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Let $c \geq 0$ and denote by $\mathcal{K}(\mathbb{H}, c)$ the set of all infinitesimal generators $G : \mathbb{H} \to \mathbb{C}$ on the upper half-plane $\mathbb{H}$ such that $\limsup_{y\to\infty} y \cdot |G(iy)| \leq c$. This class is related to univalent functions $f : \mathbb{H} \to \mathbb{H}$ with hydrodynamic normalization and appears in the so-called chordal Loewner equation.

In this paper, we generalize the class $\mathcal{K}(\mathbb{H}, c)$ and the hydrodynamic normalization to the Euclidean unit ball in $\mathbb{C}^n$. The generalization is based on the observation that $G \in \mathcal{K}(\mathbb{H}, c)$ can be characterized by an inequality for the hyperbolic length of $G(z)$.

1. Introduction

One-parameter semigroups. Let $\mathbb{B}_n = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$ be the Euclidean unit ball in $\mathbb{C}^n$. In one dimension we write $\mathbb{D} := \mathbb{B}_1$ for the unit disc.

Definition 1.1. A continuous one-real-parameter semigroup of holomorphic functions on $\mathbb{B}_n$ is a map $[0, \infty) \ni t \mapsto \Phi_t \in \mathcal{H}(\mathbb{B}_n, \mathbb{B}_n)$ satisfying the following conditions:

(1) Φ_0 is the identity.

(2) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t, s \geq 0$.

(3) Φ_t tends to the identity locally uniformly in $\mathbb{B}_n$, when t tends to 0.

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Given such a semigroup $\{\Phi_t\}_{t\geq 0}$ and a point $z \in \mathbb{B}_n$, the limit

$$G(z) := \lim_{t\to 0} \frac{\Phi_t(z) - z}{t}$$

exists and the vector field $G : \mathbb{B}_n \to \mathbb{C}^n$, called the *infinitesimal generator*[1] of Φ_t, is a holomorphic function (see, e.g., [Abate 1992]). We denote by $\mathrm{Inf}(\mathbb{B}_n)$ the set of all infinitesimal generators of semigroups in $\mathbb{B}_n$. For any $z \in \mathbb{B}_n$, the map $w(t) := \Phi_t(z)$ is the solution of the initial value problem

$$\frac{dw(t)}{dt} = G(w(t)), \quad w(0) = z. \tag{1-1}$$

There are various characterizations of holomorphic functions $G : \mathbb{B}_n \to \mathbb{C}^n$ that are infinitesimal generators; see [Reich and Shoikhet 2005, Section 7.3], [Bracci et al. 2010, Theorem 0.2], [Bracci et al. 2014, p. 193].

The set $\mathrm{Inf}(\mathbb{D})$, i.e., all infinitesimal generators in the unit disc, can be characterized completely by the Berkson–Porta representation formula [1978]

$$\mathrm{Inf}(\mathbb{D}) = \{z \mapsto (\tau - z)(1 - \bar{\tau} z)p(z) \,|\, \tau \in \overline{\mathbb{D}},\, p \in \mathcal{H}(\mathbb{D}, \mathbb{C}) \text{ with } \mathrm{Re}(p(z)) \geq 0 \text{ for all } z \in \mathbb{D}\}. \tag{1-2}$$

Remark 1.2. Let $F : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map. Recall the Denjoy–Wolff theorem (see, e.g., [Reich and Shoikhet 2005, Theorem 5.1]): If F is not an elliptic automorphism (i.e., an automorphism with exactly one fixed point in $\mathbb{D}$), then there exists one point $\tau \in \overline{\mathbb{D}}$ (the Denjoy–Wolff point of F) such that the iterates F^n converge locally uniformly in $\mathbb{D}$ to the constant map τ.

If $\{\Phi_t\}_{t\geq 0}$ is a semigroup on $\mathbb{D}$, then we call $\tau \in \overline{\mathbb{D}}$ the Denjoy–Wolff point of $\{\Phi_t\}_{t\geq 0}$ if τ is the Denjoy–Wolff point of Φ_1, which is equivalent to $\lim_{t\to\infty} \Phi_t = \tau$ locally uniformly.

If an infinitesimal generator in the unit disc does not generate a semigroup of elliptic automorphisms of $\mathbb{D}$, then the point $\tau \in \overline{\mathbb{D}}$ from formula (1-2) is exactly the Denjoy–Wolff point of the semigroup.

There are two special cases of infinitesimal generators in $\mathbb{D}$ that have been studied intensively and turned out to be quite useful in Loewner theory and its applications. The two different cases arise from certain normalizations of the Berkson–Porta data τ and p from formula (1-2). In the *radial* case, one considers those elements $G \in \mathrm{Inf}(\mathbb{D})$ whose Berkson–Porta data τ and p satisfy

$$\tau = 0 \quad \text{and} \quad p(0) = 1,$$

i.e., $G(z) = -zp(z)$.

[1] There is no standard convention in the literature and often $-G$ is called the infinitesimal generator of the semigroup.

This class plays a central role in studying the class S of all univalent functions $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$, $f'(0) = 1$, via the powerful tools of Loewner's theory, which considers a nonautonomous version of (1-1); see, e.g., [Pommerenke 1975, Chapter 6]. The class of radial generators as well as the class S have been generalized in this context to the polydisc $\mathbb{D}^n$ (see [Poreda 1987a; 1987b]), and to the unit ball $\mathbb{B}_n$ (see [Graham and Kohr 2003] for a collection of several results and references).

The second class, the set of all *chordal* generators[2], consists of all $G \in \mathrm{Inf}(\mathbb{D})$ whose Berkson–Porta data τ and p satisfy

$$\tau = 1 \quad \text{and} \quad \angle \lim_{z \to 1} \frac{p(z)}{z-1} \ \text{is finite.}$$

The aim of this paper is to introduce a generalization of the chordal class for the unit ball $\mathbb{B}_n$.

The hydrodynamic normalization in one dimension. Instead of fixing an interior point, like in the class S, it can be of interest to investigate univalent self-mappings of $\mathbb{D}$ that fix a boundary point. In this case, one usually passes from $\mathbb{D}$ to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

A class of such mappings that is easy to describe and that appears in several applications is the set of all univalent mappings $f : \mathbb{H} \to \mathbb{H}$ that fix the boundary point ∞ and have the so-called *hydrodynamic normalization*. Basic properties of this class can be found in [Goryaĭnov and Ba 1992]; see also [Bauer 2005; Contreras et al. 2010]. One of its main applications is the chordal Loewner equation; see [Abate et al. 2010, Section 4] for further references.

A univalent function $f : \mathbb{H} \to \mathbb{H}$ has *hydrodynamic normalization* (at ∞) if f has the expansion

$$f(z) = z - \frac{c}{z} + \gamma(z),$$

where $c \geq 0$, which is usually called *half-plane capacity*, and γ satisfies

$$\angle \lim_{z \to \infty} z \cdot \gamma(z) = 0.$$

We denote by $\mathfrak{P}$ the set of all these functions. Then $\mathfrak{P}$ is a semigroup and the functional $l : \mathfrak{P} \to [0, \infty)$, $l(f) = c$, is additive: if $f_1, f_2 \in \mathfrak{P}$, then $f_1 \circ f_2 \in \mathfrak{P}$ and $l(f_1 \circ f_2) = l(f_1) + l(f_2)$.

