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OF VOLUME OF REPRESENTATIONS

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Let G be a rank-1 simple Lie group and let M be a connected, orientable, aspherical, tame manifold. Assume that each end of M has amenable fundamental group. There are several definitions of volume of representations of $\pi_1(M)$ into G. We give a new definition of volume of representations and furthermore show that all definitions so far are equivalent.

1. Introduction

Let G be a semisimple Lie group and let $\mathcal{X}$ be the associated symmetric space of dimension n. Let M be a connected, orientable, aspherical, tame manifold of the same dimension as $\mathcal{X}$. First assume that M is compact. To each representation $\rho : \pi_1(M) \to G$, one can associate a volume of ρ in the following way. First, associate a flat bundle E_ρ over M with fiber $\mathcal{X}$ to ρ. Since $\mathcal{X}$ is contractible, there always exists a section $s : M \to E_\rho$. Let $\omega_\mathcal{X}$ be the Riemannian volume form on $\mathcal{X}$. One may think of $\omega_\mathcal{X}$ as a closed differential form on E_ρ by spreading $\omega_\mathcal{X}$ over the fibers of E_ρ. Then the volume of ρ is defined by

$$\mathrm{Vol}(\rho) = \int_M s^* \omega_\mathcal{X}.$$

Since any two sections are homotopic to each other, the volume $\mathrm{Vol}(\rho)$ does not depend on the choice of section.

The volume of representations has been used to characterize discrete faithful representations. Let Γ be a uniform lattice in G. Then the volume of representations satisfies a Milnor–Wood type inequality. More precisely, for any representation $\rho : \Gamma \to G$, we have

$$(1) \qquad\qquad |\mathrm{Vol}(\rho)| \leq \mathrm{Vol}(\Gamma \backslash \mathcal{X}).$$

Furthermore, equality holds in (1) if and only if ρ is discrete and faithful. This is the so-called *volume rigidity theorem*. Goldman [1982] proved the volume rigidity theorem in the higher rank case and Besson, Courtois and Gallot [Besson et al. 2007] proved the theorem in the rank-1 case.

Now assume that M is noncompact. Then the definition of volume of representations as above is not valid anymore since some problems of integrability arise. So far, three definitions of volume of representations have been given under some conditions on M. Let us first fix the following notation throughout the paper.

Setup. Let M be a noncompact, connected, orientable, aspherical, tame manifold. Denote by $\bar{M}$ the compact manifold with boundary whose interior is homeomorphic to M. Assume that each connected component of $\partial \bar{M}$ has amenable fundamental group. Let G be a rank-1 semisimple Lie group with trivial center and no compact factors. Let $\mathcal{X}$ be the associated symmetric space of dimension n. Assume that M has the same dimension as $\mathcal{X}$.

First of all, Dunfield [1999] introduced the notion of pseudodeveloping map to define the volume of representations of a nonuniform lattice Γ in $\mathrm{SO}(3, 1)$. It was successful in making an invariant associated with a representation $\rho : \Gamma \to \mathrm{SO}(3, 1)$ but he did not prove that the volume of representations does not depend on the chosen pseudodeveloping map. After that, Francaviglia [2004] proved the well-definedness of the volume of representations. Then Francaviglia and Klaff [2006] extended the definition of volume of representations and the volume rigidity theorem to general nonuniform hyperbolic lattices. We call the definition of volume of representations via pseudodeveloping map D1. For more detail about D1, see [Francaviglia and Klaff 2006] or Section 4.

The second definition D2 of volume of representations was given by Bucher, Burger and Iozzi [Bucher et al. 2013] and generalizes the one introduced in [Burger et al. 2010] for noncompact surfaces. They used the theory of bounded cohomology to make an invariant associated with a representation. Given a representation $\rho : \pi_1(M) \to G$, one cannot get any information from the pullback map in degree n in continuous cohomology, $\rho_c^* : H_c^n(G, \mathbb{R}) \to H^n(\pi_1(M), \mathbb{R})$, since $H^n(\pi_1(M), \mathbb{R}) \cong H^n(M, \mathbb{R})$ is trivial. However, the situation is different in continuous bounded cohomology. Not only may the pullback map $\rho_b^* : H_{c,b}^n(G, \mathbb{R}) \to H_b^n(\pi_1(M), \mathbb{R})$ be nontrivial but it also encodes subtle algebraic and topological properties of a representation such as injectivity and discreteness. Bucher, Burger and Iozzi gave a proof of the volume rigidity theorem for representations of hyperbolic lattices from the point of view of bounded cohomology. We refer the reader to [Bucher et al. 2013] or Section 2 for further discussion about D2.

Recently, S. Kim and I. Kim [2014] gave a new definition, called D3, of volume of representations in the case that M is a complete Riemannian manifold with

finite Lipschitz simplicial volume. See [Kim and Kim 2014] or Section 5 for the exact definition of D3. In D3, it is not necessary that each connected component of $\partial \bar{M}$ has amenable fundamental group, while the amenable condition on $\partial \bar{M}$ is necessary in D2. They only use the bounded cohomology and ℓ^1-homology of M. It is quite useful to define the volume of representations in the case that the amenable condition on $\partial \bar{M}$ does not hold. They give a proof of the volume rigidity theorem for representations of lattices in an arbitrary semisimple Lie group in their setting.

In this note, we will give another definition of volume of representations, called D4. In D4, ρ-equivariant maps are involved as in D1 and the bounded cohomology of M is involved as in D2 and D3. In fact, D4 seems to be a kind of definition connecting the other definitions D1, D2 and D3. Eventually we show that all the definitions are equivalent.

Theorem 1.1. *Let G be a rank-1 simple Lie group with trivial center and no compact factors. Let M be a noncompact, connected, orientable, aspherical, tame manifold. Suppose that each end of M has amenable fundamental group. Then all definitions D1, D2 and D3 of volume of representations of $\pi_1(M)$ into G are equivalent. Furthermore, if M admits a complete Riemannian metric with finite Lipschitz simplicial volume, all definitions D1, D2, D3 and D4 are equivalent.*

The paper is organized as follows. For our proof, we recall the definitions of volume of representations in the order D2, D4, D1, D3. In Section 2, we first recall definition D2. In Section 3, we give definition D4 and then prove that D2 and D4 are equivalent. In Section 4, after recalling definition D1, we show the equivalence of D1 and D4. Finally in Section 5, we complete the proof of Theorem 1.1 by proving that D3 and D4 are equivalent.

2. Bounded cohomology and definition D2

We choose the appropriate complexes for the continuous cohomology and continuous bounded cohomology of G for our purpose. Consider the complex $C_c^*(\mathcal{X}, \mathbb{R})_{\text{alt}}$ with the homogeneous coboundary operator, where

$$C_c^k(\mathcal{X}, \mathbb{R})_{\text{alt}} = \{f : \mathcal{X}^{k+1} \to \mathbb{R} \mid f \text{ is continuous and alternating}\}.$$

The action of G on $C_c^k(\mathcal{X}, \mathbb{R})_{\text{alt}}$ is given by

$$g \cdot f(x_0, \ldots, x_k) = f(g^{-1}x_0, \ldots, g^{-1}x_k).$$

Then the continuous cohomology $H_c^*(G, \mathbb{R})$ can be isomorphically computed by the cohomology of the G-invariant complex $C_c^*(\mathcal{X}, \mathbb{R})_{\text{alt}}^G$ (see [Guichardet 1980, Chapitre III]). According to the Van Est isomorphism [Borel and Wallach 2000, Proposition IX.5.5], the continuous cohomology $H_c^*(G, \mathbb{R})$ is isomorphic to the set

of G-invariant differential forms on $\mathcal{X}$. Hence, in degree n, $H_c^n(G, \mathbb{R})$ is generated by the Riemannian volume form $\omega_{\mathcal{X}}$ on $\mathcal{X}$.

