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EFFECTIVE DIVISORS ON THE PROJECTIVE LINE HAVING SMALL DIAGONALS AND SMALL HEIGHTS AND THEIR APPLICATION TO ADELIC DYNAMICS

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We establish a quantitative adelic equidistribution theorem for a sequence of effective divisors on the projective line over the separable closure of a product formula field having small diagonals and small g-heights with respect to an adelic normalized weight g in arbitrary characteristic and in a possibly nonseparable setting. Applying this quantitative adelic equidistribution result to adelic dynamics of f, we obtain local proximity estimates between the iterations of a rational function $f \in k(z)$ of degree > 1 and a rational function $a \in k(z)$ of degree > 0 over a product formula field k of characteristic 0.

1. Introduction

Let k be a field and denote by k_s the separable closure of k in an algebraic closure $\bar{k}$. For every $d \in \mathbb{N} \cup \{0\}$, let $k[p_0, p_1]_d$ be the set of all homogeneous polynomials in two variables over k of degree d. A *k-effective divisor* $\mathcal{Z}$ on $\mathbb{P}^1(\bar{k})$ is a divisor on $\mathbb{P}^1(\bar{k})$ defined by the zeros in $\mathbb{P}^1(\bar{k})$ of some $P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1]_d$ *taking into account their multiplicities*, and is said to be on $\mathbb{P}^1(k_s)$ if $\operatorname{supp} \mathcal{Z} \subset \mathbb{P}^1(k_s)$. The defining polynomial $P(p_0, p_1)$ of $\mathcal{Z}$ is unique up to multiplication in k^* $(= k \setminus \{0\})$, and is called a *representative* of $\mathcal{Z}$. Effective divisors include Galois conjugacy classes of algebraic numbers, and are also called *Galois stable multisets* in $\mathbb{P}^1(\bar{k})$.

Our first aim in this article is to establish a *quantitative* adelic equidistribution of sequences of k-effective divisors on $\mathbb{P}^1(k_s)$, where k is a *product formula* field, having not only small g-heights (with respect to an adelic normalized weight g) but also *small diagonals* in arbitrary characteristic and in a possibly nonseparable setting. Secondly, we contribute to the study of the local *proximities* between the iterations of a rational function $f \in k(z)$ of degree > 1 and a rational function $a \in k(z)$ of degree > 0 on a chordal disk D of radius > 0 in the projective line $\mathbb{P}^1(\mathbb{C}_v)$ for each place v of k, in the setting of adelic dynamics of characteristic 0.

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1.1. ***Arithmetic over a product formula field.*** A field k is a *product formula field* if k is equipped with

(i) a set M_k of all places of k, which are either *finite* or *infinite*,

(ii) a set $\{|\cdot|_v : v \in M_k\}$, where for each $v \in M_k$, $|\cdot|_v$ is a nontrivial absolute value of k representing v (and then by definition $|\cdot|_v$ is nonarchimedean if and only if v is finite), and

(iii) a set $\{N_v : v \in M_k\}$, where $N_v \in \mathbb{N}$ for every $v \in M_k$

such that the following *product formula* holds: if $z \in k \setminus \{0\}$ then we have $|z|_v \neq 1$ for at most finitely many $v \in M_k$ and moreover

$$\prod_{v \in M_k} |z|_v^{N_v} = 1. \tag{PF}$$

Product formula fields include number fields and function fields over curves, and a product formula field is a number field if and only if it has at least one infinite place (see, e.g., the paragraph after Definition 7.51 of [Baker and Rumely 2010]).

Let k be a product formula field. For each $v \in M_k$, let k_v be the completion of k with respect to $|\cdot|_v$ and $\mathbb{C}_v$ the completion of an algebraic closure of k_v with respect to (the extended) $|\cdot|_v$. We fix an embedding of $\bar{k}$ into $\mathbb{C}_v$ which extends that of k into k_v; by convention, the dependence on $v \in M_k$ of a local quantity induced by $|\cdot|_v$ is emphasized by adding the suffix v to it. A family $g = \{g_v : v \in M_k\}$ is an *adelic continuous weight* if

(i) for every $v \in M_k$, g_v is a continuous function on the *Berkovich* projective line $\mathsf{P}^1(\mathbb{C}_v)$ such that

$$\mu_v^g := \Delta g_v + \Omega_{\mathrm{can},v}$$

is a probability Radon measure on $\mathsf{P}^1(\mathbb{C}_v)$ (see (2-2) for the definition of the probability Radon measure $\Omega_{\mathrm{can},v}$ on $\mathsf{P}^1(\mathbb{C}_v)$, and (2-3) for the normalization of the Laplacian Δ on $\mathsf{P}^1(\mathbb{C}_v)$), and

(ii) there is a finite subset E_g in M_k such that $g_v \equiv 0$ on $\mathsf{P}^1(\mathbb{C}_v)$ for all $v \in M_k \setminus E_g$.

Moreover, g is called an *adelic normalized weight* if, in addition,

(iii) the *g_v-equilibrium energy* V_{g_v} of $\mathsf{P}^1(\mathbb{C}_v)$ vanishes for every $v \in M_k$ (see Section 2.1 for the definition of V_{g_v}).

For an adelic continuous weight $g = \{g_v : v \in M_k\}$, the family $\mu^g := \{\mu_v^g : v \in M_k\}$ is called an *adelic probability measure* (compare [Favre and Rivera-Letelier 2006, Définition 1.1]). An adelic continuous weight $g = \{g_v : v \in M_k\}$ is said to be *placewise Hölder continuous* if for every $v \in M_k$, g_v is Hölder continuous on $\mathsf{P}^1(\mathbb{C}_v)$ with respect to the small model metric d_v on $\mathsf{P}^1(\mathbb{C}_v)$ (see (3-1) for the definition of d_v).

Given $P \in \bigcup_{d\in\mathbb{N}} k[p_0, p_1]_d$ and an adelic continuous weight $g = \{g_v : v \in M_k\}$, the *g-height* of a k-effective divisor $\mathcal{Z}$ on $\mathbb{P}^1(\bar{k})$ represented by P is

$$h_g(\mathcal{Z}) := \sum_{v\in M_k} N_v \frac{M_{g_v}(P)}{\deg P}, \tag{1-1}$$

where, for every $v \in M_k$, $M_{g_v}(P)$ is the logarithmic g_v*-Mahler measure* of P (see (2-10) for the definition of $M_{g_v}(P)$ and Section 2.3 for a proof that $h_g(\mathcal{Z}) \in \mathbb{R}$); by (PF), $h_g(\mathcal{Z})$ is well defined. For every $v \in M_k$, letting $\delta_{\mathcal{S}}$ be the Dirac measure on $\mathsf{P}^1(\mathbb{C}_v)$ at a point $\mathcal{S} \in \mathsf{P}^1(\mathbb{C}_v)$, a k-effective divisor $\mathcal{Z}$ on $\mathbb{P}^1(\bar{k})$ is regarded as a positive and discrete Radon measure $\sum_{w\in\operatorname{supp}\mathcal{Z}} (\operatorname{ord}_w \mathcal{Z})\delta_w$ on $\mathsf{P}^1(\mathbb{C}_v)$, still denoted by $\mathcal{Z}$. Then the *diagonal*

$$(\mathcal{Z}\times\mathcal{Z})(\operatorname{diag}_{\mathbb{P}^1(\bar{k})}) = \sum_{w\in\operatorname{supp}\mathcal{Z}} (\operatorname{ord}_w \mathcal{Z})^2$$

of $\mathcal{Z}$ is independent of $v \in M_k$. For a sequence $(\mathcal{Z}_n)$ of k-effective divisors on $\mathbb{P}^1(\bar{k})$ satisfying $\lim_{n\to\infty} \deg \mathcal{Z}_n = \infty$, we say $(\mathcal{Z}_n)$ *has small g-heights with respect to an adelic normalized weight g* if $\limsup_{n\to\infty} h_g(\mathcal{Z}_n) \le 0$, and we say $(\mathcal{Z}_n)$ *has small diagonals* if $\lim_{n\to\infty}((\mathcal{Z}_n \times \mathcal{Z}_n)(\operatorname{diag}_{\mathbb{P}^1(\bar{k})}))/(\deg \mathcal{Z}_n)^2 = 0$.

1.2. ***Quantitative adelic equidistribution of effective divisors.*** The following is one of our main results; for the Galois conjugacy class of an algebraic number, this was due to Favre and Rivera-Letelier [2006, Théorème 7]. For the definitions of the C^1-regularity of a continuous test function ϕ on $\mathsf{P}^1(\mathbb{C}_v)$, the Lipschitz constant $\operatorname{Lip}(\phi)_v$ on $(\mathsf{P}^1(\mathbb{C}_v), \mathrm{d}_v)$, and the Dirichlet norm $\langle\phi, \phi\rangle_v$ of ϕ for each $v \in M_k$, see Section 7.

Theorem 1. *Let k be a product formula field and k_s the separable closure of k in $\bar{k}$. Let $g = \{g_v : v \in M_k\}$ be a placewise Hölder continuous adelic normalized weight. Then for every $v \in M_k$, there is $C > 0$ such that for every k-effective divisor $\mathcal{Z}$ on $\mathbb{P}^1(k_s)$ and every test function $\phi \in C^1(\mathsf{P}^1(\mathbb{C}_v))$,*

$$\left|\int_{\mathsf{P}^1(\mathbb{C}_v)} \phi \,\mathrm{d}\Big(\frac{\mathcal{Z}}{\deg \mathcal{Z}} - \mu_v^g\Big)\right| \le C\cdot\max\big\{\operatorname{Lip}(\phi)_v, \langle\phi,\phi\rangle_v^{1/2}\big\}\sqrt{\max\Big\{h_g(\mathcal{Z}), (\log\deg\mathcal{Z})\frac{(\mathcal{Z}\times\mathcal{Z})(\operatorname{diag}_{\mathbb{P}^1(k_s)})}{(\deg\mathcal{Z})^2}\Big\}}. \tag{1-2}$$

In Theorem 1, if $v \in M_k$ is an infinite place, or equivalently, $\mathbb{C}_v \cong \mathbb{C}$, then the estimate (1-2) gives a quantitative estimate of the *Kantorovich–Wasserstein metric*

$$W\Big(\frac{\mathcal{Z}}{\deg\mathcal{Z}}, \mu_v^g\Big) = \sup_\phi \left|\int_{\mathbb{P}^1(\mathbb{C})} \phi\,\mathrm{d}\Big(\frac{\mathcal{Z}}{\deg\mathcal{Z}} - \mu_v^g\Big)\right|$$

between the probability Radon measures $\mathcal{Z}/\deg\mathcal{Z}$ and μ_v^g on $\mathsf{P}^1(\mathbb{C}_v) \cong \mathbb{P}^1(\mathbb{C})$, where ϕ ranges over all Lipschitz continuous functions on $\mathbb{P}^1(\mathbb{C})$ whose Lipschitz constants equal 1 with respect to the normalized chordal metric $[z, w]$ on $\mathbb{P}^1(\mathbb{C})$ (see Remark 4.2). For the details of the metric W including its role in the optimal transportation problems, see, e.g., [Villani 2009].

The next theorem is a qualitative version of Theorem 1. For a sequence of Galois conjugacy classes of algebraic numbers, this was due to Baker and Rumely [2006, Theorem 2.3], Chambert-Loir [2006, Théorème 4.2], and Favre and Rivera-Letelier [2006, Théorème 2]; see also [Szpiro, Ullmo, and Zhang 1997; Bilu 1997; Rumely 1999; Chambert-Loir 2000; Autissier 2001; Baker and Hsia 2005; Baker and Rumely 2006; Chambert-Loir 2006; Favre and Rivera-Letelier 2006], and, most recently, [Yuan 2008] on big line bundles over arithmetic varieties.

Theorem 2 (asymptotically Fekete configuration of effective divisors). *Let k be a product formula field and k_s its separable closure in $\bar{k}$. Let $g = \{g_v : v \in M_k\}$ be an adelic normalized weight. If a sequence $(\mathcal{Z}_n)$ of k-effective divisors on $\mathbb{P}^1(k_s)$ satisfying* $\lim_{n\to\infty} \deg \mathcal{Z}_n = \infty$ *has both small diagonals and small g-heights, then for every $v \in M_k$, $(\mathcal{Z}_n)$ is an asymptotically g_v-Fekete configuration on* $\mathsf{P}^1(\mathbb{C}_v)$. *In particular,* $\lim_{n\to\infty} \mathcal{Z}_n/\deg \mathcal{Z}_n = \mu_v^g$ *weakly on* $\mathsf{P}^1(\mathbb{C}_v)$.

In Theorem 2, the assertion that $(\mathcal{Z}_n)$ *is an asymptotically g_v-Fekete configuration on* $\mathsf{P}^1(\mathbb{C}_v)$ (see (2-7) for the definition), which is also called a *g_v-pseudo-equidistribution* on $\mathsf{P}^1(\mathbb{C}_v)$, is stronger than the final equidistribution assertion. For a relationship between the Kantorovich–Wasserstein metric W and (asymptotically) Fekete configurations on complex manifolds, see [Lev and Ortega-Cerdà 2012, §7]. For a recent result on the *capacity and the transfinite diameter* on complex manifolds, see [Berman and Boucksom 2010] (on $\mathbb{C}^n$, we also refer to the survey [Levenberg 2010]); for the *convergence of* (*asymptotically*) *Fekete points* on complex manifolds, see [Berman, Boucksom, and Nyström 2011].

1.3. ***Quantitative equidistribution in adelic dynamics.*** For rational functions f, a over a field k and for $n \in \mathbb{N}$, the divisor $[f^n = a]$ defined by the roots of the equation $f^n = a$ in $\mathbb{P}^1(\bar{k})$ is a k-effective divisor on $\mathbb{P}^1(\bar{k})$ if $f^n \not\equiv a$.

Let k be a product formula field. For a rational function $f \in k(z)$ of degree $d > 1$, let $\hat{g}_f := \{g_{f,v} : v \in M_k\}$ be the *adelic dynamical Green function* in the sense that for every $v \in M_k$, $g_{f,v}$ is the dynamical Green function of f on $\mathsf{P}^1(\mathbb{C}_v)$, so that $\mu_{f,v} := \mu^{g_{f,v}}$ is the f-equilibrium (or canonical) measure on $\mathsf{P}^1(\mathbb{C}_v)$ (see Section 9 for details). The family $\hat{g}_f$ is in fact an adelic normalized weight, and the $\hat{g}_f$-height function $h_{\hat{g}_f}$ coincides with the Call–Silverman *f-dynamical* (or canonical) height function. For every rational function $a \in k(z)$, the sequence $([f^n = a])$ has *strictly small* $\hat{g}_f$-heights in that $\limsup_{n\to\infty}(d^n + \deg a) \cdot h_{\hat{g}_f}([f^n = a]) < \infty$ (Lemma 9.2). Hence the following are consequences of Theorems 1 and 2, respectively.

Theorem 3. *Let k be a product formula field and k_s its separable closure in $\bar{k}$. Let $f \in k(z)$ be a rational function of degree $d > 1$ and $a \in k(z)$ a rational function.*

Then for every $v \in M_k$, *there exists a constant* $C > 0$ *such that for every test function* $\phi \in C^1(\mathsf{P}^1(\mathbb{C}_v))$ *and every* $n \in \mathbb{N}$,

$$\left|\int_{\mathsf{P}^1(\mathbb{C}_v)} \phi \,\mathrm{d}\Big(\frac{[f^n = a]}{d^n + \deg a} - \mu_{f,v}\Big)\right| \le C \cdot \max\big\{\mathrm{Lip}(\phi)_v, \langle \phi, \phi\rangle_v^{1/2}\big\} \sqrt{\frac{n \cdot ([f^n = a] \times [f^n = a])(\mathrm{diag}_{\mathbb{P}^1(k_s)})}{(d^n + \deg a)^2}} \tag{1-3}$$

if $f^n \not\equiv a$ *and the divisor* $[f^n = a]$ *on* $\mathbb{P}^1(\bar{k})$ *is on* $\mathbb{P}^1(k_s)$.

Theorem 4. *Let* k *be a product formula field and* k_s *its separable closure in* $\bar{k}$. *Let* $f \in k(z)$ *be a rational function of degree* $d > 1$ *and* $a \in k(z)$ *a rational function. If the sequence* $([f^n = a])$ *has small diagonals and the divisor* $[f^n = a]$ *is on* $\mathbb{P}^1(k_s)$ *for every sufficiently large* $n \in \mathbb{N}$, *then for every* $v \in M_k$, $([f^n = a])$ *is an asymptotically* $g_{f,v}$*-Fekete configuration on* $\mathsf{P}^1(\mathbb{C}_v)$. *In particular,*

$$\lim_{n\to\infty} \frac{[f^n = a]}{d^n + \deg a} = \mu_{f,v}$$

weakly on $\mathsf{P}^1(\mathbb{C}_v)$.