Remark 1.3. Let $f \in \mathfrak{P}$ with $l(f) = c$. If we transfer f to the unit disc by conjugation by the Cayley transform, then we obtain a function $\tilde{f} : \mathbb{D} \to \mathbb{D}$ having

[2] Note that there is no standard use of the words "radial" and "chordal" in the literature. In [Contreras et al. 2010], e.g., an element $G \in \mathrm{Inf}(\mathbb{D})$ is called *radial* if $\tau \in \mathbb{D}$ and chordal if $\tau \in \partial\mathbb{D}$.

the expansion

$$\tilde{f}(z) = z - \frac{c}{4}(z-1)^3 + \tilde{\gamma}(z),$$

where $\angle \lim_{z\to 1} \tilde{\gamma}(z)/(z-1)^3 = 0$.

If $\{\Phi_t\}_{t\geq 0}$ is a one-real-parameter semigroup contained in $\mathfrak{P}$ with $l(\Phi_1) = a$, then it is easy to see that $l(\Phi_t) = a\cdot t$. If H is the generator of this semigroup, then we also define $l(H) := a$.

We will be interested in the following set of chordal generators.

Definition 1.4. By $\mathcal{K}(\mathbb{H}, c)$ we denote the set of all infinitesimal generators H of one-real-parameter semigroups $\{\Phi_t\}_{t\geq 0}$ contained in $\mathfrak{P}$ with $l(H) \leq c$.

Remark 1.5. The set $\mathcal{K}(\mathbb{H}, c)$ can be characterized in various ways; see [Goryaĭnov and Ba 1992, Section 1] and [Maassen 1992, Proposition 2.2].

It is known that $H \in \mathcal{K}(\mathbb{H}, c)$ for some $c \geq 0$ if and only if H maps $\mathbb{H}$ into $\overline{\mathbb{H}}$ and

$$\limsup_{y\to\infty} y|H(iy)| \leq c. \tag{1-3}$$

In fact, $l(H) = \limsup_{y\to\infty} y|H(iy)|$.

Furthermore, this is equivalent to H maps $\mathbb{H}$ into $\overline{\mathbb{H}}$ and

$$|H(z)| \leq \frac{c}{\operatorname{Im}(z)} \tag{1-4}$$

for all $z \in \mathbb{H}$. The number $l(H)$ is the smallest constant such that this inequality holds.

Finally, it is known that this property is equivalent to the fact that $-G$ is the Cauchy transform of a finite, nonnegative Borel measure μ on $\mathbb{R}$, i.e.,

$$H(z) = \int_{\mathbb{R}} \frac{\mu(du)}{u-z}. \tag{1-5}$$

The number $l(H)$ can be calculated by $l(H) = \mu(\mathbb{R})$.

Remark 1.6. It is easy to see that the following holds: if $f \in \mathfrak{P}$ with $c = l(f)$, then $H := f - \mathrm{id} \in \mathcal{K}(\mathbb{H}, c)$ with $l(H) = c$.

Let $C : \mathbb{H} \to \mathbb{D}$, $C(z) = (z-i)/(z+i)$, be the Cayley map. We define $\mathcal{K}(\mathbb{D}, c)$ by

$$\mathcal{K}(\mathbb{D}, c) = \{C'(C^{-1})\cdot(H\circ C^{-1}) \mid H \in \mathcal{K}(\mathbb{H}, c)\}.[3]$$

The rest of this paper is organized as follows: In Section 2 we look for an invariant characterization of chordal generators, i.e., of the sets $\mathcal{K}(\mathbb{H}, c)$ and $\mathcal{K}(\mathbb{D}, c)$, and we introduce the class $\mathcal{K}(\mathbb{B}_n, c)$ for the higher-dimensional unit ball. It will turn out to be quite useful to study "slices" of this class, which is done in Section 3. In Section 4 we introduce and study the class $\mathfrak{P}_n$, a higher-dimensional analog of the class $\mathfrak{P}$.

[3] If $\{\Phi_t\}_{t\geq 0}$ is a semigroup in $\mathbb{H}$ with generator H, then $\{C\circ\Phi_t\circ C^{-1}\}_{t\geq 0}$ is a semigroup in $\mathbb{D}$ and its generator is given by $C'(C^{-1})\cdot(H\circ C^{-1})$.

2. Chordal generators in higher dimensions

Invariant formulation for $\mathcal{K}(\mathbb{D}, c)$ and $\mathcal{K}(\mathbb{H}, c)$. For $R > 0$, we let $E_{\mathbb{D}}(1, R)$ be the horodisc in $\mathbb{D}$ with center 1 and radius R, i.e.,

$$E_{\mathbb{D}}(1, R) = \left\{ z \in \mathbb{D} \;\middle|\; \frac{1}{|u_{\mathbb{D}}(z)|} < R \right\},$$

where $u_{\mathbb{D}}(z) = -(1 - |z|^2)/|1 - z|^2$ is the Poisson kernel in $\mathbb{D}$ with respect to 1.

By using the Cayley map, we define analogously

$$E_{\mathbb{H}}(\infty, R) = C^{-1}(E_{\mathbb{D}}(1, R)) = \left\{ z \in \mathbb{H} \;\middle|\; \frac{1}{\operatorname{Im}(z)} < R \right\}.$$

For $z \in \mathbb{D}$ and a tangent vector $v \in \mathbb{C}$, we denote by $|v|_{\mathbb{D},z}$ the hyperbolic length of v, i.e.,

$$|v|_{\mathbb{D},z} := \frac{2|v|}{1 - |z|^2}.$$

Furthermore, we let $R_{\mathbb{D}}(z)$ be the radius R of the horodisc $E_{\mathbb{D}}(1, R)$ that satisfies $z \in \partial E(1, R)$; in short, $R_{\mathbb{D}}(z) = 1/|u_{\mathbb{D}}(z)|$. Analogously, for $z \in \mathbb{H}$ and $v \in \mathbb{C}$, we define $R_{\mathbb{H}}(z) := 1/\operatorname{Im}(z)$ and the hyperbolic length $|v|_{\mathbb{H},z} := |v|/\operatorname{Im}(z)$.

According to (1-4), we know that $H \in \mathcal{K}(\mathbb{H}, c)$ if and only if H maps $\mathbb{H}$ into $\overline{\mathbb{H}}$ and $|H(z)| \le c/\operatorname{Im}(z)$ for all $z \in \mathbb{H}$. By using the Berkson–Porta formula, it is easy to see that we can rephrase this to: $H \in \mathcal{K}(\mathbb{H}, c)$ if and only if $H \in \operatorname{Inf}(\mathbb{H})$ and $|H(z)| \le c/\operatorname{Im}(z)$ for all $z \in \mathbb{H}$.