Let $C_{c,b}^k(\mathcal{X}, \mathbb{R})_{\mathrm{alt}}$ be a subcomplex of continuous, alternating, bounded real-valued functions on $\mathcal{X}^{k+1}$. The continuous bounded cohomology $H_{c,b}^*(G, \mathbb{R})$ is obtained by the cohomology of the G-invariant complex $C_{c,b}^*(\mathcal{X}, \mathbb{R})_{\mathrm{alt}}^G$ (see [Monod 2001, Corollary 7.4.10]). The inclusion of complexes $C_{c,b}^*(\mathcal{X}, \mathbb{R})_{\mathrm{alt}}^G \subset C_c^*(\mathcal{X}, \mathbb{R})_{\mathrm{alt}}^G$ induces a comparison map $H_{c,b}^*(G, \mathbb{R}) \to H_c^*(G, \mathbb{R})$.

Let Y be a countable CW-complex. Denote by $C_b^k(Y, \mathbb{R})$ the complex of bounded real-valued k-cochains on Y. For a subspace $B \subset Y$, let $C_b^k(Y, B, \mathbb{R})$ be the subcomplex of those bounded k-cochains on Y that vanish on simplices with image contained in B. The complexes $C_b^*(Y, \mathbb{R})$ and $C_b^*(Y, B, \mathbb{R})$ define the bounded cohomologies $H_b^*(Y, \mathbb{R})$ and $H_b^*(Y, B, \mathbb{R})$ respectively. For our convenience, we give another complex which computes the bounded cohomology $H_b^*(Y, \mathbb{R})$ of Y. Let $C_b^k(\widetilde{Y}, \mathbb{R})_{\mathrm{alt}}$ denote the complex of bounded, alternating real-valued Borel functions on $(\widetilde{Y})^{k+1}$. The $\pi_1(Y)$-action on $C_b^*(\widetilde{Y}, \mathbb{R})_{\mathrm{alt}}$ is defined as the G-action on $C_c^*(\mathcal{X}, \mathbb{R})$. Ivanov [1985] proved that the $\pi_1(Y)$-invariant complex $C_b^*(\widetilde{Y}, \mathbb{R})_{\mathrm{alt}}^{\pi_1(Y)}$ defines the bounded cohomology of Y.

Bucher, Burger and Iozzi [Bucher et al. 2013] used bounded cohomology to define the volume of representations. Let $\bar{M}$ be a connected, orientable, compact manifold with boundary. Suppose that each component of $\partial \bar{M}$ has amenable fundamental group. In that case, it is proved in [Bucher et al. 2012; Kim and Kuessner 2015] that the natural inclusion $i : (\bar{M}, \varnothing) \to (\bar{M}, \partial \bar{M})$ induces an isometric isomorphism in bounded cohomology,

$$i_b^* : H_b^*(\bar{M}, \partial \bar{M}, \mathbb{R}) \to H_b^*(\bar{M}, \mathbb{R}),$$

in degrees $* \geq 2$. Noting the remarkable result of Gromov [1982, Section 3.1] that the natural map $H_b^n(\pi_1(\bar{M}), \mathbb{R}) \to H_b^n(\bar{M}, \mathbb{R})$ is an isometric isomorphism in bounded cohomology, for a given representation $\rho : \pi_1(M) \to G$ we have a map

$$\rho_b^* : H_{c,b}^n(G, \mathbb{R}) \to H_b^n(\pi_1(\bar{M}), \mathbb{R}) \cong H_b^n(\bar{M}, \mathbb{R}) \cong H_b^n(\bar{M}, \partial \bar{M}, \mathbb{R}).$$

The G-invariant Riemannian volume form $\omega_{\mathcal{X}}$ on $\mathcal{X}$ gives rise to a continuous bounded cocycle $\Theta : \mathcal{X}^{n+1} \to \mathbb{R}$ defined by

$$\Theta(x_0, \ldots, x_n) = \int_{[x_0, \ldots, x_n]} \omega_{\mathcal{X}},$$

where $[x_0, \ldots, x_n]$ is the geodesic simplex with ordered vertices $x_0, \ldots, x_n$ in $\mathcal{X}$. The boundedness of Θ is due to the fact that the volume of geodesic simplices in $\mathcal{X}$ is uniformly bounded from above [Inoue and Yano 1982]. Hence the cocycle Θ induces a continuous cohomology class $[\Theta]_c \in H_c^n(G, \mathbb{R})$ and, moreover, a continuous bounded cohomology class $[\Theta]_{c,b} \in H_{c,b}^n(G, \mathbb{R})$. The image of $((i_b^*)^{-1} \circ \rho_b^*)[\Theta]_{c,b}$

via the comparison map $c : H_b^n(\bar{M}, \partial\bar{M}, \mathbb{R}) \to H^n(\bar{M}, \partial\bar{M}, \mathbb{R})$ is an ordinary relative cohomology class. Its evaluation on the relative fundamental class $[\bar{M}, \partial\bar{M}]$ gives an invariant associated with ρ.

Definition 2.1 (D2). For a representation $\rho : \pi_1(M) \to G$, define the invariant

$$\mathrm{Vol}_2(\rho) = \langle (c \circ (i_b^*)^{-1} \circ \rho_b^*)[\Theta]_{c,b}, [\bar{M}, \partial\bar{M}] \rangle.$$

In definition D2, a specific continuous bounded volume class $[\Theta]_{c,b}$ in $H_{c,b}^n(G, \mathbb{R})$ is involved. The question is naturally raised as to whether, if another continuous bounded volume class is used in D2 instead of $[\Theta]_{c,b}$, the value of the volume of representations changes or not. One could expect that definition D2 does not depend on the choice of continuous bounded volume class but it does not seem easy to get an answer directly. It turns out that D2 is independent of the choice of continuous bounded volume class. For a proof, see Section 5.

Proposition 2.2. *Definition D2 does not depend on the choice of continuous bounded volume class. That is, for any two continuous bounded volume classes* $\omega_b, \omega_b' \in H_{c,b}^n(G, \mathbb{R})$,

$$\langle (c \circ (i_b^*)^{-1} \circ \rho_b^*)(\omega_b), [\bar{M}, \partial\bar{M}] \rangle = \langle (c \circ (i_b^*)^{-1} \circ \rho_b^*)(\omega_b'), [\bar{M}, \partial\bar{M}] \rangle.$$

Bucher, Burger and Iozzi proved the volume rigidity theorem for hyperbolic lattices as follows.

Theorem 2.3 [Bucher et al. 2013]. *Let $n \geq 3$. Let $i : \Gamma \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^n)$ be a lattice embedding and let $\rho : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^n)$ be any representation. Then*

$$|\mathrm{Vol}_2(\rho)| \leq |\mathrm{Vol}_2(i)| = \mathrm{Vol}(\Gamma\backslash\mathbb{H}^n),$$

with equality if and only if ρ is conjugated to i by an isometry.

3. New definition D4

In this section we give a new definition of volume of representations. It will turn out that the new definition is useful in proving that all the definitions of volume of representations are equivalent.

End compactification. Let $\widehat{M}$ be the end compactification of M obtained by adding one point for each end of M. Let $\widetilde{M}$ denote the universal cover of M. Define $\widehat{\widetilde{M}}$ to be the space obtained by adding to $\widetilde{M}$ one point for each lift of each end of M. The points added to M are called *ideal points* of M and the points added to $\widetilde{M}$ are called *ideal points* of $\widetilde{M}$. Denote by $\partial\widehat{M}$ the set of ideal points of M and by $\partial\widehat{\widetilde{M}}$ the set of ideal points of $\widetilde{M}$. Let $p : \widetilde{M} \to M$ be the universal covering map. The covering map $p : \widetilde{M} \to M$ extends to a map $\hat{p} : \widehat{\widetilde{M}} \to \widehat{M}$ and, moreover, the action of $\pi_1(M)$

on $\widetilde{M}$ by covering transformations induces an action on $\widetilde{\widehat{M}}$. The action on $\widetilde{\widehat{M}}$ is not free because each point of $\partial\widetilde{\widehat{M}}$ is stabilized by some peripheral subgroup of $\pi_1(M)$.

