_k(G)$ *is* (AB3*), *i.e., it has arbitrary limits.*

(iii) $\mathrm{Mod}_k(G)$ *has enough injective objects.*

(iv) $\mathrm{Mod}_k(G)$ *is a Grothendieck category, i.e., it satisfies* (AB5) *and has a generator.*

(v) $V^I \neq 0$ *for any nonzero* V *in* $\mathrm{Mod}_k(G)$.

Proof. (i) This is obvious. (ii) Take the subspace of smooth vectors in the limit of k-vector spaces. (iii) This is shown in [Vignéras 1996, §I.5.9]. Alternatively, it is a consequence of (iv); compare [Kashiwara and Schapira 2006, Theorem 9.6.2]. (v) Since I is pro-p, where p is the characteristic of k, the only irreducible smooth representation of I is the trivial one.

(iv) Because of (i) it remains to exhibit a generator of $\mathrm{Mod}_k(G)$. We define

$$Y := \bigoplus_J \mathrm{ind}_J^G(1),$$

where J runs over all open subgroups in G. For any V in $\mathrm{Mod}_k(G)$, we have

$$\mathrm{Hom}_{\mathrm{Mod}_k(G)}(Y, V) = \prod_J V^J.$$

Since $V = \bigcup_J V^J$, we easily deduce that Y is a generator of $\mathrm{Mod}_k(G)$. □

As usual, let $D(G) := D(\mathrm{Mod}_k(G))$ be the derived category of unbounded complexes in $\mathrm{Mod}_k(G)$.

Remark 2. $D(G)$ has arbitrary direct sums, which can be computed as direct sums of complexes.

Proof. See the first paragraph in [Kashiwara and Schapira 2006, §14.3]. □

According to [Lazard 1965, Théorème V.2.2.8; Serre 1965], the group I has cohomological dimension d. This means that the higher derived functors of the left exact functor

$$\begin{aligned} \mathrm{Mod}_k(I) &\longrightarrow \mathrm{Vec}_k \\ E &\longmapsto E^I \end{aligned}$$

into the category Vec_k of k-vector spaces are zero in degrees $> d$. On the other hand, the restriction functor

$$\begin{aligned} \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}_k(I) \\ V &\longmapsto V|_I \end{aligned}$$

is exact and respects injective objects. The latter is a consequence of the fact that compact induction

$$\begin{aligned} \mathrm{Mod}_k(I) &\longrightarrow \mathrm{Mod}_k(G) \\ E &\longmapsto \mathrm{ind}_I^G(E) \end{aligned}$$

is an exact left adjoint; compare [Vignéras 1996, §I.5.7]. Hence the higher derived functors of the composed functor

$$\begin{aligned} H^0(I, \cdot\,) : \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Vec}_k \\ V &\longmapsto V^I \end{aligned}$$

are given by $V \longmapsto H^i(I, V|_I)$ and vanish in degrees $> d$. It follows that the total right derived functor

$$RH^0(I, \cdot\,) : D(G) \longrightarrow D(\mathrm{Vec}_k)$$

between the corresponding (unbounded) derived categories exists [Hartshorne 1966, Corollary I.5.3].

To compute $RH^0(I, \cdot\,)$, we use the formalism of K-injective complexes as developed in [Spaltenstein 1988]. We let $C(\mathrm{Mod}_k(G))$ and $K(\mathrm{Mod}_k(G))$ denote the category of unbounded complexes in $\mathrm{Mod}_k(G)$ with chain maps and homotopy classes of chain maps, respectively, as morphisms. The K-injective complexes form a full triangulated subcategory $K_{\mathrm{inj}}(\mathrm{Mod}_k(G))$ of $K(\mathrm{Mod}_k(G))$. Exactly in the same way as [op. cit., Proposition 3.11] one can show that any complex in $C(\mathrm{Mod}_k(G))$ has a right K-injective resolution (recall from Lemma 1(ii) that the category $\mathrm{Mod}_k(G)$ has inverse limits). Alternatively, one may apply [Serpé 2003, Theorem 3.13] or [Kashiwara and Schapira 2006, Theorem 14.3.1] based upon Lemma 1(iv). The existence of K-injective resolutions means that the natural functor

$$K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) \xrightarrow{\simeq} D(G)$$

is an equivalence of triangulated categories. We fix a quasi-inverse $\boldsymbol{i}$ of this functor. Then the derived functor $RH^0(I, \cdot\,)$ is naturally isomorphic to the composed functor

$$D(G) \xrightarrow{\ \boldsymbol{i}\ } K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) \longrightarrow K(\mathrm{Vec}_k) \longrightarrow D(\mathrm{Vec}_k)$$

with the middle arrow given by

$$V^\bullet \mapsto \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V^\bullet).$$

Explanation. Let $V^\bullet$ be a complex in $C(\mathrm{Mod}_k(G))$. To compute $RH^0(I, \cdot\,)$ according to [Hartshorne 1966], one chooses a quasi-isomorphism $V^\bullet \xrightarrow{\simeq} C^\bullet$ into a complex consisting of objects which are acyclic for the functor $H^0(I, \cdot\,)$. On the other hand, let $V^\bullet \xrightarrow{\simeq} A^\bullet$ be a quasi-isomorphism into a K-injective complex. By

[Spaltenstein 1988, Proposition 1.5(c)] we then have, in $K(\mathrm{Mod}_k(G))$, a unique commutative diagram:

$$\begin{tikzcd} & C^\bullet \arrow[dd] \\ V^\bullet \arrow[ur] \arrow[dr] & \\ & A^\bullet \end{tikzcd}$$

We claim that the induced map

$$(C^\bullet)^I \xrightarrow{\simeq} (A^\bullet)^I$$

is a quasi-isomorphism. Choose quasi-isomorphisms

$$A^\bullet \xrightarrow{\simeq} \tilde{C}^\bullet \xrightarrow{\simeq} \tilde{A}^\bullet$$

where $\tilde{C}^\bullet$ consists of $H^0(I, \cdot\,)$-acyclic objects and $\tilde{A}^\bullet$ is K-injective. By [Spaltenstein 1988, Proposition 1.5(b)], the composite is an isomorphism in $K(\mathrm{Mod}_k(G))$ and hence induces a quasi-isomorphism $(A^\bullet)^I \xrightarrow{\simeq} (\tilde{A}^\bullet)^I$. But by [Hartshorne 1966, Theorem I.5.1 and Corollary I.5.3(γ)], the composite $C^\bullet \xrightarrow{\simeq} A^\bullet \xrightarrow{\simeq} \tilde{C}^\bullet$ also induces a quasi-isomorphism $(C^\bullet)^I \xrightarrow{\simeq} (\tilde{C}^\bullet)^I$.

Lemma 3. *The (hyper)cohomology functor $H^\ell(I, \cdot\,)$, for any $\ell \in \mathbb{Z}$, commutes with arbitrary direct sums in $D(G)$.*

Proof. First of all we observe that the cohomology functor $H^\ell(I, \cdot\,)$ commutes with arbitrary direct sums in $\mathrm{Mod}_k(G)$ [Serre 1994, §I.2.2, Proposition 8]. This, in particular, implies that arbitrary direct sums of $H^0(I, \cdot\,)$-acyclic objects in $\mathrm{Mod}_k(G)$ again are $H^0(I, \cdot\,)$-acyclic. Now let $(V_j^\bullet)_{j \in J}$ be a family of objects in $D(G)$, where we view each $V_j^\bullet$ as an actual complex. Then, according to Remark 2, the direct sum of the $V_j^\bullet$ in $D(G)$ is represented by the direct sum complex $\bigoplus_j V_j^\bullet$. Now we choose quasi-isomorphisms $V_j^\bullet \xrightarrow{\simeq} C_j^\bullet$ in $C(\mathrm{Mod}_k(G))$, where all representations C_j^m are $H^0(I, \cdot\,)$-acyclic. By the preliminary observation, the direct sum map

$$\bigoplus_j V_j^\bullet \xrightarrow{\simeq} C^\bullet := \bigoplus_j C_j^\bullet$$

again is a quasi-isomorphism where all terms of the target complex are $H^0(I, \cdot\,)$-acyclic. We therefore obtain

$$H^\ell\big(I, \bigoplus_j V_j^\bullet\big) = h^\ell((C^\bullet)^I) = \bigoplus_j h^\ell((C_j^\bullet)^I) = \bigoplus_j H^\ell(I, V_j^\bullet). \qquad \square$$

As usual, we view $\mathrm{Mod}_k(G)$ as the full subcategory of those complexes in $D(G)$ which have zero terms outside of degree zero.

Lemma 4. $\mathrm{ind}_I^G(1)$ *is a compact object in $D(G)$.*

Proof. We have to show that the functor $\mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1), \cdot)$ commutes with arbitrary direct sums in $D(G)$. For any $V^\bullet$ in $D(G)$, we compute

$$(1)\qquad \begin{aligned}\mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1), V^\bullet) &= \mathrm{Hom}_{K(\mathrm{Mod}_k(G))}(\mathrm{ind}_I^G(1), \boldsymbol{i}(V^\bullet)) \\ &= h^0(\boldsymbol{i}(V^\bullet)^I) = H^0(I, V^\bullet),\end{aligned}$$

where the first identity uses [Spaltenstein 1988, Proposition 1.5(b)]. The claim therefore follows from Lemma 3. □

Proposition 5. *Let $E^\bullet$ be in $D(I)$. Then $E^\bullet = 0$ if and only if $H^j(I, E^\bullet) = 0$ for any $j \in \mathbb{Z}$.*

Proof. The completed group ring $\Omega := \varprojlim_N k[I/N]$ of I over k, where N runs over all open normal subgroups of I, is a pseudocompact local ring; compare [Schneider 2011, §19]. If $\mathfrak{m} \subseteq \Omega$ denotes the maximal ideal, then $\Omega/\mathfrak{m} = k$. Since Ω is noetherian — [Lazard 1965, Proposition V.2.2.4] for $k = \mathbb{F}_p$ and [Schneider 2011, Theorem 33.4] together with [Bourbaki 2006, Chapitre IX, §2.3, Proposition 5] in general — its pseudocompact topology coincides with the $\mathfrak{m}$-adic topology [Schneider 2011, Lemma 19.8]. This implies that:

- $\Omega/\mathfrak{m}^j$ lies in $\mathrm{Mod}_k(I)$ for any $j \in \mathbb{N}$.
- For any E in $\mathrm{Mod}_k(I)$, we have

$$E = \bigcup_{j \in \mathbb{N}} E^{\mathfrak{m}^j = 0} \quad \text{where } E^{\mathfrak{m}^j=0} := \{v \in E : \mathfrak{m}^j v = 0\}.$$

Because of

$$E^{\mathfrak{m}^j=0} = \mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, E),$$

we need to consider the left exact functors $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot)$ on $\mathrm{Mod}_k(I)$. Their right derived functors, of course, are $\mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot)$. In particular,

$$\mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}, \cdot) = H^i(I, \cdot).$$

For any $j \in \mathbb{N}$, we have the short exact sequence

$$0 \longrightarrow \mathfrak{m}^j/\mathfrak{m}^{j+1} \longrightarrow \Omega/\mathfrak{m}^{j+1} \longrightarrow \Omega/\mathfrak{m}^j \longrightarrow 0$$

in $\mathrm{Mod}_k(I)$. Moreover, $\mathfrak{m}^j/\mathfrak{m}^{j+1} \cong k^{n(j)}$ for some $n(j) \geq 0$ since Ω is noetherian. The associated long exact Ext-sequence therefore reads

$$\cdots \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^{j+1}, \cdot) \longrightarrow H^i(I, \cdot)^{n(j)} \longrightarrow \cdots$$

By induction with respect to j, we deduce that:

- Each functor $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot)$ has cohomological dimension $\leq d$.

– Each $H^0(I, \cdot)$-acyclic object in $\mathrm{Mod}_k(I)$ is $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot)$-acyclic for any $j \geq 1$.

It follows that the total right derived functors $R\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot)$ on $D(I)$ exist. More explicitly, let $E^\bullet$ be any complex in $D(I)$ and choose a quasi-isomorphism $E^\bullet \xrightarrow{\simeq} C^\bullet$ into a complex consisting of $H^0(I, \cdot)$-acyclic objects. It then follows that we have the short exact sequence of complexes

$$0 \to \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, C^\bullet) \to \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^{j+1}, C^\bullet) \to ((C^\bullet)^I)^{n(j)} \to 0.$$

Suppose now that $RH^0(I, E^\bullet) = 0$. This means that the complex $(C^\bullet)^I$ is exact. By induction with respect to j, we obtain the exactness of the complex

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, C^\bullet) = (C^\bullet)^{\mathfrak{m}^j=0}$$

for any $j \in \mathbb{N}$. Hence $C^\bullet$ and $E^\bullet$ are exact. □

Proposition 6. $\mathrm{ind}_I^G(1)$ *is a generator of the triangulated category* $D(G)$ *in the sense that any strictly full triangulated subcategory of* $D(G)$, *closed under all direct sums*, *which contains* $\mathrm{ind}_I^G(1)$, *coincides with* $D(G)$.

Proof. By (1) we have

$$\begin{aligned}\mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1)[j], V^\bullet) &= \mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1), V^\bullet[-j]) \\ &= H^0(I, V^\bullet[-j]) = H^{-j}(I, V^\bullet)\end{aligned}$$

for any $V^\bullet$ in $D(G)$. Hence, Proposition 5 implies that the family of shifts $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$ is a generating set of $D(G)$ in the sense of Neeman [2001, Definition 8.1.1]. On the other hand, by Lemma 4, each shift $\mathrm{ind}_I^G(1)[j]$ is a compact object. In Neeman's language this means that $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$ is an $\aleph_0$-perfect class consisting of $\aleph_0$-small objects [Neeman 2001, Remark 4.2.6 and Definition 4.2.7]. According to Neeman's Lemma 4.2.1, the class $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$ then is β-perfect for any infinite cardinal β. Hence Neeman's Theorem 8.3.3 applies and shows (see the explanations in §3.2.6–3.2.8 of that same reference) that any strictly full triangulated subcategory of $D(G)$ closed under all direct sums which contains $\mathrm{ind}_I^G(1)$, and therefore the whole class $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$, coincides with $D(G)$. □

3. The Hecke DGA

In order to also "derive" the picture on the Hecke algebra side we fix an injective resolution $\mathrm{ind}_I^G(1) \xrightarrow{\simeq} \mathcal{I}^\bullet$ in $C(\mathrm{Mod}_k(G))$ and introduce the differential graded algebra

$$\mathcal{H}_I^\bullet := \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}}$$

over k. We recall that

$$\mathcal{H}_I^n = \prod_{q\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathcal{I}^q, \mathcal{I}^{q+n})$$

with differential

$$(da)_q(x) = d(a_q(x)) - (-1)^n a_{q+1}(dx)$$

for $a = (a_q) \in \mathcal{H}_I^n$ and multiplication

$$(ba)_q := (-1)^{mn} a_{q+m} \circ b_q$$

for $a = (a_q) \in \mathcal{H}_I^n$ and $b = (b_q) \in \mathcal{H}_I^m$. The cohomology of $\mathcal{H}_I^\bullet$ is given by

$$h^*(\mathcal{H}_I^\bullet) = \mathrm{Ext}^*_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), \mathrm{ind}_I^G(1));$$

compare [Hartshorne 1966, §I.6]. In particular,

$$h^0(\mathcal{H}_I^\bullet) = \mathcal{H}_I.$$

Remark 7. $h^*(\mathcal{H}_I^\bullet) = H^*(I, \mathrm{ind}_I^G(1))$ and, in particular, $h^i(\mathcal{H}_I^\bullet) = 0$ for $i > d$.

Proof. We compute

$$\begin{aligned} h^*(\mathcal{H}_I^\bullet) &= \mathrm{Ext}^*_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), \mathrm{ind}_I^G(1)) \\ &= h^*(\mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), \mathcal{I}^\bullet)) \\ &= h^*((\mathcal{I}^\bullet)^I) = H^*(I, \mathrm{ind}_I^G(1)). \end{aligned}$$ □

Let $D(\mathcal{H}_I^\bullet)$ be the derived category of differential graded left $\mathcal{H}_I^\bullet$-modules. Note that $\mathcal{H}_I^\bullet$ is a compact generator of $D(\mathcal{H}_I^\bullet)$ [Keller 1998, §2.5]. It is well known that $\mathcal{H}_I^\bullet$ and $D(\mathcal{H}_I^\bullet)$ do not depend, up to quasi-isomorphism and equivalence, respectively, on the choice of the injective resolution $\mathcal{I}^\bullet$. For the convenience of the reader, we briefly recall the argument. Let $\mathrm{ind}_I^G(1) \xrightarrow{\simeq} \mathcal{J}^\bullet$ be a second injective resolution in $C(\mathrm{Mod}_k(G))$, and let $f : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ be a homotopy equivalence inducing the identity on $\mathrm{ind}_I^G(1)$ with homotopy inverse g. We form the differential graded algebra

$$\mathcal{A}^\bullet := \left\{(a, b) \in \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}} \times \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}} : a \circ f = f \circ a\right\}$$

(with respect to componentwise multiplication) and consider the commutative diagram

$$\begin{tikzcd} \mathcal{A}^\bullet \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}} \arrow[d, "b \mapsto f\circ b"] \\ \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}} \arrow[r, "a \mapsto a\circ f"] & \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, \mathcal{J}^\bullet). \end{tikzcd}$$

Obviously, the maps pr_i are homomorphisms of differential graded algebras (and the bottom horizontal and right perpendicular arrows are homotopy equivalences of complexes). By direct inspection, one checks that the pr_i, in fact, are quasi-isomorphisms. Hence the differential graded algebras $\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}}$ and $\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}}$ are naturally quasi-isomorphic to each other. Moreover, by appealing to [Bernstein and Lunts 1994, Theorem 10.12.5.1], we see that the functors

$$D(\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}}) \xrightarrow[(\mathrm{pr}_2)_*]{\sim} D(\mathcal{A}^\bullet) \xleftarrow[(\mathrm{pr}_1)_*]{\sim} D(\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}})$$

are equivalences of triangulated categories.

There is the following pair of adjoint functors

$$H : D(G) \longrightarrow D(\mathcal{H}_I^\bullet) \quad \text{and} \quad T : D(\mathcal{H}_I^\bullet) \longrightarrow D(G).$$

For any K-injective complex $V^\bullet$ in $\mathrm{Mod}_k(G)$, the natural chain map

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, V^\bullet) \xrightarrow{\simeq} \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V^\bullet)$$

is a quasi-isomorphism. But the left hand term is a differential graded left $\mathcal{H}_I^\bullet$-module in a natural way. In fact, we have the functor

$$\begin{aligned} K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) &\longrightarrow K(\mathcal{H}_I^\bullet) \\ V^\bullet &\longmapsto \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, V^\bullet) \end{aligned}$$

into the homotopy category $K(\mathcal{H}_I^\bullet)$ of differential graded left $\mathcal{H}_I^\bullet$-modules, which allows us to define the composed functor

$$H : D(G) \xrightarrow{i} K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) \longrightarrow K(\mathcal{H}_I^\bullet) \longrightarrow D(\mathcal{H}_I^\bullet).$$

The diagram

$$\begin{tikzcd} D(G) \arrow[r, "H"] \arrow[dr, "RH^0(I,\cdot)"'] & D(\mathcal{H}_I^\bullet) \arrow[d, "\text{forget}"] \\ & D(\mathrm{Vec}_k) \end{tikzcd} \tag{2}$$

then is commutative up to natural isomorphism.

For the functor T in the opposite direction we first note that $\mathcal{I}^\bullet$ is naturally a differential graded right $\mathcal{H}_I^\bullet$-module so that we can form the graded tensor product $\mathcal{I}^\bullet \otimes_{\mathcal{H}_I^\bullet} M^\bullet$ with any differential graded left $\mathcal{H}_I^\bullet$-module $M^\bullet$. This tensor product is naturally a complex in $C(\mathrm{Mod}_k(G))$. We now define T to be the composite

$$T : D(\mathcal{H}_I^\bullet) \xrightarrow{p} K_{\mathrm{pro},\mathcal{H}_I^\bullet} \xrightarrow{\mathcal{I}^\bullet \otimes_{\mathcal{H}_I^\bullet}} K(\mathrm{Mod}_k(G)) \longrightarrow D(G).$$

Here $K_{\mathrm{pro},\mathcal{H}_I^\bullet}$ denotes the full triangulated subcategory of $K(\mathcal{H}_I^\bullet)$ consisting of K-projective modules and $\boldsymbol{p}$ is a quasi-inverse of the equivalence of triangulated categories $K_{\mathrm{pro},\mathcal{H}_I^\bullet} \xrightarrow{\simeq} D(\mathcal{H}_I^\bullet)$; compare [Bernstein and Lunts 1994, Corollary 10.12.2.9].

The usual standard computation shows that T is left adjoint to H.

4. The main theorem

We need one more property of the derived category $D(G)$.

Lemma 8. *The triangulated category $D(G)$ is algebraic.*

Proof. The composite functor

$$D(G) \xrightarrow{i} K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) \xrightarrow{\subseteq} K(\mathrm{Mod}_k(G))$$

is a fully faithful exact functor between triangulated categories. Hence, the assertion follows from [Krause 2007, Lemma 7.5]. □

In view of Lemmas 4 and 8 and Proposition 6, all assumptions of Keller's theorem [1994, Theorem 4.3; 1998, Theorem 3.3(a)] (compare also [Bondal and van den Bergh 2003, Theorem 3.1.7]) are satisfied and we obtain our main result.

Theorem 9. *The functor H is an equivalence between triangulated categories*

$$D(G) \xrightarrow{\simeq} D(\mathcal{H}_I^\bullet).$$

Of course, it follows formally that the adjoint functor T is a left inverse of H.

Remark 10. The full subcategory $D(G)^{\mathrm{c}}$ of all compact objects in $D(G)$ is the smallest strictly full triangulated subcategory closed under direct summands which contains $\mathrm{ind}_I^G(1)$.

Proof. In view of Lemma 4 and Proposition 6 this follows from [Neeman 1992, Lemma 2.2]. □

The subcategory $D(G)^{\mathrm{c}}$ should be viewed as the analog of the subcategory of perfect complexes in the derived category of a ring; compare [Keller 1998, Lemma 1.4].

Another important subcategory of $D(G)$ is the bounded derived category

$$D^{\mathrm{b}}(G) := D^{\mathrm{b}}(\mathrm{Mod}_k(G)).$$

Correspondingly we have the full subcategory $D^{\mathrm{b}}(\mathcal{H}_I^\bullet)$ of all differential graded modules $M^\bullet$ in $D(\mathcal{H}_I^\bullet)$ such that $h^j(M^\bullet) = 0$ for all but finitely many $j \in \mathbb{Z}$. Since I has finite cohomological dimension, the commutative diagram (2) shows that H restricts to a fully faithful functor

$$D^{\mathrm{b}}(G) \longrightarrow D^{\mathrm{b}}(\mathcal{H}_I^\bullet).$$

On the other hand, the behavior of the functor T is controlled by an Eilenberg–Moore spectral sequence

$$E_2^{r,s} = \mathrm{Tor}_{-r}^{h^*(\mathcal{H}_I^\bullet)}(\mathcal{H}_I, h^*(M^\bullet))^s \Longrightarrow h^{r+s}(T(M^\bullet))$$

[May, Theorem 4.1]. This suggests that, except in very special cases, the functor T will not preserve the bounded subcategories.

5. Complements

5.1. ***The top cohomology.*** A first step in the investigation of the DGA $\mathcal{H}_I^\bullet$ might be the computation of its cohomology algebra $h^*(\mathcal{H}_I^\bullet)$. By Remark 7, the latter is concentrated in degrees 0 to d. Of course the usual Hecke algebra $\mathcal{H}_I = h^0(\mathcal{H}_I^\bullet)$ is a subalgebra of $h^*(\mathcal{H}_I^\bullet)$. We determine here the top cohomology $h^d(\mathcal{H}_I^\bullet)$ as a right $\mathcal{H}_I$-module.

Using the I-equivariant linear map

$$\begin{aligned} \pi_I : \mathrm{ind}_I^G(1) &\longrightarrow \mathrm{ind}_I^G(1)^I = \mathcal{H}_I \\ \phi &\longmapsto \Big[h \longmapsto \sum_{g \in I/I\cap hIh^{-1}} \phi(gh) \Big] \end{aligned}$$

we obtain the map

$$\pi_I^* : h^*(\mathcal{H}_I^\bullet) = H^*(I, \mathrm{ind}_I^G(1)) \xrightarrow{H^*(I,\pi_I)} H^*(I, \mathcal{H}_I) = H^*(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I.$$

The last equality in this chain comes from the universal coefficient theorem, which is applicable since I as a Poincaré group [Lazard 1965, Théorème V.2.5.8] has finite cohomology $H^*(I, \mathbb{F}_p)$. Of course, as a ring $\mathcal{H}_I$ is a right module over itself. For our purposes, we have to consider a modification of this module structure which is specific to characteristic p.

As a k-vector space $\mathrm{ind}_I^G(1)^I = \mathcal{H}_I$ has the basis $\{\chi_{IxI}\}_{x \in I\backslash G/I}$ consisting of the characteristic functions of the double cosets IxI. If we denote the multiplication in the algebra $\mathcal{H}_I$, as usual, by the symbol "$*$" for convolution, then in this basis it is given by the formula

$$\chi_{IxI} * \chi_{IhI} = \sum_{y \in I\backslash G/I} c_{x,y;h}\, \chi_{IyI},$$

where the coefficients are

$$c_{x,y;h} = (\chi_{IxI} * \chi_{IhI})(y) = \sum_{y \in G/I} \chi_{IxI}(g)\chi_{IhI}(g^{-1}y) = |IxI \cap yIh^{-1}I/I| \cdot 1_k,$$

with 1_k denoting the unit element in the field k. Of course, for fixed x and h we have $c_{x,y;h} = 0$ for all but finitely many $y \in I\backslash G/I$. But $IxI \cap yIh^{-1}I \neq \varnothing$ implies

$IxI \subseteq IyIh^{-1}I$; by compactness, the latter is a finite union of double cosets. Hence, also for fixed y and h, we have $c_{x,y;h} \neq 0$ for at most finitely many $x \in I\backslash G/I$. It follows that by combining the transpose of these coefficient matrices with the antiautomorphism

$$\begin{aligned}\mathcal{H}_I &\longrightarrow \mathcal{H}_I\\ \chi &\longmapsto \chi^*(g) := \chi(g^{-1}),\end{aligned}$$

we obtain through the formula

$$\chi_{IxI} *_\tau \chi_{IhI} := \sum_{y \in I\backslash G/I} c_{y,x;h^{-1}} \chi_{IyI}$$

a new right action of $\mathcal{H}_I$ on itself. We denote this new module by $\mathcal{H}_I^\tau$.

Remark. We compute

$$\begin{aligned}|IyI/I| \cdot c_{x,y;h} &= |IyI/I| \cdot (\chi_{IxI} * \chi_{IhI})(y)\\ &= \sum_{z \in G/I} \chi_{IyI}(z)\big(\chi_{IxI} * \chi^*_{Ih^{-1}I}\big)(z)\\ &= \big(\chi_{IyI} * (\chi_{IxI} * \chi^*_{Ih^{-1}I})^*\big)(1)\\ &= \big((\chi_{IyI} * \chi_{Ih^{-1}I}) * \chi^*_{IxI}\big)(1)\\ &= \sum_{z \in G/I} (\chi_{IyI} * \chi_{Ih^{-1}I})(z)\chi_{IxI}(z)\\ &= |IxI/I| \cdot (\chi_{IyI} * \chi_{Ih^{-1}I})(x)\\ &= |IxI/I| \cdot c_{y,x;h^{-1}}.\end{aligned}$$

This, of course, is valid with integral coefficients (instead of k). Moreover, $|IxI/I|$ is always a power of p. It follows that over any field of characteristic different from p one has $\mathcal{H}_I^\tau \cong \mathcal{H}_I$. It also follows that $c_{x,y;h} = c_{y,x;h^{-1}}$ whenever both are nonzero.

It is straightforward to check that

$$\pi_I(\phi) *_\tau \chi_{IhI} = \pi_I(\phi * \chi_{IhI})$$

holds true for any $\phi \in \mathrm{ind}_I^G(1)$ and any $h \in G$. Hence,

$$\pi_I : \mathrm{ind}_I^G(1) \longrightarrow \mathcal{H}_I^\tau \quad \text{and} \quad \pi_I^* : h^*(\mathcal{H}_I^\bullet) \longrightarrow H^*(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I^\tau$$

are maps of right $\mathcal{H}_I$-modules.

Proposition 11. *The map π_I^d is an isomorphism*

$$h^d(\mathcal{H}_I^\bullet) \xrightarrow{\cong} H^d(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I^\tau$$

of right $\mathcal{H}_I$-modules. By fixing a basis of the one dimensional $\mathbb{F}_p$-vector space $H^d(I, \mathbb{F}_p)$, we therefore obtain $h^d(\mathcal{H}_I^\bullet) \cong \mathcal{H}_I^\tau$ as right $\mathcal{H}_I$-modules.

Proof. It remains to show that π_I^d is bijective. We have the I-equivariant decomposition

$$\mathrm{ind}_I^G(1) = \bigoplus_{x \in I\backslash G/I} \mathrm{ind}^I_{I\cap xIx^{-1}}(1).$$

The map π_I restricts to

$$\begin{aligned} \pi_I : \mathrm{ind}^I_{I\cap xIx^{-1}}(1) &\longrightarrow k\cdot \chi_{IxI} \subseteq \mathcal{H}_I \\ \phi &\longmapsto \Big(\sum_{y\in I/I\cap xIx^{-1}} \phi(y)\Big)\cdot \chi_{IxI}. \end{aligned}$$

Since $H^*(I, \cdot)$ commutes with arbitrary direct sums it therefore suffices to show that the map

$$H^d\Big(I,\ \phi \longmapsto \sum_{y\in I/I\cap xIx^{-1}} \phi(y)\Big) : H^d(I, \mathrm{ind}^I_{I\cap xIx^{-1}}(1_{\mathbb{F}_p})) \longrightarrow H^d(I, \mathbb{F}_p)$$

is bijective. Using Shapiro's lemma this latter map identifies (compare [Serre 1994, §I.2.5]) with the corestriction map

$$\mathrm{Cor} : H^d(I\cap xIx^{-1}, \mathbb{F}_p) \longrightarrow H^d(I, \mathbb{F}_p),$$

which for Poincaré groups of dimension d is an isomorphism of one dimensional vector spaces [op. cit., (4) on p. 37]. □

5.2. ***The easiest example.*** As an example, we will make explicit the case where $G = I = \mathbb{Z}_p$ is the additive group of p-adic integers, which we nevertheless write multiplicatively with unit element e. In order to distinguish it from the unit element $1 \in k$ we will denote the multiplicative unit in $\mathbb{Z}_p$ by γ. Let Ω denote the completed group ring of $\mathbb{Z}_p$ over k. We have:

(a) The category $\mathrm{Mod}_k(G)$ coincides with the category of torsion Ω-modules.

(b) Sending $\gamma - 1$ to t defines an isomorphism of k-algebras $\Omega \cong k[\![t]\!]$ between Ω and the formal power series ring in one variable t over k.

For any V in $\mathrm{Mod}_k(G)$ we have the smooth G-representation $C^\infty(G, V)$ of all V-valued locally constant functions on G, where $g \in G$ acts on $f \in C^\infty(G, V)$ by ${}^g f(h) := g(f(g^{-1}h))$. One easily checks:

(c) $C^\infty(G, V) = C^\infty(G, k) \otimes_k V$ with the diagonal G-action on the right hand side.

(d) The map $\mathrm{Hom}_{\mathrm{Mod}_k(G)}(W, C^\infty(G, V)) \xrightarrow{\cong} \mathrm{Hom}_k(W, V)$ sending F to $[w \mapsto F(w)(e)]$ is an isomorphism for any W in $\mathrm{Mod}_k(G)$. It follows that $C^\infty(G, V)$ is an injective object in $\mathrm{Mod}_k(G)$.

(e) The short exact sequence

$$0 \longrightarrow V \longrightarrow C^{\infty}(G,k) \otimes_k V \xrightarrow{\gamma_* - 1 \otimes \mathrm{id}} C^{\infty}(G,k) \otimes_k V \longrightarrow 0, \tag{3}$$

where $\gamma_*(\phi)(h) = \phi(h\gamma)$ is an injective resolution of V in $\mathrm{Mod}_k(G)$.

