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COMPUTING HIGHER FROBENIUS–SCHUR INDICATORS IN FUSION CATEGORIES CONSTRUCTED FROM INCLUSIONS OF FINITE GROUPS

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We consider a subclass of the class of group-theoretical fusion categories: To every finite group G and subgroup H one can associate the category of G-graded vector spaces with a two-sided H-action compatible with the grading. We derive a formula that computes higher Frobenius-Schur indicators for the objects in such a category using the combinatorics and representation theory of the groups involved in their construction. We calculate some explicit examples for inclusions of symmetric groups.

1. Introduction

Higher Frobenius–Schur indicators are invariants of an object in a pivotal fusion category (and hence also invariants of that category). They generalize the degree two Frobenius-Schur indicator — which was originally defined for a representation of a finite group by its namesakes in 1906 — to higher degrees and more general objects. Categorical versions of degree two indicators were studied by Bantay [1997], as well as Fuchs, Ganchev, Szlachányi, and Vescernyés [Fuchs et al. 1999]; indicators for modules over semisimple Hopf algebras were introduced by Linchenko and Montgomery [2000] and studied in depth by Kashina, Sommerhäuser, and Zhu [2006]. The degree two indicators for modules over semisimple quasi-Hopf algebras were treated by Mason and Ng [2005]. The higher indicators for pivotal fusion categories that we deal with in the present paper were introduced in [Ng and Schauenburg 2008; 2007b; 2007a].

Frobenius–Schur indicators have become a tool for the structure theory and classification of fusion categories. The problem we deal with here, however, is simply how to calculate them in very specific examples. More concretely we will deal with a specific class of group-theoretical fusion categories [Ostrik 2003; Etingof, Nikshych and Ostrik 2005]. Degree two indicators for Hopf algebras associated

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with such categories have been studied in [Kashina, Mason and Montgomery 2002; Jedwab and Montgomery 2009]. In [Kashina, Sommerhäuser and Zhu 2006] formulas for higher indicators of smash product Hopf algebras associated to a group acting by automorphisms on another group were given. This class of examples includes the Drinfeld double of a finite group. For such doubles, the explicit formulas were used to study the question of integrality of the indicators in [Iovanov, Mason and Montgomery 2014]. Extensive computer calculations, in particular with a view towards the question of whether the indicators of the doubles of symmetric groups are positive, were conducted in [Courter 2012]; examples for certain other groups can be found in [Keilberg 2014; 2012].

Natale [2005] has derived formulas for the degree two Frobenius–Schur indicators of the objects in general group-theoretical fusion categories. Her approach is based on the fact that a group-theoretical fusion category can be written as the module category over a quasi-Hopf algebra which is known explicitly. Then the explicit definition of degree two indicators of modules over quasi-Hopf algebras in [Mason and Ng 2005] can be applied.

In principle the same approach, now using the higher indicator formula for quasi-Hopf algebras from [Ng and Schauenburg 2008], could be used to obtain higher indicator formulas for group-theoretical categories. However, those formulas involve iterated applications of the associator elements of the relevant quasi-Hopf algebra dealing with the parentheses of iterated tensor products in the category. Applying them with the explicit quasi-Hopf structure deriving from the data of a group-theoretical fusion category seems a formidable task.

We will take an entirely different approach. The formula from [Ng and Schauenburg 2007a, Theorem 4.1], generalizing the “third formula” from [Kashina, Sommerhäuser and Zhu 2006], links higher Frobenius–Schur indicators in a spherical fusion category $\mathcal{C}$ to the ribbon structure of the Drinfeld center $\mathcal{Z}(\mathcal{C})$ and the functor from $\mathcal{C}$ to $\mathcal{Z}(\mathcal{C})$ adjoint to the underlying functor. The “third formula” was used in [Shimizu 2011] to calculate indicators in Tambara–Yamagami categories; in our context the approach is aided by the fact that the centers of group-theoretical fusion categories are easy to determine: a group-theoretical fusion category is the monoidal category of bimodules over the (twisted) group algebra of a subgroup H of a finite group G inside the category Vect_G of G-graded vector spaces (twisted by a three-cocycle on G). By [Schauenburg 2001], the Drinfeld center of such a bimodule category is equivalent to the Drinfeld center of the “ambient” category. In different language this means that group-theoretical fusion categories are Morita equivalent to the category of graded vector spaces with twisted associativity; see the survey [Nikshych 2013]. We will treat the case of a group-theoretical fusion category defined without cocycles. Thus $\mathcal{C} = {}^G_H\mathcal{M}_H$, the center is $\mathcal{Z}({}^G_H\mathcal{M}_H) = \mathcal{Z}(\mathrm{Vect}_G)$, equivalent to the category of modules over the Drinfeld double of G.

In a sense, the underlying functor $\mathcal{Z}(\mathrm{Vect}_G) \to {}^G_H\mathcal{M}_H$ is already known explicitly from [Schauenburg 2001], but we need to do more. Simple objects in ${}^G_H\mathcal{M}_H$ are parametrized by group-theoretical data, namely (equivalence classes of) pairs consisting of an element of G and an irreducible representation of a certain stabilizer subgroup of H. Simple objects of $\mathcal{Z}(\mathrm{Vect}_G)$ are also classified by group-theoretical data, (equivalence classes of) pairs consisting of an element of G and an irreducible representation of its centralizer. In Section 3, we will describe the underlying functor $\mathcal{Z}(\mathrm{Vect}_G) \to {}^G_H\mathcal{M}_H$ on the level of simple objects by a formula involving only the combinatorics and representation theory of subgroups of G. Given this description, one can turn things around and describe the adjoint functor ${}^G_H\mathcal{M}_H \to \mathcal{Z}(\mathrm{Vect}_G)$ equally explicitly. Admittedly the resulting description, while completely explicit and entirely on the level of groups, subgroups, and group representations, is quite unwieldy — this is perhaps natural, since one has to deal with how conjugacy classes and centralizers (involved in the description of modules over the Drinfeld double) relate to double cosets of a chosen subgroup, and stabilizers of one-sided cosets under the regular action (involved in the description of ${}^G_H\mathcal{M}_H$).

In Section 4, we will use the description of the adjoint functor and the "third formula" to obtain a formula for the higher indicators of the simple objects of ${}^G_H\mathcal{M}_H$. Luckily we do not need complete information about the adjoint, but only the traces of the ribbon structure on the images under the adjoint. This allows us to dramatically simplify the immediate result based on the complicated description of the adjoint to obtain a surprisingly simple-looking formula for the higher indicators. It is in fact even simpler than Natale's formula for second indicators, and uses only group characters and the combinatorics of group elements and subgroups, without mentioning the associated quasi-Hopf algebra and its characters at all. One should admit, though, that characters of the associated quasi-Hopf algebra are in turn described in more "basic" terms in [Natale 2005]. Also, our results are marred by the obvious limitation that they do not treat general group-theoretical categories, but only those in whose definition the relevant group cocycles are trivial — we have amended this limitation in [Schauenburg 2015].

We also treat variants of the indicator formula that are more complicated, involving passing to orbits under the action of auxiliary subgroups, but computationally advantageous for the same reason that they pass from sums over the entire group H to sums over certain orbits.

In Section 5, we will explicitly calculate indicators in several examples of fusion categories associated to an inclusion of symmetric groups $S_{n-2} \subset S_n$. We use the "simple" version of our indicator formula for the cases $n = 4, 5$. The cases $n = 6, 7$ illustrate how the more complicated versions reduce the size of the calculations needed down to a manageable size.

2. Preliminaries

Throughout the paper, G is a finite group, and $H \subset G$ a subgroup. We denote the adjoint action of G on itself by $x \triangleright g = xgx^{-1}$. If V is a representation of a subgroup $K \subset G$, and $x \in G$, we denote by $x \triangleright V$ the twisted representation of $x \triangleright K$ with the same underlying vector space V on which $y \in x \triangleright K$ acts like $x^{-1} \triangleright y \in K$.

We work over the field $\mathbb{C}$ of complex numbers; representations are complex representations; and characters are ordinary characters.

The category ${}^{G}_{H}\mathcal{M}_H := {}^{\mathbb{C}G}_{\mathbb{C}H}\mathcal{M}_{\mathbb{C}H}$ is defined as the category of $\mathbb{C}H$-bimodules over the group algebra of H, considered as an algebra in the category of $\mathbb{C}G$-comodules, that is, of G-graded vector spaces. Thus, an object of ${}^{G}_{H}\mathcal{M}_H$ is a G-graded vector space $M \in \mathrm{Vect}_G$ with a two-sided H-action compatible with the grading in the sense that $|hmk| = h|m|k$ for $h, k \in H$ and $m \in M$.