Note that $\widehat{M}$ can be obtained by collapsing each connected component of $\partial\bar{M}$ to a point. Similarly, $\widetilde{\widehat{M}}$ can be obtained by collapsing each connected component of $\bar{p}^{-1}(\partial\bar{M})$ to a point where $\bar{p} : \widetilde{\bar{M}} \to \bar{M}$ is the universal covering map. We denote the collapsing map by $\pi : \widetilde{\bar{M}} \to \widetilde{\widehat{M}}$.

One advantage of $\widehat{M}$ is the existence of a fundamental class in singular homology. While the top dimensional singular homology of M vanishes, the top dimensional singular homology of $\widehat{M}$ with coefficients in $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$. Moreover, it can be easily seen that $H_*(\widehat{M}, \mathbb{R})$ is isomorphic to $H_*(\bar{M}, \partial\bar{M}, \mathbb{R})$ in degree $* \geq 2$. Hence the fundamental class of $\widehat{M}$ is well-defined and we denote it by $[\widehat{M}]$.

The cohomology groups. Let Y be a topological space and suppose that a group L acts continuously on Y. Then the cohomology group $H^*(Y; L, \mathbb{R})$ associated with Y and L is defined in the following way. Our main reference for this cohomology is [DuPre 1968].

For $k > 0$, define

$$F_{\mathrm{alt}}^k(Y, \mathbb{R}) = \{f : Y^{k+1} \to \mathbb{R} \mid f \text{ is alternating}\}.$$

Let $F_{\mathrm{alt}}^k(Y, \mathbb{R})^L$ denote the subspace of L-invariant functions, where the action of L on $F_{\mathrm{alt}}^k(Y, \mathbb{R})$ is given by

$$(g \cdot f)(y_0, \ldots, y_k) = f(g^{-1}y_0, \ldots, g^{-1}y_k),$$

for $f \in F_{\mathrm{alt}}^k(Y)$, $g \in L$. Define a coboundary operator $\delta_k : F_{\mathrm{alt}}^k(Y, \mathbb{R}) \to F_{\mathrm{alt}}^{k+1}(Y, \mathbb{R})$ by the usual

$$(\delta_k f)(y_0, \ldots, y_{k+1}) = \sum_{i=0}^{k+1} (-1)^i f(y_0, \ldots, \hat{y}_i, \ldots, y_{k+1}).$$

The coboundary operator restricts to the complex $F_{\mathrm{alt}}^*(Y, \mathbb{R})^L$. The cohomology $H^*(Y; L, \mathbb{R})$ is defined as the cohomology of this complex. Define $F_{\mathrm{alt},b}^*(Y, \mathbb{R})$ as the subspace of $F_{\mathrm{alt}}^*(Y, \mathbb{R})$ consisting of bounded alternating functions. Clearly the coboundary operator restricts to the complex $F_{\mathrm{alt},b}^*(Y, \mathbb{R})^L$ and so it defines a cohomology, denoted by $H_b^*(Y; L, \mathbb{R})$. In particular, for a manifold M, the cohomology $H^*(\widetilde{M}; \pi_1(M), \mathbb{R})$ is actually isomorphic to the group cohomology $H^*(\pi_1(M), \mathbb{R})$, and $H_b^*(\widetilde{M}; \pi_1(M), \mathbb{R})$ is isomorphic to the bounded cohomology $H_b^*(\pi_1(M), \mathbb{R})$.

Remark 3.1. Let L and L' be groups acting continuously on topological spaces Y and Y', respectively. Given a homomorphism $\rho : L \to L'$, any ρ-equivariant

continuous map $P : Y \to Y'$ defines a chain map,

$$P^* : F_{\text{alt}}^*(Y', \mathbb{R})^{L'} \to F_{\text{alt}}^*(Y, \mathbb{R})^L.$$

Thus it gives a morphism in cohomology. Let $Q : Y \to Y'$ be another ρ-equivariant map. For each $k > 0$, one may define

$$H_k(y_0, \ldots, y_k) = \sum_{i=0}^{i} (-1)^k (P(y_0), \ldots, P(y_i), Q(y_i), \ldots, Q(y_k)).$$

Then by a straightforward computation,

$$(\partial_{k+1} H_k + H_{k-1} \partial_k)(y_0, \ldots, y_k) = (P(y_0), \ldots, P(y_k)) - (Q(y_0), \ldots, Q(y_k)).$$

It follows from the above identity that, for any cocycle $f \in F_{\text{alt}}^k(Y', \mathbb{R})^{L'}$,

$$(P^*f - Q^*f)(y_0, \ldots, y_k) = \delta_k(f \circ H_{k-1})(y_0, \ldots, y_k).$$

From this usual process in cohomology theory, one could expect that P and Q induce the same morphism in cohomology. However, since $f \circ H_{k-1}$ may not be alternating, P and Q may not induce the same morphism in cohomology.

Since $\Theta : \mathcal{X}^{n+1} \to \mathbb{R}$ is a G-invariant continuous bounded alternating cocycle, it yields a bounded cohomology class $[\Theta]_b \in H_b^n(\mathcal{X}; G, \mathbb{R})$. Let $\bar{\mathcal{X}}$ be the compactification of $\mathcal{X}$ obtained by adding the ideal boundary $\partial \mathcal{X}$. Extending the G-action on $\mathcal{X}$ to $\bar{\mathcal{X}}$, we can define a cohomology $H^*(\bar{\mathcal{X}}; G, \mathbb{R})$ and bounded cohomology $H_b^*(\bar{\mathcal{X}}; G, \mathbb{R})$. In the rank-1 case, since the geodesic simplex is well-defined for any $(n + 1)$-tuple of points of $\bar{X}$, the cocycle Θ can be extended to a G-invariant alternating bounded cocycle $\bar{\Theta} : \bar{X}^{n+1} \to \mathbb{R}$. Hence $\bar{\Theta}$ determines a cohomology class $[\bar{\Theta}] \in H^n(\bar{\mathcal{X}}; G, \mathbb{R})$ and $[\bar{\Theta}]_b \in H_b^n(\bar{\mathcal{X}}; G, \mathbb{R})$.

Let $\widehat{D} : \widehat{\widetilde{M}} \to \bar{\mathcal{X}}$ be a ρ-equivariant continuous map whose restriction to $\widetilde{M}$ is a ρ-equivariant continuous map from $\widetilde{M}$ to $\mathcal{X}$. We will consider only such kinds of equivariant maps throughout the paper. Denote by $D : \widetilde{M} \to \mathcal{X}$ the restriction of $\widehat{D}$ to $\widetilde{M}$. Then $\widehat{D}$ induces a homomorphism in cohomology,

$$\widehat{D}^* : H^n(\bar{\mathcal{X}}; G, \mathbb{R}) \to H^n(\widehat{\widetilde{M}}; \pi_1(M), \mathbb{R}).$$

Note that the action of $\pi_1(M)$ on $\widehat{\widetilde{M}}$ is not free and hence $H^*(\widehat{\widetilde{M}}; \pi_1(M), \mathbb{R})$ may not be isomorphic to $H^*(\widehat{M}, \mathbb{R})$. Let $H_{\text{simp}}^*(\widehat{M}, \mathbb{R})$ denote the simplicial cohomology induced from a simplicial structure on $\widehat{M}$. Then there is a natural restriction map $H^*(\widehat{\widetilde{M}}; \pi_1(M), \mathbb{R}) \to H_{\text{simp}}^*(\widehat{M}, \mathbb{R}) \cong H^*(\widehat{M}, \mathbb{R})$. Thus we regard the cohomology class $\widehat{D}^*[\bar{\Theta}]$ as a cohomology class of $H^n(\widehat{M}, \mathbb{R})$. Let $[\widehat{M}]$ be the fundamental cycle in $H_n(\widehat{M}, \mathbb{R}) \cong \mathbb{R}$.

