**STRONGLY POSITIVE REPRESENTATIONS
OF EVEN GSPIN GROUPS**

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We obtain a classification of strongly positive representations of split even general spin groups over a p-adic field F using the Jacquet module method (Tadić's structure formula). Furthermore, we study discrete series representations of split even general spin groups over F.

1. Introduction

The classifications of (strongly positive) discrete series representations of metaplectic groups, classical groups, and odd GSpin groups over a p-adic field have been studied by several authors [Kim 2015b; Matić 2011; Mœglin 2002; Mœglin and Tadić 2002; Zelevinsky 1980]. The main purpose of this paper is to obtain a classification of strongly positive representations of split even GSpin groups over a nonarchimedean local field F of characteristic different from two, assuming the uniqueness of the nonnegative rank one reducibility point (see Remark 1.2 for more details about this assumption). Our results generalize Matić's algebraic approach [Matić 2011] to the case of even GSpin groups. Our results for even GSpin groups can be applied to even special orthogonal groups to classify strongly positive representations of SO_{2n}. In addition, the results are parallel to those for odd GSpin groups [Kim 2015b]. However, parts of their proofs are quite different because of differences in the group structures. For example, there are two nonconjugate standard parabolic subgroups whose Levi subgroups are of the form $GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_k}$ in the even case; therefore, we classify $D(\rho; \sigma_{\text{cusp}}, c(\sigma_{\text{cusp}}))$ instead of $D(\rho; \sigma_{\text{cusp}})$ in Section 4B, where c is an outer automorphism on the Dynkin diagram of even GSpin groups that permutes the last two simple roots.

To explain our results more precisely, let $G_n := \mathbf{GSpin}_{2n}$ denote the split even general spin group of semisimple rank n over F, and $\mathbf{GL}_m$ the general linear group of semisimple rank m. Let G_n and GL_m denote the groups of F-points of G_n and $\mathbf{GL}_m$, respectively. Let R and R_{GL} denote the Grothendieck groups of the category of all admissible representations of finite length of even GSpin and GL groups. Note that R contains two inequivalent representations of the group

$\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_k}$ (e.g., $\rho_1 \otimes \cdots \otimes \rho_k \otimes (1 \otimes e)$ and $\rho_1 \otimes \cdots \otimes \rho_k \otimes (1 \otimes c)$, following the notation in Section 2).

Let SP denote the set of all strongly positive representations in R, and let LJ denote the set of pairs (Jord, σ'), where

$$\mathrm{Jord} = \bigcup_{i=1}^{k} \bigcup_{j=1}^{k_i} \{(\rho_i, b_j^{(i)})\}$$

and σ' is an irreducible supercuspidal representation in R such that

- $\{\rho_1, \rho_2, \ldots, \rho_k\} \subset R_{\mathrm{GL}}$ is a (possibly empty) set of mutually nonisomorphic, irreducible and essentially self-dual supercuspidal unitary representations such that $\nu^{a_{\rho_i}} \rho_i \rtimes \sigma'$ is reducible for $a_{\rho_i} > 0$ (this defines a_{ρ_i} due to the uniqueness of the reducibility point; see Remark 1.2 for more details),

- $k_i = \lceil a_{\rho_i} \rceil$, and

- for each $i = 1, 2, \ldots, k$, the sequence $b_1^{(i)}, b_2^{(i)}, \ldots, b_{k_i}^{(i)}$ of real numbers is such that $-1 < b_1^{(i)} < b_2^{(i)} < \cdots < b_{k_i}^{(i)}$ and $a_{\rho_i} - b_j^{(i)} \in \mathbb{Z}$ for each $j = 1, \ldots, k_i$.

Remark 1.1. In the set LJ, the condition of "being essentially self-dual" on the representation ρ_i for each $i = 1, \ldots, k$ is due to certain Weyl group actions on the induced representation $\nu^a \rho_i \rtimes \sigma'$ (see Corollary 4.6). In the case of even special orthogonal groups, after a minor change to the set LJ, we can construct an SO version of LJ that corresponds to the set of strongly positive representations of even SO. One minor change would be the condition of "being self-dual" on the corresponding representation ρ_i in the case of even special orthogonal groups.

Remark 1.2 [Silberger 1980]. Let ρ and σ denote irreducible unitary supercuspidal representations of GL_n and G_n, respectively. In this paper, we assume that there exists a unique nonnegative real number a such that $\nu^a \rho \rtimes \sigma$ reduces. This number a is called the nonnegative rank one reducibility point determined by ρ and σ.

We construct a bijective mapping as follows (see Theorem 4.16).

Theorem A. *There exists a bijective mapping Φ between* SP *and* LJ. *More precisely, consider $\sigma \in$ SP to be the unique irreducible subrepresentation of the induced representation of the form*

$$\left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i} - k_i + j} \rho_i, \nu^{b_j^{(i)}} \rho_i] \big) \right) \rtimes \sigma'.$$

Then, we may define $\Phi(\sigma)$ as

$$\left(\bigcup_{i=1}^{k} \bigcup_{j=1}^{k_i} \{(\rho_i, b_j^{(i)})\}, \sigma' \right) \in \mathrm{LJ}.$$

To construct Φ, we first classify the special case $D(\rho; \sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}}))$ (see Section 4B), which is the set of strongly positive representations whose super-cuspidal supports are the representations $\sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}})$ of G_n and twists of the representation ρ of GL. More precisely, in Theorem 4.9 we construct a bijective mapping between $D(\rho; \sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}}))$ and the set of induced representations of the form

$$\delta\big([\nu^{a_1}\rho, \nu^{b_1}\rho]\big) \times \delta\big([\nu^{a_2}\rho, \nu^{b_2}\rho]\big) \times \cdots \times \delta\big([\nu^{a_k}\rho, \nu^{b_k}\rho]\big) \rtimes c'(\sigma_{\mathrm{cusp}}),$$

where $a_i = a - k + i$, $b_1 < \cdots < b_k$, $k \leq \lceil a \rceil$ and $c' \in C := \{e, c\}$. Here, a is the reducibility point determined by ρ and σ_{cusp}, i.e., $\nu^s \rho \rtimes \sigma_{\mathrm{cusp}}$ is reducible if and only if $|s| = a$. We then generalize this result to the set of strongly positive representations. The classification for even GSpin groups needs more work than those for odd GSpin groups since we consider two different representations $\sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}})$ in $D(\rho; \sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}}))$. (In the odd case, we only need to consider $D(\rho; \sigma_{\mathrm{cusp}})$.)

As an application, our main results give rise to a proof of the equality of L-functions through the local Langlands correspondence in the case of GSpin groups [Kim 2015a]. Furthermore, the equality of L-functions also has an application in the proof of the generic Arthur packet conjecture in our case. Briefly, the generic Arthur packet conjecture states that if the L-packet attached to the Arthur parameter has a generic member, then it is tempered [Shahidi 2011]. This conjecture can be considered a local version of the Generalized Ramanujan Conjecture.

The second purpose of the paper is to explicitly construct Tadić's structure formula for even GSpin groups. Tadić's structure formula studies the Jacquet modules of parabolically induced representations, and it is one of the main tools in the proof of Theorem A. We apply and adapt the ideas from papers of Ban [1999a] and Jantzen [2006] (Tadić's structure formula for even special orthogonal groups) to our case.

The paper is organized as follows. In Section 2, we recall the standard notation and preliminaries. In Section 3, we construct Tadić's structure formula for G_n (Theorem 3.4), which gives the explicit structure of the Jacquet modules of the parabolically induced representations of G_n. In Section 4, we construct the classification of strongly positive representations for G_n (Theorem A). In Section 5, we describe embeddings of the general discrete series representations using Casselman's square integrability criterion [Kim 2009] (Theorem 5.1). This embedding of discrete series representations, together with Theorem A, has an interesting application in the proof of the equality of L-functions through the local Langlands correspondence for GSpin groups [Kim 2015a].