The final equidistribution assertion in Theorem 4 has been established in [Brolin 1965; Ljubich 1983; Freire, Lopes, and Mañé 1983] in complex dynamics, and in [Favre and Rivera-Letelier 2010] in (not necessarily adelic) nonarchimedean dynamics (of characteristic 0 when $\deg a > 0$). For every constant $a \in \mathbb{P}^1(k)$, the estimate (1-3) in Theorem 3 has been obtained in [Okuyama 2013b, Theorems 4 and 5] in complex and (not necessarily adelic) nonarchimedean dynamics of characteristic 0. In complex dynamics, for every $f \in \mathbb{C}(z)$ of degree $d > 1$, every constant $a \in \mathbb{P}^1(\mathbb{C})$, and every $\phi \in C^2(\mathbb{P}^1(\mathbb{C}))$, a finer estimate than (1-3) has been obtained in [Drasin and Okuyama 2007, Theorem 2 and (4.2)].

1.4. ***Application to a motivating question.*** Let K be an algebraically closed field that is complete with respect to a nontrivial absolute value $|\cdot|$, and $[z, w]$ be the normalized *chordal metric* on $\mathbb{P}^1 = \mathbb{P}^1(K)$ (see (2-1)). A subset D in $\mathbb{P}^1$ is called a *chordal disk* (in $\mathbb{P}^1$) if $D = \{z \in \mathbb{P}^1 : [z, w] \le r\}$ for some $w \in \mathbb{P}^1$ and some *radius* $r \ge 0$. Even in the specific case $a = \mathrm{Id}$ (see, e.g., [Cremer 1928; Siegel 1942; Brjuno 1971; 1972; Herman and Yoccoz 1983; Yoccoz 1988; 1995; Pérez-Marco 1993; 2001]), which is one of the most interesting cases and is related to *the difficulty of small denominators* in nonarchimedean and complex dynamics, the following question has not been completely understood.

Question. How uniformly close on a chordal disk D of radius > 0 can the sequence (f^n) of the iterations of a rational function $f \in K(z)$ of degree > 1 be to a rational function $a \in K(z)$ of degree > 0?

For a study of this question on the projective space $\mathbb{P}^N(K)$, see [Okuyama 2010]. The following estimate of the *local proximity sequence* $(\sup_D [f^n, a]_v)$ is an application of Theorem 3 to this question in the setting of adelic dynamics.

Theorem 5. *Let k be a product formula field of characteristic* 0. *Let $f \in k(z)$ be a rational function of degree > 1 and $a \in k(z)$ a rational function of degree > 0. Then for every $v \in M_k$ and every chordal disk D in $\mathbb{P}^1(\mathbb{C}_v)$ of radius > 0, as $n \to \infty$,*

$$\log \sup_D [f^n, a]_v = O\Big(\sqrt{n \cdot \big([f^n = a] \times [f^n = a]\big)\big(\mathrm{diag}_{\mathbb{P}^1(\bar{k})}\big)}\Big). \tag{1-4}$$

Here, the implicit constant in $O(\,\cdot\,)$ possibly depends on f and a.

In the case that $a = \mathrm{Id}$, we will see that $\big([f^n = \mathrm{Id}] \times [f^n = \mathrm{Id}]\big)\big(\mathrm{diag}_{\mathbb{P}^1(\bar{k})}\big) = O(d^n)$ as $n \to \infty$ in Section 10. Hence Theorem 5 concludes the following.

Theorem 6. *Let k be a product formula field of characteristic* 0. *Let $f \in k(z)$ be a rational function of degree $d > 1$. Then for every $v \in M_k$ and every chordal disk D in $\mathbb{P}^1(\mathbb{C}_v)$ of radius > 0,*

$$\log \sup_D [f^n, \mathrm{Id}]_v = O\big(\sqrt{nd^n}\big) \quad \text{as } n \to \infty. \tag{1-5}$$

1.5. *The unit $D^*(p)$.* The next result generalizes the obvious fact that the discriminant of a polynomial in one variable over a field k is in k. The unit $D^*(p)$ plays an important role in the nonseparable case and might have been studied before, but we could find no relevant literature.

Theorem 7. *Let k be a field and k_s the separable closure of k in an algebraic closure $\bar{k}$ of k. For every $p(z) \in k[z]$ of degree > 0, let $\{z_1, \dots, z_m\}$ be the set of all distinct zeros of $p(z)$ in $\bar{k}$ so that $p(z) = a \cdot \prod_{j=1}^m (z - z_j)^{d_j}$ in $\bar{k}[z]$ for some $a \in k \setminus \{0\}$ and some sequence $(d_j)_{j=1}^m$ in $\mathbb{N}$. If $\{z_1, \dots, z_m\} \subset k_s$, then*

$$D^*(p) := \prod_{j=1}^m \prod_{i : i \neq j} (z_j - z_i)^{d_i d_j} \in k \setminus \{0\},$$

where, a priori, this $D^*(p)$ is always in $\bar{k} \setminus \{0\}$.

1.6. *Organization of this article.* In Section 2, we recall background from potential theory and arithmetic on the Berkovich projective line. In Section 3, we extend Favre and Rivera-Letelier's regularization $[\,\cdot\,]_\epsilon$ of discrete Radon measures and establish required estimates on them, and in Section 4 we see the negativity of regularized Fekete sums and a Cauchy–Schwarz inequality. In Sections 5 and 6, we compute the g-Fekete sums $(\mathcal{Z}, \mathcal{Z})_g$ and estimate the regularized g-Fekete sums $(\mathcal{Z}_\epsilon, \mathcal{Z}_\epsilon)_g$ with respect to a k-effective divisor $\mathcal{Z}$ on $\mathbb{P}^1(\bar{k})$. In Section 7, we prove Theorems 1 and 2; the arguments are more or less adaptions of those in the proofs of [Favre and Rivera-Letelier 2006, Théorème 7] and [Baker and Rumely 2010, Theorem 10.24], respectively. In Section 8, we review background from nonarchimedean and complex dynamics. Finally, we prove Theorems 3 and 4 in Section 9, Theorems 5 and 6 in Section 10, and Theorem 7 in Section 11.

2. Background from potential theory and arithmetic

Notation 2.1. For a field k, the origin of k^2 is also denoted by $0 = 0_k$, and we write $\pi = \pi_k : k^2 \setminus \{0\} \to \mathbb{P}^1 = \mathbb{P}^1(k)$ for the canonical projection, so that $\pi(0, 1) = \infty$ and $\pi(p_0, p_1) = p_1/p_0$ if $p_0 \neq 0$. Set the wedge product $(z_0, z_1) \wedge (w_0, w_1) := z_0 w_1 - z_1 w_0$ on k^2.

Let K be an algebraically closed field that is complete with respect to a nontrivial absolute value $|\cdot|$, which is said to be *nonarchimedean* if the strong triangle inequality $|z + w| \leq \max\{|z|, |w|\}$ holds, and *archimedean* otherwise. On K^2, let $\|(p_0, p_1)\|$ be either the maximal norm $\max\{|p_0|, |p_1|\}$ (for nonarchimedean K) or the euclidean norm $\sqrt{|p_0|^2 + |p_1|^2}$ (for archimedean K). The *normalized chordal metric* $[z, w]$ on $\mathbb{P}^1 = \mathbb{P}^1(K)$ is the function

$$(z, w) \mapsto [z, w] = |p \wedge q| / (\|p\| \cdot \|q\|) \leq 1 \tag{2-1}$$

on $\mathbb{P}^1 \times \mathbb{P}^1$, where $p \in \pi^{-1}(z)$, $q \in \pi^{-1}(w)$. The metric topology on $\mathbb{P}^1$ with respect to $[z, w]$ agrees with the relative topology on $\mathbb{P}^1$ from the *Berkovich projective line* $\mathsf{P}^1 = \mathsf{P}^1(K)$, which is a compact augmentation of $\mathbb{P}^1$ containing $\mathbb{P}^1$ as a dense subset, and is isomorphic to $\mathbb{P}^1$ if and only if K is archimedean (see Section 3.2 for more details when K is nonarchimedean). Letting $\delta_{\mathcal{S}}$ be the Dirac measure on P^1 at a point $\mathcal{S} \in \mathsf{P}^1$, set

$$\Omega_{\mathrm{can}} := \begin{cases} \delta_{\mathcal{S}_{\mathrm{can}}} & \text{for nonarchimedean } K, \\ \omega & \text{for archimedean } K, \end{cases} \tag{2-2}$$

where $\mathcal{S}_{\mathrm{can}}$ is the canonical (or Gauss) point in P^1 for nonarchimedean K (see Section 3.2 for the definition), and ω is the Fubini–Study area element on $\mathbb{P}^1$ normalized as $\omega(\mathbb{P}^1) = 1$ for archimedean K. For nonarchimedean K, the *generalized Hsia kernel* $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ on P^1 with respect to $\mathcal{S}_{\mathrm{can}}$ is the unique (jointly) upper semicontinuous and separately continuous extension of the normalized chordal metric $[z, w]$ on $\mathbb{P}^1(\times\mathbb{P}^1)$ to $\mathsf{P}^1 \times \mathsf{P}^1$ (see (3-4) for a more concrete description). By convention, for archimedean K, the kernel function $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ is defined by $[z, w]$ itself. Let $\Delta = \Delta_{\mathsf{P}^1}$ be the distributional Laplacian on P^1 normalized so that for each $\mathcal{S}' \in \mathsf{P}^1$,

$$\Delta \log [\,\cdot\,, \mathcal{S}']_{\mathrm{can}} = \delta_{\mathcal{S}'} - \Omega_{\mathrm{can}} \quad \text{on } \mathsf{P}^1. \tag{2-3}$$

For the construction of the Laplacian Δ in the nonarchimedean case, see [Baker and Rumely 2010, §5; Favre and Jonsson 2004, §7.7; Thuillier 2005, §3] and also [Jonsson 2015, §2.5]. In [Baker and Rumely 2010], the opposite sign convention for Δ is adopted.

2.1. ***Potential theory on P^1 with external fields.*** For the foundation of the potential theory on the (Berkovich) projective line, see [Baker and Rumely 2010; Favre and Rivera-Letelier 2010; Thuillier 2005], and also [Jonsson 2015; Tsuji 1959, III §11] ([Thuillier 2005] is on more general curves than lines and [Tsuji 1959, III §11] is on $\mathbb{P}^1(\mathbb{C})$). We also refer to [Saff and Totik 1997] for the generalities of *weighted* potential theory, i.e., logarithmic potential theory *with external fields*.

A *continuous weight g on* P^1 is a continuous function on P^1 such that

$$\mu^g := \Delta g + \Omega_{\mathrm{can}}$$

is a probability Radon measure on P^1. For a continuous weight g on P^1, the *g-potential kernel* on P^1 (or the negative of an Arakelov Green kernel function on P^1 relative to μ^g [Baker and Rumely 2010, §8.10]) is the function

$$\Phi_g(\mathcal{S}, \mathcal{S}') := \log[\mathcal{S}, \mathcal{S}']_{\mathrm{can}} - g(\mathcal{S}) - g(\mathcal{S}') \quad \text{on } \mathsf{P}^1 \times \mathsf{P}^1, \tag{2-4}$$

and the *g-potential* of a Radon measure ν on P^1 is the function

$$U_{g,\nu}(\,\cdot\,) := \int_{\mathsf{P}^1} \Phi_g(\,\cdot\,, \mathcal{S}')\, \mathrm{d}\nu(\mathcal{S}') \quad \text{on } \mathsf{P}^1. \tag{2-5}$$

By Fubini's theorem, $\Delta U_{g,\nu} = \nu - \nu(\mathsf{P}^1)\mu^g$ on P^1. The *g-equilibrium energy $V_g \in (-\infty, +\infty)$ of* P^1 is the supremum of the *g-energy* functional

$$\nu \mapsto \int_{\mathsf{P}^1 \times \mathsf{P}^1} \Phi_g \, \mathrm{d}(\nu \times \nu) = \int_{\mathsf{P}^1} U_{g,\nu}\, \mathrm{d}\nu \tag{2-6}$$

on the space of all probability Radon measures ν on P^1; indeed, $V_g > -\infty$ since $V_g \ge \int_{\mathsf{P}^1 \times \mathsf{P}^1} \Phi_g \, \mathrm{d}(\Omega_{\mathrm{can}} \times \Omega_{\mathrm{can}}) > -\infty$. A probability Radon measure μ on P^1 at which the g-energy functional (2-6) attains the supremum V_g is called a *g-equilibrium mass distribution on* P^1; in fact the unique g-equilibrium mass distribution on P^1 is μ^g, and moreover, $U_{g,\mu^g} \equiv V_g$ on P^1 (for nonarchimedean K, see [Baker and Rumely 2010, Theorem 8.67, Proposition 8.70]). For a discussion on such a Gauss variational problem, see [Saff and Totik 1997, Chapter 1].

A *normalized weight g on* P^1 is a continuous weight on P^1 satisfying $V_g = 0$; for every continuous weight g on P^1, $\bar{g} := g + V_g/2$ is the unique normalized weight on P^1 such that $\mu^{\bar{g}} = \mu^g$.

For a continuous weight g on P^1 and a Radon measure ν on P^1, the *g-Fekete* *sum* with respect to ν is

$$(\nu, \nu)_g := \int_{\mathsf{P}^1 \times \mathsf{P}^1 \setminus \mathrm{diag}_{\mathbb{P}^1(K)}} \Phi_g \, \mathrm{d}(\nu \times \nu),$$

which generalizes the classical *Fekete sum* associated with a finite subset in $\mathbb{C}$ (see [Fekete 1930a; 1930b; 1933]). If $\operatorname{supp} \nu$ is a discrete (so finite) subset in P^1, i.e., if ν is a *discrete* measure on P^1, then $(\nu, \nu)_g$ is always finite (even if $\operatorname{supp} \nu \subset \mathbb{P}^1$).

For a continuous weight g on P^1, a sequence (ν_n) of positive and discrete Radon measures on P^1 satisfying $\lim_{n\to\infty} \nu_n(\mathsf{P}^1) = \infty$ is called an *asymptotically g-Fekete configuration on* P^1 if the sequence (ν_n) not only has *small diagonals* in that $(\nu_n \times \nu_n)(\mathrm{diag}_{\mathbb{P}^1(K)}) = o(\nu_n(\mathsf{P}^1)^2)$ as $n \to \infty$ but also satisfies $\lim_{n\to\infty}(\nu_n, \nu_n)_g/(\nu_n(\mathsf{P}^1))^2 = V_g$; under the former small diagonals condition, the latter one is equivalent to the weaker

$$\liminf_{n\to\infty} \frac{(\nu_n, \nu_n)_g}{(\nu_n(\mathsf{P}^1))^2} \geq V_g, \tag{2-7}$$

since we always have

$$\limsup_{n\to\infty} \frac{(\nu_n, \nu_n)_g}{(\nu_n(\mathsf{P}^1))^2} \leq V_g \tag{2-8}$$

(see, e.g., [Baker and Rumely 2010, Lemma 7.54]). By a classical argument (see [Saff and Totik 1997, Theorem 1.3 in Chapter III]), if (ν_n) is an asymptotically g-Fekete configuration on P^1, then $\lim_{n\to\infty} \nu_n/\nu_n(\mathsf{P}^1) = \mu^g$ weakly on P^1.

2.2. *Local arithmetic on* P^1.

Let k be a field.

Definition 2.2. A field extension K/k is an *algebraic and metric augmentation* of k if K is algebraically closed and (topologically) complete with respect to a nontrivial absolute value $|\cdot|$ (e.g., $\mathbb{C}_v$ is an algebraic and metric augmentation of a product formula field k for every $v \in M_k$).