(f) For any $g \in G$ define the map $F_g : C^{\infty}(G,k) \to C^{\infty}(G,k)$ by $F_g(\phi)(h) := \phi(hg)$. In particular, $F_\gamma = \gamma_*$. Sending g to F_g defines an isomorphism of k-algebras

$$\Omega \xrightarrow{\cong} \mathrm{End}_{\mathrm{Mod}_k(G)}(C^{\infty}(G,k)).$$

Obviously $\mathrm{ind}_I^G(1) = k$ is the trivial G-representation. By (3) we may take for $\mathcal{I}^\bullet$ the injective resolution

$$C^{\infty}(G,k) \xrightarrow{\gamma_* - 1} C^{\infty}(G,k) \longrightarrow 0 \longrightarrow \cdots$$

Using (f) we deduce that $\mathcal{H}_I^\bullet$ is

$$\cdots \longrightarrow \mathcal{H}_I^{-1} = \Omega \xrightarrow{d^{-1}} \mathcal{H}_I^0 = \Omega \times \Omega \xrightarrow{d^0} \mathcal{H}_I^1 = \Omega \longrightarrow \cdots$$

with

$$d^{-1}a = ((\gamma - 1)a, (\gamma - 1)a) \quad \text{and} \quad d^0(a,b) = (\gamma - 1)(a - b)$$

and multiplication

$$\begin{aligned}(a_{-1}, (a_0, b_0), a_1) \cdot (a'_{-1}, (a'_0, b'_0), a'_1) \\ = (a'_0 a_{-1} + a'_{-1} b_0, (a'_0 a_0 - a'_{-1} a_1, b'_0 b_0 - a'_1 a_{-1}), a'_1 a_0 + b'_0 a_1).\end{aligned}$$

Using (b) we then identify $\mathcal{H}_I^\bullet$ with the upper row in the commutative diagram

$$\begin{tikzcd} k[[t]] \arrow[r, "a \mapsto (ta, ta)"] & k[[t]] \times k[[t]] \arrow[r, "(a,b) \mapsto t(a-b)"] & k[[t]] \\ 0 \arrow[u] \arrow[r] & k \arrow[u, "a \mapsto (a,a)"'] \arrow[r, "0"] & k \arrow[u, "\subseteq"'] \end{tikzcd}$$

We view the bottom row as the differential graded algebra of dual numbers $k[\epsilon]/(\epsilon^2)$ in degrees 0 and 1 with the zero differential. It is easy to check that the vertical arrows in the above diagram constitute a quasi-isomorphism of differential graded algebras. In particular, this says that $\mathcal{H}_I^\bullet$ is quasi-isomorphic to its cohomology algebra with zero differential (ϵ corresponds to the projection map $G = \mathbb{Z}_p \to \mathbb{F}_p \subseteq k$, as a generator of $H^1(G,k) = \mathrm{Hom}^{\mathrm{cont}}(\mathbb{Z}_p, k)$). According to our Theorem 9, we therefore obtain that H composed with the pullback along the above quasi-isomorphism is an equivalence of triangulated categories

$$D(\mathbb{Z}_p) \xrightarrow{\simeq} D(k[\epsilon]/(\epsilon^2)). \tag{4}$$

We finish by determining this functor explicitly. Let V be an object in $\mathrm{Mod}_k(G)$. Using the injective resolution (3) we can represent $H(V)$ by the complex

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}\big([C^\infty(G,k)\xrightarrow{\gamma_*-1}C^\infty(G,k)],\ [C^\infty(G,k)\otimes_k V\xrightarrow{\gamma_*-1\otimes\mathrm{id}}C^\infty(G,k)\otimes_k V]\big).$$

Furthermore, using the identifications in (c) and (d), this latter complex can be computed to be the complex

$$\mathrm{Hom}_k(C^\infty(G,k),V)\xrightarrow{d^{-1}}\mathrm{Hom}_k(C^\infty(G,k),V)\times\mathrm{Hom}_k(C^\infty(G,k),V)\xrightarrow{d^0}\mathrm{Hom}_k(C^\infty(G,k),V)$$

in degrees -1, 0, and 1 with the differentials

$$d^{-1}f=(f\circ(\gamma_*-1),\ f\circ(\gamma_*-1)+(\gamma-1)\circ f\circ\gamma_*)\quad\text{and}$$
$$d^0(f_0,f_1)=(\gamma-1)\circ f_0\circ\gamma_*+(f_0-f_1)\circ(\gamma_*-1).$$

Let $\delta_e\in\mathrm{Hom}_k(C^\infty(G,k),k)$ denote the "Dirac distribution" $\delta_e(\phi):=\phi(e)$ in the unit element. The diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{Hom}_k(C^\infty(G,k),V) \arrow[d, "d^{-1}"] \\
V \arrow[r, "v\mapsto(\delta_e(\cdot)v,\,\delta_e(\cdot)\gamma(v))"] \arrow[d, "\gamma-1"'] & \mathrm{Hom}_k(C^\infty(G,k),V)\times\mathrm{Hom}_k(C^\infty(G,k),V) \arrow[d, "d^0"] \\
V \arrow[r, "v\mapsto\delta_e(\cdot)v"] & \mathrm{Hom}_k(C^\infty(G,k),V)
\end{tikzcd}$$

is commutative. We claim that the horizontal arrows form a quasi-isomorphism $\alpha^\bullet$. In order to define a map in the opposite direction we let $\phi_1\in C^\infty(G,k)$ denote the constant function with value 1. Using that $\gamma_*(\phi_1)=\phi_1$, one checks that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_k(C^\infty(G,k),V) \arrow[r] \arrow[d, "d^{-1}"'] & 0 \arrow[d] \\
\mathrm{Hom}_k(C^\infty(G,k),V)\times\mathrm{Hom}_k(C^\infty(G,k),V) \arrow[r, "(f_0,f_1)\mapsto f_0(\phi_1)"] \arrow[d, "d^0"'] & V \arrow[d, "\gamma-1"] \\
\mathrm{Hom}_k(C^\infty(G,k),V) \arrow[r, "f\mapsto f(\phi_1)"] & V
\end{tikzcd}$$

is commutative. Hence the horizontal arrows define a homomorphism of complexes $\beta^\bullet$ such that $\beta^\bullet\circ\alpha^\bullet=\mathrm{id}$. Applying $\mathrm{Hom}_k(\cdot\,,V)$ to our injective resolution of k, we obtain the short exact sequence

$$0\longrightarrow\mathrm{Hom}_k(C^\infty(G,k),V)\xrightarrow{f\mapsto f\circ(\gamma_*-1)}\mathrm{Hom}_k(C^\infty(G,k),V)\xrightarrow{\beta^1}V\longrightarrow 0.$$

This implies that d^{-1} is injective and that $\mathrm{im}(d^0) \supseteq \ker(\beta^1)$. The former says that the cohomology in degree -1 is zero. Because of

$$\mathrm{Hom}_k(C^\infty(G,k), V) = \ker(\beta^1) \oplus \mathrm{im}(\alpha^1), \tag{5}$$

the latter shows the surjectivity of $h^1(\alpha^\bullet)$. Hence $h^1(\alpha^\bullet)$ is bijective. A pair (f_0, f_1) represents a class in $\ker(h^0(\beta^\bullet))$ if and only if $d^0(f_0, f_1) = 0$ and $\beta^0(f_0, f_1) = 0$. The first condition implies that

$$f_1 \circ (\gamma_* - 1) = (\gamma - 1) \circ f_0 \circ \gamma_* + f_0 \circ (\gamma_* - 1).$$

By (5) the second condition says that we may write $f_0 = \delta_e(\,\cdot\,)v + f \circ (\gamma_* - 1)$ for $v := f_0(\phi_1) \in V$ and some $f \in \mathrm{Hom}_k(C^\infty(G,k), V)$. Inserting this into the above equation we obtain

$$f_1 \circ (\gamma_* - 1) = \delta_e(\,\cdot\,)(\gamma(v) - v) + (\gamma \circ f \circ \gamma_* - f) \circ (\gamma_* - 1).$$

It follows that

$$\gamma(v) = v \quad \text{and} \quad f_1 = (\gamma \circ f \circ \gamma_* - f).$$

Using this last identity one checks that $(f_0, f_1) = d^{-1}f + (\delta_e(\,\cdot\,)v, 0)$. But we have $0 = d^0(\delta_e(\,\cdot\,)v, 0) = \delta_e(\gamma_* \cdot)(\gamma - 1)(v) + \delta_e((\gamma_* - 1)\cdot)v = \delta_e((\gamma_* - 1)\cdot)v$, which implies that $v = 0$. We conclude that $h^0(\beta^\bullet)$ is injective and hence bijective and that therefore $h^0(\alpha^\bullet)$ is bijective.

A differential graded $k[\epsilon]/(\epsilon^2)$-module is the same as a graded k-vector space with two anticommuting differentials ϵ and d of degree 1. Given the smooth G-representation V, we form the graded $k[\epsilon]/(\epsilon^2)$-module $k[\epsilon]/(\epsilon^2) \otimes_k V$ (sitting in degrees 0 and 1) and equip it with the differential $d_V(v_0 + v_1\epsilon) := (\gamma - 1)(v_0)\epsilon$. The above computations together with the fact that ϵ corresponds to the identity in $\mathcal{H}_I^1 = \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathcal{I}^0, \mathcal{I}^1) = \mathrm{End}_{\mathrm{Mod}_k(G)}(C^\infty(G,k))$ proves the following:

Proposition 12. *The equivalence (4) sends V in $\mathrm{Mod}_k(G)$ to the differential graded module $(k[\epsilon]/(\epsilon^2) \otimes_k V, d_V)$.*

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ON CRDAHA AND FINITE GENERAL LINEAR AND UNITARY GROUPS

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Dedicated to the memory of Robert Steinberg

We show a connection between Lusztig induction operators in finite general linear and unitary groups and parabolic induction in cyclotomic rational double affine Hecke algebras. Two applications are given: an explanation of a bijection result of Broué, Malle and Michel, and some results on modular decomposition numbers of finite general linear groups.

1. Introduction

Let Γ_n be the complex reflection group $G(e, 1, n)$, the wreath product of $\mathcal{S}_n$ and $\mathbb{Z}/e\mathbb{Z}$, where $e > 1$ is fixed for all n. Let $H(\Gamma_n)$ be the cyclotomic rational double affine Hecke algebra, or CRDAHA, associated with the complex reflection group Γ_n. The representation theory of the algebras $H(\Gamma_n)$ is related to the representation theory of the groups Γ_n, and thus to the modular representation theory of finite general linear groups $\mathrm{GL}(n, q)$ and unitary groups $U(n, q)$. In this paper we study this connection in the context of a recent paper of Shan and Vasserot [2012]. In particular we show a connection between Lusztig induction operators in general linear and unitary groups and certain operators in a Heisenberg algebra acting on a Fock space. We give two applications of this result, where ℓ is a prime not dividing q and e is the order of $q \bmod \ell$. The first is a connection via Fock spaces between an induction functor in CRDAHA described in [Shan and Vasserot 2012] and Lusztig induction, which gives an explanation for a bijection given by Broué, Malle and Michel [1993] and Enguehard [1992] between characters in an ℓ-block of a finite general linear, unitary or classical group and characters of a corresponding complex reflection group. The second is an application to the ℓ-modular theory of $\mathrm{GL}(n, q)$, describing some Brauer characters by Lusztig induction, for large ℓ.

The paper is organized as follows. In Section 3 we state the results on CRDAHA from [Shan and Vasserot 2012] that we need. We introduce the category $\mathcal{O}(\Gamma) = \bigoplus_{n\geq 0} \mathcal{O}(\Gamma_n)$ where $\mathcal{O}(\Gamma_n)$ is the category $\mathcal{O}$ of $H(\Gamma_n)$.

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In Section 4 we describe the ℓ-block theory of $\mathrm{GL}(n, q)$ and $U(n, q)$. The unipotent characters in a unipotent block are precisely the constituents of a Lusztig induced character from an e-split Levi subgroup. Complex reflection groups arise when considering the defect groups of the blocks.

In Section 5 we introduce the Fock space and the Heisenberg algebra, and describe the connection between parabolic induction in CRDAHA and a Heisenberg algebra action on a Fock space given in [Shan and Vasserot 2012]. We have a Fock space $\mathcal{F}^{(s)}_{m,\ell}$ where $m, \ell > 1$ are positive integers and (s) is an ℓ-tuple of integers. In [Shan and Vasserot 2012] a functor a^*_μ, where μ is a partition, is introduced on the Grothendieck group $[\mathcal{O}(\Gamma)]$ and is identified with an operator S_μ of a Heisenberg algebra on the above Fock space.

The case $\ell = 1$ is considered in Section 6. We consider a Fock space with a basis indexed by unipotent representations of general linear or unitary groups. We define the action of a Heisenberg algebra on this by a Lusztig induction operator $\mathcal{L}_\mu$ and prove that it can be identified with an operator S_μ defined by Leclerc and Thibon [1996]. This is one of the main results of the paper. It involves using a map introduced by Farahat [1954] on the characters of symmetric groups, which appears to be not widely known.

In Sections 7 and 8 we give applications of this result, using the results of Section 5. The first application is that parabolic induction a^*_μ in CRDAHA and Lusztig induction $\mathcal{L}_\mu$ on general linear or unitary groups can be regarded as operators arising from equivalent representations of the Heisenberg algebra. This gives an explanation for an observation of Broué, Malle and Michel on a bijection between Lusztig induced characters in a block of $\mathrm{GL}(n, q)$ and $U(n, q)$ and characters of a complex reflection group arising from the defect group of the block.

The second application deals with ℓ-decomposition numbers of the unipotent characters of $\mathrm{GL}(n, q)$ for large ℓ. Via the q-Schur algebra we can regard these numbers as arising from the coefficients of a canonical basis $G^-(\lambda)$ of a Fock space, where λ runs through all partitions, in terms of the standard basis. The $G^-(\lambda)$ then express the Brauer characters of $\mathrm{GL}(n, q)$ in terms of unipotent characters. The $G^-(\lambda)$ are also described in terms of the S_μ, and so we finally get that if $\lambda = \mu + e\alpha$ where μ' is e-regular, the Brauer character parametrized by λ is in fact a Lusztig induced generalized character.

2. Notation

We let $\mathcal{P}$, $\mathcal{P}_n$, $\mathcal{P}^\ell$, $\mathcal{P}^\ell_n$ denote the set of all partitions, the set of all partitions of $n \geq 0$, the set of all ℓ-tuples of partitions, and the set of all ℓ-tuples of partitions of integers $n_1, n_2, \ldots n_\ell$ such that $\sum n_i = n$, respectively.

If $\mathcal{C}$ is an abelian category, we write $[\mathcal{C}]$ for the complexified Grothendieck group of $\mathcal{C}$.

We write $\lambda \vdash n$ if λ is a partition of $n \geq 0$. The parts of λ are denoted by $\{\lambda_1, \lambda_2, \ldots\}$. If $\lambda = \{\lambda_i\}$, $\mu = \{\mu_i\}$ are partitions, then $\lambda + \mu = \{\lambda_i + \mu_i\}$ and $e\lambda = \{e\lambda_i\}$ where e is a positive integer.

3. CRDAHA, complex reflection groups

The main reference for this section is [Shan and Vasserot 2012]. We use the notation of Section 3.3 (page 967) of this paper.

Let $\Gamma_n = \mu_\ell \wr \mathcal{S}_n$, where μ_ℓ is the group of ℓ-th roots of unity in $\mathbb{C}$ and $\mathcal{S}_n$ is the symmetric group of degree n, so that Γ_n is a complex reflection group. The category of finite-dimensional complex representations of Γ_n is denoted by $\mathrm{Rep}(\mathbb{C}\Gamma_n)$. The irreducible modules in $\mathrm{Rep}(\mathbb{C}\Gamma_n)$ are known by a classical construction and denoted by $\bar{L}_\lambda$ where $\lambda \in \mathcal{P}_n^\ell$ (see for instance [Shan and Vasserot 2012, Equation (3.4), p. 968]). Let $R(\Gamma) = \bigoplus_{n \geq 0} [\mathrm{Rep}(\mathbb{C}\Gamma_n)]$.

Let $\mathfrak{H}$ be the reflection representation of Γ_n and $\mathfrak{H}^*$ its dual. The cyclotomic rational double affine Hecke algebra or CRDAHA associated with Γ_n is denoted by $H(\Gamma_n)$, and is the quotient of the smash product of $\mathbb{C}\Gamma_n$ and the tensor algebra of $\mathfrak{H} \oplus \mathfrak{H}^*$ by certain relations. The definition involves certain parameters (see [Shan and Vasserot 2012, p. 967]) which play a role in the results we quote from [Shan and Vasserot 2012], although we will not state them explicitly.

The category $\mathcal{O}$ of $H(\Gamma_n)$ is denoted by $\mathcal{O}(\Gamma_n)$. This is the category of $H(\Gamma_n)$-modules whose objects are finitely generated as $\mathbb{C}[\mathfrak{H}]$-modules and are $\mathfrak{H}$-locally nilpotent. Here $\mathbb{C}[\mathfrak{H}]$ is the subalgebra of $H(\Gamma_n)$ generated by $\mathfrak{H}^*$. Then $\mathcal{O}(\Gamma_n)$ is a highest weight category (see for instance [Rouquier et al. 2013]) and its standard modules are denoted by Δ_λ where $\lambda \in \mathcal{P}_n^\ell$. Let $\mathcal{O}(\Gamma) = \bigoplus_{n \geq 0} \mathcal{O}(\Gamma_n)$. This is one of the main objects of our study.

We then have a $\mathbb{C}$-linear isomorphism $\mathrm{spe} : [\mathrm{Rep}(\mathbb{C}\Gamma_n)] \to [\mathcal{O}(\Gamma_n)]$ given by $[\bar{L}_\lambda] \to [\Delta_\lambda]$. We will from now on consider $[\mathcal{O}(\Gamma_n)]$ instead of $[\mathrm{Rep}(\mathbb{C}\Gamma_n)]$.

Let $r, m, n \geq 0$. For n, r we have a parabolic subgroup $\Gamma_{n,r} \cong \Gamma_n \times \mathcal{S}_r$ of Γ_{n+r}, and there is a canonical equivalence of categories $\mathcal{O}(\Gamma_{n,r}) = \mathcal{O}(\Gamma_n) \otimes \mathcal{O}(\mathcal{S}_r)$ (for the tensor product of categories, see for instance [Deligne 1990, Section 5.1, Proposition 5.13]). By the work of Bezrukavnikov and Etingof [2009] there are induction and restriction functors

$$^{\mathcal{O}}\mathrm{Ind}_{n,r} : \mathcal{O}(\Gamma_n) \otimes \mathcal{O}(\mathcal{S}_r) \to \mathcal{O}(\Gamma_{n+r})$$

and

$$^{\mathcal{O}}\mathrm{Res}_{n,r} : \mathcal{O}(\Gamma_{n+r}) \to \mathcal{O}(\Gamma_n) \otimes \mathcal{O}(\mathcal{S}_r).$$

For $\mu \vdash r$, Shan and Vasserot [2012, Section 5.1] defined functors $A_{\mu,!}$, A_μ^*, $A_{\mu,*}$ on the bounded derived category $\mathcal{D}^b(\mathcal{O}(\Gamma))$.

Here we will be concerned with A^*_μ, defined as follows.

$$A^*_\mu : \mathcal{D}^b(\mathcal{O}(\Gamma_n)) \to \mathcal{D}^b(\mathcal{O}(\Gamma_{n+mr})), \qquad M \to {}^{\mathcal{O}}\mathrm{Ind}_{n,mr}(M \otimes L_{m\mu}) \tag{3-1}$$

Then a^*_μ is defined as the endomorphism of $[\mathcal{O}(\Gamma)]$ induced by A^*_μ.

4. Finite general linear and unitary groups

In this section we describe a connection between the block theory of $\mathrm{GL}(n,q)$ or $U(n,q)$, and complex reflection groups. This was first observed by Broué, Malle and Michel [1993] and Enguehard [1992] for arbitrary finite reductive groups.

Let $G_n = \mathrm{GL}(n,q)$ or $U(n,q)$. The unipotent characters of G_n are indexed by partitions of n. Using the description in [Broué et al. 1993, p. 45] we denote the character corresponding to $\lambda \vdash n$ of $\mathrm{GL}(n,q)$ or the character, up to sign, corresponding to $\lambda \vdash n$ of $U(n,q)$ as in [Fong and Srinivasan 1982] by χ_λ.

Let ℓ be a prime not dividing q and e the order of $q \bmod \ell$. The ℓ-modular representations of G_n have been studied by various authors (see for instance [Cabanes and Enguehard 2004]) since they were introduced in [Fong and Srinivasan 1982]. The partition of the unipotent characters of G_n into ℓ-blocks is described in the following theorem from [Fong and Srinivasan 1982]. This classification depends only on e, so we can refer to an ℓ-block as an e-block, e.g., in Section 7.

Theorem 4.1. *The unipotent characters χ_λ and χ_μ of G_n are in the same e-block if and only if the partitions λ and μ of n have the same e-core.*

There are subgroups of G_n called e-split Levi subgroups ([Cabanes and Enguehard 2004, p. 190]). In the case of $G_n = \mathrm{GL}(n,q)$ an e-split Levi subgroup L is of the form a product of smaller general linear groups over F_{q^e} and G_k with $k \le n$. In the case of $G_n = U(n,q)$, L is of the form a product of smaller general linear groups or of smaller unitary groups over F_{q^e} and G_k with $k \le n$. Then a pair (L, χ_λ) is an e-cuspidal pair if L is e-split of the form a product of copies of tori, all of order $q^e - 1$ in the case of $\mathrm{GL}(n,q)$, or all of orders $q^e - 1$, $q^{2e} - 1$ or $q^{e/2} + 1$ in the case of $U(n,q)$ and G_k, where G_k has an e-cuspidal unipotent character χ_λ ([Broué et al. 1993, p. 18, p. 27; Enguehard 1992, p. 42]). Here a character of L is e-cuspidal if it is not a constituent of a character obtained by Lusztig induction R^L_M from a proper e-split Levi subgroup M of L.

The unipotent blocks, i.e., blocks containing unipotent characters, are classified by e-cuspidal pairs up to G_n-conjugacy. Let B be a unipotent block corresponding to (L, χ_λ). Then if $\mu \vdash n$, $\chi_\mu \in B$ if and only if $\langle R^{G_n}_L(\chi_\lambda), \chi_\mu\rangle \neq 0$. As above, $R^{G_n}_L$ is Lusztig induction.

The defect group of a unipotent block is contained in $N_{G_n}(T)$ for a maximal torus T of G_n such that $N_{G_n}(T)/T$ is isomorphic to a complex reflection group

$W_{G_n}(L,\lambda) = \mathbb{Z}_e \wr S_k$ for some $k \geq 1$. Here $\mathbb{Z}_e = \mathbb{Z}/e\mathbb{Z}$. Thus the irreducible characters of $W_{G_n}(L,\lambda)$ are parametrized by $\mathcal{P}_k^e$.

Let B be a unipotent block of G_n and $W_{G_n}(L,\lambda)$ as above. We then have the following theorem due to Broué, Malle and Michel [1993, Section 3.2] and to Enguehard [1992, Theorem B].

Theorem 4.2 (Global to local bijection for G_n). *Let M be an e-split Levi subgroup containing L and let $W_M(L,\lambda)$ be defined as above for M. Let μ be a partition, and let I_L^M be the isometry mapping the character of $W_M(L,\lambda)$ parametrized by the e-quotient of μ to the unipotent character χ_μ of M (up to sign) which is a constituent of $R_M^{G_n}(\lambda)$. Then we have $R_M^{G_n} I_L^M = I_L^{G_n} \operatorname{Ind}_{W_M(L,\lambda)}^{W_{G_n}(L,\lambda)}$.*

The theorem is proved case by case for "generic groups", and thus for finite reductive groups. We have stated it only for G_n.

We state a refined version of the theorem involving CRDAHA and prove it in Section 7.

5. Heisenberg algebra, Fock space

Throughout this section we use the notation of [Shan and Vasserot 2012, Sections 4.2, 4.5, 4.6].

The affine Kac–Moody algebra $\widehat{\mathfrak{sl}_\ell}$ is generated by elements e_p, f_p for $p = 0, \dots, \ell-1$, satisfying Serre relations ([Shan and Vasserot 2012, Section 3.4]). We have $\widehat{\mathfrak{sl}_\ell} = \mathfrak{sl}_\ell \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\mathbf{1}$, where $\mathbf{1}$ is central.

The Heisenberg algebra is the Lie algebra $\mathfrak{H}$ generated by 1, b_r, b'_r for $r \geq 0$, with relations $[b'_r, b'_s] = [b_r, b_s] = 0$, $[b'_r, b_s] = r1\delta_{r,s}$ for $r, s \geq 0$ ([Shan and Vasserot 2012, Section 4.2]). In $U(\mathfrak{H})$ we then have elements $b_{r_1}, b_{r_2}, \dots$ with $\sum_i r_i = r$. If $\lambda \in \mathcal{P}$ we then have the element $b_\lambda = b_{\lambda_1} b_{\lambda_2} \dots$, and then for any symmetric function f the element b_f equals $\sum_{\lambda\in\mathcal{P}} z_\lambda^{-1} \langle P_\lambda, f\rangle b_\lambda$. Here P_λ is a power sum symmetric function and $z_\lambda = \prod_i i^{m_i} m_i!$ where m_i is the number of parts of λ equal to i. The scalar product $\langle\cdot,\cdot\rangle$ is the one used in symmetric functions, where the Schur functions form an orthonormal basis (see [Macdonald 1995]).

We now define Fock spaces $\mathcal{F}_m$, $\mathcal{F}_{m,\ell}^{(d)}$ and $\mathcal{F}_{m,\ell}^{(s)}$, where $m > 1$. Choose a basis $(\epsilon_1, \dots, \epsilon_m)$ of $\mathbb{C}^m$. If $d \in \mathbb{Z}$, let $\mathcal{F}_m^{(d)}$ be the space of elements of the form $u_{i_1} \wedge u_{i_2} \dots$ for $i_1 > i_2 \dots$, where $u_{i-jm} = \epsilon_i \otimes t^j$ with $i_k = d-k+1$ for $k \gg 0$. If we set $|\lambda, d\rangle = u_{i_1} \wedge u_{i_2} \dots$ for $i_k = \lambda_k + d - k + 1$, the elements $|\lambda, d\rangle$ with $\lambda \in \mathcal{P}$ form a basis of $\mathcal{F}_m^{(d)}$. The Fock space $\mathcal{F}_m$ is defined as the space of semi-infinite wedges of the $\mathbb{C}$-vector space $\mathbb{C}^m \otimes \mathbb{C}[t,t^{-1}]$, and we have $\mathcal{F}_m = \bigoplus_{d\in\mathbb{Z}} \mathcal{F}_m{}^{(d)}$. Then $\widehat{\mathfrak{sl}_m}$ acts on $\mathcal{F}_m{}^{(d)}$. This setup has been studied by Leclerc and Thibon [1996; 2000].

Similarly choose a basis $(\epsilon_1, \dots \epsilon_m)$ of $\mathbb{C}^m$ and a basis $(\epsilon'_1, \dots, \epsilon'_\ell)$ of $\mathbb{C}^\ell$.

The Fock space $\mathcal{F}_{m,\ell}$ of rank m and level ℓ is defined as the space of semi-infinite wedges, i.e., elements of the form $u_{i_1} \wedge u_{i_1} \dots$ with $i_1 > i_2 > \dots$, where the u_j are vectors in a $\mathbb{C}$-vector space $\mathbb{C}^m \otimes \mathbb{C}^\ell \otimes \mathbb{C}[z, z^{-1}]$ given by $u_{i+(j-1)m-km\ell} = \epsilon_i \otimes \epsilon'_j \otimes z^k$, with $i = 1, 2, \ldots m$, $j = 1, 2, \ldots \ell$, $k \in \mathbb{Z}$. Then $\widehat{\mathfrak{sl}_\ell}$, $\widehat{\mathfrak{sl}_m}$ and $\mathfrak{H}$ act on the space ([Shan and Vasserot 2012, Section 4.6]), and these actions are pairwise commuting.

Let $d \in \mathbb{Z}$. There is a space $\Lambda^{d+\infty/2}$ defined by Uglov [2000, Section 4.1]. This space has a basis which Uglov indexes by $\mathcal{P}$ or by pairs (λ, s) where $\lambda \in \mathcal{P}^m$ and $s = (s_p)$ is an m-tuple of integers with $\sum_p s_p = d$. There is a bijection between the two index sets given by $\lambda \to (\lambda^*, s)$ where λ^* is the m-quotient of λ and s is a particular labeling of the m-core of λ ([Uglov 2000, Sections 4.1, 4.2]).

We have a decomposition $\mathcal{F}_{m,\ell} = \bigoplus_{d \in \mathbb{Z}} \mathcal{F}^{(d)}_{m,\ell}$ defined using semi-infinite wedges, as in the case of $\mathcal{F}_m$. Then $\mathcal{F}^{(d)}_{m,\ell}$ can be identified with the space defined by Uglov.

There is a subspace $\mathcal{F}^{(s)}_{m,\ell}$ of $\mathcal{F}^{(d)}_{m,\ell}$, the Fock space associated with (s), which is a weight space for the $\widehat{\mathfrak{sl}_\ell}$ action ([Shan and Vasserot 2012, p. 982]). We have $\mathcal{F}^{(d)}_{m,\ell} = \bigoplus \mathcal{F}^{(s)}_{m,\ell}$, the sum of weight spaces. Here we can define a basis $\{|\lambda, s\rangle\}$ with $\lambda \in \mathcal{P}^\ell$ of ${\mathcal{F}_{m,\ell}}^{(s)}$. The spaces $\mathcal{F}^{(s)}_{m,\ell}$ were also studied by Uglov.

The endomorphism of $\mathbb{C}^m \otimes \mathbb{C}[t, t^{-1}]$ induced by multiplication by t^r gives rise to a linear operator b_r and its adjoint b'_r on ${\mathcal{F}_m}^{(d)}$, and thus to an action of $\mathfrak{H}$ on ${\mathcal{F}_m}^{(d)}$. We also have an action of $\mathfrak{H}$ by operators b_r, b'_r on ${\mathcal{F}_{m,\ell}}^{(s)}$, and this is the main result that we need ([Shan and Vasserot 2012, p. 982]).

We now choose a fixed ℓ-tuple s. With suitable parameters of $H(\Gamma_n)$ for each n, the $\mathbb{C}$-vector space $[\mathcal{O}(\Gamma)]$ is then canonically isomorphic to $\mathcal{F}^{(s)}_{m,\ell}$. We then have the following $\mathbb{C}$-linear isomorphisms ([Shan and Vasserot 2012, Equation (5.20), p. 990]):

$$\begin{aligned} [\mathcal{O}(\Gamma)] &\to R(\Gamma) \to \mathcal{F}^{(s)}_{m,\ell}, \\ \Delta_\lambda &\to \bar{L}_\lambda \to |\lambda, s\rangle. \end{aligned} \tag{5-1}$$

Consider the Fock space $\mathcal{F}^{(s)}_{m,\ell}$ with basis indexed by $\{|\lambda, s\rangle\}$ where $\lambda \in \mathcal{P}^\ell$. The element $b_{s_\mu} \in \mathfrak{H}$, i.e., b_f where $f = s_\mu$, a Schur function, acts by an operator S_μ on the space. The functor a^*_μ on $[\mathcal{O}(\Gamma)]$ (see Section 3) is now identified with S_μ by [Shan and Vasserot 2012, Proposition 5.13, p. 990].

Remark. The bijection between m-core partitions and the m-tuples (s) as above has been studied by combinatorialists (see for instance [Garvan et al. 1990]).

6. Fock space revisited

References for the combinatorial definitions in this section are [Leclerc and Thibon 1996; 2000]. Given a partition μ we introduce three operators on a Fock space: an operator S_μ defined by Leclerc and Thibon [1996], an operator $\mathcal{F}^*_\mu$ defined by Farahat [1954] on representations of the symmetric groups $\mathcal{S}_n$, and the operators $\mathcal{L}_\mu$ of Lusztig induction on G_n. The algebra of symmetric functions in $\{x_1, x_2, \ldots\}$ is denoted by Λ.

Integers ℓ, m were introduced in Section 5. For the rest of the paper we set $\ell = m = e$, where e is a positive integer which was used in the context of blocks of G_n. Thus $\Gamma_n = \mu_e \wr \mathcal{S}_n$.

First consider the space $\mathcal{F}_e^{(d)}$ where $d \in \mathbb{Z}$, with basis elements $\{|\lambda, d\rangle\}$ where $\lambda \in \mathcal{P}$. Leclerc and Thibon [1996] introduced elements in $U(\mathfrak{H})$ which we write in our previous notation as b_{h_ρ} and b_{s_μ}, acting as operators V_ρ and S_μ on $\mathcal{F}_e^{(d)}$ where $\rho, \mu \in \mathcal{P}$ and h_ρ is a homogeneous symmetric function. These operators have a combinatorial description as follows. Here we will write $|\lambda\rangle$ for $|\lambda, d\rangle$.

First they define commuting operators V_k for $k \geq 1$ on $\mathcal{F}_e^{(d)}$ defined by

$$V_k(|\lambda\rangle) = \sum_\mu (-1)^{-s(\mu/\lambda)} |\mu\rangle,$$

where the sum is over all μ such that μ/λ is a horizontal n-ribbon strip of weight k, and $s(\mu/\lambda)$ is the "spin" of the strip.

Remark. The minus sign in the exponent in the formula is not necessary, but appears because it is a special case of a quantized formula.

Here a ribbon is the same as a rim-hook, i.e., a skew partition which does not contain a 2×2 square. The spin is the leg length of the ribbon, i.e., the number of rows -1.

Definition. (see [Lam 2005]) A horizontal n-ribbon strip of weight k is a tiling of a skew partition by k n-ribbons such that the top rightmost square of every ribbon touches the northern edge of the shape. The spin of the strip is the sum of the spins of all the ribbons.

It can be shown that a tiling of a skew partition as above is unique. More generally we can then define V_ρ where ρ is a composition. If $\rho = \{\rho_1, \rho_2, \ldots\}$ then $V_\rho = V_{\rho_1}.V_{\rho_2} \ldots$. Finally we define operators S_μ acting on $\mathcal{F}_e^{(d)}$ which we connect to Lusztig induction. They coincide with the operators mentioned at the end of the last section.

Definition. We have $S_\mu = \sum_\rho \kappa_{\mu\rho} V_\rho$ where the $\kappa_{\mu\rho}$ are inverse Kostka numbers ([Leclerc and Thibon 1996, p. 204; Lam 2005, p. 8]).

Remark. Let $p_e(f)$ denote the plethysm by the power function in Λ, i.e.,

$$p_e(f(x_1, x_2, \ldots)) = f(x_1^e, x_2^e, \ldots).$$

(This is related to a Frobenius morphism; see [Leclerc and Thibon 2000, p. 171].) In fact in [Leclerc and Thibon 1996] $\mathfrak{H}$ is regarded as a $\mathbb{C}(v)$-space where v is an indeterminate. Then V_ρ and S_μ are v-analogs of multiplication by $p_e(h_\rho)$ and $p_e(s_\mu)$ in Λ.

Next, let $\mathcal{A}_n$ be the category of unipotent representations of G_n. Let $\mathcal{A} = \bigoplus_{n\geq 0}[\mathcal{A}_n]$. We recall from Section 4 that the unipotent characters of G_n are denoted by $\{\chi_\lambda\}$ where $\lambda \vdash n$. We now regard $\mathcal{A}$ as having a basis $[\chi_\lambda]$ where λ runs through all partitions. Then $\mathcal{A}$ is isomorphic to $\mathcal{F}_e^{(d)}$ as a $\mathbb{C}$-vector space, since $\mathcal{A}$ also has a basis indexed by partitions.

We now define Lusztig operators $\mathcal{L}_\mu$ on $\mathcal{A}$ and then relate them to the S_μ.

Definition. Let $\mu \vdash k$. The Lusztig map $\mathcal{L}_\mu : \mathcal{A} \to \mathcal{A}$ is as follows. Define $\mathcal{L}_\mu : [\mathcal{A}_n] \to [\mathcal{A}_{n+ke}]$ by $[\chi_\lambda] \to [R_L^{G_{n+ke}}(\chi_\lambda \times \chi_\mu)]$, where $L = G_n \times \mathrm{GL}(k, q^e)$ or $L = G_n \times U(k, q^e)$, an e-split Levi subgroup of G_{n+ke}.

Finally, consider the characters of $\mathcal{S}_n$. We denote the character corresponding to $\lambda \in \mathcal{P}_n$ as ϕ_λ. We also use $\lambda \in \mathcal{P}_n$ to denote representatives of conjugacy classes of $\mathcal{S}_n$. Let $\mathcal{C}_n$ be the category of representations of $\mathcal{S}_n$ and $\mathcal{C} = \bigoplus_{n\geq 0}[\mathcal{C}_n]$.

Given partitions $\nu \vdash (n+ke)$, $\lambda \vdash n$ such that ν/λ is defined, Farahat [1954] has defined a character $\hat{\phi}_{\nu/\lambda}$ of S_k, as follows. Let the e-tuples $(\nu^{(i)})$, $(\lambda^{(i)})$ be the e-quotients of ν and λ. Then $\epsilon \prod_i \phi_{(\nu^{(i)}/\lambda^{(i)})}$, where $\epsilon = \pm 1$ is a character of a Young subgroup of S_k, which induces up to the character $\hat{\phi}_{\nu/\lambda}$ of S_k.

We will instead use an approach of Enguehard ([1992, p. 37]) which is more conceptual and convenient for our purpose.

Definition. The Farahat map $\mathcal{F} : [\mathcal{C}_{ek}] \to [\mathcal{C}_k]$ is defined by $(\mathcal{F}\chi)(\mu) = \chi(e\mu)$, where $\mu \vdash k$.

Let $\mu \vdash k$. Taking adjoints and denoting $\mathcal{F}^*$ by $\mathcal{F}^*_\mu$ we then have, for $\lambda \vdash n$:

Definition. Define $\mathcal{F}^*_\mu : [\mathcal{C}_n] \to [\mathcal{C}_{n+ek}]$ by $\phi_\lambda \to \mathrm{Ind}_{\mathcal{S}_{ek}\times\mathcal{S}_n}^{\mathcal{S}_{n+ek}}(\mathcal{F}^*(\phi_\mu) \times \phi_\lambda)$.