2. Notation and preliminaries

Let F be a nonarchimedean local field of characteristic different from two. For a connected reductive group $\boldsymbol{G}$ defined over F, we let G be the group of F-points of

G. Let G_n be a split even general spin group of semisimple rank n defined over F, as defined by Asgari [2002]. A split even GSpin group $G_n := \mathbf{GSpin}_{2n}$ is a split reductive linear algebraic group of type D_n whose derived group is a double covering of a split special orthogonal group and whose connected component of the Langlands dual group is $^L G^0 = \mathrm{GSO}_{2n}(\mathbb{C})$. Therefore, the based root datum of G_n is the dual based root datum of GSO_{2n}.

The following proposition describes the structure of GSpin groups as studied by Asgari [2002].

Proposition 2.1. *The root datum* (X^*, R^*, X_*, R_*) *of* G_n *can be described as*

$$X^* = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n,$$
$$X_* = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*$$

(there is a standard $\mathbb{Z}$-pairing $\langle \, , \, \rangle$ on $X^* \times X_*$), with R^* and R_* generated by

$$\Delta^* = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\},$$
$$\Delta_* = \{\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_{n-1}^* + e_n^* - e_0^*\}.$$

Let $s = (n_1, n_2, \ldots, n_k; n')$ be an ordered partition of n. Let $P_s = M_s N_s$ denote the standard parabolic subgroup of G_n that corresponds to the partition s. The Levi factor M_s is isomorphic to $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_k} \times G_{n'}$ [Asgari 2002, Theorem 2.7]. When $n' = 0$ and $n_k > 0$, we have two nonconjugate standard parabolic subgroups whose Levi subgroups are of the form $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_k}$. In this case, the corresponding set of simple roots contains exactly one of α_{n-1}, α_n. The corresponding Levi factor is denoted by $M_{(n_1,\ldots,n_k;0)} = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_k}$, if the corresponding set of simple roots contains α_{n-1}; or by $c(M_{(n_1,\ldots,n_k;0)})$ otherwise. Here, we let c be the outer automorphism on the Dynkin diagram of G_n that permutes α_{n-1} and α_n and fixes other simple roots.

For representations $\rho_1, \ldots, \rho_k$ of $\mathrm{GL}_{n_1}, \ldots, \mathrm{GL}_{n_k}$, we let $\rho_1 \otimes \cdots \otimes \rho_k \otimes (1 \otimes e)$ denote a representation of $M_{(n_1,\ldots,n_k;0)}$, and $\rho_1 \otimes \cdots \otimes \rho_k \otimes (1 \otimes c))$ a representation of $c(M_{(n_1,\ldots,n_k;0)})$. Let ν be a character of GL_n defined by $|\det|_F$. We denote the induced representation $\mathrm{Ind}_P^{G_n}(\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma)$ by

$$\rho_1 \times \cdots \times \rho_k \rtimes \sigma$$

where each ρ_i is a representation of some GL_{n_i}, and σ is a representation of G_n. We also write $r_s(\sigma)$ for the normalized Jacquet module of the representation σ with respect to P_s. In other words, r_s is a functor from admissible representations of G_n to admissible representations of M_s. In particular, for a subquotient σ of $\rho \rtimes \sigma_{\mathrm{cusp}}$, where ρ is a representation of GL_k and σ_{cusp} is a supercuspidal representation of G_n, we define $r_{\mathrm{GL}}(\sigma) = r_{(k;n)}(\sigma)$.

In the case of GL, for $P' = M'N'$ the standard parabolic subgroup of GL_n with $M' \cong GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_k}$, we denote the induced representation $Ind_{P'}^{GL_n}(\rho_1 \otimes \cdots \otimes \rho_k)$ by

$$\rho_1 \times \cdots \times \rho_k,$$

with each ρ_i a representation of some GL_{n_i}.

We also follow the notation introduced in [Bernstein and Zelevinsky 1977]. Let ρ be an irreducible unitary supercuspidal representation of some GL_p. We define the segment $\Delta := [\nu^a \rho, \nu^{a+k} \rho] = \{\nu^a \rho, \nu^{a+1} \rho, \ldots, \nu^{a+k} \rho\}$, where $a \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$. If $a > 0$, we call the segment Δ strongly positive. We note that the induced representation $\nu^{a+k} \rho \times \nu^{a+k-1} \rho \times \cdots \times \nu^a \rho$ has a unique irreducible subrepresentation, which we denote by $\delta(\Delta)$. Then $\delta(\Delta)$ is an essentially square-integrable representation attached to Δ (Section 3.1 of [Zelevinsky 1980]).

Let us briefly review the Langlands classification for general linear groups. For every irreducible essentially square-integrable representation δ of some GL_n, there exists a unique $e(\delta) \in \mathbb{R}$ such that the representation $\nu^{-e(\delta)} \delta$ is unitarizable. When $\delta = \delta(\Delta)$, we simply denote $e(\delta(\Delta))$ by $e(\Delta)$. Suppose $\delta_1, \delta_2, \ldots, \delta_k$ are irreducible essentially square-integrable representations of $GL_{n_1}, GL_{n_2}, \ldots, GL_{n_k}$ with $e(\delta_1) \leq e(\delta_2) \leq \cdots \leq e(\delta_k)$. Then, the induced representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k$ has a unique irreducible subrepresentation, which we denote by $L(\delta_1, \delta_2, \ldots, \delta_k)$. This irreducible subrepresentation is called the Langlands subrepresentation, and it appears with the multiplicity one in $\delta_1 \times \delta_2 \times \cdots \times \delta_k$. Every irreducible representation ρ of GL_n is isomorphic to some $L(\delta_1, \delta_2, \ldots, \delta_k)$. Given ρ, the representations $\delta_1, \delta_2, \ldots, \delta_k$ are unique up to a permutation. If $i_1, i_2, \ldots, i_k$ is a permutation of $1, 2, \ldots, k$ such that the representations $\delta_{i_1} \times \cdots \times \delta_{i_k}$ and $\delta_1 \times \cdots \times \delta_k$ are isomorphic, we also write $L(\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_k})$ for $L(\delta_1, \delta_2, \ldots, \delta_k)$.

The Grothendieck group of the category of all finite length admissible representations of G_n (resp. GL_n), a free abelian group over the set of all irreducible representations of G_n (resp. GL_n), is denoted by $R(n)$ (resp. $R_{GL}(n)$). We set

$$R = \bigoplus_{n \geq 0} R(n),$$

$$R_{GL} = \bigoplus_{n \geq 0} R_{GL}(n).$$

The strongly positive representations of G_n are defined as follows.

Definition 2.2 (strongly positive). An irreducible representation σ of G_n is called strongly positive if for each representation $\nu^{s_1} \rho_1 \times \nu^{s_2} \rho_2 \times \cdots \times \nu^{s_k} \rho_k \rtimes \sigma_{\mathrm{cusp}}$, where each ρ_i ($i = 1, 2, \ldots, k$) is an irreducible supercuspidal unitary representation of some GL_{n_i}, $\sigma_{\mathrm{cusp}} \in R$ is an irreducible supercuspidal representation of $G_{n'}$ and

$s_i \in \mathbb{R}$ $(i = 1, 2, \ldots, k)$ is such that

$$\sigma \hookrightarrow \nu^{s_1} \rho_1 \times \nu^{s_2} \rho_2 \times \cdots \times \nu^{s_k} \rho_k \rtimes \sigma_{\text{cusp}},$$

we have $s_i > 0$ for each i.

Finally, the next proposition recalls the properties of discrete series representations studied in [Asgari 2002].