Definition 3.2 (D4). Let $D : \widetilde{M} \to \mathcal{X}$ be a ρ-equivariant continuous map which is extended to a ρ-equivariant map $\widehat{D} : \widehat{\widetilde{M}} \to \bar{\mathcal{X}}$. Then we define the invariant

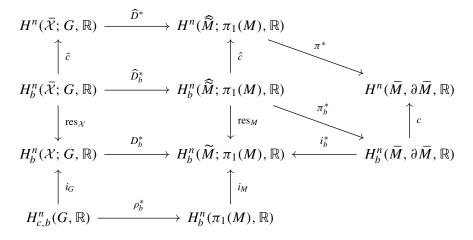$$\mathrm{Vol}_4(\rho, D) = \langle \widehat{D}^*[\bar{\Theta}], [\widehat{M}] \rangle.$$

As observed before, $\widehat{D}^*[\bar{\Theta}]$ may depend on the choice of ρ-equivariant map. However, it turns out that the value $\mathrm{Vol}_4(\rho, D)$ is independent of the choice of ρ-equivariant continuous map as follows.

Proposition 3.3. *Let $\rho : \pi_1(M) \to G$ be a representation. Then*

$$\mathrm{Vol}_2(\rho) = \mathrm{Vol}_4(\rho, D).$$

Proof. Since the continuous bounded cohomology $H^*_{c,b}(G, \mathbb{R})$ can be computed isomorphically from the complex $C^*_{c,b}(\mathcal{X}, \mathbb{R})_{\mathrm{alt}}$, there is the natural inclusion $C^*_{c,b}(\mathcal{X}, \mathbb{R})_{\mathrm{alt}} \subset F^*_{\mathrm{alt},b}(\mathcal{X}, \mathbb{R})$. Denote the homomorphism in cohomology induced from the inclusion by $i_G : H^k_{c,b}(G, \mathbb{R}) \to H^k_b(\mathcal{X}; G, \mathbb{R})$. Clearly, $i_G([\Theta]_{c,b}) = [\Theta]_b$.

The bounded cohomology $H^*_b(\pi_1(M), \mathbb{R})$ is obtained by the cohomology of the complex $C^*_b(\widetilde{M}, \mathbb{R})^{\pi_1(M)}_{\mathrm{alt}}$. Since $C^*_b(\widetilde{M}, \mathbb{R})_{\mathrm{alt}} = F^*_{\mathrm{alt},b}(\widetilde{M}, \mathbb{R})$, the induced map $i_M : H^k_b(\pi_1(M), \mathbb{R}) \to H^k_b(\widetilde{M}; \pi_1(M), \mathbb{R})$ is the identity map. Let $\widehat{D} : \widehat{\widetilde{M}} \to \bar{\mathcal{X}}$ be a ρ-equivariant map which maps $\widetilde{M}$ to $\mathcal{X}$. Then consider the following commutative diagram, where $\pi : \widetilde{M} \to \widehat{\widetilde{M}}$ is the collapsing map:

Note that the map ρ^*_b in the bottom of the diagram is actually induced from the restriction map $D : \widetilde{M} \to \mathcal{X}$. However, it does not depend on the choice of equivariant map but only on the homomorphism ρ. In other words, any continuous equivariant map from $\widetilde{M}$ to $\mathcal{X}$ gives rise to the same map, $\rho^*_b : H^*_{c,b}(G, \mathbb{R}) \to H^*_b(\pi_1(M), \mathbb{R})$. For this reason, we denote it by ρ^*_b instead of $D^*_{c,b}$.

Note that π induces a map $\pi^* : F^*_{\mathrm{alt}}(\widehat{\widetilde{M}}, \mathbb{R}) \to F^*_{\mathrm{alt}}(\widetilde{M}, \mathbb{R})$. It follows from the alternating property that the image of π^* is contained in $C^*(\bar{M}, \partial\bar{M}, \mathbb{R})$. Hence the

map $\pi^* : H^n(\widehat{\overline{M}}; \pi_1(M), \mathbb{R}) \to H^n(\overline{M}, \partial \overline{M}, \mathbb{R})$ makes sense. One can understand $\pi_b^* : H_b^n(\widehat{\overline{M}}; \pi_1(M), \mathbb{R}) \to H_b^n(\overline{M}, \partial \overline{M}, \mathbb{R})$ in a similar way.

Noting that $\bar{c}([\overline{\Theta}]_b) = [\overline{\Theta}]$ and $\mathrm{res}_{\mathcal{X}}([\overline{\Theta}]_b) = [\Theta]_b$, it follows from the above commutative diagram that

$$((i_b^*)^{-1} \circ i_M \circ \rho_b^*)[\Theta]_{c,b} = ((i_b^*)^{-1} \circ D_b^* \circ i_G)[\Theta]_{c,b} = ((i_b^*)^{-1} \circ D_b^* \circ \mathrm{res}_{\mathcal{X}})[\overline{\Theta}]_b$$
$$= ((i_b^*)^{-1} \circ \mathrm{res}_M \circ \widehat{D}_b^*)[\overline{\Theta}]_b = (\pi_b^* \circ \widehat{D}_b^*)[\overline{\Theta}]_b.$$

Hence

$$\begin{aligned}
\mathrm{Vol}_2(\rho) &= \langle (c \circ (i_b^*)^{-1} \circ i_M \circ \rho_b^*)[\Theta]_{c,b}, [\overline{M}, \partial \overline{M}] \rangle \\
&= \langle (c \circ \pi_b^* \circ \widehat{D}_b^*)[\overline{\Theta}]_b, [\overline{M}, \partial \overline{M}] \rangle = \langle (\pi^* \circ \widehat{D}^* \circ \bar{c})[\overline{\Theta}]_b, [\overline{M}, \partial \overline{M}] \rangle \\
&= \langle (\pi^* \circ \widehat{D}^*)[\overline{\Theta}], [\overline{M}, \partial \overline{M}] \rangle = \langle \widehat{D}^*[\overline{\Theta}], \pi_*[\overline{M}, \partial \overline{M}] \rangle \\
&= \langle \widehat{D}^*[\overline{\Theta}], [\widehat{M}] \rangle = \mathrm{Vol}_4(\rho, D). \qquad \qquad \square
\end{aligned}$$

Proposition 3.3 implies that the value $\mathrm{Vol}_4(\rho, D)$ does not depend on the choice of continuous equivariant map. Hence from now on we will use the notation $\mathrm{Vol}_4(\rho) := \mathrm{Vol}(\rho, D)$. Furthermore, Proposition 3.3 allows us to interpret the invariant $\mathrm{Vol}_2(\rho)$ in terms of a pseudodeveloping map via $\mathrm{Vol}_4(\rho)$ in the next section. Note that a pseudodeveloping map for ρ is a specific kind of ρ-equivariant continuous map $\widehat{\widetilde{M}} \to \overline{\mathcal{X}}$.

4. Pseudodeveloping map and definition D1

Dunfield [1999] introduced the notion of pseudodeveloping map in order to define the volume of representations $\rho : \pi_1(M) \to \mathrm{SO}(3, 1)$ for a noncompact complete hyperbolic 3-manifold M of finite volume. We start by recalling the definition of pseudodeveloping map.

Definition 4.1 (cone map). Let $\mathcal{A}$ be a set, let $t_0 \in \mathbb{R}$, and let $\mathrm{cone}(\mathcal{A})$ be the cone obtained from $\mathcal{A} \times [t_0, \infty]$ by collapsing $\mathcal{A} \times \{\infty\}$ to a point, called ∞. A map $\widehat{D} : \mathrm{cone}(\mathcal{A}) \to \overline{\mathcal{X}}$ is a *cone map* if $\widehat{D}(\mathrm{cone}(\mathcal{A})) \cap \partial \mathcal{X} = \{\widehat{D}(\infty)\}$ and for all $a \in \mathcal{A}$ the map $\widehat{D}|_{a \times [t_0, \infty]}$ is either the constant to $\widehat{D}(\infty)$ or the geodesic ray from $\widehat{D}(a, t_0)$ to $\widehat{D}(\infty)$, parametrized in such a way that the parameter $(t - t_0)$, $t \in [t_0, \infty]$, is the arc length.