By the standard classification of maximal tori in G_n we can denote a set of representatives of the G_n-conjugacy classes of the tori by $\{T_w\}$, where w runs over a set of representatives for the conjugacy classes of S_n. We then have that the unipotent character $\chi_\lambda = \frac{1}{|S_n|}\sum_{w\in S_n}\lambda(w)R_{T_w}^{G_n}(1)$ (see for instance [Fong and Srinivasan 1982, Equation (1.13)]). Here, as before, $R_{T_w}^{G_n}(1)$ is Lusztig induction.

We assume in the proposition below that when $G_n = U(n, q)$ that $e \equiv 0 \bmod 4$. This is the case that is analogous to the case of $\mathrm{GL}(n, q)$. The other cases for e require some straightforward modifications which we mention below. The proof of the proposition has been sketched by Enguehard ([1992, p. 37]) when $G_n = \mathrm{GL}(n, q)$.

Let M be the e-split Levi subgroup of G_n isomorphic to $\mathrm{GL}(k, q^e) \times GL_\ell$. We denote by ${}^*R_M^{G_n}$ the adjoint of the Lusztig map $R_M^{G_n}$. It is an analogue of the map $\mathcal{F}^*$, and this is made precise below.

If $\lambda \vdash n$, we have a bijection $\phi_\lambda \leftrightarrow \chi_\lambda$ between $[\mathcal{C}_n]$ and $[\mathcal{A}_n]$. We then have an obvious bijection $\psi : \phi_\lambda \leftrightarrow \chi_\lambda$ between $\mathcal{C}$ and $\mathcal{A}$.

Proposition. *Let $G = \mathrm{GL}(ek, q)$ or $U(ek, q)$. In the case of $U(ek, q)$ we assume $e \equiv 0 \bmod 4$. Let $M \cong \mathrm{GL}(k, q^e)$, a subgroup of G. Let $\psi : \phi_\lambda \leftrightarrow \chi_\lambda$ between $\mathcal{C}$ and $\mathcal{A}$ be as above. Then*:

(i) *If $\lambda \vdash ek$, then $\psi(\mathcal{F}(\phi_\lambda)) = {}^*R_M^G(\chi_\lambda)$.*

(ii) *If $\mu \vdash k$, then $\psi(\mathcal{F}^*(\phi_\mu)) = R_M^G(\chi_\mu)$.*

Proof. We have

$$\psi(\mathcal{F}(\phi_\lambda)) = \frac{1}{|\mathcal{S}_k|} \sum_{w \in \mathcal{S}_k} (\mathcal{F}\phi_\lambda)(w) R_{T_w}^M(1) = \frac{1}{|\mathcal{S}_k|} \sum_{w \in \mathcal{S}_k} \phi_\lambda(ew) R_{T_w}^M(1).$$

Since the torus parametrized by w in M is parametrized by ew in G, we can write this as $\frac{1}{|\mathcal{S}_k|} \sum_{w \in \mathcal{S}_k} \phi_\lambda(ew) R_{T_{ew}}^M(1)$.

On the other hand, we have (see [Fong and Srinivasan 1982, Lemma 2B]), using the parametrization of tori in M, ${}^*R_M^G(\chi_\lambda) = \frac{1}{|\mathcal{S}_k|} \sum_{w \in \mathcal{S}_k} \phi_\lambda(w) R_{T_w}^M(1)$. This proves (i). Then (ii) follows by taking adjoints. □

The proposition clearly generalizes to the subgroup $M \cong \mathrm{GL}(k, q^e) \times G_\ell$ of G_n where $n = ek + \ell$. In the case of $U(n, q)$, if e is odd we replace e by e' where $e' = 2e$ with $M \cong \mathrm{GL}(k, q^{e'})$, and if $e \equiv 2 \bmod 4$ by e' where $e' = e/2$ with $M \cong U(k, q^{e'})$, the proof being similar.

Using the isomorphisms between the spaces $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{F}_e^{(d)}$, we now regard the operators $\mathcal{L}_\mu$, $\mathcal{F}_\mu^*$ and $\mathcal{S}_\mu$ as acting on $\mathcal{F}_e^{(d)}$.

We now prove one of the main results in this paper.

Theorem 6.1. *The operators $\mathcal{L}_\mu$ and S_μ on $\mathcal{F}_e^{(d)}$ coincide.*

Proof. We note that $\mathcal{F}_\mu^* = \mathcal{L}_\mu$. This follows from the previous proposition, generalized to G_n, and the fact that parabolic induction in symmetric groups is compatible with Lusztig induction in G_n, using the combinatorial description of both functors. We will now show that $\mathcal{F}_\mu^* = S_\mu$.

More generally we consider the character $\hat{\phi}_{\nu/\lambda}$ of S_k defined by Farahat, where $\nu \vdash (n + ke)$ and $\mu \vdash n$, and describe it using $\mathcal{F}$. The restriction of ϕ_ν to $S_n \times S_{ke}$ can be written as a sum of $\phi_\lambda \times \phi_{\nu/\lambda}$ where $\phi_{\nu/\lambda}$ is a (reducible) character of S_{ke}, and characters not involving ϕ_λ. We then define $\hat{\phi}_{\nu/\lambda} = \mathcal{F}(\phi_{\nu/\lambda})$, a character of S_k. We then note that $\hat{\phi}_{\nu/\lambda}(u) = \phi_{\nu/\lambda}(eu)$. Using the characteristic map we get a corresponding skew symmetric function s_{ν^*/λ^*}. This is precisely the function which has been described in [Macdonald 1995, p. 91], since it is derived from the usual symmetric function $s_{\nu/\lambda}$ by taking e-th roots of variables. Using the plethysm function p_e and its adjoint ψ_e ([Lascoux et al. 1997, p. 1048]) we get $s_{\nu^*/\lambda^*} = \psi_e(s_{\nu/\lambda})$.

By the above facts we get

$$\begin{aligned}(\hat{\phi}_{\nu/\lambda}, \phi_\mu) &= (s_{\nu^*/\lambda^*}, s_\mu) = (\psi_e(s_{\nu/\lambda}), s_\mu)\\ &= (s_{\nu/\lambda}, p_e(s_\mu)) = (p_e(s_\mu).s_\lambda, s_\nu)\\ &= (S_\mu[\chi_\lambda], [\chi_\nu]).\end{aligned}$$

The last equality can be seen as follows. There is a $\mathbb{C}$-linear isomorphism between the algebra Λ and $\mathcal{F}_e^{(d)}$, since both have bases indexed by $\mathcal{P}$. Under this isomorphism multiplication by the symmetric function $p_e(s_\mu)$ on Λ corresponds to the operator S_μ on a Fock space (see [Leclerc and Thibon 1996, p. 6]).

This proves that $\mathcal{L}_\mu = S_\mu$. □

We recall that $\widehat{sl}_e$ acts on $\mathcal{F}_e^{(d)}$ and hence on $\mathcal{A}$.

Corollary. *The highest weight vectors $V_\rho\varnothing$ of the irreducible components of the $\widehat{sl}_e$-module $\mathcal{A}$* ([Lascoux et al. 1997, p. 1054]) *can be described by Lusztig induction.*

Remark. In fact Leclerc and Thibon also have a parameter q in their definition of S_μ, since they deal with a deformed Fock space. Thus S_μ can be regarded as a quantized version of a Lusztig operator $\mathcal{L}_\mu$.

Remark. In the notation of [Leclerc and Thibon 2000, p. 173] we have

$$(s_{\nu^*/\lambda^*}, s_\mu) = (s_{\nu_0/\lambda_0} s_{\nu_1/\lambda_1} \cdots s_{\nu_{e-1}/\lambda_{e-1}}, s_\mu) = c^\mu_{\nu/\lambda}.$$

In this equation, the $c^\mu_{\nu/\lambda}$ are Littlewood–Richardson coefficients. We now have $(\chi_\nu, R^{G_n}_M(\chi_\lambda \times \chi_\mu)) = \epsilon c^\mu_{\nu/\lambda}$, where $\epsilon = \pm 1$. In particular $c^{(k)}_{\nu/\lambda}$ is the number of tableaux of shape ν such that ν/λ is a horizontal e-ribbon of weight k. Thus the Lusztig operator $\mathcal{L}_k$ can be described in terms of e-ribbons of weight k, similar to the case of $k = 1$ which classically is described by e-hooks.

7. CRDAHA and Lusztig induction

The main reference for parabolic induction in this section is [Shan and Vasserot 2012].

In this section we show a connection between the parabolic induction functor a^*_μ on $[\mathcal{O}(\Gamma)]$ and the Lusztig induction functor $\mathcal{L}_\mu$ in $\mathcal{A}$ using Fock spaces. In particular this gives an explanation of the global to local bijection for G_n given in Theorem 4.2. This can be regarded as a local, block-theoretic version of Theorem 6.1.

As mentioned in Section 4, the unipotent characters χ_λ in an e-block of G_n are constituents of the Lusztig character $R^{G_n}_L(\lambda)$ where (L, λ) is an e-cuspidal pair. Up to sign, they are in bijection with the characters of $W_{G_n}(L, \lambda)$, and they all have the same e-core.

For our result we can assume $d=0$, which we do from now on. We set $\ell=m=e$ as in Section 6. We have spaces $\mathcal{F}_e^{(0)}$ and $\mathcal{F}_{e,e}^{(0)}=\bigoplus_s \mathcal{F}_{e,e}^{(s)}$ where $s=(s_p)$ is an e-tuple of integers with $\sum_p s_p=0$. We now fix such an s.

By [Shan and Vasserot 2012, Sections 6.17, 6.22, p. 1010] we have an $U(\mathfrak{H})$-isomorphism between $\mathcal{F}_e^{(0)}$ and $\mathcal{F}_{e,e}^{(0)}$. Let $\mathcal{F}_e^{(s)}$ be the inverse image of $\mathcal{F}_{e,e}^{(s)}$ under this isomorphism. We then have $\mathbb{C}$-isomorphisms from $\mathcal{F}_{e,e}^{(s)}$ to $[\mathcal{O}(\Gamma)]$, and from $\mathcal{F}_e^{(s)}$ to $\mathcal{A}^{(s)}$, where $\mathcal{A}^{(s)}$ is the subspace of $\mathcal{A}$ spanned by $[\chi_\lambda]$ where the χ_λ are in e-blocks parametrized by e-cores labeled by (s) (see Section 5).

The spaces $\mathcal{F}_e^{(s)}$, $\mathcal{F}_{e,e}^{(s)}$, $[\mathcal{O}(\Gamma)]$, $\mathcal{A}^{(s)}$ have bases $\{|\lambda,s\rangle:\lambda\in\mathcal{P}\}$, $\{|\lambda,s\rangle:\lambda\in\mathcal{P}^e\}$, $\{\Delta_\lambda:\lambda\in\mathcal{P}^e\}$ and $[\chi_\lambda]$ where λ has e-core labeled by s, respectively.

We have maps $S_\mu:\mathcal{F}_{e,e}^{(s)}\to\mathcal{F}_{e,e}^{(s)}$ for $\mu\in\mathcal{P}^e$, $S_\mu:\mathcal{F}_e^{(s)}\to\mathcal{F}_e^{(s)}$ for $\mu\in\mathcal{P}$, $\mathcal{L}_\mu:\mathcal{A}^{(s)}\to\mathcal{A}^{(s)}$ and $a_\mu^*:[\mathcal{O}(\Gamma)]\to[\mathcal{O}(\Gamma)]$.

Here we note that Lusztig induction preserves e-cores, and thus $\mathcal{L}_\mu$ fixes $\mathcal{A}^{(s)}$.

The following theorem can be regarded as a refined version of the global to local bijection of [Broué et al. 1993]. The case $e=1$ is due to Enguehard ([1992, p. 37]), where the proof is a direct verification of the theorem from the definition of the Farahat map $\mathcal{F}$ in $\mathcal{S}_n$ (see Section 6) and Lusztig induction in G_n.

Theorem 7.1. *Under the isomorphism $\mathcal{A}^{(s)}\cong[\mathcal{O}(\Gamma)]$ given by $[\chi_\lambda]\to[\Delta_{\lambda^*}]$ where λ^* is the e-quotient of λ, Lusztig induction $\mathcal{L}_\mu$ on $\mathcal{A}^{(s)}$ with $\mu\in\mathcal{P}$ corresponds to parabolic induction a_μ^* on $[\mathcal{O}(\Gamma)]$ with $\mu\in\mathcal{P}^e$.*

Proof. Consider the action of $b_{s_\mu}\in U(\mathfrak{H})$ on $\mathcal{F}_{e,e}^{(s)}$. The operator S_μ acting on $\mathcal{F}_{e,e}^{(s)}$ can be identified with a_μ^* acting on $[\mathcal{O}(\Gamma)]$, with the basis element $|\lambda,s\rangle$ corresponding to $[\Delta_\lambda]$ ([Shan and Vasserot 2012, Equation (5.20)]).

On the other hand, $b_{s_\mu}\in U(\mathfrak{H})$ acts as S_μ on the space $\mathcal{F}_e^{(s)}$ and thus, by Theorem 6.1 as $\mathcal{L}_\mu$ on $\mathcal{A}^{(s)}$ with the basis element $|\lambda,s\rangle$ corresponding to $[\chi_\lambda]$.

Now $\mathcal{F}_e^{(s)}$ is isomorphic to $\mathcal{A}^{(s)}$ and $\mathcal{F}_{e,e}^{(s)}$ is isomorphic to $[\mathcal{O}(\Gamma)]$. Thus we have shown that a_μ^* and $\mathcal{L}_\mu$ correspond under two equivalent representations of $U(\mathfrak{H})$. □

Corollary. *The BMM-bijection of Theorem 4.2 between the constituents of the Lusztig map $R_L^{G_n}(\lambda)$ where (L,λ) is an e-cuspidal pair and the characters of $W_{G_n}(L,\lambda)$ is described via equivalent representations of $U(\mathfrak{H})$ on Fock spaces.*

This follows from the theorem, using the map spe (see Section 3).

8. Decomposition numbers

References for this section are [Dipper and James 1989; Leclerc and Thibon 1996; 2000]. In this section we assume $G_n=\mathrm{GL}(n,q)$, since we will be using the connection with q-Schur algebras. We describe connections between weight spaces of $\widehat{\mathfrak{sl}_e}$ on Fock spaces, blocks of q-Schur algebras, and blocks of G_n. We show that some Brauer characters of G_n can be described by Lusztig induction.

The ℓ-decomposition numbers of the groups G_n have been studied by Dipper and James and by Geck, Gruber, Hiss and Malle. The latter have also studied the classical groups, using modular Harish-Chandra induction. One of the key ideas in these papers is to compare the decomposition matrices of the groups with those of q-Schur algebras.

We have the Dipper–James theory over a field of characteristic 0 or ℓ. Dipper and James define ([1989, Section 2.9]) the q-Schur algebra $\mathcal{S}_q(n)$, endomorphism algebra of a sum of permutation representations of the Hecke algebra $\mathcal{H}_n$ of type A_{n-1}.The unipotent characters and the ℓ-modular Brauer characters of G_n are both indexed by partitions of n (see [Fong and Srinivasan 1982]). Similarly the Weyl modules and the simple modules of $\mathcal{S}_q(n)$ are both indexed by partitions of n (see [Dipper and James 1989]).

For $\mathcal{S}_q(n)$ over k of characteristic ℓ, $q \in k$, one can define the decomposition matrix of $\mathcal{S}_q(n)$, where q is an e-th root of unity, where as before e is the order of q mod ℓ. By the above, this is a square matrix whose entries are the multiplicities of simple modules in Weyl modules. Dipper and James ([1989, Theorem 4.9]) showed that this matrix, up to reordering the rows and columns, is the same as the unipotent part of the ℓ-decomposition matrix of G_n, the transition matrix between the ordinary (complex) characters and the ℓ-modular Brauer characters. The rows and columns of the matrices are indexed by partitions of n.

We consider the Fock space $\mathcal{F} = \mathcal{F}_e{}^{(d)}$ for a fixed d, which as in Section 6 is isomorphic to $\mathcal{A}$, and has the standard basis $\{|\lambda\rangle : \lambda \in \mathcal{P}\}$. It also has two canonical bases $G^+(\lambda)$ and $G^-(\lambda)$ for $\lambda \in \mathcal{P}$ ([Leclerc and Thibon 1996; 2000]). There is a recursive algorithm to determine these two bases.

We fix an s as in Section 6. The algebra $\widehat{\mathfrak{sl}_e}$ acts on $\mathcal{F}_{e,e}^{(s)}$ and hence on $\mathcal{F}_e^{(s)}$. The connection between $\widehat{\mathfrak{sl}_e}$-weight spaces and blocks of the q-Schur algebras and hence blocks of $\mathrm{GL}(n, q)$ with $n \geq 0$ is known, and we describe it below. We denote the Weyl module of $\mathcal{S}_q(n)$ parametrized by λ by $W(\lambda)$.

We need to introduce a function res on $\mathcal{P}$. If $\lambda \in \mathcal{P}$, the e-residue of the (i, j)-node of the Young diagram of λ is the nonnegative integer r given by $r \equiv j - i \mod e$ for $0 \leq r < e$, denoted $\mathrm{res}_{i,j}(\lambda)$. Then $\mathrm{res}(\lambda) = \bigcup_{(i,j)}(\mathrm{res}_{i,j}(\lambda))$.

Proposition. *A weight space for $\widehat{\mathfrak{sl}_e}$ on $\mathcal{F}_e^{(s)}$ can be regarded as a union of blocks of q-Schur algebras with q a primitive e-th root of unity.*

Proof. The fact that res defines a weight space follows for instance from [Rouquier et al. 2013, p. 60]. Two Weyl modules $W(\lambda)$, $W(\mu)$ are in the same block if and only if $\mathrm{res}(\lambda) = \mathrm{res}(\mu)$ (see for instance [Mathas 2004, Theorem 5.5, (i) $\Leftrightarrow$ (iv)]. $\square$

Thus a weight space determines a set of partitions of a fixed $n \geq 0$.

Corollary. *A weight space for $\widehat{\mathfrak{sl}_e}$ on $\mathcal{F}_e^{(s)}$ can be regarded as a union of blocks of groups* $\mathrm{GL}(n, q)$, *where the n are determined from the weight space.*

We now have the following theorem which connects the ℓ-decomposition numbers of G_n with $n \geq 0$ with Fock spaces.

Theorem 8.1. *Let ϕ_μ be the Brauer character of G_n indexed by $\mu \in \mathcal{P}_n$. Let $\lambda \in \mathcal{P}_n$. Then, for large ℓ, $(\chi_\mu, \phi_\lambda) = (G^-(\lambda), |\mu\rangle)$.*

Proof. The decomposition matrix of $\mathcal{S}_q(n)$ over a field of characteristic 0, with q a root of unity, is known by Varagnolo–Vasserot [1999]. By their work the coefficients in the expansion of the $G^+(\lambda)$ in terms of the standard basis give the decomposition numbers for the algebras $\mathcal{S}_q(n)$ for $n \geq 0$, with q specialized at an e-th root of unity.

By an asymptotic argument of Geck [2001] we can pass from the decomposition matrices of q-Schur algebras in characteristic 0 to those in characteristic ℓ, where ℓ is large. Then by the Dipper–James theorem we can pass to the decomposition matrices of the groups G_n over a field of characteristic ℓ with q an e-th root of unity in the field.

Let D_n be the unipotent part of the ℓ-decomposition matrix of G_n and E_n its inverse transpose. Thus D_n has columns $G^+(\lambda)$ and E_n has rows given by $G^-(\lambda)$ (see [Leclerc and Thibon 1996, Section 4]). The rows of E_n also give the Brauer characters of G_n, in terms of unipotent characters. These two descriptions of the rows of E_n then give the result. □

The following analog of Steinberg's tensor product theorem is proved for the canonical basis $G^-(\lambda)$ in [Leclerc and Thibon 1996].

Theorem 8.2. *Let λ be a partition such that λ' is e-singular, so that $\lambda = \mu + e\alpha$ where μ' is e-regular. Then $G^-(\lambda) = S_\alpha G^-(\mu)$.*

We now show that the rows indexed by partitions λ as in the above theorem can be described by Lusztig induction. By replacing S_α by $\mathcal{L}_\alpha$ and using Theorem 6.1 it follows that in these cases, Lusztig-induced characters coincide with Brauer characters.

Theorem 8.3. *Let $\lambda = \mu + e\alpha$ where μ' is e-regular. Then, for sufficiently large ℓ, the Brauer character represented by $G^-(\lambda)$ is equal to the Lusztig generalized character $R_L^{G_n}(G^-(\mu) \times \chi_\alpha)$, where $L = G_m \times \mathrm{GL}(k, q^e)$, $n = m + ke$, and $\alpha \vdash k$.*

By using the BMM bijection, Theorem 4.2, we have the following corollary.

Corollary. *Let $\mu = \phi$, so that $\lambda = e\alpha$. Then the Brauer character represented by $G^-(\lambda)$ can be calculated from an induced character in a complex reflection group.*

Some tables giving the basis vectors $G^-(\lambda)$ for $e = 2$ are given in [Leclerc and Thibon 2000]. In our examples we use transpose partitions of the partitions in these tables, and rows instead of columns.

We first give an example of a weight space for $\widehat{sl}_e$, which is also a block for G_n, with $n=4$, $e=4$. This is an example of a decomposition matrix D for $n=4$, $e=4$. This matrix occurs in a paper of Ariki [2011] as a decomposition matrix of a q-Schur algebra.

$$\begin{pmatrix} 4|| & 1 & 0 & 0 & 0 \\ 31|| & 1 & 1 & 0 & 0 \\ 211|| & 0 & 1 & 1 & 0 \\ 1111|| & 0 & 0 & 1 & 1 \end{pmatrix}$$

The following example is to illustrate Theorem 8.3. It was calculated using a GAP [2015] program for decomposition matrices of q-Schur algebras. It is an example of the inverse of a decomposition matrix for $n=6$, $e=2$. Here ℓ is large, because of the comparison with q-Schur algebras.

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1
\end{pmatrix}$$

Here the rows are indexed as: $6, 51, 42, 41^2, 3^2, 31^3, 2^3, 2^21^2, 21^4, 1^6$. In the above matrix:

(1) The rows indexed by $1^6, 2^21^2, 3^2, 21^4, 41^2$ have interpretations as Brauer characters, in terms of $R_L^{G_n}$, with L an e-split Levi of the form $\mathrm{GL}(3, q^2)$ for $\lambda = 1^6, 2^21^2, 3^2$, of the form $\mathrm{GL}(2, q) \times \mathrm{GL}(2, q^2)$ for $\lambda = 21^4$, and of the form $\mathrm{GL}(4, q) \times \mathrm{GL}(1, q^2)$ for $\lambda = 41^2$.

(2) Put $L = \mathrm{GL}(3, q^2)$. Then:

   (a) the row indexed by 3^2 is $R_L^G(\chi_3) = \chi_{3^2} - \chi_{42} + \chi_{51} - \chi_6$,
   (b) the row indexed by 2^21^2 is $R_L^G(\chi_{21})$ and
   (c) the row indexed by 1^6 is $R_L^G(\chi_{1^3})$.

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WEIL REPRESENTATIONS OF FINITE GENERAL LINEAR GROUPS AND FINITE SPECIAL LINEAR GROUPS

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Dedicated to the memory of Professor R. Steinberg

Let q be a prime power and let G_n be the general linear group $\mathrm{GL}_n(q)$ or the special linear group $\mathrm{SL}_n(q)$, with $n \geq 4$. We prove two characterization theorems for the Weil representations of G_n in any characteristic coprime to q, one in terms of the restriction to a standard subgroup G_{n-1}, and another in terms of the restriction to a maximal parabolic subgroup of G_n.

1. Introduction

The so-called *Weil representations* were introduced by A. Weil [1964] for classical groups over local fields. Weil mentioned that the finite field case may be considered analogously. This was developed in detail by R. E. Howe [1973] and P. Gérardin [1977] for characteristic zero representations. The same representations, still in characteristic zero, were introduced independently by I. M. Isaacs [1973] and H. N. Ward [1972] for symplectic groups $\mathrm{Sp}_{2n}(q)$ with q odd, and by G. M. Seitz [1975] for unitary groups. (These representations for $\mathrm{Sp}_{2n}(p)$ were also constructed in [Bolt et al. 1961].) Weil representations of finite symplectic groups $\mathrm{Sp}_{2n}(q)$ with $2 \mid q$ were constructed by R. M. Guralnick and the author in [Guralnick and Tiep 2004]. Weil representations attract much attention because of their many interesting features; see, for instance, [Dummigan 1996; Dummigan and Tiep 1999; Gow 1989; Gross 1990; Scharlau and Tiep 1997; 1999; Tiep 1997a; 1997b, Zalesski 1988].

The construction of the Weil representations may be found in [Howe 1973; Gérardin 1977; Guralnick and Tiep 2004; Seitz 1975], etc. In particular, in the case of general and special linear groups, they can be constructed as follows. Let $W = \mathbb{F}_q^n$ with $n \geq 3$, and let $\tilde{\zeta} \in \mathbb{C}$, respectively $\zeta \in \mathbb{F}_q$, be a fixed primitive $(q-1)$-th root of unity. Then $\mathrm{SL}(W) = \mathrm{SL}_n(q)$ has $q-1$ complex Weil representations, which are

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the nontrivial irreducible constituents of the permutation representation of $\mathrm{SL}(W)$ on $W\setminus\{0\}$. The characters of these representations are $\tau_{n,q}^{i}$, where $0\le i\le q-2$ and

$$\tau_{n,q}^{i}(g)=\frac{1}{q-1}\sum_{k=0}^{q-2}\tilde{\xi}^{ik}q^{\dim\mathrm{Ker}(g-\xi^{k}\cdot 1_W)}-2\delta_{0,i}. \tag{1-1}$$

Similarly, $\mathrm{GL}(W)=\mathrm{GL}_n(q)$ has $(q-1)^2$ complex Weil representations, which are the $q-1$ nontrivial irreducible constituents of the permutation representation of $\mathrm{GL}(W)$ on $W\setminus\{0\}$, tensored with one of the $q-1$ representations of degree 1 of $\mathrm{GL}_n(q)$. If we fix a character α of order $q-1$ of $\mathrm{GL}_n(q)$, then the characters of these representations are $\tau_{n,q}^{i,j}$, where $0\le i,j\le q-2$ and

$$\tau_{n,q}^{i,j}(g)=\left(\frac{1}{q-1}\sum_{k=0}^{q-2}\tilde{\xi}^{ij}q^{\dim\mathrm{Ker}(g-\xi^{k}\cdot 1_W)}-2\delta_{0,i}\right)\cdot\alpha^{j}(g). \tag{1-2}$$

Note that $\tau_{n,q}^{i,j}$ restricts to $\tau_{n,q}^{i}$ over $\mathrm{SL}_n(q)$.

From now on, let us fix a prime p, a power q of p, and an algebraically closed field $\mathbb{F}$ of characteristic ℓ not equal to p. If G is a finite (general or special) linear group, unitary group, or symplectic group, then by a *Weil representation* of G over $\mathbb{F}$, we mean any composition factor of degree >1 of a reduction modulo ℓ of a complex Weil representation of G. As it turns out, another important feature of the Weil representations is that, with very few small exceptions, Weil representations are precisely the irreducible $\mathbb{F}G$-representations of the first few smallest degrees (larger than 1); see [Brundan and Kleshchev 2000; Guralnick et al. 2002; 2006, Guralnick and Tiep 1999, 2004, Hiss and Malle 2001; Tiep and Zalesski 1996]. Aside from this characterization by degree, Weil representations can also be recognized by various conditions imposed on their restrictions to standard subgroups, or parabolic subgroups. This was done in the case where G is a symplectic group or a unitary group, in [Tiep and Zalesski 1997] for complex representations and in [Guralnick et al. 2002, 2006, Guralnick and Tiep 2004] for modular representations (in cross characteristics). However, the case where G is a general or special linear group has not been treated. Perhaps one of the reasons for the absence until now of such characterizations (as regards the restriction to a parabolic subgroup) is that the obvious analogue of [Guralnick et al. 2002, Corollary 12.4] fails in this case; see Example 4.1.

We now fix $W=\mathbb{F}_q^n$ with a basis $(e_1,\dots,e_n)$, and consider $G_n=G=\mathrm{GL}(W)$ or $\mathrm{SL}(W)$. By a *standard subgroup* G_m in G_n, where $1\le m\le n-1$, we mean (any G-conjugate of) the subgroup

$$G_m=\mathrm{Stab}_{G_n}\big(\langle e_1,\dots,e_m\rangle_{\mathbb{F}_q},e_{m+1},\dots,e_n\big).$$

Next, we fix a primitive p-th root of unity $\epsilon\in\mathbb{C}$. A maximal parabolic subgroup of G is conjugate to

$$P=\mathrm{Stab}_G\big(\langle e_1,\dots,e_k\rangle_{\mathbb{F}_q}\big)$$

for some $1 \le k \le n-1$. The unipotent radical Q of P is

$$\left\{ [X] := \begin{pmatrix} I_k & X \\ 0 & I_{n-k} \end{pmatrix} \;\middle|\; X \in M_{k,n-k}(\mathbb{F}_q) \right\},$$

where $M_{a,b}(\mathbb{F}_q)$ is the space of all $a \times b$ matrices over $\mathbb{F}_q$. Any irreducible Brauer character of Q can then be written in the form

$$\beta_Y : [X] \mapsto \epsilon^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} \mathrm{tr}(XY)}$$

for a unique $Y \in M_{n-k,k}(\mathbb{F}_q)$. We define the *rank* of β_Y to be the rank of the matrix Y.

The main results of the paper are the following theorems, which characterize the Weil representations of finite general and special linear groups in terms of their restrictions to a standard or maximal parabolic subgroup.

Theorem A. *Let q be a prime power, $n \ge 5$, and for any integer $m \ge 4$, let G_m denote the general linear group $\mathrm{GL}_m(q)$ or the special linear group $\mathrm{SL}_m(q)$. Consider the standard embedding of G_m in G_n when $n > m$. Let $\mathbb{F}$ be an algebraically closed field of characteristic zero or characteristic coprime to q. Then for any finite-dimensional irreducible $\mathbb{F}G_n$-representation Φ, the following statements are equivalent*:

(i) *Either* $\deg \Phi = 1$, *or Φ is a Weil representation of G_n.*

(ii) *Φ has property $(\mathcal{W})$; that is, if Ψ is any composition factor of the restriction $\Phi|_{G_{n-1}}$, then either* $\deg \Psi = 1$, *or Ψ is a Weil representation of G_{n-1}.*

(iii) *For some m with $4 \le m \le n-1$, every composition factor of the restriction $\Phi|_{G_m}$ either is a Weil representation or has degree* 1.

(iv) *For every m with $4 \le m \le n-1$, every composition factor of the restriction $\Phi|_{G_m}$ either is a Weil representation or has degree* 1.

Theorem B. *Let q be a prime power, $n \ge 4$, and let G denote the general linear group $\mathrm{GL}_n(q)$ or the special linear group $\mathrm{SL}_n(q)$. Let $\mathbb{F}$ be an algebraically closed field of characteristic zero or characteristic coprime to q. Let $W = \mathbb{F}_q^n$ denote the natural G-module, and let P_k be the stabilizer of a k-dimensional subspace of W in G, where $2 \le k \le n-2$. Then for any finite-dimensional irreducible $\mathbb{F}G_n$-representation Φ, the following statements are equivalent*:

(i) *Either* $\deg \Phi = 1$, *or Φ is a Weil representation of G_n.*

(ii) *Φ has property $(\mathcal{P}_k)$; that is, the restriction $\Phi|_Q$ to the unipotent radical Q of P_k contains only irreducible $\mathbb{F}Q$-representations of rank* ≤ 1.

Note that the definitions of properties $(\mathcal{W})$ and $(\mathcal{P}_k)$ do not depend on the choice of the particular standard or parabolic subgroup. Our subsequent proofs also make use of another local property $(\mathcal{Z})$, which is defined by the condition (2-1) in Section 2.

Theorem B has already been used by A. E. Zalesski [≥ 2015] in his recent work.

Throughout the paper, we will say that an ordinary or Brauer character φ of $\mathrm{GL}_n(q)$ or $\mathrm{SL}_n(q)$ has property $(\mathcal{W})$, or $(\mathcal{P}_k)$, if a representation affording φ does as well. The notation $\mathrm{IBr}(X)$ denotes the set of irreducible Brauer characters of a finite group X in characteristic ℓ. If Y is a subgroup of a finite group X, and Φ is an $\mathbb{F}X$-representation and Ψ is an $\mathbb{F}Y$-representation, then $\Phi|_Y$ is the restriction of Φ to Y, and Ψ^Y is the $\mathbb{F}X$-representation induced from Ψ, with similar notation for ordinary and Brauer characters, as well as for modules. If χ is a complex character of X, then χ° denotes the restriction of χ to the set of ℓ'-elements in X.

2. Local properties and Weil representations

A key ingredient of our inductive approach is the following statement:

Proposition 2.1. *Let $G = G_n = \mathrm{GL}_n(q)$ with $n \geq 5$ and let Φ be an $\mathbb{F}G$-representation. Let $K = G_4$ be a standard subgroup of G. Then the following statements hold.*

(i) *If Φ has property $(\mathcal{P}_k)$ for some $2 \leq k \leq n-2$, then the $\mathbb{F}K$-representation $\Phi|_K$ has property $(\mathcal{P}_2)$.*

(ii) *If $\Phi|_K$ has property $(\mathcal{P}_2)$, then Φ has property $(\mathcal{P}_k)$ for all $2 \leq k \leq n-2$.*

Proof. Consider

$$P = \mathrm{Stab}_G\big(\langle e_1, \dots, e_k\rangle_{\mathbb{F}_q}\big),$$
$$K = \mathrm{Stab}_G\big(\langle e_1, e_2, e_{k+1}, e_{k+2}\rangle_{\mathbb{F}_q}, e_3, \dots, e_k, e_{k+3}, \dots, e_n\big).$$

Then $P_2 := P \cap K$ plays the role of the second parabolic subgroup of K, the stabilizer in K of the plane $\langle e_1, e_2\rangle_{\mathbb{F}_q}$, with unipotent radical $Q_2 := Q \cap K$. Let φ denote the Brauer character of Φ.

(i) Suppose that Φ has property $(\mathcal{P}_k)$. Consider any irreducible constituent $\beta = \beta_Y$ of $\varphi|_Q$. By the assumption, $\mathrm{rank}(Y) \leq 1$. Writing

$$Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix},$$

where Y_1 is 2×2, it is easy to see that the restriction $\beta|_{Q_2}$ is just the character β_{Y_1} of Q_2. Certainly, $\mathrm{rank}(Y_1) \leq \mathrm{rank}(Y)$. It follows that $\Phi|_K$ has property $(\mathcal{P}_2)$.