Proposition 2.3. *Let M be $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_k} \times G_{n'} \subset G_n$. Let ρ_i be a supercuspidal representation of GL_{n_i} and σ a supercuspidal representation of $G_{n'}$. Write $\rho_i = \nu^{e(\rho_i)} \rho_i^u$, where $e(\rho_i) \in \mathbb{R}$ and ρ_i^u is a unitary supercuspidal representation for $i = 1, \ldots, k$. If $\rho_1 \times \cdots \times \rho_k \rtimes \sigma$ has a discrete series subrepresentation, then $\rho_i^u \cong \tilde{\rho}_i^u \otimes (\omega_\sigma \circ \det)$ for $i = 1, \ldots, k$.*

3. Tadić's structure formula for even GSpin groups

Tadić's structure formulae for Sp_{2n}, SO_{2n+1}, SO_{2n} and GSpin_{2n+1} in [Tadić 1995; Jantzen 2006; Kim 2015b] are based on the geometric lemma (2.11 in [Bernstein and Zelevinsky 1977] or Section 6 in [Casselman 1995]). Briefly, the geometric lemma explicitly calculates the Jacquet modules of induced representations ($r_{G,N} \circ i_{G,N}$ in [Bernstein and Zelevinsky 1977]) and it depends on the double coset representative Weyl group elements ($W^{M,N}$ in [Bernstein and Zelevinsky 1977]) and their actions on the simple roots and representations of Levi subgroups. In this section, we explicitly construct the structure of Jacquet modules of parabolically induced representations of G_n (Tadić's structure formula for G_n, Theorem 3.4) using the geometric lemma. We will adapt and follow the results in [Ban 1999a; Jantzen 2006], i.e., the case of SO_{2n}. Ban characterizes the double coset representative Weyl group elements ($[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}]$ in [Ban 1999a, Section 5]) and its action on the simple roots in the cases of D_n-type groups, which include SO_{2n} and G_n. Jantzen [2006] constructs Tadić's structure formula for SO_{2n} using Ban's results. Therefore, once we calculate the Weyl group action on the representations in our case, we are ready to adapt Jantzen's calculation [2006] to construct Tadić's structure formula for G_n.

Let $(p, \epsilon) \in S_n \rtimes \{\pm 1\}^n$ be an element in the Weyl group W_{G_n} with $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n$ such that $\prod_{i=1}^n \epsilon_i = 1$. We can identify $(p, \epsilon) \in W_{G_n}$ with $p \cdot \epsilon \in W_{\mathrm{SO}_{2n}}$ where the action by conjugation of p and $\epsilon \in W_{\mathrm{SO}_{2n}}$ on the standard maximal torus in SO_{2n} can be defined by

$$p \cdot \mathrm{diag}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}) = \mathrm{diag}(x_{p^{-1}(1)}, \ldots, x_{p^{-1}(n)}, x_{p^{-1}(n)}^{-1}, \ldots, x_{p^{-1}(1)}^{-1}),$$

$$\epsilon \cdot \mathrm{diag}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}) = \mathrm{diag}(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}, x_n^{-\epsilon_n}, \ldots, x_1^{-\epsilon_1}).$$

We can get the action of those on the roots (see also [Hundley and Sayag 2012]).

Lemma 3.1. *Let $e_0, e_1, \ldots, e_n$ and $e'_0, e'_1, \ldots, e'_n$ be the standard bases of the character lattice and the cocharacter lattice of G_n as in Proposition 2.1. Let $(p, \epsilon) \in S_n \ltimes \{\pm 1\}^n$ be as above.*

Then

$$(p, \epsilon) \cdot e_i = \begin{cases} e_{p(i)} & \text{for } i > 0, \epsilon_i = 1, \\ -e_{p(i)} & \text{for } i > 0, \epsilon_i = -1, \\ e_0 + \displaystyle\sum_{\epsilon_i = -1} e_{p(i)} & \text{for } i = 0, \end{cases}$$

$$(p, \epsilon) \cdot e'_i = \begin{cases} e'_{p(i)} & \text{for } i > 0, \epsilon_i = 1, \\ e'_0 - e'_{p(i)} & \text{for } i > 0, \epsilon_i = -1, \\ e'_0 & \text{for } i = 0. \end{cases}$$

Proof. The calculations of $(p, \epsilon) \cdot e_i$ for $i > 0$ are done in [Ban 1999a, Section 5]. We can also calculate $(p, \epsilon) \cdot e'_i$ directly from the matrix calculation since $e'_0, \ldots, e'_n$ comprise the character lattice of GSO. For $(p, \epsilon) \cdot e_0$, we need to use the duality of e_i and e'_i. $\qquad\square$

Let $\Delta^* = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$ be a simple root for G_n as explained in Section 2. From Lemma 3.1, we can calculate the action of (p, ϵ) on the set of simple roots in R^* (see also [Ban 1999a]).

Corollary 3.2. *With notation as in Lemma 3.1,*

$$(p, \epsilon) \cdot \alpha_i = \begin{cases} \epsilon_i e_{p(i)} - \epsilon_{i+1} e_{p(i+1)} & \text{for } 0 \leq i \leq n - 1, \\ \epsilon_{n-1} e_{p(n-1)} + \epsilon_n e_{p(n)} & \text{for } i = n. \end{cases}$$

Now we are ready to construct Tadić's structure formula for G_n. Let ρ_i be an irreducible smooth representation of GL_{n_i} for $i = 1, 2, 3, 4$. Let σ be an irreducible smooth representation in R and let ω_σ be the central character of σ. For any element $c_1 \in C = \{e, c\}$, we define $\widetilde{\rtimes}$ as follows:

(3-1) $(\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes c_1) \widetilde{\rtimes} (\rho_4 \otimes \sigma) = (\tilde{\rho}_1 \otimes (\omega_\sigma \circ \det)) \times \rho_2 \times \rho_4 \otimes \rho_3 \rtimes c_1(\sigma).$

One extends $\widetilde{\rtimes}$ to a $\mathbb{Z}$-bilinear mapping

$$\widetilde{\rtimes} : (R_{\mathrm{GL}} \otimes R_{\mathrm{GL}} \otimes R_{\mathrm{GL}} \otimes \mathbb{Z}[C]) \times (R_{\mathrm{GL}} \otimes R) \to R_{\mathrm{GL}} \otimes R.$$

We denote by $\boldsymbol{m}$ the linear extension to $R_{\mathrm{GL}} \otimes R_{\mathrm{GL}}$ of parabolic induction from a maximal parabolic subgroup.

Let

$$\Omega_k = \begin{cases} \Delta & \text{if } k = 0, \\ \Delta \setminus \{\alpha_k\} & \text{if } k \leq n - 2, \\ \Delta \setminus \{\alpha_{n-1}, \alpha_n\} & \text{if } k = n - 1, \\ \Delta \setminus \{\alpha_n\} & \text{if } k = n. \end{cases}$$

and $\overline{\Omega}_n = \Delta \setminus \{\alpha_{n-1}\} = c(\Omega_n)$. We define $\mu^*(\sigma)$ as follows. For $0 \leq k \leq n$, write

$$r_{M_{\Omega_k}, G_n}(\sigma) = \sum_{i \in I_k} \rho_{i,k} \otimes \sigma_{i,k},$$

$$r_{M_{\overline{\Omega}_n}, G_n}(\sigma) = \sum_{j \in J} \rho_j \otimes (1 \otimes c),$$

the normalized Jacquet modules of σ with respect to the standard maximal parabolic subgroups $P_{\Omega_k} = M_{\Omega_k} N_{\Omega_k}$ and $P_{\overline{\Omega}_n} = M_{\overline{\Omega}_n} N_{\overline{\Omega}_n}$, respectively. For such σ, we can define $\mu^*(\sigma) \in R_{\mathrm{GL}} \otimes R$ as

$$\mu^*(\sigma) = \sum_{k=0}^{n} \sum_{i \in I_k} \rho_{i,k} \otimes \sigma_{i,k} + \sum_{j \in J} \rho_j \otimes (1 \otimes c).$$