For every $P \in \bigcup_{d\in\mathbb{N}} k[p_0, p_1]_d$, there is a sequence $(q_j^P)_{j=1}^{\deg P}$ in $\bar{k}^2 \setminus \{0\}$ giving a factorization

$$P(p_0, p_1) = \prod_{j=1}^{\deg P} \left((p_0, p_1) \wedge q_j^P\right) \tag{2-9}$$

of P in $\bar{k}[p_0, p_1]$. Set $z_j^P := \pi(q_j^P) \in \mathbb{P}^1(\bar{k})$ for each $j \in \{1, 2, \dots, \deg P\}$. Although the sequence $(q_j^P)_{j=1}^{\deg P}$ is not unique, the sequence $(z_j^P)_{j=1}^{\deg P}$ in $\mathbb{P}^1(\bar{k})$ is independent of the choice of $(q_j^P)_{j=1}^{\deg P}$ up to permutations. Let in addition K be an algebraic and metric completion of k. Then the sum $M^\#(P) := \sum_{j=1}^{\deg P} \log \|q_j^P\|$ is also independent of the choice of $(q_j^P)_{j=1}^{\deg P}$, and for every continuous weight g on $\mathsf{P}^1 = \mathsf{P}^1(K)$, the *logarithmic g-Mahler measure* of P is

$$M_g(P) := \sum_{j=1}^{\deg P} g(z_j^P) + M^\#(P). \tag{2-10}$$

The function $S_P := |P(\,\cdot\,/\|\cdot\|)|$ on $K^2 \setminus \{0\}$ descends to $\mathbb{P}^1(K)$ and in turn extends continuously to P^1 so that $\log S_P = \sum_{j=1}^{\deg P} \log[\,\cdot\,, z_j^P]_{\mathrm{can}} + M^\#(P)$ on P^1, which can be rewritten as $\log S_P - (\deg P)g = \sum_{j=1}^{\deg P} \Phi_g(\,\cdot\,, z_j^P) + M_g(P)$ on P^1. Integrating both sides against $\mathrm{d}\mu^g$ over P^1, by $U_{g,\mu^g} \equiv V_g$ on P^1, we have the *Jensen-type* formula

$$M_g(P) = \int_{\mathsf{P}^1} \bigl(\log S_P - (\deg P)g\bigr)\,\mathrm{d}\mu^g - (\deg P)V_g. \tag{2-11}$$

2.3. ***A lemma on global arithmetic.*** Let k be a product formula field. The proof of the next result is not based on a field extension of k.

Lemma 2.3. *For every $P \in \bigcup_{d\in\mathbb{N}} k[p_0, p_1]_d$, we have $\sum_{v\in M_k} N_v \cdot M^\#(P)_v \in \mathbb{R}_{\geq 0}$.*

Proof. Let $(q_j^P)_{j=1}^{\deg P}$ be a sequence in $\bar{k}^2 \setminus \{0\}$ giving a factorization (2-9) of P, and let $L(P(1,\cdot)) \in k \setminus \{0\}$ be the coefficient of the maximal degree term of $P(1,z) \in k[z]$. Setting $q_j^P = ((q_j^P)_0, (q_j^P)_1)$, for each $j \in \{1,2,\dots,\deg P\}$, we have

$$L(P(1,\cdot)) = (-1)^{\deg P - \deg_\infty P}\Biggl(\prod_{j:\pi(q_j^P)=\infty} (q_j^P)_1\Biggr)\Biggl(\prod_{j:\pi(q_j^P)\neq\infty} (q_j^P)_0\Biggr)$$

since for each $j \in \{1,2,\dots,\deg P\}$,

$$q_j^P = \begin{cases} (q_j^P)_0 \cdot (1, \pi(q_j^P)) & \text{if } \pi(q_j^P) \neq \infty, \\ (q_j^P)_1 \cdot (0,1) & \text{if } \pi(q_j^P) = \infty. \end{cases}$$

Thus we have $\sum_{v\in M_k} N_v \cdot M^\#(P)_v \geq \sum_{v\in M_k} N_v \log|L(P(1,\cdot))|_v = 0$, where the final equality is by (PF).

For each $i, j \in \mathbb{N}\cup\{0\}$ satisfying $i + j = \deg P$, if the coefficient $a_{i,j} \in k$ of the expansion $P(p_0,p_1) = \sum_{i+j=\deg P} a_{i,j} p_0^i p_1^j$ in $k[p_0,p_1]_{\deg P}$ does not vanish, then by (PF), there is a finite subset $E_{i,j}$ in M_k such that $|a_{i,j}|_v = 1$ for every $v \in M_k \setminus E_{i,j}$. Set $E_P := \{\text{infinite places of } k\} \cup \bigcup_{i,j\in\mathbb{N}\cup\{0\}: a_{i,j}\neq 0} E_{i,j}$. For every $v \in M_k \setminus E_P$, by the strong triangle inequality, $|P(p_0,p_1)|_v$ is bounded above by

$$\max\{\max\{|p_0|_v, |p_1|_v\}^{i+j} : i,j \in \mathbb{N}\cup\{0\},\, i+j = \deg P\} = \|(p_0,p_1)\|_v^{\deg P}$$

on $\mathbb{C}_v^2$, so that $\log S_{P,v} \leq 0$ on $\mathbb{P}^1(\mathbb{C}_v)$ and in turn on $\mathsf{P}^1(\mathbb{C}_v)$. Set $g^0 := \{g_v^0 : v \in M_k\}$ with $g_v^0 \equiv 0$ on $\mathsf{P}^1(\mathbb{C}_v)$ for every $v \in M_k$; then g^0 is an adelic continuous weight. For every finite $v \in M_k$, we have $\mu_v^{g^0} = \delta_{\mathcal{S}_{\mathrm{can},v}}$ on $\mathsf{P}^1(\mathbb{C}_v)$ and moreover $V_{g_v^0} = \log[\mathcal{S}_{\mathrm{can},v}, \mathcal{S}_{\mathrm{can},v}]_{\mathrm{can},v} = 0$, so that by the Jensen-type formula (2-11), we have $M^\#(P)_v = M_{g_v^0}(P) = \log S_{P,v}(\mathcal{S}_{\mathrm{can},v})$. Hence, $M^\#(P)_v \leq 0$ for every $v \in M_k \setminus E_P$, and we conclude that $\sum_{v\in M_k} N_v \cdot M^\#(P)_v < \infty$ since $\#E_P < \infty$. □

3. Regularization of discrete Radon measures whose supports are in $\mathbb{P}^1$

Let K be an algebraically closed field complete with respect to a nontrivial absolute value $|\cdot|$.

3.1. ***The small model metric* d *and the Hsia kernel* $|\mathcal{S}-\mathcal{S}'|_\infty$.** The kernel function $[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$ is not necessarily a metric on $\mathsf{P}^1=\mathsf{P}^1(K)$; indeed, for every $\mathcal{S}\in\mathsf{P}^1$, $[\mathcal{S},\mathcal{S}]_{\mathrm{can}}$ vanishes if and only if $\mathcal{S}\in\mathbb{P}^1=\mathbb{P}^1(K)$. The *small model metric* d on P^1 is the function

$$\mathsf{d}(\mathcal{S},\mathcal{S}'):=[\mathcal{S},\mathcal{S}']_{\mathrm{can}}-\frac{[\mathcal{S},\mathcal{S}]_{\mathrm{can}}+[\mathcal{S}',\mathcal{S}']_{\mathrm{can}}}{2}\quad\text{on } \mathsf{P}^1\times\mathsf{P}^1, \tag{3-1}$$

which extends the normalized chordal metric $[z,w]$ on $\mathbb{P}^1$ (but this d does not induce the topology of P^1; see [Baker and Rumely 2010, §2.7; Favre and Rivera-Letelier 2006, §4.7] for details). On the other hand, the *Hsia kernel* $|\mathcal{S}-\mathcal{S}'|_\infty$ on the *Berkovich affine line* $\mathsf{A}^1=\mathsf{A}^1(K)=\mathsf{P}^1\setminus\{\infty\}$ is the function

$$|\mathcal{S}-\mathcal{S}'|_\infty:=[\mathcal{S},\mathcal{S}']_{\mathrm{can}}\cdot[\mathcal{S},\infty]_{\mathrm{can}}^{-1}\cdot[\mathcal{S}',\infty]_{\mathrm{can}}^{-1}\quad\text{on } \mathsf{A}^1\times\mathsf{A}^1, \tag{3-2}$$

although the difference $\mathcal{S}-\mathcal{S}'$ itself is not defined unless both $\mathcal{S},\mathcal{S}'\in K$ (for details, see [Baker and Rumely 2010, Chapter 4]). The kernel $|\mathcal{S}-\mathcal{S}'|_\infty$ is the unique (jointly) upper semicontinuous and separately continuous extension of the function $|z-w|$ on $K\times K$ to $\mathsf{A}^1\times\mathsf{A}^1$.

3.2. ***A short description of* P^1 *for nonarchimedean* K.** Suppose that K is nonarchimedean. A subset B in K is called a (K-closed) *disk* in K if it has the form $B=\{z\in K:|z-a|\le r\}$ for some $a\in K$ and some *radius* $r\ge 0$. By the strong triangle inequality, *two disks in K either nest or are disjoint*. This alternative extends to any two decreasing infinite sequences of disks in K such that they either *infinitely nest* or *are eventually disjoint*, and so induces a *cofinal equivalence relation* among them.

Example 3.1. Instead of giving a formal definition of the cofinal equivalence class $\mathcal{S}$ of a decreasing infinite sequence (B_n) of disks in K, let us be practical: each $z\in K$ is regarded as the cofinal equivalence class of the constant sequence (B_n) of the disks $B_n\equiv\{z\}$ in K (of radii $\equiv 0$). More generally, for every cofinal equivalence class $\mathcal{S}$ of a decreasing infinite sequence (B_n) of disks in K, the intersection $B_{\mathcal{S}}:=\bigcap_{n\in\mathbb{N}}B_n$ is independent of the choice of the representatives (B_n) of $\mathcal{S}$, and if $B_{\mathcal{S}}\neq\varnothing$, then $B_{\mathcal{S}}$ is still a disk in K and the $\mathcal{S}$ is represented by the constant sequence $(\tilde{B}_n)$ of the disks $\tilde{B}_n\equiv B_{\mathcal{S}}$ in K.

As a set, the set of all cofinal equivalence classes $\mathcal{S}$ of decreasing infinite sequences (B_n) of disks in K and in addition $\infty\in\mathbb{P}^1$ is nothing but P^1 ([Berkovich 1990, p. 17]; see also [Baker and Rumely 2010, §2; Favre and Rivera-Letelier 2006, §3; Benedetto 2010, §6.1]): for example, the *canonical* (or *Gauss*) point $\mathcal{S}_{\mathrm{can}}$ in

P^1 is represented by the ring of K-integers $\mathcal{O}_K := \{z \in K : |z| \le 1\}$, which is a disk in K. The above alternative induces a partial ordering $\succeq$ on P^1 such that for every $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$ satisfying $B_{\mathcal{S}}, B_{\mathcal{S}'} \neq \varnothing$, we have $\mathcal{S} \succeq \mathcal{S}'$ if and only if $B_{\mathcal{S}} \supset B_{\mathcal{S}'}$ (the description is a little complicated when one of $B_{\mathcal{S}}, B_{\mathcal{S}'}$ equals $\varnothing$). For every $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$ satisfying $\mathcal{S} \succeq \mathcal{S}'$, the *segment* between $\mathcal{S}$ and $\mathcal{S}'$ in P^1 is the set of all points $\mathcal{S}'' \in \mathsf{P}^1$ satisfying $\mathcal{S} \succeq \mathcal{S}'' \succeq \mathcal{S}'$, which can be equipped with either the ordering induced by $\succeq$ on P^1 or its opposite. All those (oriented) segments make P^1 a *tree* in the sense of Jonsson [2015, §2, Definition 2.2]. The (Gelfand) topology of P^1 coincides with the (weak) topology of P^1 as a tree.

For each $\mathcal{S} \in \mathsf{P}^1 \setminus \{\infty\}$ represented by (B_n), set

$$\operatorname{diam} \mathcal{S} := \lim_{n\to\infty} \operatorname{diam} B_n \quad (= \operatorname{diam} B_{\mathcal{S}} \text{ if } B_{\mathcal{S}} \neq \varnothing),$$

where $\operatorname{diam} B$ denotes the diameter of a disk B in K with respect to $|\cdot|$; by convention, for $\mathcal{S} = \infty$, we set $B_\infty := K$ and $\operatorname{diam} \infty := +\infty$. The *hyperbolic space* is $\mathsf{H}^1 = \mathsf{H}^1(K) := \mathsf{P}^1 \setminus \mathbb{P}^1 = \{\mathcal{S} \in \mathsf{P}^1 : \operatorname{diam} \mathcal{S} \in (0, +\infty)\}$. The *big model* (or *hyperbolic*) *metric* ρ on H^1 is a path metric on H^1 (but does not induce the relative topology of H^1 induced by P^1) so that for every $\mathcal{S}, \mathcal{S}' \in \mathsf{H}^1$ satisfying $\mathcal{S} \succeq \mathcal{S}'$,

$$\rho(\mathcal{S}, \mathcal{S}') = \log(\operatorname{diam} \mathcal{S} / \operatorname{diam} \mathcal{S}') \tag{3-3}$$

(see, e.g., [Baker and Rumely 2010, §2.7]). In terms of ρ, the generalized Hsia kernel $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ with respect to $\mathcal{S}_{\mathrm{can}}$ is interpreted as a Gromov product

$$\log\,[\mathcal{S}, \mathcal{S}']_{\mathrm{can}} = -\rho(\mathcal{S}'', \mathcal{S}_{\mathrm{can}}) \quad \text{on } \mathsf{H}^1 \times \mathsf{H}^1, \tag{3-4}$$

where $\mathcal{S}''$ is the unique point in H^1 lying between $\mathcal{S}$ and $\mathcal{S}'$, between $\mathcal{S}'$ and $\mathcal{S}_{\mathrm{can}}$, and between $\mathcal{S}_{\mathrm{can}}$ and $\mathcal{S}$ (see [Favre and Rivera-Letelier 2006, §3.4]). Similarly, for every $\mathcal{S}, \mathcal{S}' \in \mathsf{A}^1$,

$$|\mathcal{S} - \mathcal{S}'|_\infty = \operatorname{diam} \mathcal{S}'', \tag{3-5}$$

where $\mathcal{S}''$ is the smallest point in A^1 satisfying both $\mathcal{S}'' \succeq \mathcal{S}$ and $\mathcal{S}'' \succeq \mathcal{S}'$ with respect to the partial ordering $\succeq$ on P^1.

For every $\epsilon > 0$, a continuous mapping

$$\pi_\epsilon : \mathsf{A}^1 \to \mathsf{A}^1$$

is defined by $\pi_\epsilon(\mathcal{S}) := \mathcal{S}''$ for every $\mathcal{S} \in \mathsf{A}^1$, where $\mathcal{S}'' \in \{\mathcal{S} \in \mathsf{P}^1 : \operatorname{diam} \mathcal{S} \in [\epsilon, +\infty)\}$ is the unique point between ∞ and $\mathcal{S}$ satisfying $\operatorname{diam} \mathcal{S}'' = \max\{\epsilon, \operatorname{diam} \mathcal{S}\}$ (see [Favre and Rivera-Letelier 2006, §4.6] for details).

3.3. *Regularization on* P^1. When K is archimedean, fix a nonnegative smooth decreasing function $\xi : [0, \infty) \to [0, 1]$ such that $\operatorname{supp} \xi \subset [0, 1]$ and $\int_0^\infty \xi(x)\, dx = 1$, and set $\xi_\epsilon(x) := \xi(x/\epsilon)/\epsilon$ on $[0, +\infty)$ for each $\epsilon > 0$. For every $z \in K$ and every

$\epsilon > 0$, the *ϵ-regularization* $[z]_\epsilon$ of δ_z is the convolution $\xi_\epsilon * \delta_z$ on $\mathbb{P}^1$, i.e., for any continuous test function ϕ on $\mathbb{P}^1$,

$$(\xi_\epsilon * \delta_z)(\phi) = \int_0^\epsilon \xi_\epsilon(r)\,\mathrm{d}r \int_0^{2\pi} \phi(z + re^{i\theta})\,\frac{\mathrm{d}\theta}{2\pi}.$$

When K is nonarchimedean, for every $z \in K$ and every $\epsilon > 0$, the *ϵ-regularization* $[z]_\epsilon$ of δ_z is defined by $[z]_\epsilon := (\pi_\epsilon)_*\delta_z = \delta_{\pi_\epsilon(z)}$ on P^1 [Favre and Rivera-Letelier 2006, p. 343]. In both cases, $[z]_\epsilon$ is a probability Radon measure on P^1, the *chordal* potential $\mathsf{P}^1 \ni \mathcal{S} \mapsto \int_{\mathsf{P}^1} \log[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}\,\mathrm{d}[z]_\epsilon(\mathcal{S}')$ of $[z]_\epsilon$ is a continuous function on P^1, and for every $z, w \in K$ and every $\epsilon > 0$, the estimate

$$\int_{\mathsf{A}^1\times\mathsf{A}^1} \log|\mathcal{S} - \mathcal{S}'|_\infty\,\mathrm{d}([z]_\epsilon \times [w]_\epsilon)(\mathcal{S}, \mathcal{S}') \ge \begin{cases} \log|z - w| & \text{if } z \ne w, \\ C_{\mathrm{abs}} + \log\epsilon & \text{if } z = w \end{cases} \tag{3-6}$$

holds, where $C_{\mathrm{abs}} \le 0$ is an absolute constant and in fact $C_{\mathrm{abs}} = 0$ for nonarchimedean K [Favre and Rivera-Letelier 2006, Lemmes 2.10, 4.11, and their proofs].