The last inequality is equivalent to $|H(z)|/\operatorname{Im}(z) \le c/\operatorname{Im}(z)^2$ or

$$|H(z)|_{\mathbb{H},z} \le \frac{c}{\operatorname{Im}(z)^2} = c \cdot R_{\mathbb{H}}(z)^2.$$

If we pass from $\mathbb{H}$ to $\mathbb{D}$ and transform H into $G = C'(C^{-1}) \cdot (H \circ C^{-1})$, then G satisfies $|G(C(z))|_{\mathbb{D},C(z)} = |H(z)|_{\mathbb{H},z}$ and we immediately get the following characterization.

Proposition 2.1. *Let $G \in \operatorname{Inf}(\mathbb{D})$. Then*

$$G \in \mathcal{K}(\mathbb{D}, c) \quad \Longleftrightarrow \quad |G(z)|_{\mathbb{D},z} \le c \cdot R_{\mathbb{D}}(z)^2 \text{ for all } z \in \mathbb{D}.$$

Let $H \in \operatorname{Inf}(\mathbb{H})$. Then

$$H \in \mathcal{K}(\mathbb{H}, c) \quad \Longleftrightarrow \quad |H(z)|_{\mathbb{H},z} \le c \cdot R_{\mathbb{H}}(z)^2 \text{ for all } z \in \mathbb{H}.$$

Chordal generators in the unit ball. For $n \in \mathbb{N}$, let u_n be the pluricomplex Poisson kernel in $\mathbb{B}_n$ with pole at $e_1 := (1, 0, \ldots, 0)$, i.e.,

$$u_{\mathbb{B}_n,p} = -\frac{1 - \|z\|^2}{|1 - z_1|^2}.$$

The level sets of $u_{\mathbb{B}_n}$ are exactly the boundaries of horospheres with center e_1; more precisely, the set

$$E_{\mathbb{B}_n}(e_1, R) := \{z \in \mathbb{B}_n \mid |u_{\mathbb{B}_n}(z)|^{-1} < R\}, \quad R > 0,$$

is the horosphere with center e_1 and radius R.

Furthermore, for $z \in \mathbb{B}_n$ and $v \in \mathbb{C}^n$, we denote by $\|v\|_{\mathbb{B}_n,z}$ the Kobayashi-hyperbolic length of the vector v with respect to z.

Motivated by Proposition 2.1, we make the following definition.

Definition 2.2. Let $c \geq 0$. We define the class $\mathcal{K}(\mathbb{B}_n, c)$ to be the set of all infinitesimal generators G on $\mathbb{B}_n$ such that, for all $z \in \mathbb{B}_n$,

$$\|G(z)\|_{\mathbb{B}_n,z} \leq \frac{c}{u_{\mathbb{B}_n}(z)^2}. \tag{2-1}$$

Remark 2.3. $\mathcal{K}(\mathbb{B}_n, c)$ is a compact family: Montel's theorem and the definition of $\mathcal{K}(\mathbb{B}_n, c)$ immediately imply that it is a normal family. If a sequence $(G_n) \subset \mathcal{K}(\mathbb{B}_n, c)$ converges locally uniformly to $G : \mathbb{B}_n \to \mathbb{C}^n$, then G is holomorphic and also an infinitesimal generator, which can be seen by using the characterization given in [Bracci et al. 2010, Theorem 0.2]. Of course, G also satisfies (2-1) and we conclude $G \in \mathcal{K}(\mathbb{B}_n, c)$.

Just as we passed from $\mathbb{D}$ to $\mathbb{H}$ in one dimension, we can pass from the unit ball $\mathbb{B}_n$ to the Siegel upper half-space $\mathbb{H}_n = \{(z_1, \tilde{z}) \in \mathbb{C}^n \mid \operatorname{Im}(z_1) > \|\tilde{z}\|^2\}$ in order to get simpler formulas:

The Cayley map

$$C : \mathbb{H}_n \to \mathbb{B}_n, \quad C(z) = (C_1(z), \ldots, C_n(z)) = \left(\frac{z_1 - i}{z_1 + i}, \frac{2z_2}{z_1 + i}, \ldots, \frac{2z_n}{z_1 + i}\right),$$

maps $\mathbb{H}_n$ biholomorphically onto $\mathbb{B}_n$. It extends to a homeomorphism from the one-point compactification $\widehat{\mathbb{H}}_n = \mathbb{H}_n \cup \partial\mathbb{H}_n \cup \{\infty\}$ of $\mathbb{H}_n \cup \partial\mathbb{H}_n$ to the closure of $\mathbb{B}^n$.

The pluricomplex Poisson kernel transforms as follows:

$$u_{\mathbb{H}_n}(z) := u_{\mathbb{B}_n}(C(z)) = -\operatorname{Im}(z_1) + \|\tilde{z}\|^2.$$

Thus, we define the horosphere $E_{\mathbb{H}_n}(\infty, R)$ with center ∞ and radius $R > 0$ by

$$E_{\mathbb{H}_n}(\infty, R) := \left\{z \in \mathbb{H}_n \;\middle|\; \operatorname{Im}(z_1) - \|\tilde{z}\|^2 > \frac{1}{R}\right\}.$$

For $v \in \mathbb{C}^n$ and $z \in \mathbb{H}_n$, we let $\|v\|_{\mathbb{H}_n,z}$ be the Kobayashi hyperbolic length of v.

Let $c \geq 0$. We define the class $\mathcal{K}(\mathbb{H}_n, c)$ to be the set of all infinitesimal generators H on $\mathbb{H}_n$ satisfying the inequality

$$\|H(z)\|_{\mathbb{H}_n,z} \leq \frac{c}{u_{\mathbb{H}_n}(z)^2}$$

for all $z \in \mathbb{H}_n$. Then we have

$$\mathcal{K}(\mathbb{B}_n, c) = \{C'(C^{-1}) \cdot (H \circ C^{-1}) \mid H \in \mathcal{K}(\mathbb{H}_n, c)\}.$$

From now on we will stay in the upper half-space $\mathbb{H}_n$, where most of the computations we need take a simpler form.

3. Slices

Normalized geodesics and slices. For any $H \in \mathrm{Inf}(\mathbb{H}_n)$, one can consider one-dimensional slices by using the so-called *Lempert projection devices*; see [Bracci and Shoikhet 2014, Section 3].

If $w \in \mathbb{H}_n$, then there exists a unique complex geodesic passing through w and ∞. Let us choose a parametrization $\varphi : \mathbb{H} \to \mathbb{H}_n$ of this geodesic. There exists a unique holomorphic map $P : \mathbb{H}_n \to \mathbb{H}_n$ with $P^2 = P$ and $P \circ \varphi = \varphi$. Define $\widetilde{P} = \varphi^{-1} \circ P$. Then

$$h_\varphi : \mathbb{H} \to \mathbb{C}, \quad h_\varphi(\zeta) = d\widetilde{P}(\varphi(\zeta)) \cdot H(\varphi(\zeta)),$$

is an infinitesimal generator on $\mathbb{H}$; see [Bracci and Shoikhet 2014, p. 6].