Using Jacquet modules with respect to the maximal parabolic subgroups of GL_n, we can also define

$$\boldsymbol{m}^*(\rho) = \sum_{k=0}^{n} s.s.(r_k(\rho)) \in R_{\mathrm{GL}} \otimes R_{\mathrm{GL}}$$

for an irreducible representation ρ of GL_n, and then extend $\boldsymbol{m}^*$ linearly to the whole of R_{GL}. Here, $r_k(\rho)$ denotes the Jacquet module of the representation ρ with respect to the parabolic subgroup whose Levi subgroup is $\mathrm{GL}_k \times \mathrm{GL}_{n-k}$, and $s.s.$ denotes so-called semisimplification, i.e., a canonical map from objects of the category of all smooth finite length representations of GL into the Grothendieck group of this category. We define $\boldsymbol{s} : R_{\mathrm{GL}} \otimes R_{\mathrm{GL}} \to R_{\mathrm{GL}} \otimes R_{\mathrm{GL}}$ by $\boldsymbol{s}(x \otimes y) = y \otimes x$. Let $\mathfrak{M}_C^* : R_{\mathrm{GL}} \to R_{\mathrm{GL}} \otimes R_{\mathrm{GL}} \otimes R_{\mathrm{GL}} \otimes \mathbb{Z}[C]$ be the map $(1 \otimes \boldsymbol{m}_C^*) \circ \boldsymbol{s} \circ \boldsymbol{m}^*$, where

$$1 \otimes \boldsymbol{m}_C^* : R_{\mathrm{GL}} \otimes R_{\mathrm{GL}} \to R_{\mathrm{GL}} \otimes R_{\mathrm{GL}} \otimes R_{\mathrm{GL}} \otimes \mathbb{Z}[C]$$

$$\rho_1 \otimes \rho_2 \mapsto \rho_1 \otimes \boldsymbol{m}^*(\rho_2) \otimes c^{n_1},$$

for representations ρ_1 of GL_{n_1} and ρ_2 of GL_{n_2}.

Remark 3.3. In our case of even GSpin groups, we have c^{n_1} when we calculate μ^*, while we don't have such action in the case of odd GSpin groups. This is due to the differences of the Weyl group actions on the simple roots. In our case, the corresponding Weyl group element acts on e_n by $(-1)^{n_1}$. Therefore, if n_1 is odd, we need to permute $\alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$. In other words, we have the outer automorphism c on the representation σ.

The following theorem is called Tadić's structure formula for even GSpin groups.

Theorem 3.4. *For $\rho \in R_{\mathrm{GL}}(i)$ and $\sigma \in R(n-i)$, we have*

$$\mu^*(\rho \rtimes \sigma) = \mathfrak{M}_C^*(\rho) \widetilde{\rtimes} \mu^*(\sigma).$$

Proof. We sketch the proof and explain how we can adapt the approach in [Jantzen 2006] to our case. Let us explicitly calculate the left hand side of the equation in the theorem. Since GSpin_{2n} is also of D_n-type, we can apply the double coset representative Weyl group elements, which are studied in [Ban 1999a], to our case. Therefore, we have $\mu^*(\rho \rtimes \sigma)$ as in [Jantzen 2006, pp. 811–812]. Now, we calculate the action of the double coset representative Weyl group elements (q_n in [Jantzen 2006]) on the representations. The actions of q_n produce contragredient of $\tau_s^{(3)}(d)$ with $(\omega_\theta \circ \det)$, i.e., $(\omega_\theta \circ \det)\tilde{\tau}_s^{(3)}(d)$ (see p. 812 for the notation of $\tau_s^{(3)}(d)$ and θ). Accordingly, we need to define $\widetilde{\rtimes}$ as (3-1). This forces $\mu^*(\rho \rtimes \sigma)$ to be equal to $\mathfrak{M}_C^*(\rho)\widetilde{\rtimes}\mu^*(\sigma)$ after changing index several times as in the proof of [Jantzen 2006, Chapter 4]. $\qquad\square$

4. Classification of strongly positive representations for even GSpin groups

We give the classification of strongly positive representations of even GSpin groups in this section. We apply and adapt some proofs from Matić's results [2011] for metaplectic groups to our case. Therefore, we mostly focus on the following cases which are quite different from [Matić 2011]. For example, when the reducibility point is $\frac{1}{2}$, we follow the idea of [Kim 2015b] instead of [Matić 2011]. We also emphasize the difference between the even case and the odd case, and omit the proof if it is similar to the case of either metaplectic groups or odd GSpin groups. For example, in the even case, we need to add $c(\sigma_{\mathrm{cusp}})$ when we classify the special case $D(\rho; \sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}}))$ (in the odd case, we classify $D(\rho; \sigma_{\mathrm{cusp}})$).

4A. *Several lemmas.* We recall several lemmas which are essential in this section. Let us first recall the half integer conjecture in the case of GSpin groups. Let σ be an irreducible supercuspidal representation of $G_{n'}$ and let ρ be an irreducible supercuspidal unitary representation of GL_k. The following is a recent result of Mœglin which is called the half integer conjecture for GSpin groups [Mœglin 2014, Theorem 3.1.1]:

Lemma 4.1 [Mœglin 2014]. *Let $a \in \mathbb{R}$ be a nonnegative real number such that $\nu^a \rho \rtimes \sigma$ reduces. Then, $a \in \frac{1}{2}\mathbb{Z}$.*

Remark 4.2. In [Kim 2015b], we classified the strongly positive representations of odd GSpin groups assuming the half integer conjecture. Due to Mœglin's results (Lemma 4.1), we can completely remove the assumption in the odd case.

Remark 4.3. When we further assume that σ is a generic representation, Shahidi [1990] proved that $a \in \{0, \frac{1}{2}, 1\}$.

Let us calculate $\mu^*\big(\big(\prod_{j=1}^n \delta([\nu^{a_j}\rho_j, \nu^{b_j}\rho_j])\big) \rtimes \sigma\big)$, where each ρ_j is an irreducible supercuspidal representation of GL_k, the real numbers a_j and b_j are such

that $b_j - a_j \in \mathbb{Z}_{\geq 0}$ for each $j = 1, \ldots, n$ and σ is an irreducible supercuspidal representation of finite length of $G_{n'}$.

Example 4.4. Let us first calculate the case when $n = 1$. Since σ is supercuspidal, $\mu^*(\sigma) = 1 \otimes \sigma$. Following the definition of $\mathfrak{M}_C^*$, we have

$$\mathfrak{M}_C^*\big(\delta\big([\nu^a \rho_1, \nu^b \rho_1]\big)\big)$$
$$= \sum_{i=a-1}^{b} \sum_{j=i}^{b} \delta\big([\nu^a \rho_1, \nu^i \rho_1]\big) \otimes \delta\big([\nu^{j+1}\rho_1, \nu^b \rho_1]\big) \otimes \delta\big([\nu^{i+1}\rho_1, \nu^j \rho_1]\big) \otimes c^{k(i-a+1)}.$$

Therefore,

$$\mu^*\big(\delta\big([\nu^a \rho_1, \nu^b \rho_1]\big) \rtimes \sigma\big) = \sum_{i=a-1}^{b} \sum_{j=i}^{b} \delta\big([\nu^{-i}\tilde{\rho}_1 \otimes (\omega_\sigma \circ \det), \nu^{-a}\tilde{\rho}_1 \otimes (\omega_\sigma \circ \det)]\big)$$
$$\times \delta\big([\nu^{j+1}\rho_1, \nu^b \rho_1]\big) \otimes \delta\big([\nu^{i+1}\rho_1, \nu^j \rho_1]\big) \rtimes c^{k(i-a+1)}(\sigma).$$

We omit $\delta\big([\nu^x \rho_1, \nu^y \rho_1]\big)$ if $x > y$. Then, to calculate

$$\mu^*\left(\left(\prod_{j=1}^{n} \delta\big([\nu^{a_j}\rho_j, \nu^{b_j}\rho_j]\big)\right) \rtimes \sigma\right),$$

we use (1.3) of [Tadić 1998]:

$$m^*\left(\prod_{j=1}^{n} \delta\big([\nu^{a_j}\rho_j, \nu^{b_j}\rho_j]\big)\right) = \prod_{j=1}^{n}\left(\sum_{i_j=a_j-1}^{b_j} \delta\big([\nu^{i_j+1}\rho_j, \nu^{b_j}\rho_j]\big) \otimes \delta\big([\nu^{a_j}\rho_j, \nu^{i_j}\rho_j]\big)\right).$$

The Weyl group elements are essential objects when we define the intertwining operators between two induced representations [Shahidi 2010, Chapter 4]. We recall the action of the Weyl group elements on the induced representations. Let M_θ be a Levi subgroup isomorphic to $\mathrm{GL}_k \times G_{n-k}$ for $\theta = \Delta \setminus \alpha_k$. There is a unique Weyl group element w_0 such that $w_0(\alpha_k) < 0$ and $w_0(\theta) \subset \Delta$.

