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Let Σ be a compact immersed stable capillary hypersurface in a wedge bounded by two hyperplanes in $\mathbb{R}^{n+1}$. Suppose that Σ meets those two hyperplanes in constant contact angles $\geq \pi/2$ and is disjoint from the edge of the wedge, and suppose that $\partial\Sigma$ consists of two smooth components with one in each hyperplane of the wedge. It is proved that if $\partial\Sigma$ is embedded for $n = 2$, or if each component of $\partial\Sigma$ is convex for $n \geq 3$, then Σ is part of the sphere. The same is true for Σ in the half-space of $\mathbb{R}^{n+1}$ with connected boundary $\partial\Sigma$.

1. Introduction

The isoperimetric inequality says that among all domains of fixed volume in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ the one with least boundary area is the round ball. What happens if the boundary area is a critical value instead of the minimum? For this question the more general domains enclosed by the immersed hypersurfaces have to be considered, hence one needs to introduce the oriented volume (as defined in (1)). Then the answer to the question is that given a compact immersed hypersurface Σ in $\mathbb{R}^{n+1}$, its area is critical among all variations of Σ preserving the oriented volume enclosed by Σ if and only if Σ has constant mean curvature (CMC).

So, H. Hopf [1989, p. 131] raised the question as to whether there exist closed surfaces with CMC which are not spheres. To this question, W.-Y. Hsiang [1982] obtained a counterexample, a CMC immersion of $\mathbb{S}^3$ in $\mathbb{R}^4$ which is not round, and Wente [1986] constructed a CMC immersion of a torus in $\mathbb{R}^3$.

Is there an extra condition on a CMC surface Σ which guarantees that Σ is a sphere? There are some affirmative results in this regard:

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(i) Aleksandrov [1962a; 1962b] showed that every compact *embedded* hypersurface of CMC in $\mathbb{R}^{n+1}$ is a sphere,

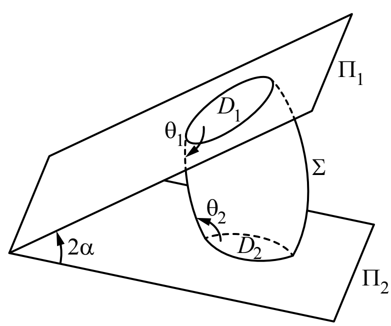
(ii) Hopf himself [1989] proved that an immersed CMC 2-*sphere* is round, and

(iii) Barbosa and do Carmo [1984] showed that the only compact immersed *stable* CMC hypersurface of $\mathbb{R}^{n+1}$ is the sphere.

A CMC hypersurface Σ is said to be stable if the second variation of the n-dimensional area of Σ is nonnegative for all $(n+1)$-dimensional volume-preserving perturbations of Σ.

A CMC surface with nonempty boundary along which it makes a constant contact angle with a prescribed supporting surface is called a capillary surface. It is an equilibrium surface of the sum of the area and the wetting energy on the supporting surface (we call it the total energy of the surface) for volume-preserving variations (see Section 2). Such a surface is said to be stable if the second variation of the total energy is nonnegative for all volume-preserving variations. In this paper, we prove the following uniqueness result (Section 4, Theorem 1) which is a generalization of the theorem by Barbosa and do Carmo [1984] mentioned above:

> *Let Σ be a compact immersed stable capillary hypersurface in a wedge bounded by two hyperplanes in $\mathbb{R}^{n+1}$, $n \geq 2$. Suppose that Σ meets those two hyperplanes in constant contact angles $\geq \pi/2$ and does not hit the edge of the wedge. We also assume that $\partial\Sigma$ consists of two smooth embedded $(n-1)$-dimensional manifolds, one in each hyperplane of the wedge, and that each component of $\partial\Sigma$ is convex when $n \geq 3$* (see figure). *Then Σ is part of the sphere. Also, the same conclusion holds if Σ is in the half-space of $\mathbb{R}^{n+1}$ and $\partial\Sigma$ is connected.*



We emphasize that there is a stable capillary surface between two parallel planes which is not part of the sphere [Vogel 1989]. Our result shows that, if the initial supporting surface is the union of two parallel planes and we consider a stable nonspherical capillary surface, then the configuration changes discontinuously on infinitesimal tilting of one of the planes. Such discontinuity was pointed out already in [Concus et al. 2001] without the stability of the surface.

The idea of our proof is motivated by Wente [1991]. He simplified Barbosa and do Carmo's proof by using the parallel hypersurfaces and the homothetic contraction. We have found that Wente's method carries over nicely to our capillary hypersurfaces in a wedge and in the half-space. On the other hand, the Minkowski inequality for $\partial\Sigma$ is indispensable in our arguments. Wente informed us that recently Marinov [2012] obtained the same result when Σ is in $\mathbb{R}^3$ and $\partial\Sigma$ is in a plane.

