EFFECTIVE DIVISORS ON THE PROJECTIVE LINE HAVING SMALL DIAGONALS AND SMALL HEIGHTS AND THEIR APPLICATION TO ADELIC DYNAMICS

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We establish a quantitative adelic equidistribution theorem for a sequence of effective divisors on the projective line over the separable closure of a product formula field having small diagonals and small $g$-heights with respect to an adelic normalized weight $g$ in arbitrary characteristic and in a possibly nonseparable setting. Applying this quantitative adelic equidistribution result to adelic dynamics of $f$, we obtain local proximity estimates between the iterations of a rational function $f \in k(z)$ of degree $> 1$ and a rational function $a \in k(z)$ of degree $> 0$ over a product formula field $k$ of characteristic $0$.

1. Introduction

Let $k$ be a field and denote by $k_s$ the separable closure of $k$ in an algebraic closure $\overline{k}$. For every $d \in \mathbb{N} \cup \{0\}$, let $k[p_0, p_1]_d$ be the set of all homogeneous polynomials in two variables over $k$ of degree $d$. A $k$-effective divisor $Z$ on $\mathbb{P}^1(\overline{k})$ is a divisor on $\mathbb{P}^1(\overline{k})$ defined by the zeros in $\mathbb{P}^1(\overline{k})$ of some $P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1]_d$ taking into account their multiplicities, and is said to be on $\mathbb{P}^1(k_s)$ if $\text{supp} \ Z \subset \mathbb{P}^1(k_s)$. The defining polynomial $P(p_0, p_1)$ of $Z$ is unique up to multiplication in $k^* (= k \setminus \{0\})$, and is called a representative of $Z$. Effective divisors include Galois conjugacy classes of algebraic numbers, and are also called Galois stable multisets in $\mathbb{P}^1(\overline{k})$.

Our first aim in this article is to establish a quantitative adelic equidistribution of sequences of $k$-effective divisors on $\mathbb{P}^1(k_s)$, where $k$ is a product formula field, having not only small $g$-heights (with respect to an adelic normalized weight $g$) but also small diagonals in arbitrary characteristic and in a possibly nonseparable setting. Secondly, we contribute to the study of the local proximities between the iterations of a rational function $f \in k(z)$ of degree $> 1$ and a rational function $a \in k(z)$ of degree $> 0$ on a chordal disk $D$ of radius $> 0$ in the projective line $\mathbb{P}^1(\mathbb{C}_v)$ for each place $v$ of $k$, in the setting of adelic dynamics of characteristic $0$.

Keywords: product formula field, effective divisor, small diagonals, small heights, quantitative equidistribution, asymptotically Fekete configuration, local proximity sequence, adelic dynamics.
1.1. **Arithmetic over a product formula field.** A field \( k \) is a **product formula field** if \( k \) is equipped with

(i) a set \( M_k \) of all places of \( k \), which are either **finite** or **infinite**, 
(ii) a set \( \{ | \cdot |_v : v \in M_k \} \), where for each \( v \in M_k \), \( | \cdot |_v \) is a nontrivial absolute value of \( k \) representing \( v \) (and then by definition \( | \cdot |_v \) is nonarchimedean if and only if \( v \) is finite), and 
(iii) a set \( \{ N_v : v \in M_k \} \), where \( N_v \in \mathbb{N} \) for every \( v \in M_k \) such that the following **product formula** holds: if \( z \in k \setminus \{0\} \) then we have \( |z|_v \neq 1 \) for at most finitely many \( v \in M_k \) and moreover

\[
\prod_{v \in M_k} |z|_v^{N_v} = 1.
\]

Product formula fields include number fields and function fields over curves, and a product formula field is a number field if and only if it has at least one infinite place (see, e.g., the paragraph after Definition 7.51 of [Baker and Rumely 2010]).

Let \( k \) be a product formula field. For each \( v \in M_k \), let \( k_v \) be the completion of \( k \) with respect to \( | \cdot |_v \) and \( \mathbb{C}_v \) the completion of an algebraic closure of \( k_v \) with respect to (the extended) \( | \cdot |_v \). We fix an embedding of \( k \) into \( \mathbb{C}_v \) which extends that of \( k \) into \( k_v \); by convention, the dependence on \( v \in M_k \) of a local quantity induced by \( | \cdot |_v \) is emphasized by adding the suffix \( v \) to it. A family \( g = \{ g_v : v \in M_k \} \) is an **adelic continuous weight** if

(i) for every \( v \in M_k \), \( g_v \) is a continuous function on the Berkovich projective line \( \mathbb{P}^1(\mathbb{C}_v) \) such that

\[
\mu_v^g := \Delta g_v + \Omega_{\text{can},v}
\]

is a probability Radon measure on \( \mathbb{P}^1(\mathbb{C}_v) \) (see (2-2) for the definition of the probability Radon measure \( \Omega_{\text{can},v} \) on \( \mathbb{P}^1(\mathbb{C}_v) \), and (2-3) for the normalization of the Laplacian \( \Delta \) on \( \mathbb{P}^1(\mathbb{C}_v) \)), and 
(ii) there is a finite subset \( E_g \) in \( M_k \) such that \( g_v \equiv 0 \) on \( \mathbb{P}^1(\mathbb{C}_v) \) for all \( v \in M_k \setminus E_g \).

Moreover, \( g \) is called an **adelic normalized weight** if, in addition,

(iii) the \( g_v \)-**equilibrium energy** \( V_{g_v} \) of \( \mathbb{P}^1(\mathbb{C}_v) \) vanishes for every \( v \in M_k \) (see Section 2.1 for the definition of \( V_{g_v} \)).

For an adelic continuous weight \( g = \{ g_v : v \in M_k \} \), the family \( \mu^g := \{ \mu_v^g : v \in M_k \} \) is called an **adelic probability measure** (compare [Favre and Rivera-Letelier 2006, Définition 1.1]). An adelic continuous weight \( g = \{ g_v : v \in M_k \} \) is said to be **placewise Hölder continuous** if for every \( v \in M_k \), \( g_v \) is Hölder continuous on \( \mathbb{P}^1(\mathbb{C}_v) \) with respect to the small model metric \( d_v \) on \( \mathbb{P}^1(\mathbb{C}_v) \) (see (3-1) for the definition of \( d_v \)).
Given $P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1]_d$ and an adelic continuous weight $g = \{g_v : v \in M_k\}$, the $g$-height of a $k$-effective divisor $Z$ on $\mathbb{P}^1(\bar{k})$ represented by $P$ is

\[
h_g(Z) := \sum_{v \in M_k} N_v \frac{M_{g_v}(P)}{\deg P},
\]

where, for every $v \in M_k$, $M_{g_v}(P)$ is the logarithmic $g_v$-Mahler measure of $P$ (see (2-10) for the definition of $M_{g_v}(P)$ and Section 2.3 for a proof that $h_g(Z) \in \mathbb{R}$); by (PF), $h_g(Z)$ is well defined. For every $v \in M_k$, letting $\delta_S$ be the Dirac measure on $\mathbb{P}^1(\mathbb{C}_v)$ at a point $S \in \mathbb{P}^1(\mathbb{C}_v)$, a $k$-effective divisor $Z$ on $\mathbb{P}^1(\bar{k})$ is regarded as a positive and discrete Radon measure $\sum_{w \in \text{supp} Z} (\text{ord}_w Z) \delta_w$ on $\mathbb{P}^1(\mathbb{C}_v)$, still denoted by $Z$. Then the diagonal

\[
(Z \times Z)(\text{diag}_{\mathbb{P}^1(\bar{k})}) = \sum_{w \in \text{supp} Z} (\text{ord}_w Z)^2
\]

of $Z$ is independent of $v \in M_k$. For a sequence $(Z_n)$ of $k$-effective divisors on $\mathbb{P}^1(\bar{k})$ satisfying $\lim_{n \to \infty} \deg Z_n = \infty$, we say $(Z_n)$ has small $g$-heights with respect to an adelic normalized weight $g$ if $\limsup_{n \to \infty} h_g(Z_n) \leq 0$, and we say $(Z_n)$ has small diagonals if $\lim_{n \to \infty} ((Z_n \times Z_n)(\text{diag}_{\mathbb{P}^1(\bar{k})}))/((\deg Z_n)^2 = 0$.

1.2. Quantitative adelic equidistribution of effective divisors. The following is one of our main results; for the Galois conjugacy class of an algebraic number, this was due to Favre and Rivera-Letelier [2006, Théorème 7]. For the definitions of the $C^1$-regularity of a continuous test function $\phi$ on $\mathbb{P}^1(\mathbb{C}_v)$, the Lipschitz constant $\text{Lip}(\phi)_v$ on $(\mathbb{P}^1(\mathbb{C}_v), d_v)$, and the Dirichlet norm $\langle \phi, \phi \rangle_v$ of $\phi$ for each $v \in M_k$, see Section 7.

**Theorem 1.** Let $k$ be a product formula field and $k_s$ the separable closure of $k$ in $\bar{k}$. Let $g = \{g_v : v \in M_k\}$ be a placewise Hölder continuous adelic normalized weight. Then for every $v \in M_k$, there is $C > 0$ such that for every $k$-effective divisor $Z$ on $\mathbb{P}^1(k_s)$ and every test function $\phi \in C^1(\mathbb{P}^1(\mathbb{C}_v))$,

\[
(1-2) \left| \int_{\mathbb{P}^1(\mathbb{C}_v)} \phi \, d\left(\frac{Z}{\deg Z} - \mu_v^g\right) \right| \leq C \cdot \max\{\text{Lip}(\phi)_v, \langle \phi, \phi \rangle_v^{1/2}\} \sqrt{\max\left\{h_g(Z), (\log \deg Z) \frac{(Z \times Z)(\text{diag}_{\mathbb{P}^1(\bar{k})})}{(\deg Z)^2}\right\}}.
\]

In Theorem 1, if $v \in M_k$ is an infinite place, or equivalently, $\mathbb{C}_v \cong \mathbb{C}$, then the estimate (1-2) gives a quantitative estimate of the Kantorovich–Wasserstein metric

\[
W\left(\frac{Z}{\deg Z}, \mu_v^g\right) = \sup_{\phi} \left| \int_{\mathbb{P}^1(\mathbb{C})} \phi \, d\left(\frac{Z}{\deg Z} - \mu_v^g\right) \right|
\]

between the probability Radon measures $Z/\deg Z$ and $\mu_v^g$ on $\mathbb{P}^1(\mathbb{C}_v) \cong \mathbb{P}^1(\mathbb{C})$, where $\phi$ ranges over all Lipschitz continuous functions on $\mathbb{P}^1(\mathbb{C})$ whose Lipschitz
constants equal 1 with respect to the normalized chordal metric \([z, w]\) on \(\mathbb{P}^1(\mathbb{C})\) (see Remark 4.2). For the details of the metric \(W\) including its role in the optimal transportation problems, see, e.g., [Villani 2009].

The next theorem is a qualitative version of Theorem 1. For a sequence of Galois conjugacy classes of algebraic numbers, this was due to Baker and Rumely [2006, Theorem 2.3], Chambert-Loir [2006, Théorème 4.2], and Favre and Rivera-Letelier [2006, Théorème 2]; see also [Szpiro, Ullmo, and Zhang 1997; Bilu 1997; Rumely 1999; Chambert-Loir 2000; Autissier 2001; Baker and Hsia 2005; Baker and Rumely 2006; Chambert-Loir 2006; Favre and Rivera-Letelier 2006], and, most recently, [Yuan 2008] on big line bundles over arithmetic varieties.

**Theorem 2** (asymptotically Fekete configuration of effective divisors). Let \(k\) be a product formula field and \(k_s\) its separable closure in \(\overline{k}\). Let \(g = \{g_v : v \in M_k\}\) be an adelic normalized weight. If a sequence \((\mathbb{Z}_n)\) of \(k\)-effective divisors on \(\mathbb{P}^1(k_s)\) satisfying \(\lim_{n \to \infty} \deg \mathbb{Z}_n = \infty\) has both small diagonals and small \(g\)-heights, then for every \(v \in M_k\), \((\mathbb{Z}_n)\) is an asymptotically \(g_v\)-Fekete configuration on \(\mathbb{P}^1(\mathbb{C}_v)\). In particular, \(\lim_{n \to \infty} \mathbb{Z}_n / \deg \mathbb{Z}_n = \mu_v^g\) weakly on \(\mathbb{P}^1(\mathbb{C}_v)\).

In Theorem 2, the assertion that \((\mathbb{Z}_n)\) is an asymptotically \(g_v\)-Fekete configuration on \(\mathbb{P}^1(\mathbb{C}_v)\) (see (2-7) for the definition), which is also called a \(g_v\)-pseudoequidistribution on \(\mathbb{P}^1(\mathbb{C}_v)\), is stronger than the final equidistribution assertion. For a relationship between the Kantorovich–Wasserstein metric \(W\) and (asymptotically) Fekete configurations on complex manifolds, see [Lev and Ortega-Cerdà 2012, §7]. For a recent result on the capacity and the transfinite diameter on complex manifolds, see [Berman and Boucksom 2010] (on \(\mathbb{C}^n\), we also refer to the survey [Levenberg 2010]); for the convergence of (asymptotically) Fekete points on complex manifolds, see [Berman, Boucksom, and Nyström 2011].

1.3. **Quantitative equidistribution in adelic dynamics.** For rational functions \(f, a\) over a field \(k\) and for \(n \in \mathbb{N}\), the divisor \([f^n = a]\) defined by the roots of the equation \(f^n = a\) in \(\mathbb{P}^1(\overline{k})\) is a \(k\)-effective divisor on \(\mathbb{P}^1(\overline{k})\) if \(f^n \neq a\).

Let \(k\) be a product formula field. For a rational function \(f \in k(z)\) of degree \(d > 1\), let \(\hat{g}_f := \{g_{f,v} : v \in M_k\}\) be the adelic dynamical Green function in the sense that for every \(v \in M_k\), \(g_{f,v}\) is the dynamical Green function of \(f\) on \(\mathbb{P}^1(\mathbb{C}_v)\), so that \(\mu_{f,v} := \mu^{g_{f,v}}\) is the \(f\)-equilibrium (or canonical) measure on \(\mathbb{P}^1(\mathbb{C}_v)\) (see Section 9 for details). The family \(\hat{g}_f\) is in fact an adelic normalized weight, and the \(\hat{g}_f\)-height function \(h_{\hat{g}_f}\) coincides with the Call–Silverman \(f\)-dynamical (or canonical) height function. For every rational function \(a \in k(z)\), the sequence \([f^n = a]\) has strictly small \(\hat{g}_f\)-heights in that \(\limsup_{n \to \infty} (d^n + \deg a) \cdot h_{\hat{g}_f}([f^n = a]) < \infty\) (Lemma 9.2). Hence the following are consequences of Theorems 1 and 2, respectively.

**Theorem 3.** Let \(k\) be a product formula field and \(k_s\) its separable closure in \(\overline{k}\). Let \(f \in k(z)\) be a rational function of degree \(d > 1\) and \(a \in k(z)\) a rational function.
Then for every \( v \in M_k \), there exists a constant \( C > 0 \) such that for every test function \( \phi \in C^1(\mathbb{P}^1(\mathbb{C}_v)) \) and every \( n \in \mathbb{N} \),

\[
\left| \int_{\mathbb{P}^1(\mathbb{C}_v)} \phi \left( \frac{[f^n = a]}{d^n + \deg a} - \mu_{f,v} \right) \right| \leq C \cdot \max \{ \text{Lip}(\phi)_v, \langle \phi, \phi \rangle_v^{1/2} \} \sqrt{n \cdot ([f^n = a] \times [f^n = a]) \text{diag}_{\mathbb{P}^1(k_v)}}
\]

if \( f^n \not\equiv a \) and the divisor \([f^n = a]\) on \( \mathbb{P}^1(\mathbb{C}_v) \) is on \( \mathbb{P}^1(k_v) \).