(ii) Suppose that Φ does not possess property $(\mathcal{P}_k)$ for some $2 \leq k \leq n-2$. Then we can find an irreducible constituent $\beta = \beta_Y$ of $\varphi|_Q$, where $\mathrm{rank}(Y) =: r \geq 2$. Note that the conjugation by the element $\mathrm{diag}(A, B)$ in the Levi subgroup

$$L = \mathrm{GL}_k(q) \times \mathrm{GL}_{n-k}(q)$$

of P sends β_Y to $\beta_{B^{-1}YA}$. Replacing β by a suitable L-conjugate, we may assume that the principal $r \times r$ submatrix of Y is the identity matrix I_r. Now, in the notation of (i), we have $\beta|_{Q_2} = \beta_{Y_1}$, where $Y_1 = I_2$. But this violates property $(\mathcal{P}_2)$ for $\Phi|_K$. $\square$

Corollary 2.2. *Let $G = G_n = \mathrm{GL}_n(q)$ with $n \geq 5$ and let Φ be an $\mathbb{F}G$-representation. Let $H = G_m$ be a standard subgroup of G for some $4 \leq m \leq n-1$. Then the following statements hold.*

(i) *If Φ has property $(\mathcal{P}_k)$ for some $2 \leq k \leq n-2$, then the $\mathbb{F}H$-representation $\Phi|_H$ has property $(\mathcal{P}_j)$ for all $2 \leq j \leq m-2$.*

(ii) *If Φ_H has property $(\mathcal{P}_j)$ for some $2 \leq j \leq m-2$, then Φ has property $(\mathcal{P}_k)$ for all $2 \leq k \leq n-2$.*

Proof. Consider a standard subgroup $K = G_4$ of H.

(i) By applying Proposition 2.1(i) to Φ, $\Phi|_K$ has property $(\mathcal{P}_2)$. Hence, by applying Proposition 2.1(ii) to $\Phi|_H$, $\Phi|_H$ has property $(\mathcal{P}_j)$ for all $2 \leq j \leq m-2$.

(ii) By applying Proposition 2.1(i) to $\Phi|_H$, $\Phi|_K$ has property $(\mathcal{P}_2)$. Hence, by applying Proposition 2.1(ii) to Φ, Φ has property $(\mathcal{P}_k)$ for all $2 \leq k \leq n-2$. $\square$

We will also fix the following elements in $\mathrm{GL}_n(q)$:

$$\boldsymbol{x} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}, \quad \boldsymbol{y} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-4}, \quad \boldsymbol{z} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-3}.$$

If φ is a Brauer character of G, we define

$$\varphi[1] := \varphi(1) - (q+1)\varphi(\boldsymbol{x}) + q\varphi(\boldsymbol{y}), \quad \varphi[2] := \varphi(\boldsymbol{y}) - \varphi(\boldsymbol{z}).$$

We will furthermore say that φ (or any representation affording it) has *property* $(\mathcal{Z})$ if

$$\varphi[1] = \varphi[2] = 0. \tag{2-1}$$

Corollary 2.3. *Let $G = \mathrm{GL}_n(q)$ or $\mathrm{SL}_n(q)$ with $n \geq 4$ and let Φ be a Weil representation of G over $\mathbb{F}$. Then Φ has property $(\mathcal{P}_k)$ for all $2 \leq k \leq n-2$. If $n \geq 5$, then Φ has property $(\mathcal{W})$.*

Proof. It suffices to prove the statement in the case where $\mathbb{F} = \mathbb{C}$ and furthermore $G = \mathrm{GL}_n(q)$, as $\tau_{n,q}^{i,j}$ restricts to $\tau_{n,q}^{i}$ over $\mathrm{SL}_n(q)$.

(i) First we consider the case $G = GL_4(q)$ and consider the parabolic subgroup $P = \mathrm{Stab}_G(\langle e_1, e_2 \rangle_{\mathbb{F}_q})$, with unipotent radical Q. It is easy to check that $\mathrm{IBr}(Q)$ consists of three P-orbits: $\{1_Q\}$, $\mathcal{O}_1$ of characters of rank 1, and $\mathcal{O}_2$ of characters of rank 2; moreover,

$$|\mathcal{O}_1| = (q^2-1)(q+1), \quad |\mathcal{O}_2| = (q^2-1)(q^2-q).$$

If $q \geq 3$, then $|\mathcal{O}_2| > \tau_{4,q}^{i,j}(1)$ for all i, j, whence $(\tau_{4,q}^{i,j})|_Q$ can afford only characters of rank ≤ 1, and so we are done.

Suppose that $q = 2$, in which case there is only one complex Weil character $\chi = \tau_{4,2}^{1,1}$ of degree 14. We also consider the irreducible complex character ρ of $G = \mathrm{GL}_4(2) \cong \mathsf{A}_8$ of degree 7. As $|\mathcal{O}_1| = 9$, $|\mathcal{O}_2| = 6$, and $Q \not\leq \mathrm{Ker}(\rho)$, we must have that

$$\rho|_Q = 1_Q + \sum_{\lambda \in \mathcal{O}_2} \lambda.$$

For the aforementioned involutions $\boldsymbol{x}$, $\boldsymbol{y}$, we have that

$$\chi(\boldsymbol{x}) = 6, \quad \chi(\boldsymbol{y}) = 2,$$

whence $\boldsymbol{x}$ belongs to class $2A$ and $\boldsymbol{y}$ belongs to class $2B$ in the notation of [Conway et al. 1985]. It follows that

$$\rho(\boldsymbol{x}) = -1, \quad \rho(\boldsymbol{y}) = 3,$$

and so

$$\sum_{\lambda \in \mathcal{O}_2} \lambda(\boldsymbol{x}) = -2, \quad \sum_{\lambda \in \mathcal{O}_2} \lambda(\boldsymbol{y}) = 2, \quad \sum_{\lambda \in \mathcal{O}_1} \lambda(\boldsymbol{x}) = 1, \quad \sum_{\lambda \in \mathcal{O}_1} \lambda(\boldsymbol{x}) = -3.$$

Since

$$\chi|_Q = a \cdot 1_Q + b \sum_{\lambda \in \mathcal{O}_1} \lambda + c \sum_{\lambda \in \mathcal{O}_2} \lambda$$

for some nonnegative integers a, b, c, we conclude that $(a, b, c) = (5, 1, 0)$, i.e., χ has property $(\mathcal{P}_2)$, as desired.

(ii) Now we consider the general case of $G = \mathrm{GL}_n(q)$ with $n \geq 5$, and consider a standard subgroup $H = \mathrm{GL}_4(q)$ and a standard subgroup $L = \mathrm{GL}_{n-1}(q)$ in G. Let τ_n denote the permutation character of G on $\mathbb{F}_q^n$, so that

$$\tau_n = \sum_{i=0}^{q-2} \tau_{n,q}^{i,0} + 2 \cdot 1_G.$$

Note that $(\tau_n)|_L = q\tau_{n-1}$, and so Φ has property $(\mathcal{W})$. Similarly, $(\tau_n)|_H = q^{n-4}\tau_4$. Furthermore, according to (i), τ_4 has property $(\mathcal{P}_2)$. It follows that $(\tau_n)|_H$ also has property $(\mathcal{P}_2)$, and so, by Corollary 2.2(ii), τ_n has property $(\mathcal{P}_k)$. Consequently, $\tau_{n,q}^{i,0}$ and $\tau_{n,q}^{i,j}$ also possess property $(\mathcal{P}_k)$. ☐

Lemma 2.4. *Let $G = \mathrm{GL}_n(q)$ or $\mathrm{SL}_n(q)$, where $n \geq 4$, and let Φ be a Weil representation of G over $\mathbb{F}$. Then Φ has property $(\mathcal{Z})$.*

Proof. Let φ be the Brauer character of Φ. It is well known, see [Guralnick and Tiep 1999] for instance, that φ is a linear combination of the reduction modulo ℓ of some $\chi = \tau_{n,q}^{i,j}$ or $\tau_{n,q}^{i}$ (note that such reductions need not be irreducible) and a linear character of G. As any linear character has property $(\mathcal{Z})$, it suffices to show that χ has property $(\mathcal{Z})$. According to (1-2), we have

$$\chi(1)=\frac{q^n-1}{q-1}-\delta_{0,i},\quad \chi(x)=\frac{q^{n-1}-1}{q-1}-\delta_{0,i},\quad \chi(y)=\chi(z)=\frac{q^{n-2}-1}{q-1}-\delta_{0,i},$$

which implies property $(\mathcal{Z})$ for χ. $\square$

Proposition 2.5. *Let $G=\mathrm{GL}_n(q)$ and $S=\mathrm{SL}_n(q)\leq G$ with $n\geq 4$. Let Φ be an $\mathbb{F}G$-representation and let Ψ be an $\mathbb{F}S$-representation. Also, let $\mathcal{P}=(\mathcal{P}_k)$ for some $2\leq k\leq n-2$, or $\mathcal{P}=(\mathcal{W})$.*

(i) *Φ has property $\mathcal{P}$ if and only if $\Phi|_S$ has property $\mathcal{P}$.*

(ii) *Ψ has property $\mathcal{P}$ if and only if Ψ^G has property $\mathcal{P}$.*

Proof. (a) First we consider the case $\mathcal{P}=(\mathcal{P}_k)$ and let P be the stabilizer in G of a k-space in the natural module $W=\mathbb{F}_q^n$, with unipotent radical Q. Note that $Q<P\cap S$. Furthermore, if $\mathcal{O}_1$ denotes the set of all Brauer irreducible characters of Q of rank 1, then $\mathcal{O}_1$ forms a single P-orbit and also a single $P\cap S$-orbit.

It is clear that Φ has property $(\mathcal{P}_k)$ if and only if $\Phi|_S$ has property $(\mathcal{P}_k)$, since $Q<S$. It is also clear that Ψ has property $(\mathcal{P}_k)$ whenever Ψ^G has property $(\mathcal{P}_k)$, since $\Psi|_Q$ is a subquotient of $(\Psi^G)|_Q$. Assume now that Ψ has property $(\mathcal{P}_k)$ and affords the Brauer character ψ. Then, by the aforementioned discussion, $\psi|_Q=a\cdot 1_Q+b\gamma$ for some integers $a,b\geq 0$ and $\gamma:=\sum_{\lambda\in\mathcal{O}_1}\lambda$. Note that we can find a cyclic subgroup $C\cong C_{q-1}$ of P such that $G=S\rtimes C$, and again by the aforementioned discussion, C preserves γ. As

$$(\psi^G)|_Q=\sum_{c\in C}(\psi^c)|_Q,$$

we conclude that $(\psi^G)|_Q=(q-1)(a\cdot 1_Q+b\gamma)$, and so Ψ^G has property $(\mathcal{P}_k)$.
(b) Next we consider the case $\mathcal{P}=(\mathcal{W})$ and let $H\cong\mathrm{GL}_{n-1}(q)$ be a standard subgroup of G, so that $H\cap S\cong\mathrm{SL}_{n-1}(q)$ is also a standard subgroup of S. We already mentioned that Weil representations of H restrict irreducibly to Weil representations of $H\cap S$. In particular, if Φ has property $(\mathcal{W})$ then so does $\Phi|_S$. Similarly, as the composition factors of Ψ are among the composition factors of $(\Psi^G)|_S$, if Ψ^G has property $(\mathcal{W})$ then so does Ψ.

Conversely, observe that

$$(\tau^i_{m,q})^G=\sum_{j=0}^{q-2}\tau^{i,j}_{m,q}.$$

Hence, if Θ is a Weil representation of $\mathrm{SL}_m(q)$ and $m\geq 3$, then every composition factor of $\Theta^{\mathrm{GL}_m(q)}$ is a Weil representation of $\mathrm{GL}_m(q)$ or a representation of degree 1. Moreover, we can find a cyclic subgroup $C\cong C_{q-1}$ of H such that $G=S\rtimes C$ and

$H = (H \cap S) \rtimes C$. It follows that if Ψ has property $(\mathcal{W})$ then so does Ψ^G. Finally, as the composition factors of Φ are among the composition factors of $(\Phi|_S)^G$, if $\Phi|_S$ has property $(\mathcal{W})$ then so do $(\Phi|_S)^G$ and Φ. $\square$

Corollary 2.6. *Let $G = \mathrm{GL}_n(q)$ or $\mathrm{SL}_n(q)$ and let Φ be an $\mathbb{F}G$-representation.*

(i) *If Φ possesses property $(\mathcal{W})$ and $n \geq 5$, then Φ has property $(\mathcal{Z})$.*

(ii) *If Φ has property $(\mathcal{P}_k)$ for some $2 \leq k \leq n-2$ and $n \geq 4$, then the Brauer character φ of Φ satisfies $\varphi[1] = 0$.*

(iii) *Suppose that either the assumption of* (i) *or of* (ii) *holds. If P_1 is the stabilizer in G of a 1-space of the natural module $W = \mathbb{F}_q^n$ of G and V is an $\mathbb{F}G$-module affording Φ, then $\boldsymbol{C}_V(Q_1) \neq 0$ for $Q_1 := \boldsymbol{O}_p(P_1)$.*

Proof. (i) Suppose that $n \geq 5$ and Φ has property $(\mathcal{W})$. Then we can choose $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ from a standard subgroup $H = \mathrm{GL}_{n-1}(q)$ or $\mathrm{SL}_{n-1}(q)$ of G. By Lemma 2.4, $\Phi|_H$ has property $(\mathcal{Z})$, and so does Φ.

(ii) Suppose now that $n \geq 4$ and Φ has property $(\mathcal{P}_k)$. Then we can choose $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ from a standard subgroup $H = \mathrm{GL}_4(q)$ or $\mathrm{SL}_4(q)$ of G. First we consider the case $n = 4$, so that $k = 2$, and let $\chi := \tau_{4,q}^{0,0} \in \mathrm{Irr}(H)$; in particular, $\chi(1) = (q^4 - q)/(q-1)$. By Lemma 2.4, χ has property $(\mathcal{Z})$. On the other hand, in the notation of the proof of Corollary 2.3, it follows from the arguments in that proof that

$$\chi|_Q = \sum_{\lambda \in \mathcal{O}_1} \lambda + (2q+1) \cdot 1_Q.$$

Hence, choosing $\boldsymbol{x}, \boldsymbol{y} \in Q$, we see that

$$\alpha[1] = \gamma[1] = 0 \tag{2-2}$$

for $\alpha := 1_Q$ and $\gamma := \sum_{\lambda \in \mathcal{O}_1} \lambda$. Also note that P always acts transitively on $\mathcal{O}_1$, no matter if $H = \mathrm{GL}_4(q)$ or $\mathrm{SL}_4(q)$. It follows that $\varphi|_Q = a\alpha + b\gamma$ for some integers $a, b \geq 0$, and so (2-2) implies that $\varphi[1] = 0$.

Now we consider the case $n \geq 5$. If $G = \mathrm{GL}_n(q)$, then by Proposition 2.1(i), $\Phi|_H$ has property $(\mathcal{P}_2)$ and so $\varphi[1] = 0$ by the case $n = 4$. Suppose now that $G = \mathrm{SL}_n(q)$ and set $M = \mathrm{GL}_n(q) \geq G$. We can choose a standard subgroup $L \cong \mathrm{GL}_4(q)$ in M such that $L \cap G = H \cong \mathrm{SL}_4(q)$. By Proposition 2.5(ii) applied to (M, G), Φ^M has property $(\mathcal{P}_k)$, and so $(\Phi^M)|_L$ has property $(\mathcal{P}_2)$ by Proposition 2.1(i). But then $(\Phi^M)|_H$ has property $(\mathcal{P}_2)$ by Proposition 2.5(i). As we can find a cyclic subgroup $C \cong C_{q-1}$ of L such that $L = H \rtimes C$ and $M = G \rtimes C$, we see that $(\Phi^M)|_H = ((\Phi|_H)^L)|_H$. We can now conclude that Φ_H has property $(\mathcal{P}_2)$ and so $\varphi[1] = 0$ again by the case $n = 4$.

(iii) By the results of (i) and (ii), we may assume that $\varphi[1] = 0$ for the Brauer character φ of Φ. Assume the contrary that $\boldsymbol{C}_V(Q_1) = 0$. Note that P_1 acts transitively on $\mathrm{IBr}(Q_1) \setminus \{1_{Q_1}\}$. It follows that we can write

$$\varphi_{Q_1} = c \sum_{1_{Q_1} \neq \lambda \in \mathrm{IBr}(Q_1)} \lambda \tag{2-3}$$

for some integer $c > 0$. In particular, taking $\boldsymbol{x} \in Q_1$ we get

$$\varphi(1) = c(q^{n-1} - 1), \quad \varphi(\boldsymbol{x}) = -c,$$

and so the relation $\varphi[1] = 0$ implies that

$$\varphi(\boldsymbol{y}) = -c(q^{n-2} + 1). \tag{2-4}$$

On the other hand, as $n \geq 4$, we can choose an $\mathrm{SL}_n(q)$-conjugate $\boldsymbol{y}_1 \in P_1$ of $\boldsymbol{y}$ that projects onto a transvection in $\mathrm{GL}_{n-1}(q)$ under the embedding

$$P_1/Q_1 \hookrightarrow \mathrm{GL}_1(q) \times \mathrm{GL}_{n-1}(q).$$

Such an element $\boldsymbol{y}_1$ acts on $\mathrm{IBr}(Q_1) \setminus \{1_{Q_1}\}$ with exactly $q^{n-2} - 1$ fixed points. Coupled with (2-3), this implies that

$$|\varphi(\boldsymbol{y})| = |\varphi(\boldsymbol{y}_1)| \leq (q^{n-2} - 1)c,$$

contradicting (2-4). □

3. The general linear groups

For $G = \mathrm{GL}_n(q)$ and V an irreducible $\mathbb{F}G$-module, we will use James' parametrization [1986]

$$V = \big(D(s_1, \lambda_1) \circ D(s_2, \lambda_2) \circ \cdots \circ D(s_t, \lambda_t)\big) \uparrow G \tag{3-1}$$

for V as given in [Guralnick and Tiep 1999, Proposition 2.4]. Here, $s_i \in \overline{\mathbb{F}}^\times$ has degree d_i over $\mathbb{F}_q$ and is ℓ-regular, $\lambda_i \vdash k_i$, and $n = \sum_{i=1}^t k_i d_i$. Moreover, for any $i \neq j$, s_i and s_j do not have the same minimal polynomial over $\mathbb{F}_q$. Each $D(s_i, \lambda_i)$ is an irreducible module for $\mathrm{GL}_{k_i d_i}(q)$, and if $t > 1$ then V is Harish-Chandra-induced from the Levi subgroup

$$L = \mathrm{GL}_{k_1 d_1}(q) \times \mathrm{GL}_{k_2 d_2}(q) \times \cdots \times \mathrm{GL}_{k_t s_t}(q)$$

of a certain parabolic subgroup $P = QL$ of G. Namely, consider

$$D(s_1, \lambda_1) \otimes D(s_2, \lambda_2) \otimes \cdots \otimes D(s_t, \lambda_t)$$

as an irreducible L-module, inflate it to an irreducible $\mathbb{F}P$-module U, and then induce to G to get V: $V = U^G$. Note that the Harish-Chandra induction is commutative (with respect to the factors $D(s_i, \lambda_i)$) and transitive. Also, note that the Weil modules of G are precisely $D(a, (n-1, 1))$ and $\big(D(a, (n-1)) \circ D(b, (1))\big) \uparrow G$, with $a, b \in \mathbb{F}_q^\times$ being ℓ-regular and $a \neq b$. For the irreducible *complex* G-modules, we will instead use the notation

$$\big(S(s_1, \lambda_1) \circ S(s_2, \lambda_2) \circ \cdots \circ S(s_t, \lambda_t)\big) \uparrow G.$$

Proposition 3.1. *Theorem B holds for $G = \mathrm{GL}_4(q)$.*

Proof. The implication "(i) $\Rightarrow$ (ii)" follows from Corollary 2.3. For the other implication, let Φ be an irreducible $\mathbb{F}G$-representation possessing property $(\mathcal{P}_2)$ and let φ denote its Brauer character. By Corollary 2.6(ii)–(iii), we have $\varphi[1] = 0$ and $\boldsymbol{C}_V(Q_1) \neq 0$, if V is an $\mathbb{F}G$-module affording Φ.

(i) First we consider the case $\ell = 0$, or more generally, φ lifts to a complex character of G. Using the character table of G given, e.g., in [Geck et al. 1996], one can check that the relation $\varphi[1] = 0$ implies that φ is a Weil character. (Note that the character table of G was first determined in [Steinberg 1951].)

(ii) From now on we may assume that $p \neq \ell \mid |G|$ and that φ does *not* lift to a complex character. We use the label for V given in (3-1). Since any irreducible $\mathbb{F}X$-module of $X \in \{\mathrm{GL}_1(q), \mathrm{GL}_2(q)\}$ lifts to a complex module, we see that $D(s_i, \lambda_i)$ lifts to a complex $\mathrm{GL}_{k_i d_i}(q)$-module if $k_i d_i \leq 2$. The same also happens if $\lambda_i = (k_i)$; see [Guralnick and Tiep 1999, Corollary 2.6]. Hence, the condition that V does not lift to a complex module implies that one of the following cases must occur for V as labeled in (3-1):

(a) $t = 1$, $V = D(s, \lambda)$.

(b) $t = 2$, $V = \big(D(a, \lambda) \circ D(b, (1))\big) \uparrow G$, $\deg(a) = \deg(b) = 1$, $\lambda \vdash 3$, and $a \neq b$.

Let e be the smallest positive integer such that $\ell \mid (1 + q + q^2 + \cdots + q^{e-1})$.

(iii) Suppose we are in case (a). Then, by [Kleshchev and Tiep 2010, Theorem 5.4], $\boldsymbol{C}_V(Q_1) = 0$ whenever $\deg(s) > 1$. Hence we must have that $\deg(s) = 1$. By [Guralnick and Tiep 1999, Lemma 2.9] without loss we may assume that $s = 1$, i.e., Φ is a unipotent representation. Note that Φ has degree 1 if $\lambda = (4)$ and is a Weil representation if $\lambda = (3, 1)$. Now we let χ_j, $1 \leq j \leq 5$, denote the complex unipotent character of G labeled by (4), $(3, 1)$, $(2, 2)$, $(2, 1^2)$, and (1^4), respectively. Similarly, we let φ_j, $1 \leq j \leq 5$, denote the Brauer unipotent character of G labeled by (4), $(3, 1)$, $(2, 2)$, $(2, 1^2)$, and (1^4), respectively. Then

$$\chi_1[1]=\chi_2[1]=\varphi_1[1]=\varphi_2[1]=0, \quad \chi_3[1]=q^4, \quad \chi_4[1]=q^5, \quad \chi_5[1]=q^6. \tag{3-2}$$

Recall that we use the notation χ° to denote the restriction of a complex character χ to the set of ℓ'-elements of G, and that $\varphi[1] = 0$.

By the results of [James 1990], there are integers x_1, x_2 such that

$$\chi_3^\circ = \varphi_3 + x_1\varphi_1 + x_2\varphi_2.$$

It follows by (3-2) that $\varphi_3[1] = \chi_3[1] = q^4$; in particular, $\lambda \neq (2, 2)$.

Next, by [Guralnick and Tiep 1999, Proposition 3.1], there are nonnegative integers y_1, y_2, y_3 such that $y_3 \leq 1$ and

$$\chi_4^\circ = \varphi_4 + y_1\varphi_1 + y_2\varphi_2 + y_3\varphi_3.$$

It follows by (3-2) that

$$\varphi_4[1] = \chi_4[1] - y_3\varphi_3[1] = q^4(q - y_3) > 0;$$

in particular, $\lambda \neq (2, 1^2)$.

Finally, let $\lambda = (1^4)$, i.e., $\varphi = \varphi_5$. Then λ is 2-divisible and 4-divisible in the sense of [Kleshchev and Tiep 2010, Definition 4.3]. Since $\boldsymbol{C}_V(Q_1) \neq 0$, it follows by [loc. cit., Theorem 5.4] that $e \neq 2, 4$. If $e = 3$, then by [James 1990], we have

$$\chi_5^\circ = \varphi + \varphi_3,$$

and so $\chi_5[1] = \varphi_3[1] = q^4$, contrary to (3-2). Thus we must have $e \geq 5 > n$, and so $\varphi_5 = \chi_5^\circ$ by [James 1990, Theorem 6.4], whence $\chi_5[1] = \varphi[1] = 0$, again a contradiction.

(iv) Suppose we are in case (b). By [Guralnick and Tiep 1999, Lemma 2.9], we may assume that $a = 1$. Let ρ_1, ρ_2, and ρ_3 denote the (ordinary) characters of the irreducible $\mathbb{C}G$-modules $\big(S(1,\mu) \circ S(b,(1))\big) \uparrow G$, with $\mu = (3)$, $(2, 1)$, and (1^3), respectively. Similarly, let ψ_1, ψ_2, and ψ_3 denote the Brauer characters of the irreducible $\mathbb{F}G$-modules $\big(D(1,\mu) \circ D(b,(1))\big) \uparrow G$, with $\mu = (3)$, $(2, 1)$, and (1^3), respectively. Using [Geck et al. 1996], we can compute

$$\rho_1[1] = 0, \quad \rho_2[1] = q^4(q+1), \quad \rho_3[1] = q^5(q+1). \tag{3-3}$$

Note that ψ_1 is a Weil character, and so $\psi_1[1] = 0$. Using the decomposition matrix for $\mathrm{GL}_3(q)$ [James 1990], we get a nonnegative integer x such that

$$\rho_2^\circ = \psi_2 + x\psi_1.$$

It follows by (3-3) that $\psi_2[1] = \rho_2[1] = q^4(q+1)$. Furthermore, there are integers $0 \leq y, z \leq 1$ such that

$$\rho_3^\circ = \psi_3 + y\psi_1 + z\psi_2.$$

It follows by (3-3) that

$$\psi_3[1] = \rho_3[1] - z\psi_2[1] \geq q^4(q^2 - 1).$$

We have therefore shown that $\varphi = \psi_1$, a Weil character. □

Proposition 3.2. *Let $G = \mathrm{GL}_n(q)$ with $n \geq 5$ and let V be an irreducible $\mathbb{F}G$-module. Suppose that V has property* ($\mathcal{W}$). *Then in the label* (3-1) *for V*, $\deg(s_i) = 1$ *for all i. Furthermore, either V is a Weil module, or $t = 1$.*

Proof. (i) First we consider the case $t = 1$. By Corollary 2.6, $\boldsymbol{C}_V(Q_1) \neq 0$, whence $\deg(s_1) = 1$ by [Kleshchev and Tiep 2010, Theorem 5.4].

(ii) From now on we may assume that $t \geq 2$, so that V is Harish-Chandra-induced. Next we show that $\deg(s_i) = 1$ for all i. Assume for instance that

$d_1 = \deg(s_1) > 1$. Then we consider a standard subgroup $H = \mathrm{GL}_m(q)$ of G with $2 \le m := n - k_t d_t \le n-1$. Property $(\mathcal{W})$ implies that all composition factors of $V|_H$ are Weil modules or of dimension 1. We can find a parabolic subgroup $P = QL$ with Levi subgroup $L = H \times \mathrm{GL}_{k_t d_t}(q)$. Then note that V can be obtained by inflating the irreducible $\mathbb{F}L$-module $A \otimes D(s_t, \lambda_t)$ to P, where

$$A := \big(D(s_1,\lambda_1) \circ D(s_2,\lambda_2) \circ \cdots \circ D(s_{t-1},\lambda_{t-1})\big) \uparrow H,$$

and then induce to G. In particular, A is a simple submodule of $V|_H$, and clearly A is not a Weil module as $\deg(s_1) > 1$. Suppose that $\dim(A) = 1$. Since $\deg(s_1) > 1$, we must have that $(m,q) = (2,2)$, $\deg(s_1) = 2$, $\lambda_1 = (1)$, and $t = 2$. Applying property $(\mathcal{W})$ to a standard subgroup $\mathrm{GL}_3(2)$ containing $H = \mathrm{GL}_2(2)$ (as its standard subgroup) and then restricting further down to H, we see, however, that $A = \deg(s_1,(1))$ cannot occur in $V|_H$, a contradiction.

(iii) Here we show that one of the following holds:

(a) $t = 3$, $\lambda_i = (k_i)$ for all i, and $\{k_1,k_2,k_3\} = \{n-2,1,1\}$.

(b) $t = 2$ and $\lambda_i \in \{(k_i),(k_i-1,1)\}$ for all i.

The arguments in (ii) show that A is a simple submodule of $V|_H$. Again by property $(\mathcal{W})$, A is either a Weil module or of dimension 1. First we consider the case $t \ge 3$. Then note that $\dim(A)$ is at least the index of a proper parabolic subgroup of H and so can never be equal to 1. Thus A is a Weil module. The identification of Weil modules among the ones labeled in (3-1) now implies that in fact $t = 3$, $\lambda_i = (k_i)$ for all $i = 1,2$, and $1 \in \{k_1,k_2\}$. Interchanging k_3 with k_1 or k_2 in the above construction and noting that $n \ge 5$, we then get $k_3 = 1$ and $\lambda_3 = (1)$ as well.

Suppose now that $t = 2$. The claim is obvious if $k_i \le 2$, so we consider the case where $k_1 \ge 3$, say. Then $A = D(s_1,\lambda_1)$ is a Weil module of $\mathrm{GL}_{k_1}(q)$ or of dimension 1. It follows that $\lambda_1 \in \{(k_1),(k_1-1,1)\}$, as stated.

(iv) Abusing the notation, now we use H to denote the standard subgroup

$$H = \mathrm{Stab}_G\big(\langle e_1,e_2,\ldots,e_{n-1}\rangle_{\mathbb{F}_q}, e_n\big) \cong \mathrm{GL}_{n-1}(q),$$

and $P = Q \rtimes L$ to denote the parabolic subgroup

$$P = \mathrm{Stab}_G\big(\langle e_1,\ldots,e_k\rangle_{\mathbb{F}_q}\big),$$

where $k := k_1$ and

$$L = \mathrm{Stab}_G\big(\langle e_1,\ldots,e_k\rangle_{\mathbb{F}_q}, \langle e_{k+1},\ldots,e_n\rangle_{\mathbb{F}_q}\big).$$

Then we can obtain V by inflating the irreducible $\mathbb{F}L$-module B, where

$$B := D(s_1,\lambda_1) \otimes \big(D(s_2,\lambda_2) \circ \cdots \circ D(s_t,\lambda_t)\big) \uparrow \mathrm{GL}_{n-k}(q),$$

to an irreducible $\mathbb{F}P$-module U and then induce to G. Note that $P_H = Q_H \rtimes L_H$ is also a parabolic subgroup of H, where

$$P_H := P \cap H, \quad Q_H := Q \cap H, \quad L_H := L \cap H.$$

Mackey's formula implies that $V|_H$ contains a subquotient $V' := (U')^H$, where $U' := U|_{P_H}$. But note that $Q_H \leq Q$ acts trivially on U, so in fact V' is Harish-Chandra-induced from the $\mathbb{F}L_H$-module B', where $B' := B|_{L_H}$.

(v) Now we can complete the case $t = 3$. As shown in (iii), in this case we have that $\lambda_i = (k_i)$ for all i and we may furthermore assume that $(k_1, k_2, k_3) = (1, 1, n-2)$. Repeating the argument in (iv) and using the notation therein, we see that B' contains a simple subquotient isomorphic to

$$D(s_1, (1)) \otimes \big(D(s_2, (1)) \circ D(s_3, (n-3))\big) \uparrow \mathrm{GL}_{n-2}(q).$$

It follows that $V|_H$ contains a subquotient isomorphic to

$$\big(D(s_1, (1)) \circ D(s_2, (1)) \circ D(s_3, (n-3))\big) \uparrow H,$$

which is irreducible, but not a Weil module nor of dimension 1. This contradiction shows that the case $t = 3$ is impossible.

(vi) Finally, we consider the case $t = 2$. As shown in (iii), $\lambda_i \in \{(k_i), (k_i - 1, 1)\}$; also, recall that $n = k_1 + k_2 \geq 5$. Suppose first that $k_2 \geq k_1 \geq 2$. Then, in the notation of (iv), we see that $B = D(s_1, \lambda_1) \otimes D(s_2, \lambda_2)$ and so B' contains a simple subquotient isomorphic to $D(s_1, \lambda_1) \otimes D(s_2, \mu)$ with $\mu \in \{(k_2 - 1), (k_2 - 2, 1)\}$. It follows that $V|_H$ contains a subquotient isomorphic to

$$\big(D(s_1, \lambda_1) \circ D(s_2, \mu)\big) \uparrow H,$$

which is irreducible, but not a Weil module nor of dimension 1, a contradiction.

Hence we may assume that $(k_1, k_2) = (1, n-1)$. If $\lambda_2 = (n-1)$, then V is a Weil module. Assume that $\lambda_2 = (n-2, 1)$. Then, in the notation of (iv), we see that $B = D(s_1, \lambda_1) \otimes D(s_2, (n-2, 1))$ and so B' contains a simple subquotient isomorphic to $D(s_1, \lambda_1) \otimes D(s_2, (n-3, 1))$. It follows that $V|_H$ contains a subquotient isomorphic to

$$\big(D(s_1, (1)) \circ D(s_2, (n-3, 1))\big) \uparrow H,$$

which is irreducible, but not a Weil module nor of dimension 1, again contradicting property $(\mathcal{W})$. □

Proposition 3.3. *Let $G = \mathrm{GL}_n(q)$ with $n \geq 5$ and let V be an irreducible $\mathbb{F}G$-module. Suppose that V has property $(\mathcal{W})$. Then either V is a Weil module, or* $\dim V = 1$.

Proof. (i) Consider the label (3-1) for V. By Proposition 3.2 and [Guralnick and Tiep 1999, Lemma 2.9], we may assume that $V = D(1, \lambda)$, a unipotent representation; furthermore, $\boldsymbol{C}_V(Q_1) \neq 0$ by Corollary 2.6. Now V is a subquotient of the reduction modulo ℓ of the unipotent $\mathbb{C}G$-module $S(1,\lambda)$. We can find a standard subgroup $H = \mathrm{GL}_{n-1}(q)$ of G as a direct factor of the Levi subgroup $L_1 = H \times \mathrm{GL}_1(q)$ of $P_1 = Q_1 L_1$. By the Howlett–Lehrer comparison theorem [1983, Theorem 5.9], the Harish-Chandra restriction ${}^*R^G_{L_1}$ of unipotent characters of G can be computed inside the Weyl group S_n of G, and it is similar for the Harish-Chandra induction $R^G_{L_1}$. For brevity, we denote the character of $S(1,\lambda)$ by χ^λ and the Brauer character of $D(1,\lambda)$ by φ^λ, with similar notation for other partitions. In this notation, ${}^*R^G_{L_1}(\chi^\lambda)$ is the sum of unipotent characters χ^μ of L_1 labeled by $\mu \vdash (n-1)$, where the Young diagram $Y(\mu)$ of μ is obtained from the Young diagram $Y(\lambda)$ of λ by removing one removable node. For instance,

$$\begin{aligned} {}^*R^G_{L_1}(\chi^{(n)}) &= \chi^{(n-1)}, \\ {}^*R^G_{L_1}(\chi^{(n-1,1)}) &= \chi^{(n-1)} + \chi^{(n-2,1)}, \\ {}^*R^G_{L_1}(\chi^{(n-2,2)}) &= \chi^{(n-3,2)} + \chi^{(n-2,1)}, \\ {}^*R^G_{L_1}(\chi^{(n-2,1^2)}) &= \chi^{(n-3,1^2)} + \chi^{(n-2,1)}. \end{aligned} \tag{3-4}$$

It is similar for the Harish-Chandra induction; in particular,

$$\begin{aligned} R^G_{L_1}(\chi^{(n-1)}) &= \chi^{(n)} + \chi^{(n-1,1)}, \\ R^G_{L_1}(\chi^{(n-2,1)}) &= \chi^{(n-1,1)} + \chi^{(n-2,2)} + \chi^{(n-2,1^2)}. \end{aligned} \tag{3-5}$$

Let ψ be the Brauer character of a simple submodule of the L_1-module $\boldsymbol{C}_V(Q_1)$. Then ψ is an irreducible constituent of ${}^*R^G_{L_1}(\varphi^\lambda)$, the Brauer L_1-character of $\boldsymbol{C}_V(Q_1)$. The above arguments show that ψ is an irreducible constituent of $(\chi^\mu)^\circ$ for some $\mu \vdash (n-1)$, whence $\psi = \varphi^\nu$ for some $\nu \vdash (n-1)$. On the other hand, property $(\mathcal{W})$ implies that ψ is a Weil character (while restricted to H), or has degree 1. As $n \geq 5$, it follows that $\nu = (n-1)$ or $(n-2,1)$. By Frobenius' reciprocity, φ^λ is an irreducible constituent of $R^G_{L_1}(\varphi^\nu)$, and so of $(R^G_{L_1}(\chi^\nu))^\circ$ as well. Restricting (3-5) to ℓ'-elements, we see by [Guralnick and Tiep 1999, Proposition 3.1] that λ is (n), $(n-1,1)$, $(n-2,2)$, or $(n-2,1^2)$. The first possibility leads to the principal character, and the second one yields a Weil character.