The category ${}^{G}_{H}\mathcal{M}_H$ is a fusion category. The tensor product is the tensor product of $\mathbb{C}H$-bimodules. Simple objects are parametrized by irreducible representations of the stabilizers of right cosets of H in G. More precisely, let $D \in H\backslash G/H$ be a double coset of H in G, let $d \in D$, and let $S = \mathrm{Stab}_H(dH) = H \cap (d \triangleright H)$ be the stabilizer in H of the right coset dH under the action of H on its right cosets in G. Then the subcategory ${}^{D}_{H}\mathcal{M}_H \subset {}^{G}_{H}\mathcal{M}_H$, defined to contain those objects the degrees of all of whose homogeneous elements lie in D, is equivalent to the category $\mathrm{Rep}(S)$ of representations of S. The equivalence ${}^{D}_{H}\mathcal{M}_H \to \mathrm{Rep}(S)$ takes M to $(M_{dH})/H \cong (M/H)_{dH/H}$, the space of those vectors in the quotient of M by the right action of H whose degrees lie in the right coset of d. Details are in [Zhu 2001; Schauenburg 2002a]. We will denote the inverse equivalence by $\mathcal{F}_d : \mathrm{Rep}(\mathrm{Stab}_H(dH)) \to {}^{HdH}_{H}\mathcal{M}_H$, so that we have a category equivalence

$$\bigoplus_d \mathrm{Rep}(\mathrm{Stab}_H(dH)) \xrightarrow{(\mathcal{F}_d)_d} {}^{G}_{H}\mathcal{M}_H$$

in which the sum runs over a set of representatives of the double cosets of H in G. Of course ${}^{D}_{H}\mathcal{M}_H$ can be described by choosing a different representative of D. If $h \in H$, then dh has the same right coset as d, and $\mathcal{F}_{dh} = \mathcal{F}_d$, while $\mathrm{Stab}_H(hdH) = h \triangleright \mathrm{Stab}_H(dH)$ and $\mathcal{F}_d(W) = \mathcal{F}_{hd}(h \triangleright W)$ for $W \in \mathrm{Rep}(\mathrm{Stab}_H(dh))$.

In the special case $H = G$, the above description, with the neutral element representing the sole class of G in G, amounts to the (well-known) equivalence $\mathrm{Rep}(G) \cong {}^{G}_{G}\mathcal{M}_G$ sending $V \in \mathrm{Rep}(G)$ to $V \otimes \mathbb{C}G$ with the regular right G-action and the diagonal left G-action. This is a monoidal category equivalence.

The category ${}^{G}_{G}\mathcal{YD} = {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ of (left-left) Yetter–Drinfeld modules over $\mathbb{C}G$ has objects the G-graded vector spaces with a left G-action compatible with the grading in the sense that $|gv| = g|v|g^{-1}$ for $g \in G$ and $v \in V \in {}^{G}_{G}\mathcal{YD}$. The category ${}^{G}_{G}\mathcal{YD}$ is the (right) center of the category ${}^{G}\mathcal{M}$ of G-graded vector spaces: the half-braiding

$c : U \otimes V \to V \otimes U$ between a graded vector space U and a Yetter–Drinfeld module V is given by $u \otimes v \mapsto |u|v \otimes u$. To calculate indicators using the "third formula" we also need the fact that the canonical pivotal structure of ${}^G_G\mathcal{YD}$ is given by the ordinary vector space isomorphism $V \to V^{**}$, so that pivotal trace and ordinary trace coincide. Finally, the ribbon automorphism θ of an object $V \in {}^G_G\mathcal{YD}$ is given by $\theta(v) = |v|v$.

Simple objects of ${}^G_G\mathcal{YD}$ are parametrized by irreducible representations of the centralizers in G of elements of G. (In fact, this can be viewed as a special case of the description of graded bimodules above, as we shall review in Example 4.7 below). More precisely, let $g \in G$ and $C_G(g)$, the centralizer of g in G. Then, a functor

$$\mathcal{G}_g : \operatorname{Rep}(C_G(g)) \to {}^G_G\mathcal{YD}$$

can be defined by sending $V \in \operatorname{Rep}(C_G(g))$ to the $\mathbb{C}G$-module

$$\operatorname{Ind}^G_{C_G(g)} V = \mathbb{C}G \otimes_{\mathbb{C}C_G(g)} V,$$

endowed with the grading given by $|x \otimes v| = xgx^{-1}$ for $x \in G$ and $v \in V$. We note the special case $g = 1$ which recovers the canonical (monoidal) inclusion functor $\operatorname{Rep}(G) \to {}^G_G\mathcal{YD}$. Summing over different elements, we obtain a category equivalence

$$\bigoplus_g \operatorname{Rep}(C_G(g)) \xrightarrow{(\mathcal{G}_g)_g} {}^G_G\mathcal{YD}.$$

The sum runs over a set of representatives of the conjugacy classes of G, and the image of the functor $\mathcal{G}_g$ consists of those Yetter–Drinfeld modules, the degrees of whose homogeneous elements lie in the conjugacy class of g. We note for later use that the ribbon automorphism of $\mathcal{G}_g(V)$ is $\theta(x \otimes v) = (x \triangleright g)(x \otimes v) = xg \otimes v = x \otimes gv$; the trace of θ^m is therefore $[G : C_G(g)]\,\chi(g^m)$, where χ denotes the character of V.

As a final piece of notation, we will write $\langle M, N\rangle := \dim_{\mathbb{C}}(\operatorname{Hom}_{\mathcal{C}}(M, N))$ for objects M, N in a semisimple category.

3. The center and the adjoint

By a result of Müger [2003], the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a pivotal fusion category $\mathcal{C}$ is a modular category, and the underlying functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ has a two-sided adjoint $\mathcal{K}$. To handle the center of ${}^G_H\mathcal{M}_H$ and the adjoint functor $\mathcal{K}$ we use the fact [Schauenburg 2001] that the center of a category of bimodules in a tensor category $\mathcal{C}$ coincides, in many cases including the present one, with the center of $\mathcal{C}$ itself.

To be precise, we will use the "right center" $\overline{\mathcal{Z}}(\mathcal{C})$ whose objects are pairs (V, c) in which $c : X \otimes V \to V \otimes X$ is a half-braiding defined for any $X \in \mathcal{C}$, and we denote by $\overline{\mathcal{K}}$ the adjoint functor of the underlying functor $\overline{\mathcal{Z}}(\mathcal{C}) \to \mathcal{C}$.

Then, writing $\mathcal{C} = {}^G\mathcal{M} = \mathrm{Vect}_G$ for the category of G-graded vector spaces, we have a category equivalence

$$ {}^G_G\mathcal{YD} \cong \bar{\mathcal{Z}}(\mathcal{C}) \longrightarrow \bar{\mathcal{Z}}({}_{\mathbb{C}H}\mathcal{C}_{\mathbb{C}H}) = \bar{\mathcal{Z}}\big({}^G_H\mathcal{M}_H\big) $$

which sends $(N, c) \in \bar{\mathcal{Z}}(\mathcal{C})$ to an object of $\bar{\mathcal{Z}}({}_{\mathbb{C}H}\mathcal{C}_{\mathbb{C}H})$ whose underlying right $\mathbb{C}H$-module is $N \otimes \mathbb{C}H$, whose left $\mathbb{C}H$-module structure is given by

$$ \mathbb{C}H \otimes N \otimes \mathbb{C}H \xrightarrow{c\otimes\mathbb{C}H} N \otimes \mathbb{C}H \otimes \mathbb{C}H \xrightarrow{N\otimes\nabla} N \otimes \mathbb{C}H, $$

and whose half-braiding (which we do not need) is induced by the half-braiding of N.

Thus, we identify $\bar{\mathcal{Z}}({}^G_H\mathcal{M}_H) = {}^G_G\mathcal{YD}$, and we identify the underlying functor $\bar{\mathcal{Z}}({}^G_H\mathcal{M}_H) \to {}^G_H\mathcal{M}_H$ with the functor

$$ \mathcal{U} : {}^G_G\mathcal{YD} \ni N \longrightarrow N \otimes \mathbb{C}H \in {}^G_H\mathcal{M}_H, $$

where the obvious right $\mathbb{C}H$-module $N \otimes \mathbb{C}H$ has left module structure given by $a(n \otimes b) = an \otimes ab$ and grading given by $|n \otimes b| = |n|b$.