Here we mention some additional related results. McCuan [1997] and Park [2005] proved that an embedded annular capillary surface in a wedge in $\mathbb{R}^3$ is necessarily part of the sphere. The question then arises whether one can extend the theorems of Aleksandrov, Hopf, and Barbosa–do Carmo to the case of capillary surfaces in a wedge or in the half-space. That is:

(i) Does there exist no compact embedded capillary surface of genus ≥ 1 in a wedge (or in the half-space) of $\mathbb{R}^3$?

(ii) Is there a compact immersed annular capillary surface of genus 0 (or higher) in a wedge (or in the half-space) which is not part of the sphere?

(iii) Which hypothesis of McCuan's and Park's can be dropped or generalized if the capillary surface is stable?

As mentioned above, in this paper we give an answer to (iii). To question (i), McCuan [1997] gave an affirmative answer with the contact angle condition $\theta_i \leq \pi/2$. In relation to question (ii), Wente [1995] constructed noncompact capillary surfaces bifurcating from the cylinder in a wedge.

Finally, it should be mentioned that the stable capillary surfaces in a ball also have been studied very actively. To begin with, Nitsche [1985] showed that a capillary disk in a ball $\subset \mathbb{R}^3$ is a spherical cap (for a simpler proof, see [Finn and McCuan 2000, Appendix]). Ros and Souam [1997] proved that a stable capillary surface of genus 0 in a ball in $\mathbb{R}^3$ is a spherical cap. They also proved that a stable minimal surface with constant contact angle in a ball $\subset \mathbb{R}^3$ is a flat disk or a surface of genus 1 with at most three boundary components. Moreover, Ros and Vergasta [1995] showed that a stable minimal hypersurface in a ball $B \subset \mathbb{R}^n$ which is orthogonal to ∂B is totally geodesic, and that a stable capillary surface in a ball $\subset \mathbb{R}^3$ and orthogonal to ∂B is a spherical cap or a surface of genus 1 with at most two boundary components.

2. Preliminaries

Let Π_1 and Π_2 be two hyperplanes in $\mathbb{R}^{n+1}$ containing the $(n-1)$-plane $\{x_n = 0, x_{n+1} = 0\}$ and making angles α and $-\alpha$ (with $0 < \alpha < \pi/2$) with the horizontal hyperplane $\{x_{n+1} = 0\}$, respectively. Let $\Omega \subset \{x_n > 0\}$ be the wedge-shaped domain bounded by Π_1 and Π_2. We denote by $\bar{\Omega}$ the closure of Ω. Denote by $X : (\Sigma, \partial\Sigma) \to (\bar{\Omega}, \partial\Omega)$ an immersion of an n-dimensional oriented compact connected C^∞ manifold Σ with nonempty boundary into Ω such that $X(\Sigma^\circ) \subset \Omega$ and $X(\partial\Sigma) \subset \partial\Omega$, where $\Sigma^\circ := \Sigma - \partial\Sigma$. The $(n-1)$-plane

$$\Pi_0 := \Pi_1 \cap \Pi_2 = \{x_n = 0, x_{n+1} = 0\}$$

is called the edge of the wedge Ω. In this paper we are concerned only with the immersed surfaces $X(\Sigma)$ which connect Π_1 to Π_2 without intersecting Π_0.

For the immersion $X : (\Sigma, \partial\Sigma) \to (\bar{\Omega}, \partial\Omega)$, the n-dimensional area $\mathcal{H}^n(X)$ is written as

$$\mathcal{H}^n(X) = \int_\Sigma dS,$$

where dS is the volume form of Σ induced by X. The $(n+1)$-dimensional *oriented volume* $V(X)$ enclosed by $X(\Sigma)$ is defined by

$$V(X) = \frac{1}{n+1} \int_\Sigma \langle X, \nu \rangle \, dS, \tag{1}$$

where the Gauss map ν is the unit normal vector field along X with orientation determined as follows. Let $\{e_1, \ldots, e_n\}$ be an oriented frame on the tangent space $T_p(\Sigma)$, $p \in \Sigma$. Then $\{dX_p(e_1), \ldots, dX_p(e_n), \nu\}$ is a frame of $\mathbb{R}^{n+1}$ with positive orientation.