For each ideal point v of M, fix a product structure $T_v \times [0, \infty)$ on the end relative to v. The fixed product structure induces a cone structure on a neighborhood of v in $\widehat{M}$, which is obtained from $T_v \times [0, \infty]$ by collapsing $T_v \times \{\infty\}$ to a point v. We lift such structures to the universal cover. Let $\tilde{v}$ be an ideal point of $\widetilde{M}$ that projects to the ideal point v. Denote by $E_{\tilde{v}}$ the cone at $\tilde{v}$ that is homeomorphic to $P_{\tilde{v}} \times [0, \infty]$, where $P_{\tilde{v}}$ covers T_v and $P_{\tilde{v}} \times \{\infty\}$ is collapsed to $\tilde{v}$.

Definition 4.2 (pseudodeveloping map). Let $\rho : \pi_1(M) \to G$ be a representation. A *pseudodeveloping map* for ρ is a piecewise-smooth ρ-equivariant map $D : \widetilde{M} \to \mathcal{X}$. Moreover, D is required to extend to a continuous map $\widehat{D} : \widehat{\widetilde{M}} \to \bar{\mathcal{X}}$ with the property that there exists a $t \in \mathbb{R}^+$ such that, for each end $E_{\tilde{v}} = P_{\tilde{v}} \times [0, \infty]$ of $\widehat{\widetilde{M}}$, the restriction of $\widehat{D}$ to $P_{\tilde{v}} \times [t, \infty]$ is a cone map.

Definition 4.3. A *triangulation* of $\widehat{M}$ is an identification of $\widehat{M}$ with a complex obtained by gluing together with simplicial attaching maps. It is not required for the complex to be simplicial, but it is required that open simplices embed.

Note that a triangulation of $\widehat{M}$ always exists and it lifts uniquely to a triangulation of $\widehat{\widetilde{M}}$. Given a triangulation of $\widehat{M}$, one can define the straightening of pseudodeveloping maps as follows.

Definition 4.4 (straightening map). Let $\widehat{M}$ be triangulated. Let $\rho : \pi_1(M) \to G$ be a representation and $D : \widetilde{M} \to \mathcal{X}$ a pseudodeveloping map for ρ. A straightening of D is a continuous piecewise-smooth ρ-equivariant map $\mathrm{Str}(D) : \widehat{\widetilde{M}} \to \bar{\mathcal{X}}$ such that

- for each simplex σ of the triangulation, $\mathrm{Str}(D)$ maps $\tilde{\sigma}$ to $\mathrm{Str}(D \circ \tilde{\sigma})$,
- for each end $E_{\tilde{v}} = P_{\tilde{v}} \times [0, \infty]$, there exists a $t \in \mathbb{R}$ such that $\mathrm{Str}(D)$ restricted to $P_{\tilde{v}} \times [t, \infty]$ is a cone map,

where $\tilde{\sigma}$ is a lift of σ to $\widehat{\widetilde{M}}$ and $\mathrm{Str}(D \circ \tilde{\sigma})$ is the geodesic straightening of the map $D \circ \tilde{\sigma} : \Delta^n \to \bar{\mathcal{X}}$.

Note that any straightening of a pseudodeveloping map is also a pseudodeveloping map.

Lemma 4.5. *Let $\widehat{M}$ be triangulated. Let $\rho : \pi_1(M) \to G$ be a representation and $D : \widetilde{M} \to \mathcal{X}$ a pseudodeveloping map for ρ. Then a straightening $\mathrm{Str}(D)$ of D exists and, furthermore, $\mathrm{Str}(D) : \widehat{\widetilde{M}} \to \bar{\mathcal{X}}$ is always equivariantly homotopic to $\widehat{D}$ via a homotopy that fixes the vertices of the triangulation.*

Proof. First, set $\mathrm{Str}(D)(V) = f(V)$ for every vertex V of the triangulation. Then extend $\mathrm{Str}(D)$ to a map which is piecewise-straight with respect to the triangulation. This is always possible because $\mathcal{X}$ is contractible. Note that $\widehat{D}$ and $\mathrm{Str}(D)$ agree on the ideal vertices of $\widehat{\widetilde{M}}$ and are equivariantly homotopic via the straight-line homotopy between them. Hence it can be easily seen that the extension is a straightening of D. $\square$

For any pseudodeveloping map $D : \widetilde{M} \to \mathcal{X}$, for ρ,

$$\int_M D^* \omega_{\mathcal{X}}$$

is always finite. This can be seen as follows. We stick to the notation used in Definition 4.2. We may assume that the restriction of $\widehat{D}$ to each $E_{\tilde{v}} = P_{\tilde{v}} \times [0, \infty]$

is a cone map. Choose a fundamental domain F_0 of T_v in $P_{\tilde{v}}$. Then there exists a $t \in \mathbb{R}^+$ such that

$$\left| \int_{T_v \times [t, \infty)} D^* \omega_{\mathcal{X}} \right| = \mathrm{Vol}_n \big(\mathrm{cone}(D(F_0 \times \{t\})) \big) \leq \frac{1}{n-1} \mathrm{Vol}_{n-1}(D(F_0 \times \{t\})),$$

where Vol_{n-1} denotes the $(n-1)$-dimensional volume. The last inequality holds for any Hadamard manifold with sectional curvature at most -1. See [Gromov 1982, Section 1.2]. Hence the integral of $D^* \omega_{\mathcal{X}}$ over M is finite.

Definition 4.6 (D1). Let $D : \widetilde{M} \to \mathcal{X}$ be a pseudodeveloping map for a representation $\rho : \pi_1(M) \to G$. Define the invariant

$$\mathrm{Vol}_1(\rho, D) = \int_M D^* \omega_{\mathcal{X}}.$$

In the case that $G = \mathrm{SO}(n, 1)$, Francaviglia [2004] showed that definition D1 does not depend on the choice of pseudodeveloping map. We give a self-contained proof for this in the rank-1 case.

Proposition 4.7. *Let $\rho : \pi_1(M) \to G$ be a representation. Then, for any pseudodeveloping map $D : \widetilde{M} \to \mathcal{X}$,*

$$\mathrm{Vol}_1(\rho, D) = \mathrm{Vol}_4(\rho).$$

Thus, $\mathrm{Vol}_1(\rho, D)$ does not depend on the choice of pseudodeveloping map.

Proof. Let $\mathcal{T}$ be a triangulation of $\widehat{M}$ with simplices $\sigma_1, \ldots, \sigma_N$. Then the triangulation gives rise to a fundamental cycle $\sum_{i=1}^N \sigma_i$ of $\widehat{M}$. Let $\mathrm{Str}(D)$ be a straightening of D with respect to the triangulation $\mathcal{T}$. Since $\mathrm{Str}(D)$ is a ρ-equivariant continuous map, we have

$$\mathrm{Vol}_4(\rho) := \mathrm{Vol}_4(\rho, D) = \langle \mathrm{Str}(D)^* [\bar{\Theta}], [\widehat{M}] \rangle = \langle \bar{\Theta}, \sum_{i=1}^N \mathrm{Str}(\widehat{D}(\sigma_i)) \rangle$$

$$= \sum_{i=1}^N \int_{\mathrm{Str}(\widehat{D}(\sigma_i))} \omega_{\mathcal{X}} = \int_M \mathrm{Str}(D)^* \omega_{\mathcal{X}}.$$

Since both $\mathrm{Str}(D)$ and $\widehat{D}$ are pseudodeveloping maps for ρ that agree on the ideal points of $\widehat{\widetilde{M}}$, it can be proved, using the same arguments as the proof of [Dunfield 1999, Lemma 2.5.1], that

$$\int_M \mathrm{Str}(D)^* \omega_{\mathcal{X}} = \int_M D^* \omega_{\mathcal{X}} = \mathrm{Vol}_1(\rho, D). \qquad \square$$

Remark 4.8. While D1 is defined with only a pseudodeveloping map, definition D4 is defined with any equivariant map. This is one advantage of definition D4. By Proposition 4.7, the notation $\mathrm{Vol}_1(\rho) := \mathrm{Vol}_1(\rho, D)$ makes sense.