**Theorem 4.** Let \( k \) be a product formula field and \( k_s \) its separable closure in \( \overline{k} \). Let \( f \in k(z) \) be a rational function of degree \( d > 1 \) and \( a \in k(z) \) a rational function. If the sequence \([f^n = a]\) has small diagonals and the divisor \([f^n = a]\) is on \( \mathbb{P}^1(k_s) \) for every sufficiently large \( n \in \mathbb{N} \), then for every \( v \in M_k \), \([f^n = a]\) is an asymptotically \( g_{f,v}\)-Fekete configuration on \( \mathbb{P}^1(\mathbb{C}_v) \). In particular,

\[
\lim_{n \to \infty} \frac{[f^n = a]}{d^n + \deg a} = \mu_{f,v}
\]

weakly on \( \mathbb{P}^1(\mathbb{C}_v) \).

The final equidistribution assertion in Theorem 4 has been established in [Brolin 1965; Ljubitsch 1983; Freire, Lopes, and Mañé 1983] in complex dynamics, and in [Favre and Rivera-Letelier 2010] in (not necessarily adelic) nonarchimedean dynamics (of characteristic 0 when \( \deg a > 0 \)). For every constant \( a \in \mathbb{P}^1(k) \), the estimate (1-3) in Theorem 3 has been obtained in [Okuyama 2013b, Theorems 4 and 5] in complex and (not necessarily adelic) nonarchimedean dynamics of characteristic 0. In complex dynamics, for every \( f \in \mathbb{C}(z) \) of degree \( d > 1 \), every constant \( a \in \mathbb{P}^1(\mathbb{C}) \), and every \( \phi \in C^2(\mathbb{P}^1(\mathbb{C})) \), a finer estimate than (1-3) has been obtained in [Drasin and Okuyama 2007, Theorem 2 and (4.2)].

### 1.4. Application to a motivating question.

Let \( K \) be an algebraically closed field that is complete with respect to a nontrivial absolute value \(|\cdot|\), and \([z, w]\) be the normalized chordal metric on \( \mathbb{P}^1 = \mathbb{P}^1(K) \) (see (2-1)). A subset \( D \) in \( \mathbb{P}^1 \) is called a chordal disk (in \( \mathbb{P}^1 \)) if \( D = \{ z \in \mathbb{P}^1 : [z, w] \leq r \} \) for some \( w \in \mathbb{P}^1 \) and some radius \( r \geq 0 \). Even in the specific case \( a = \text{Id} \) (see, e.g., [Cremer 1928; Siegel 1942; Brjuno 1971; 1972; Herman and Yoccoz 1983; Yoccoz 1988; 1995; Pérez-Marco 1993; 2001]), which is one of the most interesting cases and is related to the difficulty of small denominators in nonarchimedean and complex dynamics, the following question has not been completely understood.

**Question.** How uniformly close on a chordal disk \( D \) of radius \( > 0 \) can the sequence \((f^n)\) of the iterations of a rational function \( f \in K(z) \) of degree \( > 1 \) be to a rational function \( a \in K(z) \) of degree \( > 0 \)?
For a study of this question on the projective space $\mathbb{P}^N(K)$, see [Okuyama 2010]. The following estimate of the local proximity sequence (sup$_D$[$f^n, a$]) is an application of Theorem 3 to this question in the setting of adelic dynamics.

**Theorem 5.** Let $k$ be a product formula field of characteristic $0$. Let $f \in k(z)$ be a rational function of degree $> 1$ and $a \in k(z)$ a rational function of degree $> 0$. Then for every $v \in M_k$ and every chordal disk $D$ in $\mathbb{P}^1(\mathbb{C}_v)$ of radius $> 0$, as $n \to \infty$,

\[ (1-4) \log \sup_D [f^n, a]_v = O\left( \sqrt{n \cdot \left( [f^n = a] \times [f^n = a] \right)(\text{diag}_{\mathbb{P}^1(\bar{k})})} \right). \]

Here, the implicit constant in $O(\cdot)$ possibly depends on $f$ and $a$.

In the case that $a = \text{Id}$, we will see that $([f^n = \text{Id}] \times [f^n = \text{Id}]) = O(d^n)$ as $n \to \infty$ in Section 10. Hence Theorem 5 concludes the following.

**Theorem 6.** Let $k$ be a product formula field of characteristic $0$. Let $f \in k(z)$ be a rational function of degree $d > 1$. Then for every $v \in M_k$ and every chordal disk $D$ in $\mathbb{P}^1(\mathbb{C}_v)$ of radius $> 0$,

\[ (1-5) \log \sup_D [f^n, \text{Id}]_v = O\left( \sqrt{nd^n} \right) \quad \text{as } n \to \infty. \]

**1.5. The unit $D^*(p)$.** The next result generalizes the obvious fact that the discriminant of a polynomial in one variable over a field $k$ is in $k$. The unit $D^*(p)$ plays an important role in the nonseparable case and might have been studied before, but we could find no relevant literature.

**Theorem 7.** Let $k$ be a field and $k_s$ the separable closure of $k$ in an algebraic closure $\bar{k}$ of $k$. For every $p(z) \in k[z]$ of degree $> 0$, let $\{z_1, \ldots, z_m\}$ be the set of all distinct zeros of $p(z)$ in $\bar{k}$ so that $p(z) = a \cdot \prod_{j=1}^m (z - z_j)^{d_j}$ in $\bar{k}[z]$ for some $a \in k \setminus \{0\}$ and some sequence $(d_j)_{j=1}^m$ in $\mathbb{N}$. If $\{z_1, \ldots, z_m\} \subset k_s$, then

\[ D^*(p) := \prod_{j=1}^m \prod_{i:i \neq j} (z_j - z_i)^{d_{i,j}} \in k \setminus \{0\}, \]

where, a priori, this $D^*(p)$ is always in $\bar{k} \setminus \{0\}$.

**1.6. Organization of this article.** In Section 2, we recall background from potential theory and arithmetic on the Berkovich projective line. In Section 3, we extend Favre and Rivera-Letelier’s regularization $[\cdot]_\epsilon$ of discrete Radon measures and establish required estimates on them, and in Section 4 we see the negativity of regularized Fekete sums and a Cauchy–Schwarz inequality. In Sections 5 and 6, we compute the $g$-Fekete sums $(Z, Z)_g$ and estimate the regularized $g$-Fekete sums $(Z_\epsilon, Z_\epsilon)_g$ with respect to a $k$-effective divisor $Z$ on $\mathbb{P}^1(\bar{k})$. In Section 7, we prove Theorems 1 and 2; the arguments are more or less adaptions of those in the
proofs of [Favre and Rivera-Letelier 2006, Théorème 7] and [Baker and Rumely 2010, Theorem 10.24], respectively. In Section 8, we review background from nonarchimedean and complex dynamics. Finally, we prove Theorems 3 and 4 in Section 9, Theorems 5 and 6 in Section 10, and Theorem 7 in Section 11.

2. Background from potential theory and arithmetic

Notation 2.1. For a field $k$, the origin of $k^2$ is also denoted by $0 = 0_k$, and we write $\pi = \pi_k : k^2 \setminus \{0\} \to \mathbb{P}^1 = \mathbb{P}^1(k)$ for the canonical projection, so that $\pi(0, 1) = \infty$ and $\pi(p_0, p_1) = p_1/p_0$ if $p_0 \neq 0$. Set the wedge product $(z_0, z_1) \wedge (w_0, w_1) := z_0 w_1 - z_1 w_0$ on $k^2$.

Let $K$ be an algebraically closed field that is complete with respect to a nontrivial absolute value $| \cdot |$, which is said to be nonarchimedean if the strong triangle inequality $|z + w| \leq \max\{|z|, |w|\}$ holds, and archimedean otherwise. On $K^2$, let $\| (p_0, p_1) \|$ be either the maximal norm $\max\{|p_0|, |p_1|\}$ (for nonarchimedean $K$) or the euclidean norm $\sqrt{|p_0|^2 + |p_1|^2}$ (for archimedean $K$). The **normalized chordal metric** $[z, w]$ on $\mathbb{P}^1 = \mathbb{P}^1(K)$ is the function

$$
(z, w) \mapsto [z, w] = |p \wedge q|/\left(\|p\| \cdot \|q\|\right) \leq 1
$$
on $\mathbb{P}^1 \times \mathbb{P}^1$, where $p \in \pi^{-1}(z), q \in \pi^{-1}(w)$. The metric topology on $\mathbb{P}^1$ with respect to $[z, w]$ agrees with the relative topology on $\mathbb{P}^1$ from the Berkovich projective line $\mathbb{P}^1 = \mathbb{P}^1(K)$, which is a compact augmentation of $\mathbb{P}^1$ containing $\mathbb{P}^1$ as a dense subset, and is isomorphic to $\mathbb{P}^1$ if and only if $K$ is archimedean (see Section 3.2 for more details when $K$ is nonarchimedean). Letting $\delta_S$ be the Dirac measure on $\mathbb{P}^1$ at a point $S \in \mathbb{P}^1$, set

$$
\Omega_{\text{can}} := \begin{cases} 
\delta_{S_{\text{can}}} & \text{for nonarchimedean } K, \\
\omega & \text{for archimedean } K,
\end{cases}
$$
where $S_{\text{can}}$ is the canonical (or Gauss) point in $\mathbb{P}^1$ for nonarchimedean $K$ (see Section 3.2 for the definition), and $\omega$ is the Fubini–Study area element on $\mathbb{P}^1$ normalized as $\omega(\mathbb{P}^1) = 1$ for archimedean $K$. For nonarchimedean $K$, the **generalized Hsia kernel** $[S, S']_{\text{can}}$ on $\mathbb{P}^1$ with respect to $S_{\text{can}}$ is the unique (jointly) upper semicontinuous and separately continuous extension of the normalized chordal metric $[z, w]$ on $\mathbb{P}^1(\times \mathbb{P}^1)$ to $\mathbb{P}^1 \times \mathbb{P}^1$ (see (3-4) for a more concrete description). By convention, for archimedean $K$, the kernel function $[S, S']_{\text{can}}$ is defined by $[z, w]$ itself. Let $\Delta = \Delta_{\mathbb{P}^1}$ be the distributional Laplacian on $\mathbb{P}^1$ normalized so that for each $S' \in \mathbb{P}^1$,

$$
\Delta \log [\cdot, S']_{\text{can}} = \delta_{S'} - \Omega_{\text{can}} \quad \text{on } \mathbb{P}^1.
$$
For the construction of the Laplacian $\Delta$ in the nonarchimedean case, see [Baker and Rumely 2010, §5; Favre and Jonsson 2004, §7.7; Thuillier 2005, §3] and also [Jonsson 2015, §2.5]. In [Baker and Rumely 2010], the opposite sign convention for $\Delta$ is adopted.

2.1. Potential theory on $\mathbb{P}^1$ with external fields. For the foundation of the potential theory on the (Berkovich) projective line, see [Baker and Rumely 2010; Favre and Rivera-Letelier 2010; Thuillier 2005], and also [Jonsson 2015; Tsuji 1959, III §11] ([Thuillier 2005] is on more general curves than lines and [Tsuji 1959, III §11] is on $\mathbb{P}^1(\mathbb{C})$). We also refer to [Saff and Totik 1997] for the generalities of weighted potential theory, i.e., logarithmic potential theory with external fields.

A continuous weight $g$ on $\mathbb{P}^1$ is a continuous function on $\mathbb{P}^1$ such that
$$\mu^g := g + \Sigma_{\text{can}}$$
is a probability Radon measure on $\mathbb{P}^1$. For a continuous weight $g$ on $\mathbb{P}^1$, the $g$-potential kernel on $\mathbb{P}^1$ (or the negative of an Arakelov Green kernel function on $\mathbb{P}^1$ relative to $\mu^g$ [Baker and Rumely 2010, §8.10]) is the function
\begin{align}
\Phi_g(S, S') := \log [S, S']_{\text{can}} - g(S) - g(S') & \quad \text{on } \mathbb{P}^1 \times \mathbb{P}^1,
\end{align}
and the $g$-potential of a Radon measure $\nu$ on $\mathbb{P}^1$ is the function
\begin{align}
U_{g, \nu}(\cdot) := \int_{\mathbb{P}^1} \Phi_g(\cdot, S') \, d\nu(S') & \quad \text{on } \mathbb{P}^1.
\end{align}
By Fubini’s theorem, $\Delta U_{g, \nu} = \nu - \nu(\mathbb{P}^1)\mu^g$ on $\mathbb{P}^1$. The $g$-equilibrium energy $V_g \in (-\infty, +\infty)$ of $\mathbb{P}^1$ is the supremum of the $g$-energy functional
\begin{align}
\nu \mapsto \int_{\mathbb{P}^1 \times \mathbb{P}^1} \Phi_g \, d(\nu \times \nu) = \int_{\mathbb{P}^1} U_{g, \nu} \, d\nu
\end{align}
on the space of all probability Radon measures $\nu$ on $\mathbb{P}^1$; indeed, $V_g > -\infty$ since $V_g \geq \int_{\mathbb{P}^1 \times \mathbb{P}^1} \Phi_g \, d(\Sigma_{\text{can}} \times \Sigma_{\text{can}}) > -\infty$. A probability Radon measure $\mu$ on $\mathbb{P}^1$ at which the $g$-energy functional (2-6) attains the supremum $V_g$ is called a $g$-equilibrium mass distribution on $\mathbb{P}^1$; in fact the unique $g$-equilibrium mass distribution on $\mathbb{P}^1$ is $\mu^g$, and moreover, $U_{g, \mu^g} \equiv V_g$ on $\mathbb{P}^1$ (for nonarchimedean $K$, see [Baker and Rumely 2010, Theorem 8.67, Proposition 8.70]). For a discussion on such a Gauss variational problem, see [Saff and Totik 1997, Chapter 1].

A normalized weight $g$ on $\mathbb{P}^1$ is a continuous weight on $\mathbb{P}^1$ satisfying $V_g = 0$; for every continuous weight $g$ on $\mathbb{P}^1$, $\bar{g} := g + V_g/2$ is the unique normalized weight on $\mathbb{P}^1$ such that $\mu^\bar{g} = \mu^g$.

For a continuous weight $g$ on $\mathbb{P}^1$ and a Radon measure $\nu$ on $\mathbb{P}^1$, the $g$-Fekete
\( \text{sum} \) with respect to \( \nu \) is

\[
(v, \nu)_g := \int_{\mathbb{P}^1 \times \mathbb{P}^1 \setminus \text{diag}_{\mathbb{P}^1(K)}} \Phi_g \, d(\nu \times \nu),
\]

which generalizes the classical \textit{Fekete sum} associated with a finite subset in \( \mathbb{C} \) (see [Fekete 1930a; 1930b; 1933]). If \( \text{supp} \nu \) is a discrete (so finite) subset in \( \mathbb{P}^1 \), i.e., if \( \nu \) is a \textit{discrete} measure on \( \mathbb{P}^1 \), then \( (v, \nu)_g \) is always finite (even if \( \text{supp} \nu \subset \mathbb{P}^1 \)).