Next, let $g \in G$, set $C := C_G(g)$, and let $V \in \mathrm{Rep}(C)$. We consider

$$ \mathcal{U}\mathcal{G}_g(V) = \mathbb{C}G \underset{\mathbb{C}C}{\otimes} V \otimes \mathbb{C}H \in {}^G_H\mathcal{M}_H. $$

Let $\mathfrak{X}_g$ be a set of representatives of the double cosets in $H\backslash G/C$, giving the decomposition $G = \bigsqcup_{x\in\mathfrak{X}_g} HxC$. Then each $\mathbb{C}HxC \underset{\mathbb{C}C}{\otimes} V \otimes \mathbb{C}H \subset \mathbb{C}G \underset{\mathbb{C}C}{\otimes} V \otimes \mathbb{C}H$ is a subobject in ${}^G_H\mathcal{M}_H$, and we have

$$ \mathcal{U}\mathcal{G}_g(V) = \bigoplus_{x\in\mathfrak{X}_g} \mathbb{C}HxC \underset{\mathbb{C}C}{\otimes} V \otimes \mathbb{C}H. $$

Note that the degrees of the homogeneous elements of $\mathbb{C}HxC \otimes_{\mathbb{C}C} V \otimes H$ lie in the double coset $H(x \triangleright g)H$, so that $\mathbb{C}HxC \otimes_{\mathbb{C}C} V \otimes \mathbb{C}H$ is in the image of the functor $\mathcal{F}_{x\triangleright g}$. To calculate the preimage, observe first that the degree of $hxc \otimes v \otimes h' \in \mathbb{C}HxC \otimes_{\mathbb{C}C} V \otimes \mathbb{C}H$ is $(hx \triangleright g)h'$, and thus is in $(x \triangleright g)H$ if and only if $h \in \mathrm{Stab}_H((x \triangleright g)H) =: J$. Hence

$$ \mathbb{C}HxC \underset{\mathbb{C}C}{\otimes} V \otimes \mathbb{C}H = \mathcal{F}_{x\triangleright g}(\mathbb{C}JxC \underset{\mathbb{C}C}{\otimes} V). $$

Next, observe that for $j, \tilde{j} \in J$ and $c, \tilde{c} \in C$, we have $jxc = \tilde{j}x\tilde{c}$ if and only if $\tilde{j}^{-1}j = x \triangleright (\tilde{c}c^{-1})$, which implies that we have an isomorphism

$$ \mathbb{C}JxC \underset{\mathbb{C}C}{\otimes} V \ni jxc \otimes v \mapsto j \otimes cv \in \mathbb{C}J \underset{\mathbb{C}[J\cap(x\triangleright C)]}{\otimes} (x \triangleright V). $$

Note that $J \cap (x \triangleright C) = \mathrm{Stab}_H((x \triangleright g)H) \cap C_G(x \triangleright g) = H \cap C_G(x \triangleright g) = H \cap x \triangleright C$.

We have shown:

$$\mathbb{C}HxC \underset{\mathbb{C}C}{\otimes} V \otimes \mathbb{C}H = \mathcal{F}_{x\triangleright g}\big(\operatorname{Ind}_{H\cap(x\triangleright C)}^{\operatorname{Stab}_H((x\triangleright g)H)} \operatorname{Res}^{x\triangleright C}_{H\cap(x\triangleright C))}(x\triangleright V)\big),$$

whence

$$\mathcal{U}\mathcal{G}_g(V) = \bigoplus_{x\in\mathfrak{X}_g} \mathcal{F}_{x\triangleright g}\big(\operatorname{Ind}_{H\cap(x\triangleright C)}^{\operatorname{Stab}_H((x\triangleright g)H)} \operatorname{Res}^{x\triangleright C}_{H\cap(x\triangleright C)}(x\triangleright V)\big).$$

Let $d\in G$ and $S=\operatorname{Stab}_H(dH)$. Let $\mathfrak{H}_d$ be a set of representatives of H/S. Thus the double coset HdH is the disjoint union $HdH=\bigsqcup_{h\in\mathfrak{H}_d} hdH$; that is, $\mathfrak{H}_d d$ is a set of representatives of the right cosets contained in HdH.

If $x\triangleright g\in HdH$, then there is a unique $h\in\mathfrak{H}_d$ such that $(x\triangleright g)H=hdH$; thus, $\operatorname{Stab}_H((x\triangleright g)H)=\operatorname{Stab}_H(hdH)=h\triangleright S$, and for a representation N of $\operatorname{Stab}_H((x\triangleright g)H)$ we have $\mathcal{F}_{x\triangleright g}N=\mathcal{F}_{hd}N=\mathcal{F}_d(h^{-1}\triangleright N)$. Again $H\cap(x\triangleright C)=(h\triangleright S)\cap(x\triangleright C)$. Therefore,

$$\begin{aligned}(\mathcal{U}\mathcal{G}_g(V))_{HdH} &= \bigoplus \mathcal{F}_{hd}\big(\operatorname{Ind}^{h\triangleright S}_{(h\triangleright S)\cap(x\triangleright C)} \operatorname{Res}^{x\triangleright C}_{(h\triangleright S)\cap(x\triangleright C)}(x\triangleright V)\big)\\ &= \bigoplus \mathcal{F}_d\big(h^{-1}\triangleright\big(\operatorname{Ind}^{h\triangleright S}_{(h\triangleright S)\cap(x\triangleright C)} \operatorname{Res}^{x\triangleright C}_{(h\triangleright S)\cap(x\triangleright C)}(x\triangleright V)\big)\big),\end{aligned}$$

where the sum is over all $x\in\mathfrak{X}_g$ and $h\in\mathfrak{H}_d$ such that $x\triangleright g\in hdH$, and if $W\in\operatorname{Irr}(S)$, then

$$\begin{aligned}\big\langle\mathcal{U}\mathcal{G}_g(V),\mathcal{F}_d(W)\big\rangle &= \sum\big\langle h^{-1}\triangleright\big(\operatorname{Ind}^{h\triangleright S}_{(h\triangleright S)\cap(x\triangleright C)} \operatorname{Res}^{x\triangleright C}_{(h\triangleright S)\cap(x\triangleright C)}(x\triangleright V)\big), W\big\rangle\\ &= \sum\big\langle\operatorname{Ind}^{h\triangleright S}_{(h\triangleright S)\cap(x\triangleright C)} \operatorname{Res}^{x\triangleright C}_{(h\triangleright S)\cap(x\triangleright C)}(x\triangleright V), h\triangleright W\big\rangle.\end{aligned}$$

For the adjoint $\overline{\mathcal{K}}$ of $\mathcal{U}$, this implies, by Frobenius reciprocity:

$$\big\langle\overline{\mathcal{K}}\mathcal{F}_d(W),\mathcal{G}_g(V)\big\rangle = \sum\big\langle x^{-1}\triangleright\big(\operatorname{Ind}^{x\triangleright C}_{(h\triangleright S)\cap(x\triangleright C)} \operatorname{Res}^{h\triangleright S}_{(h\triangleright S)\cap(x\triangleright C)}(h\triangleright W)\big), V\big\rangle.$$

This means that we have calculated a formula for the adjoint $\overline{\mathcal{K}}$: denoting by $\mathfrak{C}$ a system of representatives for the conjugacy classes of G, we have

$$\begin{aligned}\overline{\mathcal{K}}\mathcal{F}_d(W) &= \sum \mathcal{G}_g\big(x^{-1}\triangleright\big(\operatorname{Ind}^{x\triangleright C_G(g)}_{(h\triangleright S)\cap(x\triangleright C_G(g))} \operatorname{Res}^{h\triangleright S}_{(h\triangleright S)\cap(x\triangleright C_G(g))}(h\triangleright W)\big)\big)\\ &= \sum \mathcal{G}_{x\triangleright g}\big(\operatorname{Ind}^{x\triangleright C_G(g)}_{(h\triangleright S)\cap(x\triangleright C_G(g))} \operatorname{Res}^{h\triangleright S}_{(h\triangleright S)\cap(x\triangleright C_G(g))}(h\triangleright W)\big),\end{aligned}$$

where the sum is over all $g\in\mathfrak{C}$, $x\in\mathfrak{X}_g$, and $h\in\mathfrak{H}_d$ such that $x\triangleright g\in hdH$. While this is clearly not a particularly pleasant or practical formula, we can say something in its favor: It expresses the functor $\overline{\mathcal{K}}$ entirely in terms of the groups involved and their representations, using, of course, the translation of group representations to objects in the two categories involved via the functors $\mathcal{F}$ and $\mathcal{G}$.

4. Indicator formulas for group inclusions

We retain the notations of the previous section, and proceed to calculate the higher Frobenius–Schur indicators of objects in ${}^G_H\mathcal{M}_H$. This is based on the categorical version of the "third formula" in [Kashina, Sommerhäuser and Zhu 2006, §6.4] that calculates indicators in a fusion category $\mathcal{C}$ through the adjoint $\overline{\mathcal{K}}$.

The formula obtained above for the adjoint $\overline{\mathcal{K}} : {}^G_H\mathcal{M}_H \to {}^G_G\mathcal{YD}$ yields, via [Ng and Schauenburg 2007a, Theorem 4.1], a formula for the higher indicators of the simple objects of ${}^G_H\mathcal{M}_H$. Since we are dealing with the right center, the relevant formula [op. cit., Remark 4.3] is

$$\nu_m(X) = \frac{1}{|G|}\operatorname{Tr}\big(\theta^{-m}_{\overline{\mathcal{K}}(X)}\big).$$

We proceed to use the information available on $\overline{\mathcal{K}}$ to apply it.

First, let η' be a character of $(h \triangleright S) \cap (x \triangleright C)$, and $\chi = \operatorname{Ind}^{x\triangleright C}_{(h\triangleright S)\cap(x\triangleright C)}(\eta')$. Then by a standard formula for induced characters,

$$\begin{aligned}\chi(x \triangleright g^m) &= \frac{1}{|(h \triangleright S)\cap(x \triangleright C)|}\sum_{\substack{y\in x\triangleright C\\ y\triangleright x\triangleright g^m \in h\triangleright S}} \eta'(y \triangleright x \triangleright g^m)\\ &= \begin{cases}[x\triangleright C : (h\triangleright S)\cap(x\triangleright C)]\,\eta'(x\triangleright g^m) & \text{if } x\triangleright g^m \in h\triangleright S,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

as elements in $x \triangleright C$ commute with $x \triangleright g^m$.