We will need special parametrizations of these geodesics: In [Bracci and Patrizio 2005, p. 516], it is shown that for any complex geodesic $\varphi : \mathbb{H} \to \mathbb{H}_n$ with $\varphi(\infty) = \infty$, there exists $a_\varphi > 0$ such that

$$u_{\mathbb{H}_n}(\varphi(\zeta)) = a_\varphi \cdot u_{\mathbb{H}}(\zeta)$$

for all $\zeta \in \mathbb{H}$. Call a geodesic $\varphi : \mathbb{H} \to \mathbb{H}_n$ *normalized* if $\varphi(\infty) = \infty$ and $a_\varphi = 1$.

Lemma 3.1. *Let $a \in \mathbb{C}$ and $\gamma \in \mathbb{C}^{n-1}$ such that $(a, \gamma) \in \mathbb{H}_n$. Then the map*

$$\varphi_\gamma : \mathbb{H} \to \mathbb{H}_n, \quad \varphi_\gamma(\zeta) := (\zeta + i\|\gamma\|^2, \gamma),$$

is a normalized geodesic through (a, γ). Furthermore, if $H = (H_1, \widetilde{H}) \in \mathrm{Inf}(\mathbb{H}_n)$, then the slice $h_\gamma := h_{\varphi_\gamma}$ of H with respect to φ_γ is given by

$$h_\gamma(\zeta) = H_1(\varphi_\gamma(\zeta)) - 2i\bar{\gamma}^T \cdot \widetilde{H}(\varphi_\gamma(\zeta)). \tag{3-1}$$

Proof. Let $\psi : \mathbb{D} \to \mathbb{B}_n$ be a complex geodesic with $\psi(1) = e_1$. As a parametrization for ψ, one can choose (see [Bracci and Shoikhet 2014, Section 3])

$$\psi(\zeta) = (\alpha^2(\zeta - 1) + 1, \alpha(\zeta - 1)\beta),$$

where $\alpha > 0$ and $\beta \in \mathbb{C}^{n-1}$ such that $\|\beta\|^2 = 1 - \alpha^2$. Then

$$C^{-1}(\psi(\zeta)) = \left(i\frac{2 + \alpha^2(\zeta - 1)}{\alpha^2(1 - \zeta)}, i\beta/\alpha \right)$$

and

$$\zeta \mapsto C^{-1}\big(\psi(C_1(\zeta))\big) = \left(-i + \frac{\zeta+i}{\alpha^2}, i\beta/\alpha\right)$$
$$= \left(\frac{\zeta}{\alpha^2} + i\frac{1-\alpha^2}{\alpha^2}, i\beta/\alpha\right) = \left(\frac{\zeta}{\alpha^2} + i\left\|\frac{\beta}{\alpha}\right\|^2, i\beta/\alpha\right)$$

is a complex geodesic from $\mathbb{H}$ to $\mathbb{H}_n$. A reparametrization (ζ/α^2 to ζ) and setting $\gamma = i\beta/\alpha$ gives the geodesic

$$\varphi_\gamma(\zeta) = (\zeta + i\|\gamma\|^2, \gamma). \tag{3-2}$$

This complex geodesic is normalized because it satisfies $\varphi_\gamma(\infty) = \infty$ and

$$u_{\mathbb{H}_n}(\varphi_\gamma(\zeta)) = \operatorname{Im}(\zeta + i\|\gamma\|^2) - \|\gamma\|^2 = \operatorname{Im}(\zeta) = u_{\mathbb{H}}(\zeta).$$

The projection onto $\varphi_\gamma(\mathbb{H})$ is given by

$$P(z_1, \tilde{z}) = (z_1 - 2i\bar{\gamma}^T \cdot \tilde{z} + 2i\|\gamma\|^2, \gamma). \tag{3-3}$$

Clearly, P is holomorphic and maps $\mathbb{H}_n$ onto $\varphi_\gamma(\mathbb{H})$ because

$$\begin{aligned}\operatorname{Im}(z_1 - 2i\bar{\gamma}^T \cdot \tilde{z} + 2i\|\gamma\|^2) &= \operatorname{Im}(z_1) - 2\operatorname{Im}(i\bar{\gamma}^T \cdot \tilde{z}) + 2\|\gamma\|^2 \\ &\geq \|\tilde{z}\|^2 - 2\|\gamma\|\,\|\tilde{z}\| + \|\gamma\|^2 + \|\gamma\|^2 \\ &= (\|\gamma\| - \|\tilde{z}\|)^2 + \|\gamma\|^2 \geq \|\gamma\|^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}(P \circ P)(z_1, \tilde{z}) &= (z_1 - 2i\bar{\gamma}^T\tilde{z} + 2i\|\gamma\|^2 - 2i\bar{\gamma}^T\gamma + 2i\|\gamma\|^2, \gamma) \\ &= (z_1 - 2i\bar{\gamma}^T\tilde{z} + 2i\|\gamma\|^2, \gamma) = P(z_1, \tilde{z}).\end{aligned}$$

Thus, the inverse $\widetilde{P} : \mathbb{H}_2 \to \mathbb{H}$, $\widetilde{P} = \varphi_\gamma^{-1} \circ P$, is given by

$$\widetilde{P}(z_1, \tilde{z}) = (z_1 - 2i\bar{\gamma}^T\tilde{z} + i\|\gamma\|^2).$$

If $H(z) = (H_1(z), \widetilde{H}(z))$ is a generator on $\mathbb{H}_n$, we get the slice reduction

$$\begin{aligned}h_{\varphi_\gamma}(\zeta) &= d\widetilde{P}(\varphi_\gamma(\zeta)) \cdot H(\varphi_\gamma(\zeta)) \\ &= H_1(\varphi_\gamma(\zeta)) - 2i\bar{\gamma}^T \cdot \widetilde{H}(\varphi_\gamma(\zeta)).\end{aligned}$$ □

Some explicit formulas. Later on we will need explicit formulas of the Kobayashi norms of $dP(z)H(z)$ and $H(z) - dP(z) \cdot H(z)$. The following lemma is proven in the Appendix.

Lemma 3.2. *Let $a \in \mathbb{C}$, $p, v \in \mathbb{C}^{n-1}$ and $z = (z_1, \tilde{z}) \in \mathbb{H}_n$. Then the following formulas hold*:

$$\left\| \begin{pmatrix} a \\ 0 \end{pmatrix} \right\|_{\mathbb{H}_n, z} = \frac{|a|}{|u_{\mathbb{H}_n}(z)|}, \tag{3-4}$$

$$\left\| \begin{pmatrix} 2i\,\bar{p}^T v \\ v \end{pmatrix} \right\|_{\mathbb{H}_n, z} = 2\frac{\sqrt{\|v\|^2 |u_{\mathbb{H}_n}(z)| + |\overline{(p-\tilde{z})}^T v|^2}}{|u_{\mathbb{H}_n}(z)|}, \tag{3-5}$$

$$\left\| \begin{pmatrix} a - 2i\bar{\tilde{z}}^T v \\ 0 \end{pmatrix} + \begin{pmatrix} 2i\bar{\tilde{z}}^T v \\ v \end{pmatrix} \right\|^2_{\mathbb{H}_n, z} = \left\| \begin{pmatrix} a - 2i\bar{\tilde{z}}^T v \\ 0 \end{pmatrix} \right\|^2_{\mathbb{H}_n, z} + \left\| \begin{pmatrix} 2i\bar{\tilde{z}}^T v \\ v \end{pmatrix} \right\|^2_{\mathbb{H}_n, z}. \tag{3-6}$$

By using Lemma 3.2 we obtain the following explicit expressions.