Let us extend the ϵ-regularization $[\,\cdot\,]_\epsilon$ and the estimate (3-6) to P^1. Set $\iota(z) := 1/z \in \mathrm{PGL}(2, K)$, which extends to an automorphism on P^1 (see Fact 8.2), so that $\iota^2 = \mathrm{Id}$ on P^1 and $[\iota(\mathcal{S}), \iota(\mathcal{S}')]_{\mathrm{can}} = [\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ (so $\mathsf{d}(\iota(\mathcal{S}), \iota(\mathcal{S}')) = \mathsf{d}(\mathcal{S}, \mathcal{S}')$) on $\mathsf{P}^1 \times \mathsf{P}^1$. For every $\epsilon > 0$, set $[\infty]_\epsilon := \iota_*[0]_\epsilon$.

For every $z \in \mathbb{P}^1$ and every $\epsilon > 0$, we have

$$\operatorname{supp}[z]_\epsilon \subset \{\mathcal{S} \in \mathsf{P}^1 : \mathsf{d}(\mathcal{S}, z) \le \epsilon\}, \tag{3-7}$$

as follows immediately from the definitions of $|\mathcal{S} - \mathcal{S}'|_\infty$ (and (3-5)), d, and $[z]_\epsilon$ when $z \in K$, and from (3-7) applied to $z = 0$ and the invariance of d under ι when $z = \infty$. Moreover, for every $z \in K$ and every $\epsilon > 0$,

$$\sup_{\mathcal{S} \in \operatorname{supp}[z]_\epsilon} |\log[\mathcal{S}, \infty]_{\mathrm{can}} - \log[z, \infty]| \le \epsilon \tag{3-8}$$

by a direct computation of $\log[\,\cdot\,, \infty]_{\mathrm{can}} - \log[z, \infty]$ on K, using that $\operatorname{supp}[z]_\epsilon \subset \{\mathcal{S} \in \mathbb{P}^1 : |\mathcal{S} - z|_\infty \le \epsilon\}$ and the density of K in A^1.

Lemma 3.2. *Let g be a continuous weight on P^1 having a modulus of continuity η on $(\mathsf{P}^1, \mathsf{d})$. Then for every $\epsilon > 0$ and every $z, w \in \mathbb{P}^1$,*

$$\int_{\mathsf{P}^1\times\mathsf{P}^1} \Phi_g\,\mathrm{d}([z]_\epsilon \times [w]_\epsilon) \ge \begin{cases} \Phi_g(z, w) - 2\epsilon - 2\eta(\epsilon) & \textit{if } z \ne w, \\ C_{\mathrm{abs}} + \log\epsilon - 2\epsilon + 2\log[z, \infty] - 2\eta(\epsilon) - 2g(z) & \textit{if } z = w \in K, \\ C_{\mathrm{abs}} + \log\epsilon - 2\epsilon - 2\eta(\epsilon) - 2g(\infty) & \textit{if } z = w = \infty. \end{cases} \tag{3-9}$$

Proof. Since $\Phi_g(\mathcal{S}, \mathcal{S}') = \log[\mathcal{S}, \mathcal{S}']_{\mathrm{can}} - g(\mathcal{S}) - g(\mathcal{S}')$ on $\mathsf{P}^1 \times \mathsf{P}^1$, by (3-7), we can assume $g \equiv 0$ (and $\eta \equiv 0$) on P^1 without loss of generality. For every $z, w \in K$, by the definition (3-2) of $|\mathcal{S}-\mathcal{S}'|_\infty$ and (3-8),

$$\int_{\mathsf{P}^1\times\mathsf{P}^1}\log[\mathcal{S},\mathcal{S}']_{\mathrm{can}}\,\mathrm{d}([z]_\epsilon\times[w]_\epsilon)(\mathcal{S},\mathcal{S}')\\ \geq\int_{\mathsf{A}^1\times\mathsf{A}^1}\log|\mathcal{S}-\mathcal{S}'|_\infty\,\mathrm{d}([z]_\epsilon\times[w]_\epsilon)(\mathcal{S},\mathcal{S}')-2\epsilon+\log[z,\infty]+\log[w,\infty],$$

which with the estimate (3-6) yields (3-9) (for $g\equiv\eta\equiv 0$) in this case. The estimate (3-9) (for $g\equiv\eta\equiv 0$) in the case $z=w=\infty$ follows from $[\infty]_\epsilon=\iota_*[0]_\epsilon$, $[\iota(\mathcal{S}),\iota(\mathcal{S}')]_{\mathrm{can}}=[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$, and the estimate (3-9) for $z=w=0$.

There remains the case that $z=\infty$ and $w\in K$ (so $z\neq w$). If K is nonarchimedean, then for every $w\in K$ and $\epsilon>0$, the equalities $[\infty]_\epsilon=\iota_*[0]_\epsilon$ and $[\iota(\mathcal{S}),\iota(\mathcal{S}')]_{\mathrm{can}}=[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$, together with the interpretation (3-4) of $[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$, yield

$$\int_{\mathsf{P}^1\times\mathsf{P}^1}\log[\mathcal{S},\mathcal{S}']_{\mathrm{can}}\,\mathrm{d}([\infty]_\epsilon\times[w]_\epsilon)(\mathcal{S},\mathcal{S}')\\ =\int_{\mathsf{P}^1\times\mathsf{P}^1}\log[\mathcal{S},\mathcal{S}']_{\mathrm{can}}\,\mathrm{d}([0]_\epsilon\times\iota_*[w]_\epsilon)(\mathcal{S},\mathcal{S}')=\log[\pi_\epsilon(0),\iota(\pi_\epsilon(w))]_{\mathrm{can}}\\ \geq\log[0,\iota(w)]=\log[\infty,w]\geq\log[\infty,w]-2\epsilon,$$

which implies the estimate (3-9) (for $g\equiv\eta\equiv 0$) in the case $z=\infty$ and $w\in K$ when K is nonarchimedean. If K is archimedean, then for every $w\in K$ and every $r,r'>0$, we have

$$\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\int_0^{2\pi}\log\Big|(0+re^{i\theta})-\frac{1}{w+r'e^{i\phi}}\Big|\frac{\mathrm{d}\theta}{2\pi}\\ =\int_0^{2\pi}\max\{-\log|w+r'e^{i\phi}|,\log r\}\frac{\mathrm{d}\phi}{2\pi}\geq-\int_0^{2\pi}\log\big|(w+r'e^{i\phi})-0\big|\frac{\mathrm{d}\phi}{2\pi},$$

so that for every $w\in K\cong\mathsf{A}^1$ and every $\epsilon>0$,

$$\int_{\mathsf{A}^1\times\mathsf{A}^1}\log|\mathcal{S}-\mathcal{S}'|_\infty\,\mathrm{d}([0]_\epsilon\times\iota_*[w]_\epsilon)(\mathcal{S},\mathcal{S}')\\ =\int_{\mathsf{A}^1\times\mathsf{A}^1}\log|\mathcal{S}-\iota(\mathcal{S}')|_\infty\,\mathrm{d}([0]_\epsilon\times[w]_\epsilon)(\mathcal{S},\mathcal{S}')\geq-\int_{\mathsf{A}^1}\log|\mathcal{S}'-0|_\infty\,\mathrm{d}[w]_\epsilon(\mathcal{S}').$$

On the other hand, for every $w\in K$ and every $\epsilon>0$, by the definition (2-1) of the chordal metric $[z,w]$ on $\mathbb{P}^1\cong\mathsf{P}^1$ (and $[0,\infty]=1$),

$$\int_{\mathsf{P}^1}\log[\mathcal{S}',\infty]_{\mathrm{can}}\,\mathrm{d}(\iota_*[w]_\epsilon)(\mathcal{S}')=\int_{\mathsf{P}^1}\log[\mathcal{S}',0]_{\mathrm{can}}\,\mathrm{d}[w]_\epsilon(\mathcal{S}')\\ =\int_{\mathsf{A}^1}\log|\mathcal{S}'-0|_\infty\,\mathrm{d}[w]_\epsilon(\mathcal{S}')+\int_{\mathsf{P}^1}\log[\mathcal{S}',\infty]_{\mathrm{can}}\,\mathrm{d}[w]_\epsilon(\mathcal{S}').$$

From these computations and (3-8), for every $w\in K$ and every $\epsilon>0$, we get

$$\int_{\mathsf{P}^1\times\mathsf{P}^1}\log[\mathcal{S},\mathcal{S}']_{\mathrm{can}}\,\mathrm{d}([\infty]_\epsilon\times[w]_\epsilon)(\mathcal{S},\mathcal{S}')$$
$$\begin{aligned}&=\int_{\mathsf{P}^1\times\mathsf{P}^1}\log[\mathcal{S},\mathcal{S}']_{\mathrm{can}}\,\mathrm{d}([0]_\epsilon\times\iota_*[w]_\epsilon)(\mathcal{S},\mathcal{S}')\\&\geq\int_{\mathsf{P}^1}\log[\mathcal{S},\infty]_{\mathrm{can}}\,\mathrm{d}[0]_\epsilon(\mathcal{S})+\int_{\mathsf{P}^1}\log[\mathcal{S}',\infty]_{\mathrm{can}}\,\mathrm{d}[w]_\epsilon(\mathcal{S}')\\&\geq\log[0,\infty]+\log[w,\infty]-2\epsilon=\log[w,\infty]-2\epsilon,\end{aligned}$$

which implies the estimate (3-9) (for $g\equiv\eta\equiv 0$) in the case $z=\infty$ and $w\in K$ when K is archimedean. □

4. The negativity of regularized Fekete sums and a Cauchy–Schwarz inequality

Let K be an algebraically closed field that is complete with respect to a nontrivial absolute value $|\cdot|$. For every $\epsilon>0$ and every discrete measure ν on $\mathsf{P}^1=\mathsf{P}^1(K)$ whose support is in $\mathbb{P}^1=\mathbb{P}^1(K)$, the *$\epsilon$-regularization* of ν is

$$\nu_\epsilon:=\sum_{w\in\operatorname{supp}\nu}\nu(\{w\})[w]_\epsilon\quad\text{on }\mathsf{P}^1.$$

For every continuous weight g on P^1, let us call $(\nu_\epsilon,\nu_\epsilon)_g$ the *ϵ-regularized g-Fekete sum* with respect to this ν.

4.1. *C^1-regularity and the Dirichlet norm.* Recall the description of P^1 given in Section 3.2. For nonarchimedean K, a function ϕ on $\mathsf{P}^1=\mathsf{P}^1(K)$ is *in* $C^1(\mathsf{P}^1)$ if

(i) ϕ is continuous on P^1 and locally constant except for a union $\mathcal{T}$ of at most finitely many segments in $\mathsf{H}^1=\mathsf{H}^1(K)$, which are oriented by the partial ordering $\succeq$ on P^1, and

(ii) the derivative ϕ' with respect to the length parameter induced by the hyperbolic metric ρ on each segment in $\mathcal{T}$ exists and is continuous on $\mathcal{T}$.

The *Dirichlet norm* of $\phi\in C^1(\mathsf{P}^1)$ is defined by $\langle\phi,\phi\rangle^{1/2}:=\left(\int_{\mathcal{T}}(\phi')^2\,\mathrm{d}\rho\right)^{1/2}$, where $\mathrm{d}\rho$ is the 1-dimensional Hausdorff measure on H^1 with respect to ρ (for details, see [Favre and Rivera-Letelier 2006, §5.5]). When K is archimedean, the C^1-regularity and the Dirichlet norm of a function ϕ on $\mathsf{P}^1\cong\mathbb{P}^1$ is defined with respect to the complex (or differentiable) structure of $\mathbb{P}^1$. For completeness, we include a proof of the following.

Proposition 4.1. *Every ϕ in $C^1(\mathsf{P}^1)$ is Lipschitz continuous on* $(\mathsf{P}^1,\mathsf{d})$.

Proof. When K is archimedean, this is obvious. Suppose that K is nonarchimedean and let $\phi\in C^1(\mathsf{P}^1)$. By definition, ϕ is locally constant on P^1 except for a union

$\mathcal{T}$ of at most finitely many segments in H^1, and is Lipschitz continuous on $\mathcal{T}$ with respect to ρ. The set $\mathcal{T}$ is compact in (H^1,ρ), and for every $\mathcal{S},\mathcal{S}'\in\mathsf{H}^1$, by the definition (3-1) of d, (3-4), and (3-3), if $\mathcal{S}_{\mathrm{can}}\succeq\mathcal{S}\succeq\mathcal{S}'$, then

$$\mathsf{d}(\mathcal{S},\mathcal{S}')=\operatorname{diam}\mathcal{S}-\frac{\operatorname{diam}\mathcal{S}+\operatorname{diam}\mathcal{S}'}{2}=\frac{\operatorname{diam}\mathcal{S}-\operatorname{diam}\mathcal{S}'}{2}\geq\frac{\operatorname{diam}\mathcal{S}'}{2}\rho(\mathcal{S},\mathcal{S}'),$$

and similarly, if $\mathcal{S}_{\mathrm{can}}\preceq\mathcal{S}\preceq\mathcal{S}'$, then $\mathsf{d}(\mathcal{S},\mathcal{S}')\geq\rho(\mathcal{S},\mathcal{S}')/(2\operatorname{diam}\mathcal{S}')$. Hence we conclude that ϕ is also Lipschitz continuous on $\mathcal{T}$ with respect to d, and in turn on the whole P^1 with respect to d. □

The Lipschitz constant of a Lipschitz continuous function ϕ on $(\mathsf{P}^1,\mathsf{d})$ is denoted by $\operatorname{Lip}(\phi)$.

Remark 4.2. When K is archimedean (so $\mathsf{P}^1\cong\mathbb{P}^1$), we have $\langle\phi,\phi\rangle^{1/2}\leq\operatorname{Lip}(\phi)$ for every $\phi\in C^1(\mathbb{P}^1)$. Moreover, every Lipschitz continuous function ϕ on $(\mathbb{P}^1,[z,w])$ is approximated by functions in $C^1(\mathbb{P}^1)$ in the Lipschitz norm.

4.2. ***The negativity of*** $(\nu_\epsilon,\nu_\epsilon)_g$ ***and a Cauchy–Schwarz inequality.*** For every Radon measure μ on P^1 satisfying $\mu(\mathsf{P}^1)=0$, if the chordal potential of μ, which is defined by $\mathcal{S}\mapsto\int_{\mathsf{P}^1}\log[\mathcal{S},\mathcal{S}']_{\mathrm{can}}\,d\mu(\mathcal{S}')$, is continuous on P^1, then we have the *positivity* property $\int_{\mathsf{P}^1\times\mathsf{P}^1}(-\log|\mathcal{S}-\mathcal{S}'|_\infty)\,d(\mu\times\mu)(\mathcal{S},\mathcal{S}')\geq 0$ (see [Favre and Rivera-Letelier 2006, §2.5 and §4.5]) and in fact the *Cauchy–Schwarz inequality*

$$\left|\int_{\mathsf{P}^1}\phi\,d\mu\right|^2\leq\langle\phi,\phi\rangle\cdot\int_{\mathsf{P}^1\times\mathsf{P}^1}(-\log|\mathcal{S}-\mathcal{S}'|_\infty)\,d(\mu\times\mu)(\mathcal{S},\mathcal{S}')\tag{4-1}$$

for every test function $\phi\in C^1(\mathsf{P}^1)$ (see [Favre and Rivera-Letelier 2006, (32) and (33)]).

In particular, for every $\epsilon>0$, every normalized weight g on P^1, every test function $\phi\in C^1(\mathsf{P}^1)$, and every discrete measure ν on P^1 whose support is in $\mathbb{P}^1$, the computation

$$\begin{aligned}0&\leq\int_{\mathsf{P}^1\times\mathsf{P}^1}(-\log|\mathcal{S}-\mathcal{S}'|_\infty)\,d((\nu_\epsilon-(\nu(\mathsf{P}^1))\mu^g)\times(\nu_\epsilon-(\nu(\mathsf{P}^1))\mu^g))(\mathcal{S},\mathcal{S}')\\&=\int_{\mathsf{P}^1\times\mathsf{P}^1}(-\Phi_g)\,d((\nu_\epsilon-(\nu(\mathsf{P}^1))\mu^g)\times(\nu_\epsilon-(\nu(\mathsf{P}^1))\mu^g))=-(\nu_\epsilon,\nu_\epsilon)_g\end{aligned}$$

(recalling $U_{g,\mu^g}\equiv 0$ on P^1) yields not only the *negativity* $(\nu_\epsilon,\nu_\epsilon)_g\leq 0$ but, with the Cauchy–Schwarz inequality (4-1) and the triangle inequality, also the estimate

$$\begin{aligned}\left|\int_{\mathsf{P}^1}\phi\,d\big(\nu-\nu(\mathsf{P}^1)\mu^g\big)\right|&=\left|\int_{\mathsf{P}^1}\phi\,d\big((\nu-\nu_\epsilon)+(\nu_\epsilon-(\deg\nu)\mu^g)\big)\right|\\&\leq(\deg\nu)\operatorname{Lip}(\phi)\epsilon+\langle\phi,\phi\rangle^{1/2}\cdot(-(\nu_\epsilon,\nu_\epsilon)_g)^{1/2}.\end{aligned}\tag{4-2}$$

5. Computations of Fekete sums $(\mathcal{Z}, \mathcal{Z})_g$

Let k be a field. For a k-effective divisor $\mathcal{Z}$ on $\mathbb{P}^1(\bar{k})$, set

$$D^*(\mathcal{Z}|\bar{k}) := \prod_{w \in \operatorname{supp} \mathcal{Z} \setminus \{\infty\}} \prod_{w' \in \operatorname{supp} \mathcal{Z} \setminus \{w, \infty\}} (w - w')^{(\operatorname{ord}_w \mathcal{Z})(\operatorname{ord}_{w'} \mathcal{Z})} \in \bar{k} \setminus \{0\},$$

which is in fact in $k \setminus \{0\}$ by Theorem 7 if $\mathcal{Z}$ is on $\mathbb{P}^1(k_s)$. For every $P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1]_d$, let $L(P(1, \cdot)) \in k \setminus \{0\}$ be the coefficient of the maximal degree term of $P(1, z) \in k[z]$ (appearing in Section 2.3).