In this paper $X(\Sigma)$ is immersed while $X(\partial\Sigma)$ is assumed to be embedded. $X(\partial\Sigma)$ influences the area $\mathcal{H}^n(X)$ through the *wetting energy*. Set $C_i = X(\partial\Sigma) \cap \Pi_i$ and let $D_i \subset \Pi_i$ be the domain bounded by C_i. The wetting energy $\mathcal{W}(X)$ of X is defined by

$$\mathcal{W}(X) = \omega_1 \mathcal{H}^n(D_1) + \omega_2 \mathcal{H}^n(D_2),$$

where ω_i is a constant with $|\omega_i| < 1$ and $\mathcal{H}^n(D_i)$ is the n-dimensional area of D_i. Then we define the *total energy* $E(X)$ of the immersion X by

$$E(X) = \mathcal{H}^n(X) + \mathcal{W}(X).$$

Note that $\Sigma \cup D_1 \cup D_2$ is a piecewise smooth hypersurface without boundary. We can extend $\nu : \Sigma \to S^n$ to the Gauss map $\nu : \Sigma \cup D_1 \cup D_2 \to S^n$. Since the origin of $\mathbb{R}^{n+1}$ is on the edge Π_0 of Ω, $\langle X, \nu \rangle = 0$ on $D_1 \cup D_2$. Hence the oriented volume

$$\widehat{V}(X) = \frac{1}{n+1} \int_{\Sigma \cup D_1 \cup D_2} \langle X, \nu \rangle \, dS \tag{2}$$

coincides with $V(X)$.

Let $X_t : (\Sigma, \partial\Sigma) \to (\bar{\Omega}, \partial\Omega)$ be a 1-parameter family of immersions with $X_0 = X$. It is well known [Finn 1986, Chapter 1] that a necessary and sufficient condition for X to be a critical point of the total energy for all variations X_t for which the volume

$\widehat{V}(X_t)$ is constant is that the immersed surface have constant mean curvature H and that the contact angle θ_i of $X(\Sigma)$ with Π_i (measured between $X(\Sigma)$ and D_i) be constant along C_i (see figure on page 2). More precisely,

$$\cos\theta_i = -\omega_i \quad \text{on } C_i.$$

The hypersurface $X(\Sigma)$ of constant mean curvature with constant contact angle along C_i will be called a *capillary* hypersurface. A capillary hypersurface is said to be stable if the second variation of $E(X_t)$ at $t = 0$ is nonnegative for all volume-preserving perturbations $X_t : (\Sigma, \partial\Sigma) \to (\overline{\Omega}, \partial\Omega)$ of $X(\Sigma)$.

A capillary hypersurface $X(\Sigma)$ in $\overline{\Omega}$ has a nice property called the *balancing formula* [Choe 2002; Concus et al. 2001; Korevaar et al. 1989]:

Lemma 1. *We have*

$$nH\mathcal{H}^n(D_i) = -(\sin\theta_i)\mathcal{H}^{n-1}(C_i), \quad i = 1, 2. \tag{3}$$

Proof. We first remark the following fact. Let $\hat{\Sigma}$ be an m-dimensional oriented compact connected C^∞ manifold, and $Y : \hat{\Sigma} \to \mathbb{R}^{m+1}$ a continuous map which is a piecewise C^∞ immersion. Also let $\hat{\nu}$ be the Gauss map of Y. Then, by using the divergence theorem, we obtain

$$\int_{\hat{\Sigma}} \hat{\nu}\, dS = 0.$$

Now integrate

$$\Delta_\Sigma X = nH\nu$$

on Σ to get

$$\sum_{i=1}^{2} \int_{C_i} \eta\, ds = nH \int_\Sigma \nu\, d\Sigma,$$

where η is the outward-pointing unit conormal to $\partial\Sigma$ on X. Then, use the above remark to obtain

$$\sum_{i=1}^{2} \int_{C_i} \eta\, ds = -nH \sum_{i=1}^{2} \int_{D_i} \nu\, dS. \tag{4}$$

Denote by N_i the unit normal to Π_i that points outward from Ω. Denote by n_i the inward pointing unit normal to C_i in Π_i. Set

$$\epsilon_i := \begin{cases} 1 & \text{if } \nu = N_i \text{ on } D_i, \\ -1 & \text{if } \nu = -N_i \text{ on } D_i. \end{cases} \tag{5}$$

Then from (4) we obtain

$$\sum_{i=1}^{2}\int_{C_i}\big((\sin\theta_i)\epsilon_i N_i-(\cos\theta_i)n_i\big)\,ds+\sum_{i=1}^{2} nH\mathcal{H}^n(D_i)\epsilon_i N_i=0,$$

that is, for the $(n-1)$-dimensional area $\mathcal{H}^{n-1}(C_i)$,

$$\sum_{i=1}^{2}(\sin\theta_i)\epsilon_i\mathcal{H}^{n-1}(C_i)N_i-\sum_{i=1}^{2}(\cos\theta_i)\int_{C_i}n_i\,ds+\sum_{i=1}^{2}nH\mathcal{H}^n(D_i)\epsilon_i N_i=0.$$