Let η be the character of $W \in \operatorname{Rep}(S)$, and let χ be the character of $V := \operatorname{Ind}^{x\triangleright C}_{h\triangleright S\cap x\triangleright C}\operatorname{Res}^{h\triangleright S}_{h\triangleright S\cap x\triangleright C}(h\triangleright\eta)$. Then

$$\begin{aligned}\operatorname{Tr}\big(\theta^m_{\mathcal{G}_{x\triangleright g}(V)}\big) &= [G : x\triangleright C]\,\chi(x\triangleright g^m)\\ &= \begin{cases}[G : (h\triangleright S)\cap(x\triangleright C)]\,\eta(h^{-1}x\triangleright g^m) & \text{if } x\triangleright g^m\in h\triangleright S,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

By the formula for $\overline{\mathcal{K}}(\mathcal{F}_d(W))$ obtained in the previous section, this finally implies (using $|(h\triangleright S)\cap(x\triangleright C_G(g))| = |S\cap(h^{-1}x\triangleright C_G(g)| = |S\cap C_G(h^{-1}x\triangleright g)|$) that

$$\nu_m(\mathcal{F}_d(W)) = \sum \frac{1}{|S\cap C_G(h^{-1}x\triangleright g)|}\,\overline{\eta}(h^{-1}x\triangleright g^m), \tag{1}$$

where the sum is over $g\in\mathfrak{C}$, $x\in\mathfrak{X}_g$, and $h\in\mathfrak{H}_d$ such that $x\triangleright g\in hdH$ and $x\triangleright g^m\in h\triangleright S$. Surely this sum is not pleasant to work with; it involves summing over all conjugacy classes of the group and all representatives of certain double cosets, as well as over the coset representatives in $\mathfrak{H}_d$, albeit that last sum involves either no summand (for many combinations of g and x we might have $x\triangleright g\notin HdH$), or just one summand (the representative of the unique right coset containing $x\triangleright g$).

We shall process it further using the observation

$$HdH = \bigsqcup_{\substack{g\in\mathfrak{C},\ x\in\mathfrak{X}_g \\ x\triangleright g\in HdH}} H \triangleright (x \triangleright g) = \bigsqcup_{\substack{g\in\mathfrak{C},\ x\in\mathfrak{X}_g \\ h\in\mathfrak{H}_d \\ x\triangleright g\in hdH}} H \triangleright (h^{-1}x \triangleright g). \tag{2}$$

For the first equality, one has to check when $x \triangleright g$ and $y \triangleright g$, for $x, y \in G$, are in the same orbit of the action of H on G by conjugation:

$$\begin{aligned} \exists h \in H : h \triangleright (x \triangleright g) = y \triangleright g \quad &\Longleftrightarrow \quad \exists h \in H : hxgx^{-1}h^{-1} = ygy^{-1} \\ &\Longleftrightarrow \quad \exists h \in H : y^{-1}hx \in C_G(g) \\ &\Longleftrightarrow \quad x \in HyC_G(g), \end{aligned}$$

while the second is an obvious reparametrization.

Thus, the set

$$\mathfrak{R}_d = \{h^{-1}x \triangleright g \mid g \in \mathfrak{C}, x \in \mathfrak{X}_g, h \in \mathfrak{H}_d, x \triangleright g \in hdH\} \tag{3}$$

is a set of representatives of the orbits of the action of H on HdH by conjugation. Moreover, $\mathfrak{R}_d \subset dH$. Thus, $\mathfrak{R}_d$ is a set of representatives of the orbits of the action of S on dH by conjugation. We have very nearly proved the main result of the paper:

Theorem 4.1. *Let G be a finite group, $H \subset G$ a subgroup, $d \in G$, $S = \mathrm{Stab}_H(dH)$, $W \in \mathrm{Rep}(S)$ with character η, and $\mathcal{F}_d(W)$ the object of ${}^G_H\mathcal{M}_H$ corresponding to W. Then*

$$\nu_m(\mathcal{F}_d(W)) = \frac{1}{|S|} \sum_{\substack{r\in dH \\ r^m\in S}} \overline{\eta}(r^m) = \frac{1}{|S|} \sum_{\substack{h\in H \\ (dh)^m\in S}} \overline{\eta}((dh)^m). \tag{4}$$

Proof. Substituting (3) in the indicator formula (1) yields

$$\nu_m(\mathcal{F}_d(W)) = \sum_{\substack{r\in\mathfrak{R}_d \\ r^m\in S}} \frac{1}{|S\cap C_G(r)|} \overline{\eta}(r^m). \tag{5}$$

But for $s \in S$ we have $(s \triangleright r)^m \in S \Longleftrightarrow r^m \in S$, and $\eta((s \triangleright r)^m) = \eta(r^m)$ whenever $r^m \in S$. Since $S \cap C_G(r)$ is the stabilizer of r under the adjoint action of S, the first equality in (4) follows. The second equality is a trivial reparametrization. □

In the following we keep the notations of Theorem 4.1.

Remark 4.2. Note that for $r \in dH$ we have $r^m \in S \Longleftrightarrow r^m \in H$. Thus we could modify the conditions in the sums (4) and subsequent similar sums, but in the examples that we treated, it seemed easier to check whether an element is in S than to check whether it is in H.

Remark 4.3. For $m \in \mathbb{N}$, the elements

$$\mu_m(d) := \frac{1}{|S|} \sum_{\substack{r \in dH \\ r^m \in S}} r^m = \frac{1}{|S|} \sum_{\substack{h \in H \\ (dh)^m \in S}} (dh)^m \in \mathbb{C}S \tag{6}$$

are central in the group algebra $\mathbb{C}S$, and $\nu_m(\mathcal{F}_d(W)) = \eta(\mu_m(d))$.

Remark 4.4. If $d \in C_G(H)$, then $S = H$, and for $h \in H$ we have $(dh)^m = d^m h^m \in H$ if and only if $d^m \in H$, so that

$$\mu_m(d) = \begin{cases} d^m \dfrac{1}{|H|} \displaystyle\sum_{h \in H} h^m & \text{if } d^m \in H, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

and therefore, since $d^m \in H$ is in the center of H,

$$\nu_m(\mathcal{F}_d(W)) = \begin{cases} \dfrac{\bar{\eta}(d^m)}{\eta(1)} \nu_m(W) & \text{if } d^m \in H, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The most obvious case of this is when $d = 1$; the image of $\mathcal{F}_1$ is the monoidal subcategory ${}^H_H\mathcal{M}_H \subset {}^G_H\mathcal{M}_H$, which is monoidally equivalent to $\operatorname{Rep}(H)$. The formula (8) can also be used to easily obtain examples where the higher indicators are not real: the cyclic group G of order 9, its generator d, its subgroup H of order 3, and a nontrivial irreducible character of the latter will do to obtain $\nu_3(\mathcal{F}_d(W))$, a nontrivial third root of unity.

Lemma 4.5. *Let $y \in S$. Then*

$$\sum_{\chi \in \operatorname{Irr}(S)} \nu_m(\mathcal{F}_d(\chi))\, \chi(y) = |\{h \in H \mid (dh)^m = y\}|. \tag{9}$$

In fact the function $\zeta_m(y) = |\{h \in H \mid (dh)^m = y\}|$ is easily seen to be a class function on S, so one can verify (9) by taking its scalar product with an irreducible character η. The left hand side gives the m-th indicator by the orthogonality relations, the right hand side by (4).

Remark 4.6. Assume that $H \subset G$ is part of an exact factorization, i.e., there exists a subgroup $L \subset G$ such that $LH = G$ and $L \cap G = \{1\}$. As pointed out in [Schauenburg 2002b], the category ${}^G_H\mathcal{M}_H$ is then equivalent to the category of modules over a bismash product Hopf algebra $\mathbb{C}^L \# \mathbb{C}H$. Thus, our results comprise a method to calculate indicators for bismash product Hopf algebras (of which the double below is a special case).

Example 4.7. Let Γ be a finite group, $G = \Gamma \times \Gamma$ with diagonal embedding $\Delta : \Gamma \to \Gamma \times \Gamma$, and $H = \Delta(\Gamma)$. It is well known that the category ${}^G_H\mathcal{M}_H \cong {}^\Gamma_\Gamma\mathcal{M}^\Gamma_\Gamma$ is equivalent to the module category of the Drinfeld double of Γ (in fact this is a special case of [Schauenburg 1994]).

Let $\mathfrak{G}$ be a cross section of the conjugacy classes of Γ. Then $\{(\gamma, 1) \mid \gamma \in \mathfrak{G}\}$ is a cross section of the double cosets of H in G. Let $d = (\gamma, 1)$. Then $S = \mathrm{Stab}_H(dH) = \Delta(C_\Gamma(\gamma))$. Let $h = \Delta(\theta) \in H$ and $m \in \mathbb{N}$. Then $(dh)^m = (\gamma\theta, \theta)^m = ((\gamma\theta)^m, \theta^m)$, thus $(dh)^m \in S$ if and only if $(\gamma\theta)^m = \theta^m$. Therefore, our indicator formula yields

$$\nu_m(\mathcal{F}_d(W)) = \frac{1}{|C_\Gamma(\gamma)|} \sum_{\substack{\theta\in\Gamma \\ (\gamma\theta)^m = \theta^m}} \overline{\eta}(\theta^m). \tag{10}$$

This formula was obtained in [Kashina, Sommerhäuser and Zhu 2006]; see also [Iovanov, Mason and Montgomery 2014], where the corresponding special case of (9) can be found. Note that we can replace $\overline{\eta}$ by η since the indicators in this case are known to be real.