For a continuous weight \( g \) on \( \mathbb{P}^1 \), a sequence \( (v_n) \) of positive and discrete Radon measures on \( \mathbb{P}^1 \) satisfying \( \lim_{n \to \infty} v_n(\mathbb{P}^1) = \infty \) is called an \textit{asymptotically g-Fekete configuration on} \( \mathbb{P}^1 \) if the sequence \( (v_n) \) not only has \textit{small diagonals} in that \( (v_n \times v_n)(\text{diag}_{\mathbb{P}^1(K)}) = o(v_n(\mathbb{P}^1)^2) \) as \( n \to \infty \) but also satisfies

\[
\lim_{n \to \infty} (v_n, v_n)_g/(v_n(\mathbb{P}^1))^2 = V_g\!, \quad \text{under the former small diagonals condition, the latter one is equivalent to the weaker}
\]

\[
(2-7) \quad \lim_{n \to \infty} \inf \frac{(v_n, v_n)_g}{(v_n(\mathbb{P}^1))^2} \geq V_g,
\]

since we always have

\[
(2-8) \quad \lim_{n \to \infty} \sup \frac{(v_n, v_n)_g}{(v_n(\mathbb{P}^1))^2} \leq V_g
\]

(see, e.g., [Baker and Rumely 2010, Lemma 7.54]). By a classical argument (see [Saff and Totik 1997, Theorem 1.3 in Chapter III]), if \( (v_n) \) is an asymptotically g-Fekete configuration on \( \mathbb{P}^1 \), then \( \lim_{n \to \infty} v_n/\nu_n(\mathbb{P}^1) = \mu^g \) weakly on \( \mathbb{P}^1 \).

\[\text{2.2. Local arithmetic on } \mathbb{P}^1.\] Let \( k \) be a field.

**Definition 2.2.** A field extension \( K/k \) is an \textit{algebraic and metric augmentation} of \( k \) if \( K \) is algebraically closed and (topologically) complete with respect to a nontrivial absolute value \( | \cdot | \) (e.g., \( C_v \) is an algebraic and metric augmentation of a product formula field \( k \) for every \( v \in M_k \)).

For every \( P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1], \) there is a sequence \( (q_j^P)_{j=1}^{\deg P} \) in \( \bar{k}^2 \setminus \{0\} \) giving a factorization

\[
(2-9) \quad P(p_0, p_1) = \prod_{j=1}^{\deg P} ((p_0, p_1) \cap q_j^P)
\]

of \( P \) in \( \bar{k}[p_0, p_1] \). Set \( z_j^P := \pi(q_j^P) \in \mathbb{P}^1(\bar{k}) \) for each \( j \in \{1, 2, \ldots, \deg P\} \). Although the sequence \( (q_j^P)_{j=1}^{\deg P} \) is not unique, the sequence \( (z_j^P)_{j=1}^{\deg P} \) in \( \mathbb{P}^1(\bar{k}) \) is independent of the choice of \( (q_j^P)_{j=1}^{\deg P} \) up to permutations. Let in addition \( K \) be an algebraic and metric completion of \( k \). Then the sum \( \sum_{j=1}^{\deg P} \log \|q_j^P\| \) is also independent of the choice of \( (q_j^P)_{j=1}^{\deg P} \), and for every continuous weight \( g \) on \( \mathbb{P}^1 = \mathbb{P}^1(K) \), the \textit{logarithmic g-Mahler measure} of \( P \) is
\( (2-10) \)
\[
M_g(P) := \sum_{j=1}^{\deg P} g(z_j^P) + M^#(P).
\]

The function \( S_P := |P(\cdot/\| \cdot \|)| \) on \( K^2 \setminus \{0\} \) descends to \( \mathbb{P}^1(K) \) and in turn extends continuously to \( \mathbb{P}^1 \) so that \( \log S_P = \sum_{j=1}^{\deg P} \log [\cdot, z_j^P]_{\text{can}} + M^#(P) \) on \( \mathbb{P}^1 \), which can be rewritten as \( \log S_P - (\deg P)g = \sum_{j=1}^{\deg P} \Phi_g(\cdot, z_j^P) + M_g(P) \) on \( \mathbb{P}^1 \). Integrating both sides against \( d\mu^g \) over \( \mathbb{P}^1 \), by \( U_{g, \mu^g} \equiv V_g \) on \( \mathbb{P}^1 \), we have the Jensen-type formula
\[
(2-11) \quad M_g(P) = \int_{\mathbb{P}^1} (\log S_P - (\deg P)g) \, d\mu^g - (\deg P) V_g.
\]

### 2.3. A lemma on global arithmetic.

Let \( k \) be a product formula field. The proof of the next result is not based on a field extension of \( k \).

**Lemma 2.3.** For every \( P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1]^d \), we have \( \sum_{v \in M_k} N_v \cdot M^#(P)_v \in \mathbb{R}_{\geq 0} \).

**Proof.** Let \( (q_j^P)^{\deg P}_{j=1} \) be a sequence in \( \mathbf{k}^2 \setminus \{0\} \) giving a factorization \((2-9)\) of \( P \), and let \( L(P(1, \cdot)) \in k \setminus \{0\} \) be the coefficient of the maximal degree term of \( P(1, z) \in k[z] \). Setting \( q_j^P = ((q_j^P)_0, (q_j^P)_1) \), for each \( j \in \{1, 2, \ldots, \deg P\} \), we have
\[
L(P(1, \cdot)) = (-1)^{\deg P - \deg \infty} P\left( \prod_{j: \pi(q_j^P) = \infty} (q_j^P)_1 \right) \left( \prod_{j: \pi(q_j^P) \neq \infty} (q_j^P)_0 \right)
\]
since for each \( j \in \{1, 2, \ldots, \deg P\} \),
\[
q_j^P = \begin{cases} 
(q_j^P)_0 \cdot (1, \pi(q_j^P)) & \text{if } \pi(q_j^P) \neq \infty, \\
(q_j^P)_1 \cdot (0, 1) & \text{if } \pi(q_j^P) = \infty.
\end{cases}
\]
Thus we have \( \sum_{v \in M_k} N_v \cdot M^#(P)_v \geq \sum_{v \in M_k} N_v \log |L(P(1, \cdot))|_v = 0 \), where the final equality is by (PF).

For each \( i, j \in \mathbb{N} \cup \{0\} \) satisfying \( i + j = \deg P \), if the coefficient \( a_{i,j} \in k \) of the expansion \( P(p_0, p_1) = \sum_{i+j=\deg P} a_{i,j} p_0^i p_1^j \) in \( k[p_0, p_1]^{\deg P} \) does not vanish, then by (PF), there is a finite subset \( E_{i,j} \) in \( M_k \) such that \( |a_{i,j}|_v = 1 \) for every \( v \in M_k \setminus E_{i,j} \). Set \( E_P := \{\text{finite places of } k\} \cup \bigcup_{i,j \in \mathbb{N} \cup \{0\}: |a_{i,j}|_v = 1} E_{i,j} \). For every \( v \in M_k \setminus E_P \), by the strong triangle inequality, \( |P(p_0, p_1)|_v = \|\geq \) is bounded above by
\[
\max\{\max\{|p_0|_v, |p_1|_v\}^{i+j}: i, j \in \mathbb{N} \cup \{0\}, i + j = \deg P\} = \|\geq (p_0, p_1)|_v \]
on \( \mathbb{C}_v^2 \), so that \( \log S_{P,v} \leq 0 \) on \( \mathbb{P}^1(\mathbb{C}_v) \) and in turn on \( \mathbb{P}^1(\mathbb{C}_v) \). Set \( g^0_v := \{g^0_v : v \in M_k\} \) with \( g^0_v \equiv 0 \) on \( \mathbb{P}^1(\mathbb{C}_v) \) for every \( v \in M_k \); then \( g^0 \) is an adelic continuous weight. For every finite \( v \in M_k \), we have \( \mu_{g^0_v} = \delta_{\text{can}, v} \) on \( \mathbb{P}^1(\mathbb{C}_v) \) and moreover \( V_{g^0_v} = \log [S_{\text{can}, v}, S_{\text{can}, v}]_{\text{can}, v} = 0 \), so that by the Jensen-type formula \((2-11)\), we have \( M^#(P)_v = M_{g^0_v}(P) = \log S_{P,v}(S_{\text{can}, v}) \). Hence, \( M^#(P)_v \leq 0 \) for every \( v \in M_k \setminus E_P \), and we conclude that \( \sum_{v \in M_k} N_v \cdot M^#(P)_v < \infty \) since \#\(E_P\) < \(\infty \). \( \square \)
3. Regularization of discrete Radon measures whose supports are in \( \mathbb{P}^1 \)

Let \( K \) be an algebraically closed field complete with respect to a nontrivial absolute value \( | \cdot | \).

3.1. The small model metric \( d \) and the Hsia kernel \( |S - S'|_\infty \). The kernel function \( [S, S']_{\text{can}} \) is not necessarily a metric on \( \mathbb{P}^1 = \mathbb{P}^1(K) \); indeed, for every \( S \in \mathbb{P}^1 \), \([S, S]_{\text{can}}\) vanishes if and only if \( S \in \mathbb{P}^1 = \mathbb{P}^1(K) \). The small model metric \( d \) on \( \mathbb{P}^1 \) is the function

\[
(3-1) \quad d(S, S') := [S, S']_{\text{can}} - \frac{[S, S]_{\text{can}} + [S', S']_{\text{can}}}{2} \quad \text{on } \mathbb{P}^1 \times \mathbb{P}^1,
\]

which extends the normalized chordal metric \([z, w]\) on \( \mathbb{P}^1 \) (but this \( d \) does not induce the topology of \( \mathbb{P}^1 \); see [Baker and Rumely 2010, §2.7; Favre and Rivera-Letelier 2006, §4.7] for details). On the other hand, the Hsia kernel \( |S - S'|_\infty \) on the Berkovich affine line \( \mathbb{A}^1 = \mathbb{A}^1(K) = \mathbb{P}^1 \setminus \{ \infty \} \) is the function

\[
(3-2) \quad |S - S'|_\infty := [S, S']_{\text{can}} \cdot [S, \infty]_{\text{can}}^{-1} \cdot [S', \infty]_{\text{can}}^{-1} \quad \text{on } \mathbb{A}^1 \times \mathbb{A}^1,
\]

although the difference \( S - S' \) itself is not defined unless both \( S, S' \in K \) (for details, see [Baker and Rumely 2010, Chapter 4]). The kernel \( |S - S'|_\infty \) is the unique (jointly) upper semicontinuous and separately continuous extension of the function \( |z - w| \) on \( K \times K \) to \( \mathbb{A}^1 \times \mathbb{A}^1 \).

3.2. A short description of \( \mathbb{P}^1 \) for nonarchimedean \( K \). Suppose that \( K \) is nonarchimedean. A subset \( B \) in \( K \) is called a (\( K \)-closed) disk in \( K \) if it has the form \( B = \{ z \in K : |z - a| \leq r \} \) for some \( a \in K \) and some radius \( r \geq 0 \). By the strong triangle inequality, two disks in \( K \) either nest or are disjoint. This alternative extends to any two decreasing infinite sequences of disks in \( K \) such that they either infinitely nest or are eventually disjoint, and so induces a cofinal equivalence relation among them.

Example 3.1. Instead of giving a formal definition of the cofinal equivalence class \( S \) of a decreasing infinite sequence \( (B_n) \) of disks in \( K \), let us be practical: each \( z \in K \) is regarded as the cofinal equivalence class of the constant sequence \( (B_n) \) of the disks \( B_n \equiv \{ z \} \) in \( K \) (of radii \( \equiv 0 \)). More generally, for every cofinal equivalence class \( S \) of a decreasing infinite sequence \( (B_n) \) of disks in \( K \), the intersection \( B_S := \bigcap_{n \in \mathbb{N}} B_n \) is independent of the choice of the representatives \( (B_n) \) of \( S \), and if \( B_S \neq \emptyset \), then \( B_S \) is still a disk in \( K \) and the \( S \) is represented by the constant sequence \( (\tilde{B}_n) \) of the disks \( \tilde{B}_n \equiv B_S \) in \( K \).

As a set, the set of all cofinal equivalence classes \( S \) of decreasing infinite sequences \( (B_n) \) of disks in \( K \) and in addition \( \infty \in \mathbb{P}^1 \) is nothing but \( \mathbb{P}^1 \) ([Berkovich 1990, p. 17]; see also [Baker and Rumely 2010, §2; Favre and Rivera-Letelier 2006, §3; Benedetto 2010, §6.1]): for example, the canonical (or Gauss) point \( S_{\text{can}} \) in
\( \mathbb{P}^1 \) is represented by the ring of \( K \)-integers \( \mathcal{O}_K := \{ z \in K : |z| \leq 1 \} \), which is a disk in \( K \). The above alternative induces a partial ordering \( \geq \) on \( \mathbb{P}^1 \) such that for every \( S, S' \in \mathbb{P}^1 \) satisfying \( B_S, B_{S'} \neq \emptyset \), we have \( S \geq S' \) if and only if \( B_S \supset B_{S'} \) (the description is a little complicated when one of \( B_S, B_{S'} \) equals \( \emptyset \)). For every \( S, S' \in \mathbb{P}^1 \) satisfying \( S \geq S' \), the segment between \( S \) and \( S' \) in \( \mathbb{P}^1 \) is the set of all points \( S'' \in \mathbb{P}^1 \) satisfying \( S \geq S'' \geq S' \), which can be equipped with either the ordering induced by \( \geq \) on \( \mathbb{P}^1 \) or its opposite. All those (oriented) segments make \( \mathbb{P}^1 \) a tree in the sense of Jonsson [2015, §2, Definition 2.2]. The (Gelfand) topology of \( \mathbb{P}^1 \) coincides with the (weak) topology of \( \mathbb{P}^1 \) as a tree.

For each \( S \in \mathbb{P}^1 \setminus \{ \infty \} \) represented by \((B_n)\), set
\[
\text{diam } S := \lim_{n \to \infty} \text{diam } B_n \quad (= \text{diam } B_S \text{ if } B_S \neq \emptyset),
\]
where \( \text{diam } B \) denotes the diameter of a disk \( B \) in \( K \) with respect to \( | \cdot | \); by convention, for \( S = \infty \), we set \( B_\infty := K \) and \( \text{diam } \infty := +\infty \). The hyperbolic space is \( H^1 = H^1(K) := \mathbb{P}^1 \setminus \{ \infty \} = \{ S \in \mathbb{P}^1 : \text{diam } S \in (0, +\infty) \} \). The big model (or hyperbolic) metric \( \rho \) on \( H^1 \) is a path metric on \( H^1 \) (but does not induce the relative topology of \( H^1 \) induced by \( \mathbb{P}^1 \)) so that for every \( S, S' \in H^1 \) satisfying \( S \geq S' \),
\[
(3-3) \quad \rho(S, S') = \log(\text{diam } S / \text{diam } S')
\]
(see, e.g., [Baker and Rumely 2010, §2.7]). In terms of \( \rho \), the generalized Hsia kernel \([S, S']_{\text{can}}\) with respect to \( S_{\text{can}} \) is interpreted as a Gromov product
\[
(3-4) \quad \log [S, S']_{\text{can}} = -\rho(S'', S_{\text{can}}) \quad \text{on } H^1 \times H^1,
\]
where \( S'' \) is the unique point in \( H^1 \) lying between \( S \) and \( S' \), between \( S' \) and \( S_{\text{can}} \), and between \( S_{\text{can}} \) and \( S \) (see [Favre and Rivera-Letelier 2006, §3.4]). Similarly, for every \( S, S' \in A^1 \),
\[
(3-5) \quad |S - S'|_\infty = \text{diam } S'',
\]
where \( S'' \) is the smallest point in \( A^1 \) satisfying both \( S'' \geq S \) and \( S'' \geq S' \) with respect to the partial ordering \( \geq \) on \( \mathbb{P}^1 \).

For every \( \epsilon > 0 \), a continuous mapping
\[
\pi_\epsilon : A^1 \to A^1
\]
is defined by \( \pi_\epsilon(S) := S'' \) for every \( S \in A^1 \), where \( S'' \in \{ S \in \mathbb{P}^1 : \text{diam } S \in [\epsilon, +\infty) \} \) is the unique point between \( \infty \) and \( S \) satisfying \( \text{diam } S'' = \max\{ \epsilon, \text{diam } S \} \) (see [Favre and Rivera-Letelier 2006, §4.6] for details).