Using the above remark again, we obtain

$$\sum_{i=1}^{2}\big(nH\mathcal{H}^n(D_i)+(\sin\theta_i)\mathcal{H}^{n-1}(C_i)\big)N_i=0.$$

Since N_1 and N_2 are linearly independent, we obtain the formula (3). □

Another tool that will be essential in this paper is the formula for the volume of tubes due to H. Weyl [1939]. Given an immersion X of a compact oriented n-manifold M into $\mathbb{R}^{n+1}$, let $X_t=X+t\nu$ be the one-parameter family of parallel hypersurfaces to X. Thanks to the parallelness of X_t one can easily see that X_t has the same unit normal vector field as X and that the area $\mathcal{H}^n(X_t)$ is a polynomial of degree n in t. Namely, if $k_1,\ldots,k_n$ are the principal curvatures of X, then

$$\begin{aligned}\mathcal{H}^n(X_t)&=\int_M\prod_{i=1}^{n}(1-k_it)\,dS\\&=a_0+a_1t+a_2t^2+\cdots+a_nt^n,\\a_0&=\mathcal{H}^n(X_0),\\a_1&=-\int_M nH\,dS,\\a_2&=\int_M\sum_{i<j}k_ik_j\,dS,\\a_\ell&=(-1)^\ell\int_M\sum_{i_1<\cdots<i_\ell}k_{i_1}k_{i_2}\cdots k_{i_\ell}\,dS.\end{aligned}\tag{6}$$

Moreover, the oriented volume $V(X_t)$ satisfies

$$\frac{d}{dt}V(X_t)=\mathcal{H}^n(X_t).$$

Hence

$$V(X_t)=v_0+v_1t+v_2t^2+\cdots+v_{n+1}t^{n+1},$$
$$v_1=a_0,\quad 2v_2=a_1,\quad \ldots.$$

3. Admissible variations

Here we assume that our capillary hypersurface $X : (\Sigma, \partial\Sigma) \to (\bar{\Omega}, \partial\Omega)$ has a nonempty boundary component on each Π_i, $i = 1, 2$. But the case when Σ is in the half-space and $\partial\Sigma$ is connected can be treated similarly.

To check the stability of X one needs to deal with its volume-preserving variations $X_t : (\Sigma, \partial\Sigma) \to (\bar{\Omega}, \partial\Omega)$. The specific variation that we use arises from the parallel hypersurfaces

$$X_t^1 = X + t\nu.$$

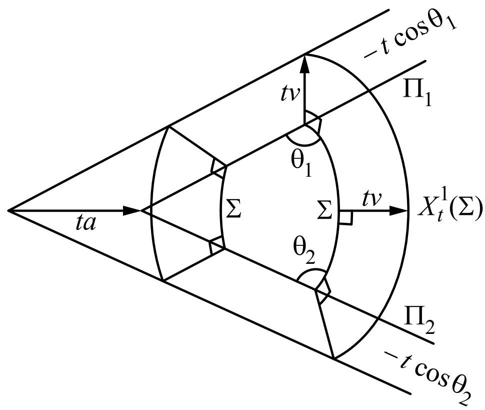
But X_t^1 does not satisfy the boundary condition $X_t^1(\partial\Sigma) \subset \partial\Omega$ unless $\theta_i = \pi/2$. To move the boundary to a desired place in $\partial\Omega$, we apply a translation

$$X_t^2(p) = p + ta$$

for some $a \in \mathbb{R}^{n+1}$. The vector a is determined in such a way that

$$X_t^2 \circ X_t^1(\partial\Sigma) \subset \partial\Omega.$$

Clearly such a vector uniquely exists as can be seen in the figure.