In the proof of Theorem 4.1 we have obtained the simple looking indicator formula (4) via the more complicated formula (5). But in fact the latter is, in some respects, better than the former: it involves a sum over fewer terms, namely orbits of the adjoint action of S instead of individual elements of dH. Of course, for this simplification we could have taken any section of the orbits on dH instead of $\mathfrak{R}_d$. In fact, we can also pass to orbits over a group different from S; also, it may be convenient to take orbits in H of the action on H corresponding to the adjoint action on dH:

Proposition 4.8. *In the notation of Theorem 4.1, set $E = C_G(d) \cap SC_G(S) \cap N_G(H)$. Then, $SE = ES$ is a subgroup of G. Let $S' \subset SE$ be a subgroup, and let $\mathfrak{R}'_d$ be a section of the orbits of dH under the adjoint action of S' on dH. Then,*

$$\nu_m(\mathcal{F}_d(W)) = \frac{1}{|S|} \sum_{\substack{r\in\mathfrak{R}'_d \\ r^m \in S}} \frac{|S'|}{|S' \cap C_G(r)|} \overline{\eta}(r^m). \tag{11}$$

Alternatively, let S' act on H by "twisted conjugation" defined by the formula $s \,\tilde{\triangleright}\, h = (d^{-1} \triangleright s)hs^{-1}$. Let $\mathfrak{T}'_d$ be a system of representatives of the orbits. Then,

$$\nu_m(\mathcal{F}_d(W)) = \frac{1}{|S|} \sum_{\substack{h\in\mathfrak{T}'_d \\ (dh)^m \in S}} \frac{|S'|}{|S' \cap C_G(dh)|} \overline{\eta}((dh)^m). \tag{12}$$

Proof. Let $x \in E$ and $u \in S = H \cap (d \triangleright H)$. Then $x \triangleright u \in (x \triangleright H) \cap (xd \triangleright H) = H \cap d \triangleright H = S$ since $x \triangleright H = H$ and $xd = dx$ by hypothesis. Thus E normalizes S, and $SE = ES$ is a subgroup of G. Now let $x \in E$ and $h \in H$. Since $x \in SC_G(S)$, we have $(dh)^m \in S$ if and only if $x \triangleright (dh)^m \in S$; in fact, these two elements are then conjugate in S. The condition $x \in C_G(d)$ implies $x \triangleright (dh)^m = (d(x \triangleright h))^m$, and $x \in N_G(H)$ implies $x \triangleright h \in H$. Thus the action of S' on dH is well defined, and the condition $r^m \in S$ is invariant along the orbits, as well as the values $\eta(r^m)$

along those orbits where $r^m \in S$. This implies (11), since $S' \cap C_G(r)$ is the stabilizer of r. Since $s \triangleright (dh) = d(s \tilde{\triangleright} h)$ for $s \in S$ and $h \in H$, we obtain (12) by a simple reparametrization. □

Remark 4.9. The previous result is perhaps the most useful if $S' \subset C_G(d)$, so that the twisted adjoint action coincides with the adjoint action. At any rate, it allows us to replace H by a set of orbit representatives before passing to the nastier part of the calculations involved in applying the indicator formula to concrete examples.

To set notation for subsequent calculations, let $\overline{G}$ be the set of orbits of G under the adjoint action of S', and $\overline{S}$ the image of S in $\overline{G}$. We do not distinguish notationally elements of $\overline{G}$ from those of G. We also let $\tilde{H}$ be the set of orbits of the twisted adjoint action of S' on H, and $Q(d) := \sum_{h \in H} h \in \mathbb{C}\tilde{H}$. Set

$$T(d) := \sum_{h \in \mathfrak{T}'_d} [S' : S' \cap C_G(dh)]dh = dQ(d) \in \mathbb{C}\overline{G}. \tag{13}$$

Let $\mathbb{C}\overline{G} \ni x \mapsto x^{[m]} \in \mathbb{C}\overline{G}$ be the linear map induced by taking m-th powers of group elements. Let $\pi : \mathbb{C}\overline{G} \to \mathbb{C}\overline{S}$ be the linear projection annihilating $\overline{G} \setminus \overline{S}$. Then

$$\nu_m(\mathcal{F}_d(W)) = \overline{\eta}(\bar{\mu}_m(d)) \quad \text{with} \quad \bar{\mu}_m(d) = \frac{1}{|S|}\pi(T(d)^{[m]}). \tag{14}$$

Of course $\bar{\mu}_m(d)$ is just the image of $\mu_m(d)$ in $\mathbb{C}\overline{S}$.

5. Example calculations

Consider the symmetric group S_n and the subgroup $S_m \subset S_n$ for $m < n$. For $d \in S_n$ the stabilizer $\mathrm{Stab}_{S_m}(dS_n) = S_m \cap d \triangleright S_m$ consists of those permutations $\sigma \in S_m$ for which $d^{-1} \triangleright \sigma \in S_m$. For $d^{-1} \triangleright \sigma$ to fix every element greater than m it is necessary and sufficient that σ fix every element k with $d^{-1}(k) \notin \{1, \dots, m\}$. Thus $\mathrm{Stab}_{S_m}(dS_m) = S_{\{1,\dots,m\} \cap \{d(1),\dots,d(m)\}}$ is a symmetric group. We have seen that in general higher indicators for the objects of ${}^G_H\mathcal{M}_H$ are nonnegative rational linear combinations of character values of the stabilizers $\mathrm{Stab}_H dH$. Moreover, higher indicators for any pivotal fusion category are cyclotomic integers.

Proposition 5.1. *Let $m < n$. Then all values of the higher Frobenius–Schur indicators for the objects of ${}^{S_n}_{S_m}\mathcal{M}_{S_m}$ are integers.*

The following example shows that this can fail if we embed S_m into S_n in a different fashion.

Example 5.2. Consider

$$G = S_9 \supset H = \{\sigma \in S_9 \mid i \equiv j \pmod 3 \Rightarrow \sigma(i) \equiv \sigma(j) \pmod 3\},$$

so H is the subgroup of those permutations in S_9 that preserve conjugacy modulo 3. Thus $H \cong S_3$ is generated by $t = (123)(456)(789)$ and $s = (12)(45)(78)$.

The element $d = (147258369) \in S_9$ satisfies $d^3 = t$, so in particular $d^{-1} \triangleright t \in H$. On the other hand $d^{-1} \triangleright s = (12)(45)(79)$, so $d^{-1} \triangleright s \notin H$ because $1 \equiv 7 \pmod 3$ while $2 \not\equiv 9 \pmod 3$. It follows that $S = \mathrm{Stab}_H(dH) = \langle t \rangle$.

To compute $\mu_3(d)$, first observe that $d^3 = (dt)^3 = (dt^2)^3 = t$. The computation $ds = (157369)(248)$ and $(ds)^3 = (13)(56)(79) \notin S$ shows that $(dh)^3 \notin S$ for $h \in H \setminus \{1, t, t^2\}$, since such h are conjugate to s by powers of t, which commute with d. Thus $\mu_3(d) = t$.

In particular $\nu_3(\mathcal{F}_d(\eta)) = \zeta^{-1}$ is not real when $\eta(t) = \zeta$ is a nontrivial third root of unity.

We will now compute some of the indicator values for the canonically embedded subgroups $S_{n-2} \subset S_n$ (as we shall see, this contains, in a sense, the case $S_{n-1} \subset S_n$, or rather $S_{n-2} \subset S_{n-1}$). We note already that all the indicator values we will find are nonnegative.

For $n \geq 4$, it is easy to check that S_{n-2} has seven double cosets in S_n:

$$\begin{aligned}
&\{\sigma \in S_n \mid \sigma(n-1) = n-1, \sigma(n) = n)\} = S_{n-2},\\
&\{\sigma \in S_n \mid \sigma(n-1) \neq n-1, \sigma(n) = n)\},\\
&\{\sigma \in S_n \mid \sigma(n-1) = n-1, \sigma(n) \neq n)\},\\
&\{\sigma \in S_n \mid \sigma(n-1) = n, \sigma(n) = n-1)\},\\
&\{\sigma \in S_n \mid \sigma(n-1) = n, \sigma(n) \neq n-1)\},\\
&\{\sigma \in S_n \mid \sigma(n-1) \neq n, \sigma(n) = n-1)\},\\
&\{\sigma \in S_n \mid \{\sigma(n-1), \sigma(n)\} \cap \{n-1, n\} = \varnothing\}.
\end{aligned}$$

A convenient set of double coset representatives is $d_1 = ()$, $d_2 = (n-2, n-1)$, $d_3 = (n-2, n)$, $d_4 = (n-1, n)$, $d_5 = (n-2, n-1, n)$, $d_6 = (n-2, n, n-1)$, and $d_7 = (n-3, n-1)(n-2, n)$.