Lemma 4.5. *Let ρ and σ be irreducible supercuspidal representations of GL_k and G_{n-k}, respectively. Then*

$$(\rho \otimes \sigma)^{w_0} = (\tilde{\rho} \otimes (\omega_\sigma \circ \det)) \otimes c^k(\sigma),$$

where ω_σ is the central character of σ.

Proof. Since $w_0(\alpha_k) < 0$ and $w_0(\theta) \subset \Delta$, we can explicitly calculate its action on the simple roots. Let us identify w_0 as $(p, \epsilon) \in S_n \rtimes \{\pm 1\}^n$ with $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n$ such that $\prod_{i=1}^{n} \epsilon_i = 1$. Then

$$p(i) = \begin{cases} k+1-i & \text{for } 1 \leq i \leq k, \\ i & \text{for } k+1 \leq i \leq n, \end{cases} \quad \text{and} \quad \epsilon_i = \begin{cases} -1 & \text{for } 1 \leq i \leq k, \\ 1 & \text{for } k+1 \leq i \leq n, \\ (-1)^k & \text{for } i = n, \end{cases}$$

for $k \neq n-1, n$, and

$$p(i) = n+1-i \quad \text{and} \quad \epsilon_i = \begin{cases} (-1)^{k-1} & \text{for } i = 1, \\ -1 & \text{for } 2 \leq i \leq n-1, \\ (-1)^{k+n-1} & \text{for } i = n, \end{cases}$$

for $k = n-1, n$. Using this identification, the lemma follows from Lemma 3.1. $\square$

Corollary 4.6. *Let ρ and σ be as in Lemma 4.5. Then $\rho \rtimes \sigma$ and $(\tilde{\rho} \otimes (\omega_\sigma \circ \det)) \rtimes c^k(\sigma)$ are associate. Therefore, Lemma 5.4 (iii) of* [Bernstein et al. 1986] *implies that the sets of irreducible composition factors of $\rho \rtimes \sigma$ and $(\tilde{\rho} \otimes (\omega_\sigma \circ \det)) \rtimes c^k(\sigma)$ are the same. Furthermore, if we assume that $\rho \rtimes \sigma$ is irreducible, then $\rho \rtimes \sigma \cong (\tilde{\rho} \otimes (\omega_\sigma \circ \det)) \rtimes c^k(\sigma)$.*

Now we show that strongly positive representations can be embedded into parabolically induced representations of special type. More precisely, we consider parabolically induced representations of the form

$$\text{(4-1)} \qquad \delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes c'(\sigma_{\text{cusp}}),$$

where $\Delta_1, \Delta_2, \ldots, \Delta_k$ is a sequence of strongly positive segments satisfying $0 < e(\Delta_1) \leq e(\Delta_2) \leq \cdots \leq e(\Delta_k)$ (we allow $k = 0$ here), σ_{cusp} is an irreducible supercuspidal representation of $G_{n'}$ and $c' \in C$. Note that the idea of such embeddings of representations was initiated in [Muić 2006] and further refined in [Hanzer and Muić 2008].

Lemma 4.7. *Let $\Delta_1, \ldots, \Delta_k$ and σ_{cusp} be as above. Then the induced representation $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes c'(\sigma_{\text{cusp}})$ has a unique irreducible subrepresentation, which we denote by $\delta(\Delta_1, \ldots, \Delta_k; c'(\sigma_{\text{cusp}}))$.*

Proof. We briefly explain the main ideas of the proof and how we adapt the proof from [Matić 2011] to the case of even GSpin groups. The case $k = 0$ is clear. Let $j_1 < j_2 < \cdots < j_s$ be the positive integers such that

$$e(\Delta_1) = \cdots = e(\Delta_{j_1}) < e(\Delta_{j_1+1}) = \cdots = e(\Delta_{j_2}) < \cdots < e(\Delta_{j_s+1}) = \cdots = e(\Delta_k).$$

Then Theorem 3.4 implies that an irreducible representation

$$\delta(\Delta_1) \times \cdots \times \delta(\Delta_{j_1}) \otimes \delta(\Delta_{j_1+1}) \times \cdots \times \delta(\Delta_{j_2}) \otimes \cdots \otimes c'(\sigma_{\text{cusp}})$$

appears with multiplicity one in the Jacquet module of $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes c'(\sigma_{\text{cusp}})$ with respect to the appropriate parabolic subgroup. Therefore, since this irreducible representation is contained in the Jacquet module of any

subrepresentation of $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes c'(\sigma_{\mathrm{cusp}})$ with respect to the appropriate parabolic subgroup, it follows that the induced representation $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes c'(\sigma_{\mathrm{cusp}})$ has a unique irreducible subrepresentation. $\square$

Now, as in the odd case [Kim 2015b], we can show that strongly positive representations can be embedded into induced representations of the form (4-1).

Lemma 4.8. *Let $\sigma \in R$ denote a strongly positive representation. Then σ can be embedded into certain induced representations of the form* (4-1).

4B. *Classification of strongly positive representations: the $D(\rho; \sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}}))$ case.* Let ρ be an essentially self-dual irreducible supercuspidal representation of GL_{n_ρ} and σ_{cusp} an irreducible supercuspidal representation of $G_{n'}$. Also, let $D(\rho; \sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}}))$ be the set of strongly positive representations whose supercuspidal supports are the representations $\sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}})$ and twists of the representation ρ. We assume that there exists a unique nonnegative real number a such that $\nu^a \rho \rtimes \sigma_{\mathrm{cusp}}$ reduces [Silberger 1980]. The half integer conjecture for GSpin groups (Lemma 4.1) implies that $a \in \frac{1}{2}\mathbb{Z}$.

In this section, we construct the classification of strongly positive representations in $D(\rho; \sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}}))$. Lemma 4.8 implies that there exists a mapping from the set of strongly positive representations of G_n into the set of induced representations of the form (4-1). We first refine the image of this map when we restrict the map to $D(\rho; \sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}}))$.

Theorem 4.9. *Suppose that σ is an irreducible strongly positive representation in $D(\rho; \sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}}))$, taken as the unique irreducible subrepresentation of an induced representation of the form* (4-1). *Write $\Delta_i = [\nu^{a_i}\rho, \nu^{b_i}\rho]$ for $i = 1, \ldots, k$. Then $a_i = a - k + i$ for each i, $b_1 < \cdots < b_k$ and $k \leq \lceil a \rceil$.*

Proof. We consider the case when $a = \frac{1}{2}$. We first show, by induction on k, that $a_i = a$ for each $i = 1, \ldots, k$ and that $b_1 \leq \cdots \leq b_k$ when $a = \frac{1}{2}$.

For the case $k = 1$, note that if $a_i \neq a$, then

$$\nu^{a_i}\rho \rtimes \sigma_{\mathrm{cusp}} \cong (\nu^{-a_i}\tilde{\rho} \otimes (\omega_{\sigma_{\mathrm{cusp}}} \circ \det)) \rtimes c^{n_\rho}(\sigma_{\mathrm{cusp}}) \cong \nu^{-a_i}\rho \rtimes \sigma_{\mathrm{cusp}}.$$

Therefore, we have the embedding

$$\sigma \hookrightarrow \nu^{b_1}\rho \times \cdots \times \nu^{a_1+1}\rho \times \nu^{-a_1}\rho \rtimes \sigma_{\mathrm{cusp}}$$

which contradicts the strong positivity of σ.