However, $X_t^2 \circ X_t^1$ is not volume-preserving. One way of making it into a volume-preserving variation is to deform it by a homothetic contraction

$$X_t := s(t) X_t^2 \circ X_t^1, \tag{7}$$

where $s(t)$ satisfies

$$\widehat{V}(X_t) = \widehat{V}(X_0) = v_0. \tag{8}$$

In order to compute $\widehat{V}(X_t)$ we first must consider the oriented volume $\widehat{V}(X_t^2 \circ X_t^1)$ enclosed by $X_t^2 \circ X_t^1(\Sigma) \cup D_1^t \cup D_2^t$, where $D_i^t \subset \Pi_i$ is the domain bounded by $\Pi_i \cap X_t^2 \circ X_t^1(\partial\Sigma)$. Note here that since $X_t^2 \circ X_t^1(\Sigma) \cup D_1^t \cup D_2^t$ is closed, the oriented volume $\widehat{V}(X_t^2 \circ X_t^1)$ as computed by (2) is independent of the translation X_t^2. While

t increases by Δt, the oriented volume $\widehat{V}(X_t^2 \circ X_t^1)$ increases by $\mathcal{H}^n(X_t^2 \circ X_t^1)\Delta t$ on $X_t^2 \circ X_t^1(\Sigma)$ and by $-\cos\theta_i \mathcal{H}^n(D_i^t)\Delta t$ on D_i^t. Hence

$$\frac{d}{dt}\widehat{V}(X_t^2 \circ X_t^1) = \mathcal{H}^n(X_t^2 \circ X_t^1) - \sum_i \cos\theta_i \mathcal{H}^n(D_i^t). \tag{9}$$

Calling $-\sum_i \cos\theta_i \mathcal{H}^n(D_i^t)$ the wetting energy $\mathcal{W}(X_t^2 \circ X_t^1)$ of $X_t^2 \circ X_t^1(\Sigma)$, we define the total energy by

$$E(X_t^2 \circ X_t^1) = \mathcal{H}^n(X_t^2 \circ X_t^1) + \mathcal{W}(X_t^2 \circ X_t^1).$$

The tube formula (6) for the capillary hypersurface Σ yields

$$\mathcal{H}^n(X_t^2 \circ X_t^1) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n,$$

$$a_0 = \mathcal{H}^n(\Sigma), \quad a_1 = -nHa_0, \quad a_2 = \int_\Sigma \sum_{i<j} k_i k_j \, dS,$$

$$\frac{d}{dt}\widehat{V}(X_t^2 \circ X_t^1) = E(X_t^2 \circ X_t^1). \tag{10}$$

Recall $C_i = X(\partial\Sigma) \cap \Pi_i$. Since $X_t^2 \circ X_t^1(\Sigma)$ has constant contact angle with $\partial\Omega$ for all t, $X_t^2 \circ X_t^1(C_i)$ are the parallel hypersurfaces of $p_{\Pi_i}(X_t^2(C_i))$, where p_{Π_i} denotes the projection of $\mathbb{R}^{n+1}$ onto Π_i. Also recall $\partial D_i = C_i$, $D_i = D_i^0$. The distance between $X_t^2 \circ X_t^1(C_i)$ and $p_{\Pi_i}(X_t^2(C_i))$ is $t\sin\theta_i$. Hence again by the tube formula for $\mathcal{H}^{n-1}(X_t^2 \circ X_t^1(C_i))$, we obtain

$$\mathcal{H}^n(D_i^t) = \mathcal{H}^n(D_i) + \mathcal{H}^{n-1}(C_i)\, t\sin\theta_i - \frac{1}{2}\left(\int_{C_i}(n-1)\overline{H}\, d\overline{S}\right)t^2 \sin^2\theta_i + \cdots + (-1)^{n-1}\frac{1}{n}\left(\int_{C_i} \bar{k}_1 \bar{k}_2 \cdots \bar{k}_{n-1}\, d\overline{S}\right)t^n \sin^n\theta_i,$$

where $\overline{H}$ and $\bar{k}_i$ are, respectively, the mean curvature and the principal curvature of C_i in Π_i with respect to the outward unit normal, and $d\overline{S}$ is the $(n-1)$-dimensional volume form of C_i.

Then (9) gives

$$\frac{d}{dt}\widehat{V}(X_t^2 \circ X_t^1) = a_0 - \sum_i \cos\theta_i \mathcal{H}^n(D_i) - \left(nHa_0 + \sum_i \cos\theta_i \sin\theta_i\, \mathcal{H}^{n-1}(C_i)\right)t + \left(\int_\Sigma \sum_{i<j} k_i k_j \, dS + \frac{1}{2}\sum_i \cos\theta_i \sin^2\theta_i \int_{C_i}(n-1)\overline{H}\, d\overline{S}\right)t^2 + \cdots .$$

Hence if we write

$$E(X_t^2 \circ X_t^1) = e_0 + e_1 t + \cdots + e_n t^n,$$

then (10) yields

$$
\begin{aligned}
e_0 &= a_0 - \sum_i \cos\theta_i \mathcal{H}^n(D_i), \\
e_1 &= -nHa_0 - \sum_i \cos\theta_i \sin\theta_i \mathcal{H}^{n-1}(C_i), \\
e_2 &= \int_\Sigma \sum_{i<j} k_i k_j \, dS + \frac{1}{2}\sum_i \cos\theta_i \sin^2\theta_i \int_{C_i} (n-1)\overline{H}\, d\overline{S}.
\end{aligned} \tag{11}
$$