### 3.3. Regularization on \( \mathbb{P}^1 \)

When \( K \) is archimedean, fix a nonnegative smooth decreasing function \( \xi : [0, \infty) \to [0, 1] \) such that \( \text{supp } \xi \subset [0, 1] \) and \( \int_0^\infty \xi(x) \, dx = 1 \), and set \( \xi_\epsilon(x) := \xi(x/\epsilon)/\epsilon \) on \([0, +\infty)\) for each \( \epsilon > 0 \). For every \( z \in K \) and every
\( \epsilon > 0 \), the \( \epsilon \)-regularization \([z]_\epsilon\) of \( \delta_z\) is the convolution \( \xi_\epsilon \ast \delta_z\) on \( \mathbb{P}^1 \), i.e., for any continuous test function \( \phi \) on \( \mathbb{P}^1 \),

\[
(\xi_\epsilon \ast \delta_z)(\phi) = \int_0^\epsilon \xi_\epsilon(r) \, dr \int_0^{2\pi} \phi(z + re^{i\theta}) \, \frac{d\theta}{2\pi}.
\]

When \( K \) is nonarchimedean, for every \( z \in K \) and every \( \epsilon > 0 \), the \( \epsilon \)-regularization \([z]_\epsilon\) of \( \delta_z\) is defined by \([z]_\epsilon := (\pi_\epsilon)_* \delta_z = \delta_{\pi_\epsilon(z)}\) on \( \mathbb{P}^1 \) [Favre and Rivera-Letelier 2006, p. 343]. In both cases, \([z]_\epsilon\) is a probability Radon measure on \( \mathbb{P}^1 \), the chordal potential \( \mathbb{P}^1 \ni S \mapsto \int_{\mathbb{P}^1} \log [S, S']_{\text{can}} \, d[z]_\epsilon(S') \) of \([z]_\epsilon\) is a continuous function on \( \mathbb{P}^1 \), and for every \( z, w \in K \) and every \( \epsilon > 0 \), the estimate

\[
(3-6) \quad \int_{A^1 \times A^1} \log |S - S'|_\infty \, d([z]_\epsilon \times [w]_\epsilon)(S, S') \geq \begin{cases} 
\log |z - w| & \text{if } z \neq w, \\
C_{\text{abs}} + \log \epsilon & \text{if } z = w
\end{cases}
\]

holds, where \( C_{\text{abs}} \leq 0 \) is an absolute constant and in fact \( C_{\text{abs}} = 0 \) for nonarchimedean \( K \) [Favre and Rivera-Letelier 2006, Lemmas 2.10, 4.11, and their proofs].

Let us extend the \( \epsilon \)-regularization \([\cdot]_\epsilon\) and the estimate (3-6) to \( \mathbb{P}^1 \). Set \( \iota(z) := 1/z \in \text{PGL}(2, K) \), which extends to an automorphism on \( \mathbb{P}^1 \) (see Fact 8.2), so that \( \iota^2 = \text{Id} \) on \( \mathbb{P}^1 \) and \([\iota(S), \iota(S')]_{\text{can}} = [S, S']_{\text{can}} \) (so \( d(\iota(S), \iota(S')) = d(S, S') \)) on \( \mathbb{P}^1 \times \mathbb{P}^1 \). For every \( \epsilon > 0 \), set \([\infty]_\epsilon := \iota_*[0]_\epsilon\).

For every \( z \in \mathbb{P}^1 \) and every \( \epsilon > 0 \), we have

\[
(3-7) \quad \text{supp } [z]_\epsilon \subset \{ S \in \mathbb{P}^1 : d(S, z) \leq \epsilon \},
\]

as follows immediately from the definitions of \( |S - S'|_\infty \) (and (3-5)), \( d \), and \([z]_\epsilon\) when \( z \in K \), and from (3-7) applied to \( z = 0 \) and the invariance of \( d \) under \( \iota \) when \( z = \infty \). Moreover, for every \( z \in K \) and every \( \epsilon > 0 \),

\[
(3-8) \quad \sup_{S \in \text{supp } [z]_\epsilon} |\log [S, \infty]_{\text{can}} - \log [z, \infty]| \leq \epsilon
\]

by a direct computation of \( \log [\cdot, \infty]_{\text{can}} - \log [z, \infty] \) on \( K \), using that \( \text{supp } [z]_\epsilon \subset \{ S \in \mathbb{P}^1 : |S - z|_\infty \leq \epsilon \} \) and the density of \( K \) in \( A^1 \).

**Lemma 3.2.** Let \( g \) be a continuous weight on \( \mathbb{P}^1 \) having a modulus of continuity \( \eta \) on \( (\mathbb{P}^1, d) \). Then for every \( \epsilon > 0 \) and every \( z, w \in \mathbb{P}^1 \),

\[
(3-9) \quad \int_{\mathbb{P}^1 \times \mathbb{P}^1} \Phi_g \, d([z]_\epsilon \times [w]_\epsilon)
\]

\[
\geq \begin{cases} 
\Phi_g(z, w) - 2\epsilon - 2\eta(\epsilon) & \text{if } z \neq w, \\
C_{\text{abs}} + \log \epsilon - 2\epsilon + 2 \log |z| - 2\eta(\epsilon) - 2g(z) & \text{if } z = w \in K, \\
C_{\text{abs}} + \log \epsilon - 2\epsilon - 2\eta(\epsilon) - 2g(\infty) & \text{if } z = w = \infty.
\end{cases}
\]

**Proof.** Since \( \Phi_g(S, S') = \log [S, S']_{\text{can}} - g(S) - g(S') \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \), by (3-7), we can assume \( g \equiv 0 \) (and \( \eta \equiv 0 \)) on \( \mathbb{P}^1 \) without loss of generality. For every \( z, w \in K \),
by the definition (3-2) of $|S - S'|_\infty$ and (3-8),

$$
\int_{\mathbb{P}^1 \times \mathbb{P}^1} \log [S, S'] \text{can} d([z]_e \times [w]_e)(S, S')
\geq \int_{A^1 \times A^1} \log |S - S'|_\infty d([z]_e \times [w]_e)(S, S') - 2 \epsilon + \log |z, \infty| + \log |w, \infty|,
$$

which with the estimate (3-6) yields (3-9) (for $g \equiv \eta \equiv 0$) in this case. The estimate (3-9) (for $g \equiv \eta \equiv 0$) in the case $z = w = \infty$ follows from $[\infty]_e = \iota_*[0]_e$, $[\iota(S), \iota(S') \text{can} = [S, S'] \text{can}$, and the estimate (3-9) for $z = w = 0$.

There remains the case that $z = \infty$ and $w \in K$ (so $z \neq w$). If $K$ is nonarchimedean, then for every $w \in K$ and $\epsilon > 0$, the equalities $[\infty]_e = \iota_*[0]_e$ and $[\iota(S), \iota(S') \text{can} = [S, S'] \text{can}$, together with the interpretation (3-4) of $[S, S'] \text{can}$, yield

$$
\int_{\mathbb{P}^1 \times \mathbb{P}^1} \log [S, S'] \text{can} d([\infty]_e \times [w]_e)(S, S')
= \int_{\mathbb{P}^1 \times \mathbb{P}^1} \log [S, S'] \text{can} d([0]_e \times \iota_*[w]_e)(S, S') = \log [\pi_\epsilon(0), \iota(\pi_\epsilon(w))] \text{can}
\geq \log [0, \iota(w)] = \log \{\infty, w\} \geq \log \{\infty, w\} - 2 \epsilon,
$$

which implies the estimate (3-9) (for $g \equiv \eta \equiv 0$) in the case $z = \infty$ and $w \in K$ when $K$ is nonarchimedean. If $K$ is archimedean, then for every $w \in K$ and every $r, r' > 0$, we have

$$
\int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \log \left| (0 + re^{i\theta}) - \frac{1}{w + r'e^{i\phi}} \right| \frac{d\theta}{2\pi}
= \int_0^{2\pi} \max\{-\log |w + r'e^{i\phi}|, \log r\} \frac{d\phi}{2\pi} \geq -\int_0^{2\pi} \log \left| (w + r'e^{i\phi}) - 0 \right| \frac{d\phi}{2\pi},
$$

so that for every $w \in K \cong A^1$ and every $\epsilon > 0$,

$$
\int_{A^1 \times A^1} \log |S - S'|_\infty d([0]_e \times \iota_*[w]_e)(S, S')
= \int_{A^1 \times A^1} \log |S - \iota(S')|_\infty d([0]_e \times [w]_e)(S, S') \geq -\int_{A^1} \log |S' - 0|_\infty d[w]_e(S').
$$

On the other hand, for every $w \in K$ and every $\epsilon > 0$, by the definition (2-1) of the chordal metric $[z, w]$ on $\mathbb{P}^1 \cong \mathbb{P}^1$ (and $[0, \infty] = 1$),

$$
\int_{\mathbb{P}^1} \log [S', \infty] \text{can} d(\iota_*[w]_e)(S') = \int_{\mathbb{P}^1} \log [S', 0] \text{can} d[w]_e(S')
= \int_{A^1} \log |S' - 0|_\infty d[w]_e(S') + \int_{\mathbb{P}^1} \log [S', \infty] \text{can} d[w]_e(S').
$$

From these computations and (3-8), for every $w \in K$ and every $\epsilon > 0$, we get
\[
\int_{P^1 \times P^1} \log [S, S']_{\text{can}} \, d([\infty]_{\epsilon} \times [w]_{\epsilon})(S, S') \\
= \int_{P^1 \times P^1} \log [S, S']_{\text{can}} \, d([0]_{\epsilon} \times \iota_* [w]_{\epsilon})(S, S') \\
\geq \int_{P^1} \log [S, \infty]_{\text{can}} \, d[0]_{\epsilon}(S) + \int_{P^1} \log [S', \infty]_{\text{can}} \, d[w]_{\epsilon}(S') \\
\geq \log [0, \infty] + \log [w, \infty] - 2\epsilon = \log [w, \infty] - 2\epsilon,
\]
which implies the estimate (3-9) (for \( g \equiv \eta \equiv 0 \)) in the case \( z = \infty \) and \( w \in K \) when \( K \) is archimedean. \( \square \)

4. The negativity of regularized Fekete sums and a Cauchy–Schwarz inequality

Let \( K \) be an algebraically closed field that is complete with respect to a nontrivial absolute value \( | \cdot | \). For every \( \epsilon > 0 \) and every discrete measure \( \nu \) on \( P^1 = P^1(K) \) whose support is in \( \mathbb{P}^1 = P^1(K) \), the \( \epsilon \)-regularization of \( \nu \) is

\[
\nu_{\epsilon} := \sum_{w \in \text{supp } \nu} \nu(\{w\})[w]_{\epsilon} \quad \text{on } P^1.
\]

For every continuous weight \( g \) on \( P^1 \), let us call \( (\nu_{\epsilon}, \nu_{\epsilon})_g \) the \( \epsilon \)-regularized \( g \)-Fekete sum with respect to this \( \nu \).

4.1. \( C^1 \)-regularity and the Dirichlet norm. Recall the description of \( P^1 \) given in Section 3.2. For nonarchimedean \( K \), a function \( \phi \) on \( P^1 = P^1(K) \) is in \( C^1(P^1) \) if

(i) \( \phi \) is continuous on \( P^1 \) and locally constant except for a union \( T \) of at most finitely many segments in \( H^1 = H^1(K) \), which are oriented by the partial ordering \( \succeq \) on \( P^1 \), and

(ii) the derivative \( \phi' \) with respect to the length parameter induced by the hyperbolic metric \( \rho \) on each segment in \( T \) exists and is continuous on \( T \).

The Dirichlet norm of \( \phi \in C^1(P^1) \) is defined by \( \langle \phi, \phi \rangle^{1/2} := \left( \int_T (\phi')^2 \, d\rho \right)^{1/2} \), where \( d\rho \) is the 1-dimensional Hausdorff measure on \( H^1 \) with respect to \( \rho \) (for details, see [Favre and Rivera-Letelier 2006, §5.5]). When \( K \) is archimedean, the \( C^1 \)-regularity and the Dirichlet norm of a function \( \phi \) on \( P^1 \simeq \mathbb{P}^1 \) is defined with respect to the complex (or differentiable) structure of \( \mathbb{P}^1 \). For completeness, we include a proof of the following.

**Proposition 4.1.** Every \( \phi \) in \( C^1(P^1) \) is Lipschitz continuous on \( (P^1, d) \).

**Proof.** When \( K \) is archimedean, this is obvious. Suppose that \( K \) is nonarchimedean and let \( \phi \in C^1(P^1) \). By definition, \( \phi \) is locally constant on \( P^1 \) except for a union
The set $T$ of at most finitely many segments in $H^1$, and is Lipschitz continuous on $T$ with respect to $\rho$. The set $T$ is compact in $(H^1, \rho)$, and for every $S, S' \in H^1$, by the definition (3-1) of $d$, (3-4), and (3-3), if $S_{\text{can}} \geq S \geq S'$, then

$$d(S, S') = \text{diam } S - \frac{\text{diam } S + \text{diam } S'}{2} = \frac{\text{diam } S - \text{diam } S'}{2} \geq \frac{\text{diam } S'}{2} \rho(S, S'),$$

and similarly, if $S_{\text{can}} \leq S \leq S'$, then $d(S, S') \geq \rho(S, S')/(2 \text{diam } S')$. Hence we conclude that $\phi$ is also Lipschitz continuous on $T$ with respect to $d$, and in turn on the whole $P^1$ with respect to $d$. \hfill \Box

The Lipschitz constant of a Lipschitz continuous function $\phi$ on $(P^1, d)$ is denoted by $\text{Lip}(\phi)$.

**Remark 4.2.** When $K$ is archimedean (so $P^1 \cong P^1$), we have $\langle \phi, \phi \rangle^{1/2}_g \leq \text{Lip}(\phi)$ for every $\phi \in C^1(P^1)$. Moreover, every Lipschitz continuous function $\phi$ on $(P^1, [z, w])$ is approximated by functions in $C^1(P^1)$ in the Lipschitz norm.

**4.2. The negativity of $(v_\varepsilon, v_\varepsilon)_g$ and a Cauchy–Schwarz inequality.** For every Radon measure $\mu$ on $P^1$ satisfying $\mu(P^1) = 0$, if the chordal potential of $\mu$, which is defined by $S \mapsto \int_{P^1} \log |S, S'_\text{can}| d\mu(S')$, is continuous on $P^1$, then we have the positivity property $\int_{P^1 \times P^1} (-\log |S - S'|_\infty) d(\mu \times \mu)(S, S') \geq 0$ (see [Favre and Rivera-Letelier 2006, §2.5 and §4.5]) and in fact the Cauchy–Schwarz inequality

$$\langle \phi, \phi \rangle_g \leq (\phi, \phi) \cdot \int_{P^1 \times P^1} (-\log |S - S'|_\infty) d(\mu \times \mu)(S, S')$$

for every test function $\phi \in C^1(P^1)$ (see [Favre and Rivera-Letelier 2006, (32) and (33)]).