Note that d_2 and d_3 are conjugate by $(n-1, n)$. The same holds for d_5 and d_6. We have $\mathrm{Stab}_{S_{n-2}}(d_2 S_{n-2}) = \mathrm{Stab}_{S_{n-2}}(d_5 S_{n-2}) = S_{n-3}$, $\mathrm{Stab}_{S_{n-2}}(d_7 S_{n-2}) = S_{n-4}$, and $\mathrm{Stab}_{S_{n-2}}(d_4 S_{n-2}) = S_{n-2}$.

Note that every d_i commutes with the elements in $\mathrm{Stab}_{S_{n-2}}(d_i S_{n-2})$; this is particular to our choice of representatives. It implies that the twisted conjugation action of the stabilizers on the group S_{n-2} from Proposition 4.8 is the ordinary adjoint action.

Note further that d_4 commutes with the elements of S_{n-2}. By Remark 4.4 it follows that

$$\nu_m(\mathcal{F}_{(n-1,n)}(W)) = \begin{cases} \nu_m(W) & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,} \end{cases} \tag{15}$$

for any $W \in \mathrm{Rep}(S_{n-2})$, while $\nu_m(\mathcal{F}_{()}(W)) = \nu_m(W)$.

Note also that $d_2 \in S_{n-1}$. Thus, the indicators for objects in $\mathcal{F}_{d_2}(\mathrm{Rep}(S_{n-2}))$ can also be viewed as indicators in the subcategory ${}^{S_{n-1}}_{S_{n-2}}\mathcal{M}_{S_{n-2}}$. The subgroup $S_{n-2} \subset S_{n-1}$ is part of an exact factorization, $S_{n-1} = C_{n-1} \cdot S_{n-2}$, where C_{n-1} denotes the cyclic group generated by the $(n-1)$-cycle $(1, 2, \dots, n-1)$. As remarked already, these indicators are indicators for modules over a bismash product Hopf algebra $\mathbb{C}^{C_{n-2}} \# \mathbb{C}S_{n-1}$. Observe that the exact factorization suggests a different choice of coset representative, namely the $(n-1)$-cycle instead of d_2. We have the feeling that d_2 is the better choice since the $(n-1)$-cycle does not commute with elements in the corresponding stabilizer.

Since the images of $\mathcal{F}_{d_2}$ and $\mathcal{F}_{d_3}$ are mapped to each other by an autoequivalence, as well as the images of $\mathcal{F}_{d_5}$ and $\mathcal{F}_{d_6}$, we can concentrate on the indicators of the objects in the images of $\mathcal{F}_{d_i}$ for $i = 2, 5, 7$. We will treat some of them below for small values of n.

$S_2 \subset S_4$. Consider $H = \langle (1\ 2) \rangle \subset G = S_4$. We have the following double coset representatives, with their right cosets and double cosets:

i	d_i	$d_i H \setminus \{d_i\}$	$H d_i H \setminus d_i H$	$\mathrm{Stab}_H(d_i H)$
1	$()$	$(1\ 2)$		H
2	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3), (1\ 3\ 2)$	$\{()\}$
3	$(2\ 4)$	$(1\ 2\ 4)$	$(1\ 4), (1\ 4\ 2)$	$\{()\}$
4	$(3\ 4)$	$(1\ 2)(3\ 4)$		H
5	$(2\ 3\ 4)$	$(1\ 2\ 3\ 4)$	$(1\ 3\ 4), (1\ 3\ 4\ 2)$	$\{()\}$
6	$(2\ 4\ 3)$	$(1\ 2\ 4\ 3)$	$(1\ 4\ 3), (1\ 4\ 3\ 2)$	$\{()\}$
7	$(2\ 3)(1\ 4)$	$(1\ 4\ 2\ 3)$	$(1\ 4\ 2\ 3), (1\ 3\ 2\ 4)$	$\{()\}$

We proceed to list the sequences of the higher Frobenius–Schur indicators for all the simple objects of ${}^G_H\mathcal{M}_H$ in the images of the functors $\mathcal{F}_{d_i}$. These sequences are periodic, and we list them for one complete period.

For d_1, they are the sequences of the higher Frobenius–Schur indicators of the representations of H, namely $(1, \dots)$ with period one for the trivial representation, and $(1, 0, \dots)$ with period two for the nontrivial representation.

In all other cases, the only powers of the elements of $d_i H$ that lie in the stabilizer $\mathrm{Stab}_H(d_i H)$ are identity elements. (This requires only a glance for d_4, as the stabilizer itself is trivial in the other cases.) Thus, regardless of the choice of representation also in the d_4 case, the indicator ν_m counts how many of the two m-th powers of the two elements of $d_i H$ are trivial; the count is then divided by two in the d_4 case. Thus the indicator sequences, up to a full period, are

$$(\nu_m(\mathcal{F}_{d_i}(W)))_m = \begin{cases} (0, 1, 1, 1, 0, 2, \dots) & \text{for } i = 2, 3, \\ (0, 1, \dots) & \text{for } i = 4, \\ (0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 2, \dots) & \text{for } i = 5, 6, \\ (0, 1, 0, 2, \dots) & \text{for } i = 7. \end{cases}$$

(Note that the case d_4 was already treated above using Remark 4.4.)

$S_3 \subset S_5$. In this case we have the right cosets

i	d_i	$d_i S_3 \setminus \{d_i\}$
1	()	(12), (13), (23), (123), (132)
2	(34)	(12)(34), (143), (243), (1243), (1432)
3	conjugate preceding row by (45)	
4	(45)	(12)(45), (13)(45), (23)(45), (123)(45), (132)(45)
5	(345)	(12)(345), (1453), (2453), (12453), (14532)
6	conjugate preceding row by (45)	
7	(2435)	(14352), (15243), (25)(34), (143)(25), (152)(34)

We have

$$\mathrm{Stab}_{S_3}(d_i S_3) = \begin{cases} S_3 & \text{for } i = 1, 4, \\ S_2 = \langle (1\ 2) \rangle & \text{for } i = 2, 3, 5, 6, \\ \{()\} & \text{for } i = 7. \end{cases}$$

As indicated above, we will only treat the indicators for d_2, d_5, and d_7.

One sees that for $i = 2$, the only possibility for a power of an element of $d_i S_3$ to be in $\mathrm{Stab}_{S_3}(d_i S_3)$ is if that power is trivial. The same is of course true for $i = 7$. So the m-th indicators for the simple objects in the images of $\mathcal{F}_{d_i}$ for $i = 2, 7$ do not "see" the representations of $\mathrm{Stab}_{S_3}(d_i S_3)$, but only count the number of elements whose orders divide m; the count has to be divided by 2 if $i = 2$. We have

$$\nu_m(\mathcal{F}_{d_2}(W)) = \begin{cases} 0, & (m, 12) = 1, \\ 1, & (m, 12) = 2, 3, \\ 2, & (m, 12) = 4, 6, \\ 3, & (m, 12) = 12; \end{cases} \qquad \nu_m(\mathcal{F}_{d_7}(W)) = \begin{cases} 0, & (m, 60) = 1, 3, \\ 1, & (m, 60) = 2, \\ 2, & (m, 60) = 4, 5, 15, \\ 3, & (m, 60) = 6, 10, \\ 4, & (m, 60) = 12, 20, \\ 5, & (m, 60) = 30, \\ 6, & (m, 60) = 60. \end{cases}$$

Finally $d_5 S_3$ contains one element, $(1\ 2)(3\ 4\ 5)$, whose third power is in $\mathrm{Stab}_{S_3}(d_5 S_3) \setminus \{()\}$. Powers of the other elements are only in the stabilizer when they are trivial. Thus, we obtain

$$\mu_m(d_5) = \mu_m(d_6) = \begin{cases} 0 & \text{when } (m, 60) = 1, 2, \\ \frac{1}{2}(() + (1\ 2)) & \text{when } (m, 60) = 3, \\ () & \text{when } (m, 60) = 4, 5, 6, 10, \\ 2() & \text{when } (m, 60) = 12, 20, 30, \\ \frac{1}{2}(3() + (1\ 2)) & \text{when } (m, 60) = 15, \\ 3() & \text{when } (m, 60) = 60. \end{cases}$$

For the trivial representation W_0 of $\langle(1\ 2)\rangle$, this yields

$$\nu_m(\mathcal{F}_{d_5}(W_0)) = \nu_m(\mathcal{F}_{d_6}(W_0)) = \begin{cases} 0 & \text{when } (m, 60) = 1, 2, \\ 1 & \text{when } (m, 60) = 3, 4, 5, 6, 10, \\ 2 & \text{when } (m, 60) = 12, 15, 20, 30, \\ 3 & \text{when } (m, 60) = 60. \end{cases}$$

For the nontrivial irreducible representation W_1 of $\langle(1\ 2)\rangle$, we obtain

$$\nu_m(\mathcal{F}_{d_5}(W_1)) = \nu_m(\mathcal{F}_{d_6}(W_1)) = \begin{cases} 0 & \text{when } (m, 60) = 1, 2, 3, \\ 1 & \text{when } (m, 60) = 4, 5, 6, 10, 15, \\ 2 & \text{when } (m, 60) = 12, 20, 30, \\ 3 & \text{when } (m, 60) = 60. \end{cases}$$

$S_4 \subset S_6$. Since $|S_4| = 24$, it seems worth reducing the size of calculations in this case by considering orbits of S_4 as outlined in Proposition 4.8. We will use $S' = \mathrm{Stab}_{S_4}(d_i S_4)$.