On the other hand, if we let

$$\widehat{V}(X_t^2 \circ X_t^1) = v_0 + v_1 t + v_2 t^2 + \cdots + v_{n+1} t^{n+1},$$

then it follows from (7), (8), and the binomial series that

$$
\begin{aligned}
s(t)^n &= v_0^{n/(n+1)} (v_0 + v_1 t + v_2 t^2 + \cdots + v_{n+1} t^{n+1})^{-n/(n+1)} \\
&= 1 - \frac{n}{n+1}\Big(\frac{v_1}{v_0}\Big) t + \bigg(\frac{n(2n+1)}{2(n+1)^2}\Big(\frac{v_1}{v_0}\Big)^2 - \frac{n}{n+1}\Big(\frac{v_2}{v_0}\Big)\bigg) t^2 + \cdots .
\end{aligned}
$$

Thus

$$
\begin{aligned}
E(X_t) &= s(t)^n E(X_t^2 \circ X_t^1(\Sigma)) \\
&= e_0 + \bigg(e_1 - \frac{n}{n+1}\Big(\frac{v_1}{v_0}\Big) e_0\bigg) t \\
&\quad + \bigg(e_2 - \frac{n}{n+1}\Big(\frac{v_1}{v_0}\Big) e_1 + \frac{n(2n+1)}{2(n+1)^2}\Big(\frac{v_1}{v_0}\Big)^2 e_0 - \frac{n}{n+1}\Big(\frac{v_2}{v_0}\Big) e_0\bigg) t^2 \\
&\quad + \cdots .
\end{aligned} \tag{12}
$$

From (10) we have

$$v_1 = e_0, \quad 2v_2 = e_1, \tag{13}$$

and the fact that $E'(0) = 0$ in (12) implies

$$v_0 = \frac{n}{n+1}\, \frac{e_0^2}{e_1}. \tag{14}$$

Substituting the identities of (13) and (14) into the coefficient of t^2 in (12) yields

$$E''(0)/2 = \frac{1}{2ne_0}\big(2ne_0e_2 - (n-1)e_1^2\big).$$

Hence from (11) we get

$$ne_0E''(0) = 2n\Big(a_0 - \sum_i \cos\theta_i \mathcal{H}^n(D_i)\Big) \times \Big(\int_\Sigma \sum_{i<j} k_ik_j\, dS + \frac{1}{2}\sum_i \cos\theta_i \sin^2\theta_i \int_{C_i}(n-1)\overline{H}\, d\overline{S}\Big) - (n-1)\Big(nHa_0 + \sum_i \cos\theta_i \sin\theta_i\, \mathcal{H}^{n-1}(C_i)\Big)^2.$$

Then the balancing formula (3) yields

$$\Big(nHa_0 + \sum_i \cos\theta_i \sin\theta_i\, \mathcal{H}^{n-1}(C_i)\Big)^2 = n^2H^2\Big(a_0 - \sum_i \cos\theta_i \mathcal{H}^n(D_i)\Big)^2.$$

Therefore,

$$\begin{aligned} ne_0E''(0) &= \Big(a_0 - \sum_i \cos\theta_i \mathcal{H}^n(D_i)\Big) \\ &\quad\times \Big(2n\int_\Sigma \sum_{i<j} k_ik_j\, dS + n\sum_i \cos\theta_i \sin^2\theta_i \int_{C_i}(n-1)\overline{H}\, d\overline{S} \\ &\quad - \int_\Sigma n^2(n-1)H^2\, dS + n^2(n-1)H^2\sum_i \cos\theta_i \mathcal{H}^n(D_i)\Big) \\ &= \Big(a_0 - \sum_i \cos\theta_i \mathcal{H}^n(D_i)\Big) \\ &\quad\times \Big(-\int_\Sigma \sum_{i<j}(k_i-k_j)^2\, dS + n\sum_i \cos\theta_i \sin^2\theta_i \int_{C_i}(n-1)\overline{H}\, d\overline{S} \\ &\quad + n^2(n-1)H^2\sum_i \cos\theta_i \mathcal{H}^n(D_i)\Big). \end{aligned}$$

Applying the balancing formula (3) again, this gives

$$ne_0E''(0) = \Big(a_0 - \sum_i \cos\theta_i \mathcal{H}^n(D_i)\Big)\Big(-\int_\Sigma \sum_{i<j}(k_i-k_j)^2\, dS + (n-1)\sum_i \cos\theta_i \sin^2\theta_i\Big(n\int_{C_i}\overline{H}\, d\overline{S} + \frac{\mathcal{H}^{n-1}(C_i)^2}{\mathcal{H}^n(D_i)}\Big)\Big). \tag{15}$$

We shall see in the next section that

$$n\int_{\partial D_i}\overline{H}\, d\overline{S} + \frac{\mathcal{H}^{n-1}(\partial D_i)^2}{\mathcal{H}^n(D_i)} \geq 0.$$

4. Theorem

We are now ready to state the theorem of this paper.