In particular, for every $\varepsilon > 0$, every normalized weight $g$ on $P^1$, every test function $\phi \in C^1(P^1)$, and every discrete measure $\nu$ on $P^1$ whose support is in $P^1$, the computation

$$0 \leq \int_{P^1 \times P^1} (-\log |S - S'|_\infty) d((v_\varepsilon - (v(P^1))\mu^g) \times (v_\varepsilon - (v(P^1))\mu^g))(S, S')$$

$$= \int_{P^1 \times P^1} (-\Phi_g^\varepsilon) d((v_\varepsilon - (v(P^1))\mu^g) \times (v_\varepsilon - (v(P^1))\mu^g)) = -(v_\varepsilon, v_\varepsilon)_g$$

(recalling $U_{g, \mu^g} \equiv 0$ on $P^1$) yields not only the negativity $(v_\varepsilon, v_\varepsilon)_g \leq 0$ but, with the Cauchy–Schwarz inequality (4-1) and the triangle inequality, also the estimate

$$\langle \phi, \phi \rangle_g \leq \text{deg } \nu \cdot \text{Lip}(\phi)\varepsilon + (\phi, \phi)^{1/2} \cdot (-\varepsilon)^{1/2}.$$
5. Computations of Fekete sums \((\mathcal{Z}, \mathcal{Z})_g\)

Let \(k\) be a field. For a \(k\)-effective divisor \(\mathcal{Z}\) on \(\mathbb{P}^1(\bar{k})\), set

\[
D^*(\mathcal{Z}|\bar{k}) := \prod_{w \in \text{supp } \mathcal{Z}\setminus \{\infty\}} \prod_{w' \in \text{supp } \mathcal{Z}\setminus \{w, \infty\}} (w - w')^{(\text{ord}_w \mathcal{Z})(\text{ord}_{w'} \mathcal{Z})} \in \bar{k} \setminus \{0\},
\]

which is in fact in \(k\) by Theorem 7 if \(\mathcal{Z}\) is on \(\mathbb{P}^1(k_s)\). For every \(P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1]_d\), let \(L(P(1, \cdot)) \in k \setminus \{0\}\) be the coefficient of the maximal degree term of \(P(1, z) \in k[z]\) (appearing in Section 2.3).

**Lemma 5.1.** Let \(k\) be a field. Let \(\mathcal{Z}\) be a \(k\)-effective divisor on \(\mathbb{P}^1(\bar{k})\) represented by \(P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1]_d\), and let \((q_j^P)_{j=1}^{\text{deg } P}\) be a sequence in \(\bar{k}^2 \setminus \{0\}\) giving a factorization \((2\,-9)\) of \(P\). For each \(j \in \{1, 2, \ldots, \text{deg } P\}\), set \(q_j^P = ((q_j^P)_0, (q_j^P)_1)\) and \(z_j := \pi(q_j^P) \in \mathbb{P}^1(\bar{k})\). Suppose \((q_j^P)_{j=1}^{\text{deg } P}\) is normalized with respect to a distinguished zero \(w_0 \in \mathbb{P}^1(\bar{k})\) of \(P\) so that for each \(j \in \{1, 2, \ldots, \text{deg } P\}\),

\[
\begin{align*}
(q_j^P)_0 &= 1 \quad \text{if } z_j \not\in \{w_0, \infty\}, \\
(q_j^P)_1 &= 1 \quad \text{if } w_0 \not= z_j = \infty.
\end{align*}
\]

Then

\[
L(P(1, \cdot)) = (-1)^{\text{deg } P - \text{deg}_{\infty} P} \cdot \left\{ \prod_{j : z_j = w_0} (q_j^P)_0 \right\} \left\{ \prod_{j : z_j = w_0} (q_j^P)_1 \right\} \text{ if } w_0 \not= \infty,
\]

and

\[
\prod_{j=1}^{\text{deg } P} \prod_{j : z_j \not= z_j} (q_i^P \wedge q_j^P) = (-1)^{\text{deg}_{\infty} P \cdot (\text{deg } P - \text{deg}_{\infty} P)} \cdot L(P(1, \cdot))^{2(\text{deg } P - \text{deg}_{w_0} P)} \cdot D^*(\mathcal{Z}|\bar{k}).
\]

**Proof.** Without normalizing the sequence \((q_j^P)_{j=1}^{\text{deg } P}\) we have, by direct computation,

\[
\prod_{j=1}^{\text{deg } P} \prod_{i : z_i \not= z_j} (q_i^P \wedge q_j^P)
\]

\[
= \prod_{j : z_j = \infty} ((q_j^P)_0(q_j^P)_1) \cdot \prod_{j : z_j \not= \infty} ((q_j^P)_1(q_j^P)_0) \cdot \prod_{j : z_j \not= \infty, i : z_i \not= z_j} ((q_j^P)_0(q_j^P)_0(z_j - z_i))
\]

\[
= (-1)^{\text{deg}_{\infty} P \cdot (\text{deg } P - \text{deg}_{\infty} P)} \cdot \left( \prod_{j : z_j = \infty} ((q_j^P)_1^{\text{deg } P - \text{deg}_{\infty} P}) \cdot \prod_{i : z_i \not= z_j} ((q_i^P)_0) \right)^2
\]

\[
\cdot \left( \prod_{j : z_j \not= \infty} ((q_j^P)_0^{\text{deg } P - \text{deg}_{z_j} P}) \cdot \prod_{i : z_i \not= z_j} ((q_i^P)_0) \right) \cdot D^*(\mathcal{Z}|\bar{k}).
\]

Let us normalize \((q_j^P)\) so that the normalization \((5-1)\) holds with respect to a
distinguished zero $w_0 \in \mathbb{P}^1(\bar{k})$ of $P$. Then (5-2) follows from
\[
L(P(1, \cdot)) = (-1)^{\deg P - \deg_\infty P} \left( \prod_{j: z_j = \infty} (q_j^P)_{1} \right) \left( \prod_{j: z_j \neq \infty} (q_j^P)_{0} \right)
\]
and the normalization (5-1).

Let us show (5-3). If $w_0 = \infty$, then under the normalization (5-1), the equality (5-4) yields
\[
\deg P \prod_{j: z_j = \infty} \left( q_j^P \right)^{1} = \left( -1 \right)^{\deg_\infty P (\deg P - \deg_\infty P)} \cdot \left( \prod_{j: z_j = \infty} (q_j^P)_{1} \right)^{2 (\deg P - \deg_\infty P)} \cdot 1 \cdot D^*(Z | \bar{k}),
\]
which with (5-2) implies (5-3) when $w_0 = \infty$. If $w_0 \neq \infty$, then under the normalization (5-1), the equality (5-4) yields
\[
\deg P \prod_{j: z_j \neq \infty} \left( q_j^P \right)^{1} = \left( -1 \right)^{\deg_\infty P (\deg P - \deg_\infty P)} \cdot \left( \prod_{j: z_j = \infty} (q_j^P)_{1} \right)^{2 (\deg P - \deg_\infty P)} \cdot D^*(Z | \bar{k}),
\]
which with (5-2) implies (5-3) when $w_0 \neq \infty$.

\[\square\]

**Lemma 5.2** (local computation). Let $k$ be a field and $K$ an algebraic and metric augmentation of $k$ (see Section 2.2). For every continuous weight $g$ on $P^1 = P^1(K)$ and every $k$-effective divisor $Z$ on $P^1(\bar{k})$ represented by a homogeneous polynomial $P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1]_d$, we have
\[
(5-5) \quad (Z, Z)_g + 2 \cdot \sum_{w \in \supp Z \setminus \{\infty\}} (\ord_w Z)^2 \log [w, \infty] - 2 \cdot \sum_{w \in \supp Z} (\ord_w Z)^2 g(w)
\]
\[
= 2 (\deg Z) \log |L(P(1, \cdot))| + \log |D^*(Z | \bar{k})| - 2 (\deg Z) M_g(P).
\]

**Proof.** Let $Z$ and $P$ be as in the statement and let $(q_j^P)_{j=1}^{\deg P}$ be a sequence in $\bar{k}^2 \setminus \{0\}$ giving a factorization (2-9) of $P$ and satisfying the normalization (5-1) with
respect to a distinguished zero \( w_0 \in \mathbb{P}^1(\bar{k}) \) of \( P \). Set \( z_j := \pi(q_j^P) \in \mathbb{P}^1(\bar{k}) \) for each \( j \in \{1, 2, \ldots, \deg P\} \). Since by definition

\[
\Phi_g(z, z') = \log [z, z'] - g(z) - g(z')
\]

on \( \mathbb{P}^1(K) \times \mathbb{P}^1(K) \), we have

\[
(Z, Z)_g = \log \left( \prod_{j=1}^{\deg P} \prod_{i: z_i \neq z_j} |q_i \wedge q_j|^P \right) - 2 \cdot \sum_{j=1}^{\deg P} \sum_{i: z_i \neq z_j} (g(z_i) + \log \|q_i^P\|);
\]

by (5-3),

\[
\log \left( \prod_{j=1}^{\deg P} \prod_{i: z_i \neq z_j} |q_i \wedge q_j|^P \right) = 2(\deg P - \deg_{w_0} P) \log |L(P(1, \cdot))| + \log |D^*(Z|\bar{k})|,
\]

and we also have

\[
\sum_{j=1}^{\deg P} \sum_{i: z_i \neq z_j} (g(z_i) + \log \|q_i^P\|)
\]

\[
= \sum_{j=1}^{\deg P} \sum_{i=1}^{\deg P} (g(z_i) + \log \|q_i^P\|) - \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} (g(z_i) + \log \|q_i^P\|)
\]

\[
= (\deg P)M_g(P) - \sum_{j=1}^{\deg P} (\deg_{w_j} P)g(z_j) - \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} \log \|q_i^P\|,
\]

where the final equality is by the definition (2-10) of \( M_g(P) \). Hence

\[
(Z, Z)_g = 2(\deg P) \log |L(P(1, \cdot))| + \log |D^*(Z|\bar{k})| - 2(\deg P)M_g(P)
\]

\[
+ 2 \sum_{w \in \text{supp } Z} (\text{ord}_w Z)^2 g(w) - 2 \left( (\deg_{w_0} P) \log |L(P(1, \cdot))| - \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} \log \|q_i^P\| \right).
\]

For each \( j \in \{1, 2, \ldots, \deg P\} \), also set \( q_j^P = ((q_j^P)_0, (q_j^P)_0) \). If \( \infty \notin \text{supp } Z \), then \( w_0 \neq \infty \), and by the normalization (5-1) and the equality (5-2),

\[
(\deg_{w_0} P) \log |L(P(1, \cdot))| - \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} \log \|q_i^P\|
\]

\[
= - \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} (\log \|q_i^P\| + \log |(q_i^P)_0|) = \sum_{j=1}^{\deg P} \sum_{i: z_i = z_j} \log [z_i, \infty]
\]

\[
= \sum_{w \in \text{supp } Z} (\text{ord}_w Z)^2 \log [w, \infty] = \sum_{w \in \text{supp } Z \setminus \{\infty\}} (\text{ord}_w Z)^2 \log [w, \infty].
\]
If \( \infty \in \text{supp} \mathcal{Z} \), then we can set \( w_0 = \infty \), and by the normalization \((5-1)\) and the equality \((5-2)\) (and \( q_i^P = (q_i^P)_1 \cdot (0, 1) \) when \( z_i = \infty \)),

\[
(\deg w_0 P) \log |L(P(1, \cdot))| - \sum_{j=1}^{\deg P} \sum_{i : z_i = z_j} \log \|q_i^P\| = - \sum_{j : z_j = \infty} \sum_{i : z_i = z_j} (\log \|q_i^P\| - \log \|(q_i^P)_1\|) - \sum_{j : z_j \neq \infty} \sum_{i : z_i = z_j} (\log \|q_i^P\| - \log \|(q_i^P)_0\|)
\]

\[= \sum_{j : z_j \neq \infty} \sum_{i : z_i = z_j} \log [z_i, \infty] = \sum_{w \in \text{supp} \mathcal{Z} \setminus \{\infty\}} (\text{ord}_w \mathcal{Z})^2 \log [w, \infty].\]

This completes the proof.

**Lemma 5.3** (global computation). Let \( k \) be a product formula field and \( k_s \) the separable closure of \( k \) in \( \bar{k} \). Then for every adelic continuous weight \( g = \{ g_v : v \in M_k \} \) and every \( k \)-effective divisor \( \mathcal{Z} \) on \( \mathbb{P}^1(k_s) \),

\[
(5-6) \quad \sum_{v \in M_k} N_v (\mathcal{Z}, \mathcal{Z})_{g_v} + 2 \sum_{w \in \text{supp} \mathcal{Z} \setminus \{\infty\}} (\text{ord}_w \mathcal{Z})^2 \log [w, \infty]_v = -2(\deg \mathcal{Z})^2 h_g(\mathcal{Z}) + 2 \sum_{v \in M_k} N_v \sum_{w \in \text{supp} \mathcal{Z}} (\text{ord}_w \mathcal{Z})^2 g_v(w).
\]

**Proof.** Let \( P \in \bigcup_{d \in \mathbb{N}} k[p_0, p_1]_d \) be a representative of \( \mathcal{Z} \). Summing up the product of \( N_v \) and \((5-5)\) (for this \( P \)) over all \( v \in M_k \), we have

\[
\sum_{v \in M_k} N_v (\mathcal{Z}, \mathcal{Z})_{g_v} + 2 \sum_{w \in \text{supp} \mathcal{Z} \setminus \{\infty\}} (\text{ord}_w \mathcal{Z})^2 \log [w, \infty]_v - 2 \sum_{w \in \text{supp} \mathcal{Z}} (\text{ord}_w \mathcal{Z})^2 g_v(w)
\]

\[= -2(\deg \mathcal{Z})^2 h_g(\mathcal{Z})\]

by the product formula (PF) (since \( L(P(1, \cdot)) \in k \setminus \{0\} \) and, under the assumption that \( \mathcal{Z} \) is on \( \mathbb{P}^1(k_s) \), \( D^*(\mathcal{Z}|\bar{k}) \in k \setminus \{0\} \)) and the definition \((1-1)\) of \( h_g(\mathcal{Z}) \).

\[\square\]

**6. Estimates of regularized Fekete sums \((\mathcal{Z}_e, \mathcal{Z}_e)_g\)**

**6.1. Local estimate.** Let \( k \) be a field and \( K \) an algebraic and metric augmentation of \( k \). Let \( \mathcal{Z} \) be a \( k \)-effective divisor on \( \mathbb{P}^1(\bar{k}) \), which we regard as the Radon measure

\[
\sum_{w \in \text{supp} \mathcal{Z}} (\text{ord}_w \mathcal{Z}) \delta_w
\]

on \( \mathbb{P}^1 = \mathbb{P}^1(K) \), and let \( g \) be a continuous weight on \( \mathbb{P}^1 \) such that \( g \) is a \( 1/\kappa \)-Hölder continuous function on \((\mathbb{P}^1, d)\) for some \( \kappa \geq 1 \) having the \( 1/\kappa \)-Hölder constant \( C(g) \geq 0 \).
Lemma 6.1. For every $\epsilon > 0$,

$$(Z_{\epsilon}, Z_{\epsilon})_g \geq (Z, Z)_g + 2 \sum_{w \in \text{supp } Z \setminus \{\infty\}} (\text{ord}_w Z)^2 \log [w, \infty] - 2 \sum_{w \in \text{supp } Z} (\text{ord}_w Z)^2 g(w)$$

$$+ (C_{\text{abs}} + \log \epsilon) \cdot (Z \times Z)(\text{diag}_{\mathbb{P}^1(k)}) - 2(\deg Z)^2 (\epsilon + C(g)\epsilon^{1/\kappa}).$$

Proof. Set $\eta(\epsilon) = C(g)\epsilon^{1/\kappa}$. For every $\epsilon > 0$, using (3-9),

$$(Z_{\epsilon}, Z_{\epsilon})_g - (Z, Z)_g$$

$$= \int_{\mathbb{P}^1 \times \mathbb{P}^1} \Phi_g d(Z_{\epsilon} \times Z_{\epsilon}) - \int_{\mathbb{P}^1 \times \mathbb{P}^1 \setminus \text{diag}_{\mathbb{P}^1(k)}} \Phi_g d(Z \times Z)$$

$$= \sum_{w \in \text{supp } Z} (\text{ord}_w Z)^2 \int_{\mathbb{P}^1 \times \mathbb{P}^1} \Phi_g d([w]_\epsilon \times [w]_\epsilon)$$

$$+ \sum_{(z, w) \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus \text{diag}_{\mathbb{P}^1(k)}} \left( \int_{\mathbb{P}^1 \times \mathbb{P}^1} \Phi_g(S, S') d([z]_\epsilon \times [w]_\epsilon)(S, S') - \Phi_g(z, w) \right)$$

$$\geq \sum_{w \in \text{supp } Z \setminus \{\infty\}} (\text{ord}_w Z)^2 (C_{\text{abs}} + \log \epsilon - 2\epsilon + 2 \log [w, \infty] - 2\eta(\epsilon) - 2g(w))$$

$$+ (Z(\{\infty\}))^2 (C_{\text{abs}} + \log \epsilon - 2\epsilon - 2\eta(\epsilon) - 2g(\infty))$$

$$+ ((\deg Z)^2 - (Z \times Z)(\text{diag}_{\mathbb{P}^1(k)}))(-2\epsilon - 2\eta(\epsilon))$$

$$= ((Z \times Z)(\text{diag}_{\mathbb{P}^1(k)}))(C_{\text{abs}} + \log \epsilon - 2\epsilon - 2\eta(\epsilon))$$

$$+ 2 \sum_{w \in \text{supp } Z \setminus \{\infty\}} (\text{ord}_w Z)^2 \log [w, \infty] - 2 \sum_{w \in \text{supp } Z} (\text{ord}_w Z)^2 g(w)$$

$$+ ((\deg Z)^2 - (Z \times Z)(\text{diag}_{\mathbb{P}^1(k)}))(-2\epsilon - 2\eta(\epsilon)),$$

which completes the proof. \hfill \Box

6.2. Global estimate. Let $k$ be a product formula field, and $Z$ a $k$-effective divisor on $\mathbb{P}^1(k_s)$. Let $g = \{g_v : v \in M_k\}$ be a placewise Hölder continuous adelic normalized weight, so for every $v \in M_k$, $g_v$ is a normalized weight on $\mathbb{P}^1(\mathbb{C}_v)$ and is a $1/\kappa_v$-Hölder continuous function on $(\mathbb{P}^1(\mathbb{C}_v), d_v)$ for some $\kappa_v \geq 1$ having the $1/\kappa_v$-Hölder constant $C(g_v) \geq 0$.