For $i = 2, 5$ the stabilizer is S_3. The orbits of S_4 under the adjoint action of S_3 are obtained by subdividing the well-known conjugacy classes of S_4 according to the placement of the letter 4 in the respective cycle structure. Trusting details to the reader, we state:

$$\begin{aligned} Q(d_i) = {} & () + 3(12) + 3(14) \\ & + 2(123) + 6(124) \\ & + 3(12)(34) \\ & + 6(1234). \end{aligned}$$

From this we obtain

$$\begin{aligned} T((45)) = (45)Q((45)) = {} & (45) + 3(12)(45) + 3(154) \\ & + 2(123)(45) + 6(1254) \\ & + 3(12)(354) \\ & + 6(12354) \end{aligned}$$

and

$$\begin{aligned}T((456)) = (456)\,Q((456)) = (456) &+ 3(12)(456) + 3(1564)\\ &+ 2(123)(456) + 6(12564)\\ &+ 3(12)(3564)\\ &+ 6(123564).\end{aligned}$$

Thus (omitting the neutral element and writing $3 := 3(\,) \in \mathbb{C}\bar{S}$, etc.),

$$\begin{aligned}
\bar{\mu}_2((45)) &= \tfrac{1}{6}(1+3+2(123)) = \tfrac{1}{3}(2+(123)),\\
\bar{\mu}_3((45)) &= \tfrac{1}{6}(3+3(12)) = \tfrac{1}{2}(1+(12)),\\
\bar{\mu}_4((45)) &= \tfrac{1}{6}(1+3+2(123)+6) = \tfrac{1}{3}(5+(123)),\\
\bar{\mu}_5((45)) &= 1,\\
\bar{\mu}_6((45)) &= \tfrac{1}{6}(1+3+3+2+3) = 2,\\
\bar{\mu}_{10}((45)) &= \tfrac{1}{6}(1+3+2(123)+6) = \tfrac{1}{3}(5+(123)) = \bar{\mu}_4((45)),\\
\bar{\mu}_{12}((45)) &= \tfrac{1}{6}(1+3+3+2+6+3) = 3,\\
\bar{\mu}_{15}((45)) &= \tfrac{1}{6}(3+3(12)+6) = \tfrac{1}{2}(3+(12)),\\
\bar{\mu}_{30}((45)) &= \tfrac{1}{6}(1+3+3+2+3+6) = 3 = \bar{\mu}_{12}((45)),\\
\bar{\mu}_{20}((45)) &= \tfrac{1}{6}(1+3+2(123)+6+6) = \tfrac{1}{3}(8+(123)),\\
\bar{\mu}_{60}((45)) &= 4,\\
\bar{\mu}_2((456)) &= 0,\\
\bar{\mu}_3((456)) &= \tfrac{1}{6}(1+3(12)+2) = \tfrac{1}{2}(1+(12)),\\
\bar{\mu}_4((456)) &= \tfrac{1}{6}(3+3) = 1,\\
\bar{\mu}_5((456)) &= 1,\\
\bar{\mu}_6((456)) &= \tfrac{1}{6}(1+3+2+6) = 2,\\
\bar{\mu}_{10}((456)) &= 1,\\
\bar{\mu}_{12}((456)) &= \tfrac{1}{6}(1+3+3+2+3+6) = 3,\\
\bar{\mu}_{15}((456)) &= \tfrac{1}{6}(1+3(12)+2+6) = \tfrac{1}{2}(3+(12)),\\
\bar{\mu}_{20}((456)) &= \tfrac{1}{6}(3+6+3) = 2\\
\bar{\mu}_{30}((456)) &= \tfrac{1}{6}(1+3+2+6+6) = 3,\\
\bar{\mu}_{60}((456)) &= 4.
\end{aligned}$$

For d_7, the calculations are even more tedious; we now need the S_2-orbits of S_4, that is, the subdivision of the conjugacy classes of S_4 according to the placement of the letters 3, 4 in the cycle structure. Thus,

$$\begin{aligned} Q((35)(46)) = {} & () + (12) + 2(13) + 2(14) + (34) \\ & + 2(123) + 2(124) + 2(134) + 2(143) \\ & + (12)(34) + 2(13)(24) \\ & + 2(1234) + 2(1243) + 2(1324) \end{aligned}$$

and

$$\begin{aligned} T((35)(46)) = {} & (35)(46) + (12)(35)(46) + 2(153)(46) + 2(164)(35) + (3645) \\ & + 2(1253)(46) + 2(1264)(35) + 2(15364) + 2(16453) \\ & + (12)(3645) + 2(153)(264) \\ & + 2(125364) + 2(126453) + 2(153264), \end{aligned}$$

giving

$$\begin{aligned} \bar\mu_2((35)(46)) = \bar\mu_3((35)(46)) = 1, \qquad & \bar\mu_{12}((35)(46)) = 10, \\ \bar\mu_4((35)(46)) = 4, \qquad & \bar\mu_{15}((35)(46)) = 3, \\ \bar\mu_5((35)(46)) = 2, \qquad & \bar\mu_{20}((35)(46)) = 6, \\ \bar\mu_6((35)(46)) = 7, \qquad & \bar\mu_{30}((35)(46)) = 9, \\ \bar\mu_{10}((35)(46)) = 3, \qquad & \bar\mu_{60}((35)(46)) = 12. \end{aligned}$$

In particular, the indicators of the two simples in the image of $\mathcal{F}_{(35)(46)}$ are identical; while for the other cases, we have to distinguish between the three irreducible representations of S_3, to wit, the trivial representation W_0, the sign representation W_1, and the two-dimensional irreducible W_2. We obtain:

object		m (for ν_m)										
d_i	W_j	2	3	4	5	6	10	12	15	20	30	60
(45)	W_0	1	1	2	1	2	2	3	2	3	3	4
	W_1	1	0	2	1	2	2	3	1	3	3	4
	W_2	1	1	3	2	4	3	6	3	5	6	8
(456)	W_0	0	1	1	1	2	1	3	2	2	3	4
	W_1	0	0	1	1	2	1	3	1	2	3	4
	W_2	0	1	2	2	4	2	6	3	4	6	8
(35)(46)	any	1	1	4	2	7	3	10	3	6	9	12

$S_5 \subset S_7$. If we want to deal with the representations associated to $d_7 = (46)(57)$ as in the preceding example, we calculate with a sum $Q((46)(57))$ with as many terms as there are orbits in S_5 of the adjoint action of S_3. One can check that there are 28 orbits. But we can reduce the task considerably (if not quite by half) by extending the stabilizer to a larger group S' as indicated in Proposition 4.8. As the element $(45)(67)$ commutes with d_7 and $\mathrm{Stab}_{S_5}(d_7S_5)$, and normalizes S_5, we can choose $S' = S_3 \cdot \langle (45)(67) \rangle$. Thus, we get

$$\begin{aligned} Q((46)(57)) = {} & () + 3(12) + 6(14) + (45) \\ & + 2(123) + 12(124) + 6(145) \\ & + 6(12)(34) + 3(12)(45) + 6(14)(25) \\ & + 12(1234) + 12(1245) + 6(1425) \\ & + 6(12)(345) + 12(14)(235) + 2(45)(123) \\ & + 12(12345) + 12(12435) \end{aligned}$$

with "only" 18 terms. We calculate

$$\begin{aligned} T((46)(57)) = {} & (46)(57) + 3(12)(46)(57) + 6(164)(57) + (4756) \\ & + 2(123)(46)(57) + 12(1264)(57) + 6(16475) \\ & + 6(12)(364)(57) + 3(12)(4756) + 6(164)(275) \\ & + 12(12364)(57) + 12(126475) + 6(164275) \\ & + 6(12)(36475) + 12(164)(2375) + 2(4756)(123) \\ & + 12(1236475) + 12(1264375). \end{aligned}$$

From here, we can go through all the divisors m of the exponent 420 of S_7 to obtain the elements $\bar{\mu}_m$ and the indicators for the three irreducible representations of S_3. The Table 1 calculates $\bar{\mu}_m$ in two stages, giving first an "unsimplified" version of $\pi(T^{[m]})$ in an attempt to hint at how this intermediate result can really be read off quite directly from the expression for T obtained above.