(ii) Here we consider the case $\lambda = (n-2,2)$. Then applying [loc. cit., Proposition 3.1] to G, we can write

$$(\chi^\lambda)^\circ = \varphi^\lambda + x_1(\chi^{(n)})^\circ + x_2(\chi^{(n-1,1)})^\circ$$

for some integers x_1, x_2. It follows by (3-4) that

$${}^*R^G_{L_1}(\varphi^\lambda) = \big(\chi^{(n-3,2)} + (1-x_2)\chi^{(n-2,1)} - (x_1+x_2)\chi^{(n-1)}\big)^\circ.$$

Applying [loc. cit., Proposition 3.1] to H, we then get

$${}^*R^G_{L_1}(\varphi^\lambda) = \varphi^{(n-3,2)} + x_1'\varphi^{(n-1)} + x_2'\varphi^{(n-2,1)}$$

for some integers x'_1, x'_2. By the linear independence of irreducible Brauer characters, $\varphi^{(n-3,2)}$ is an irreducible constituent of $(\varphi^\lambda)|_H$, contradicting $(\mathcal{W})$.

(iii) Finally, assume that $\lambda = (n-2, 1^2)$. Then applying [loc. cit., Proposition 3.1] to G, we have

$$(\chi^\lambda)^\circ = \varphi^\lambda + y_1(\chi^{(n)})^\circ + y_2(\chi^{(n-1,1)})^\circ + y_3(\chi^{(n-2,2)})^\circ$$

for some integers y_1, y_2, y_3. It follows by (3-4) that

$${}^*R^G_{L_1}(\varphi^\lambda) = \big(\chi^{(n-3,1^2)} - y_3\chi^{(n-3,2)} + (1-y_2-y_3)\chi^{(n-2,1)} - (y_1+y_2)\chi^{(n-1)}\big)^\circ.$$

Applying [loc. cit., Proposition 3.1] to H, we then get

$${}^*R^G_{L_1}(\varphi^\lambda) = \varphi^{(n-3,1^2)} + y'_1\varphi^{(n-1)} + y'_2\varphi^{(n-2,1)} + y'_3\varphi^{(n-2,2)}$$

for some integers y'_1, y'_2, y'_3. By linear independence of irreducible Brauer characters, $\varphi^{(n-3,1^2)}$ is an irreducible constituent of $(\varphi^\lambda)|_H$, again contradicting $(\mathcal{W})$. $\square$

We note (without giving proof, since we do not need it subsequently) that property $(\mathcal{Z})$ can also be used to characterize Weil representations of $\mathrm{GL}_n(q)$ with $n \geq 5$.

Proposition 3.4. *Theorem A holds for $G = \mathrm{GL}_n(q)$ with $n \geq 5$.*

Proof. The implication "(i) ⇒ (ii)" follows from Corollary 2.3. In fact, by applying Corollary 2.3 successively, we see that (i) also implies (iii) and (iv). Also, note that (iv) obviously implies (iii), and (ii) implies (i) by Proposition 3.3.

It remains to show that (iii) implies (i). We proceed by induction on $n \geq 5$, with the induction base $n = 5$ (so that $m = 4$) already established in Proposition 3.3. For the induction step $n \geq 6$, consider a chain of standard subgroups

$$H = \mathrm{GL}_m(q) \leq L = \mathrm{GL}_{n-1}(q) < G = \mathrm{GL}_n(q).$$

Let Ψ be any composition factor of $\Phi|_L$. According to (iii), every composition factor of $\Psi|_H$ is either a Weil representation or has dimension 1. Hence, by the induction hypothesis applied to L, we see that either Ψ is a Weil representation or $\deg\Psi = 1$. Thus Φ has property $(\mathcal{W})$, and so we are done by Proposition 3.3. $\square$

Proposition 3.5. *Theorem B holds for $G = \mathrm{GL}_n(q)$ with $n \geq 4$.*

Proof. The implication "(i) ⇒ (ii)" follows from Corollary 2.3. For the other implication, let $G = \mathrm{GL}_n(q)$ with $n \geq 4$ and let Φ be an irreducible $\mathbb{F}G$-representation with property $(\mathcal{P}_k)$ for some $2 \leq k \leq n-2$. By Proposition 3.1, we may assume that $n \geq 5$ and consider a standard subgroup $H \cong \mathrm{GL}_4(q)$ of G. By Corollary 2.2(i), every composition factor Ψ of $\Phi|_H$ has property $(\mathcal{P}_2)$. By Proposition 3.1, either Ψ is a Weil representation or $\deg\Psi = 1$. Thus Φ fulfills condition (iii) of Theorem A for G and with $m = 4$. Hence Φ is either a Weil representation or has degree 1 by Proposition 3.4. $\square$

4. The special linear groups

Proof of Theorems A and B. By Propositions 3.4 and 3.5, it suffices to prove the theorems for $S = \mathrm{SL}_n(q)$. Also, by Corollary 2.3, it suffices to prove the implication "(ii) $\Rightarrow$ (i)" (as the implication "(iii) $\Rightarrow$ (i)" of Theorem A can then be proved using the same arguments as in the proof of Proposition 3.4). Let $\mathcal{P} \in \{(\mathcal{P}_k), (\mathcal{W})\}$ and let U be an irreducible $\mathbb{F}S$-module with property $\mathcal{P}$. We consider S as the derived subgroup of $G = \mathrm{GL}_n(q)$. By Proposition 2.5(ii), a simple submodule V of U^G has property $\mathcal{P}$. Applying Proposition 3.5 if $\mathcal{P} = (\mathcal{P}_k)$, respectively Proposition 3.4 if $\mathcal{P} = (\mathcal{W})$, to V, we see that V is a Weil module or $\dim V = 1$. As U is an irreducible constituent of $V|_S$, we conclude that U is a Weil module or has dimension 1. □

We note that one could try to prove the complex case of Theorem A for $\mathrm{GL}_n(q)$ using the results of [Zelevinsky 1981] or [Thoma 1971]. We conclude by the following example showing that Weil representations of $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$ do not admit a "middle-free" characterization in the spirit of [Guralnick et al. 2002, Corollary 12.4].

Example 4.1. Let $G = \mathrm{GL}_n(q)$ or $\mathrm{SL}_n(q)$ with $n \geq 3$. Consider the natural module $W = \langle e_1, \dots, e_n \rangle_{\mathbb{F}_q}$ and let

$$P = \mathrm{Stab}_G\big(\langle e_1 \rangle_{\mathbb{F}_q}, \langle e_1, \dots, e_{n-1} \rangle_{\mathbb{F}_q}\big).$$

Note that $P = \boldsymbol{N}_G(Z)$ for a long-root subgroup Z of G. Also, we may assume that (a G-conjugate of) the long-root element $\boldsymbol{x}$ defined in Section 2 is contained in Z. Suppose that an irreducible $\mathbb{F}G$-module V has *no middle with respect to* $Q = \boldsymbol{O}_p(P)$ as in [Guralnick et al. 2002, Corollary 12.4], i.e.,

$$\boldsymbol{C}_V(Z) = \boldsymbol{C}_V(Q).$$

It follows that

$$V = \boldsymbol{C}_V(Q) \oplus [V, Z].$$

Let φ denote the Brauer character of V. Then we have

$$\varphi(\boldsymbol{x}) = a - q^{n-2} b,$$

where $\dim \boldsymbol{C}_V(Q) = a$ and $\dim[V, Z] = q^{n-2}(q-1)b$. But note that $Q \setminus Z$ contains a G-conjugate $\boldsymbol{x}'$ of $\boldsymbol{x}$, and

$$\varphi(\boldsymbol{x}') = a.$$

It follows that $b = 0$, $V = \boldsymbol{C}_V(Q)$, Q acts trivially on V, and so $\dim V = 1$.

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THE PRO-p IWAHORI HECKE ALGEBRA OF A REDUCTIVE p-ADIC GROUP, V (PARABOLIC INDUCTION)

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I dedicate this work to the memory of Robert Steinberg, having in mind both a nice encounter in Los Angeles and the representations named after him, which play such a fundamental role in the representation theory of reductive p-adic groups.

We give basic properties of the parabolic induction and coinduction functors associated to R-algebras modelled on the pro-p Iwahori Hecke R-algebras $\mathcal{H}_R(G)$ and $\mathcal{H}_R(M)$ of a reductive p-adic group G and of a Levi subgroup M when R is a commutative ring. We show that the parabolic induction and coinduction functors are faithful, have left and right adjoints that we determine, respect finitely generated R-modules, and that the induction is a twisted coinduction.

1. Introduction

We give basic properties of the parabolic induction and coinduction functors associated to R-algebras modelled on the pro-p Iwahori Hecke R-algebras $\mathcal{H}_R(G)$ and $\mathcal{H}_R(M)$ of a reductive p-adic group G and of a Levi subgroup M when R is a commutative ring. We show that the parabolic induction and coinduction functors are faithful, have left and right adjoints that we determine, respect finitely generated R-modules, and that the induction is a twisted coinduction.

When R is an algebraically closed field of characteristic p, Abe [2014, §4] proved that the induction is a twisted coinduction when he classified the simple $\mathcal{H}_R(G)$-modules in terms of supersingular simple $\mathcal{H}_R(M)$-modules. In two forthcoming articles [Ollivier and Vignéras $\geq$ 2015; Abe et al. $\geq$ 2015], we will use this paper

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to compute the images of an irreducible admissible R-representation of G by the basic functors: invariants by a pro-p-Iwahori subgroup, left or right adjoint of the parabolic induction.

Let R be a commutative ring and let $\mathcal{H}$ be a pro-p Iwahori Hecke R-algebra, associated to a pro-p Iwahori Weyl group $W(1)$ and parameter maps $\mathfrak{S} \xrightarrow{\mathfrak{q}} R$, $\mathfrak{S}(1) \xrightarrow{\mathfrak{c}} R[Z_k]$ [Vignéras 2013a, §4.3; 2015b].

For the reader unfamiliar with these definitions, we recall them briefly. The pro-p Iwahori Weyl group $W(1)$ is an extension of an Iwahori–Weyl group W by a finite commutative group Z_k, and $X(1)$ denotes the inverse image in $W(1)$ of a subset X of W. The Iwahori–Weyl group contains a normal affine Weyl subgroup W^{aff}; $\mathfrak{S}$ is the set of all affine reflections of W^{aff}, and $\mathfrak{q}$ is a W-equivariant map $\mathfrak{S} \to R$, with W acting by conjugation on $\mathfrak{S}$ and trivially on R; $\mathfrak{c}$ is a $(W(1) \times Z_k)$-equivariant map $\mathfrak{S}(1) \to R[Z_k]$, with $W(1)$ acting by conjugation and Z_k by multiplication on both sides.

The Iwahori–Weyl group is a semidirect product $W = \Lambda \rtimes W_0$, where Λ is the (commutative finitely generated) subgroup of translations and W_0 is the finite Weyl subgroup of W^{aff}.

Let S^{aff} be a set of generators of W^{aff} such that $(W^{\text{aff}}, S^{\text{aff}})$ is an affine Coxeter system and $(W_0, S := S^{\text{aff}} \cap W_0)$ is a finite Coxeter system. The Iwahori–Weyl group is also a semidirect product $W = W^{\text{aff}} \rtimes \Omega$, where Ω denotes the normalizer of S^{aff} in W. Let ℓ denote the length of $(W^{\text{aff}}, S^{\text{aff}})$ extended to W and then inflated to $W(1)$ such that $\Omega \subset W$ and $\Omega(1) \subset W(1)$ are the subsets of length-0 elements.

Let $\tilde{w} \in W(1)$ denote a fixed but arbitrary lift of $w \in W$.

The subset $\mathfrak{S} \subset W^{\text{aff}}$ of all affine reflections is the union of the W^{aff}-conjugates of S^{aff} and the map $\mathfrak{q}$ is determined by its values on S^{aff}; the map $\mathfrak{c}$ is determined by its values on any set $\tilde{S}^{\text{aff}} \subset S^{\text{aff}}(1)$ of lifts of S^{aff} in $W(1)$.

Definition 1.1. The R-algebra $\mathcal{H}$ associated to $(W(1), \mathfrak{q}, \mathfrak{c})$ and S^{aff} is the free R-module of basis $(T_{\tilde{w}})_{\tilde{w} \in W(1)}$ and relations generated by the braid and quadratic relations

$$T_{\tilde{w}} T_{\tilde{w}'} = T_{\tilde{w}\tilde{w}'}, \quad T_{\tilde{s}}^2 = \mathfrak{q}(s)(\tilde{s})^2 + \mathfrak{c}(\tilde{s}) T_{\tilde{s}}$$

for all $\tilde{w}, \tilde{w}' \in W(1)$ with $\ell(w) + \ell(w') = \ell(ww')$ and all $\tilde{s} \in S^{\text{aff}}(1)$.

By the braid relations, the map $R[\Omega(1)] \to \mathcal{H}$ sending $\tilde{u} \in \Omega(1)$ to $T_{\tilde{u}}$ identifies $R[\Omega(1)]$ with a subring of $\mathcal{H}$ containing $R[Z_k]$. This identification is used in the quadratic relations. The isomorphism class of $\mathcal{H}$ is independent of the choice of S^{aff}.

Let S_M be a subset of S. We recall the definitions of the pro-p Iwahori Weyl group $W_M(1)$, the parameter maps $\mathfrak{S}_M \xrightarrow{\mathfrak{q}_M} R$, $\mathfrak{S}_M(1) \xrightarrow{\mathfrak{c}_M} R[Z_k]$ and S_M^{aff} given in [Vignéras 2015b].

The set S_M generates a finite Weyl subgroup $W_{M,0}$ of W_0, $W_M := \Lambda \rtimes W_{M,0}$ is a subgroup of W, $W_M(1)$ is the inverse image of W_M in $W(1)$, $\mathfrak{S}_M(1) =$

$\mathfrak{S}(1) \cap W_M(1)$, $\mathfrak{q}_M$ is the restriction of $\mathfrak{q}$ to $\mathfrak{S}_M$, and $\mathfrak{c}_M$ is the restriction of $\mathfrak{c}$ to $\mathfrak{S}_M(1)$. The subgroup $W_M^{\mathrm{aff}} := W^{\mathrm{aff}} \cap W_M \subset W_M$ is an affine Weyl group and S_M^{aff} denotes the set of generators of W_M^{aff} containing S_M such that $(W_M^{\mathrm{aff}}, S_M^{\mathrm{aff}})$ is an affine Coxeter system.

Definition 1.2. For $S_M \subset S$, the R-algebra $\mathcal{H}_M$ associated to $(W_M(1), \mathfrak{q}_M, \mathfrak{c}_M)$ and S_M^{aff} is called a Levi algebra of $\mathcal{H}$.

Let $(T_{\tilde{w}}^M)_{\tilde{w} \in W_M(1)}$ denote the basis of $\mathcal{H}_M$ associated to $(W_M(1), \mathfrak{q}_M, \mathfrak{c}_M)$ and S_M^{aff} and ℓ_M the length of $W_M(1)$ associated to S_M^{aff}.

Remark 1.3. When $S_M = S$, we have $\mathcal{H}_M = \mathcal{H}$, and when $S_M = \varnothing$, we have $\mathcal{H}_M = R[\Lambda(1)]$.

In general when $S_M \neq S$, S_M^{aff} is not $W_M \cap S^{\mathrm{aff}}$, and $\mathcal{H}_M$ is not a subalgebra of $\mathcal{H}$; it embeds in $\mathcal{H}$ only when the parameters $\mathfrak{q}(s) \in R$ for $s \in S^{\mathrm{aff}}$ are invertible.

As in the theory of Hecke algebras associated to types, one introduces the subalgebra $\mathcal{H}_M^+ \subset \mathcal{H}_M$ of basis $(T_{\tilde{w}}^M)_{\tilde{w} \in W_{M^+}(1)}$ associated to the positive monoid

$$W_{M^+} := \{w \in W_M \mid w(\Sigma^+ - \Sigma_M^+) \subset \Sigma^{\mathrm{aff},+}\},$$

where $\Sigma_M \subset \Sigma$ are the reduced root systems defining $W_M^{\mathrm{aff}} \subset W^{\mathrm{aff}}$, the upper index indicates the positive roots with respect to S^{aff}, S_M^{aff}, and Σ^{aff} is the set of affine roots of Σ. One chooses an element $\tilde{\mu}_M$ central in $W_M(1)$, in particular of length $\ell_M(\tilde{\mu}_M) = 0$, lifting a strictly positive element μ_M in $\Lambda_{M^+} := \Lambda \cap W_{M^+}$. The element $T_{\tilde{\mu}_M}^M$ of $\mathcal{H}_M$ is invertible of inverse $T_{\tilde{\mu}_M^{-1}}^M$, but in general $T_{\tilde{\mu}_M}$ is not invertible in $\mathcal{H}$.

Theorem 1.4. (i) *The R-submodule $\mathcal{H}_{M^+}$ of basis $(T_{\tilde{w}}^M)_{\tilde{w} \in W_{M^+}(1)}$ is a subring of $\mathcal{H}_M$, called the positive subalgebra of $\mathcal{H}_M$.*

(ii) *The R-algebra $\mathcal{H}_M = \mathcal{H}_{M^+}[(T_{\tilde{\mu}_M}^M)^{-1}]$ is a localization of $\mathcal{H}_{M^+}$ at $T_{\tilde{\mu}_M}^M$.*

(iii) *The injective linear map $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$ sending $T_{\tilde{w}}^M$ to $T_{\tilde{w}}$ for $\tilde{w} \in W_M(1)$ restricted to $\mathcal{H}_{M^+}$ is a ring homomorphism.*

(iv) *As a $\theta(\mathcal{H}_{M^+})$-module, $\mathcal{H}$ is the almost localization of a left free $\theta(\mathcal{H}_{M^+})$-module $\mathcal{V}_{M^+}$ at $T_{\tilde{\mu}_M}$.*

The theorem was known in special cases. Part (iv) means that $\mathcal{H}$ is the union over $r \in \mathbb{N}$ of

$${}_r\mathcal{V}_{M^+} := \{x \in \mathcal{H} \mid T_{\tilde{\mu}_M}^r x \in \mathcal{V}_{M^+}\}, \quad \mathcal{V}_{M^+} = \oplus_{d \in {}^M W_0} \theta(\mathcal{H}_{M^+}) T_{\tilde{d}}.$$

Here ${}^M W_0$ is the set of elements of minimal lengths in the cosets $W_{M,0} \backslash W_0$ and $\tilde{d} \in W(1)$ is an arbitrary lift of d. The theorem admits a variant for the subalgebra $\mathcal{H}_{M^-} \subset \mathcal{H}_M$ associated to the negative submonoid W_{M^-}, inverse of W_{M^+}, for the linear map $\mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}$ sending $(T^M_{\tilde{w}})^*$ to $T^*_{\tilde{w}}$ for $\tilde{w} \in W_M(1)$ [Vignéras 2013a, Proposition 4.14], and with *left* replaced by *right* in (iv): $\mathcal{H}_M = \mathcal{H}_{M^-}[T^M_{\tilde{\mu}_M}]$, θ^* restricted to $\mathcal{H}_{M^-}$ is a ring homomorphism, and the right $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{H}$ is the almost localisation at $T^*_{\tilde{\mu}_M^{-1}}$ of a right free $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{V}^*_{M^-}$ of rank $|W_{M,0}|^{-1}|W_0|$, meaning that $\mathcal{H}$ is the union over $r \in \mathbb{N}$ of

$$_r\mathcal{V}^*_{M^-} := \{x \in \mathcal{H} \mid x(T^*_{\tilde{\mu}_M^{-1}})^r \in \mathcal{V}^*_{M^-}\}, \quad \mathcal{V}^*_{M^-} := \sum_{d \in W_0^M} T^*_{\tilde{d}}\,\theta^*(\mathcal{H}_{M^-}).$$

Here W_0^M is the inverse of ${}^MW_0$.

For a ring A, let Mod_A denote the category of right A-modules and ${}_A\mathrm{Mod}$ the category of left A-modules. Given two rings $A \subset B$, the induction $- \otimes_A B$ and the coinduction $\mathrm{Hom}_A(B, -)$ from Mod_A to Mod_B are the left and the right adjoint of the restriction Res^B_A. The ring B is considered as a left A-module for the induction, and as a right A-module for the coinduction.

Property (iv) and its variant describe $\mathcal{H}$ as a left $\theta(\mathcal{H}_{M^+})$-module and as a right $\theta^*(\mathcal{H}_{M^-})$-module. The linear maps θ and θ^* identify the subalgebras $\mathcal{H}_{M^+}$, $\mathcal{H}_{M^-}$ of $\mathcal{H}_M$ with the subalgebras $\theta(\mathcal{H}_{M^+})$, $\theta^*(\mathcal{H}_{M^-})$ of $\mathcal{H}$.

Definition 1.5. The parabolic induction and coinduction from $\mathrm{Mod}_{\mathcal{H}_M}$ to $\mathrm{Mod}_{\mathcal{H}}$ are the functors $I^{\mathcal{H}}_{\mathcal{H}_M} = - \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}$ and $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M} = \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, -)$.

We show the following:

Theorem 1.6. *The parabolic induction $I^{\mathcal{H}}_{\mathcal{H}_M}$ is faithful, transitive, respects finitely generated R-modules, and admits a right adjoint* $\mathrm{Hom}_{\mathcal{H}_{M^+}}(\mathcal{H}_M, -)$.

If R is a field, the right adjoint functor respects finite dimension.

The transitivity of the parabolic induction means that for $S_M \subset S_{M'} \subset S$,

$$I^{\mathcal{H}}_{\mathcal{H}_M} = I^{\mathcal{H}}_{\mathcal{H}_{M'}} \circ I^{\mathcal{H}_{M'}}_{\mathcal{H}_M} : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}_{M'}} \to \mathrm{Mod}_{\mathcal{H}}.$$

Let w_0 denote the longest element of W_0, $S_{w_0(M)}$ the subset $w_0 S_M w_0$ of S, and $w_0^M := w_0 w_{M,0}$, where $w_{M,0}$ is the longest element of $W_{M,0}$. A lift $\tilde{w}_0^M \in W_0(1)$ of w_0^M defines an R-algebra isomorphism

$$\mathcal{H}_M \to \mathcal{H}_{w_0(M)}, \qquad T^M_{\tilde{w}} \mapsto T^{w_0(M)}_{\tilde{w}_0^M \tilde{w} (\tilde{w}_0^M)^{-1}} \quad \text{for } \tilde{w} \in W_M(1), \tag{1}$$

inducing an equivalence of categories

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_0^M} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}}$$

of inverse $\tilde{\mathfrak{w}}_0^{w_0(M)}$ defined by the lift $(\tilde{w}_0^M)^{-1} \in W_0(1)$ of $w_0^{w_0(M)} = (w_0^M)^{-1}$.

Definition 1.7. The w_0-twisted parabolic induction and coinduction from $\mathrm{Mod}_{\mathcal{H}_M}$ to $\mathrm{Mod}_{\mathcal{H}}$ are the functors $I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M$ and $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M$.

Up to modulo equivalence, these functors do not depend on the choice of the lift of w_0^M used for their construction.

Theorem 1.8. *The parabolic induction* (*resp. coinduction*) *is equivalent to the* w_0*-twisted parabolic coinduction* (*resp. induction*):

$$\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}} \simeq I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}} \circ \tilde{\mathfrak{w}}_0^M, \qquad I_{\mathcal{H}_M}^{\mathcal{H}} \simeq \mathbb{I}_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}} \circ \tilde{\mathfrak{w}}_0^M.$$

Using that the coinduction admits a left adjoint and that the induction is a twisted coinduction, one proves the following:

Theorem 1.9. *The parabolic induction $I_{\mathcal{H}_M}^{\mathcal{H}}$ admits a left adjoint equivalent to*

$$\tilde{\mathfrak{w}}_0^{w_0(M)} \circ (- \otimes_{\mathcal{H}_{w_0(M)^-},\theta^*} \mathcal{H}_{w_0(M)}) : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \to \mathrm{Mod}_{\mathcal{H}_M}.$$

When R is a field, *the left adjoint functor respects finite dimension.*

The coinduction satisfies the same properties as the induction:

Corollary 1.10. *The coinduction $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}$ is faithful*, *transitive*, *respects finitely generated R-modules*, *and admits a left and a right adjoint. When R is a field*, *the left and right adjoint functors respect finite dimension.*

Note that the induction and the coinduction are exact functors, as they admit a left and a right adjoint.

We prove Theorem 1.4 in Section 2, and Theorems 1.6, 1.8 and 1.9 in Section 4.

Remark 1.11. One cannot replace $(\mathcal{H}, \mathcal{H}_M, \mathcal{H}_M^+)$ by $(\mathcal{H}, \mathcal{H}_M, \mathcal{H}_M^-)$ to define the induction $I_{\mathcal{H}_M}^{\mathcal{H}}$.

When no nonzero element of the ring R is infinitely p-divisible, is the parabolic induction functor

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{I_{\mathcal{H}_M}^{\mathcal{H}}} \mathrm{Mod}_{\mathcal{H}}$$

fully faithful? The answer is yes for the parabolic induction functor

$$\mathrm{Mod}_R^\infty(M) \xrightarrow{\mathrm{Ind}_P^G} \mathrm{Mod}_R^\infty(G)$$

when M is a Levi subgroup of a parabolic subgroup P of a reductive p-adic group G and $\mathrm{Mod}_R^\infty(G)$ the category of smooth R-representations of G [Vignéras 2014, Theorem 5.3].

2. Levi algebra

We prove Theorem 1.4 and its variant on the subalgebra $\mathfrak{H}_M^\epsilon \subset \mathfrak{H}_M$, its image in $\mathcal{H}$, on $\mathfrak{H}_M$ as a localisation of $\mathfrak{H}_M^\epsilon$ and on $\mathcal{H}$ as an almost left localisation of $\theta(\mathfrak{H}_M^+)$, and almost left localisation of $\theta^*(\mathfrak{H}_M^-)$.

2A. ***Monoid*** $\boldsymbol{W_{M^\epsilon}}$**.** Let $S_M \subset S$ and $\epsilon \in \{+,-\}$. To S^{aff} is associated a submonoid $W_{M^\epsilon} \subset W_M$ defined as follows.

Let Σ denote the reduced root system of affine Weyl group W^{aff}, V the real vector space of dual generated by Σ, $\Sigma^{\mathrm{aff}} = \Sigma + \mathbb{Z}$ the set of affine roots of Σ and $\mathfrak{H} = \{\mathrm{Ker}_V(\gamma) \mid \gamma \in \Sigma^{\mathrm{aff}}\}$ the set of kernels of the affine roots in V. We fix a W_0-invariant scalar product on V. The affine Weyl group W^{aff} identifies with the group generated by the orthogonal reflections with respect to the affine hyperplanes of $\mathfrak{H}$.

Let $\mathfrak{A}$ denote the alcove of vertex 0 of $(V, \mathfrak{H})$ such that S^{aff} is the set of orthogonal reflections with respect to the walls of $\mathfrak{A}$ and S is the subset associated to the walls containing 0. An affine root which is positive on $\mathfrak{A}$ is called positive. Let $\Sigma^{\mathrm{aff},+}$ denote the set of positive affine roots, $\Sigma^+ := \Sigma \cap \Sigma^+_{\mathrm{aff}}$, $\Sigma^{\mathrm{aff},-} := -\Sigma^{\mathrm{aff},-}$, and $\Sigma^- := -\Sigma^+$.

Let Δ_M denote the set of positive roots $\alpha \in \Sigma^+$ such that $\mathrm{Ker}\,\alpha$ is a wall of $\mathfrak{A}$ and the orthogonal reflection s_α of V with respect to $\mathrm{Ker}\,\alpha$ belongs to S_M, $\Sigma_M \subset \Sigma$ the reduced root system generated by Δ_M, and $\Sigma^\epsilon_M := \Sigma_M \cap \Sigma^\epsilon_{\mathrm{aff}}$.

Definition 2.1. The positive monoid $W_{M^+} \subset W_M$ is

$$\{w \in W_M \mid w(\Sigma^+ - \Sigma^+_M) \subset \Sigma^{\mathrm{aff},+}\}.$$

The negative monoid $W_{M^-} := \{w \in W_M \mid w^{-1} \in W_{M^+}\}$ is the inverse monoid.

It is well known that the finite Weyl group $W_{M,0}$ is the W_0-stabilizer of $\Sigma^\epsilon - \Sigma^\epsilon_M$. This implies

$$W_{M^\epsilon} = \Lambda_{M^\epsilon} \rtimes W_{M,0}, \qquad \text{where } \Lambda_{M^\epsilon} := \Lambda \cap W_{M^\epsilon}.$$

Let $\Lambda \xrightarrow{\nu} V$ denote the homomorphism such that $\lambda \in \Lambda$ acts on V by translation by $\nu(\lambda)$.

Lemma 2.2. $\Lambda_{M^\epsilon} = \{\lambda \in \Lambda \mid -(\gamma \circ \nu)(\lambda) \geq 0$ *for all* $\gamma \in \Sigma^\epsilon - \Sigma^\epsilon_M\}$.

Proof. Let $\lambda \in \Lambda$. By definition, $\lambda \in \Lambda_{M^+}$ if and only if $\lambda(\gamma)$ is positive for all $\gamma \in \Sigma^+ - \Sigma^+_M$. We have $\lambda(\gamma) = \gamma - \nu(\lambda)$. The minimum of the values of γ on $\mathfrak{A}$ is 0 [Vignéras 2013a, (35)]. So $\gamma(v - \nu(\lambda)) \geq 0$ for $\gamma \in \Sigma^+ - \Sigma^+_M$ and $v \in \mathfrak{A}$ is equivalent to $-(\gamma \circ \nu)(\lambda) \geq 0$ for all $\gamma \in \Sigma^+ - \Sigma^+_M$. $\square$

When $S_M \subset S_{M'} \subset S$, we have the inclusion $\Sigma^\epsilon_M \subset \Sigma^\epsilon_{M'}$, the inverse inclusion $\Sigma^\epsilon - \Sigma^\epsilon_M \subset \Sigma^\epsilon - \Sigma^\epsilon_{M'}$, and the inclusions $W_M \subset W_{M'}$ and $W_{M^\epsilon} \subset W^\epsilon_{M'}$.

Remark 2.3. Set $\mathcal{D}^\epsilon := \{v \in V \mid \gamma(v) \geq 0 \text{ for } \gamma \in \Sigma^\epsilon\}$ and $\Lambda^\epsilon := (-\nu)^{-1}(\mathcal{D}^\epsilon)$. The antidominant Weyl chamber of V is $\mathcal{D}^-$ and the dominant Weyl chamber is $\mathcal{D}^+$. Careful: [Vignéras 2015a, §1.2(v)] uses a different notation: $\Lambda^\epsilon = (\nu)^{-1}(\mathcal{D}^\epsilon)$.

The Bruhat order $\leq$ of the affine Coxeter system $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ extends to W: for $w_1, w_2 \in W^{\mathrm{aff}}$, $u_1, u_2 \in \Omega$, we have $w_1u_1 \leq w_2u_2$ if $u_1 = u_2$ and $w_1 \leq w_2$ [Vignéras 2006, Appendice]. We write $w < w'$ if $w \leq w'$ and $w \neq w'$ for $w, w' \in W$. Careful:

the Bruhat order $\leq_M$ on W_M associated to $(W_M^{\mathrm{aff}}, S_M^{\mathrm{aff}})$ is not the restriction of $\leq$ when S_M^{aff} is not contained in S^{aff} [Vignéras 2015b].

Remark 2.4. The basic properties of $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ extend to W:

(i) If $x \leq y$ for $x, y \in W$ and $s \in S^{\mathrm{aff}}$,

$$sx \leq (y \text{ or } sy), \quad xs \leq (y \text{ or } ys), \quad (x \text{ or } sx) \leq sy, \quad (x \text{ or } xs) \leq ys$$

[Vignéras 2015a, Lemma 3.1, Remark 3.2].

(ii) $W = \bigsqcup_{\lambda \in \Lambda^{\epsilon}} W_0 \lambda W_0$ [Henniart and Vignéras 2015, §6.3, Lemma].

(iii) For $\lambda \in \Lambda^+$, $W_0 \lambda W_0$ admits a unique element of maximal length $w_\lambda = w_0 \lambda$, where w_0 is the unique element of maximal length in W_0, and $\ell(w_\lambda) = \ell(w_0) + \ell(\lambda)$ [Vignéras 2015a, Lemma 3.5].

(iv) For $\lambda \in \Lambda^+$, $\{w \in W \mid w \leq w_\lambda\} \supset \bigsqcup_{\mu \in \Lambda^+, \mu \leq \lambda} W_0 \mu W_0$ [Vignéras 2015a, Lemma 3.5].

Remark 2.5. The set $\{w \in W \mid w \leq w_\lambda\}$ is a union of (W_0, W_0)-classes only if $\lambda, \mu \in \Lambda^+$, $\mu \leq w_0 \lambda$ implies $\mu \leq \lambda$. I see no reason for this to be true.

Lemma 2.6. *The monoid W_{M^ϵ} is a lower subset of W_M for the Bruhat order $\leq_M$: for $w \in W_{M^\epsilon}$, any element $v \in W_M$ such that $v \leq_M w$ belongs to W_{M^ϵ}.*

Proof. See [Abe 2014, Lemma 4.1]. □

An element $w \in W$ admits a reduced decomposition in (W, S^{aff}), $w = s_1 \cdots s_r u$ with $s_i \in S^{\mathrm{aff}}$, $u \in \Omega$. As in [Vignéras 2013a], we set for $w, w' \in W$,

$$q_w := \mathfrak{q}(s_1) \cdots \mathfrak{q}(s_r), \quad q_{w,w'} := (q_w q_{w'} q_{ww'}^{-1})^{1/2}. \tag{2}$$

This is independent of the choice of the reduced decomposition. For $w, w' \in W_M$ and $s_i \in S_M^{\mathrm{aff}}$, $u \in \Omega_M$, let $q_{M,w}$, $q_{M,w,w'}$ denote the similar elements. They may be different from q_w, $q_{w,w'}$.

Lemma 2.7. *We have $S_M^{\mathrm{aff}} \cap W_{M^\epsilon} \subset S^{\mathrm{aff}}$ and $q_{w,w'} = q_{M,w,w'}$ if $w, w' \in W_{M^\epsilon}$.*

In particular, $\ell_M(w) + \ell_M(w') - \ell_M(ww') = \ell(w) + \ell(w') - \ell(ww')$ if $w, w' \in W_{M^\epsilon}$.