Lemma 5.1. *Let k be a field. Let $\mathcal{Z}$ be a k-effective divisor on $\mathbb{P}^1(\bar{k})$ represented by $P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1]_d$, and let $(q_j^P)_{j=1}^{\deg P}$ be a sequence in $\bar{k}^2 \setminus \{0\}$ giving a factorization* (2-9) *of P. For each $j \in \{1, 2, \dots, \deg P\}$, set $q_j^P = ((q_j^P)_0, (q_j^P)_0)$ and $z_j := \pi(q_j^P) \in \mathbb{P}^1(\bar{k})$. Suppose $(q_j^P)_{j=1}^{\deg P}$ is normalized with respect to a distinguished zero $w_0 \in \mathbb{P}^1(\bar{k})$ of P so that for each $j \in \{1, 2, \dots, \deg P\}$,*

$$\begin{cases} (q_j^P)_0 = 1 & \text{if } z_j \notin \{w_0, \infty\}, \\ (q_j^P)_1 = 1 & \text{if } w_0 \neq z_j = \infty. \end{cases} \tag{5-1}$$

Then

$$L(P(1, \cdot)) = (-1)^{\deg P - \deg_\infty P} \cdot \begin{cases} \prod_{j : z_j = w_0} (q_j^P)_0 & \text{if } w_0 \neq \infty, \\ \prod_{j : z_j = w_0} (q_j^P)_1 & \text{if } w_0 = \infty, \end{cases} \tag{5-2}$$

and

$$\prod_{j=1}^{\deg P} \prod_{i : z_i \neq z_j} (q_i^P \wedge q_j^P) = (-1)^{\deg_\infty P (\deg P - \deg_\infty P)} \cdot L(P(1, \cdot))^{2(\deg P - \deg_{w_0} P)} \cdot D^*(\mathcal{Z}|\bar{k}). \tag{5-3}$$

Proof. Without normalizing the sequence $(q_j^P)_{j=1}^{\deg P}$ we have, by direct computation,

$$\begin{aligned} \prod_{j=1}^{\deg P} \prod_{i : z_i \neq z_j} (q_i^P \wedge q_j^P) &= \prod_{\substack{j : z_j = \infty \\ i : z_i \neq \infty}} \big((q_i^P)_0 (q_j^P)_1\big) \cdot \prod_{\substack{j : z_j \neq \infty \\ i : z_i = \infty}} \big(-(q_i^P)_1 (q_j^P)_0\big) \cdot \prod_{\substack{j : z_j \neq \infty \\ i : z_i \notin \{z_j, \infty\}}} \big((q_i^P)_0 (q_j^P)_0 (z_j - z_i)\big) \\ &= (-1)^{\deg_\infty P (\deg P - \deg_\infty P)} \cdot \Bigg(\prod_{j : z_j = \infty} \Bigg((q_j^P)_1^{\deg P - \deg_\infty P} \cdot \prod_{i : z_i \neq \infty} (q_i^P)_0 \Bigg) \Bigg)^2 \\ &\quad \cdot \Bigg(\prod_{j : z_j \neq \infty} \Bigg((q_j^P)_0^{\deg P - \deg_\infty P - \deg_{z_j} P} \cdot \prod_{i : z_i \notin \{z_j, \infty\}} (q_i^P)_0 \Bigg) \Bigg) \cdot D^*(\mathcal{Z}|\bar{k}). \end{aligned} \tag{5-4}$$

Let us normalize (q_j^P) so that the normalization (5-1) holds with respect to a distinguished zero $w_0 \in \mathbb{P}^1(\bar{k})$ of P. Then (5-2) follows from

$$L(P(1,\cdot)) = (-1)^{\deg P - \deg_\infty P} \cdot \Bigl(\prod_{j:z_j=\infty} (q_j^P)_1\Bigr)\Bigl(\prod_{j:z_j\neq\infty} (q_j^P)_0\Bigr)$$

and the normalization (5-1).

Let us show (5-3). If $w_0 = \infty$, then under the normalization (5-1), the equality (5-4) yields

$$\prod_{j=1}^{\deg P} \prod_{i:z_i\neq z_j} (q_i^P \wedge q_j^P) = (-1)^{\deg_\infty P(\deg P - \deg_\infty P)} \cdot \Bigl(\prod_{j:z_j=\infty} (q_j^P)_1\Bigr)^{2(\deg P - \deg_\infty P)} \cdot 1 \cdot D^*(\mathcal{Z}|\bar{k}),$$

which with (5-2) implies (5-3) when $w_0 = \infty$. If $w_0 \neq \infty$, then under the normalization (5-1), the equality (5-4) yields

$$\begin{aligned}\prod_{j=1}^{\deg P} \prod_{i:z_i\neq z_j} (q_i^P \wedge q_j^P) &= (-1)^{\deg_\infty P(\deg P - \deg_\infty P)} \cdot \Bigl(\prod_{i:z_i=w_0} (q_i^P)_0\Bigr)^{2\deg_\infty P} \\ &\quad \cdot \Bigl(\prod_{j:z_j=w_0} \bigl((q_j^P)_0^{\deg P - \deg_\infty P - \deg_{z_j} P} \cdot 1\bigr)\Bigr) \\ &\quad \cdot \Bigl(\prod_{j:z_j\notin\{w_0,\infty\}} \Bigl(1 \cdot \prod_{i:z_i=w_0} (q_i^P)_0\Bigr)\Bigr) \cdot D^*(\mathcal{Z}|\bar{k}) \\ &= (-1)^{\deg_\infty P(\deg P - \deg_\infty P)} \cdot \Bigl(\prod_{i:z_i=w_0} (q_i^P)_0\Bigr)^{2\deg_\infty P + 2(\deg P - \deg_\infty P - \deg_{w_0} P)} \cdot D^*(\mathcal{Z}|\bar{k}),\end{aligned}$$

which with (5-2) implies (5-3) when $w_0 \neq \infty$. □

Lemma 5.2 (local computation). *Let k be a field and K an algebraic and metric augmentation of k* (see Section 2.2). *For every continuous weight g on $\mathsf{P}^1 = \mathsf{P}^1(K)$ and every k-effective divisor $\mathcal{Z}$ on $\mathbb{P}^1(\bar{k})$ represented by a homogeneous polynomial $P \in \bigcup_{d\in\mathbb{N}} k[p_0, p_1]_d$, we have*

$$(\text{5-5})\quad (\mathcal{Z},\mathcal{Z})_g + 2 \cdot \sum_{w\in\operatorname{supp}\mathcal{Z}\setminus\{\infty\}} (\operatorname{ord}_w \mathcal{Z})^2 \log[w,\infty] - 2 \cdot \sum_{w\in\operatorname{supp}\mathcal{Z}} (\operatorname{ord}_w \mathcal{Z})^2 g(w) = 2(\deg\mathcal{Z})\log|L(P(1,\cdot))| + \log|D^*(\mathcal{Z}|\bar{k})| - 2(\deg\mathcal{Z})M_g(P).$$

Proof. Let $\mathcal{Z}$ and P be as in the statement and let $(q_j^P)_{j=1}^{\deg P}$ be a sequence in $\bar{k}^2\setminus\{0\}$ giving a factorization (2-9) of P and satisfying the normalization (5-1) with respect to a distinguished zero $w_0 \in \mathbb{P}^1(\bar{k})$ of P. Set $z_j := \pi(q_j^P) \in \mathbb{P}^1(\bar{k})$ for each $j \in \{1, 2, \dots, \deg P\}$. Since by definition

$$\Phi_g(z, z') = \log [z, z'] - g(z) - g(z')$$

on $\mathbb{P}^1(K) \times \mathbb{P}^1(K)$, we have

$$(\mathcal{Z}, \mathcal{Z})_g = \log\Bigg(\prod_{j=1}^{\deg P} \prod_{i: z_i \neq z_j} |q_i^P \wedge q_j^P| \Bigg) - 2 \cdot \sum_{j=1}^{\deg P} \sum_{i: z_i \neq z_j} (g(z_i) + \log \|q_i^P\|);$$

by (5-3),

$$\log\Bigg(\prod_{j=1}^{\deg P} \prod_{i: z_i \neq z_j} |q_i^P \wedge q_j^P| \Bigg) = 2(\deg P - \deg_{w_0} P) \log \big|L(P(1, \cdot))\big| + \log \big|D^*(\mathcal{Z}|\bar{k})\big|,$$

and we also have

$$\begin{aligned} &\sum_{j=1}^{\deg P} \sum_{i: z_i \neq z_j} \big(g(z_i) + \log \|q_i^P\|\big) \\ &\quad = \sum_{j=1}^{\deg P} \sum_{i=1}^{\deg P} \big(g(z_i) + \log \|q_i^P\|\big) - \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} \big(g(z_i) + \log \|q_i^P\|\big) \\ &\quad = (\deg P) M_g(P) - \sum_{j=1}^{\deg P} (\deg_{z_j} P) g(z_j) - \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} \log \|q_i^P\|, \end{aligned}$$

where the final equality is by the definition (2-10) of $M_g(P)$. Hence

$$\begin{aligned} (\mathcal{Z}, \mathcal{Z})_g &= 2(\deg P) \log \big|L(P(1, \cdot))\big| + \log \big|D^*(\mathcal{Z}|\bar{k})\big| - 2(\deg P) M_g(P) \\ &+ 2 \sum_{w \in \operatorname{supp} \mathcal{Z}} (\operatorname{ord}_w \mathcal{Z})^2 g(w) - 2 \Bigg((\deg_{w_0} P) \log |L(P(1, \cdot))| - \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} \log \|q_i^P\| \Bigg). \end{aligned}$$

For each $j \in \{1, 2, \dots, \deg P\}$, also set $q_j^P = ((q_j^P)_0, (q_j^P)_0)$. If $\infty \notin \operatorname{supp} \mathcal{Z}$, then $w_0 \neq \infty$, and by the normalization (5-1) and the equality (5-2),

$$\begin{aligned} &(\deg_{w_0} P) \log \big|L(P(1, \cdot))\big| - \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} \log \|q_i^P\| \\ &\quad = - \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} \big(\log \|q_i^P\| - \log |(q_i^P)_0|\big) = \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} \log [z_i, \infty] \\ &\quad = \sum_{w \in \operatorname{supp} \mathcal{Z}} (\operatorname{ord}_w \mathcal{Z})^2 \log [w, \infty] = \sum_{w \in \operatorname{supp} \mathcal{Z} \setminus \{\infty\}} (\operatorname{ord}_w \mathcal{Z})^2 \log [w, \infty]. \end{aligned}$$

If $\infty \in \operatorname{supp} \mathcal{Z}$, then we can set $w_0 = \infty$, and by the normalization (5-1) and the equality (5-2) (and $q_i^P = (q_i^P)_1 \cdot (0, 1)$ when $z_i = \infty$),

$$
\begin{aligned}
&(\deg_{w_0} P) \log\bigl|L(P(1,\cdot))\bigr| - \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} \log \|q_i^P\| \\
&= -\sum_{j: z_j = \infty} \sum_{i: z_i = z_j} \bigl(\log \|q_i^P\| - \log |(q_i^P)_1|\bigr) - \sum_{j: z_j \neq \infty} \sum_{i: z_i = z_j} \bigl(\log \|q_i^P\| - \log |(q_i^P)_0|\bigr) \\
&\qquad = \sum_{j: z_j \neq \infty} \sum_{i: z_i = z_j} \log [z_i, \infty] = \sum_{w \in \operatorname{supp} \mathcal{Z} \setminus \{\infty\}} (\operatorname{ord}_w \mathcal{Z})^2 \log [w, \infty].
\end{aligned}
$$

This completes the proof. □

Lemma 5.3 (global computation). *Let k be a product formula field and k_s the separable closure of k in $\bar{k}$. Then for every adelic continuous weight $g = \{g_v : v \in M_k\}$ and every k-effective divisor $\mathcal{Z}$ on $\mathbb{P}^1(k_s)$,*

$$
\begin{aligned}
\text{(5-6)}\quad &\sum_{v \in M_k} N_v \Bigl((\mathcal{Z}, \mathcal{Z})_{g_v} + 2 \sum_{w \in \operatorname{supp} \mathcal{Z} \setminus \{\infty\}} (\operatorname{ord}_w \mathcal{Z})^2 \log [w, \infty]_v \Bigr) \\
&\qquad = -2(\deg \mathcal{Z})^2 h_g(\mathcal{Z}) + 2 \sum_{v \in M_k} N_v \sum_{w \in \operatorname{supp} \mathcal{Z}} (\operatorname{ord}_w \mathcal{Z})^2 g_v(w).
\end{aligned}
$$

Proof. Let $P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1]_d$ be a representative of $\mathcal{Z}$. Summing up the product of N_v and (5-5) (for this P) over all $v \in M_k$, we have

$$
\begin{aligned}
&\sum_{v \in M_k} N_v \Bigl((\mathcal{Z}, \mathcal{Z})_{g_v} + 2 \sum_{w \in \operatorname{supp} \mathcal{Z} \setminus \{\infty\}} (\operatorname{ord}_w \mathcal{Z})^2 \log [w, \infty]_v - 2 \sum_{w \in \operatorname{supp} \mathcal{Z}} (\operatorname{ord}_w \mathcal{Z})^2 g_v(w) \Bigr) \\
&\qquad = -2(\deg \mathcal{Z})^2 h_g(\mathcal{Z})
\end{aligned}
$$

by the product formula (PF) (since $L(P(1,\cdot)) \in k \setminus \{0\}$ and, under the assumption that $\mathcal{Z}$ is on $\mathbb{P}^1(k_s)$, $D^*(\mathcal{Z}|\bar{k}) \in k \setminus \{0\}$) and the definition (1-1) of $h_g(\mathcal{Z})$. □

6. Estimates of regularized Fekete sums $(\mathcal{Z}_\epsilon, \mathcal{Z}_\epsilon)_g$

6.1. *Local estimate.* Let k be a field and K an algebraic and metric augmentation of k. Let $\mathcal{Z}$ be a k-effective divisor on $\mathbb{P}^1(\bar{k})$, which we regard as the Radon measure

$$
\sum_{w \in \operatorname{supp} \mathcal{Z}} (\operatorname{ord}_w \mathcal{Z}) \delta_w
$$

on $\mathsf{P}^1 = \mathsf{P}^1(K)$, and let g be a continuous weight on P^1 such that g is a $1/\kappa$-Hölder continuous function on $(\mathsf{P}^1, \mathsf{d})$ for some $\kappa \geq 1$ having the $1/\kappa$-Hölder constant $C(g) \geq 0$.