For good measure, we shall also finish the calculations for $d_2 = (56)$ and $d_5 = (567)$. In each case $\mathrm{Stab}_{S_5}(d_iS_5) = S_4$, and

$$\begin{aligned} Q(d_i) = {} & () + 6(12) + 4(15) + 8(123) + 12(125) \\ & + 3(12)(34) + 12(12)(35) + 6(1234) + 24(1235) \\ & + 8(123)(45) + 12(125)(34) + 24(12345). \end{aligned}$$

Thus,

$$\begin{aligned} T((56)) = {} & (56) + 6(12)(56) + 4(165) + 8(123)(56) + 12(1265) \\ & + 3(12)(34)(56) + 12(12)(365) + 6(1234)(56) + 24(12365) \\ & + 8(123)(465) + 12(1265)(34) + 24(123465), \end{aligned}$$

m	$\pi(T((46)(57))^{[m]})$	$\bar{\mu}_m((46)(57))$	$\nu_m(\mathcal{F}_{(46)(57)}(W_j))$ W_0	W_1	W_2
2	$1+3+2(123)$	$\frac{1}{3}(2+(123))$	1	1	1
3	6	1	1	1	2
4	$1+3+1+2(123)+12+3+2(123)$	$\frac{1}{3}(10+2(123))$	4	4	6
5	$6+6(12)$	$1+(12)$	2	0	2
6	$1+3+6+2+6+6+12+6$	7	7	7	14
7	$12+12$	4	4	4	8
10	$1+3+2(123)+6+12+6$	$\frac{1}{3}(14+(123))$	5	5	9
12	$1+3+6+1+2+12+6+3+6+12+6+12+2$	12	12	12	24
14	$1+3+2(123)+12+12$	$\frac{1}{3}(14+(123))$	5	5	9
15	$6+6+6(12)$	$2+(12)$	3	1	4
20	$1+3+1+2(123)+12+6+3+12+6+2(123)$	$\frac{1}{3}(22+2(123))$	8	8	14
21	$6+12+12$	5	5	5	10
28	$1+3+1+12+3+2(123)+12+12$	$\frac{1}{3}(22+2(123))$	8	8	14
30	$1+3+6+2+6+6+6+12+12+6+6$	11	11	11	22
35	$6+6(12)+12+12$	$5+(12)$	6	4	10
42	$1+3+6+2+6+6+12+6+12+12$	11	11	11	22
60	$1+3+6+1+2+12+6+6+3+6+12+12+6+6+12+2$	16	16	16	32
70	$1+3+2(123)+6+12+6+12+12$	$\frac{1}{3}(26+(123))$	9	9	17
84	$1+3+6+1+2+12+6+3+6+12+6+12+2+12+12$	16	16	16	32
105	$6+6+6(12)+12+12$	$6+(12)$	7	5	12
140	$1+3+1+2(123)+12+6+3+12+6+2(123)+12+12$	$\frac{1}{3}(34+2(123))$	12	12	22
210	$1+3+6+2+6+6+6+12+12+6+6+12+12$	15	15	15	30
420		20	20	20	40

Table 1. Indicator calculations on $\mathrm{Im}(\mathcal{F}_{(46)(57)}) \subset {}^{S_7}_{S_5}\mathcal{M}_{S_5}$.

and

$$\begin{aligned} T((567)) = {} & (567) + 6(12)(567) + 4(1675) + 8(123)(567) + 12(12675) \\ & + 3(12)(34)(567) + 12(12)(3675) + 6(1234)(567) + 24(123675) \\ & + 8(123)(4675) + 12(12675)(34) + 24(1234675). \end{aligned}$$

Thus, we obtain the elements $\bar{\mu}_m((56))$ and $\bar{\mu}_m((567))$ listed in Table 2.

From this information, together with the character table of S_4 given in Table 3, one can then calculate all the indicator values for the simples in the images of $\mathcal{F}_{(56)}$ and $\mathcal{F}_{(567)}$; see Table 4.

m	$\bar{\mu}_m((56))$	$\bar{\mu}_m((567))$
2	$\frac{1}{14}(5+4(123)+3(12)(34))$	0
3	$\frac{1}{2}(1+(12))$	$\frac{1}{8}(3+2(12)+(12)(34)+2(1234))$
4	$\frac{1}{3}(5+(123))$	$\frac{1}{6}(4+2(123))$
5	1	$\frac{1}{2}(1+(12))$
6	$\frac{1}{4}(11+(12)(34))$	$\frac{1}{4}(7+(12)(34))$
7	0	1
10	$\frac{1}{12}(17+4(123)+3(12)(34))$	1
12	4	3
14	$\frac{1}{12}(5+4(123)+3(12)(34))$	1
15	$\frac{1}{2}(3+(12))$	$\frac{1}{8}(7+6(12)+(12)(34)+2(1234))$
20	$\frac{1}{3}(8+(123))$	$\frac{1}{6}(10+2(123))$
21	$\frac{1}{2}(1+(12))$	$\frac{1}{8}(11+2(12)+(12)(34)+2(1234))$
28	$\frac{1}{6}(10+2(123))$	$\frac{1}{3}(5+(123))$
30	$\frac{1}{4}(15+(12)(34))$	$\frac{1}{4}(11+(12)(34))$
35	1	$\frac{1}{2}(3+(12))$
42	$\frac{1}{4}(11+(12)(34))$	$\frac{1}{4}(11+(12)(34))$
60	5	4
70	$\frac{1}{12}(17+4(123)+3(12)(34))$	2
84	4	4
105	$\frac{1}{2}(3+(12))$	$\frac{1}{8}(15+6(12)+(12)(34)+2(1234))$
210	$\frac{1}{4}(15+(12)(34))$	$\frac{1}{4}(15+(12)(34))$
420	5	5

Table 2. $\bar{\mu}_m((56)), \bar{\mu}_m((567)) \in \mathbb{C}S_4$ for indicators in ${}^{S_7}_{S_5}\mathcal{M}_{S_5}$.

	()	(12)	(123)	(12)(34)	(1234)
η_0	1	1	1	1	1
η_1	1	−1	1	1	−1
η_2	2	0	−1	2	0
η_3	3	1	0	−1	−1
η_4	3	−1	0	−1	1

Table 3. Character table of S_4.

	$\nu_m(\mathcal{F}_{(56)}(W_i))$					$\nu_m(\mathcal{F}_{(567)}(W_i))$				
m	W_0	W_1	W_2	W_3	W_4	W_0	W_1	W_2	W_3	W_4
2	1	1	1	1	1	0	0	0	0	0
3	1	0	1	2	1	1	0	1	1	1
4	2	2	3	5	5	1	1	1	2	2
5	1	1	2	3	3	1	0	1	2	1
6	3	3	6	8	8	2	2	4	5	5
7	0	0	0	0	0	1	1	2	3	3
10	2	2	3	4	4	1	1	2	3	3
12	4	4	8	12	12	3	3	6	9	9
14	1	1	1	1	1	1	1	2	3	3
15	2	1	3	5	4	2	0	2	3	2
20	3	3	5	8	8	2	2	3	5	5
21	1	0	1	2	1	2	1	3	4	4
28	2	2	3	5	5	2	2	3	5	5
30	4	4	8	11	11	3	3	6	8	8
35	1	1	2	3	3	2	1	3	5	4
42	3	3	6	8	8	3	3	6	8	8
60	5	5	10	15	14	4	4	8	12	12
70	2	2	3	4	4	2	2	4	6	6
84	4	4	8	12	12	4	4	8	12	12
105	2	1	3	5	4	3	1	4	6	5
210	4	4	8	11	11	4	4	8	11	11
420	5	5	10	15	15	5	5	10	15	15

Table 4. Indicators on $\mathrm{Im}(\mathcal{F}_{(56)})$, $\mathrm{Im}(\mathcal{F}_{(567)}) \subset {}^{S_7}_{S_5}\mathcal{M}_{S_5}$.

The GAP [2014] code on the next page can be used to calculate the higher indicators for objects in ${}^{G}_{H}\mathcal{M}_{H}$ for any finite group G and subgroup H available to GAP. It uses the simple but inefficient formula (4). Moreover it is written in the most

```
IndicatorForOneRep:=function(m,G,H,d,S,eta)
  local h,sum;
  sum:=0;
  for h in H do
    if (d*h)^m in S
      then sum:=sum+((d*h)^(-m))^eta;
    fi;
  od;
  return(sum/Size(S));
end;

IndicatorsForDoubleCoset:=function(G,H,d)
  local S,eta,irreps,m;
  S:=Intersection(H,H^(d^(-1)));
  irreps:=Irr(S);
  for m in DivisorsInt(Exponent(G)) do
    Print(m,":");
    for eta in irreps do
      Print(IndicatorForOneRep(m,G,H,d,S,eta),",");
    od;
    Print("\n");
  od;
end;
```

GAP code to compute indicators in ${}^{G}_{H}\mathcal{M}_{H}$.

straightforward manner, makes hardly any attempt to reduce the load of calculations, and blindly repeats the same steps several times instead. For the moment, we do not pursue the quest to write better code (storing intermediate results such as the elements μ_m instead of recalculating them for each representation), nor the task to make use of the improved formula in Proposition 4.8 to speed up matters. The clumsy code is sufficient to do any of the calculations done above "by hand" again in seconds. Thus it *could* have been used to verify these results *if* the author *had* had any reason to mistrust his capability to perform flawless computations. Also, *if* the original calculations *had* contained errors, the GAP code *could* have been used to track those down and possibly correct them.

As it stands, the code was also sufficiently efficient to check that the inclusions $S_6 \subset S_8$ as well as $S_7 \subset S_9$ continue to produce only nonnegative indicator values.

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