Lemma 3.3. *Let $H = (H_1, \widetilde{H}) \in \mathrm{Inf}(\mathbb{H}_n)$ and fix $z \in \mathbb{H}_n$. Denote by P the projection onto the complex geodesic through z and ∞. Then the following formulas hold*:

$$\begin{aligned} dP(z) \cdot H(z) &= (H_1(z) - 2i\bar{\tilde{z}}^T \widetilde{H}(z), 0), \\ H(z) - dP(z) \cdot H(z) &= (2i\bar{\tilde{z}}^T \widetilde{H}(z), \widetilde{H}(z)). \end{aligned} \tag{3-7}$$

Furthermore,

$$\|H(z)\|^2_{\mathbb{H}_n, z} = \|dP(z) \cdot H(z)\|^2_{\mathbb{H}_n, z} + \|H(z) - dP(z) \cdot H(z)\|^2_{\mathbb{H}_n, z}, \tag{3-8}$$

$$\|dP(z) H(z)\|_{\mathbb{H}_n, z} = \frac{|H_1(z) - 2i\bar{\tilde{z}}^T \widetilde{H}(z)|}{|u_{\mathbb{H}_n}(z)|}, \tag{3-9}$$

$$\|H(z) - dP(z) \cdot H(z)\|_{\mathbb{H}_n, z} = 2\frac{\|\widetilde{H}(z)\|}{\sqrt{|u_{\mathbb{H}_n}(z)|}}. \tag{3-10}$$

Proof. The formulas for $dP(z)H(z)$ and $H(z) - dP(z)H(z)$ follow from the explicit form (3-3).

Equation (3-8) follows from (3-6) with $a = H_1(z)$ and $v = \widetilde{H}(z)$.

Furthermore, (3-9) follows directly from (3-4) with $a = H_1(z) - 2i\bar{\tilde{z}}^T \widetilde{H}(z)$ and (3-10) from (3-5) by setting $p = \tilde{z}$ and $v = \widetilde{H}$. □

Slices of generators in $\mathcal{K}(\mathbb{H}_n, c)$ and examples.

Proposition 3.4. *Let $c \geq 0$ and $H \in \mathcal{K}(\mathbb{H}_n, c)$. Then every normalized slice h_γ of H belongs to $\mathcal{K}(\mathbb{H}, c)$.*

Proof. Fix $\gamma \in \mathbb{C}^{n-1}$ and $\zeta \in \mathbb{H}$ and let $z = \varphi_\gamma(\zeta)$.

Furthermore, let P be the projection onto $\varphi_\gamma(\mathbb{H})$. Now we write $H(z)$ as

$$H(z) = dP(z) \cdot H(z) + (H(z) - dP(z)H(z)).$$

As $H \in \mathcal{K}(\mathbb{H}_n, c)$, equation (3-8) implies

$$\|H(z)\|^2_{\mathbb{H}_n,z} = \|dP(z)\cdot H(z)\|^2_{\mathbb{H}_n,z} + \|H(z) - dP(z)H(z)\|^2_{\mathbb{H}_n,z} \le \frac{c^2}{u_{\mathbb{H}_n}(z)^4}.$$

In particular,

$$\|dP(z)\cdot H(z)\|_{\mathbb{H}_n,z} \le \frac{c}{u_{\mathbb{H}_n}(z)^2}. \tag{3-11}$$

By the definition of the slice h_γ, we have

$$dP(\varphi_\gamma(\zeta))\cdot H(\varphi_\gamma(\zeta)) = (d\varphi_\gamma)(\zeta)\cdot h_\gamma(\zeta),$$

and consequently

$$\|dP(\varphi_\gamma(\zeta))\cdot H(\varphi_\gamma(z))\|_{\mathbb{H}_n,\varphi_\gamma(\zeta)} = \|(d\varphi_\gamma)(\zeta)\cdot h_\gamma(\zeta)\|_{\mathbb{H}_n,\varphi_\gamma(\zeta)} = |h_\gamma(\zeta)|_{\mathbb{H},\zeta}.$$

The last equality holds as φ_γ is a complex geodesic. Equation (3-11) implies

$$|h_\gamma(\zeta)|_{\mathbb{H},\zeta} \le \frac{c}{u_{\mathbb{H}_n}(\varphi_\gamma(\zeta))^2} = \frac{c}{u_{\mathbb{H}}(\zeta)^2},$$

where the last equality holds as φ_γ is normalized. Hence, $h_\gamma \in \mathcal{K}(\mathbb{H}, c)$. □

Remark 3.5. If two holomorphic functions $H_1, H_2 : \mathbb{H}_n \to \mathbb{C}^n$ have the same slices, i.e., $dP(z)H_1(z) = dP(z)H_2(z)$ for all $z \in \mathbb{H}_n$, then $H_1 = H_2$; see the proof of Theorem 3.2 in [Casavecchia 2010].

Example 3.6. The family $\{\Phi_t(z) = (z_1, e^{-it/z_1}z_2)\}_{t\ge 0}$ is a semigroup on $\mathbb{H}_2$. Its generator H is given by

$$H(z_1, z_2) = \left(0, -i\frac{z_2}{z_1}\right).$$

Thus, for $\gamma \in \mathbb{C}$, the slice h_γ has the form

$$h_\gamma(z) = -2i\bar{\gamma}\cdot -i\frac{\gamma}{z + i|\gamma|^2} = \frac{-2|\gamma|^2}{z + i|\gamma|^2}.$$

Consequently, the limit $\lim_{y\to\infty} y\cdot|h(iy)| = 2|\gamma|^2$ exists, but does not have an upper bound that is independent of γ. Proposition 3.4 implies that for any $c \ge 0$, $H \notin \mathcal{K}(\mathbb{H}_2, c)$.