Lemma 6.1. *For every* $\epsilon > 0$,

$$(\mathcal{Z}_\epsilon, \mathcal{Z}_\epsilon)_g \geq (\mathcal{Z}, \mathcal{Z})_g + 2 \sum_{w \in \operatorname{supp} \mathcal{Z} \setminus \{\infty\}} (\operatorname{ord}_w \mathcal{Z})^2 \log [w, \infty] - 2 \sum_{w \in \operatorname{supp} \mathcal{Z}} (\operatorname{ord}_w \mathcal{Z})^2 g(w) + (C_{\text{abs}} + \log \epsilon) \cdot (\mathcal{Z} \times \mathcal{Z})(\operatorname{diag}_{\mathbb{P}^1(\bar{k})}) - 2(\deg \mathcal{Z})^2 (\epsilon + C(g)\epsilon^{1/\kappa}).$$

Proof. Set $\eta(\epsilon) = C(g)\epsilon^{1/\kappa}$. For every $\epsilon > 0$, using (3-9),

$$\begin{aligned}
&(\mathcal{Z}_\epsilon, \mathcal{Z}_\epsilon)_g - (\mathcal{Z}, \mathcal{Z})_g \\
&= \int_{\mathsf{P}^1 \times \mathsf{P}^1} \Phi_g \, \mathrm{d}(\mathcal{Z}_\epsilon \times \mathcal{Z}_\epsilon) - \int_{\mathsf{P}^1 \times \mathsf{P}^1 \setminus \operatorname{diag}_{\mathbb{P}^1(K)}} \Phi_g \, \mathrm{d}(\mathcal{Z} \times \mathcal{Z}) \\
&= \sum_{w \in \operatorname{supp} \mathcal{Z}} (\operatorname{ord}_w \mathcal{Z})^2 \int_{\mathsf{P}^1 \times \mathsf{P}^1} \Phi_g \, \mathrm{d}([w]_\epsilon \times [w]_\epsilon) \\
&\quad + \sum_{(z,w) \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus \operatorname{diag}_{\mathbb{P}^1}} \left(\int_{\mathsf{P}^1 \times \mathsf{P}^1} \Phi_g(\mathcal{S}, \mathcal{S}') \, \mathrm{d}([z]_\epsilon \times [w]_\epsilon)(\mathcal{S}, \mathcal{S}') - \Phi_g(z, w) \right) \\
&\geq \sum_{w \in \operatorname{supp} \mathcal{Z} \setminus \{\infty\}} (\operatorname{ord}_w \mathcal{Z})^2 \big(C_{\text{abs}} + \log \epsilon - 2\epsilon + 2 \log [w, \infty] - 2\eta(\epsilon) - 2g(w) \big) \\
&\qquad + \big(\mathcal{Z}(\{\infty\}) \big)^2 (C_{\text{abs}} + \log \epsilon - 2\epsilon - 2\eta(\epsilon) - 2g(\infty)) \\
&\qquad + \big((\deg \mathcal{Z})^2 - (\mathcal{Z} \times \mathcal{Z})(\operatorname{diag}_{\mathbb{P}^1(\bar{k})}) \big) (-2\epsilon - 2\eta(\epsilon)) \\
&= \big((\mathcal{Z} \times \mathcal{Z})(\operatorname{diag}_{\mathbb{P}^1(\bar{k})}) \big) \big(C_{\text{abs}} + \log \epsilon - 2\epsilon - 2\eta(\epsilon) \big) \\
&\qquad + 2 \sum_{w \in \operatorname{supp} \mathcal{Z} \setminus \{\infty\}} (\operatorname{ord}_w \mathcal{Z})^2 \log [w, \infty] - 2 \sum_{w \in \operatorname{supp} \mathcal{Z}} (\operatorname{ord}_w \mathcal{Z})^2 g(w) \\
&\qquad + \big((\deg \mathcal{Z})^2 - (\mathcal{Z} \times \mathcal{Z})(\operatorname{diag}_{\mathbb{P}^1(\bar{k})}) \big) (-2\epsilon - 2\eta(\epsilon)),
\end{aligned}$$

which completes the proof. $\square$

6.2. ***Global estimate.*** Let k be a product formula field, and $\mathcal{Z}$ a k-effective divisor on $\mathbb{P}^1(k_s)$. Let $g = \{g_v : v \in M_k\}$ be a placewise Hölder continuous adelic normalized weight, so for every $v \in M_k$, g_v is a normalized weight on $\mathsf{P}^1(\mathbb{C}_v)$ and is a $1/\kappa_v$-Hölder continuous function on $(\mathsf{P}^1(\mathbb{C}_v), \mathsf{d}_v)$ for some $\kappa_v \geq 1$ having the $1/\kappa_v$-Hölder constant $C(g_v) \geq 0$.

Lemma 6.2. *For every* $v_0 \in M_k$ *and every* $\epsilon > 0$,

$$N_{v_0}(\mathcal{Z}_\epsilon, \mathcal{Z}_\epsilon)_{g_{v_0}} \geq -2(\deg \mathcal{Z})^2 h_g(\mathcal{Z}) + (C_{\text{abs}} + \log \epsilon) \cdot (\mathcal{Z} \times \mathcal{Z})(\operatorname{diag}_{\mathbb{P}^1(k_s)}) \cdot \sum_{v \in E_g \cup \{v_0\}} N_v - 2 (\deg \mathcal{Z})^2 \sum_{v \in E_g \cup \{v_0\}} N_v (\epsilon + C(g_v)\epsilon^{1/\kappa_{v_0}}).$$

Proof. Fix $v_0 \in M_k$. We use, for every $v \in M_k$, the notation

$$W_v := (\mathcal{Z},\mathcal{Z})_{g_v} + 2\sum_{w\in\operatorname{supp}\mathcal{Z}\setminus\{\infty\}}(\operatorname{ord}_w\mathcal{Z})^2\log[w,\infty]_v - 2\sum_{w\in\operatorname{supp}\mathcal{Z}}(\operatorname{ord}_w\mathcal{Z})^2 g_v(w).$$

Since $(\mathcal{Z}_\epsilon,\mathcal{Z}_\epsilon)_{g_v}\le 0$ for every $\epsilon>0$ and every $v\in M_k$ (see Section 4.2), using also Lemma 6.1, we have

$$\begin{aligned}N_{v_0}(\mathcal{Z}_\epsilon,\mathcal{Z}_\epsilon)_{g_{v_0}} &\ge \sum_{v\in E_g\cup\{v_0\}} N_v(\mathcal{Z}_\epsilon,\mathcal{Z}_\epsilon)_{g_{v_0}}\\ &\ge \sum_{v\in E_g\cup\{v_0\}} N_v W_v + (C_{\mathrm{abs}}+\log\epsilon)\cdot(\mathcal{Z}\times\mathcal{Z})(\operatorname{diag}_{\mathbb{P}^1(k_s)})\cdot\sum_{v\in E_g\cup\{v_0\}}N_v\\ &\qquad - 2(\deg\mathcal{Z})^2\sum_{v\in E_g\cup\{v_0\}}N_v(\epsilon + C(g_v)\epsilon^{1/\kappa_{v_0}}).\end{aligned}$$

Moreover, since for every $v\in M_k\setminus E_g$, $g_v\equiv 0$ on $\mathsf{P}^1(\mathbb{C}_v)$ and $(\mathcal{Z},\mathcal{Z})_{g_v}\le 0$, using also (5-6), we have

$$\sum_{v\in E_g\cup\{v_0\}} N_v W_v \ge \sum_{v\in M_k} N_v W_v = -2(\deg\mathcal{Z})^2 h_g(\mathcal{Z}),$$

which completes the proof. □

7. Proofs of Theorems 1 and 2

Proof of Theorem 1. Fix $v_0\in M_k$. For every $v\in M_k$, g_v is a $1/\kappa_v$-Hölder continuous function on $(\mathsf{P}^1(\mathbb{C}_v),\mathsf{d}_v)$ for some $\kappa_v\ge 1$ having the $1/\kappa_v$-Hölder constant $C(g_v)\ge 0$. Set $\epsilon = 1/(\deg\mathcal{Z})^{2\kappa_{v_0}}$. For every test function $\phi\in C^1(\mathsf{P}^1(\mathbb{C}_{v_0}))$, by (4-2) and Lemma 6.2,

$$\begin{aligned}\left|\int_{\mathsf{P}^1(\mathbb{C}_{v_0})}\phi\,\mathrm{d}\Big(\frac{\mathcal{Z}}{\deg\mathcal{Z}}-\mu^g_{v_0}\Big)\right| &\le \frac{\operatorname{Lip}(\phi)_{v_0}}{(\deg\mathcal{Z})^{2\kappa_0}} + \frac{\langle\phi,\phi\rangle_{v_0}^{1/2}}{N_{v_0}^{1/2}}\\ &\cdot\Bigg(2\cdot h_g(\mathcal{Z}) + (-C_{\mathrm{abs}}+2\kappa_{v_0}\log\deg\mathcal{Z})\cdot\frac{(\mathcal{Z}\times\mathcal{Z})(\operatorname{diag}_{\mathbb{P}^1(k_s)})}{(\deg\mathcal{Z})^2}\cdot\sum_{v\in E_g\cup\{v_0\}}N_v\\ &\qquad + 2\sum_{v\in E_g\cup\{v_0\}}N_v\Big(\frac{1}{(\deg\mathcal{Z})^{2\kappa_0}}+\frac{C(g_v)}{(\deg\mathcal{Z})^2}\Big)\Bigg)^{1/2},\end{aligned}$$

which completes the proof. □

Proof of Theorem 2. Fix $v_0\in M_k$. For every $n\in\mathbb{N}$, we have $(\mathcal{Z}_n,\mathcal{Z}_n)_{g_v}\le 0$ if $v\in M_k\setminus E_g$. Hence by (2-8), (5-6), and the assumption that $V_{g_v}=0$ for every

$v \in M_k$, we obtain

$$N_{v_0}\frac{(\mathcal{Z}_n,\mathcal{Z}_n)_{g_{v_0}}}{(\deg\mathcal{Z}_n)^2}+\#E_g\cdot o(1)\ \ge \sum_{v\in M_k}N_v\frac{(\mathcal{Z}_n,\mathcal{Z}_n)_{g_v}}{(\deg\mathcal{Z}_n)^2}$$
$$\ge -2\cdot h_g(\mathcal{Z}_n)-2\frac{(\mathcal{Z}_n\times\mathcal{Z}_n)(\mathrm{diag}_{\mathbb{P}^1(k_s)})}{(\deg\mathcal{Z}_n)^2}\sum_{v\in E_g}N_v\sup_{\mathsf{P}^1(\mathbb{C}_v)}|g_v|\quad\text{as } n\to\infty;$$

thus, under the assumption that $(\mathcal{Z}_n)$ has both small diagonals and small g-heights, we have $\liminf_{n\to\infty}(\mathcal{Z}_n,\mathcal{Z}_n)_{g_{v_0}}/(\deg\mathcal{Z}_n)^2\ge 0=V_{g_{v_0}}$. Hence (2-7) holds for g_{v_0} and $(\mathcal{Z}_n)$, and the proof is complete. □

8. Nonarchimedean and complex dynamics

Fact 8.1. Let k be a field. For a rational function $\phi\in k(z)$, we call

$$F_\phi=((F_\phi)_0,(F_\phi)_1)\ \in \bigcup_{d\in\mathbb{N}\cup\{0\}}(k[p_0,p_1]_d\times k[p_0,p_1]_d)$$

a *lift* of ϕ if $\pi\circ F_\phi=\phi\circ\pi$ on $k^2\setminus\{0\}$ and, in addition, $F_\phi^{-1}(0)=\{0\}$ when $\deg\phi>0$. The latter nondegeneracy condition is equivalent to the nonvanishing of $\mathrm{Res}(F_\phi):=\mathrm{Res}((F_\phi)_0,(F_\phi)_1)$; for the definition of the homogeneous resultant $\mathrm{Res}(P,Q)\in k$ for $P,Q\in\bigcup_{d\in\mathbb{N}\cup\{0\}}k[p_0,p_1]_d$, see, e.g., [Silverman 2007, §2.4]. Such a lift F_ϕ of ϕ is unique up to multiplication in k^*, and is in fact in $k[p_0,p_1]_{\deg\phi}\times k[p_0,p_1]_{\deg\phi}$.

Let K be an algebraically closed field that is complete with respect to a nontrivial absolute value $|\cdot|$.

8.1. *The dynamical Green function g_f on P^1.*

For the foundation of a potential-theoretical study of dynamics on the Berkovich projective line, see [Baker and Rumely 2010; Favre and Rivera-Letelier 2010] for nonarchimedean K and, e.g., [Berteloot and Mayer 2001, §VIII] for archimedean K ($\cong\mathbb{C}$).

Fact 8.2. Let $\phi\in K(z)$ be a rational function of degree $d_0\in\mathbb{N}\cup\{0\}$. The action of ϕ on $\mathbb{P}^1=\mathbb{P}^1(K)$ uniquely extends to a continuous endomorphism on $\mathsf{P}^1=\mathsf{P}^1(K)$. When $d_0>0$, the extended ϕ is surjective, open, and discrete and preserves $\mathbb{P}^1$ and $\mathsf{H}^1=\mathsf{H}^1(K)$, the local degree function $z\mapsto\deg_z\phi$ on $\mathbb{P}^1$ also canonically extends to P^1, and the (mapping) degree of the extended $\phi:\mathsf{P}^1\to\mathsf{P}^1$ still equals d_0 (see [Baker and Rumely 2010, §2.3, §9; Benedetto 2010, §6.3]): in particular, the extended action of ϕ on P^1 induces a push-forward ϕ_* and a pullback ϕ^* on the spaces of continuous functions and of Radon measures on P^1. When $d_0=0$, the extended ϕ is still constant, and we set $\phi^*\mu:=0$ on P^1 for every Radon measure μ on P^1 by convention. Let $F_\phi\in K[p_0,p_1]_{\deg\phi}\times K[p_0,p_1]_{\deg\phi}$ be a lift of ϕ. The function

$$T_{F_\phi} := \log\big\|F_\phi(\,\cdot\,/\|\cdot\|)\big\| = \log\|F_\phi\| - (\deg\phi)\log\|\cdot\| \tag{8-1}$$

on $K^2 \setminus \{0\}$ descends to $\mathbb{P}^1$ and in turn extends continuously to P^1, satisfying $\Delta T_{F_\phi} = \phi^*\Omega_{\mathrm{can}} - (\deg\phi)\Omega_{\mathrm{can}}$ on P^1 (see, e.g., [Okuyama 2013a, Definition 2.8]). Moreover, ϕ is a Lipschitz continuous endomorphism on $(\mathsf{P}^1, \mathsf{d})$ and T_{F_ϕ} is a Lipschitz continuous function on $(\mathsf{P}^1, \mathsf{d})$ (for nonarchimedean K, see [Baker and Rumely 2010, Proposition 9.37]). For every $n \in \mathbb{N}$, the homogeneous polynomial $F_\phi^n \in K[p_0, p_1]_{\deg\phi^n} \times K[p_0, p_1]_{\deg\phi^n}$ is a lift of ϕ^n.

Let $f \in K(z)$ be a rational function of degree $d > 1$, and consider a lift $F \in K[p_0, p_1]_d \times K[p_0, p_1]_d$ of f. The uniform limit $g_F := \lim_{n\to\infty} T_{F^n}/d^n$ on P^1 exists, and more precisely, for every $n \in \mathbb{N}$,

$$\sup_{\mathsf{P}^1}\left|g_F - \frac{T_{F^n}}{d^n}\right| \le \frac{\sup_{\mathsf{P}^1}|T_F|}{d^n(d-1)}. \tag{8-2}$$

The limit g_F is called the *dynamical Green function of F* on P^1 and is a continuous weight on P^1. The probability Radon measure

$$\mu_f := \mu^{g_F} = \Delta g_F + \Omega_{\mathrm{can}} = \lim_{n\to\infty} \frac{(f^n)^*\Omega_{\mathrm{can}}}{d^n} \quad \text{weakly on } \mathsf{P}^1$$

is independent of the choice of F and satisfies $f^*\mu_f = d \cdot \mu_f$ on P^1. It is called the *f-equilibrium* (*or canonical*) *measure* on P^1. Moreover, g_F is a Hölder continuous function on $(\mathsf{P}^1, \mathsf{d})$ (for nonarchimedean K, see [Favre and Rivera-Letelier 2006, §6.6]). The remarkable *energy formula*

$$V_{g_F} = -\frac{\log|\mathrm{Res}\, F|}{d(d-1)} \tag{8-3}$$

was first established by DeMarco [2003] for archimedean K and was generalized to rational functions defined over a number field by Baker and Rumely [2006] (for a simple proof of (8-3) which also works for general K, see [Baker 2009, Appendix A] or [Okuyama and Stawiska 2011, Appendix]). The *dynamical Green function g_f of f on* P^1 is the unique normalized weight on P^1 such that $\mu^{g_f} = \mu_f$, i.e., for any lift F of f, $g_f \equiv g_F + V_{g_F}/2$ on P^1.

8.2. ***A Berkovich space version of the quasiperiodicity region $\mathcal{E}_f$.*** For nonarchimedean dynamics, see [Baker and Rumely 2010, §10; Favre and Rivera-Letelier 2010, §2.3; Benedetto 2010, §6.4]. For complex dynamics, see, e.g., [Milnor 2006].