Proof. See [Abe 2014, Lemma 4.4, proof of Lemma 4.5]. □

An element $\lambda \in \Lambda_{M^\epsilon}$ such that all the inequalities in Lemma 2.2 are strict is called strictly positive if $\epsilon = +$, and strictly negative if $\epsilon = +$. We choose

a central element $\tilde{\mu}_M$ of $W_M(1)$ lifting a strictly positive element μ_M of Λ.

We set $\tilde{\mu}_{M^+} := \tilde{\mu}_M$ and $\tilde{\mu}_{M^-} := \tilde{\mu}_M^{-1}$. The center of the pro-$p$ Iwahori Weyl group $W_M(1)$ is the set of elements in the center of $\Lambda(1)$ fixed by the finite Weyl group $W_{M,0}$ [Vignéras 2014]. Hence $\tilde{\mu}_{M^\epsilon}$ is an element of the center of $\Lambda(1)$ fixed by $W_{M,0}$ and $-\gamma\circ\nu(\mu_{M^\epsilon})>0$ for all $\gamma\in\Sigma^\epsilon-\Sigma^\epsilon_M$. We have $\gamma\circ\nu(\mu_{M^\epsilon})=0$ for $\gamma\in\Sigma_M$. The length of μ_{M^ϵ} is 0 in W_M, and is positive in W when $S_M\neq S$.

Let $\mathcal{H}_{M^\epsilon}$ denote the R-submodule of the Iwahori–Hecke R-algebra $\mathcal{H}_M$ of M of basis $(T^M_{\tilde w})_{\tilde w\in W_{M^\epsilon}(1)}$, and $\mathcal{H}_M\xrightarrow{\theta}\mathcal{H}$ the linear map sending $T^M_{\tilde w}$ to $T_{\tilde w}$ for $\tilde w\in W_M(1)$.

The proofs of the properties (i), (ii), (iii) of Theorem 1.4 and its variant are as follows:

(i) $\mathcal{H}_{M^\epsilon}$ is a subring of $\mathcal{H}_M$, because $T^M_{\tilde w}T^M_{\tilde w'}$ is a linear combination of elements $T_{\tilde v}$ such that $v\leq_M ww'$ [Vignéras 2013a].

(iii) We have $\theta(T^M_{\tilde w_1}T^M_{\tilde w_2})=T_{\tilde w_1}T_{\tilde w_2}$ and $\theta^*((T^M_{\tilde w_1})^*(T^M_{\tilde w_2})^*)=T^*_{\tilde w_1}T^*_{\tilde w_2}$ for $w_1,w_2\in W_{M^\epsilon}$. This follows from the braid relations if $\ell_M(w_1)+\ell_M(w_2)=\ell_M(w_1w_2)$ because $\ell(w_1)+\ell(w_2)=\ell(w_1w_2)$ (Lemma 2.7). If $w_2=s\in S^{\mathrm{aff}}_M$ with $\ell_M(w_1)-1=\ell_M(w_1s)$, this follows from the quadratic relations

$$T_{\tilde w_1}T_{\tilde s}=T_{\tilde w_1\tilde s^{-1}}\big(\mathfrak{q}(s)(\tilde s)^2+T_{\tilde s}\mathfrak{c}(\tilde s)\big)=\mathfrak{q}(s)T_{\tilde w_1\tilde s}+T_{\tilde w_1}\mathfrak{c}(\tilde s),$$
$$T^*_{\tilde w_1}T^*_{\tilde s}=\mathfrak{q}(s)T^*_{\tilde w_1\tilde s}-T^*_{\tilde w_1}\mathfrak{c}(\tilde s),$$

$s\in S^{\mathrm{aff}}$, $\ell(w_1)-1=\ell(w_1s)$ (Lemma 2.7) and $\mathfrak{q}(s)=\mathfrak{q}_M(s)$, $\mathfrak{c}(\tilde s)=\mathfrak{c}_M(\tilde s)$ [Vignéras 2015b]. In general the formula is proved by induction on $\ell_M(w_2)$ [Abe 2014, §4.1]. The proof of [Abe 2014, Lemma 4.5] applies.

(ii) $\mathcal{H}_M=\mathcal{H}_{M^\epsilon}[(T^M_{\tilde\mu_{M^\epsilon}})^{-1}]$, because for $w\in W_M$, there exists $r\in\mathbb{N}$ such that $\mu^{\epsilon r}_M w\in W_{M^\epsilon}$.

Remark 2.8. If the parameters $\mathfrak{q}(s)$ are invertible in R, then $\mathcal{H}_{M^+}\xrightarrow{\theta}\mathcal{H}$ extends uniquely to an algebra homomorphism $\mathcal{H}_M\hookrightarrow\mathcal{H}$, sending $T^M_{\tilde\mu_M^{-\epsilon r}\tilde w}$ to $T^{-r}_{\tilde\mu_{M^\epsilon}}T_{\tilde w}$ for $\tilde w\in W_{M^+}(1)$, $r\in\mathbb{N}$.

Remark 2.9. The trivial character $\chi_1:\mathcal{H}\to R$ of $\mathcal{H}$ is defined by

$$\chi_1(T_{\tilde w})=q_w\quad(\tilde w\in W(1)).$$

When $\mathcal{H}$ is the Hecke algebra of the pro-p-Iwahori subgroup of a reductive p-adic group G, we know that $\mathcal{H}$ acts on the trivial representation of G by χ_1. Note that the restriction of the trivial character of $\mathcal{H}_M$ to $\theta(\mathcal{H}_{M^+})$ is not equal to $\chi_1\circ\theta$ when $\ell_M(\mu_M)=0$, $\ell(\mu_M)\neq 0$.

2B. ***An anti-involution ζ.*** The R-linear bijective map

$$\mathcal{H}\xrightarrow{\zeta}\mathcal{H}\quad\text{such that}\quad\zeta(T_{\tilde w})=T_{\tilde w^{-1}}\quad\text{for }\tilde w\in W(1)\tag{3}$$

is an anti-involution when $\zeta(h_1h_2)=\zeta(h_2)\zeta(h_1)$ for $h_1,h_2\in\mathcal{H}$ because $\zeta\circ\zeta=\mathrm{id}$. For $S_M\subset S$, let $\mathcal{H}\xrightarrow{\zeta_M}\mathcal{H}_M$ denote the linear map such that $\zeta(T^M_{\tilde w})=T^M_{\tilde w^{-1}}$ for $\tilde w\in W_M(1)$.

Lemma 2.10. 1. *The following properties are equivalent*:

(i) ζ *is an anti-involution.*

(ii) $\zeta(\mathfrak{c}(\tilde{s})) = c_{(\tilde{s})^{-1}}$ *for* $\tilde{s} \in S^{\mathrm{aff}}(1)$.

(iii) $\zeta \circ \mathfrak{c} = \mathfrak{c} \circ (-)^{-1}$, *where* $\mathfrak{S}(1) \xrightarrow{\mathfrak{c}} R[Z_k]$ *is the parameter map.*

2. *If* ζ *is an anti-involution then* ζ_M *is an anti-involution.*

Proof. Let $\tilde{w} = \tilde{s}_1 \cdots \tilde{s}_{\ell(w)}\tilde{u}$ be a reduced decomposition, $\tilde{s}_i \in S^{\mathrm{aff}}(1)$, $\tilde{u} \in W(1)$, $\ell(\tilde{u}) = 0$ and let $\tilde{s} \in S^{\mathrm{aff}}(1)$. Then,

$$\zeta(T_{\tilde{w}}) = T_{(\tilde{w})^{-1}} = T_{(\tilde{u})^{-1}} T_{\tilde{s}_{\ell(w)}^{-1}} \cdots T_{\tilde{s}_1^{-1}} = \zeta(T_{\tilde{u}})\zeta(T_{\tilde{s}_{\ell(w)}}) \cdots \zeta(T_{\tilde{s}_1}),$$

$$(\zeta(T_{\tilde{s}}))^2 = T_{\tilde{s}^{-1}}^2 = \mathfrak{q}(s)\tilde{s}^{-2} + \mathfrak{c}(\tilde{s}^{-1})T_{\tilde{s}^{-1}}.$$

The map ζ is an antiautomorphism if and only if $\zeta(\mathfrak{c}(\tilde{s})) = \mathfrak{c}(\tilde{s}^{-1})$ for $\tilde{s} \in S^{\mathrm{aff}}(1)$. This is equivalent to $\zeta \circ \mathfrak{c} = \mathfrak{c} \circ (-)^{-1}$ because $\mathfrak{S}(1)$ is the union of the $W(1)$-conjugates of $S^{\mathrm{aff}}(1)$, $\mathfrak{c}$ is $W(1)$-equivariant and ζ commutes with the conjugation by $W(1)$.

If $\mathfrak{c}$ satisfies (iii), its restriction $\mathfrak{c}_M$ to $\mathfrak{S}_M(1)$ satisfies (iii). □

Lemma 2.11. *When* $\mathcal{H} = \mathcal{H}(G)$ *is the pro-p Iwahori Hecke R-algebra of a reductive p-adic group G, we have that* ζ *is an anti-involution.*

Proof. Let $s \in \mathfrak{S}$, $\tilde{s}$ be an admissible lift and $t \in Z_k$. Then $\mathfrak{c}(\tilde{s})$ is invariant by ζ [Vignéras 2013a, Proposition 4.4]. If $u \in U_\gamma^*$ for $\gamma = \alpha + r \in \Phi_{\mathrm{red}}^{\mathrm{aff}}$, then $u^{-1} \in U_\gamma^*$ and $m_\alpha(u)^{-1} = m_\alpha(u^{-1})$. Hence the set of admissible lifts of s is stable by the inverse map. As the group Z_k is commutative, we have

$$(\zeta \circ c)(t\tilde{s}) = \zeta(tc(s)) = t^{-1}c(s) = c(s)t^{-1} = c(t\tilde{s})^{-1}. \qquad \square$$

From now on, we suppose that ζ *is an anti-involution.* We recall the involutive automorphism [Vignéras 2013a, Proposition 4.24]

$$\mathcal{H} \xrightarrow{\iota} \mathcal{H} \quad \text{such that} \quad \iota(T_{\tilde{w}}) = (-1)^{\ell(w)} T_{\tilde{w}}^* \quad \text{for } \tilde{w} \in W(1),$$

and [Vignéras 2013a, Proposition 4.13 2)]:

$$(4) \qquad T_{\tilde{s}}^* := T_{\tilde{s}} - \mathfrak{c}(\tilde{s}) \quad \text{for } \tilde{s} \in S^{\mathrm{aff}}(1), \qquad T_{\tilde{w}}^* := T_{\tilde{s}_1}^* \cdots T_{\tilde{s}_r}^* T_{\tilde{u}} \quad \text{for } \tilde{w} \in W(1)$$

of reduced decomposition $\tilde{w} = \tilde{s}_1 \cdots \tilde{s}_{\ell(w)}\tilde{u}$.

Remark 2.12. We have $\zeta(T_{\tilde{w}}^*) = T_{(\tilde{w})^{-1}}^*$ for $\tilde{w} \in W(1)$, ζ and ι commute, $\zeta_M(\mathcal{H}_{M^\epsilon}) = \mathcal{H}_M^{-\epsilon}$ and $\theta \circ \zeta_M = \zeta \circ \theta$, $\theta^* \circ \zeta_M = \zeta \circ \theta^*$.

2C. ϵ-*alcove walk basis.* We define a basis of $\mathcal{H}$ associated to $\epsilon \in \{+, -\}$ and an orientation o of $(V, \mathfrak{H})$, which we call an ϵ-alcove walk basis associated to o.

For $s \in S^{\mathrm{aff}}$, let α_s denote the positive affine root such that s is the orthogonal reflection with respect to $\operatorname{Ker} \alpha_s$. For an orientation o of $(V, \mathfrak{H})$, let $\mathcal{D}_o$ denote the corresponding (open) Weyl chamber in $(V, \mathfrak{H})$, $\mathfrak{A}_o$ the (open) alcove of vertex 0 contained in $\mathcal{D}_o$, and $o.w$ the orientation of Weyl chamber $w^{-1}(\mathfrak{D}_o)$ for $w \in W$. We recall [Vignéras 2013a]:

Definition 2.13. The following properties determine uniquely elements $E_o(\tilde{w}) \in \mathcal{H}$ for any orientation o of $(V, \mathfrak{H})$ and $\tilde{w} \in W(1)$. For $\tilde{w} \in W(1)$, $\tilde{s} \in S^{\text{aff}}(1)$, $\tilde{u} \in \Omega(1)$,

$$E_o(\tilde{s}) = \begin{cases} T_{\tilde{s}} & \text{if } \alpha_s \text{ is negative on } \mathfrak{A}_o, \\ T^*_{\tilde{s}} = T_{\tilde{s}} - \mathfrak{c}(\tilde{s}) & \text{if } \alpha_s \text{ is positive on } \mathfrak{A}_o, \end{cases} \tag{5}$$

$$E_o(\tilde{u}) = T_{\tilde{u}}, \tag{6}$$

$$E_o(\tilde{s})E_{o.s}(\tilde{w}) = q_{s,w}E_o(\tilde{s}\tilde{w}). \tag{7}$$

They imply, for $w' \in W$, $\lambda \in \Lambda$,

$$E_o(\tilde{w}')E_{o.w'}(\tilde{w}) = q_{w',w}E_o(\tilde{w}'\tilde{w}), \quad E_o(\tilde{\lambda})E_o(\tilde{w}) = q_{\lambda,w}E_o(\tilde{\lambda}\tilde{w}). \tag{8}$$

We recall that λ acts on V by translation by $\nu(\lambda)$. The Weyl chamber $\mathcal{D}_o$ of the orientation o is characterized by

$$E_o(\tilde{\lambda}) = T_{\tilde{\lambda}} \text{ when } \nu(\lambda) \text{ belongs to the closure of } \mathcal{D}_o. \tag{9}$$

The alcove walk basis of $\mathcal{H}$ associated to o is $(E_o(\tilde{w}))_{\tilde{w}\in W(1)}$ [Vignéras 2013a]. The Bernstein basis $(E(\tilde{w}))_{\tilde{w}\in W(1)}$ is the alcove walk basis associated to the antidominant orientation (of Weyl chamber $\mathcal{D}^-$). By Remark 2.3 and [Vignéras 2013a],

$$E(\tilde{w}) = T_{\tilde{w}} \quad \text{for } w \in \Lambda^+ \cup W_0, \qquad E(\tilde{w}) = T^*_{\tilde{w}} \quad \text{for } w \in \Lambda^-.$$

Definition 2.14. The ϵ-alcove walk basis $(E^\epsilon_o(\tilde{w}))_{\tilde{w}\in W(1)}$ of $\mathcal{H}$ associated to o is

$$E^\epsilon_o(\tilde{w}) := \begin{cases} E_o(\tilde{w}) & \text{if } \epsilon = +, \\ \zeta(E_o(\tilde{w}^{-1})) & \text{if } \epsilon = -. \end{cases} \tag{10}$$

Lemma 2.15. *The elements $E^-_o(\tilde{w})$ for any orientation o of $(V, \mathcal{H})$ and $\tilde{w} \in W(1)$ are determined by the following properties. For $\tilde{w} \in W(1)$, $\tilde{s} \in S^{\text{aff}}(1)$, $\tilde{u} \in \Omega(1)$,*

$$E^-_o(\tilde{s}) = E_o(\tilde{s}), \quad E^-_o(\tilde{u}) = E_o(\tilde{u}), \tag{11}$$

$$E^-_{o.s}(\tilde{w})E^-_o(\tilde{s}) = q_{w,s}E^-_o(\tilde{w}\tilde{s}). \tag{12}$$

They imply, for $w' \in W$, $\lambda \in \Lambda$,

$$E^-_{o.w'^{-1}}(\tilde{w})E^-_o(\tilde{w}') = q_{w,w'}E^-_o(\tilde{w}\tilde{w}'), \quad E^-_o(\tilde{w})E^-_o(\tilde{\lambda}) = q_{w,\lambda}E^-_o(\tilde{w}\tilde{\lambda}). \tag{13}$$

Proof.

$$\begin{aligned} E^-_o(\tilde{s}) &= \zeta(E_o((\tilde{s})^{-1})) = E_o(\tilde{s}), \\ E^-_o(\tilde{w}\tilde{u}) &= \zeta(E_o((\tilde{w}\tilde{u})^{-1})) = \zeta(E_o((\tilde{u})^{-1}(\tilde{w})^{-1})) = \zeta(T_{(\tilde{u})^{-1}}E_o((\tilde{w})^{-1})) \\ &= \zeta(E_o((\tilde{w})^{-1}))T_{\tilde{u}} = E^-_o(\tilde{w})T_{\tilde{u}}, \end{aligned}$$

$$\begin{aligned} E^-_{o.s}(\tilde w)E^-_o(\tilde s) &= \zeta(E_{o.s}((\tilde w)^{-1}))\zeta(E_o((\tilde s)^{-1})) = \zeta(E_o((\tilde s)^{-1})E_{o.s}((\tilde w)^{-1})) \\ &= q_{s,w^{-1}}\zeta(E_o((\tilde s)^{-1}(\tilde w)^{-1})) = q_{w,s}\zeta(E_o((\tilde w\tilde s)^{-1})) = q_{w,s}E^-_o(\tilde w\tilde s). \end{aligned}$$

We used that $q_w = q_{w^{-1}}$ implies

$$q_{w_1^{-1},w_2^{-1}} = (q_{w_1^{-1}}q_{w_2^{-1}}q^{-1}_{w_1^{-1}w_2^{-1}})^{1/2} = (q_{w_1}q_{w_2}q^{-1}_{w_2w_1})^{1/2} = q_{w_2,w_1}$$

for $w_1, w_2 \in W$. □

The ϵ-alcove walk bases satisfy the triangular decomposition

$$E^\epsilon_o(\tilde w) - T_{\tilde w} \in \sum_{\tilde w' \in W(1),\, \tilde w' < \tilde w} RT_{\tilde w'}. \tag{14}$$

Remark 2.16. The basis $E_-(\tilde w)$ introduced in [Abe 2014] is the $-$ alcove walk basis associated to the dominant Weyl chamber, satisfying $E_-(\tilde w) = T^*_{\tilde w}$ if $w \in W_0$ and $E_-(\tilde\lambda) = T_{\tilde\lambda}$ if $\lambda \in \Lambda^-$.

Let V_M be the real vector space of dual generated by Σ_M with a $W_{M,0}$-invariant scalar product and the corresponding set $\mathfrak{H}_M$ of affine hyperplanes.

Lemma 2.17. *If $\epsilon, \epsilon' \in \{+,-\}$ and o_M is any orientation of $(V_M, \mathfrak{H}_M)$, then $(E^{\epsilon'}_{o_M}(\tilde w))_{\tilde w \in W_{M^\epsilon}(1)}$ is a basis of $\mathcal{H}_{M^\epsilon}$.*

When $\mathfrak{q}(s) = 0$, see [Abe 2014, Lemma 4.2].

Proof. A basis of $\mathcal{H}_{M^\epsilon}$ is $(T^M_{\tilde w})_{\tilde w \in W_{M^\epsilon}(1)}$. As $w <_M w'$ and $w' \in W_{M^\epsilon}$ implies $w \in W_{M^\epsilon}$ (Lemma 2.6), the triangular decomposition (14) implies that $(E^{\epsilon'}_{o_M}(\tilde w))_{\tilde w \in W_{M^\epsilon}(1)}$ is a basis of $\mathcal{H}_{M^\epsilon}$. □

Lemma 2.18. *The ϵ-Bernstein basis satisfies $E^\epsilon(\tilde w) = T_{\tilde w}$ if $w \in \Lambda^\epsilon \cup W_0$.*

Proof. The inverse of $\Lambda^+ \cup W_0$ is $\Lambda^- \cup W_0$; hence

$$E^-(\tilde w) = \zeta(E((\tilde w)^{-1})) = \zeta(T_{(\tilde w)^{-1}}) = T_{\tilde w}. \qquad \square$$

The ϵ-Bernstein elements on $W_{M^\epsilon}(1)$ are compatible with θ and θ^*:

Proposition 2.19 [Ollivier 2010, Proposition 4.7; 2014, Lemma 3.8; Abe 2014, Lemma 4.5].

$$\theta(E^\epsilon_M(\tilde w)) = \theta^*(E^\epsilon_M(\tilde w)) = E^\epsilon(\tilde w) \quad \textit{for } \tilde w \in W_{M^\epsilon}(1).$$

Proof. It suffices to prove the proposition when the $\mathfrak{q}(s)$ are invertible. Let $\tilde w \in W(1)$. We write $\tilde w = \tilde\lambda\tilde u = \tilde\lambda_1(\tilde\lambda_2)^{-1}\tilde u$ with $u \in W_0$, and λ_1, λ_2 in Λ^ϵ. We have

$$E(\tilde\lambda_1)E((\tilde\lambda_2)^{-1}) = q_{\lambda_1,\lambda_2^{-1}}E(\tilde\lambda), \qquad E(\tilde\lambda_2)E((\tilde\lambda_2)^{-1}) = q_{\lambda_2,\lambda_2^{-1}} = q_{\lambda_2},$$
$$E(\tilde\lambda_1)E((\tilde\lambda_2)^{-1})E(\tilde u) = q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u}E(\tilde w).$$

We suppose the $\mathfrak{q}(s)$ are invertible. Then,

$$E(\tilde{w}) = q_{\lambda_2}(q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}E(\tilde{\lambda}_1)E(\tilde{\lambda}_2)^{-1}E(\tilde{u}), \tag{15}$$
$$= q_{\lambda_2}(q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}\begin{cases} T_{\tilde{\lambda}_1}T_{\tilde{\lambda}_2}^{-1}T_{\tilde{u}} & \text{if } \epsilon = +, \\ T^*_{\tilde{\lambda}_1}(T^*_{\tilde{\lambda}_2})^{-1}T_{\tilde{u}} & \text{if } \epsilon = -. \end{cases}$$

We suppose now $w \in W_{M^\epsilon}$, that is, $\lambda \in \Lambda_{M^\epsilon}$, $u \in W_{M,0}$. Note $\Lambda^\epsilon \subset \Lambda_{M^\epsilon}$ and $q_{M,\lambda,u} = q_{\lambda,u}$ (Lemma 2.7). If $\epsilon = +$, we have

$$E_M(\tilde{w}) = q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}T^M_{\tilde{\lambda}_1}(T^M_{\tilde{\lambda}_2})^{-1}T^M_{\tilde{u}},$$

and

$$\begin{aligned}\theta(E_M(\tilde{w})) &= q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}T_{\tilde{\lambda}_1}T_{\tilde{\lambda}_2}^{-1}T_{\tilde{u}} \\ &= q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}q_{\lambda_2}^{-1}q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u}E(\tilde{w}) \\ &= q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda_2})^{-1}q_{\lambda_1,\lambda_2^{-1}}E(\tilde{w}).\end{aligned}$$

The triangular decomposition of $E_M(\tilde{w})$ and $E(\tilde{w})$ implies

$$q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda_2})^{-1}q_{\lambda_1,\lambda_2^{-1}} = 1$$

and $\theta(E_M(\tilde{w})) = E(\tilde{w})$ for $w \in W_{M^+}$. If $\epsilon = -$, the same argument applied to θ^* gives $\theta^*(E_M(\tilde{w})) = E(\tilde{w})$ for $w \in W_{M^-}$.

By Remark 2.12, $\zeta \circ \theta = \theta \circ \zeta_M$, $\zeta \circ \theta^* = \theta \circ \zeta^*_M$, $W_{M^{-\epsilon}}$ is the inverse of W_{M^ϵ} and $E^-(\tilde{w}) = \zeta(E((\tilde{w})^{-1}))$. Hence for $w \in W_{M^-}$,

$$E^-(\tilde{w}) = (\zeta \circ \theta)(E_M((\tilde{w})^{-1})) = (\theta \circ \zeta_M)(E_M((\tilde{w})^{-1})) = \theta(E^-_M(\tilde{w})).$$

Similarly, for $w \in W_{M^+}$, we have $E^-(\tilde{w}) = \theta^*(E^-_M(\tilde{w}))$. □

2D. *w_0-twist.* Let $S_M \subset S$, w_0 denote the longest element of W_0 and $S_{w_0(M)} = w_0 S_M w_0 \subset w_0 S w_0 = S$. The longest element $w_{M,0}$ of $W_{M,0}$ satisfies $w_{M,0}(\Sigma^\epsilon_M) = \Sigma^{-\epsilon}_M$, and $w_{M,0}(\Sigma^\epsilon - \Sigma^\epsilon_M) = \Sigma^\epsilon - \Sigma^\epsilon_M$. The longest element $w_{w_0(M),0}$ of $W_{w_0(M),0}$ is $w_0 w_{M,0} w_0$.

Let $w_0^M := w_0 w_{M,0}$. Its inverse ${}^M w_0 := w_{M,0} w_0$ is $w_0^{w_0(M)}$ and $w_0^M(\Sigma^\epsilon_M) = \Sigma^\epsilon_{w_0(M)}$. This implies that $w_0^M(\Sigma^{\mathrm{aff},\epsilon}_M) = \Sigma^{\mathrm{aff},\epsilon}_{w_0(M)}$. Indeed the image by w_0^M of the simple roots of Σ_M is the set of simple roots of $\Sigma_{w_0(M)}$, and this remains true for the simple affine roots which are not roots. Note that the irreducible components $\Sigma_{M,i}$ of Σ_M have a unique highest root $a_{M,i}$, and that the $-a_{M,i}+1$ are the simple affine roots of Σ which are not roots. We have $w_0^M(-a_{M,i}+1) = w_0 w_{M,0}(-a_{M,i}+1) = w_0(a_{M,i})+1$. The irreducible components of $\Sigma_{w_0(M)}$ are the $w_0(\Sigma_{M,i})$ and $-w_0(a_{M,i})$ is the highest root of $w_0(\Sigma_{M,i})$.

We deduce

$$w_0^M S_M^{\mathrm{aff}} (w_0^M)^{-1} = S_{w_0(M)}^{\mathrm{aff}},$$
$$w_0^M W_{M,0}^{\mathrm{aff}} (w_0^M)^{-1} = W_{w_0(M,)0}^{\mathrm{aff}}, \quad w_0^M W_{M,0} (w_0^M)^{-1} = W_{w_0(M,)0}.$$

We have $\Lambda = w_0^M \Lambda (w_0^M)^{-1}$ and $w_0^M \Lambda_M^{\epsilon} (w_0^M)^{-1} = \Lambda_{w_0(M)}^{-\epsilon}$. Recalling $W_M = \Lambda \rtimes W_{M,0}$, $W_{M^\epsilon} = \Lambda_{M^\epsilon} \rtimes W_{M,0}$ and the group Ω_M of elements which stabilize $\mathfrak{A}_M$, we deduce

$$\begin{gathered} w_0^M W_M (w_0^M)^{-1} = W_{w_0(M)}, \\ w_0^M \Omega_M (w_0^M)^{-1} = \Omega_{w_0(M)}, \quad w_0^M W_{M^\epsilon} (w_0^M)^{-1} = W_{w_0(M)}^{-\epsilon}. \end{gathered} \tag{16}$$

Let ν_M denote the action of W_M on V_M and $\mathfrak{A}_M$ the dominant alcove of $(V_M, \mathfrak{H}_M)$. The linear isomorphism

$$V_M \xrightarrow{w_0^M} V_{w_0(M)}, \qquad \langle \alpha, x \rangle = \langle w_0^M(\alpha), w_0^M(x) \rangle \quad \text{for } \alpha \in \Sigma_M,$$

satisfies

$$w_0^M \circ \nu_M(w) = \nu_{w_0(M)}(w_0^M w (w_0^M)^{-1}) \circ w_0^M \quad \text{for } w \in W_M.$$

It induces a bijection $\mathfrak{H}_M \to \mathfrak{H}_{w_0(M)}$ sending $\mathfrak{A}_M$ to $\mathfrak{A}_{w_0(M)}$, a bijection $\mathfrak{D}_M \mapsto w_0^M(\mathfrak{D}_M)$ between the Weyl chambers, and a bijection $o_M \mapsto w_0^M(o_M)$ between the orientations such that $\mathfrak{D}_{w_0^M(o_M)} = w_0^M(\mathfrak{D}_{o_M})$.

Proposition 2.20. *Let $\tilde{w}_0^M \in W_0(1)$ be a lift of w_0^M. The R-linear map*

$$\mathcal{H}_M \xrightarrow{j} \mathcal{H}_{w_0(M)}, \qquad T_{\tilde{w}}^M \mapsto T_{\tilde{w}_0^M \tilde{w} (\tilde{w}_0^M)^{-1}}^{w_0(M)} \quad \textit{for } \tilde{w} \in W_M(1),$$

is an R-algebra isomorphism sending $\mathcal{H}_{M^\epsilon}$ onto $\mathcal{H}_{w_0(M)^{-\epsilon}}$ and respecting the ϵ'-alcove walk basis

$$j(E_{o_M}^{\epsilon'}(\tilde{w})) = E_{w_0^M(o_M)}^{\epsilon'}(\tilde{w}_0^M \tilde{w} (\tilde{w}_0^M)^{-1}) \quad \textit{for } \tilde{w} \in W_M(1)$$

for any orientation o_M of $(V_M, \mathfrak{H}_M)$ and $\epsilon, \epsilon' \in \{+, -\}$.

Proof. The proof is formal using the properties given above the proposition and the characterization of the elements in the ϵ'-alcove walks bases given by (5), (6), (7) if $\epsilon' = +$ and (11), (12) if $\epsilon' = -$. □

We study now the transitivity of the w_0-twist. Let $S_M \subset S_{M'} \subset S$. We have the subset $w_{M',0} S_M w_{M',0} = S_{w_{M',0}(M)}$ of S and we associate to the conjugation by a lift $\tilde{w}_{M',0}$ of $w_{M',0}$ in $W(1)$ an isomorphism $\mathcal{H}_M \xrightarrow{j'} \mathcal{H}_{w_{M',0}(M)}$ similar to $\mathcal{H}_M \xrightarrow{j} \mathcal{H}_{w_0(M)}$ in Proposition 2.20. We will show that j factorizes by j'.

We have $w_0^M = w_0^{M'} w_{M'}^M$, where $w_{M'}^M := w_{M',0} w_{M,0}$ (equal to w_0^M if $S = S_{M'}$),

$$W_{w_{M',0}(M)} = w_{M'}^M W_M (w_{M'}^M)^{-1},$$
$$W_{w_0(M)} = w_0^{M'} W_{w_{M',0}(M)} (w_0^{M'})^{-1} = w_0^M W_M (w_0^M)^{-1}.$$

For $S_{M_1} \subset S_{M'}$, let $W_{M_1^{\epsilon,M'}} \subset W_{M_1}$ denote the submonoid associated to $S_{M'}^{\mathrm{aff}}$ as in Definition 2.1 and replace the pair $(\Sigma^+ - \Sigma^+_{M_1}, \Sigma^{\mathrm{aff},+})$ by $(\Sigma^+_{M'} - \Sigma^+_{M_1}, \Sigma^{\mathrm{aff},+}_{M'})$. We note that

$$W_{w_{M',0}(M)^{-\epsilon,M'}} = w_{M'}^M W_{M^\epsilon} (w_{M'}^M)^{-1},$$
$$W_{w_0(M)^{-\epsilon}} = w_0^{M'} W_{w_{M',0}(M)^{-\epsilon,M'}} (w_0^{M'})^{-1} = w_0^M W_{M^\epsilon} (w_0^M)^{-1}.$$

Let $\tilde{w}_0^M, \tilde{w}_0^{M'}, \tilde{w}_{M'}^M$ be in $W_0(1)$ lifting $w_0^M, w_0^{M'}, w_{M'}^M$ and satisfying $\tilde{w}_0^M = \tilde{w}_0^{M'} \tilde{w}_{M'}^M$. The algebra isomorphisms

$$\mathcal{H}_M \xrightarrow{j'} \mathcal{H}_{w_{M',0}(M)}, \quad \mathcal{H}_{M'} \xrightarrow{j''} \mathcal{H}_{w_0(M')}, \quad \mathcal{H}_M \xrightarrow{j} \mathcal{H}_{w_0(M)}$$

defined by $\tilde{w}_{M'}^M, \tilde{w}_0^{M'}, \tilde{w}_0^M$ respectively, as in Proposition 2.20, send the ϵ-subalgebra to the $-\epsilon$-subalgebra and are compatible with the ϵ'-Bernstein bases. We cannot compose j' with the map j'' defined by $\tilde{w}_0^{M'}$, but we can compose j' with the bijective R-linear map defined by the conjugation by $\tilde{w}_0^{M'}$ in $W(1)$

$$\mathcal{H}_{w_{M',0}(M)} \xrightarrow{k''} \mathcal{H}_{w_0(M)}, \qquad T_{\tilde{w}}^{w_{M',0}(M)} \mapsto T^{w_0(M)}_{\tilde{w}_0^{M'} \tilde{w} (\tilde{w}_0^{M'})^{-1}} \quad \text{for } \tilde{w} \in W_{w_{M',0}(M)}(1).$$

Proposition 2.21. *We have $j = k'' \circ j'$ and k'' is an R-algebra isomorphism respecting the ϵ-subalgebras and the ϵ-Bernstein bases*: $k''(\mathcal{H}_{w_{M',0}(M)^\epsilon}) = \mathcal{H}_{w_0(M)^\epsilon}$ *and* $k''(E^\epsilon_{w_{M',0}(M)}(\tilde{w})) = E^\epsilon_{w_0(M)}(\tilde{w}_0^{M'} \tilde{w} (\tilde{w}_0^{M'})^{-1})$ *for* $\epsilon \in \{+,-\}$, $w \in W_{w_{M',0}(M)}$.

Proof. The relations between the groups W_* and W_{*^ϵ} imply obviously that $j = k'' \circ j'$ and that k'' respects the ϵ-subalgebras.

Now, k'' is an algebra isomorphism respecting the ϵ'-Bernstein bases because j, j' are algebra isomorphisms respecting the ϵ'-Bernstein bases and $k'' = j \circ (j')^{-1}$. □

2E. *Distinguished representatives of W_0 modulo $W_{M,0}$*. The classical set ${}^M W_0$ of representatives on $W_{M,0} \backslash W_0$ is equal to ${}_M D_1 = {}_M D_2$, where

$$(17) \qquad {}_M D_1 := \{d \in W_0 \mid d^{-1}(\Sigma_M^+) \in \Sigma^+\},$$
$$(18) \qquad {}_M D_2 := \{d \in W_0 \mid \ell(wd) = \ell(w) + \ell(d) \text{ for all } w \in W_{M,0}\}$$

[Carter 1985, §2.3.3]. The properties of ${}^M W_0$ used in this article that we are going to prove are probably well known. Note that the classical set of representatives of $W_0 \backslash W$ is studied in [Vignéras 2015a], that $+$ can be replaced by $\epsilon \in \{+,-\}$ in the definition of ${}_M D_1$, that ${}^M w_0 = w_{M,0} w_0 \in {}^M W_0$ and that ${}^M W_0 \cap S = S - S_M$.