Example 3.7. Let

$$H : \mathbb{H}_2 \to \mathbb{C}^2, \qquad H(z_1, z_2) = \begin{pmatrix} -1/z_1 \\ z_2/2z_1^2 \end{pmatrix}.$$

For $\gamma \in \mathbb{C}$, the slice h_γ is given by

$$h_\gamma(\zeta) = \frac{-1}{\zeta + i|\gamma|^2} - 2i\bar{\gamma} \cdot \frac{\gamma}{2(\zeta + i|\gamma|^2)^2}$$
$$= \frac{-\zeta - 2i|\gamma|^2}{(\zeta + i|\gamma|^2)^2} = \frac{(-\zeta - 2i|\gamma|^2)(\bar{\zeta}^2 - 2i|\gamma|^2\bar{\zeta} - |\gamma|^4)}{\left|\zeta + i|\gamma|^2\right|^4}.$$

Let us write $\zeta = x + iy$, $x \in \mathbb{R}$, $y \in (0, \infty)$. Then a small calculation gives

$$\operatorname{Im}(h_\gamma(\zeta)) = \frac{y(x^2 + y^2) + 4y^2|\gamma|^2 + 5y|\gamma|^4 + 2|\gamma|^6}{\left|\zeta + i|\gamma|^2\right|^4} > 0.$$

Furthermore,

$$\limsup_{y\to\infty} y|h_\gamma(iy)| = 1.$$

Hence, $h_\gamma \in \mathcal{K}(\mathbb{H}, 1)$. So each slice is an infinitesimal generator in $\mathbb{H}$ and by [Bracci and Shoikhet 2014, Proposition 3.8], the function H is an infinitesimal generator in $\mathbb{H}_2$.

Now let $(z_1, z_2) \in \mathbb{H}_2$ and write $z_1 = x + iy$, $x, y \in \mathbb{R}$. Then we get

$$u_{\mathbb{H}_2}(z)^4 \cdot \|H(z)\|^2_{\mathbb{H}_2,z} = (y - |z_2|^2)^2 \cdot \frac{x^2 + y^2 + 3|z_2|^2 y}{(x^2 + y^2)^2}$$
$$\underset{y \geq |z_2|^2}{\leq} y^2 \cdot \frac{x^2 + y^2 + 3y^2}{(x^2 + y^2)^2} \leq \frac{x^2 + 4y^2}{x^2 + y^2} \leq 4$$

(an explicit formula of the Kobayashi metric is given in the Appendix). Consequently, $H \in \mathcal{K}(\mathbb{H}_2, 2)$.

Question 3.8. Let $H : \mathbb{H}_n \to \mathbb{C}^n$ be an infinitesimal generator. Assume there exists $c \geq 0$ such that $h_\gamma \in \mathcal{K}(\mathbb{H}, c)$ for every $\gamma \in \mathbb{C}^{n-1}$. Does this imply that $H \in \mathcal{K}(\mathbb{H}_n, C)$ for some $C \geq c$?

4. Univalent functions with hydrodynamic normalization

Motivated by Remark 1.6, we define the following generalization of the class $\mathfrak{P}$, where id stands for the identity mapping on $\mathbb{H}_n$.

Definition 4.1.

$$\mathfrak{P}_n := \left\{ f : \mathbb{H}_n \to \mathbb{H}_n \mid f \text{ is univalent and } f - \mathrm{id} \in \mathcal{K}(\mathbb{H}_n, c) \text{ for some } c \geq 0 \right\}.$$

Remark 4.2. It is important to note that if $f : \mathbb{H}_n \to \mathbb{H}_n$ is a holomorphic self-mapping, then the map $f - \mathrm{id}$ is automatically an infinitesimal generator; see [Reich and Shoikhet 2005, p. 207].

Basic properties of $\mathfrak{P}_n$**.** The following proposition summarizes some basic properties of $\mathfrak{P}_n$.

Proposition 4.3. (a) *$\mathfrak{P}_n$ contains no automorphism of $\mathbb{H}_n$ except the identity.*

(b) *Let $\alpha : \mathbb{H}_n \to \mathbb{H}_n$ be an automorphism of $\mathbb{H}_n$ with $\alpha(\infty) = \infty$. If $f \in \mathfrak{P}_n$, then $\alpha^{-1} \circ f \circ \alpha \in \mathfrak{P}_n$.*

(c) *Let $f \in \mathfrak{P}_n$. Then $f(E_{\mathbb{H}_n}(\infty, R)) \subset E_{\mathbb{H}_n}(\infty, R)$ for every $R > 0$.*

(d) *Let $f \in \mathfrak{P}_n$ and write $f(z) = z + H(z)$ with $H = (H_1, \tilde{H}) \in \mathcal{K}(\mathbb{H}_n, c)$. Then*

$$\|\tilde{H}(z)\|^2 \leq |H_1(z) - 2i\bar{\tilde{z}}^T \tilde{H}| \quad \text{for all } z = (z_1, \tilde{z}) \in \mathbb{H}_n. \tag{4-1}$$

(e) *Let $f \in \mathfrak{P}_n$. Then there exists $R > 0$ such that $E_{\mathbb{H}_n}(\infty, R) \subset f(\mathbb{H}_n)$.*

Proof. The statements (a) and (b) can easily be shown by using the explicit form of automorphisms of $\mathbb{H}_n$; see [Abate 1989, Proposition 2.2.4].

The statement (c) is just Julia's lemma: Write $f(z) = z + H(z)$ and let us pass to the unit ball and define $\tilde{f} : \mathbb{B}_n \to \mathbb{B}_n$, $\tilde{f} = C \circ f \circ C^{-1}$. Then

$$\tilde{f} = \frac{1}{2i + H_1(C^{-1}(z)) - z_1 H_1(C^{-1}(z))}\left(\begin{pmatrix}(1-z_1)H_1(C^{-1}(z))\\ 2(1-z_1)\tilde{H}(C^{-1}(z))\end{pmatrix} + 2iz\right).$$

By taking the sequence $z_n = (1 - 1/n, 0)$, it is easy to see that

$$\lim_{n\to\infty} \tilde{f}(z_n) = e_1 \quad \text{and} \quad \lim_{n\to\infty} \frac{1 - \|\tilde{f}(z_n)\|}{1 - \|z_n\|} = 1,$$

i.e., e_1 is a boundary regular fixed point of $\tilde{f}$ with boundary dilatation coefficient ≤ 1. Julia's lemma (see [Abate 1989, Theorem 2.2.21]) implies that $\tilde{f}(E_{\mathbb{B}_n}(e_1, R)) \subset E_{\mathbb{B}_n}(e_1, R)$ for any $R > 0$.

Inequality (d) follows directly from (c): Let $z = (z_1, \tilde{z}) \in \mathbb{H}_n$. Another formulation of (c) is $-u_{\mathbb{H}_n}(z + H(z)) \geq -u_{\mathbb{H}_n}(z)$, or more explicitly

$$\begin{aligned}\operatorname{Im}(z_1) + \operatorname{Im}(H_1(z)) - \|\tilde{z} + \tilde{H}(z)\|^2 &\geq \operatorname{Im}(z_1) - \|\tilde{z}\|^2\\ \iff \operatorname{Im}(H_1(z)) &\geq \|\tilde{z} + \tilde{H}(z)\|^2 - \|\tilde{z}\|^2 = 2\operatorname{Re}(\bar{\tilde{z}}^T \tilde{H}(z)) + \|\tilde{H}(z)\|^2\\ \iff \operatorname{Im}(H_1(z) - 2i\bar{\tilde{z}}^T \tilde{H}(z)) &\geq \|\tilde{H}(z)\|^2.\end{aligned}$$

From this inequality it follows that $\|\tilde{H}(z)\|^2 \leq |H_1(z) - 2i\bar{\tilde{z}}^T \tilde{H}|$ for all $z \in \mathbb{H}_n$.