Now suppose the theorem holds for all $m \in \mathbb{Z}$ such that $0 \leq m < k$. We prove it for k. As in the case when $k = 1$, we show $a_k = a$. We know that σ embeds in $\delta(\Delta_1) \rtimes \delta(\Delta_2, \ldots, \Delta_k; \sigma_{\mathrm{cusp}})$, since σ is the unique irreducible subrepresentation of $\delta(\Delta_1) \times \cdots \delta(\Delta_k) \rtimes \sigma_{\mathrm{cusp}}$. This implies that each $a_i = a$ for $2 \leq i \leq k$ and $b_2 \leq \cdots \leq b_k$. It remains to show that $a_1 = \frac{1}{2}$ and $b_1 \leq b_2$.

Suppose that $a_1 \notin \frac{1}{2} + \mathbb{Z}$. Then $\nu^{a_1}\rho \times \delta(\Delta_i)$ is irreducible for $i \geq 2$. Therefore, we have the embedding

$$\sigma \hookrightarrow \delta\big([\nu^{a_1+1}\rho, \nu^{b_1}\rho]\big) \times \nu^{a_1}\rho \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{\mathrm{cusp}}$$
$$\cong \delta\big([\nu^{a_1+1}\rho, \nu^{b_1}\rho]\big) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \times \nu^{a_1}\rho \rtimes \sigma_{\mathrm{cusp}}$$
$$\cong \delta\big([\nu^{a_1+1}\rho, \nu^{b_1}\rho]\big) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \times \nu^{-a_1}\rho \rtimes \sigma_{\mathrm{cusp}}$$

which contradicts the strong positivity of σ. Therefore, $a_1 \in \frac{1}{2} + \mathbb{Z}$. If $a_1 \neq \frac{1}{2}$, then $a_1 \geq \frac{3}{2}$. This implies that $\delta(\Delta_1) \times \delta(\Delta_i)$ is irreducible for $i \geq 2$ since $b_1 \leq b_i$. Therefore, we have the embedding

$$\sigma \hookrightarrow \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \times \delta(\Delta_1) \rtimes \sigma_{\mathrm{cusp}}$$
$$\cong \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \times \delta\big([\nu^{a_1+1}\rho, \nu^{b_1}\rho]\big) \times \nu^{a_1}\rho \rtimes \sigma_{\mathrm{cusp}}$$
$$\cong \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \times \delta\big([\nu^{a_1+1}\rho, \nu^{b_1}\rho]\big) \times \nu^{-a_1}\rho \rtimes \sigma_{\mathrm{cusp}}$$

which again contradicts the strong positivity of σ. Thus, $a_1 = \frac{1}{2}$. Also, $b_1 \leq b_2$ follows from $e(\Delta_1) \leq e(\Delta_2)$. The following lemma finishes the proof of the theorem in the case when $a = \frac{1}{2}$.

Lemma 4.10. *The unique irreducible subrepresentation of*

$$\delta\big([\nu^{1/2}\rho, \nu^{b_1'}\rho]\big) \times \delta\big([\nu^{1/2}\rho, \nu^{b_2'}\rho]\big) \times \cdots \times \delta\big([\nu^{1/2}\rho, \nu^{b_k'}\rho]\big) \rtimes \sigma_{\mathrm{cusp}},$$

denoted $\sigma^*_{(b_1',\ldots,b_k';1/2)}$, is not strongly positive when $k \geq 2$.

Proof. We first show the case when $k = 2$. The embedding of $\nu^{1/2}\rho \rtimes \delta(\nu^{1/2}\rho, \sigma_{\mathrm{cusp}})$ into $\nu^{1/2}\rho \times \nu^{1/2}\rho \rtimes \sigma_{\mathrm{cusp}}$ implies the embedding

$$\sigma^*_{(1/2,1/2;1/2)} \hookrightarrow \nu^{1/2}\rho \rtimes \delta(\nu^{1/2}\rho, \sigma_{\mathrm{cusp}}).$$

Using Lemma 3.8 (b) as well as Remark 3.2 of [Tadić 1998], we can show that $\delta\big([\nu^{-1/2}\rho, \nu^{1/2}\rho]\big) \rtimes \sigma_{\mathrm{cusp}}$ is a direct sum of two irreducible nonisomorphic representations, say τ_1 and τ_2, in the same way as in the proof of Sublemma 5.8 of [Kim 2015b]. From Frobenius reciprocity and

$$r_{\mathrm{GL}}\big(\delta\big([\nu^{-1/2}\rho, \nu^{1/2}\rho]\big) \rtimes \sigma_{\mathrm{cusp}}\big) =$$
$$2\delta\big([\nu^{-1/2}\rho, \nu^{1/2}\rho]\big) \otimes \sigma_{\mathrm{cusp}} + \nu^{1/2}\rho \times \nu^{1/2}\rho \otimes c^{n_\rho}(\sigma_{\mathrm{cusp}}),$$

it follows that $r_{\mathrm{GL}}(\tau_1) = \delta\big([\nu^{-1/2}\rho, \nu^{1/2}\rho]\big) \otimes \sigma_{\mathrm{cusp}} + \nu^{1/2}\rho \times \nu^{1/2}\rho \otimes c^{n_\rho}(\sigma_{\mathrm{cusp}})$ and $r_{\mathrm{GL}}(\tau_2) = \delta\big([\nu^{-1/2}\rho, \nu^{1/2}\rho]\big) \otimes \sigma_{\mathrm{cusp}}$. We also obtain that $\sigma^*_{(1/2,1/2;1/2)} \cong \tau_1$, which is a subrepresentation of $\delta\big([\nu^{-1/2}\rho, \nu^{1/2}\rho]\big) \rtimes \sigma_{\mathrm{cusp}}$, in the same way as in the proof of Sublemma 5.9 of [Kim 2015b]. Therefore, we have the embedding

$$\sigma^*_{(1/2,1/2;1/2)} \hookrightarrow \delta\big([\nu^{-1/2}\rho, \nu^{1/2}\rho]\big) \rtimes \sigma_{\mathrm{cusp}} \hookrightarrow \nu^{1/2}\rho \times \nu^{-1/2}\rho \rtimes \sigma_{\mathrm{cusp}}.$$

We conclude that $\sigma^*_{(1/2,1/2;1/2)}$ is not strongly positive.

Next we consider the case $k \geq 2$. Suppose that $\sigma^*_{(b'_1,\ldots,b'_k;1/2)}$ is strongly positive. Since each representation $\nu^{1/2}\rho \times \delta([\nu^{1/2}\rho, \nu^{b'_i}\rho])$ is irreducible for all $i = 1, \ldots, k$, we have an embedding of $\sigma^*_{(b'_1,\ldots,b'_k;1/2)}$ into

$$\delta([\nu^{3/2}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{3/2}\rho, \nu^{b'_2}\rho]) \times$$
$$\cdots \times \delta([\nu^{3/2}\rho, \nu^{b'_k}\rho]) \times \nu^{1/2}\rho \times \cdots \times \nu^{1/2}\rho \rtimes \sigma_{\mathrm{cusp}}.$$

Furthermore, since we know $\sigma^*_{(1/2,1/2;1/2)}$ is the unique irreducible subrepresentation of $\nu^{1/2}\rho \times \nu^{1/2}\rho \rtimes \sigma_{\mathrm{cusp}}$, we have the embedding

$$\sigma^*_{(b'_1,\ldots,b'_k;1/2)} \hookrightarrow \delta([\nu^{3/2}\rho, \nu^{b'_1}\rho]) \times \cdots \times \nu^{1/2}\rho \rtimes \sigma^*_{(1/2,1/2;1/2)}.$$

This contradicts the strong positivity of $\sigma^*_{(1/2,1/2;1/2)}$. $\square$

Returning to the proof of Theorem 4.9, it remains to prove the case when $a \neq \frac{1}{2}$. This case is similar to the proof in [Matić 2011] and we skip the proof here. $\square$

We also show that the map from $D(\rho; \sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}}))$ to the set of induced representations of the form (4-1) is well defined in the following theorem.