Taking inverses, we get the classical set W_0^M of representatives on $W_0/W_{M,0}$ equal to $D_{M,1} = D_{M,2}$, where

(19) $$D_{M,1} := \{d \in W_0 \mid d(\Sigma_M^+) \subset \Sigma^+\},$$

(20) $$D_{M,2} := \{d \in W_0 \mid \ell(dw) = \ell(d) + \ell(w) \text{ for all } w \in W_{M,0}\}.$$

The length of an element of W is equal to the length of its inverse, and [Vignéras 2013a, Corollary 5.10] gives that for $\lambda \in \Lambda$, $w \in W_0$,

(21) $$\ell(\lambda w) = \sum_{\beta \in \Sigma^+ \cap w(\Sigma^+)} |\beta \circ \nu(\lambda)| + \sum_{\beta \in \Phi_w} |-\beta \circ \nu(\lambda) + 1|,$$

where $\Phi_w := \Sigma^+ \cap w(\Sigma^-)$. If $w = s_1 \cdots s_{\ell(w)}$ is a reduced decomposition in (W_0, S), $\Phi_w = \{\alpha_{s_1}\} \cup s_1(\Phi_{s_1 w})$ and $\ell(w)$ is the order of Φ_w. If $w \in W_{M,0}$, we have $\Phi_w \subset \Sigma_M^+$. Let $\ell_\beta(\lambda w)$ denote the contribution of $\beta \in \Sigma^+$ to the right side of (21).

We show now that $W_{M,0}$ can be replaced by W_{M^+} in (18) and by W_{M^-} in (20) (taking the inverses). It is also a variant of the equivalence $\ell(\lambda w) < \ell(\lambda) + \ell(w) \Leftrightarrow \beta \circ \nu(\lambda) > 0$ for some $\beta \in \Phi_w$ for λ, w as in (21).

Lemma 2.22.

(i) $$\ell(wd) = \ell(w) + \ell(d) \quad \textit{for } w \in W_{M^+} \textit{ and } d \in {}^M W_0,$$
$$\ell(dw) = \ell(d) + \ell(w) \quad \textit{for } w \in W_{M^-} \textit{ and } d \in W_0^M.$$

(ii) *If* $\lambda \in \Lambda$, $w \in W_{M,0}$, $d \in {}^M W_0$, *then* $\ell(\lambda w d) < \ell(\lambda w) + \ell(d)$ *is equivalent to*
$$w(\beta) \circ \nu(\lambda) > 0 \quad \textit{and} \quad d^{-1}(\beta) \in \Sigma^- \quad \textit{for some } \beta \in \Sigma^+ - \Sigma_M^+.$$

Proof. [Ollivier 2010, Lemma 2.3; Abe 2014, Lemma 4.8]. Let $\lambda \in \Lambda$, $w \in W_{M,0}$, $d \in {}^M W_0$ and $\beta \in \Sigma^+$.

Suppose $\beta \in \Sigma_M^+$. Then $\ell_\beta(d) = 0$, $\Phi_d = \varnothing$ because $d^{-1}(\Sigma_M^\epsilon) \subset \Sigma^\epsilon$ by (17), and $\ell_\beta(\lambda w d) = \ell_\beta(\lambda w)$ because $w^{-1}(\beta) \in \Sigma^\epsilon \Leftrightarrow w^{-1}(\beta) \in \Sigma_M^\epsilon \Rightarrow d^{-1} w^{-1}(\beta) \in \Sigma^\epsilon$ by (17).

Suppose $\beta \in \Sigma^+ - \Sigma_M^+$. Then $w^{-1}(\beta) \in \Sigma^+ - \Sigma_M^+$ and $\ell_\beta(\lambda w) = |\beta \circ \nu(\lambda)|$.

The number $\ell(d)$ of $\beta \in \Sigma^+ - \Sigma_M^+$ such that $d^{-1}(\beta) \in \Sigma^-$ is equal to the number of $\beta \in \Sigma^+ - \Sigma_M^+$ such that $(wd)^{-1}(\beta) \in \Sigma^-$.

When $\lambda \in \Lambda_{M^+}$ and $(wd)^{-1}(\beta) \in \Sigma^-$, we have $\beta \circ \nu(\lambda) \le 0$ and $\ell_\beta(\lambda w d) = |\beta \circ \nu(\lambda)| + 1$. Therefore $\ell(\lambda w d) = \ell(\lambda w) + \ell(d)$, which gives (i).

When $\lambda \notin \Lambda - \Lambda_{M^+}$, $\ell(\lambda w d) < \ell(\lambda w) + \ell(d)$ if and only if there exists $\beta \in \Sigma^+ - \Sigma_M^+$ such that $\beta \circ \nu(\lambda) > 0$ and $d^{-1} w^{-1}(\beta) \in \Sigma^-$. This gives (ii) because $\beta \mapsto w^{-1}(\beta)$ is a permutation map of $\Sigma^+ - \Sigma_M^+$. □

Lemma 2.23. (i) *For* $\lambda \in \Lambda$, $w \in W_0$, *we have* $q_\lambda = q_{w\lambda w^{-1}}$, $q_w = q_{w_0 w w_0}$, *and*
$$\ell(w_0) = \ell(w) + \ell(w^{-1} w_0) = \ell(w_0 w^{-1}) + \ell(w).$$

(ii) *For* $w \in W_{M,0}$, *we have* $q_w = q_{w_0^M w (w_0^M)^{-1}}$.

Proof. (i) See [Vignéras 2013a, Proposition 5.13]. The length on W_0 is invariant by inverse and by conjugation by w_0 because $w_0 S w_0 = S$ and by [Bourbaki 1968, VI, §1, Corollaire 3].

(ii) We have $q_w = q_{w_{M,0} w w_{M,0}^{-1}} = q_{w_0^M w (w_0^M)^{-1}}$ for $w \in W_{M,0}$. □

Lemma 2.24. $$W_0^M = W_0^{w_0(M)} w_0^M = w_0 W_0^M w_{M,0}.$$

Proof. By (19),

$$d \in W_0^M \Longleftrightarrow d(\Sigma_M^+) \subset \Sigma^+ \Longleftrightarrow d(w_0^M)^{-1}(\Sigma_{w_0(M)}^+) \subset \Sigma^+ \Longleftrightarrow d(w_0^M)^{-1} \in W_0^{w_0(M)}.$$

This proves the equality $W_0^M = W_0^{w_0(M)} w_0^M$. The equality $W_0^M = w_0 W_0^M w_{M,0}$, follows from

$$\begin{aligned} d(w_0^M)^{-1}(\Sigma_{w_0(M)}^+) \subset \Sigma^+ &\Longleftrightarrow w_0 d w_{M,0} w_0(\Sigma_{w_0(M)}^+) \subset \Sigma^- \\ &\Longleftrightarrow w_0 d w_{M,0}(\Sigma_M^-) \subset \Sigma^- \Longleftrightarrow w_0 d w_{M,0} \in W_0^M. \end{aligned}$$ □

Remark 2.25. $W_M = \Lambda \rtimes W_{M,0}$ but $q_{\lambda w} = q_{w_0^M \lambda w (w_0^M)^{-1}}$ could be false for $\lambda \in \Lambda$, $w \in W_{M,0}$ such that $\ell(\lambda w) < \ell(\lambda) + \ell(w)$.

Lemma 2.26. *We have $\ell(w_0^M) = \ell(w_0^M d^{-1}) + \ell(d)$ for any $d \in W_0^M$.*

Proof. For $d \in W_0^M$, we have $\ell(d w_{M,0}) = \ell(d) + \ell(w_{M,0})$ by (20) and $w = w_0^M d^{-1}$ satisfies $w_0 = w d w_{M,0}$ and $\ell(w_0) = \ell(w) + \ell(d w_{M,0})$. We have $w_0^M = w_0 w_{M,0} = wd$ and $\ell(w_0^M) = \ell(w_0) - \ell(w_{M,0}) = \ell(w) + \ell(d)$. □

The Bruhat order $x \le x'$ in W_0 is defined by the following equivalent two conditions:

(i) There exists a reduced decomposition of x' such that by omitting some terms one obtains a reduced decomposition of x.

(ii) For any reduced decomposition of x', by omitting some terms one obtains a reduced decomposition of x.

A reduced decomposition of $w \in W_0$ followed by a reduced decomposition of $w' \in W_0$ is a reduced decomposition of ww' if and only $\ell(ww') = \ell(w) + \ell(w')$. A reduced decomposition of $d \in W_0^M$ cannot end by a nontrivial element $w \in W_{M,0}$.

Lemma 2.27. *For $w, w' \in W_{M,0}$, $d, d' \in W_0^M$, we have $dw \le d'w'$ if and only if there exists a factorisation $w = w_1 w_2$ such that $\ell(w) = \ell(w_1) + \ell(w_2)$, $dw_1 \le d'$ and $w_2 \le w'$.*

Proof. We prove the direction "only if" (the direction "if" is obvious). If $dw \le d'w'$, a reduced decomposition of dw is obtained by omitting some terms of the product of a reduced decomposition of d' and of a reduced decomposition of w'. We have $dw = d_1 w_2$ with $d_1 \le d'$, $w_2 \le w'$ and $\ell(d_1 w_2) = \ell(d_1) + \ell(w_2)$. We have $d_1 =$

dw_1, $w_1 := ww_2^{-1}$. As $w, w_2 \in w_{M,0}$ and $d \in W_0^M$, we have $\ell(dw_1) = \ell(d) + \ell(w_1)$ and $\ell(dw) = \ell(d) + \ell(w)$. Hence $\ell(w_1) + \ell(w_2) = \ell(w)$. □

Lemma 2.28. *Let $d' \in {}^{w_0(M)}W_0$, $d \in W_0^M$.*

(i) *If there exists $u \in W_{M,0}$, $u' \in W_0^M$ such that $v = w_0^M u \leq w = du'$, then $d = w_0^M$.*

(ii) *We have $d'd \in w_0^M W_{M,0}$ if and only if $d'd = w_0^M$.*

Proof. (i) As $\ell(w) = \ell(d) + \ell(u')$, we have $u = u_1 u_2$ with $w_0^M u_1 \leq d$, $u_2 \leq u'$ and $u_1, u_2 \in W_{M,0}$ (Lemma 2.27). We have

$$\ell(w_0^M u_1) = \ell(w_0^M) + \ell(u_1) = \ell(w_0^M d^{-1}) + \ell(d) + \ell(u_1)$$

(Lemma 2.26). Hence $d = w_0^M$, $u_1 = 1$.

(ii) If there exists $u \in W_{M,0}$ such that $d = d'^{-1} w_0^M u$, we have $d = d'^{-1} w_0^M$ because $d'^{-1} w_0^M \in W_0^M$ (Lemma 2.24). □

2F. *$\mathcal{H}$ as a left $\theta(\mathcal{H}_{M^+})$-module and as a right $\theta^*(\mathcal{H}_{M^-})$-module.* We prove Theorem 1.4(iv) on the structure of the left $\theta(\mathcal{H}_{M^+})$-module $\mathcal{H}$ and its variant for the right $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{H}$. We suppose $S_M \neq S$.

Recalling the properties (i), (ii), (iii) of Theorem 1.4, $\mathcal{H}_M = \mathcal{H}_{M^+}[(T^M_{\tilde{\mu}_M})^{-1}]$ is the localisation of the subalgebra $\mathcal{H}_{M^+}$ at the central element $T^M_{\tilde{\mu}_M}$. The algebra $\mathcal{H}_{M^+}$ embeds in $\mathcal{H}$ by θ. Recalling (17), (18) we choose a lift $\tilde{d} \in W(1)$ for any element d in the classical set of representatives ${}^M W_0$ of $W_{M,0} \backslash W_0$. We define

$$\mathcal{V}_{M^+} = \sum_{d \in {}^M W_0} \theta(\mathcal{H}_{M^+}) T_{\tilde{d}}. \tag{22}$$

Proposition 2.29. (i) *$\mathcal{V}_{M^+}$ is a free left $\theta(\mathcal{H}_{M^+})$-module of basis $(T_{\tilde{d}})_{d \in {}^M W_0}$.*

(ii) *For any $h \in \mathcal{H}$, there exists $r \in \mathbb{N}$ such that $T^r_{\tilde{\mu}_M} h \in \mathcal{V}_{M^+}$.*

(iii) *If $\mathfrak{q} = 0$, $T_{\tilde{\mu}_M}$ is a left and right zero divisor in $\mathcal{H}$.*

For $\mathrm{GL}(n, F)$, (ii) is proved in [Ollivier 2010, Proposition 4.7] for $(\mathfrak{q}(s)) = (0)$. When the $\mathfrak{q}(s)$ are invertible, $T_{\tilde{w}}$ is invertible in $\mathcal{H}$ for $\tilde{w} \in W(1)$.

Proof. (i) As ${}^M W_0$ is a set of representatives of $W_{M^+} \backslash W$, a set of representatives of $W_{M^+}(1) \backslash W(1)$ is the set $\{\tilde{d} \mid d \in {}^M W_0\}$ of lifts of ${}^M W_0$ in $W(1)$. The canonical bases of $\mathcal{H}_{M^+}$ and of $\mathcal{H}$ are respectively $(T_{\tilde{w}})_{(\tilde{w}) \in W_{M^+}(1)}$ and $(T_{\tilde{w}\tilde{d}})_{(\tilde{w},d) \in W_{M^+}(1) \times {}^M W_0}$, and $T_{\tilde{w}\tilde{d}} = T_{\tilde{w}} T_{\tilde{d}}$ by the additivity of lengths (Lemma 2.22).

(ii) We can suppose that h runs over in a basis of $\mathcal{H}$. We cannot take the Iwahori–Matsumoto basis $(T_{\tilde{w}})_{\tilde{w} \in W(1)}$ and we explain why. For $\tilde{w} = \tilde{w}_M \tilde{d}$ with $\tilde{w}_M \in W_{M^+}(1)$, $d \in {}^M W_0$, we choose $r \in \mathbb{N}$ such that $\tilde{\mu}^r_M \tilde{w}_M \in W_{M^+}(1)$. By the length additivity (Lemma 2.22) $T_{\tilde{\mu}^r_M \tilde{w}} = T_{\tilde{\mu}^r_M \tilde{w}_M} T_{\tilde{d}}$ lies in $\theta(\mathcal{H}_{M^+}) T_{\tilde{d}}$, but we cannot deduce that $T_{\tilde{\mu}^r_M} T_{\tilde{w}}$ lies in $\theta(\mathcal{H}_{M^+}) T_{\tilde{d}}$.

We take the Bernstein basis satisfying Lemma 2.18 and we suppose that $\mathfrak{q}(s)=\boldsymbol{q}_s$ is indeterminate (but not invertible) with the same arguments as in [Ollivier 2010, Proposition 4.8]. Then $E(\tilde{d}) = T_{\tilde{d}}$ for $d \in {}^M W_0$. If we prove that $E(\tilde{\mu}_M^r \tilde{w})$ lies in $\theta(\mathcal{H}_{M^+})T_{\tilde{d}}$ then $E(\tilde{\mu}_M)^r E_o(\tilde{w}) = \boldsymbol{q}_{\mu_M^r, w} E(\tilde{\mu}_M^r \tilde{w})$ lies also in $\theta(\mathcal{H}_{M^+})T_{\tilde{d}}$. This implies $T_{\tilde{\mu}_M}^r E_o(\tilde{w}) \in \theta(\mathcal{H}_{M^+})T_{\tilde{d}}$.

Now we prove $E(\tilde{\mu}_M^r \tilde{w}) \in \theta(\mathcal{H}_{M^+})T_{\tilde{d}}$. We write $\tilde{w}_M = \tilde{\lambda}\tilde{w}_{M,0}$, $\tilde{\lambda} \in \Lambda(1)$, $\tilde{w}_{M,0} \in W_{M,0}(1)$. Recalling $E(*) = T_*$ for $* \in W_0(1)$ and the additivity of the length (Lemma 2.22),

$$\begin{aligned}\boldsymbol{q}_{\mu_M^r\lambda, w_{M,0}d} E(\tilde{\mu}_M^r \tilde{w}) = E(\tilde{\mu}_M^r \tilde{\lambda}) E(\tilde{w}_{M,0}\tilde{d}) = E(\tilde{\mu}_M^r \tilde{\lambda}) T_{\tilde{w}_{M,0}\tilde{d}} = E(\tilde{\mu}_M^r \tilde{\lambda}) T_{\tilde{w}_{M,0}} T_{\tilde{d}} \\ = \boldsymbol{q}_{\mu_M^r\lambda, w_{M,0}} E(\tilde{\mu}_M^r \tilde{w}_M) T_{\tilde{d}}.\end{aligned}$$

The monoid W_{M^ϵ} is a lower subset of $(W_M, \leq_M)$ (Lemma 2.6). The triangular decomposition (14) implies $E_M(\tilde{\mu}_M^r \tilde{w}_M) \in \mathcal{H}_{M^+}$. By Proposition 2.19, $E(\tilde{\mu}_M^r \tilde{w}_M) \in \theta(\mathcal{H}_{M^+})$ and by the additivity of the length (Lemma 2.22),

$$\boldsymbol{q}_{w_{M,0}d} = \boldsymbol{q}_{w_{M,0}} \boldsymbol{q}_d, \quad \boldsymbol{q}_{\mu_M^r \lambda w_{M,0} d} = \boldsymbol{q}_{\mu_M^r \lambda w_{M,0}} \boldsymbol{q}_d,$$

implying

$$\boldsymbol{q}_{\mu_M^r\lambda} \boldsymbol{q}_{w_{M,0}d} \boldsymbol{q}^{-1}_{\mu_M^r \lambda w_{M,0} d} = \boldsymbol{q}_{\mu_M^r\lambda} \boldsymbol{q}_{w_{M,0}} \boldsymbol{q}^{-1}_{\mu_M^r \lambda w_{M,0}};$$

hence $\boldsymbol{q}_{\mu_M^r\lambda, w_{M,0}d} = \boldsymbol{q}_{\mu_M^r\lambda, w_{M,0}}$.

(iii) We have $\ell(\mu_M) \neq 0$ and equivalently, $\nu(\mu_M) \neq 0$ in V. We choose $w \in W_0$ with $w(\nu(\mu_M)) \neq \nu(\mu_M)$. Then $\nu(w\mu_M w^{-1}) = w(\nu(\mu_M))$ and $\nu(\mu_M)$ belong to different Weyl chambers. The alcove walk basis $(E_o(\tilde{w}))_{\tilde{w}\in W(1)}$ of $\mathcal{H}$ associated to an orientation o of V of Weyl chamber containing $\nu(\mu_M)$ satisfies

$$\begin{aligned} E_o(\tilde{\mu}_M) &= T_{\tilde{\mu}_M}, \\ E_o(\tilde{\mu}_M) E_o(\tilde{w}\tilde{\mu}_M\tilde{w}^{-1}) &= E_o(\tilde{w}\tilde{\mu}_M\tilde{w}^{-1}) E_o(\tilde{\mu}_M) = 0. \end{aligned} \tag{23}$$

□

The properties of the left $\theta(\mathcal{H}_{M^+})$-module $\mathcal{H}$ transfer to properties of the right $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{H}$, with the involutive antiautomorphism $\zeta \circ \iota$ of $\mathcal{H}$ (Remark 2.12) exchanging $T_{\tilde{w}}$ and $(-1)^{\ell(w)} T^*_{(\tilde{w})^{-1}}$ for $\tilde{w} \in W(1)$, $\theta(\mathcal{H}_{M^+})$ and $\theta^*(\mathcal{H}_{M^-})$, $\mathcal{V}_{M^+}$ and

$$\mathcal{V}^*_{M^-} := \sum_{d \in W_0^M} T^*_{\tilde{d}} \theta^*(\mathcal{H}_{M^-}), \tag{24}$$

where $W_0^M = \{d'^{-1} \mid d' \in {}^M W_0\}$ is the set of classical representatives of $W_0/W_{M,0}$ (19), and $\tilde{d} = (\tilde{d}')^{-1}$ if $d = d'^{-1}$.

Corollary 2.30. (i) *$\mathcal{V}^*_{M^-}$ is a free right $\theta^*(\mathcal{H}_{M^-})$-module of basis $(T^*_{\tilde{d}})_{d \in W_0^M}$.*

(ii) *For any $h \in \mathcal{H}$, there exists $r \in \mathbb{N}$ such that $h(T^*_{(\tilde{\mu}_M)^{-1}})^r \in \mathcal{V}^*_{M^-}$.*

(iii) *If $\mathfrak{q} = 0$, $T^*_{\tilde{\mu}_M^{-1}}$ is a left and right zero divisor in $\mathcal{H}$.*

3. Induction and coinduction

3A. ***Almost localisation of a free module.*** In this chapter, all rings have unit elements.

Definition 3.1. Let A be a ring and $a \in A$ a central nonzero divisor. We say that a left A-module B is an almost a-localisation of a left A-module $B_D \subset B$ of basis D when:

(i) D is a finite subset of B, and the map $\oplus_{d\in D} A \to B$, $(x_d) \to \sum x_d d$, is injective,

(ii) for any $b \in B$, there exists $r \in \mathbb{N}$ such that $a^r b$ lies in $B_D := \sum_{d\in D} Ad$.

Example 3.2. Our basic example is $(A, a, B, D) = (\mathcal{H}_{M^+}, T_{\mu_M}, \mathcal{H}, (T_{\tilde{d}})_{d\in {}^M W_0})$ (Proposition 2.29).

As a is central and not a zero divisor in A, the a-localisation of A is ${}_aA = A_a = \cup_{n\in\mathbb{N}} Aa^{-n}$. The left multiplication by a in A is an injective A-linear endomorphism $A \to A$, $x \mapsto ax$, and the left multiplication by a in B is an A-linear endomorphism $a_B : x \mapsto ax$ of B which may be not injective; hence B may be not a flat A-module. The ring B is the union for $r \in \mathbb{N}$ of the A-submodules

$$ {}_rB_D := \{b \in B \mid a^r b \in B_D\}, $$

and looks like a localisation of B_D at a.

Definition 3.3. Let A be a ring and $a \in A$ a central nonzero divisor. We say that a right A-module B is an almost a-localisation of a right A-module ${}_DB$ of basis D if:

(i) D is a finite subset of B, and the map $\oplus_{d\in D} A \to B$, $(x_d) \to \sum d\, x_d$, is injective,

(ii) for any $b \in B$, there exists $r \in \mathbb{N}$ such that $ba^r \in {}_DB := \sum_{d\in D} dA$.

The ring B is the union for $r \in \mathbb{N}$ of the A-submodules

$$ {}_DB_r = \{b \in B \mid ba^r \in {}_DB\}. $$

Example 3.4. Our basic example is $(A, a, B, D) = (\mathcal{H}_{M^-}, T_{\mu_M^{-1}}, \mathcal{H}, (T_{\tilde{d}})_{d\in W_0^M})$ (Corollary 2.30).

We note that $(A_a, B) = (\mathcal{H}_M, \mathcal{H})$ in Example 3.2 and in Example 3.4.

3B. ***Induction and coinduction.***

3B1. For a ring A, let Mod_A denote the category of right A-modules, and ${}_A\mathrm{Mod}$ the category of left A-modules. The A-duality $X \mapsto X^* := \mathrm{Hom}_A(X, A)$ exchanges left and right A-modules.

A functor from Mod_A to a category admits a left adjoint if and only if it is left exact and commutes with small direct products (small projective limits); it admits a right adjoint if and only if it is right exact and commutes with small direct sums (small injective limits) [Vignéras 2013b, Proposition 2.10].

For two rings $A \subset B$, we define two functors

$$\text{the induction } I_A^B := - \otimes_A B,$$
$$\text{the coinduction } \mathbb{I}_A^B := \operatorname{Hom}_A(B, -) : \operatorname{Mod}_A \to \operatorname{Mod}_B,$$

where B is seen as an (A, B)-module for the induction, and as a (B, A)-module for the coinduction. For $\mathcal{M} \in \operatorname{Mod}_A$, we have $(m \otimes x)b = m \otimes xb$, $(fb)(x) = f(bx)$ if $x, b \in B$ and $m \in \mathcal{M}$, $f \in \operatorname{Hom}_A(B, \mathcal{M})$.

The restriction $\operatorname{Res}_A^B : \operatorname{Mod}_B \to \operatorname{Mod}_A$ is equal to $\operatorname{Hom}_B(B, -) = - \otimes_B B$, where B is seen first as an (A, B)-module and then as a (B, A)-module. The induction and the coinduction are the left and right adjoints of the restriction [Benson 1998, §2.8.2].

For two rings A and B and an (A, B)-module $\mathcal{J}$, the functor

$$- \otimes_A \mathcal{J} : \operatorname{Mod}_A \to \operatorname{Mod}_B \text{ is left adjoint to } \operatorname{Hom}_B(\mathcal{J}, -) : \operatorname{Mod}_B \to \operatorname{Mod}_A.$$

Let $\mathcal{M} \in \operatorname{Mod}_A$, $\mathcal{N} \in \operatorname{Mod}_B$. The adjunction is given by the functorial isomorphism

$$\operatorname{Hom}_B(\mathcal{M} \otimes_A \mathcal{J}, \mathcal{N}) \xrightarrow{\alpha} \operatorname{Hom}_A(\mathcal{M}, \operatorname{Hom}_B(\mathcal{J}, \mathcal{N})), \quad f(m \otimes x) = \alpha(f)(m)(x),$$

for $f \in \operatorname{Hom}_B(\mathcal{M} \otimes_A \mathcal{J}, \mathcal{N})$, $m \in \mathcal{M}$, $x \in \mathcal{J}$ [Benson 1998, Lemma 2.8.2].

For three rings $A \subset B$, $A \subset C$, the isomorphism α applied to $\mathcal{M} = C$, $\mathcal{J} = B$ gives an isomorphism

$$\operatorname{Hom}_B(C \otimes_A B, -) \simeq \operatorname{Hom}_A(C, -) : \operatorname{Mod}_B \to \operatorname{Mod}_C .$$

3B2. Let $A \subset B$ be two rings and $a \in A$ a central nonzero divisor. Let $A_a = A[a^{-1}]$ denote the localisation of A at a. There is a natural inclusion $A \subset A_a$. The restriction $\operatorname{Mod}_{A_a} \to \operatorname{Mod}_A$ identifies Mod_{A_a} with the A-modules where the action of a is invertible. For $\mathcal{M}$, $\mathcal{M}'$ in Mod_{A_a}, we have

$$(25) \qquad \operatorname{Hom}_{A_a}(\mathcal{M}, \mathcal{M}') = \operatorname{Hom}_A(\mathcal{M}, \mathcal{M}'), \quad \mathcal{M} \otimes_{A_a} \mathcal{M}' = \mathcal{M} \otimes_A \mathcal{M}'.$$

For $f \in \operatorname{Hom}_A(\mathcal{M}, \mathcal{M}')$, $m \in \mathcal{M}$, $m' \in \mathcal{M}'$, we have $f(aa^{-1}m) = af(a^{-1}m) \Rightarrow a^{-1}f(m) = f(a^{-1}m)$, and $m \otimes a^{-1}m' = ma^{-1}a \otimes a^{-1}m' = ma^{-1} \otimes m'$ in $\mathcal{M} \otimes_A \mathcal{M}'$. We view Mod_{A_a} as a full subcategory of Mod_A.

The restriction followed by the induction, respectively the coinduction, $\operatorname{Mod}_A \to \operatorname{Mod}_B$ defines an induction, respectively coinduction,

$$I_{A_a}^B = I_A^B \circ \operatorname{Res}_A^{A_a} = - \otimes_A B, \quad \mathbb{I}_{A_a}^B = \mathbb{I}_A^B \circ \operatorname{Res}_A^{A_a} = \operatorname{Hom}_A(B, -) : \operatorname{Mod}_{A_a} \to \operatorname{Mod}_B,$$

even when A_a is not contained in B. The induction $I_{A_a}^B$ admits a right adjoint

$$\mathbb{I}_A^{A_a} \circ \operatorname{Res}_A^B = \operatorname{Hom}_A(A_a, -) : \operatorname{Mod}_B \to \operatorname{Mod}_{A_a}$$

because the restriction $\mathrm{Res}_A^{A_a}$ and the induction I_A^B admit a right adjoint: the coinduction $\mathbb{I}_A^{A_a}$ and the restriction Res_A^B. The coinduction $\mathbb{I}_{A_a}^B$ admits a left adjoint

$$I_A^{A_a} \circ \mathrm{Res}_A^B = - \otimes_A A_a : \mathrm{Mod}_B \to \mathrm{Mod}_{A_a}$$

because the restriction $\mathrm{Res}_A^{A_a}$ and the induction I_A^B admit a left adjoint: the induction $I_A^{A_a}$ and the corestriction Res_A^B.

When a is invertible in B, we have $A_a \subset B$ and they coincide with the induction and coinduction from A_a to B.

The induction and the coinduction of A_a seen as a right A_a-module, are the (A_a, B)-modules

$$I_{A_a}^B(A_a) = A_a \otimes_A B, \quad \mathbb{I}_{A_a}^B(A_a) = \mathrm{Hom}_A(B, A_a). \tag{26}$$

Lemma 3.5. *Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$. Then $I_{A_a}^B(\mathcal{M}) = \mathcal{M} \otimes_{A_a} I_{A_a}^B(A_a)$ in Mod_B.*

Proof. $\mathcal{M} \otimes_A B = (\mathcal{M} \otimes_{A_a} A_a) \otimes_A B = \mathcal{M} \otimes_{A_a} (A_a \otimes_A B)$. □

3B3. Let (A, a, B, D) satisfy Definition 3.1. Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$. As R-modules,

$$I_{A_a}^B(\mathcal{M}) = \mathcal{M} \otimes_A B_D \tag{27}$$

because the action of a on $\mathcal{M}$ is invertible; hence $\mathcal{M} \otimes_A {}_rB_D = \mathcal{M} \otimes_A B_D$ for $r \in \mathbb{N}$. In particular, we have the following:

Lemma 3.6. *The left A_a-module $I_{A_a}^B(A_a)$ is free of basis $(1 \otimes d)_{d \in D}$.*

Remark 3.7. The A-dual $(B_D)^*$ of the left A-module B_D is the right A-module $\oplus_{d \in D} d^* A$ of basis the dual basis $D^* = \{d^* \mid d \in D\}$ of D. Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$. We have canonical isomorphisms of R-modules

$$\oplus_{d \in D} \mathcal{M} \xrightarrow{\simeq} \mathcal{M} \otimes_A B_D \xrightarrow{\simeq} \mathrm{Hom}_A((B_D)^*, \mathcal{M}),$$
$$(x_d) \mapsto \sum_{d \in D} x_d \otimes d \mapsto (d^* \mapsto x_d)_{d \in D}.$$

The tensor product over A by a free A-module is exact and faithful; hence the induction is exact and faithful.

Let $R \subset A$ be a subring central in B. The ring R is automatically commutative and a central subring of the localisation A_a of A. The modules over A_a or B are naturally R-modules.

Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$ be a finitely generated R-module. The R-module $\mathcal{M} \otimes_{A_a} I_{A_a}^B(A_a)$ is finitely generated.

Let $\mathcal{N} \in \mathrm{Mod}_B$ be a finitely generated R-module. The R-module $\mathrm{Hom}_A(A_a, \mathcal{N})$ is finitely generated if R is a field by the Fitting lemma applied to the action of a on $\mathcal{N}$. There exists a positive integer n such that $\mathcal{N}$ is a direct sum $\mathcal{N} = \mathcal{N}_a \oplus \mathcal{N}'_a$, where a^n acts on $\mathcal{N}_a$ as an automorphism and a^n is 0 on $\mathcal{N}'_a$. Then, $\mathrm{Hom}_A(A_a, \mathcal{N}) \simeq \mathcal{N}_a$ is finite-dimensional.

We obtain the following:

Proposition 3.8. *Let (A, a, B, D) satisfy Definition 3.1. The induction functor*

$$I_{A_a}^B = - \otimes_A B : \mathrm{Mod}_{A_a} \to \mathrm{Mod}_B$$

is exact, faithful and admits a right adjoint $R_{A_a}^B := \mathrm{Hom}_A(A_a, -)$.

Let $R \subset A$ be a subring central in B. Then $I_{A_a}^B$ respects finitely generated R-modules. If R is a field, $R_{A_a}^B$ respects finite dimension over R.

3B4. Let (A, a, B, D) satisfy Definition 3.3.

For $\mathcal{M} \in \mathrm{Mod}_A$, the set $\mathcal{M}_d$ of $f \in \mathrm{Hom}_A({}_D B, \mathcal{M})$ vanishing on $D - \{d\}$ is isomorphic to $\mathcal{M}$ by the value at d. The A-dual $({}_D B)^*$ of ${}_D B$ is a free left A-module of basis D^*. We have

$$\mathrm{Hom}_A({}_D B, \mathcal{M}) = \oplus_{d \in D} \mathcal{M}_d \simeq \oplus_{d^* \in D^*} \mathcal{M} \otimes d^* = \mathcal{M} \otimes_A ({}_D B)^*. \tag{28}$$

The A-modules $\mathcal{M}_d$ and $\mathcal{M} \otimes d^*$ are isomorphic by $f \mapsto f(d) \otimes d^*$.

For $\mathcal{M} \in \mathrm{Mod}_{A_a}$, we have linear isomorphisms

$$\mathbb{I}_{A_a}^B(\mathcal{M}) = \mathrm{Hom}_A(B, \mathcal{M}) \simeq \mathrm{Hom}_A({}_D B, \mathcal{M}), \quad \mathcal{M} \otimes_A ({}_D B)^* = \mathcal{M} \otimes_A A_a \otimes_A ({}_D B)^*.$$

For $d \in D$, let $f_d \in \mathrm{Hom}_A(B, A_a)$ equal to 1 on d and 0 on $D - \{d\}$. We deduce from these arguments:

Lemma 3.9. *Let (A, a, B, D) satisfy Definition 3.3. The left A_a-module $\mathbb{I}_{A_a}^B(A_a)$ is free of basis $(f_d)_{d \in D}$ and $\mathbb{I}_{A_a}^B(\mathcal{M}) \simeq \mathcal{M} \otimes_{A_a} \mathbb{I}_A^B(A_a)$.*

Let $R \subset A$ be a subring central in B. Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$ be a finitely generated R-module. The R-module $\mathcal{M} \otimes_{A_a} \mathbb{I}_{A_a}^B(A_a)$ is finitely generated. If R is a field, and the dimension of $\mathcal{N} \in \mathrm{Mod}_B$ is finite over R, then $\mathcal{N} \otimes_A A_a = \mathcal{N}_a \otimes_A A_a \simeq \mathcal{N}_a$ has finite dimension over R by the Fitting lemma, as in the proof of Proposition 3.8. We obtain the following:

Proposition 3.10. *Let (A, a, B, D) satisfy Definition 3.3. The coinduction*

$$\mathbb{I}_{A_a}^B = \mathrm{Hom}_A(B, -) : \mathrm{Mod}_{A_a} \to \mathrm{Mod}_B$$

is exact, faithful, and admits a left adjoint $L_{A_a}^B = - \otimes_A A_a$.

Let $R \subset A$ be a subring central in B. Then $\mathbb{I}_{A_a}^B$ respects finitely generated R-modules. If R is a field, $L_{A_a}^B$ respects finite dimension over R.

4. Parabolic induction and coinduction from $\mathcal{H}_M$ to $\mathcal{H}$

We prove Theorems 1.6, 1.8 and 1.9 giving the properties of the parabolic induction from $\mathcal{H}_M$ to $\mathcal{H}$.