Theorem 1. *Let W be a wedge in $\mathbb{R}^{n+1}$ bounded by two hyperplanes Π_1 and Π_2. Let $\Sigma \subset W$ be a compact oriented immersed hypersurface that is disjoint from the edge $\Pi_1 \cap \Pi_2$ of W, having smooth embedded boundary $\partial\Sigma \subset \Pi_1 \cup \Pi_2$, and satisfying $\partial\Sigma \cap \Pi_i = \partial D_i$ for a nonempty bounded connected domain D_i in Π_i. Suppose that Σ is a stable capillary hypersurface in W. In other words, Σ is an immersed constant mean curvature hypersurface making a constant contact angle $\theta_i \geq \pi/2$ with D_i such that for all volume-preserving perturbations (for the oriented volume enclosed by $\Sigma \cup D_1 \cup D_2$), the second variation of the total energy*

$$E(\Sigma) = \mathcal{H}^n(\Sigma) - \cos\theta_1 \mathcal{H}^n(D_1) - \cos\theta_2 \mathcal{H}^n(D_2)$$

is nonnegative.

(i) *If $n = 2$, then Σ is part of the 2-sphere.*

(ii) *If $n \geq 3$ and D_1 and D_2 are convex, then Σ is part of the n-sphere.*

Conversely, if Σ is part of the n-sphere, then it is stable.

Moreover, the same conclusion holds when Σ is in the half-space of $\mathbb{R}^{n+1}$ and $\partial\Sigma$ is connected.

Proof. We prove the theorem for Σ in a wedge, and the proof for Σ in the half-space is similar.

When $n = 2$, (15) becomes

$$2e_0 E''(0) = \Big(a_0 - \sum_i \cos\theta_i \mathcal{H}^2(D_i)\Big)\Big(-\int_\Sigma (k_1 - k_2)^2\, dS + \sum_i \cos\theta_i \sin^2\theta_i \Big(2\int_{\partial D_i} k\, ds + \frac{\mathcal{H}^1(\partial D_i)^2}{\mathcal{H}^2(D_i)}\Big)\Big),$$

where k is the geodesic curvature of ∂D_i with respect to the outward unit normal along ∂D_i. Note that on the smooth Jordan curve ∂D_i, $\int_{\partial D_i} k\, ds = -2\pi$. Hence the isoperimetric inequality of D_i and the angle condition $\cos\theta_i \leq 0$ yield

$$E''(0) \leq 0.$$

Therefore Σ needs to be umbilic everywhere if it is stable.

When $n \geq 3$, Minkowski showed that for a convex domain $D \subset \mathbb{R}^n$ with mean curvature H on ∂D,

$$n\int_{\partial D} |H|\, dS \leq \frac{\mathcal{H}^{n-1}(\partial D)^2}{\mathcal{H}^n(D)}$$

[Osserman 1978, p. 1191]. Hence it follows from (15) that the stable Σ is all umbilic.

If Σ is part of the n-sphere, then Σ is the minimizer of the energy E among all embedded hypersurfaces in Ω enclosing the same volume [Zia et al. 1988]. The proof is similar to that of Theorem 4.1 in [Koiso and Palmer 2007]; the method is essentially the same as in [Winterbottom 1967]. Hence Σ is stable for all $n \geq 2$. □

Remark 1. Our contact angle condition $\theta_i \geq \pi/2$ is quite natural because McCuan [1997] proved the nonexistence of embedded capillary surfaces with $\theta_i \leq \pi/2$ in a wedge of $\mathbb{R}^3$. Also it had been experimentally observed that a wedge forces the liquid drops (bridges) with $\theta_i \leq \pi/2$ to move toward its edge.

5. Minkowski's inequality

The Minkowski inequality is not well known among geometers and its proof is not easily available in the literature. So in this section we sketch a proof of it. First we need to introduce the mixed volume [Schneider 1993].