Theorem 4.11. *Suppose that σ is an irreducible strongly positive representation in $D(\rho; \sigma_{\mathrm{cusp}}, c(\sigma_{\mathrm{cusp}}))$. Then there exist a unique set of strongly positive segments $\Delta_1, \Delta_2, \ldots, \Delta_k$ with $0 < e(\Delta_1) \leq e(\Delta_2) \leq \cdots \leq e(\Delta_k)$, and a unique irreducible supercuspidal representation $\sigma' \in R$ such that $\sigma \simeq \delta(\Delta_1, \Delta_2, \ldots, \Delta_k; \sigma')$.*

Proof. We first show the uniqueness of the partial supercuspidal support σ'. Suppose that there are two sets of strongly positive segments and representations in R, $\{\Delta_1, \Delta_2, \ldots, \Delta_k, \sigma_{\mathrm{cusp}}\}$ and $\{\Delta'_1, \Delta'_2, \ldots, \Delta'_l, c(\sigma_{\mathrm{cusp}})\}$, which satisfy the conditions in Theorem 4.9. Then we have the two embeddings

$$(4\text{-}2) \qquad\qquad \sigma \hookrightarrow \left(\prod_{i=1}^k \delta(\Delta_i) \right) \rtimes \sigma_{\mathrm{cusp}},$$

$$(4\text{-}3) \qquad\qquad \sigma \hookrightarrow \left(\prod_{j=1}^l \delta(\Delta'_j) \right) \rtimes c(\sigma_{\mathrm{cusp}}).$$

These embeddings imply that

$$\left(\prod_{j=1}^l \delta(\Delta'_j) \right) \otimes c(\sigma_{\mathrm{cusp}}) \leq r_{\mathrm{GL}}(\sigma) \leq r_{\mathrm{GL}}\left(\left(\prod_{i=1}^k \delta(\Delta_i) \right) \rtimes \sigma_{\mathrm{cusp}} \right).$$

However, Theorem 3.4 implies that $r_{\mathrm{GL}}\left(\left(\prod_{i=1}^k \delta(\Delta_i) \right) \rtimes \sigma_{\mathrm{cusp}} \right)$ can contain the support $c(\sigma_{\mathrm{cusp}})$ only if the corresponding GL part, $\prod_{j=1}^l \delta(\Delta'_j)$, has negative exponent. This is a contradiction since each Δ'_i is strongly positive for all i.

It remains to show the uniqueness of strongly positive segments. The situation becomes similar to the odd case [Kim 2015b] since we show the uniqueness of σ'. Therefore, we can apply the idea of [Matić 2011] (for $a \neq \frac{1}{2}$) and [Kim 2015b] (for $a = \frac{1}{2}$) to complete the proof, which we omit here.   □

In Theorem 4.9 and Theorem 4.11, we construct an injective mapping from $D(\rho; \sigma_{\text{cusp}}, c(\sigma_{\text{cusp}}))$ into the set of induced representations of the form

$$\delta\big([\nu^{a-k+1}\rho, \nu^{b'_1}\rho]\big) \times \delta\big([\nu^{a-k+2}\rho, \nu^{b'_2}\rho]\big) \times \cdots \times \delta\big([\nu^a\rho, \nu^{b'_k}\rho]\big) \rtimes c'(\sigma_{\text{cusp}})$$

((4-1) with refinement on the unitary exponents as in Theorem 4.9). In other words, if we let $\text{Jord}^{c'}_{(\rho,a)}$ be the set of $(c'; b_1, b_2, \ldots, b_{k_\rho})$, where $c' \in C$ and $b_i \in \mathbb{R}$ are such that $b_i - a + k_\rho - i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, k_\rho$ and $-1 < b_1 < b_2 < \cdots < b_{k_\rho}$, we construct the injective mapping

$$D(\rho; \sigma_{\text{cusp}}, c(\sigma_{\text{cusp}})) \hookrightarrow \text{Jord}^e_{(\rho,a)} \cup \text{Jord}^c_{(\rho,a)}.$$

It remains to show that this injective mapping is also surjective. For an element $(c'; b_1, b_2, \ldots, b_{k_\rho}) \in \text{Jord}^e_{(\rho,a)} \cup \text{Jord}^c_{(\rho,a)}$, let $\sigma_{(c'; b_1, \ldots, b_{k_\rho}; a)}$ be a unique irreducible subrepresentation of

$$\delta\big([\nu^{a-k_\rho+1}\rho, \nu^{b_1}\rho]\big) \times \delta\big([\nu^{a-k_\rho+2}\rho, \nu^{b_2}\rho]\big) \times \cdots \times \delta\big([\nu^a\rho, \nu^{b_{k_\rho}}\rho]\big) \rtimes c'(\sigma_{\text{cusp}}).$$

To show the surjectivity, we apply the induction argument in [Matić 2011] to show that the above subrepresentation $\sigma_{(c'; b_1, \ldots, b_{k_\rho}; a)}$ is strongly positive. We don't repeat the argument here.

Theorem 4.12. *The representation $\sigma_{(c'; b_1, \ldots, b_{k_\rho}; a)}$ is strongly positive.*

Remark 4.13. In the case of odd GSpin groups, we classify the special case $D(\rho; \sigma_{\text{cusp}})$ in [Kim 2015b]. In the even case, we need to consider $c(\sigma_{\text{cusp}})$ as a support in $D(\rho; \sigma_{\text{cusp}}, c(\sigma_{\text{cusp}}))$ as well, since the action of certain Weyl group elements makes σ into $c(\sigma)$ (e.g., Lemma 4.5).

4C. *Classification of strongly positive representations.* Let ρ_i be an essentially self-dual irreducible supercuspidal representation of $\text{GL}_{n_{\rho_i}}$ for $i = 1, \ldots, k$, and σ_{cusp} an irreducible supercuspidal representation of $G_{n'}$. We consider the set $D(\rho_1, \rho_2, \ldots, \rho_k; \sigma_{\text{cusp}}, c(\sigma_{\text{cusp}}))$ of strongly positive representations whose supercuspidal supports are the representations $\sigma_{\text{cusp}}, c(\sigma_{\text{cusp}})$ and twists of the representations $\rho_1, \ldots, \rho_k$. We assume that there exists a unique nonnegative real number a_{ρ_i} such that $\nu^{a_{\rho_i}} \rho_i \rtimes \sigma_{\text{cusp}}$ reduces for each i [Silberger 1980]. The half integer conjecture for GSpin groups (i.e., Lemma 4.1) implies that each $a_{\rho_i} \in \frac{1}{2}\mathbb{Z}$.

Theorem 4.14. *Let σ be a strongly positive representation in $D(\rho_1, \rho_2, \ldots, \rho_k;$ $\sigma_{\text{cusp}}, c(\sigma_{\text{cusp}}))$. Then σ can be considered to be the unique irreducible subrepresentation of the induced representation*

$$(4\text{-}4) \qquad \left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta\big([v^{a_{\rho_i} - k_i + j} \rho_i, v^{b_j^{(i)}} \rho_i]\big) \right) \rtimes c'(\sigma_{\text{cusp}}),$$

where $c' \in C = \{e, c\}$ and for $i = 1, \ldots, k$ and $j = 1, \ldots, k_i$, each $k_i \in \mathbb{Z}_{\geq 0}$ satisfies $k_i \leq \lceil a_{\rho_i} \rceil$ and each $b_j^{(i)} > 0$ is such that $b_j^{(i)} - a_{\rho_i} \in \mathbb{Z}_{\geq 0}$. Also, $b_j^{(i)} < b_{j+1}^{(i)}$ for $1 \leq j \leq k_i - 1$.

Proof. The proof is exactly the same as the odd case, and so is omitted. □

We also show that the map from $D(\rho_1, \rho_2, \ldots, \rho_k; \sigma_{\text{cusp}}, c(\sigma_{\text{cusp}}))$ to the set of induced representations of the form (4-4) is well defined in the following theorem.