5. Lipschitz simplicial volume and definition D3

In this section, M is assumed to be a Riemannian manifold with finite Lipschitz simplicial volume. Gromov [1982, Section 4.4] introduced the Lipschitz simplicial volume of Riemannian manifolds. One can define the Lipschitz constant for each singular simplex in M by giving the Euclidean metrics on the standard simplices. Then the Lipschitz constant of a locally finite chain c of M is defined as the supremum of the Lipschitz constants of all singular simplices occurring in c. The Lipschitz simplicial volume of M is defined by the infimum of the ℓ^1-norms of all locally finite fundamental cycles with finite Lipschitz constant. Let $[M]_{\text{Lip}}^{\ell^1}$ be the set of all locally finite fundamental cycles of M with finite ℓ^1-seminorm and finite Lipschitz constant. If $[M]_{\text{Lip}}^{\ell^1} = \varnothing$, the Lipschitz simplicial volume of M is infinite.

In the case that $[M]_{\text{Lip}}^{\ell^1} \neq \varnothing$, we gave a new definition of volume of representations in [Kim and Kim 2014] as follows. A representation $\rho : \pi_1(M) \to G$ induces a canonical pullback map $\rho_b^* : H_{c,b}^*(G, \mathbb{R}) \to H_b^*(\pi_1(M), \mathbb{R}) \cong H_b^*(M, \mathbb{R})$ in continuous bounded cohomology. Hence, for any continuous bounded volume class $\omega_b \in H_{c,b}^n(G, \mathbb{R})$, we obtain a bounded cohomology class $\rho_b^*(\omega_b) \in H_b^n(M, \mathbb{R})$. Then, the bounded cohomology class $\rho_b^*(\omega_b)$ can be evaluated on ℓ^1-homology classes in $H_n^{\ell^1}(M, \mathbb{R})$ by the Kronecker products,

$$\langle \cdot, \cdot \rangle : H_b^*(M, \mathbb{R}) \otimes H_*^{\ell^1}(M, \mathbb{R}) \to \mathbb{R}.$$

For more details about this, see [Kim and Kim 2014].

Definition 5.1 (D3). We define the invariant

$$\text{Vol}_3(\rho) = \inf \langle \rho_b^*(\omega_b), \alpha \rangle,$$

where the infimum is taken over all $\alpha \in [M]_{\text{Lip}}^{\ell^1}$ and all $\omega_b \in H_{c,b}^n(G, \mathbb{R})$ with $c(\omega_b) = \omega_\mathcal{X}$.

One advantage of D3 is that the isomorphism $H_b^n(\bar{M}, \partial \bar{M}, \mathbb{R}) \to H_b^n(\bar{M}, \mathbb{R})$ is not needed. When M admits the isomorphism above, we will verify that definition D3 is eventually equivalent to the other definitions of volume of representations.

Lemma 5.2. *Suppose that M is a noncompact, connected, orientable, aspherical, tame Riemannian manifold with finite Lipschitz simplicial volume and that each end of M has amenable fundamental group. Then, for any $\alpha \in [M]_{\text{Lip}}^{\ell^1}$ and any continuous bounded volume class ω_b,*

$$\langle \rho_b^*(\omega_b), \alpha \rangle = \langle (c \circ (i_b^*)^{-1} \circ \rho_b^*)(\omega_b), [\bar{M}, \partial \bar{M}] \rangle.$$

Proof. When M is a 2-dimensional manifold, the proof is given in [Kim and Kim 2014]. Actually the proof in the general case is the same. We sketch the proof here for the reader's convenience. Let K be a compact core of M. Note that K is a

compact submanifold with boundary that is a deformation retract of M. Consider the following commutative diagram, where every map is the map induced from the canonical inclusion:

$$
\begin{array}{ccccc}
C_b^*(M, \mathbb{R}) & \xleftarrow{\;\;j_b^*\;\;} & C_b^*(\bar{M}, \mathbb{R}) & \xleftarrow{\;\;i_b^*\;\;} & C_b^*(\bar{M}, \partial\bar{M}, \mathbb{R}) \\
& & \Big\uparrow{\scriptstyle l_b^*} & \nearrow{\scriptstyle q_b^*} & \\
& & C_b^*(\bar{M}, \bar{M} - K, \mathbb{R}) & &
\end{array}
$$

Every map in the diagram induces an isomorphism in bounded cohomology in $* \geq 2$. Thus, there exists a cocycle $z_b \in C_b^n(\bar{M}, \bar{M} - K, \mathbb{R})$ such that $l_b^*([z_b]) = \rho_b^*(\omega_b)$.

Let $c = \sum_{i=1}^{\infty} a_i \sigma_i$ be a locally finite fundamental ℓ^1-cycle with finite Lipschitz constant representing $\alpha \in [M]_{\text{Lip}}^{\ell^1}$. Then we have

$$
\langle \rho_b^*(\omega_b), \alpha \rangle = \langle l_b^*([z_b]), \alpha \rangle = \langle z_b, c \rangle.
$$

Since z_b vanishes on simplices with image contained in $\bar{M} - K$, we have the equality $\langle z_b, c \rangle = \langle z_b, c|_K \rangle$, where $c|_K = \sum_{\text{im}\,\sigma_i \cap K \neq \varnothing} a_i \sigma_i$. It is a standard fact that the sum $c|_K$ represents the relative fundamental class $[\bar{M}, \bar{M} - K]$ in $H_n(\bar{M}, \bar{M} - K, \mathbb{R})$ (see [Löh 2007, Theorem 5.3]). On the other hand, we have

$$
\begin{aligned}
\langle (c \circ (i_b^*)^{-1} \circ \rho_b^*)(\omega_b), [\bar{M}, \partial\bar{M}] \rangle &= \langle (c \circ q_b^*)([z_b]), [\bar{M}, \partial\bar{M}] \rangle \\
&= \langle [z_b], q_*[\bar{M}, \partial\bar{M}] \rangle \\
&= \langle [z_b], [\bar{M}, \bar{M} - K] \rangle = \langle z_b, c|_K \rangle. \qquad \square
\end{aligned}
$$

By Lemma 5.2 we can reformulate definition D3 as

$$
\text{Vol}_3(\rho) = \inf_{\omega_b} \langle (c \circ (i_b^*)^{-1} \circ \rho_b^*)(\omega_b), [\bar{M}, \partial\bar{M}] \rangle,
$$

where the infimum is taken over all continuous bounded volume classes. Noting that $[\Theta]_{c,b} \in H_{c,b}^n(G, \mathbb{R})$ is a continuous bounded volume class, it is clear that

$$
\text{Vol}_3(\rho) \leq \text{Vol}_2(\rho).
$$

It is conjecturally true that the comparison map $H_{c,b}^n(G, \mathbb{R}) \to H_c^n(G, \mathbb{R})$ is an isomorphism for any connected semisimple Lie group G with finite center. Hence, conjecturally, $\text{Vol}_2(\rho) = \text{Vol}_3(\rho)$. In spite of the absence of a proof of the conjecture, we will give a proof for $\text{Vol}_2(\rho) = \text{Vol}_3(\rho)$ by using definition D4.