Lemma 6.2. For every $v_0 \in M_k$ and every $\epsilon > 0$,

$$N_{v_0}(Z_{\epsilon}, Z_{\epsilon})_{g_{v_0}} \geq -2(\deg Z)^2 h_g(Z) + (C_{\text{abs}} + \log \epsilon) \cdot (Z \times Z)(\text{diag}_{\mathbb{P}^1(k)}) \cdot \sum_{v \in E_{v_k \cup \{v_0\}}} N_v$$

$$- 2(\deg Z)^2 \sum_{v \in E_{v_k \cup \{v_0\}}} N_v(\epsilon + C(g_v)\epsilon^{1/\kappa_v}).$$

Proof. Fix $v_0 \in M_k$. We use, for every $v \in M_k$, the notation
\[ W_v := (Z, Z)_g + 2 \sum_{w \in \text{supp } Z \setminus \{\infty\}} (\text{ord}_w Z)^2 \log |w, \infty|_v - 2 \sum_{w \in \text{supp } Z} (\text{ord}_w Z)^2 g_v(w). \]

Since \((Z_\epsilon, Z_\epsilon)_g \leq 0\) for every \(\epsilon > 0\) and every \(v \in M_k\) (see Section 4.2), using also Lemma 6.1, we have

\[ N_{v_0}(Z_\epsilon, Z_\epsilon)_g \geq \sum_{v \in E_g \cup \{v_0\}} N_v \epsilon + C(g_v) \epsilon^{1/\kappa_v}. \]

Moreover, since for every \(v \in M_k \setminus E_g\), \(g_v \equiv 0\) on \(P^1(C_v)\) and \((Z, Z)_g \leq 0\), using also (5-6), we have

\[ \sum_{v \in E_g \cup \{v_0\}} N_v \epsilon \geq \sum_{v \in M_k} N_v \epsilon = -2(\deg Z)^2 h_g(Z), \]

which completes the proof.

### 7. Proofs of Theorems 1 and 2

**Proof of Theorem 1.** Fix \(v_0 \in M_k\). For every \(v \in M_k\), \(g_v\) is a \(1/\kappa_v\)-Hölder continuous function on \((P^1(C_v), d_v)\) for some \(\kappa_v \geq 1\) having the \(1/\kappa_v\)-Hölder constant \(C(g_v) \geq 0\). Set \(\epsilon = 1/(\deg Z)^{2\kappa_v}\). For every test function \(\phi \in C^1(P^1(C_{v_0}))\), by (4-2) and Lemma 6.2,

\[
\left| \int_{P^1(C_{v_0})} \phi \ d\left( \frac{Z}{\deg Z} - \mu^g_{v_0} \right) \right| \leq \frac{\text{Lip}(\phi)_{v_0}}{(\deg Z)^{2\kappa_0}} + \frac{\langle \phi, \phi \rangle_{v_0}^{1/2}}{N_{v_0}^{1/2}} \cdot (Z \times Z)(\text{diag}_{P^1(k_v)}) \cdot \sum_{v \in E_g \cup \{v_0\}} N_v \cdot \left( 2 \cdot h_g(Z) + (-C_{\text{abs}} + 2\kappa_{v_0} \log \deg Z) \cdot \frac{(Z \times Z)(\text{diag}_{P^1(k_v)})}{(\deg Z)^2} \right) \cdot \sum_{v \in E_g \cup \{v_0\}} N_v \cdot \left( \frac{1}{(\deg Z)^{2\kappa_0}} + \frac{C(g_v)}{(\deg Z)^2} \right)^{1/2},
\]

which completes the proof.

**Proof of Theorem 2.** Fix \(v_0 \in M_k\). For every \(n \in \mathbb{N}\), we have \((Z_n, Z_n)_g \leq 0\) if \(v \in M_k \setminus E_g\). Hence by (2-8), (5-6), and the assumption that \(V_{g_v} = 0\) for every
\( v \in M_k \), we obtain
\[
N_{v_0}(Z_n, Z_n)_{g_0} \left( \frac{\deg Z_n}{2} \right) + \#E_g \cdot o(1) \geq \sum_{v \in M_k} N_v(Z_n, Z_n)_{g_v} \left( \frac{\deg Z_n}{2} \right)
\]
\[
\geq -2 \cdot h_g(Z_n) - 2 \left( \frac{(Z_n \times Z_n)(\text{diag}_{P^1(k_v)})}{(\deg Z_n)^2} \right) \sum_{v \in E_g} \sup_{P^1(C_v)} \|g_v\| \quad \text{as } n \to \infty;
\]
thus, under the assumption that \((Z_n)\) has both small diagonals and small \(g\)-heights, we have \(\liminf_{n \to \infty} (Z_n, Z_n)_{g_0} / (\deg Z_n)^2 \geq 0 = V_{g_0} \). Hence (2-7) holds for \(g_{v_0}\) and \((Z_n)\), and the proof is complete.

\[\square\]

8. Nonarchimedean and complex dynamics

**Fact 8.1.** Let \(k\) be a field. For a rational function \(\phi \in k(z)\), we call
\[
F_\phi = ((F_\phi)_0, (F_\phi)_1) \in \bigcup_{d \in \mathbb{N} \cup \{0\}} (k[p_0, p_1]_d \times k[p_0, p_1]_d)
\]
a lift of \(\phi\) if \(\pi \circ F_\phi = \phi \circ \pi\) on \(k^2 \setminus \{0\}\) and, in addition, \(F_\phi^{-1}(0) = \{0\}\) when \(\deg \phi > 0\). The latter nondegeneracy condition is equivalent to the nonvanishing of \(\text{Res}(F_\phi) := \text{Res}((F_\phi)_0, (F_\phi)_1)\); for the definition of the homogeneous resultant \(\text{Res}(P, Q) \in k\) for \(P, Q \in \bigcup_{d \in \mathbb{N} \cup \{0\}} k[p_0, p_1]_d\), see, e.g., [Silverman 2007, §2.4]. Such a lift \(F_\phi\) of \(\phi\) is unique up to multiplication in \(k^*\), and is in fact in \(k[p_0, p_1]_{\deg \phi} \times k[p_0, p_1]_{\deg \phi}\).

Let \(K\) be an algebraically closed field that is complete with respect to a nontrivial absolute value \(|\cdot|\).

8.1. The dynamical Green function \(g_f\) on \(P^1\). For the foundation of a potential-theoretical study of dynamics on the Berkovich projective line, see [Baker and Rumely 2010; Favre and Rivera-Letelier 2010] for nonarchimedean \(K\) and, e.g., [Berteloot and Mayer 2001, §VIII] for archimedean \(K \equiv \mathbb{C}\).

**Fact 8.2.** Let \(\phi \in K(z)\) be a rational function of degree \(d_0 \in \mathbb{N} \cup \{0\}\). The action of \(\phi\) on \(P^1 = P^1(K)\) uniquely extends to a continuous endomorphism on \(P^1 = P^1(K)\). When \(d_0 > 0\), the extended \(\phi\) is surjective, open, and discrete and preserves \(P^1\) and \(H^1 = H^1(K)\), the local degree function \(z \mapsto \deg_z \phi\) on \(P^1\) also canonically extends to \(P^1\), and the (mapping) degree of the extended \(\phi : P^1 \to P^1\) still equals \(d_0\) (see [Baker and Rumely 2010, §2.3, §9; Benedetto 2010, §6.3]): in particular, the extended action of \(\phi\) on \(P^1\) induces a push-forward \(\phi_*\) and a pullback \(\phi^*\) on the spaces of continuous functions and of Radon measures on \(P^1\). When \(d_0 = 0\), the extended \(\phi\) is still constant, and we set \(\phi^* \mu := 0\) on \(P^1\) for every Radon measure \(\mu\) on \(P^1\) by convention. Let \(F_\phi \in K[p_0, p_1]_{\deg \phi} \times K[p_0, p_1]_{\deg \phi}\) be a lift of \(\phi\). The function
on \( K^2 \setminus \{0\} \) descends to \( \mathbb{P}^1 \) and in turn extends continuously to \( \mathbb{P}^1 \), satisfying \( \Delta T_{F_\phi} = \phi^* \Omega_{\text{can}} - (\deg \phi) \Omega_{\text{can}} \) on \( \mathbb{P}^1 \) (see, e.g., [Okuyama 2013a, Definition 2.8]). Moreover, \( \phi \) is a Lipschitz continuous endomorphism on \( (\mathbb{P}^1, d) \) and \( T_{F_\phi} \) is a Lipschitz continuous function on \( (\mathbb{P}^1, d) \) (for nonarchimedean \( K \), see [Baker and Rumely 2010, Proposition 9.37]). For every \( n \in \mathbb{N} \), the homogeneous polynomial \( F^n_\phi \in K[p_0, p_1]_{\deg \phi^n} \times K[p_0, p_1]_{\deg \phi^n} \) is a lift of \( \phi^n \).

Let \( f \in K(z) \) be a rational function of degree \( d > 1 \), and consider a lift \( F \in K[p_0, p_1]_d \times K[p_0, p_1]_d \) of \( f \). The uniform limit \( g_F := \lim_{n \to \infty} T_{F^n} / d^n \) on \( \mathbb{P}^1 \) exists, and more precisely, for every \( n \in \mathbb{N} \),

\[
\sup_{\mathbb{P}^1} \left| g_F - \frac{T_{F^n}}{d^n} \right| \leq \sup_{\mathbb{P}^1} \left| \frac{T_F}{d^n} \right| \left| \frac{1}{d^n(d-1)} \right|.
\]

The limit \( g_F \) is called the dynamical Green function of \( F \) on \( \mathbb{P}^1 \) and is a continuous weight on \( \mathbb{P}^1 \). The probability Radon measure

\[
\mu_f := \mu^{g_F} = \Delta g_F + \Omega_{\text{can}} = \lim_{n \to \infty} \frac{(f^n)^* \Omega_{\text{can}}}{d^n}
\]

is independent of the choice of \( F \) and satisfies \( f^* \mu_f = d \cdot \mu_f \) on \( \mathbb{P}^1 \). It is called the \( f \)-equilibrium (or canonical) measure on \( \mathbb{P}^1 \). Moreover, \( g_F \) is a Hölder continuous function on \( (\mathbb{P}^1, d) \) (for nonarchimedean \( K \), see [Favre and Rivera-Letelier 2006, §6.6]). The remarkable energy formula

\[
V_{g_F} = -\log |\text{Res } F| - \frac{\log d}{d(d-1)}
\]

was first established by DeMarco [2003] for archimedean \( K \) and was generalized to rational functions defined over a number field by Baker and Rumely [2006] (for a simple proof of (8-3) which also works for general \( K \), see [Baker 2009, Appendix A] or [Okuyama and Stawiska 2011, Appendix]). The dynamical Green function \( g_f \) of \( f \) on \( \mathbb{P}^1 \) is the unique normalized weight on \( \mathbb{P}^1 \) such that \( \mu^{g_f} = \mu_f \), i.e., for any lift \( F \) of \( f \), \( g_f \equiv g_F + V_{g_F} / 2 \) on \( \mathbb{P}^1 \).

### 8.2. A Berkovich space version of the quasiperiodicity region \( \mathcal{E}_f \)

For nonarchimedean dynamics, see [Baker and Rumely 2010, §10; Favre and Rivera-Letelier 2010, §2.3; Benedetto 2010, §6.4]. For complex dynamics, see, e.g., [Milnor 2006].

Let \( f \in K(z) \) be a rational function of degree \( > 1 \). The Berkovich Julia set of \( f \) is

\[
J(f) := \left\{ S \in \mathbb{P}^1 : \bigcap_{U \text{ open in } \mathbb{P}^1} \mathbf{S} \left( \bigcup_{n \in \mathbb{N}} f^n(U) \right) = \mathbb{P}^1 \setminus E(f) \right\},
\]

where \( E(f) := \{ a \in \mathbb{P}^1 : \# \bigcup_{n \in \mathbb{N}} f^{-n}(a) < \infty \} \) is the exceptional set of \( f \). The
Berkovich Fatou set is $F(f) := \mathbb{P}^1 \setminus J(f)$. By definition, $J(f)$ is closed and $F(f)$ is open in $\mathbb{P}^1$, both $J(f)$ and $F(f)$ are totally invariant under $f$, and $J(f)$ has no interior point unless $J(f) = \mathbb{P}^1$. The classical Julia set $J(f) \cap \mathbb{P}^1$ (resp. the Berkovich Fatou set $F(f) \cap \mathbb{P}^1$) coincides with the set of all nonequicontinuity points (resp. the region of local equicontinuity) of the family $\{f^n : n \in \mathbb{N}\}$ as a family of endomorphisms on $(\mathbb{P}^1, [z, w])$.

A component $U$ of $F(f)$ is called a Berkovich Fatou component of $f$, and is said to be cyclic under $f$ if $f^n(U) = U$ for some $n \in \mathbb{N}$, which is called a period of $U$ under $f$. Following [Fatou 1920, §28], a cyclic Berkovich Fatou component $U$ of $f$ having a period $n \in \mathbb{N}$ is called a singular domain of $f$ if $f^n : U \to U$ is injective. Let $E_f$ be the set of all points $S \in \mathbb{P}^1$ having an open neighborhood $V$ in $\mathbb{P}^1$ such that $\liminf_{n \to \infty} \sup_{V \cap \mathbb{P}^1} |f^n| = 0$, which is a Berkovich space version of Rivera-Letelier’s quasiperiodicity region of $f$. When $K$ is archimedean, $E_f$ coincides with the union of all singular domains of $f$, and when $K$ is nonarchimedean, $E_f$ is still open and forward invariant under $f$ and is contained in the union of all singular domains of $f$ (see [Okuyama 2013a, Lemma 4.4]).