Let $f \in K(z)$ be a rational function of degree > 1. The *Berkovich Julia set* of f is

$$\mathsf{J}(f) := \left\{\mathcal{S} \in \mathsf{P}^1 : \bigcap_{U \text{ open in } \mathsf{P}^1 \text{ containing } \mathcal{S}} \Big(\bigcup_{n\in\mathbb{N}} f^n(U)\Big) = \mathsf{P}^1 \setminus E(f)\right\},$$

where $E(f) := \left\{a \in \mathbb{P}^1 : \#\bigcup_{n\in\mathbb{N}} f^{-n}(a) < \infty\right\}$ is the *exceptional set* of f. The

Berkovich Fatou set is $\mathsf{F}(f) := \mathsf{P}^1 \setminus \mathsf{J}(f)$. By definition, $\mathsf{J}(f)$ is closed and $\mathsf{F}(f)$ is open in P^1, both $\mathsf{J}(f)$ and $\mathsf{F}(f)$ are totally invariant under f, and $\mathsf{J}(f)$ has no interior point unless $\mathsf{J}(f) = \mathsf{P}^1$. The *classical Julia set* $\mathsf{J}(f) \cap \mathbb{P}^1$ (resp. the *classical Fatou set* $\mathsf{F}(f) \cap \mathbb{P}^1$) coincides with the set of all nonequicontinuity points (resp. the region of local equicontinuity) of the family $\{f^n : n \in \mathbb{N}\}$ as a family of endomorphisms on $(\mathbb{P}^1, [z, w])$.

A component U of $\mathsf{F}(f)$ is called a *Berkovich Fatou component* of f, and is said to be *cyclic* under f if $f^n(U) = U$ for some $n \in \mathbb{N}$, which is called a *period* of U under f. Following [Fatou 1920, §28], a cyclic Berkovich Fatou component U of f having a period $n \in \mathbb{N}$ is called a *singular domain* of f if $f^n : U \to U$ is injective. Let $\mathcal{E}_f$ be the set of all points $\mathcal{S} \in \mathsf{P}^1$ having an open neighborhood V in P^1 such that $\liminf_{n\to\infty} \sup_{V\cap\mathbb{P}^1} [f^n, \mathrm{Id}] = 0$, which is a Berkovich space version of Rivera-Letelier's *quasiperiodicity region* of f. When K is archimedean, $\mathcal{E}_f$ *coincides with* the union of all singular domains of f, and when K is nonarchimedean, $\mathcal{E}_f$ is still open and forward invariant under f and *is contained in* the union of all singular domains of f (see [Okuyama 2013a, Lemma 4.4]).

The following function T_* is Rivera-Letelier's *iterative logarithm* of f on $\mathcal{E}_f \cap \mathbb{P}^1$, which is a nonarchimedean counterpart of the uniformization of a Siegel disk or a Herman ring of f.

Theorem 8.3 ([Rivera-Letelier 2003, §3.2, §4.2]. See also [Favre and Rivera-Letelier 2010, Théorème 2.15]). *Suppose that K is nonarchimedean and has characteristic* 0 *and residual characteristic p. Let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree* > 1 *and suppose that $\mathcal{E}_f \neq \varnothing$, which implies $p > 0$ by* [Favre and Rivera-Letelier 2010, Lemme 2.14]. *Then for every component Y of $\mathcal{E}_f$ not containing ∞, there are $k_0 \in \mathbb{N}$, a continuous action $T : \mathbb{Z}_p \times (Y \cap K) \ni (\omega, y) \mapsto T^\omega(y) \in Y \cap K$, and a nonconstant K-valued holomorphic function T_* on $Y \cap K$ such that for every $m \in \mathbb{Z}$, $(f^{k_0})^m = T^m$ on $Y \cap K$, that for every $\omega \in \mathbb{Z}_p$, T^ω is a biholomorphism on $Y \cap K$, and that for every $\omega_0 \in \mathbb{Z}_p$,*

$$\lim_{\mathbb{Z}_p \ni \omega \to \omega_0} \frac{T^\omega - T^{\omega_0}}{\omega - \omega_0} = T_* \circ T^{\omega_0} \quad \text{locally uniformly on } Y \cap K. \tag{8-4}$$

8.3. ***The fundamental relationship between μ_f and $\mathsf{J}(f)$.*** If K is archimedean, the inclusion $\operatorname{supp} \mu_f \subset \mathsf{J}(f)$ is classical, but it is not trivial from the definition of $\mathsf{J}(f)$ when K is nonarchimedean. For an elementary proof, see [Okuyama 2013a, proof of Theorem 2.18]. Actually the equality $\operatorname{supp} \mu_f = \mathsf{J}(f)$ holds, but we will dispense with the reverse (and easier) inclusion $\mathsf{J}(f) \subset \operatorname{supp} \mu_f$.

9. Proofs of Theorems 3 and 4

Let k be a product formula field. The proof of the following is based not only on (PF) but also on elimination theory (and the strong triangle inequality).

Theorem 9.1 [Baker and Rumely 2006, Lemma 3.1]. *Let k be a product formula field. For every $\phi \in k(z)$ and every lift $F_\phi \in k[p_0, p_1]_{\deg\phi} \times k[p_0, p_1]_{\deg\phi}$ of ϕ, there exists a finite subset E_{F_ϕ} in M_k containing all the infinite places of k such that for every $v \in M_k \setminus E_{F_\phi}$, we have $|\mathrm{Res}\, F_\phi|_v = 1$ and $\|F_\phi(\cdot)\|_v = \|\cdot\|_v^{\deg\phi}$ on $\mathbb{C}_v^2$.*

Let $f \in k(z)$ be a rational function of degree > 1 and $F \in k[p_0, p_1]_d \times k[p_0, p_1]_d$ a lift of f. Then the family $\hat{g}_f = \{g_{f,v} : v \in M_k\}$ is an adelic normalized weight, where $g_{f,v}$ is the dynamical Green function of f on $\mathsf{P}^1(\mathbb{C}_v)$ for every $v \in M_k$. Indeed, letting $g_{F,v}$ be the dynamical Green function of F on $\mathsf{P}^1(\mathbb{C}_v)$ for each $v \in M_k$ and E_F be a finite subset in M_k obtained by Theorem 9.1 applied to F, for every $v \in M_k \setminus E_F$ we have $T_{F^n,v} \equiv 0$ on $\mathsf{P}^1(\mathbb{C}_v)$ for every $n \in \mathbb{N}$, giving $g_{f,v} \equiv g_{F,v} \equiv 0$ on $\mathsf{P}^1(\mathbb{C}_v)$. We call the adelic normalized weight $\hat{g}_f = \{g_{f,v} : v \in M_k\}$ and the adelic probability measure $\hat{\mu}_f := \mu^{\hat{g}_f}$ the *adelic dynamical Green function* of f and the *adelic f-equilibrium (or canonical) measure*, respectively. Here, for every $v \in M_k$, $\mu_{f,v} := \mu^{g_{f,v}} = \mu_v^{\hat{g}_f}$ (as in Section 1) is the f-equilibrium (or canonical) measure on $\mathsf{P}^1(\mathbb{C}_v)$.

Lemma 9.2. *Let k be a product formula field. Let $f, a \in k(z)$ be rational functions and suppose $d := \deg f > 1$. Then the sequence $([f^n = a])$ of k-effective divisors on $\mathbb{P}^1(\bar{k})$ has strictly small $\hat{g}_f$-heights in that*

$$\limsup_{n\to\infty} (d^n + \deg a) \cdot h_{\hat{g}_f}([f^n = a]) < \infty.$$

Proof. Let $F \in k[p_0, p_1]_d \times k[p_0, p_1]_d$ and $A \in k[p_0, p_1]_{\deg a} \times k[p_0, p_1]_{\deg a}$ be lifts of f and a, respectively. Then $F^n \wedge A \in k[p_0, p_1]_{d^n+\deg a} \times k[p_0, p_1]_{d^n+\deg a}$ is a representative of $[f^n = a]$ for every $n \in \mathbb{N}$ such that $f^n \not\equiv a$. Let E_F, E_A be finite subsets in M_k obtained by applying Theorem 9.1 to F, A, respectively, so that for every $v \in M_k \setminus (E_F \cup E_A)$ and every $n \in \mathbb{N}$, we have $T_{F^n,v} \equiv T_{A,v} \equiv 0$ and $g_{F,v} \equiv 0$ on $\mathsf{P}^1(\mathbb{C}_v)$. For every $v \in M_k$ and every sufficiently large $n \in \mathbb{N}$, since $|F^n \wedge A|_v \le \|F^n\|_v \|A\|_v$ on $\mathbb{C}_v^2 \setminus \{0\}$, we have $\log S_{F^n\wedge A,v} \le T_{F^n,v} + T_{A,v}$ on $\mathbb{P}^1(\mathbb{C}_v)$ and in turn on $\mathsf{P}^1(\mathbb{C}_v)$ (recalling that $S_{F^n\wedge A,v} = |(F^n \wedge A)(\cdot/\|\cdot\|_v)|_v$ on $\mathbb{P}^1(\mathbb{C}_v)$), so using also $g_{f,v} \equiv g_{F,v} + V_{g_{F,v}}/2$ on $\mathsf{P}^1(\mathbb{C}_v)$, we obtain

$$\frac{\log S_{F^n\wedge A,v}}{d^n + \deg a} - g_{f,v} \le \frac{T_{F^n,v} + T_{A,v}}{d^n + \deg a} - \Big(g_{F,v} + \frac{1}{2} V_{g_{F,v}}\Big) \quad \text{on } \mathsf{P}^1(\mathbb{C}_v).$$

Hence, by the definition (1-1) of $h_{\hat{g}_f}$, the Jensen-type formula (2-11), the energy formula (8-3) (with $\mathrm{Res}\, F \in k \setminus \{0\}$), and (PF), we have

$$\begin{aligned} h_{\hat{g}_f}([f^n = a]) &\le \sum_{v\in M_k} N_v \int_{\mathsf{P}^1(\mathbb{C}_v)} \Big(\frac{T_{F^n,v}+T_{A,v}}{d^n+\deg a} - g_{F,v}\Big)\, \mathrm{d}\mu_{f,v} - \frac{3}{2} \sum_{v\in M_k} N_v \cdot V_{g_{F,v}} \\ &= \sum_{v\in E_F\cup E_A} N_v \int_{\mathsf{P}^1(\mathbb{C}_v)} \Big(\frac{T_{F^n,v}+T_{A,v}}{d^n+\deg a} - g_{F,v}\Big)\, \mathrm{d}\mu_{f,v} \\ &= O(d^{-n}) \quad \text{as } n \to \infty, \end{aligned}$$

where the final order estimate is by (8-2) and $\#(E_F \cup E_A) < \infty$. □

With the help of Lemma 9.2, Theorems 3 and 4 follow from Theorems 1 and 2, respectively.

We omit the proof of the following characterization of $h_{\hat{g}_f}$, which we will dispense with in this article.

Lemma 9.3. *Let k be a product formula field. Then for every rational function $f \in k(z)$ of degree $d > 1$, the $\hat{g}_f$-height function $h_{\hat{g}_f}$ coincides with the* Call–Silverman f-dynamical (or canonical) height function *in that for every k-effective divisor $\mathcal{Z}$ on $\mathbb{P}^1(\bar{k})$, ($f_*\mathcal{Z}$ is also a k-effective divisor on $\mathbb{P}^1(\bar{k})$, and) the equality $(h_{\hat{g}_f} \circ f_*)(\mathcal{Z}) = (d \cdot h_{\hat{g}_f})(\mathcal{Z})$ holds.*

10. Proofs of Theorems 5 and 6

Let K be an algebraically closed field that is complete with respect to a nontrivial absolute value $|\cdot|$. For subsets $A, B \subset \mathbb{P}^1$, set $[A, B] := \inf_{z \in A, z' \in B}[z, z']$.

Let $f, a \in K(z)$ be rational functions and suppose that $d := \deg f > 1$. Let $N \in \mathbb{N}$ be so large that $f^n \not\equiv a$ if $n > N$. Then $\big(\overline{\bigcup_{n>N} \operatorname{supp}[f^n = a]} \cup \mathsf{J}(f)\big) \cap \mathbb{P}^1$ is closed in $\mathbb{P}^1$.

Lemma 10.1. *Suppose that K has characteristic* 0. *Let D be a chordal disk in $\mathbb{P}^1$ of radius > 0 satisfying $\liminf_{n\to\infty} \sup_D [f^n, a] = 0$. Then*:

(i) $a(D) \subset \mathcal{E}_f$.

(ii) $D \setminus \big(\overline{\bigcup_{n>N} \operatorname{supp}[f^n = a]} \cup \mathsf{J}(f)\big) \neq \varnothing$.

(iii) *There is a chordal disk D' in $\mathbb{P}^1 \setminus \mathsf{J}(f)$ of radius > 0 such that*

$$\liminf_{n\to\infty} [f^n(D'), a(D')] > 0.$$

Proof of (i). Since $\liminf_{n\to\infty} \sup_D [f^n, a] = 0$, there is a sequence (n_j) in $\mathbb{N}$ such that $\lim_{j\to\infty} \sup_D [f^{n_j}, a] = 0$ and $\lim_{j\to\infty}(n_{j+1} - n_j) = \infty$. For every $z \in D$, set $D'' := \{w \in \mathbb{P}^1 : [w, a(z)] \le r\}$ in $a(D)$ for $r > 0$ small enough. Then $\liminf_{j\to\infty} \sup_{D''}[f^{n_{j+1}-n_j}, \mathrm{Id}] \le \limsup_{j\to\infty} \sup_D [f^{n_{j+1}}, f^{n_j}] = 0$, so that $a(z) \in \mathcal{E}_f$. Hence $a(D) \subset \mathcal{E}_f$. □

Proof of (ii). When K is archimedean, let Y be the component of $\mathcal{E}_f$ containing $a(D)$, which is by the first assertion either a Siegel disk or a Herman ring of f. Setting $k_0 := \min\{n \in \mathbb{N} : f^n(Y) = Y\}$, there are a sequence (n_j) and an N in $\mathbb{N}$ with the properties that $f^{n_N}(D) \subset Y$, that $k_0 \mid (n_j - n_N)$ for every $j \ge N$, and that $a = \lim_{j\to\infty} (f^{k_0})^{(n_j - n_N)/k_0} \circ f^{n_N}$ uniformly on D. Then $D \cap \mathsf{J}(f) = \varnothing$. Let $\lambda \in \mathbb{C}$ be the rotation number of Y, so that there exists a holomorphic injection $h : Y \to \mathbb{C}$ such that $h \circ f^{k_0} = \lambda \cdot h$ on Y. Then $|\lambda| = 1$ but λ is not a root of unity (by $d > 1$). Choosing a subsequence of (n_j) if necessary, $\lambda_a := \lim_{j\to\infty} \lambda^{(n_j - n_N)/k_0} \in \mathbb{C}$

exists. For every $n \geq n_N$, if $k_0 \nmid (n-n_N)$, then $D \cap \operatorname{supp}[f^n = a] = \varnothing$, whereas if $k_0 \mid (n-n_N)$, then $h \circ f^n - h \circ a = (\lambda^{(n-n_N)/k_0} - \lambda_a) \cdot (h \circ f^{n_N})$ on D, so $\bigl(D \setminus (h \circ f^{n_N})^{-1}(0)\bigr) \cap \operatorname{supp}[f^n = a] = \varnothing$ if n is large enough.