Lemma 5.3. *Let $\omega_b \in H_{c,b}^n(G, \mathbb{R})$ be a continuous bounded volume class, and let $f_b : \mathcal{X}^{n+1} \to \mathbb{R}$ be a continuous bounded alternating G-invariant cocycle representing ω_b. Then f_b is extended to a bounded alternating G-invariant cocycle $\bar{f}_b : \bar{\mathcal{X}}^{n+1} \to \mathbb{R}$. Furthermore, $\bar{f}_b$ is uniformly continuous on $\mathcal{X}^n \times \{\xi\}$ for any $\xi \in \partial\mathcal{X}$.*

Proof. For any $(\bar{x}_0, \ldots, \bar{x}_n) \in \bar{\mathcal{X}}^{n+1}$, define

$$\bar{f}_b(\bar{x}_0, \ldots, \bar{x}_n) = \lim_{t \to \infty} f_b(c_0(t), \ldots, c_n(t)),$$

where each $c_i(t)$ is a geodesic ray toward $\bar{x}_i$. Here, for $x \in \mathcal{X}$, we say that $c : [0, \infty) \to \mathcal{X}$ is a geodesic ray toward x if there exists a $t \in [0, \infty)$ such that the restriction map $c|_{[0,t]}$ of c to $[0, t]$ is a geodesic with $c(t) = x$ and $c|_{[t,\infty)}$ is constant to x. Then it is clear that $\bar{f}_b(x_0, \ldots, x_n) = f_b(x_0, \ldots, x_n)$ for $(x_0, \ldots, x_n) \in \mathcal{X}^{n+1}$. To see the well-definedness of $\bar{f}_b$, we need to show that, for other geodesic rays $c'_i(t)$ toward $\bar{x}_i$,

$$(2) \qquad \lim_{t \to \infty} f_b(c_0(t), \ldots, c_n(t)) = \lim_{t \to \infty} f_b(c'_0(t), \ldots, c'_n(t)).$$

Note that the limit always exists because f_b is bounded. In the rank-1 case, the distance between two geodesic rays with the same endpoint decays exponentially to 0 as they go to the endpoint. Moreover, since f_b is G-invariant and G transitively acts on $\mathcal{X}$, we have that f_b is uniformly continuous on $\mathcal{X}^{n+1}$. Thus, for any $\epsilon > 0$, there exists some number $T > 0$ such that

$$|f_b(c_0(t), \ldots, c_n(t)) - f_b(c'_0(t), \ldots, c'_n(t))| < \epsilon$$

for all $t > T$. This implies (2) and hence $\bar{f}_b$ is well-defined.

The alternating property of $\bar{f}_b$ actually comes from f_b. Due to the alternating property of f_b, we have

$$\bar{f}_b(\bar{x}_0, \ldots, \bar{x}_i, \ldots, \bar{x}_j, \ldots, \bar{x}_n) = \lim_{t \to \infty} f_b(c_0(t), \ldots, c_i(t), \ldots, c_j(t), \ldots, c_n(t))$$
$$= \lim_{t \to \infty} -f_b(c_0(t), \ldots, c_j(t), \ldots, c_i(t), \ldots, c_n(t))$$
$$= -\bar{f}_b(\bar{x}_0, \ldots, \bar{x}_j, \ldots, \bar{x}_i, \ldots, \bar{x}_n).$$

Therefore, we conclude that $\bar{f}_b$ is alternating. The boundedness and G-invariance of $\bar{f}_b$ immediately follows from the boundedness and G-invariance of f_b. Furthermore, it is easy to check that $\bar{f}_b$ is a cocycle by a direct computation.

Now it remains to prove that $\bar{f}_b$ is uniformly continuous on $\mathcal{X}^n \times \{\xi\}$. It is obvious that $\bar{f}_b$ is continuous on $\mathcal{X}^n \times \{\xi\}$. Noting that the parabolic subgroup of G stabilizing ξ acts on $\mathcal{X}$ transitively, it can be easily seen that $\bar{f}_b$ is uniformly continuous on $\mathcal{X}^n \times \{\xi\}$. $\qquad \square$

The existence of $\bar{f}_b$ allows us to reformulate Vol_3 in terms of Vol_4. Following the proof of Proposition 3.3, we get

$$(3) \qquad \langle (c \circ (i_b^*)^{-1} \circ \rho_b^*)(\omega_b), [\bar{M}, \partial\bar{M}] \rangle = \langle \widehat{D}^*[\bar{f}_b], [\widehat{M}] \rangle.$$

The last term $\langle \widehat{D}^*[\bar{f}_b], [\widehat{M}] \rangle$ above is computed by $\langle \widehat{D}^*\bar{f}_b, \hat{c} \rangle$ for any equivariant map $\widehat{D}$ and fundamental cycle $\hat{c}$ of $\widehat{M}$. By choosing the proper equivariant map

and fundamental cycle, we will show that $\langle \widehat{D}^*[\bar{f}_b], [\widehat{M}]\rangle$ does not depend on the choice of continuous bounded volume class.

Proposition 5.4. *Let ω_b and ω_b' be continuous bounded volume classes, and let $\bar{f}_b$ and $\bar{f}_b'$ be the bounded alternating cocycles in $F_{\mathrm{alt}}^n(\bar{X}; G, \mathbb{R})$ associated with ω_b and ω_b' respectively, as in [Lemma 5.3](). Then*

$$\langle \widehat{D}^*[\bar{f}_b], [\widehat{M}]\rangle = \langle \widehat{D}^*[\bar{f}_b'], [\widehat{M}]\rangle.$$

Proof. It suffices to prove that, for some ρ-equivariant map $\widehat{D} : \widehat{\widetilde{M}} \to \bar{\mathcal{X}}$ and fundamental cycle $\hat{c}$ of $\widehat{M}$,

$$\langle \widehat{D}^*\bar{f}_b, \hat{c}\rangle = \langle \widehat{D}^*\bar{f}_b', \hat{c}\rangle.$$

To show this, we will prove that, for some sequence $(\hat{c}_k)_{k\in\mathbb{N}}$ of fundamental cycles of $\widehat{M}$,

$$\lim_{k\to\infty} (\langle \widehat{D}^*\bar{f}_b, \hat{c}_k\rangle - \langle \widehat{D}^*\bar{f}_b', \hat{c}_k\rangle) = 0.$$

Let $v_1, \ldots, v_s$ be the ideal points of M. As in [Section 4](), fix a product structure $T_{v_i} \times [0, \infty]$ on the end relative to v_i for each $i = 1, \ldots, s$ and then lift such structures to the universal cover. We stick to the notation used in [Section 4](). Set

$$M_k = M - \bigcup_{i=1}^s T_{v_i} \times (k, \infty].$$

Then $(M_k)_{k\in\mathbb{N}}$ is an exhausting sequence of compact cores of M. The boundary ∂M_k of M_k consists of $\bigcup_{i=1}^s T_{v_i} \times \{k\}$. Let $\mathcal{T}_0$ be a triangulation of M_0. Then we extend it to a triangulation on $\widehat{M}$ as follows. First note that $\mathcal{T}_0$ induces a triangulation on each T_{v_i}. Let τ be an $(n-1)$-simplex of the induced triangulation on T_{v_i} for some $i \in \{1, \ldots, s\}$. Then we attach $\pi(\tau \times [0, \infty])$ to $T_{v_i} \times \{0\}$ along $\tau \times \{0\}$, where $\pi : \bar{M} \to \widehat{M}$ is the collapsing map. Since π is an embedding on $\tau \times [0, \infty)$ and π maps $\tau \times \{\infty\}$ to the ideal point v_i, it can be easily seen that $\mathrm{cone}(\tau) := \pi(\tau \times [0, \infty])$ is an n-simplex. Hence we can obtain a triangulation of $\widehat{M}$ by attaching each $\mathrm{cone}(\tau)$ to ∂M_0, which is denoted by $\widehat{\mathcal{T}_0}$.