The following function $T_*$ is Rivera-Letelier’s iterative logarithm of $f$ on $E_f \cap \mathbb{P}^1$, which is a nonarchimedean counterpart of the uniformization of a Siegel disk or a Herman ring of $f$.

**Theorem 8.3** ([Rivera-Letelier 2003, §3.2, §4.2]. See also [Favre and Rivera-Letelier 2010, Théorème 2.15]). Suppose that $K$ is nonarchimedean and has characteristic $0$ and residual characteristic $p$. Let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree $> 1$ and suppose that $E_f \neq \emptyset$, which implies $p > 0$ by [Favre and Rivera-Letelier 2010, Lemme 2.14]. Then for every component $Y$ of $E_f$ not containing $\infty$, there are $k_0 \in \mathbb{N}$, a continuous action $T : \mathbb{Z}_p \times (Y \cap K) \ni (\omega, y) \mapsto T^\omega(y) \in Y \cap K$, and a nonconstant $K$-valued holomorphic function $T_*$ on $Y \cap K$ such that for every $m \in \mathbb{Z}_p$, $(f^{k_0})^m = T^m$ on $Y \cap K$, that for every $\omega \in \mathbb{Z}_p$, $T^\omega$ is a biholomorphism on $Y \cap K$, and that for every $\omega_0 \in \mathbb{Z}_p$,

\begin{equation}
\lim_{\omega \to \omega_0} \frac{T^\omega - T^{\omega_0}}{\omega - \omega_0} = T_* \circ T^{\omega_0} \quad \text{locally uniformly on } Y \cap K.
\end{equation}

**8.3. The fundamental relationship between $\mu_f$ and $J(f)$.** If $K$ is archimedean, the inclusion $\text{supp } \mu_f \subset J(f)$ is classical, but it is not trivial from the definition of $J(f)$ when $K$ is nonarchimedean. For an elementary proof, see [Okuyama 2013a, proof of Theorem 2.18]. Actually the equality $\text{supp } \mu_f = J(f)$ holds, but we will dispense with the reverse (and easier) inclusion $J(f) \subset \text{supp } \mu_f$.

9. Proofs of Theorems 3 and 4

Let $k$ be a product formula field. The proof of the following is based not only on (PF) but also on elimination theory (and the strong triangle inequality).
Theorem 9.1 [Baker and Rumely 2006, Lemma 3.1]. Let $k$ be a product formula field. For every $\phi \in k(z)$ and every lift $F_\phi \in k[p_0, p_1]_{\deg \phi} \times k[p_0, p_1]_{\deg \phi}$ of $\phi$, there exists a finite subset $E_{F_\phi}$ in $M_k$ containing all the infinite places of $k$ such that for every $v \in M_k \setminus E_{F_\phi}$, we have $|\text{Res} \ F_\phi|_v = 1$ and $\|F_\phi(\cdot)\|_v = \|\cdot\|_v^{\deg \phi}$ on $C^2_v$.

Let $f \in k(z)$ be a rational function of degree $> 1$ and $F \in k[p_0, p_1]_d \times k[p_0, p_1]_d$ a lift of $f$. Then the family $\hat{g}_f = \{g_{f,v} : v \in M_k\}$ is an adelic normalized weight, where $g_{f,v}$ is the dynamical Green function of $f$ on $P^1(C_v)$ for every $v \in M_k$. Indeed, letting $g_{F,v}$ be the dynamical Green function of $F$ on $P^1(C_v)$ for each $v \in M_k$ and $E_F$ be a finite subset in $M_k$ obtained by Theorem 9.1 applied to $F$, for every $v \in M_k \setminus E_F$ we have $T_{F^n,v} \equiv 0$ on $P^1(C_v)$ for every $n \in \mathbb{N}$, giving $g_{F,v} \equiv g_{F,v} \equiv 0$ on $P^1(C_v)$. We call the adelic normalized weight $\hat{g}_f = \{g_{f,v} : v \in M_k\}$ and the adelic dynamical Green function of $f$ and the adelic $f$-equilibrium (or canonical) measure, respectively. Here, for every $v \in M_k$, $\mu_{f,v} := \mu_{g_{f,v}} = \mu_{\hat{g}_f,v}$ (as in Section 1) is the $f$-equilibrium (or canonical) measure on $P^1(C_v)$.

Lemma 9.2. Let $k$ be a product formula field. Let $f, a \in k(z)$ be rational functions and suppose $d := \deg f > 1$. Then the sequence $([f^n = a])$ of $k$-effective divisors on $P^1(\bar{k})$ has strictly small $\hat{g}_f$-heights in that

$$\limsup_{n \to \infty} (d^n + \deg a) \cdot h_{\hat{g}_f}([f^n = a]) < \infty.$$ 

Proof. Let $F \in k[p_0, p_1]_d \times k[p_0, p_1]_d$ and $A \in k[p_0, p_1]_{\deg a} \times k[p_0, p_1]_{\deg a}$ be lifts of $f$ and $a$, respectively. Then $F^n \wedge A \in k[p_0, p_1]_{d^n + \deg a} \times k[p_0, p_1]_{d^n + \deg a}$ is a representative of $[f^n = a]$ for every $n \in \mathbb{N}$ such that $f^n \neq a$. Let $E_F, E_A$ be finite subsets in $M_k$ obtained by applying Theorem 9.1 to $F, A$, respectively, so that for every $v \in M_k \setminus (E_F \cup E_A)$ and every $n \in \mathbb{N}$, we have $T_{F^n,v} \equiv T_{A,v} \equiv 0$ and $g_{F,v} \equiv 0$ on $P^1(C_v)$. For every $v \in M_k$ and every sufficiently large $n \in \mathbb{N}$, since $|F^n \wedge A|_v \leq \|F^n\|_v \|A\|_v$ on $C^2_v \setminus \{0\}$, we have log $S_{F^n \wedge A,v} \leq T_{F^n,v} + T_{A,v}$ on $P^1(C_v)$ and in turn on $P^1(C_v)$ (recalling that $S_{F^n \wedge A,v} = |(F^n \wedge A)(\cdot/\|\cdot\|)|_v$ on $P^1(C_v)$), so using also $g_{F,v} \equiv g_{F,v} + V_{g_{F,v}}/2$ on $P^1(C_v)$, we obtain

$$\log S_{F^n \wedge A,v} - g_{F,v} \leq \frac{T_{F^n,v} + T_{A,v}}{d^n + \deg a} - \left(g_{F,v} + \frac{1}{2} V_{g_{F,v}}\right) \quad \text{on} \quad P^1(C_v).$$

Hence, by the definition (1-1) of $h_{\hat{g}_f}$, the Jensen-type formula (2-11), the energy formula (8-3) (with $\text{Res} \ F \in k \setminus \{0\}$), and (PF), we have

$$h_{\hat{g}_f}([f^n = a]) \leq \sum_{v \in M_k} N_v \int_{P^1(C_v)} \left(\frac{T_{F^n,v} + T_{A,v}}{d^n + \deg a} - g_{F,v}\right) \, d\mu_{f,v} - \frac{3}{2} \sum_{v \in M_k} N_v \cdot V_{g_{F,v}}$$

$$= \sum_{v \in E_F \cup E_A} N_v \int_{P^1(C_v)} \left(\frac{T_{F^n,v} + T_{A,v}}{d^n + \deg a} - g_{F,v}\right) \, d\mu_{f,v}$$

$$= O(d^{-n}) \quad \text{as} \quad n \to \infty,$$
where the final order estimate is by (8-2) and \( \#(E_F \cup E_A) < \infty \).

With the help of Lemma 9.2, Theorems 3 and 4 follow from Theorems 1 and 2, respectively.

We omit the proof of the following characterization of \( \hat{h}_{gf} \), which we will dispense with in this article.

**Lemma 9.3.** Let \( k \) be a product formula field. Then for every rational function \( f \in k(z) \) of degree \( d > 1 \), the \( \hat{h}_{gf} \)-height function \( h_{\hat{g}_f} \) coincides with the Call–Silverman \( f \)-dynamical (or canonical) height function in that for every \( k \)-effective divisor \( \mathcal{D} \) on \( \mathbb{P}^1(\overline{k}) \), \( (f \circ \mathcal{D}) \) is also a \( k \)-effective divisor on \( \mathbb{P}^1(\overline{k}) \), and the equality \( (h_{\hat{g}_f} \circ f)(\mathcal{D}) = (d \cdot h_{\hat{g}_f})(\mathcal{D}) \) holds.

### 10. Proofs of Theorems 5 and 6

Let \( K \) be an algebraically closed field that is complete with respect to a nontrivial absolute value \( | \cdot | \). For subsets \( A, B \subset \mathbb{P}^1 \), set \([A, B] := \inf_{z \in A, z' \in B} |z - z'|\).

Let \( f, a \in K(z) \) be rational functions and suppose that \( d := \deg f > 1 \). Let \( N \in \mathbb{N} \) be so large that \( f^n \neq a \) if \( n > N \). Then \( \left( \bigcup_{n > N} \text{supp} [f^n = a] \cup J(f) \right) \cap \mathbb{P}^1 \) is closed in \( \mathbb{P}^1 \).

**Lemma 10.1.** Suppose that \( K \) has characteristic 0. Let \( D \) be a chordal disk in \( \mathbb{P}^1 \) of radius \( r > 0 \) satisfying \( \lim \inf_{n \to \infty} \sup_D [f^n, a] = 0 \). Then:

(i) \( a(D) \subset \mathcal{E}_f \).

(ii) \( \mathcal{D} \setminus \left( \bigcup_{n > N} \text{supp} [f^n = a] \cup J(f) \right) \not= \emptyset \).

(iii) There is a chordal disk \( D' \) in \( \mathbb{P}^1 \setminus J(f) \) of radius \( r > 0 \) such that

\[
\lim_{n \to \infty} \inf [f^n(D'), a(D')] > 0.
\]

**Proof of (i).** Since \( \lim \inf_{n \to \infty} \sup_D [f^n, a] = 0 \), there is a sequence \( (n_j) \in \mathbb{N} \) such that \( \lim_{j \to \infty} \sup_D [f^{n_j}, a] = 0 \) and \( \lim_{j \to \infty} (n_{j+1} - n_j) = \infty \). For every \( z \in D \), set \( D' := \{ w \in \mathbb{P}^1 : [w, a(z)] \leq r \} \) in \( a(D) \) for \( r > 0 \) small enough. Then \( \lim \inf [f^{n_j} \circ \text{Id}, \mathcal{D}] \leq \lim \sup [f^{n_j+1} \circ \text{Id}, \mathcal{D}] = 0 \), so that \( a(z) \in \mathcal{E}_f \). Hence \( a(D) \subset \mathcal{E}_f \).

**Proof of (ii).** When \( K \) is archimedean, let \( Y \) be the component of \( \mathcal{E}_f \) containing \( a(D) \), which is by the first assertion either a Siegel disk or a Herman ring of \( f \). Setting \( k_0 := \min \{ n \in \mathbb{N} : f^n(Y) = Y \} \), there are a sequence \( (n_j) \) and an \( N \) in \( \mathbb{N} \) with the properties that \( f^{n_j} \subset Y \), that \( k_0 \) \( (n_j - n_N) \) for every \( j \geq N \), and that \( a = \lim_{j \to \infty} (f^{k_0})(n_j - n_N)/k_0 \circ f^{n_N} \) uniformly on \( D \). Then \( D \cap J(f) = \emptyset \). Let \( \lambda \in \mathbb{C} \) be the rotation number of \( Y \), so that there exists a holomorphic injection \( h : Y \to \mathbb{C} \) such that \( h \circ f^{k_0} = \lambda \cdot h \) on \( Y \). Then \( |\lambda| = 1 \) but \( \lambda \) is not a root of unity (by \( d > 1 \)). Choosing a subsequence of \( (n_j) \) if necessary, \( \lambda_a := \lim_{j \to \infty} \lambda^{(n_j - n_N)/k_0} \in \mathbb{C} \).
exists. For every \( n \geq n_N \), if \( k_0 \mid (n - n_N) \), then \( D \cap \text{supp } [f^n = a] = \emptyset \), whereas if \( k_0 \mid (n - n_N) \), then \( h \circ f^n - h \circ a = (\lambda^{(n-n_N)/k_0} - \lambda_a) \cdot (h \circ f^{nN}) \) on \( D \), so \((D \setminus (h \circ f^{nN})^{-1}(0)) \cap \text{supp } [f^n = a] = \emptyset \) if \( n \) is large enough.

When \( K \) is nonarchimedean, let \( Y \) be the component of \( \mathcal{E}_f \) containing \( a(D) \). Without loss of generality, we assume that \( \infty \not\in Y \), and then applying Theorem 8.3 to this \( Y \), we obtain \( p \in \mathbb{N}, k_0 \in \mathbb{N}, T, \) and \( T_\ast \) as in the theorem. There are a sequence \((n_j)\) and an \( N \) in \( \mathbb{N} \) such that \( f^{n_jN}(D) \subset Y, k_0 \mid (n_j - n_N) \) for every \( j \geq N \), and \( a = \lim_{j \to \infty} (f^{k_0N}(n_j - n_N)/k_0) \circ f^{nN} \) uniformly on \( D \). Then \( D \cap \text{supp } f^n = \emptyset \). Choosing a subsequence of \((n_j)\) if necessary, \( \omega_a := \lim_{j \to \infty} (n_j - n_N)/k_0 \in \mathbb{Z}_p \) exists. For every \( n \geq n_N \), if \( k_0 \mid (n - n_N) \), then \( D \cap \text{supp } [f^n = a] = \emptyset \), whereas if \( k_0 \mid (n - n_N) \), then

\[
\text{(10-1)} \\
f^n - a = (T^{(n-n_N)/k_0} - T^{\omega_a}) \circ f^{nN}
\]
on \( D \). Choose \( b \in D \setminus \{\infty\} \) and \( r \in |K^*| \) small enough that the \((K\text{-closed})\) disk \( B = \{z \in K : |z - b| \leq r\} \) is contained in \( D \), and fix \( \epsilon \in |K^*| \) so small that for \( Z_\epsilon := \bigcup_{w \in B \cap (T \circ T^{\omega_a} \circ f^{nN})^{-1}(0)} \{z \in B : |z - w| < \epsilon\} \), we have \( B \setminus Z_\epsilon \neq \emptyset \). The maximum modulus principle from rigid analysis (see [Bosch, Güntzer, and Remmert 1984, §6.2.1, §7.3.4]) gives \( \min_{z \in f^{nN}(B \setminus Z_\epsilon)} |T \circ T^{\omega_a}(z)| > 0 \), so by the uniform convergence \((8-4)\) and the equality \( \text{(10-1)} \), \( (B \setminus Z_\epsilon) \cap \text{supp } [f^n = a] = \emptyset \) if \( n \) is large enough. \( \square \)

Proof of (iii). By the first assertion, there is a unique singular domain \( U \) of \( f \) containing \( a(D) \). Fix \( n_0 \in \mathbb{N} \) such that \( f^{n_0}(U) = U \), and set \( C := \bigcup_{j=0}^{n_0-1} f^j(U) \). Then there is a component \( V \) of \( f^{-1}(C) \setminus C \) since \( f : C \to C \) is injective and \( d > 1 \). Fix a chordal disk \( D'' \) of radius \( > 0 \) in \( a^{-1}(V) \cap (\mathbb{P}^1 \setminus J(f)) \), so that \( a(D'') \subset V \subset f^{-1}(C) \setminus C \). If \( a(D'') \cap \bigcup_{n \in \mathbb{N}\cup\{0\}} f^n(D'') = \emptyset \), then we are done by setting \( D' := \{z \in \mathbb{P}^1 : |z, b| \leq r\} \) for some \( b \in D'' \) and \( r > 0 \) small enough. But if there is \( N \in \mathbb{N}\cup\{0\} \) such that \( a(D'') \cap f^N(D'') \neq \emptyset \), then by setting \( D' := \{z \in \mathbb{P}^1 : |z, b| \leq r\} \) for some \( b \in D'' \cap f^{-N}(a(D'')) \) and \( r > 0 \) small enough, we get \( \lim\inf_{n \to \infty} |a(D'), f^n(D')| > 0 \) from

\[
a(D') \cap \bigcup_{n \geq N+1} f^n(D') \subset a(D') \cap \bigcup_{n \in \mathbb{N}} f^n(a(D'')) \subset V \cap C = \emptyset. \quad \square
\]

Lemma 10.2. For every \( w_0 \in \mathbb{P}^1 \setminus \left( \bigcup_{n > N} \text{supp } [f^n = a] \cup J(f) \right) \), there is a function \( \phi_0 \in C^1(\mathbb{P}^1) \) such that \( \phi_0 \equiv \log |w_0, \cdot|_{\text{can}} \) on \( \bigcup_{n > N} \text{supp } [f^n = a] \cup J(f) \).