Finally we prove (e):

Let $f \in \mathfrak{P}_n$ and write $f(z) = z + H(z)$ with $H \in \mathcal{K}(\mathbb{H}_n, c)$. Because of (c), f maps the horosphere $E_{\mathbb{H}_n}(\infty, 1)$ into itself. Hence the statement is proven if we can show that $u_{\mathbb{H}_n}$ is bounded on $f(\partial E_{\mathbb{H}_n}(\infty, 1))$.

Let $z \in \mathbb{H}_n$ with $z \in \partial E_{\mathbb{H}_n}(\infty, 1)$, i.e., $|u_{\mathbb{H}_n}(z)| = 1$. Furthermore, we choose $\zeta \in \mathbb{H}$ and $\gamma \in \mathbb{C}$ such that $\varphi_\gamma(\zeta) = z$. Note that this implies $|u_{\mathbb{H}}(\zeta)| = \operatorname{Im}(\zeta) = 1$.

Let P be the projection onto $\varphi_\gamma(\mathbb{H})$.

Then we have

$$|u_{\mathbb{H}_n}(f(z))| = |u_{\mathbb{H}_n}(z+H(z))| = \big|u_{\mathbb{H}_n}(\underbrace{z+dP(z)H(z)}_{=:\,w} + \underbrace{H(z)-dP(z)H(z)}_{=:\,v})\big|.$$

As $dP(z)\cdot dP(z) = dP(z)$, we have $dP(z)\cdot v = 0$. A small calculation (see also [Casavecchia 2010, Lemma 3.1]) gives $v \in T_z^{\mathbb{C}}\partial E_{\mathbb{H}_n}(\infty,1)$. Furthermore, also $w \in \varphi_\gamma(\mathbb{H})$ and $dP(z) = dP(w)$ and we get $v \in T_w^{\mathbb{C}}\partial E_{\mathbb{H}_n}(\infty, |u_{\mathbb{H}_n}(w)|^{-1})$. As $E_{\mathbb{H}_n}(\infty, |u_{\mathbb{H}_n}(w)|^{-1}) = \{z \in \mathbb{H}_n \mid |u_{\mathbb{H}_n}(z)| > |u_{\mathbb{H}_n}(w)|\}$ is convex, this implies

$$\begin{aligned}|u_{\mathbb{H}_n}(w+v)| \le |u_{\mathbb{H}_n}(w)| = |u_{\mathbb{H}_n}(z+dP(z)H(z))| &\underset{\text{Lemma 3.3}}{=} |u_{\mathbb{H}_n}(z+(h_\gamma(\zeta),0))| \\ &= \operatorname{Im}(z_1) - \|\tilde z\|^2 + \operatorname{Im}(h_\gamma(\zeta)) \le \operatorname{Im}(z_1) - \|\tilde z\|^2 + |h_\gamma(\zeta)| \\ &= |u_{\mathbb{H}_n}(z)| + |h_\gamma(\zeta)| = 1 + |h_\gamma(\zeta)| \le 1 + \frac{c}{\operatorname{Im}(\zeta)} = 1+c.\end{aligned}$$

Consequently, $f(\mathbb{H}_n) \supset f(E_{\mathbb{H}_n}(\infty,1)) \supset E_{\mathbb{H}_n}(\infty, 1+c)$. □

Theorem 4.4. *$\mathfrak{P}_n$ is a semigroup: if $f, g \in \mathfrak{P}_n$, then $f \circ g \in \mathfrak{P}_n$.*

Proof. Let $f, g \in \mathfrak{P}_n$ with $F = (F_1, \widetilde F) := f - \mathrm{id}$, $G = (G_1, \widetilde G) := g - \mathrm{id}$ and

$$\|F(z)\|_{\mathbb{H}_n,z} \le \frac{c}{u_{\mathbb{H}_n}(z)^2}, \quad \|G(z)\|_{\mathbb{H}_n,z} \le \frac{d}{u_{\mathbb{H}_n}(z)^2}$$

for all $z \in \mathbb{H}_n$. Let $z = (z_1, \tilde z) \in \mathbb{H}_n$ and $p = (p_1, \tilde p) := z + G(z)$.

From Remark 4.2, we know that $f \circ g - \mathrm{id}$ is an infinitesimal generator on $\mathbb{H}_n$. It remains to estimate the hyperbolic metric of this generator. We have

$$\begin{aligned}\|(f\circ g)(z) - z\|_{\mathbb{H}_n,z} &= \|G(z) + F(z+G(z))\|_{\mathbb{H}_n,z} \\ &\le \|G(z)\|_{\mathbb{H}_n,z} + \|F(z+G(z))\|_{\mathbb{H}_n,z} \le \frac{d}{u_{\mathbb{H}_n}(z)^2} + \|F(p)\|_{\mathbb{H}_n,z} \\ &\le \frac{d}{u_{\mathbb{H}_n}(z)^2} + \|(F_1(p) - 2i\,\bar{\tilde p}^T \widetilde F(p), 0)\|_{\mathbb{H}_n,z} + \|(2i\,\bar{\tilde p}^T \widetilde F(p), \widetilde F(p))\|_{\mathbb{H}_n,z}.\end{aligned}$$

Note that $F_1(p) - 2i\,\bar{\tilde p}^T \widetilde F(p)$ corresponds to the slice of F with respect to the geodesic through p and infinity. Because of Proposition 3.4, we know that

$$|F_1(p) - 2i\,\bar{\tilde p}^T \widetilde F(p)| \le \frac{c}{|u_{\mathbb{H}_n}(p)|} \le \frac{c}{|u_{\mathbb{H}_n}(z)|},$$

where the second inequality follows from Proposition 4.3 (c). Together with (3-4), this implies

$$\text{(4-2)} \qquad \|(F_1(p) - 2i\,\bar{\tilde p}^T \widetilde F(p), 0)\|_{\mathbb{H}_n,z} = \frac{|(F_1(p) - 2i\,\bar{\tilde p}^T \widetilde F(p)|}{|u_{\mathbb{H}_n}(z)|} \le \frac{c}{u_{\mathbb{H}_n}(z)^2}.$$

It remains to show that there exists a constant $C > 0$ such that

$$\|(2i\,\bar{\tilde{p}}^T\,\widetilde{F}(p),\,\widetilde{F}(p))\|_{\mathbb{H}_n,z} \leq \frac{C}{u_{\mathbb{H}_n}(z)^2}.$$

First, (3-5) gives
(4-3)

$$\begin{aligned}\|(2i\,\bar{\tilde{p}}^T\,\widetilde{F}(p),\,\widetilde{F}(p))\|_{\mathbb{H}_n,z} &= 2\frac{\sqrt{\|\widetilde{F}(p)\|^2\,|u_{\mathbb{H}_n}(z)| + |\overline{(\tilde{p}-\tilde{z})}^T\,\widetilde{F}(p)|^2}}{|u_{\mathbb{H}_n}(z)|}\\ &\leq 2\frac{\sqrt{\|\widetilde{F}(p)\|^2\,|u_{\mathbb{H}_n}(z)| + \|(\tilde{p}-\tilde{z})\|^2\cdot\|\widetilde{F}(p)\|^2}}{|u_{\mathbb{H}_n}(z)|}\\ &= 2\frac{\|\widetilde{F}(p)\|}{|u_{\mathbb{H}_n}(z)|}\sqrt{|u_{\mathbb{H}_n}(z)| + \|\widetilde{G}(z)\|^2}.\end{aligned}$$

Now we differentiate between two cases.