4A. ***Basic properties of the parabolic induction and coinduction.*** Example 3.2 satisfies Definition 3.1 and Example 3.4 satisfies Definition 3.3. In these two examples, $(A_a, B) = (\mathcal{H}_M, \mathcal{H})$. The first one,

$$(A, a, D) = \big(\theta(\mathcal{H}_{M^+}), T_{\tilde{\mu}_M}, (T_{\tilde{d}})_{d \in {}^M W_0}\big),$$

where we identify $\mathcal{H}_{M^+}$ with $\theta(\mathcal{H}_{M^+})$, defines the parabolic induction $I_{\mathcal{H}_M}^{\mathcal{H}} = - \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H} : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}}$. The second one,

$$(A, a, D) = \big(\theta^*(\mathcal{H}_{M^-}), T^*_{(\tilde{\mu}_M)^{-1}}, (T^*_{\tilde{d}})_{d \in W_0^M}\big),$$

where we identify $\mathcal{H}_{M^-}$ with $\theta^*(\mathcal{H}_{M^-})$, defines the parabolic coinduction $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}} = \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, -) : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}}$. Propositions 3.8 and 3.10 imply:

Proposition 4.1. *The parabolic induction $I_{\mathcal{H}_M}^{\mathcal{H}}$ and the coinduction $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}$ are exact, faithful and respect finitely generated R-modules. The parabolic induction admits a right adjoint*

$$R_{\mathcal{H}_M}^{\mathcal{H}} = \mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, -) : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_M}.$$

The parabolic coinduction admits a left adjoint

$$\mathbb{L}_{\mathcal{H}_M}^{\mathcal{H}} := - \otimes_{\mathcal{H}_{M^-},\theta^*} \mathcal{H}_M : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_M}.$$

If R is a field, the adjoint functors $R_{\mathcal{H}_M}^{\mathcal{H}}$ and $\mathbb{L}_{\mathcal{H}_M}^{\mathcal{H}}$ respect finite dimension over R.

4B. ***Transitivity.*** Let $S_M \subset S_{M'} \subset S$. Let $W_{M^\epsilon,M'} = \Lambda_{M^\epsilon,M'} \rtimes W_{M,0}$ denote the submonoid of W_M associated to $S_{M'}^{\mathrm{aff}}$ as in Definition 2.1 (see before Proposition 2.21), and

$$\Lambda_{M^\epsilon,M'} = \Lambda \cap W_{M^\epsilon,M'} = \{\lambda \in \Lambda \mid -(\gamma \circ \nu)(\lambda) \geq 0 \text{ for all } \gamma \in \Sigma^\epsilon_{M'} - \Sigma^\epsilon_M\}.$$

By the properties (i), (ii), (iii) of Theorem 1.4, the R-submodule $\mathcal{H}_{M^\epsilon,M'}$ of $\mathcal{H}_M$ of basis $(T^M_{\tilde{w}})_{\tilde{w} \in W_{M^\epsilon,M'}(1)}$, is a subring of $\mathcal{H}_M$, the restriction to $\mathcal{H}_{M^\epsilon,M'}$ of the injective linear map

$$\mathcal{H}_M \xrightarrow{\theta'} \mathcal{H}_{M'}, \qquad T^M_{\tilde{w}} \mapsto T^{M'}_{\tilde{w}} \quad \text{for } \tilde{w} \in W_M(1),$$

respects the product, and $\mathcal{H}_M = \mathcal{H}_{M^\epsilon,M'}[(T^M_{\tilde{\mu}_{M^\epsilon}})^{-1}]$. Obviously, the map $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$ satisfies $\theta = \theta_{M'} \circ \theta'$ for the linear map

$$\mathcal{H}_{M'} \xrightarrow{\theta_{M'}} \mathcal{H}, \qquad T^{M'}_{\tilde{w}} \mapsto T_{\tilde{w}}, \quad \text{for } \tilde{w} \in W_{M'}(1).$$

Lemma 4.2. *We have*:

(i) $W_M \subset W_{M'}$, $W_{M^\epsilon} = W_{M^\epsilon,M'} \cap W_{M'^\epsilon}$, $\theta'(\mathcal{H}_{M^\epsilon}) = \theta'(\mathcal{H}_{M^\epsilon,M'}) \cap \mathcal{H}_{M'^\epsilon}$,

(ii) *$\tilde{\mu}_{M^\epsilon}\tilde{\mu}_{M'^\epsilon}$ is central in $W_M(1)$, satisfies $-(\gamma \circ \nu)(\mu_{M^\epsilon}\mu_{M'^\epsilon}) > 0$ for all $\gamma \in \Sigma^\epsilon - \Sigma^\epsilon_M$, and the additivity of the lengths $\ell(\mu_{M^\epsilon}\mu_{M'^\epsilon}) = \ell(\mu_{M^\epsilon}) + \ell(\mu_{M'^\epsilon})$,*

(iii) ${}^M W_0 = {}^M W_{M',0} {}^{M'} W_0$.

Proof. (i) We have $W_{M,0} \subset W_{M',0}$ and $\Lambda_{M^\epsilon} = \Lambda'_{M^\epsilon} \cap \Lambda_{M'^\epsilon}$. Therefore

$$W_M = \Lambda \rtimes W_{M,0} \subset \Lambda \rtimes W_{M',0} = W_{M'},$$

and

$$\begin{aligned} W_{M^\epsilon,M'} \cap W^\epsilon_{M'} &= (\Lambda'_{M^\epsilon} \rtimes W_{M,0}) \cap (\Lambda'_{M'^\epsilon} \rtimes W_{M',0}) \\ &= (\Lambda'_{M^\epsilon} \cap \Lambda_{M'^\epsilon}) \rtimes W_{M,0} \\ &= \Lambda_{M^\epsilon} \rtimes W_{M,0} = W_{M^\epsilon}. \end{aligned}$$

(ii) Now $\tilde{\mu}_{M'^\epsilon}$ is central in $W_{M'}(1)$, which contains $W_M(1)$, and $\tilde{\mu}_{M^\epsilon}$ is central in $W_M(1)$; hence $\tilde{\mu}_{M^\epsilon}\tilde{\mu}_{M'^\epsilon}$ is central in $W_M(1)$. We have

$$\begin{aligned} -(\gamma \circ \nu)(\mu_{M'^\epsilon}) &> 0 \quad \text{for all } \gamma \in \Sigma^\epsilon - \Sigma^\epsilon_{M'}, \\ -(\gamma \circ \nu)(\mu_{M'^\epsilon}) &= 0 \quad \text{for all } \gamma \in \Sigma_{M'}, \\ -(\gamma \circ \nu)(\mu_{M^\epsilon}\,) &> 0 \quad \text{for all } \gamma \in \Sigma^\epsilon - \Sigma^\epsilon_{M}, \\ -(\gamma \circ \nu)(\mu_{M^\epsilon}\,) &= 0 \quad \text{for all } \gamma \in \Sigma_{M}. \end{aligned}$$

Hence $-(\gamma \circ \nu)(\mu'_{M^\epsilon}\mu_{M'^\epsilon}) > 0$ for all $\gamma \in \Sigma^\epsilon - \Sigma^\epsilon_M$ and

$$\ell(\mu_{M^\epsilon}\mu_{M'^\epsilon}) = \ell(\mu_{M^\epsilon}) + \ell(\mu_{M'^\epsilon}).$$

(iii) Let $u \in {}^M W_{M',0}$, $v \in {}^{M'}W_0$ and let $w \in W_{M,0}$. We have

$$\ell(wuv) = \ell(wu) + \ell(v) = \ell(w) + \ell(u) + \ell(v) = \ell(w) + \ell(uv);$$

hence $uv \in {}^M W_0$. The injective map $(u, v) \mapsto uv : {}^M W_{M',0} \times {}^{M'}W_0 \to {}^M W_0$ is bijective because

$$|{}^M W_0| = |W_{M,0}\backslash W_0| = |W_{M,0}\backslash W_{M',0}|\,|W_{M',0}\backslash W_0| = |{}^M W_{M',0}|\,|{}^{M'}W_0|,$$

where $|X|$ denotes the number of elements of a finite set X. □

Proposition 4.3. *The induction is transitive*:

$$I^{\mathcal{H}}_{\mathcal{H}_M} = I^{\mathcal{H}}_{\mathcal{H}_{M'}} \circ I^{\mathcal{H}_{M'}}_{\mathcal{H}_M} : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}_{M'}} \to \mathrm{Mod}_{\mathcal{H}}\,.$$

The coinduction is also transitive. This is proved at the end of this paper.

Proof. By Lemma 3.5, the proposition is equivalent to

$$\mathcal{H}_M \otimes_{\mathcal{H}_{M+}} \mathcal{H} \simeq \mathcal{H}_M \otimes_{\mathcal{H}_{M+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'+}} \mathcal{H}$$

in $\mathrm{Mod}_{\mathcal{H}}$. As $\mathcal{H}_{M'} = \mathcal{H}_{M'+}[(T^{M'}_{\tilde{\mu}_{M'+}})^{-1}]$ is the localisation of the ring $\mathcal{H}_{M'+}$ at the central element $T^{M'}_{\tilde{\mu}_{M'+}} \in \mathcal{H}_{M'+}$, the right $\mathcal{H}$-module $\mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'+}} \mathcal{H}$ is the inductive limit of $(T^{M'}_{\tilde{\mu}_{M'+}})^{-r} \otimes \mathcal{H}$ for $r \in \mathbb{N}$ with the transition maps

$$(T^{M'}_{\tilde{\mu}_{M'+}})^{-r} \otimes x \mapsto (T^{M'}_{\tilde{\mu}_{M'+}})^{-r-1} \otimes T_{\tilde{\mu}_{M'+}} x \quad \text{for } x \in \mathcal{H}.$$

As $\mathcal{H}_M = \mathcal{H}_{M^+,M'}[(T^M_{\tilde{\mu}_{M^+}})^{-1}]$ is the localisation of the ring $\mathcal{H}_{M^+,M'}$ at the central element $T^M_{\tilde{\mu}_{M^+}} \in \mathcal{H}_{M^+,M'}$, the right $\mathcal{H}$-module $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ is the inductive limit of $(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ for $s \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes y \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-s-1} \otimes T^{M'}_{\tilde{\mu}_{M^+}} y \quad \text{for } y \in \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}.$$

Using that $T^{M'}_{\tilde{\mu}_{M'^+}}$ is central in $\mathcal{H}_{M'}$ and $T^{M'}_{\tilde{\mu}_{M^+}} \in \mathcal{H}_{M'^+}$, we have, for $y = (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x$,

$$T^{M'}_{\tilde{\mu}_{M^+}} y = T^{M'}_{\tilde{\mu}_{M^+}} (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x = (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} T^{M'}_{\tilde{\mu}_{M^+}} \otimes x = (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes T_{\tilde{\mu}_{M^+}} x.$$

Altogether, the right $\mathcal{H}$-module $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ is the inductive limit of $(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes \mathcal{H}$ for $r, s \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-s-1} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes T_{\tilde{\mu}_{M^+}} x,$$
$$(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r-1} \otimes T_{\tilde{\mu}_{M'^+}} x.$$

The right $\mathcal{H}$-module $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ is also the inductive limit of the modules $(T^M_{\tilde{\mu}_{M^+}})^{-r} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes \mathcal{H}$ for $r \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}})^{-r} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-r-1} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r-1} \otimes T_{\tilde{\mu}_{M^+}} T_{\tilde{\mu}_{M'^+}} x.$$

By Lemma 4.2(ii), $T_{\tilde{\mu}_{M^+}} T_{\tilde{\mu}_{M'^+}} = T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}}$. Hence, in $\mathrm{Mod}_{\mathcal{H}}$ we have

$$\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H} \simeq \varinjlim_{x \mapsto T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}} x} \mathcal{H}.$$

On the other hand, $\mathcal{H}_M = \mathcal{H}_{M^+}[(T^M_{\tilde{\mu}_{M^+}} \tilde{\mu}_{M'^+})^{-1}]$ is the localisation of $\mathcal{H}_{M^+}$ at $T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}}$ (Lemma 4.2); hence $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+}} \mathcal{H}$ is the inductive limit of $(T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}})^{-r} \otimes \mathcal{H}$ for $r \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}})^{-r-1} \otimes T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}} x.$$

We deduce that

$$\mathcal{H}_M \otimes_{\mathcal{H}_{M^+}} \mathcal{H} \simeq \varinjlim_{x \mapsto T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}} x} \mathcal{H}$$

is isomorphic to $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ in $\mathrm{Mod}_{\mathcal{H}}$. ☐

4C. *w_0-twisted induction is equal to coinduction.* We prove Theorem 1.8. When $\mathcal{H} = \mathcal{H}_R(G)$ is the pro-p Iwahori Hecke algebra of a reductive p-adic group G over an algebraically closed field R of characteristic p, Theorem 1.8 is proved by Abe [2014, Proposition 4.14]. We will extend his arguments to the general algebra $\mathcal{H}$.

Let $\tilde{w}_0^M \in W_0(1)$ lifting w_0^M. The algebra isomorphism $\mathcal{H}_M \simeq \mathcal{H}_{w_0(M)}$ defined by $\tilde{w}_0^M$ (Proposition 2.20) induces an equivalence of categories

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_0^M} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \tag{29}$$

called a w_0-twist. Let $\mathcal{M}$ be a right $\mathcal{H}_M$-module. The underlying R-module of $\tilde{\mathfrak{w}}_0^M(\mathcal{M})$ and of $\mathcal{M}$ is the same; the right action of $T^M_{\tilde{w}}$ on $\mathcal{M}$ is equal to the right action of $T^{w_0(M)}_{\tilde{w}_0^M\tilde{w}(\tilde{w}_0^M)^{-1}}$ on $\tilde{\mathfrak{w}}_0^M(\mathcal{M})$ for $\tilde{w}\in W_M(1)$. The inverse of $\tilde{\mathfrak{w}}_0^M$ is the algebra isomorphism induced by $(\tilde{w}_0^M)^{-1}$ lifting

$$^{M}w_0 := (w_0^M)^{-1} = w_{M,0}w_0 = w_0w_0w_{M,0}w_0 = w_0^{w_0(M)}.$$

Remark 4.4. The lifts of w_0^M are $t\tilde{w}_0^M = \tilde{w}_0^M t'$ with $t, t' \in Z_k$, the elements $T^M_{t'} \in \mathcal{H}_M$, $T^{w_0(M)}_t \in \mathcal{H}_{w_0(M)}$ are invertible, and the conjugation by T_t in $\mathcal{H}_M$, by $T^{w_0(M)}_t$ in $\mathcal{H}_{w_0(M)}$ induce equivalences of categories

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\mathfrak{t}'} \mathrm{Mod}_{\mathcal{H}_M}, \quad \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \xrightarrow{\mathfrak{t}} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}}$$

such that $\mathfrak{t}\tilde{\mathfrak{w}}_0^M = \mathfrak{t}\circ\tilde{\mathfrak{w}}_0^M = \tilde{\mathfrak{w}}_0^M\circ\mathfrak{t}' = \tilde{\mathfrak{w}}_0^M\mathfrak{t}'$.

Remark 4.5. The trivial characters of $\mathcal{H}_M$ and $\mathcal{H}_{w_0(M)}$ correspond by $\tilde{\mathfrak{w}}_0^M$.

We will prove that, for all $S_M \subset S$, the coinduction

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}} \mathrm{Mod}_{\mathcal{H}}$$

is equivalent to the w_0-twist induction

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_0^M} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \xrightarrow{I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}} \mathrm{Mod}_{\mathcal{H}}\,.$$

This proves Theorem 1.8 because

$$\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M} \simeq I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}\circ\tilde{\mathfrak{w}}_0^M \iff I^{\mathcal{H}}_{\mathcal{H}_M} \simeq \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}\circ\tilde{\mathfrak{w}}_0^M. \tag{30}$$

Indeed, if the left-hand side is true for all $S_M \subset S$, permuting M and $w_0(M)$ we have $\mathbb{I}_{\mathcal{H}_{w_0(M)}} \simeq I^{\mathcal{H}}_{\mathcal{H}_M}\circ\tilde{\mathfrak{w}}_0^{w_0(M)}$, and composing with $(\tilde{\mathfrak{w}}_0^{w_0(M)})^{-1}$, we get

$$I^{\mathcal{H}}_{\mathcal{H}_M} \simeq \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}\circ(\tilde{\mathfrak{w}}_0^{w_0(M)})^{-1} \simeq \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}\circ\tilde{\mathfrak{w}}_0^M$$

as $w_0^{w_0(M)} = (w_0^M)^{-1}$. The arguments can be reversed to get the equivalence.

Let $\mathcal{M}\in\mathrm{Mod}_{\mathcal{H}_M}$. We will construct an explicit functorial isomorphism in $\mathrm{Mod}_{\mathcal{H}}$:

$$(I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}\circ\tilde{\mathfrak{w}}_0^M)(\mathcal{M}) \xrightarrow{\mathfrak{b}} \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{M}). \tag{31}$$

From Lemmas 3.5, 3.6, 3.9 and Examples 3.2, 3.4, we get

(i) $I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}(\mathcal{H}_{w_0(M)}) = \mathcal{H}_{w_0(M)}\otimes_{\mathcal{H}_{w_0(M)^+},\theta}\mathcal{H}$ is a left free $\mathcal{H}_{w_0(M)}$-module of basis $1\otimes T_{\tilde{d}'}$ for $d'\in {}^{w_0(M)}W_0$, and

$$(I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}\circ\tilde{\mathfrak{w}}_0^M)(\mathcal{M}) = \tilde{\mathfrak{w}}_0^M(\mathcal{M})\otimes_{\mathcal{H}_{w_0(M)}} I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}(\mathcal{H}_{w_0(M)}).$$

(ii) $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{H}_M) = \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},\mathcal{H}_M)$, where $\mathcal{H}$ is seen as a right $\theta^*(\mathcal{H}_{M^-})$-module, is a left free $\mathcal{H}_M$-module of basis $(f^*_{\tilde{d}})_{d\in W_0^M}$, where $f^*_{\tilde{d}}(T^*_{\tilde{d}})=1$ and $f^*_{\tilde{d}}(T^*_{\tilde{x}})=0$ for $x\in W_0^M-\{d\}$, and

$$\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{M}) = \mathcal{M}\otimes_{\mathcal{H}_M}\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{H}_M).$$

It is an exercise to prove that the left $\mathcal{H}_M$-module $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{H}_M)$ admits also the basis $(f_{\tilde{d}})_{d\in W_0^M}$, where $f_{\tilde{d}}(T_{\tilde{d}})=1$ and $f_{\tilde{d}}(T_{\tilde{x}})=0$ for $x\in W_0^M-\{d\}$. We will prove that the linear map

$$m\otimes T_{\tilde{d}'}\mapsto m\otimes f_{\tilde{w}_0^M}T_{\tilde{d}'}:\oplus_{d'\in {}^{w_0(M)}W_0}\tilde{\mathfrak{w}}_0^M(\mathcal{M})\otimes T_{\tilde{d}'}\xrightarrow{\mathfrak{b}}\oplus_{d\in W_0^M}\mathcal{M}\otimes f_{\tilde{d}} \tag{32}$$

is a functorial isomorphism in $\mathrm{Mod}_{\mathcal{H}}$. The bijectivity follows from the bijectivity of the map $d'\mapsto d'^{-1}w_0^M: {}^{w_0(M)}W_0\to W_0^M$ (Lemma 2.24) and the following:

Lemma 4.6. *The map $f_{\tilde{w}_0^M}T_{\tilde{d}'}-f_{(d'^{-1}w_0^M)\tilde{}}$ lies in $\oplus_{x\in W_0^M, x<d'^{-1}w_0^M}\mathcal{M}\otimes f_{\tilde{x}}$.*

Proof. For $d\in W_0^M$, we have

$$(f_{\tilde{w}_0^M}T_{\tilde{d}'})(T_{\tilde{d}})=f_{\tilde{w}_0^M}(T_{\tilde{d}'}T_{\tilde{d}})=f_{\tilde{w}_0^M}(T_{\tilde{d}'\tilde{d}})+x,$$

where $x\in\sum Rf_{\tilde{w}_0^M}(T_{\tilde{w}})$ and the sum is over the $\tilde{w}\in W_0(1)$ with $w<d'd$ and $w\in w_0^M W_{M,0}$. If $d'd\notin w_0^M W_{M,0}$, there is no $w\in w_0^M W_{M,0}$ with $w<d'd$ (Lemma 2.26). We have $d'd\in w_0^M W_{M,0}$ if and only if $d=d'^{-1}w_0^M$ (part (ii) of Lemma 2.28). □

The restriction

$$\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)^+},\theta}:\mathrm{Mod}_{\mathcal{H}}\to\mathrm{Mod}_{\mathcal{H}_{w_0(M)^+}}$$

is left adjoint to $-\otimes_{\mathcal{H}_{w_0(M)^+},\theta}\mathcal{H}$, and the $\mathcal{H}_{w_0(M)^+}$-equivariance of the linear map

$$m\mapsto m\otimes f_{\tilde{w}_0^M}:\tilde{\mathfrak{w}}_0^M(\mathcal{M})\to\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{M}) \tag{33}$$

implies the $\mathcal{H}$-equivariance of (31), i.e., of (32). Let $\mathcal{H}_M\xrightarrow{j}\mathcal{H}_{w_0(M)}$ denote the isomorphism induced by $\tilde{w}_0^M$ (Proposition 2.20), and θ_M the linear map $\mathcal{H}_M\xrightarrow{\theta}\mathcal{H}$. The $\mathcal{H}_{w_0(M)^+}$-invariance of the map $m\mapsto m\otimes f_{\tilde{w}_0^M}$ is equivalent to

$$f_{\tilde{w}_0^M}\theta_{w_0(M)}(h)=j^{-1}(h)f_{\tilde{w}_0^M}\quad\text{for } h\in\mathcal{H}_{w_0(M)^+}. \tag{34}$$

We can suppose that h lies in the Bernstein basis of $\mathcal{H}_{w_0(M)^+}$. Let $\tilde{w}\in W_{w_0(M)^+}(1)$ and $h=E_{w_0(M)}(\tilde{w})$. As $\theta_{w_0(M)}(E_{w_0(M)}(\tilde{w}))=E(\tilde{w})$, and $j^{-1}(E_{w_0(M)}(\tilde{w}))$ is equal to $E_M((\tilde{w}_0^M)^{-1}\tilde{w}\tilde{w}_0^M)$, (34) is equivalent to the following:

Proposition 4.7. *For $w\in W_{w_0(M)^+}$, we have $f_{\tilde{w}_0^M}E(\tilde{w})=E_M((\tilde{w}_0^M)^{-1}\tilde{w}\tilde{w}_0^M)f_{\tilde{w}_0^M}$.*

Proof. By the usual reduction arguments, we suppose that the $\mathfrak{q}(s)$ are invertible in R. Using $W_{w_0(M)^+}=\Lambda_{w_0(M)^+}\rtimes W_{w_0(M),0}$, the product formula (8) and Lemma 2.23, we reduce to $w\in\Lambda_{w_0(M)^+}\cup W_{w_0(M),0}$. By induction on the length in $W_{w_0(M),0}$ with respect to $S_{w_0(M)}$, we reduce to $w\in\Lambda_{w_0(M)^+}\cup S_{w_0(M)}$.

Let $d\in W_0^M$. We have $(f_{\tilde{w}_0^M}E(\tilde{w}))(T_{\tilde{d}})=f_{\tilde{w}_0^M}(E(\tilde{w})T_{\tilde{d}})$ in $\mathcal{H}_M$. We must prove

$$f_{\tilde{w}_0^M}(E(\tilde{w})T_{\tilde{d}})=\begin{cases}0 & \text{if } d\neq w_0^M,\\ E_M((\tilde{w}_0^M)^{-1}\tilde{w}\tilde{w}_0^M) & \text{if } \tilde{d}=\tilde{w}_0^M\end{cases} \tag{35}$$

for $w\in\Lambda_{w_0(M)^+}\cup S_{w_0(M)}$.

(i) Suppose $w=\lambda\in\Lambda_{w_0(M)^+}$. Let $\mathcal{A}$ denote the subalgebra of $\mathcal{H}$ of basis $(E(\tilde{x}))_{\tilde{x}\in\Lambda(1)}$ [Vignéras 2013a, Corollary 2.8]. By the Bernstein relations [Vignéras 2013a, Theorem 2.9], we have

$$E(\tilde{\lambda})T_{\tilde{d}}=T_{\tilde{d}}E((\tilde{d})^{-1}\tilde{\lambda}\tilde{d})+\sum T_{\tilde{w}}a_{\tilde{w}},$$

where $a_{\tilde{w}}\in\mathcal{A}$ and the sum is over $\tilde{w}\in W_0(1)$, $w<d$. If $d\neq w_0^M$, the image by $f_{\tilde{w}_0^M}$ of the right-hand side vanishes because $w\in w_0^M W_{M,0}$, $w\leq d$ implies $w=d=w_0^M$; hence $f_{\tilde{w}_0^M}(E(\tilde{\lambda})T_{\tilde{d}})=0$ as we want. For $\tilde{d}=\tilde{w}_0^M$, using $(w_0^M)^{-1}\lambda\tilde{w}_0^M\in W_{w_0(M)^-}$, we have

$$\begin{aligned}f_{\tilde{w}_0^M}(E(\tilde{\lambda})T_{\tilde{w}_0^M})&=f_{\tilde{w}_0^M}(T_{\tilde{w}_0^M}E((\tilde{w}_0^M)^{-1}\tilde{\lambda}\tilde{w}_0^M)\\&=\theta^*(E((\tilde{w}_0^M)^{-1}\tilde{\lambda}\tilde{w}_0^M))\\&=E_M((\tilde{w}_0^M)^{-1}\tilde{\lambda}\tilde{w}_0^M).\end{aligned}$$

(ii) Suppose $w=s\in S_{w_0(M)}$. We have $w_0sw_0\in S_M$, $w_0sw_0w_{M,0}<w_{M,0}$ and

$$sw_0^M=sw_0w_{M,0}=w_0w_0sw_0w_{M,0}>w_0w_{M,0}=w_0^M.$$

Assume $sd<d$. We deduce $d\neq w_0^M$. Assume $\tilde{d}=\tilde{s}(\widetilde{sd})$. Then

$$E(\tilde{s})T_{\tilde{d}}=T_{\tilde{s}}T_{\tilde{d}}=T_{\tilde{s}}^2T_{(\widetilde{sd})}=(\mathfrak{q}(s)(\tilde{s})^2+\mathfrak{c}(\tilde{s})T_{\tilde{s}})T_{(\widetilde{sd})}=\mathfrak{q}(s)(\tilde{s})^2T_{(\widetilde{sd})}+\mathfrak{c}(\tilde{s})T_{\tilde{d}}.$$

We deduce that $f_{\tilde{w}_0^M}(E(\tilde{s})T_{\tilde{d}})=0$.

Assume $sd>d$. We write $\tilde{s}\,\tilde{d}=\tilde{d}_1\tilde{u}$ with $d_1\in W_0^M$, $u\in W_{M,0}$. Then $T_{\tilde{s}}T_{\tilde{d}}=T_{\tilde{s}\tilde{d}}=T_{\tilde{d}_1\tilde{u}}$. Therefore $f_{\tilde{w}_0^M}(E(\tilde{s})T_{\tilde{d}})=f_{\tilde{w}_0^M}(T_{\tilde{d}_1\tilde{u}})=0$ if $d_1\neq w_0^M$. We suppose now $d_1=w_0^M$. We have $d\leq w_0^M\leq sd$; hence $w_0^M=d$ or $w_0^M=sd$. In the latter case, a reduced decomposition of w_0^M starts by s. But this is incompatible with $s\in S_{w_0(M)}$ because $w_0^M={}^{w_0(M)}w_0$. We deduce that $d=w_0^M$. For $\tilde{d}=\tilde{w}_0^M$, we have

$$\begin{aligned}f_{\tilde{w}_0^M}(E(\tilde{s})T_{\tilde{w}_0^M})&=f_{\tilde{w}_0^M}(T_{\tilde{s}\,\tilde{w}_0^M})=f_{\tilde{w}_0^M}(T_{\tilde{w}_0^M}T_{(w_0^M)^{-1}\tilde{s}\tilde{w}_0^M})\\&=f_{\tilde{w}_0^M}(T_{\tilde{w}_0^M}E_{(w_0^M)^{-1}\tilde{s}\tilde{w}_0^M})=\theta^*(E_{(w_0^M)^{-1}\tilde{s}\tilde{w}_0^M}))\\&=E_M((\tilde{w}_0^M)^{-1}\tilde{s}\tilde{w}_0^M).\end{aligned}$$

This ends the proof of Proposition 4.7, and hence of Theorem 1.8. ☐

Corollary 4.8. *The right $\mathcal{H}$-modules $\mathcal{H}_M\otimes_{\mathcal{H}_{M^+},\theta}\mathcal{H}$ and $\mathrm{Hom}_{\mathcal{H}_{w_0(M)^-},\theta^*}(\mathcal{H},\mathcal{H}_{w_0(M)})$ are isomorphic.*

4D. ***Transitivity of the coinduction.*** Let $S_M\subset S_{M'}\subset S$. By Proposition 2.21, the algebra isomorphisms

$$\mathcal{H}_M\xrightarrow{j}\mathcal{H}_{w_0(M)},\quad \mathcal{H}_M\xrightarrow{j'}\mathcal{H}_{w_{M',0}(M)}\xrightarrow{k''}\mathcal{H}_{w_0(M)}$$

corresponding to $\tilde{w}_0^M$, $\tilde{w}_{M'}^M$, $\tilde{w}_0^{M'}$, $\tilde{w}_0^M = \tilde{w}_0^{M'} \tilde{w}_{M'}^M$, satisfy $j = k'' \circ j'$. The associated equivalences of categories, denoted by

$$(36) \qquad \mathcal{M}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_0^M} \mathcal{M}_{\mathcal{H}_{w_0(M)}}, \qquad \mathcal{M}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_{M'}^M} \mathcal{M}_{\mathcal{H}_{w_{M',0}(M)}} \xrightarrow{\tilde{\mathfrak{w}}_{0,k}^{M'}} \mathcal{M}_{\mathcal{H}_{w_0(M)}},$$

satisfy $\tilde{\mathfrak{w}}_0^M = \tilde{\mathfrak{w}}_{0,k}^{M'} \circ \tilde{\mathfrak{w}}_{M'}^M$. We refer to this as the transitivity of the w_0-twisting.

Lemma 4.9. *The functors* $\tilde{\mathfrak{w}}_0^{M'} \circ I_{\mathcal{H}_{w_{M',0}(M)}}^{\mathcal{H}_{M'}}$ *and* $I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}_{w_0(M')}} \circ \tilde{\mathfrak{w}}_{0,k}^{M'}$ *from* $\mathrm{Mod}_{\mathcal{H}_{w_{M',0}(M)}}$ *to* $\mathrm{Mod}_{\mathcal{H}_{w_0(M')}}$ *are isomorphic.*

The proof gives an explicit isomorphism.

Proof. Let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{H}_{w_{M',0}(M)}}$. The R-module $\mathcal{M} \otimes_{\mathcal{H}_{w_{M',0}(M)^+},\theta} \mathcal{H}_{M'}$ with the right action of $\mathcal{H}_{w_0(M')}$ defined by

$$(x \otimes T_{\tilde{u}}^{M'}) T^{w_0(M')}_{\tilde{w}_o^{M'} \tilde{v} (\tilde{w}_o^{M'})^{-1}} = x \otimes T_{\tilde{u}}^{M'} T_{\tilde{v}}^{M'}$$

for $x \in \mathcal{M}$, $u, v \in W_{M'}$, is $\tilde{\mathfrak{w}}_0^{M'} \circ I_{\mathcal{H}_{w_{M',0}(M)}}^{\mathcal{H}_{M'}}(\mathcal{M})$.

As $k''(\mathcal{H}_{w_{M',0}(M)^+}) = \mathcal{H}_{w_0(M)^+}$ (Proposition 2.21), the R-linear map

$$\mathcal{M} \otimes_R \mathcal{H}_{M'} \to \tilde{\mathfrak{w}}_{0,k}^{M'}(\mathcal{M}) \otimes_{\mathcal{H}_{w_0(M)^+},\theta} \mathcal{H}_{w_0(M')}$$

defined by $x \otimes T_{\tilde{u}}^{M'} \to x \otimes T^{w_0(M')}_{\tilde{w}_0^{M'} \tilde{u} (\tilde{w}_0^{M'})^{-1}}$ is the composite of the quotient map

$$\mathcal{M} \otimes_R \mathcal{H}_{M'} \to \tilde{\mathfrak{w}}_0^{M'} \circ \mathcal{M} \otimes_{\mathcal{H}_{w_{M',0}(M)^+}} \mathcal{H}_{M'},$$

and of the bijective linear map

$$\tilde{\mathfrak{w}}_0^{M'} \circ I_{\mathcal{H}_{w_{M',0}(M)}}^{\mathcal{H}_{M'}}(\mathcal{M}) \to \tilde{\mathfrak{w}}_{0,k}^{M'}(\mathcal{M}) \otimes_{\mathcal{H}_{w_0(M)^+},\theta} \mathcal{H}_{w_0(M')}.$$

The above map is clearly $\mathcal{H}_{w_0(M')}$-equivariant. □

Proposition 4.10. *The coinduction is transitive.*

Proof. By the transitivity of the w_0-twisting (36), Lemma 4.9, and the transitivity of the induction (Proposition 4.3), we have

$$\begin{aligned}
\mathbb{I}_{\mathcal{H}_{M'}}^{\mathcal{H}} \circ \mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}_{M'}} &= I_{\mathcal{H}_{w_0(M')}}^{\mathcal{H}} \circ \tilde{\mathfrak{w}}_0^{M'} \circ I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}_{w_0(M')M'}} \circ \tilde{\mathfrak{w}}_{M'}^M \\
&= I_{\mathcal{H}_{w_0(M')}}^{\mathcal{H}} \circ I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}_{w_0(M')}} \circ \tilde{\mathfrak{w}}_{0,k}^{M'} \circ \tilde{\mathfrak{w}}_{M'}^M \\
&= I_{\mathcal{H}_{w_0(M')}}^{\mathcal{H}} \circ I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}_{w_0(M')}} \circ \tilde{\mathfrak{w}}_0^M \\
&= I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}} \circ \tilde{\mathfrak{w}}_0^M = \mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}.
\end{aligned}$$ □

Proof of Theorem 1.9. The induction $I_{\mathcal{H}_M}^{\mathcal{H}}$ is equivalent to $\mathbb{I}_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}} \circ \tilde{\mathfrak{w}}_0^M$. The coinduction $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}$ is the composite of the restriction $\mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}_{M^-}}$ and of $\mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, -) : \mathrm{Mod}_{\mathcal{H}_{M^-}} \to \mathrm{Mod}_{\mathcal{H}}$. These functors admit left adjoints, the restriction $\mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_{M^-}}$ for $\mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},-)$, and the induction $-\otimes_{\mathcal{H}_{M^-}} \mathcal{H}_M : \mathrm{Mod}_{\mathcal{H}_{M^-}} \to \mathrm{Mod}_{\mathcal{H}_M}$ for the restriction $\mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}_{M^-}}$; hence $-\otimes_{\mathcal{H}_{M^-},\theta^*} \mathcal{H}_M : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_M}$ for $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}$, and

$$(\tilde{\mathfrak{w}}_0^M)^{-1} \circ (-\otimes_{\mathcal{H}_{w_0(M)^-},\theta^*} \mathcal{H}_{w_0(M)}) \simeq \tilde{\mathfrak{w}}_0^{w_0(M)} \circ (-\otimes_{\mathcal{H}_{w_0(M)^-},\theta^*} \mathcal{H}_{w_0(M)})$$

for $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M$. $\square$

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In memoriam: Robert Steinberg