Theorem 4.15. *Suppose that the representation σ is the unique irreducible subrepresentations of both representations*

$$\left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta\big([v^{a_{\rho_i} - k_i + j} \rho_i, v^{b_j^{(i)}} \rho_i]\big) \right) \rtimes \sigma_{\text{cusp}},$$

$$\left(\prod_{i=1}^{k'} \prod_{j=1}^{k_i'} \delta\big([v^{a_{\rho_i'} - k_i' + j} \rho_i', v^{c_j^{(i)}} \rho_i']\big) \right) \rtimes c'(\sigma_{\text{cusp}})$$

as in Theorem 4.14. Then $k = k'$, $\sigma_{\text{cusp}} \cong c'(\sigma_{\text{cusp}})$ and

$$\left\{ \prod_{j=1}^{k_i} \delta\big([v^{a_{\rho_i} - k_i + j} \rho_i, v^{b_j^{(i)}} \rho_i]\big) \,\Big|\, i = 1, \ldots, k \right\}$$

is a permutation of

$$\left\{ \prod_{j=1}^{k_i'} \delta\big([v^{a_{\rho_i'} - k_i' + j} \rho_i', v^{c_j^{(i)}} \rho_i']\big) \,\Big|\, i = 1, \ldots, k \right\}.$$

Proof. The arguments of the proof follow the same lines as those in the proof of Theorem 4.11. We, therefore, omit the proof here. □

Theorem 4.14 and Theorem 4.15 imply that there exists an injective mapping from $D(\rho_1, \rho_2, \ldots, \rho_k; \sigma_{\text{cusp}}, c(\sigma_{\text{cusp}}))$ into the set of induced representations of the form (4-4). Finally, it remains to show that this mapping is surjective.

Theorem 4.16. *The map described above gives a bijective correspondence between the set $D(\rho_1, \rho_2, \ldots, \rho_k; \sigma_{\text{cusp}}, c(\sigma_{\text{cusp}}))$ and the set of induced representations of the form (4-4).*

Proof. Let σ be the unique irreducible subrepresentation of the form (4-4), i.e.,

$$\left(\prod_{i=1}^{k}\prod_{j=1}^{k_i}\delta\left([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i]\right)\right)\rtimes c'(\sigma_{\text{cusp}}).$$

Since $\rho_p \not\cong \rho_q$ for $p \neq q$, we have for any $l = 1, \ldots, k$ the embedding

$$\sigma \hookrightarrow \left(\prod_{i\neq l}\prod_{j=1}^{k_i}\delta\left([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i]\right)\right)$$

$$\rtimes\delta\left([\nu^{a_{\rho_l}-k_l+l}\rho_l, \nu^{b_l^{(l)}}\rho_l], \ldots, [\nu^{a_{\rho_l}}\rho_l, \nu^{b_{k_l}^{(l)}}\rho_l]; c'(\sigma_{\text{cusp}})\right).$$

Theorem 4.12 implies that $\delta\left([\nu^{a_{\rho_l}-k_l+l}\rho_l, \nu^{b_l^{(l)}}\rho_l], \ldots, [\nu^{a_{\rho_l}}\rho_l, \nu^{b_{k_l}^{(l)}}\rho_l]; c'(\sigma_{\text{cusp}})\right)$ is strongly positive. Since l can be arbitrary, σ always has positive unitary exponents in the Jacquet module with respect to the Levi subgroup $\text{GL}_{n_{\rho_1}} \times \text{GL}_{n_{\rho_1}} \times \cdots \times \text{GL}_{n_{\rho_k}} \times G_{n'}$. Therefore, σ is strongly positive. $\square$

Since any strongly positive representation in R can be considered an element of $D(\rho_1', \rho_2', \ldots, \rho_k'; \sigma_{\text{cusp}}', c'(\sigma_{\text{cusp}}'))$ for some ρ_i' and σ_{cusp}', we can extend the bijective mapping constructed in Theorem 4.16 to any strongly positive representation in R. In sum, let SP and LJ be as defined in Section 1. Then we have a bijective correspondence between SP and LJ.

Remark 4.17. The ideas used in this section for the results of GSpin groups can be applied to even special orthogonal groups. Let us also remark that it is easier to work with even special orthogonal groups than even GSpin groups due to the results of Ban [1999a; 1999b] and Jantzen [2006]. For example, in the case of even special orthogonal groups, the Weyl group actions on the simple roots and induced representations are studied in [Ban 1999a; 1999b] and Tadić's structure formula is constructed in [Jantzen 2006, Theorem 3.4] (see also [Jantzen and Liu 2014, Theorem 3.1]). Let us remark that the classification of discrete series representations of SO_{2n} is first proved by C. Jantzen [2011] using the results for O_{2n} in [Mœglin 2002; Mœglin and Tadić 2002]. Our approach is different from [Jantzen 2011] and we generalize Matić's idea to the case of even special orthogonal groups.

5. Applications

It is easy to see that the strongly positive representations are special kinds of discrete series due to Casselman's square integrability criterion in [Kim 2009]. Furthermore, the strongly positive representations can be considered basic building blocks for discrete series representations (Theorem 5.1). The proof of the following embedding theorem is exactly the same as the case of odd GSpin groups, since the main idea of the proof depends on a slight variation of Casselman's square integrability criterion

for even GSpin groups (Proposition 3.2 in [Kim 2009]). Hence the proof is omitted. Let us remark that this idea originally comes from the results for metaplectic groups by Matić and we adapt some proofs from [Matić 2012, Chapter 3] to our situation.

Theorem 5.1. *Let σ denote a discrete series representation of G_n. Then there exists an embedding of the form*

$$\sigma \hookrightarrow \delta\big([\nu^{a_1}\rho_1, \nu^{b_1}\rho_1]\big) \times \delta\big([\nu^{a_2}\rho_2, \nu^{b_2}\rho_2]\big) \times \cdots \times \delta\big([\nu^{a_r}\rho_r, \nu^{b_r}\rho_r]\big) \rtimes \sigma_{\mathrm{sp}}$$

where $a_i \leq 0$, $a_i + b_i > 0$ and $\rho_i \in R_{\mathrm{GL}}$ is an irreducible unitary supercuspidal representation for $i = 1, \ldots, r$ (we allow $r = 0$); and $\sigma_{\mathrm{sp}} \in R$ is a strongly positive representation.

Theorem 5.1, together with our main result Theorem 4.16 giving the classification of strongly positive representations, imply the embedding

$$(5\text{-}1) \quad \sigma \hookrightarrow \delta\big([\nu^{a_1}\rho_1, \nu^{b_1}\rho_1]\big) \times \cdots \times \delta\big([\nu^{a_r}\rho_r, \nu^{b_r}\rho_r]\big)$$
$$\times \left(\prod_{i=1}^{k}\prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i'}-k_i+j}\rho_i', \nu^{b_j^{(i)}}\rho_i']\big)\right) \rtimes \sigma_{\mathrm{cusp}}.$$

where a_i, b_i and ρ_i are as in Theorem 5.1; $a_{\rho_i'}$, $b_j^{(i)}$, k_i and ρ_i' are as in Theorem 4.14; and σ_{cusp} is an irreducible supercuspidal representation of G_n.

This embedding has an interesting application in the proof of the equality of L-functions from the Langlands–Shahidi method and Artin L-functions through local Langlands correspondence (see [Shahidi 2010] for the Langlands–Shahidi method). More precisely, in [Kim 2015a] we used the following filtration of admissible representations to prove the equality of L-functions in the case of GSpin groups:

$$\text{supercuspidal} \subseteq \text{discrete series} \subseteq \text{tempered} \subseteq \text{admissible}.$$

We first showed the equality of L-functions in the supercuspidal case. Then, the above embedding (5-1) was used to generalize that result to the case of discrete series representations. Finally, that result was generalized via Langlands classification and properties of tempered representations to the cases of tempered representations and admissible representations. Furthermore, the equality of L-functions also has an interesting application in the proof of the generic Arthur packet conjecture in the case of GSpin groups [Shahidi 2011]. This conjecture can be considered a local version of the generalized Ramanujan conjecture.

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YEANSU KIM
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF IOWA
14 MACLEAN HALL
IOWA CITY, IA 52242
UNITED STATES

yeansu-kim@uiowa.edu

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Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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