Proof. Fix \( w_0 \in \mathbb{P}^1 \setminus \left( \bigcup_{n > N} \text{supp } [f^n = a] \cup J(f) \right) \). Without loss of generality, we can assume that \( w_0 \neq \infty \), and fix \( \epsilon > 0 \) so small that

\[
\{S \in \mathbb{P}^1 : |S - w_0|_{\infty} \leq \epsilon\} \subset \mathbb{P}^1 \setminus \left( \bigcup_{n > N} \text{supp } [f^n = a] \cup J(f) \right)
\]

(recall Sections 3.1 and 3.2 here).
When $K$ is nonarchimedean, by the definition of the map $\pi_\epsilon : \mathbb{A}^1 \to \mathbb{A}^1$, we have \( \{ S \in \mathbb{P}^1 \setminus \{ w_0 \} : |S - w_0|_\infty \leq \epsilon \} \). The function
\[
S \mapsto \phi_0(S) := \begin{cases} 
\log [w_0, \pi_\epsilon(w_0)]_{\text{can}} & \text{if } S \leq \pi_\epsilon(w_0), \\
\log [w_0, S]_{\text{can}} & \text{otherwise} \end{cases} \quad \text{on } \mathbb{P}^1
\]
is in $C^1(\mathbb{P}^1)$ since it is continuous on $\mathbb{P}^1$, locally constant on $\mathbb{P}^1$ except for the segment $I$ in $H^1$ joining $\pi_\epsilon(w_0)$ and $S_{\text{can}}$, and linear on $I$ with respect to the length parameter induced by the hyperbolic metric $\rho$ on $H^1$. When $K$ is archimedean (so $\mathbb{P}^1 \cong \mathbb{P}^1$), there is a function $\phi_0 \in C^1(\mathbb{P}^1)$ satisfying
\[
z \mapsto \phi_0(z) = \begin{cases} 
\int_{\mathbb{P}^1} \log [w_0, w] d[z]_{e/2}(w) & \text{if } |z - w_0| \leq \epsilon/2, \\
\log [w_0, z] & \text{if } |z - w_0| \geq \epsilon \text{ or } z = \infty. 
\end{cases}
\]
In both cases, the given $\phi_0 \in C^1(\mathbb{P}^1)$ satisfies the desired property. \qed

**Fact 10.3.** For rational functions $\phi, \psi \in K(z)$, the **chordal proximity function**
\[
S \mapsto [\phi, \psi]_{\text{can}}(S) \quad \text{on } \mathbb{P}^1
\]
between $\phi$ and $\psi$ is the unique continuous extension of the function $z \mapsto [\phi(z), \psi(z)]$ on $\mathbb{P}^1$ to $\mathbb{P}^1$ (see [Okuyama 2013a, Proposition 2.9] for its construction, as well as Remark 2.10 of the same paper), and for every continuous weight $g$ on $\mathbb{P}^1$, we also define its weighted version by $\Phi(\phi, \psi)_g := \log [\phi, \psi]_{\text{can}} - g \circ \phi - g \circ \psi$ on $\mathbb{P}^1$.

For every $n \in \mathbb{N}$ such that $f^n \not\equiv a$, recall the **Riesz decomposition**
\begin{equation}
(10-2) \quad \Phi(f^n, a)_g = U_{g.f,[f^n=a]-(d^n+\deg a)\mu_f} - U_{g.f,a^*\mu_f} + \int_{\mathbb{P}^1} \Phi(f^n, a)_g d\mu_f
\end{equation}
on $\mathbb{P}^1$, and also $U_{g.f,a^*\mu_f} = g_f \circ a + U_{g.f,a^*\Omega_{\text{can}} - \int_{\mathbb{P}^1} (g_f \circ a) d\mu_f}$ on $\mathbb{P}^1$ [Okuyama 2013a, Lemma 2.19].

**Proof of Theorem 5.** Let $k$ be a product formula field of characteristic $0$. Let $f \in k(z)$ be a rational function of degree $d > 1$ and $a \in k(z)$ a rational function of degree $0$. Let $N \in \mathbb{N}$ be so large that $f^n \not\equiv a$ if $n > N$. Fix $v \in M_k$. Let $D$ be a chordal disk in $\mathbb{P}^1(\mathbb{C}_v)$ of radius $> 0$, and assume that $\lim \inf_{n \to \infty} \sup D[f^n, a]_v = 0$; otherwise we are done. By Lemma 10.1, there are not only a point $w_0 \in D \setminus (\bigcup_{n > N} \{ f^n = a \} \cup J(f)_v$) but also a chordal disk $D'$ in $\mathbb{P}^1(\mathbb{C}_v) \setminus J(f)_v$ of radius $> 0$ such that $\lim \inf_{n \to \infty} [f^n(D'), a(D')]_v > 0$. Fix a point $w_1 \in D'$. Then also $w_1 \in \mathbb{P}^1 \setminus (\bigcup_{n > N} \{ f^n = a \} \cup J(f)_v$).

For every $n \in \mathbb{N}$ large enough and every $j \in \{0, 1\}$, by (10-2),
\begin{equation}
(10-3) \quad \log [f^n(w_j), a(w_j)]_v - g_{f,v}(f^n(w_j)) - g_{f,v}(a(w_j)) - U_{g_{f,v}^*[f^n=a]-(d^n+\deg a)\mu_{f,v}(w_j)} - U_{g_{f,v},a^*\mu_{f,v}(w_j)} + \int_{\mathbb{P}^1(\mathbb{C}_v)} \Phi(f^n, a)_{g_{f,v}} d\mu_{f,v},
\end{equation}
so that taking the difference of both sides in (10-3) for each \(j \in \{0, 1\}\) and noting that \(g_{f,v}\) and \(U_{g_{f,v},a^{*}f_{v}}\) are bounded on \(\mathbb{P}^{1}(\mathbb{C}_{v})\), we have

\[
\log [f^{n}(w_{0}), a(0)]_{v} - \log [f^{n}(w_{1}), a(1)]_{v} = \int_{\mathbb{P}^{1}(\mathbb{C}_{v})} \log [w_{0}, S'_{\text{can}, v}] d([f^{n} = a] - (d^{n} + \deg a)\mu_{f})(S') - \int_{\mathbb{P}^{1}(\mathbb{C}_{v})} \log [w_{1}, S'_{\text{can}, v}] d([f^{n} = a] - (d^{n} + \deg a)\mu_{f})(S') + O(1)
\]

as \(n \to \infty\). In the left hand side, by the choice of \(w_{0}\) and \(w_{1}\), we have

\[
\log \sup_{D} [f^{n}, a]_{v} \geq \log [f^{n}(w_{0}), a(0)]_{v}
\]

and

\[
\liminf_{n \to \infty} \log [f^{n}(w_{1}), a(1)]_{v} \geq \liminf_{n \to \infty} \log [f^{n}(D'), a(D')]_{v} > -\infty,
\]

so that as \(n \to \infty\),

\[
\log \sup_{D} [f^{n}, a]_{v} + O(1) \geq \log [f^{n}(w_{0}), a(0)]_{v} - \log [f^{n}(w_{1}), a(1)]_{v}.
\]

In the right hand side, for each \(j \in \{0, 1\}\), by Lemma 10.2 applied to \(w_{j}\), the inclusion \(\text{supp} \mu_{f} \subset J(f)\), and Theorem 3 (and \(k_{s} = \bar{k}\) in the characteristic 0 case), we have

\[
\int_{\mathbb{P}^{1}(\mathbb{C}_{v})} \log [w_{j}, S'_{\text{can}, v}] d([f^{n} = a] - (d^{n} + \deg a)\mu_{f})(S') = O\left(\sqrt{n \cdot ([f^{n} = a] \times [f^{n} = a])(\text{diag}_{\mathbb{P}^{1}(ar{k})})}\right) \text{ as } n \to \infty.
\]

These estimates complete the proof of (1-4) for this \(v \in M_{k}\).

\[\square\]

**Fact 10.4.** For a rational function \(f(z) \in k(z)\) over a field \(k\), a point \(w \in \mathbb{P}^{1}(\bar{k})\) is called a **multiple** periodic point of \(f\) if \([f^{n} = \text{Id}](\{w\}) > 1\) for some \(n \in \mathbb{N}\). For a rational function \(f(z) \in k(z)\) over a field \(k\) of characteristic 0, there are at most finitely many multiple periodic points of \(f\) in \(\mathbb{P}^{1}(\bar{k})\); this is well known in the case that \(k = \mathbb{C}\) (see, e.g., [Milnor 2006, §13]), and holds in general by the **Lefschetz principle** (see, e.g., [Eklof 1973]).

**Proof of Theorem 6.** As noted above, \(f\) has at most finitely many multiple periodic points in \(\mathbb{P}^{1}(\bar{k})\), and for every multiple periodic point \(w\) of \(f\), setting \(p = p_{w} := \min\{n \in \mathbb{N} : [f^{n} = \text{Id}](\{w\}) > 1\}\), by the (formal) power series expansion \(f^{p}(z) = w + (z - w) + C(z - w)^{[f^{p} = \text{Id}](\{w\})} + \cdots\) of \(f^{p}\) around \(w\), we also have \(\sup_{n \in \mathbb{N}} [f^{n} = \text{Id}](\{w\}) \leq [f^{p} = \text{Id}](\{w\})\) under the characteristic 0 assumption.
Hence $\sup_{n \in \mathbb{N}} \left( \sup_{w \in \text{supp} [f^n = \text{Id}]} [f^n = \text{Id}](\{w\}) \right) < \infty$, so that

$$([f^n = \text{Id}] \times [f^n = \text{Id}]) \left( \text{diag}_{\mathbb{P}^1(\bar{k})} \right) \leq (d^n + 1) \cdot \sup_{w \in \text{supp} [f^n = \text{Id}]} [f^n = \text{Id}](\{w\}) = O(d^n)$$

as $n \to \infty$. Now (1-5) follows from (1-4). \qed

11. Proof of Theorem 7

Let $k$ be a field and $k_s$ the separable closure of $k$ in $\bar{k}$. Let $p(z) \in k[z]$ be a polynomial of degree $> 0$ and $\{z_1, \ldots, z_m\}$ the set of all distinct zeros of $p(z)$ in $\bar{k}$ so that $p(z) = a \cdot \prod_{j=1}^m (z - z_j)^{d_j}$ in $k[z]$ for some $a \in k \setminus \{0\}$ and some sequence $(d_j)_{j=1}^m$ in $\mathbb{N}$. For a while, we do not assume $\{z_1, \ldots, z_m\} \subset k_s$. Let $\{p_1(z), p_2(z), \ldots, p_N(z)\}$ be the set of all mutually distinct, nonconstant, irreducible, and monic factors of $p(z)$ in $k[z]$, so that $p(z) = a \cdot \prod_{\ell=1}^N p_{\ell}(z)^{s_\ell}$ in $k[z]$ for some sequence $(s_\ell)_{\ell=1}^N$ in $\mathbb{N}$. For every $\ell \in \{1, 2, \ldots, N\}$, by the irreducibility of $p_\ell(z)$ in $k[z]$, $p_\ell(z)$ is the unique monic minimal polynomial in $k[z]$ of each zero of $p_\ell(z)$ in $\bar{k}$, so $p_\ell(z)$ and $p_n(z)$ have no common zeros in $\bar{k}$ if $\ell \neq n$. Hence for each $j \in \{1, 2, \ldots, m\}$, there is a unique $\ell =: \ell(j) \in \{1, 2, \ldots, N\}$ such that $p_\ell(z_j) = 0$.

Now suppose that $\{z_1, z_2, \ldots, z_m\} \subset k_s$. Then for every $\ell \in \{1, 2, \ldots, N\}$,

$$p_\ell(z) = \prod_{i: \ell(i) = \ell} (z - z_i)$$

in $\bar{k}[z]$, so that

$$d_i = s_{\ell(i)}$$

for every $i \in \{1, 2, \ldots, m\}$. For every distinct $\ell, n \in \{1, 2, \ldots, N\}$,

$$(11-2) \quad \prod_{j: \ell(j) = \ell} \prod_{i: \ell(i) = n} (z_j - z_i) = \prod_{j: \ell(j) = \ell} p_n(z_j) = R(p_\ell, p_n),$$

where $R(p, q) \in k$ is the (usual) resultant of $p(z), q(z) \in k[z]$. The derivation $p'_\ell(z)$ of $p_\ell(z)$ in $k[z]$ satisfies

$$p'_\ell(z) = \sum_{h: \ell(h) = \ell} \left( \prod_{i: i \neq h, \ell(i) = \ell} (z - z_i) \right)$$

in $\bar{k}[z]$. Hence for every $\ell \in \{1, 2, \ldots, N\}$,

$$(11-3) \quad \prod_{j: \ell(j) = \ell} \prod_{i: i \neq j, \ell(i) = \ell} (z_j - z_i) = \prod_{j: \ell(j) = \ell} p'_\ell(z_j) = R(p_\ell, p'_\ell).$$

By (11-1), (11-3), and (11-2), we have
\[
D^*(p) := \prod_{j=1}^{m} \prod_{i:i \neq j} (z_j - z_i)^{d_id_j} = \prod_{j=1}^{m} \prod_{i:i \neq j} (z_j - z_i)^{s_{\ell(j)}s_{\ell(j)}}
\]
\[
= \prod_{\ell=1}^{N} \left( \prod_{j: \ell(j) = \ell} \left( \prod_{i:i \neq j, \ell(i) = \ell} (z_j - z_i)^{s_{\ell}^2} \right) \left( \prod_{n:n \neq \ell, \ell(i) = n} (z_j - z_i)^{s_{n}s_{\ell}} \right) \right)
\]
\[
= \prod_{\ell=1}^{N} \left( R(p_{\ell}, p'_{\ell})^{s_{\ell}^2} \cdot \prod_{n:n \neq \ell} R(p_{\ell}, p_n)^{s_{n}s_{\ell}} \right),
\]
which is in \( k \setminus \{0\} \). Now the proof is complete. □

Acknowledgements

The author thanks the referee for their very careful scrutiny and invaluable comments, and also thanks Professors Mamoru Asada and Joe Silverman for discussions on Theorem 7 and comments. This work was partly done during the author’s visits to National Taiwan Normal University and Academia Sinica, and the author thanks the institutes and Professor Liang-Chung Hsia for their hospitality. This work was partially supported by JSPS Grant-in-Aid for Young Scientists (B), 24740087.

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Received July 29, 2014. Revised April 5, 2015.

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Correction to “Closed orbits of a charge in a weakly exact magnetic field”

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