Next, we extend $\mathcal{T}_0$ to a triangulation of M_k. In fact, M_k is decomposed as

$$M_k = M_0 \cup \bigcup_{i=1}^s T_{v_i} \times [0, k].$$

Hence we can attach each $\tau \times [0, k]$ to M_0 along $\tau \times \{0\}$ and then triangulate $\tau \times [0, k]$ by using the prism operator [Hatcher 2002, Chapter 2.1]. Via this process, we obtain a triangulation of M_k, denoted by $\mathcal{T}_k$. Note that $\mathcal{T}_0$ and $\mathcal{T}_k$ induce the same triangulation on each T_{v_i}. In addition, one can obtain a triangulation $\widehat{\mathcal{T}_k}$ of $\widehat{M}$ from $\mathcal{T}_k$ similarly to how $\widehat{\mathcal{T}_0}$ is obtained from $\mathcal{T}_0$ above.

Let c_k be the relative fundamental class of $(M_k, \partial M_k)$ induced from $\mathcal{T}_k$. Then it can be seen that

$$\hat{c}_k = c_k + (-1)^{n+1} \operatorname{cone}(\partial c_k)$$

is the fundamental cycle of $\widehat{M}$ induced from $\widehat{\mathcal{T}}_k$. Any simplex occurring in c_k is contained in M_k. Now we choose a pseudodeveloping map $\widehat{D} : \widehat{\widetilde{M}} \to \bar{X}$. Let $\tilde{v}_i$ be a lift of v_i to $\widehat{\widetilde{M}}$. Let $P_{\tilde{v}_i} \times [0, \infty]$ be the cone structure of a neighborhood of $\tilde{v}_i$, where $P_{\tilde{v}_i}$ covers T_{v_i} and $P_{\tilde{v}_i} \times \{\infty\}$ is just the ideal point $\tilde{v}_i$. We may assume that $\widehat{D}$ is a cone map on each $P_{\tilde{v}_i} \times [0, \infty]$. Let $\tilde{c}_k$ be a lift of c_k to a cochain in $\widetilde{M}$ and let $\widetilde{\partial c_k}$ be a lift of ∂c_k. Let $\tau \times \{0\}$ be an $(n-1)$-simplex in $T_{v_i} \times \{0\}$ occurring in ∂c_0 and let $\tilde{\tau}$ be a lift of τ to $P_{\tilde{v}_i}$. Then $\tilde{\tau} \times \{k\}$ is a lift of $\tau \times \{k\} \in \partial c_k$. Since $\widehat{D}$ is a cone map on $P_{\tilde{v}_i} \times [0, \infty]$, we have that $D(\tilde{\tau} \times [0, \infty])$ is the geodesic cone over $\tilde{\tau} \times \{0\}$ with top point $\tilde{v}_i$ in $\bar{\mathcal{X}}$. Hence the diameter of $D(\tilde{\tau} \times \{k\})$ decays exponentially to 0 as $k \to \infty$ for each τ.

By a direct computation, we have

$$\langle \widehat{D}^* \bar{f}_b - \widehat{D}^* \bar{f}_b', \hat{c}_k \rangle = \langle \widehat{D}^* \bar{f}_b - \widehat{D}^* \bar{f}_b', \tilde{c}_k \rangle + (-1)^{n+1} \langle \widehat{D}^* \bar{f}_b - \widehat{D}^* \bar{f}_b', \operatorname{cone}(\widetilde{\partial c_k}) \rangle$$

$$= \langle \bar{f}_b - \bar{f}_b', \widehat{D}_*(\tilde{c}_k) \rangle + (-1)^{n+1} \langle \bar{f}_b - \bar{f}_b', \widehat{D}_*(\operatorname{cone}(\widetilde{\partial c_k})) \rangle$$

$$= \langle f_b - f_b', D_*(\tilde{c}_k) \rangle + (-1)^{n+1} \langle \bar{f}_b - \bar{f}_b', \widehat{D}_*(\operatorname{cone}(\widetilde{\partial c_k})) \rangle.$$

The last equality comes from the fact that $\widehat{D}_*(\tilde{c}_k)$ is a singular chain in $\mathcal{X}$. Since f_b and f_b' are continuous bounded alternating cocycles representing the continuous volume class $\omega_{\mathcal{X}} \in H_c^n(G, \mathbb{R})$, there is a continuous alternating G-invariant function $\beta : \mathcal{X}^n \to \mathbb{R}$ such that $f_b - f_b' = \delta \beta$. Hence

$$\langle f_b - f_b', D_*(\tilde{c}_k) \rangle = \langle \delta \beta, D_*(\tilde{c}_k) \rangle = \langle \beta, \partial D_*(\tilde{c}_k) \rangle = \langle \beta, D_*(\widetilde{\partial c_k}) \rangle.$$

As observed before, since the diameter of all simplices occurring in $D_*(\widetilde{\partial c_k})$ decays to 0 as $k \to \infty$ and, moreover, β is uniformly continuous on $\mathcal{X}$, we have

$$\lim_{k \to \infty} \langle \beta, D_*(\widetilde{\partial c_k}) \rangle = 0.$$

Note that $D(\operatorname{cone}(\tilde{\tau} \times \{k\}))$ is the geodesic cone over $D(\tilde{\tau} \times \{k\})$ with top point $\tilde{v}_i$. By Lemma 5.3, both $\bar{f}_b$ and $\bar{f}_b'$ are uniformly continuous on $\mathcal{X}^n \times \{\tilde{v}_i\}$. Since the diameter of $D(\tilde{\tau} \times \{k\})$ decays to 0 as $k \to \infty$,

$$\lim_{k \to \infty} \langle \bar{f}_b, D(\operatorname{cone}(\tilde{\tau} \times \{k\})) \rangle = \lim_{k \to \infty} \langle \bar{f}_b', D(\operatorname{cone}(\tilde{\tau} \times \{k\})) \rangle = 0.$$

Applying this to each τ, we can conclude that

$$\lim_{k \to \infty} \langle \bar{f}_b, D_*(\operatorname{cone}(\widetilde{\partial c_k})) \rangle = \lim_{k \to \infty} \langle \bar{f}_b', D_*(\operatorname{cone}(\widetilde{\partial c_k})) \rangle = 0.$$

In the end, it follows that

$$\lim_{k\to\infty} \langle \widehat{D}^*\bar{f}_b - \widehat{D}^*\bar{f}_b', \hat{c}_k \rangle = 0.$$

As we mentioned, the value on the left-hand side does not depend on $\hat{c}_k$. Thus we can conclude that $\langle \widehat{D}^*\bar{f}_b - \widehat{D}^*\bar{f}_b', \hat{c}_k \rangle = 0$. This implies that $\langle \widehat{D}^*\bar{f}_b, \hat{c} \rangle = \langle \widehat{D}^*\bar{f}_b', \hat{c} \rangle$ for any fundamental cycle $\hat{c}$ of $\widehat{M}$, which completes the proof. $\square$

Combining Proposition 5.4 with (3), Proposition 2.2 immediately follows.

Proposition 5.5. *The definitions of D3 and D4 are equivalent.*

Proof. By Lemma 5.2 and Proposition 3.3, we have

$$\begin{aligned}
\text{Vol}_3(\rho) &= \inf\{\langle \rho_b^*(\omega_b), \alpha \rangle \mid c(\omega_b) = \omega_{\mathcal{X}} \text{ and } \alpha \in [M]_{\text{Lip}}^{\ell^1}\} \\
&= \inf\{\langle (c \circ (i_b^*)^{-1} \circ \rho_b^*)(\omega_b), [\bar{M}, \partial\bar{M}] \rangle \mid c(\omega_b) = \omega_{\mathcal{X}}\} \\
&= \inf\{\langle \widehat{D}^*[\bar{f}_b], [\widehat{M}] \rangle \mid c(\omega_b) = \omega_{\mathcal{X}}\} = \langle \widehat{D}^*[\bar{\Theta}], [\widehat{M}] \rangle = \text{Vol}_4(\rho). \quad \square
\end{aligned}$$

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