When K is nonarchimedean, let Y be the component of $\mathcal{E}_f$ containing $a(D)$. Without loss of generality, we assume that $\infty \notin Y$, and then applying Theorem 8.3 to this Y, we obtain $p \in \mathbb{N}$, $k_0 \in \mathbb{N}$, T, and T_* as in the theorem. There are a sequence (n_j) and an N in $\mathbb{N}$ such that $f^{n_N}(D) \subset Y$, $k_0 \mid (n_j - n_N)$ for every $j \geq N$, and $a = \lim_{j\to\infty} (f^{k_0})^{(n_j - n_N)/k_0} \circ f^{n_N}$ uniformly on D. Then $D \cap \mathsf{J}(f) = \varnothing$. Choosing a subsequence of (n_j) if necessary, $\omega_a := \lim_{j\to\infty}(n_j - n_N)/k_0 \in \mathbb{Z}_p$ exists. For every $n \geq n_N$, if $k_0 \nmid (n-n_N)$, then $D \cap \operatorname{supp}[f^n = a] = \varnothing$, whereas if $k_0 \mid (n-n_N)$, then

$$f^n - a = (T^{(n-n_N)/k_0} - T^{\omega_a}) \circ f^{n_N} \tag{10-1}$$

on D. Choose $b \in D \setminus \{\infty\}$ and $r \in |K^*|$ small enough that the (K-closed) disk $B = \{z \in K : |z-b| \leq r\}$ is contained in D, and fix $\epsilon \in |K^*|$ so small that for $Z_\epsilon := \bigcup_{w \in B \cap (T_* \circ T^{\omega_a} \circ f^{n_N})^{-1}(0)} \{z \in B : |z-w| < \epsilon\}$, we have $B \setminus Z_\epsilon \neq \varnothing$. The maximum modulus principle from rigid analysis (see [Bosch, Güntzer, and Remmert 1984, §6.2.1, §7.3.4]) gives $\min_{z \in f^{n_N}(B \setminus Z_\epsilon)} |T_* \circ T^{\omega_a}(z)| > 0$, so that by the uniform convergence (8-4) and the equality (10-1), $(B \setminus Z_\epsilon) \cap \operatorname{supp}[f^n = a] = \varnothing$ if n is large enough. □

Proof of (iii). By the first assertion, there is a unique singular domain U of f containing $a(D)$. Fix $n_0 \in \mathbb{N}$ such that $f^{n_0}(U) = U$, and set $\mathcal{C} := \bigcup_{j=0}^{n_0-1} f^j(U)$. Then there is a component V of $f^{-1}(\mathcal{C}) \setminus \mathcal{C}$ since $f : \mathcal{C} \to \mathcal{C}$ is injective and $d > 1$. Fix a chordal disk D'' of radius > 0 in $a^{-1}(V) \cap (\mathbb{P}^1 \setminus \mathsf{J}(f))$, so that $a(D'') \subset V \subset f^{-1}(\mathcal{C}) \setminus \mathcal{C}$. If $a(D'') \cap \bigcup_{n \in \mathbb{N}\cup\{0\}} f^n(D'') = \varnothing$, then we are done by setting $D' = \{z \in \mathbb{P}^1 : [z,b] \leq r\}$ for some $b \in D''$ and $r > 0$ small enough. But if there is $N \in \mathbb{N} \cup \{0\}$ such that $a(D'') \cap f^N(D'') \neq \varnothing$, then by setting $D' := \{z \in \mathbb{P}^1 : [z,b] \leq r\}$ for some $b \in D'' \cap f^{-N}(a(D''))$ and $r > 0$ small enough, we get $\liminf_{n\to\infty}[a(D'), f^n(D')] > 0$ from

$$a(D') \cap \bigcup_{n \geq N+1} f^n(D') \subset a(D'') \cap \bigcup_{n \in \mathbb{N}} f^n(a(D'')) \subset V \cap \mathcal{C} = \varnothing. \qquad \square$$

Lemma 10.2. *For every* $w_0 \in \mathbb{P}^1 \setminus \bigl(\overline{\bigcup_{n>N} \operatorname{supp}[f^n = a]} \cup \mathsf{J}(f)\bigr)$, *there is a function* $\phi_0 \in C^1(\mathsf{P}^1)$ *such that* $\phi_0 \equiv \log[w_0, \cdot]_{\mathrm{can}}$ *on* $\bigcup_{n>N} \operatorname{supp}[f^n = a] \cup \mathsf{J}(f)$.

Proof. Fix $w_0 \in \mathbb{P}^1 \setminus \bigl(\overline{\bigcup_{n>N} \operatorname{supp}[f^n = a]} \cup \mathsf{J}(f)\bigr)$. Without loss of generality, we can assume that $w_0 \neq \infty$, and fix $\epsilon > 0$ so small that

$$\bigl\{\mathcal{S} \in \mathsf{P}^1 : |\mathcal{S} - w_0|_\infty \leq \epsilon\bigr\} \subset \mathsf{P}^1 \setminus \Bigl(\overline{\bigcup_{n>N} \operatorname{supp}[f^n = a]} \cup \mathsf{J}(f)\Bigr)$$

(recall Sections 3.1 and 3.2 here).

When K is nonarchimedean, by the definition of the map $\pi_\epsilon : \mathsf{A}^1 \to \mathsf{A}^1$, we have $\{\mathcal{S} \in \mathsf{P}^1 : \mathcal{S} \preceq \pi_\epsilon(w_0)\} = \{\mathcal{S} \in \mathsf{P}^1 : |\mathcal{S} - w_0|_\infty \leq \epsilon\}$. The function

$$\mathcal{S} \mapsto \phi_0(\mathcal{S}) := \begin{cases} \log [w_0, \pi_\epsilon(w_0)]_{\mathrm{can}} & \text{if } \mathcal{S} \preceq \pi_\epsilon(w_0), \\ \log [w_0, \mathcal{S}]_{\mathrm{can}} & \text{otherwise} \end{cases} \quad \text{on } \mathsf{P}^1$$

is in $C^1(\mathsf{P}^1)$ since it is continuous on P^1, locally constant on P^1 except for the segment $\mathcal{I}$ in H^1 joining $\pi_\epsilon(w_0)$ and $\mathcal{S}_{\mathrm{can}}$, and linear on $\mathcal{I}$ with respect to the length parameter induced by the hyperbolic metric ρ on H^1. When K is archimedean (so $\mathsf{P}^1 \cong \mathbb{P}^1$), there is a function $\phi_0 \in C^1(\mathbb{P}^1)$ satisfying

$$z \mapsto \phi_0(z) = \begin{cases} \int_{\mathbb{P}^1} \log [w_0, w] \, \mathrm{d}[z]_{\epsilon/2}(w) & \text{if } |z - w_0| \leq \epsilon/2, \\ \log [w_0, z] & \text{if } |z - w_0| \geq \epsilon \text{ or } z = \infty. \end{cases}$$

In both cases, the given $\phi_0 \in C^1(\mathsf{P}^1)$ satisfies the desired property. □

Fact 10.3. For rational functions $\phi, \psi \in K(z)$, the *chordal proximity function*

$$\mathcal{S} \mapsto [\phi, \psi]_{\mathrm{can}}(\mathcal{S}) \quad \text{on } \mathsf{P}^1$$

between ϕ and ψ is the unique continuous extension of the function $z \mapsto [\phi(z), \psi(z)]$ on $\mathbb{P}^1$ to P^1 (see [Okuyama 2013a, Proposition 2.9] for its construction, as well as Remark 2.10 of the same paper), and for every continuous weight g on P^1, we also define its weighted version by $\Phi(\phi, \psi)_g := \log [\phi, \psi]_{\mathrm{can}} - g \circ \phi - g \circ \psi$ on P^1.

For every $n \in \mathbb{N}$ such that $f^n \not\equiv a$, recall the *Riesz decomposition*

$$\Phi(f^n, a)_{g_f} = U_{g_f, [f^n = a] - (d^n + \deg a)\mu_f} - U_{g_f, a^*\mu_f} + \int_{\mathsf{P}^1} \Phi(f^n, a)_{g_f} \, \mathrm{d}\mu_f \tag{10-2}$$

on P^1, and also $U_{g_f, a^*\mu_f} = g_f \circ a + U_{g_f, a^*\Omega_{\mathrm{can}}} - \int_{\mathsf{P}^1} (g_f \circ a) \, \mathrm{d}\mu_f$ on P^1 [Okuyama 2013a, Lemma 2.19].

Proof of Theorem 5. Let k be a product formula field of characteristic 0. Let $f \in k(z)$ be a rational function of degree $d > 1$ and $a \in k(z)$ a rational function of degree > 0. Let $N \in \mathbb{N}$ be so large that $f^n \not\equiv a$ if $n > N$. Fix $v \in M_k$. Let D be a chordal disk in $\mathbb{P}^1(\mathbb{C}_v)$ of radius > 0, and assume that $\liminf_{n\to\infty} \sup_D [f^n, a]_v = 0$; otherwise we are done. By Lemma 10.1, there are not only a point $w_0 \in D \setminus \left(\overline{\bigcup_{n>N}[f^n = a]} \cup \mathsf{J}(f)_v\right)$ but also a chordal disk D' in $\mathbb{P}^1(\mathbb{C}_v) \setminus \mathsf{J}(f)_v$ of radius > 0 such that $\liminf_{n\to\infty} [f^n(D'), a(D')]_v > 0$. Fix a point $w_1 \in D'$. Then also $w_1 \in \mathbb{P}^1 \setminus \left(\overline{\bigcup_{n>N}[f^n = a]} \cup \mathsf{J}(f)_v\right)$.

For every $n \in \mathbb{N}$ large enough and every $j \in \{0, 1\}$, by (10-2),

$$\begin{aligned} &\log [f^n(w_j), a(w_j)]_v - g_{f,v}(f^n(w_j)) - g_{f,v}(a(w_j)) \\ &= U_{g_{f,v}, [f^n = a] - (d^n + \deg a)\mu_{f,v}}(w_j) - U_{g_{f,v}, a^*\mu_{f,v}}(w_j) + \int_{\mathsf{P}^1(\mathbb{C}_v)} \Phi(f^n, a)_{g_{f,v}} \, \mathrm{d}\mu_{f,v}, \end{aligned} \tag{10-3}$$

so that taking the difference of both sides in (10-3) for each $j \in \{0, 1\}$ and noting that $g_{f,v}$ and $U_{g_{f,v}, a^*\mu_{f,v}}$ are bounded on $\mathsf{P}^1(\mathbb{C}_v)$, we have

$$\begin{aligned}\log\,[f^n(w_0), a(w_0)]_v - \log\,[f^n(w_1), a(w_1)]_v \\ = \int_{\mathsf{P}^1(\mathbb{C}_v)} \log\,[w_0, \mathcal{S}']_{\mathrm{can},v}\,\mathrm{d}([f^n = a] - (d^n + \deg a)\mu_f)(\mathcal{S}') \\ - \int_{\mathsf{P}^1(\mathbb{C}_v)} \log\,[w_1, \mathcal{S}']_{\mathrm{can},v}\,\mathrm{d}([f^n = a] - (d^n + \deg a)\mu_f)(\mathcal{S}') + O(1)\end{aligned}$$

as $n \to \infty$. In the left hand side, by the choice of w_0 and w_1, we have

$$\log \sup_D [f^n, a]_v \geq \log\,[f^n(w_0), a(w_0)]_v$$

and

$$\liminf_{n\to\infty} \log\,[f^n(w_1), a(w_1)]_v \geq \liminf_{n\to\infty} \log\,[f^n(D'), a(D')]_v > -\infty,$$

so that as $n \to \infty$,

$$\log \sup_D [f^n, a]_v + O(1) \geq \log\,[f^n(w_0), a(w_0)]_v - \log\,[f^n(w_1), a(w_1)]_v.$$

In the right hand side, for each $j \in \{0, 1\}$, by Lemma 10.2 applied to w_j, the inclusion $\operatorname{supp} \mu_f \subset \mathsf{J}(f)$, and Theorem 3 (and $k_s = \bar{k}$ in the characteristic 0 case), we have

$$\begin{aligned}\int_{\mathsf{P}^1(\mathbb{C}_v)} \log\,[w_j, \mathcal{S}']_{\mathrm{can},v}\,\mathrm{d}\big([f^n = a] - (d^n + \deg a)\mu_f\big)(\mathcal{S}') \\ = O\Big(\sqrt{n \cdot \big([f^n = a] \times [f^n = a]\big)\big(\mathrm{diag}_{\mathbb{P}^1(\bar{k})}\big)}\Big) \quad \text{as } n \to \infty.\end{aligned}$$

These estimates complete the proof of (1-4) for this $v \in M_k$. □

Fact 10.4. For a rational function $f(z) \in k(z)$ over a field k, a point $w \in \mathbb{P}^1(\bar{k})$ is called a *multiple* periodic point of f if $[f^n = \mathrm{Id}](\{w\}) > 1$ for some $n \in \mathbb{N}$. For a rational function $f(z) \in k(z)$ over a field k of characteristic 0, there are *at most finitely many* multiple periodic points of f in $\mathbb{P}^1(\bar{k})$; this is well known in the case that $k = \mathbb{C}$ (see, e.g., [Milnor 2006, §13]), and holds in general *by the Lefschetz principle* (see, e.g., [Eklof 1973]).

Proof of Theorem 6. As noted above, f has at most finitely many multiple periodic points in $\mathbb{P}^1(\bar{k})$, and for every multiple periodic point w of f, setting $p = p_w := \min\{n \in \mathbb{N} : [f^n = \mathrm{Id}](\{w\}) > 1\}$, by the (formal) power series expansion $f^p(z) = w + (z - w) + C(z - w)^{[f^p = \mathrm{Id}](\{w\})} + \cdots$ of f^p around w, we also have $\sup_{n\in\mathbb{N}}[f^n = \mathrm{Id}](\{w\}) \leq [f^p = \mathrm{Id}](\{w\})$ under the characteristic 0 assumption.

Hence $\sup_{n\in\mathbb{N}}\big(\sup_{w\in\operatorname{supp}[f^n=\mathrm{Id}]}[f^n=\mathrm{Id}](\{w\})\big)<\infty$, so that

$$\big([f^n=\mathrm{Id}]\times[f^n=\mathrm{Id}]\big)\big(\mathrm{diag}_{\mathbb{P}^1(\bar{k})}\big)\le(d^n+1)\cdot\sup_{w\in\operatorname{supp}[f^n=\mathrm{Id}]}[f^n=\mathrm{Id}](\{w\})=O(d^n)$$

as $n\to\infty$. Now (1-5) follows from (1-4). □

11. Proof of Theorem 7

Let k be a field and k_s the separable closure of k in $\bar{k}$. Let $p(z)\in k[z]$ be a polynomial of degree >0 and $\{z_1,\dots,z_m\}$ the set of all distinct zeros of $p(z)$ in $\bar{k}$ so that $p(z)=a\cdot\prod_{j=1}^m(z-z_j)^{d_j}$ in $\bar{k}[z]$ for some $a\in k\setminus\{0\}$ and some sequence $(d_j)_{j=1}^m$ in $\mathbb{N}$. For a while, we do not assume $\{z_1,\dots,z_m\}\subset k_s$. Let $\{p_1(z),p_2(z),\dots,p_N(z)\}$ be the set of all mutually distinct, nonconstant, irreducible, and monic factors of $p(z)$ in $k[z]$, so that $p(z)=a\cdot\prod_{\ell=1}^N p_\ell(z)^{s_\ell}$ in $k[z]$ for some sequence $(s_\ell)_{\ell=1}^N$ in $\mathbb{N}$. For every $\ell\in\{1,2,\dots,N\}$, by the irreducibility of $p_\ell(z)$ in $k[z]$, $p_\ell(z)$ is the unique monic minimal polynomial in $k[z]$ of each zero of $p_\ell(z)$ in $\bar{k}$, so $p_\ell(z)$ and $p_n(z)$ have no common zeros in $\bar{k}$ if $\ell\ne n$. Hence for each $j\in\{1,2,\dots,m\}$, there is a unique $\ell=:\ell(j)\in\{1,2,\dots,N\}$ such that $p_\ell(z_j)=0$.

Now suppose that $\{z_1,z_2,\dots,z_m\}\subset k_s$. Then for every $\ell\in\{1,2,\dots,N\}$, $p_\ell(z)=\prod_{i:\ell(i)=\ell}(z-z_i)$ in $\bar{k}[z]$, so that

$$d_i=s_{\ell(i)} \tag{11-1}$$

for every $i\in\{1,2,\dots,m\}$. For every distinct $\ell,n\in\{1,2,\dots,N\}$,

$$\prod_{j:\ell(j)=\ell}\ \prod_{i:\ell(i)=n}(z_j-z_i)=\prod_{j:\ell(j)=\ell}p_n(z_j)=R(p_\ell,p_n), \tag{11-2}$$

where $R(p,q)\in k$ is the (usual) resultant of $p(z),q(z)\in k[z]$. The derivation $p'_\ell(z)$ of $p_\ell(z)$ in $k[z]$ satisfies

$$p'_\ell(z)=\sum_{h:\ell(h)=\ell}\Bigg(\prod_{\substack{i:i\ne h,\\ \ell(i)=\ell}}(z-z_i)\Bigg)$$

in $\bar{k}[z]$. Hence for every $\ell\in\{1,2,\dots,N\}$,

$$\prod_{j:\ell(j)=\ell}\ \prod_{\substack{i:i\ne j,\\ \ell(i)=\ell}}(z_j-z_i)=\prod_{j:\ell(j)=\ell}p'_\ell(z_j)=R(p_\ell,p'_\ell). \tag{11-3}$$

By (11-1), (11-3), and (11-2), we have

$$\begin{aligned} D^*(p) &:= \prod_{j=1}^{m} \prod_{i:i\neq j} (z_j - z_i)^{d_i d_j} = \prod_{j=1}^{m} \prod_{i:i\neq j} (z_j - z_i)^{s_{\ell(i)} s_{\ell(j)}} \\ &= \prod_{\ell=1}^{N} \biggl(\prod_{j:\ell(j)=\ell} \biggl(\Bigl(\prod_{\substack{i:i\neq j,\\ \ell(i)=\ell}} (z_j - z_i)^{s_\ell^2} \Bigr) \Bigl(\prod_{n:n\neq \ell} \prod_{i:\ell(i)=n} (z_j - z_i)^{s_n s_\ell} \Bigr) \biggr) \biggr) \\ &= \prod_{\ell=1}^{N} \Bigl(R(p_\ell, p'_\ell)^{s_\ell^2} \cdot \prod_{n:n\neq \ell} R(p_\ell, p_n)^{s_n s_\ell} \Bigr), \end{aligned}$$

which is in $k \setminus \{0\}$. Now the proof is complete. □

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