The *Minkowski sum* of two sets A and B in $\mathbb{R}^n$ is the set

$$A + B = \{a + b \in \mathbb{R}^n : a \in A, b \in B\}.$$

Given convex bodies $K_1, \ldots, K_r$ in $\mathbb{R}^n$, the volume of the Minkowski sum $\lambda_1 K_1 + \cdots + \lambda_r K_r$ (for $\lambda_i \geq 0$) of the scaled convex bodies $\lambda_i K_i$ of K_i is a homogeneous polynomial of degree n given by

$$\mathcal{H}^n(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{j_1,\ldots,j_n=1}^{r} V(K_{j_1}, \ldots, K_{j_n})\lambda_{j_1} \cdots \lambda_{j_n}.$$

$V(K_{j_1}, \ldots, K_{j_n})$ is called the *mixed volume* of $K_{j_1}, \ldots, K_{j_n}$. The mixed volume is uniquely determined by the following three properties:

(i) $V(K, \ldots, K) = \mathcal{H}^n(K)$, (ii) V is symmetric, (iii) V is multilinear.

A remarkable property of the mixed volume is the *Aleksandrov–Fenchel inequality*:

$$V(K_1, K_2, K_3, \ldots, K_n)^2 \geq V(K_1, K_1, K_3, \ldots, K_n) \cdot V(K_2, K_2, K_3, \ldots, K_n).$$

For a convex body $K \subset \mathbb{R}^n$ and a unit ball $B \subset \mathbb{R}^n$, the mixed volume

$$W_j(K) := V(\overbrace{K, K, \ldots, K}^{n-j \text{ times}}, \overbrace{B, B, \ldots, B}^{j \text{ times}})$$

is called the j-th *quermassintegral* of K. The Steiner formula says that the quermassintegrals of K determine the volume of the parallel bodies of K:

$$\mathcal{H}^n(K + tB) = \sum_{j=0}^{n} \binom{n}{j} W_j(K) t^j.$$

Comparing the Steiner formula for a convex domain $D \subset \mathbb{R}^n$ with its tube formula, one can obtain

$$\begin{aligned} W_0(D) &= \mathcal{H}^n(D), \\ nW_1(D) &= \mathcal{H}^{n-1}(\partial D), \\ nW_2(D) &= \int_{\partial D} |H|\, dS, \\ n(n-1)(n-2)W_3(D) &= 2\int_{\partial D} \sum_{i<j} k_i k_j\, dS. \end{aligned}$$

The Aleksandrov–Fenchel inequality for the quermassintegrals yields

$$\begin{aligned} W_1(D)^2 &\geq W_0(D)W_2(D), \\ W_2(D)^2 &\geq W_1(D)W_3(D). \end{aligned}$$

Consequently,

$$n\int_{\partial D} |H|\, dS \leq \frac{\mathcal{H}^{n-1}(\partial D)^2}{\mathcal{H}^n(D)}, \tag{16}$$

$$\begin{aligned} \int_{\partial D} \sum_{i<j} k_i k_j\, dS &\leq \frac{(n-1)(n-2)}{2} \frac{\left(\int_{\partial D} |H|\, dS\right)^2}{\mathcal{H}^{n-1}(\partial D)} \\ &\leq \frac{(n-1)(n-2)}{2n^2} \frac{\mathcal{H}^{n-1}(\partial D)^3}{\mathcal{H}^n(D)^2}, \end{aligned} \tag{17}$$

where (16) is the desired Minkowski inequality.

Remark 2. We note that (16) is the isoperimetric inequality when D is a domain in $\mathbb{R}^2$, and so is (17) when $D \subset \mathbb{R}^3$, because

$$\int_{\partial D \subset \mathbb{R}^2} |k|\, ds = 2\pi \quad \text{and} \quad \int_{\partial D \subset \mathbb{R}^3} k_1 k_2\, dS = 4\pi.$$

Remark 3. Let $D_t \subset \mathbb{R}^n$ be the parallel domain with distance t to D. Then (16) is equivalent to

$$n\,\frac{\mathcal{H}^{n-1}(\partial D_t)'}{\mathcal{H}^{n-1}(\partial D_t)} \leq \frac{(n-1)\mathcal{H}^n(D_t)'}{\mathcal{H}^n(D_t)},$$

or equivalently,

$$\left(\frac{\mathcal{H}^{n-1}(\partial D_t)^n}{\mathcal{H}^n(D_t)^{n-1}}\right)' \leq 0.$$

Hence the isoperimetric quotient $\mathcal{H}^{n-1}(\partial D_t)^n / \mathcal{H}^n(D_t)^{n-1}$ decreases as t increases. Indeed, the parallel domain D_t becomes rounder and rounder as t increases.

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