Case 1: $|u_{\mathbb{H}_n}(z)| \geq 1$. The equations (3-8) and (3-10) imply

$$2\frac{\|\widetilde{F}(p)\|}{\sqrt{|u_{\mathbb{H}_n(p)}|}} \leq \|\widetilde{F}(p)\|_{\mathbb{H}_n,p} \leq \frac{c}{u_{\mathbb{H}_n}(p)^2};$$

thus

$$\|\widetilde{F}(p)\| \leq \frac{c}{2|u_{\mathbb{H}_n}(p)|^{3/2}} \leq \frac{c}{2|u_{\mathbb{H}_n}(z)|^{3/2}}. \tag{4-4}$$

In the same way, we get

$$\|\widetilde{G}(z)\| \leq \frac{d}{2|u_{\mathbb{H}_n}(z)|^{3/2}}. \tag{4-5}$$

Combining (4-4) with (4-3) gives

$$\begin{aligned}\|(2i\,\bar{\tilde{p}}^T\,\widetilde{F}(p),\,\widetilde{F}(p))\|_{\mathbb{H}_n,z} &\leq \frac{c}{|u_{\mathbb{H}_n}(z)|\,|u_{\mathbb{H}_n}(z)|^{3/2}}\sqrt{|u_{\mathbb{H}_n}(z)| + \|\widetilde{G}(z)\|^2}\\ &= \frac{c}{|u_{\mathbb{H}_n}(z)|^2}\sqrt{1 + \frac{\|\widetilde{G}(z)\|^2}{|u_{\mathbb{H}_n}(z)|}}\\ &\underset{(4\text{-}5)}{\leq} \frac{c}{|u_{\mathbb{H}_n}(z)|^2}\sqrt{1 + \frac{d^2}{4|u_{\mathbb{H}_n}(z)|^4}}\\ &\leq \frac{c\sqrt{1+d^2/4}}{|u_{\mathbb{H}_n}(z)|^2}.\end{aligned}$$

Case 2: $|u_{\mathbb{H}_n}(z)| \le 1$. From (4-2) we know that $|F_1(p) - 2i\,\bar{\tilde{p}}^T\,\widetilde{F}(p)| \le c/|u_{\mathbb{H}_n}(z)|$, and (4-1) implies

$$\|\widetilde{F}(p)\| \le \frac{\sqrt{c}}{\sqrt{|u_{\mathbb{H}_n}(z)|}}.$$

Similarly we get

$$\|\widetilde{G}(z)\| \le \frac{\sqrt{d}}{\sqrt{|u_{\mathbb{H}_n}(z)|}}.$$

Hence, with (4-3) we obtain

$$\begin{aligned}\|(2i\,\bar{\tilde{p}}^T\,\widetilde{F}(p), \widetilde{F}(p))\|_{\mathbb{H}_n,z} &\le 2\frac{\sqrt{c}}{|u_{\mathbb{H}_n}(z)|^{3/2}}\sqrt{|u_{\mathbb{H}_n}(z)| + \|\widetilde{G}(z)\|^2}\\ &\le 2\frac{\sqrt{c}}{|u_{\mathbb{H}_n}(z)|^{3/2}}\sqrt{|u_{\mathbb{H}_n}(z)| + \frac{d}{|u_{\mathbb{H}_n}(z)|}}\\ &= 2\frac{\sqrt{c}}{|u_{\mathbb{H}_n}(z)|^{2}}\sqrt{u_{\mathbb{H}_n}(z)^2 + d}\\ &\le 2\frac{\sqrt{c}}{|u_{\mathbb{H}_n}(z)|^{2}}\sqrt{1+d}.\end{aligned}$$ □

On the Loewner equation with a $\mathcal{K}(\mathbb{H}_n, c)$-Herglotz vector field. Let $\{\Phi_t\}_{t\ge 0}$ be a semigroup on $\mathbb{H}_n$ with generator $H \in \mathcal{K}(\mathbb{H}_n, c)$. Next we will show that this implies $\Phi_t \in \mathfrak{P}_n$ for every $t \ge 0$.

In fact we can prove a little more by considering a nonautonomous version of (1-1). To this end, let $\{H_t : \mathbb{H}_n \to \mathbb{C}^n\}_{t\ge 0}$ be a $\mathcal{K}(\mathbb{H}_n, c)$-Herglotz vector field, i.e., $H_t \in \mathcal{K}(\mathbb{H}_n, c)$ for almost every $t \ge 0$ and the map $t \mapsto H_t(z)$ is measurable for every $z \in \mathbb{H}_n$; see [Arosio and Bracci 2011, Definition 1.2]. In this case, one can solve the nonautonomous version of (1-1), namely the Loewner equation

$$\frac{\partial \varphi_t(z)}{\partial t} = H_t(\varphi_t(z)), \quad \varphi_0(z) = z \in \mathbb{H}_n, \tag{4-6}$$

which gives a family $\{\varphi_t\}_{t\ge 0}$ of univalent self-mappings of $\mathbb{H}_n$; see [Arosio and Bracci 2011, Theorem 1.4].

Theorem 4.5. *If $\{H_t\}_{t\ge 0}$ is a $\mathcal{K}(\mathbb{H}_n, c)$-Herglotz vector field and $\{\varphi_t\}_{t\ge 0}$ the solution to* (4-6), *then $\varphi_t \in \mathfrak{P}_n$ for every $t \ge 0$.*

Proof. Firstly, for every $t \ge 0$ and $R > 0$, the map φ_t maps the horosphere $E_{\mathbb{H}_n}(\infty, R)$ into itself, i.e.,

$$|u_{\mathbb{H}_n}(\varphi_t(z))| \ge |u_{\mathbb{H}_n}(z)| \tag{4-7}$$

for every $z \in \mathbb{H}_n$. This can be seen as follows:

First, consider the autonomous case $H_t(z) = J(z)$ for every $t \geq 0$ and some $J \in \mathcal{K}(\mathbb{H}_n, c)$. Let G be the corresponding generator in the unit ball, i.e., $G = C'(C^{-1}) \cdot (J \circ C^{-1})$. Then G satisfies the inequality

$$\|G(z)\| \leq \|G(z)\|_{\mathbb{B}_n,z} \leq \frac{c}{u_{\mathbb{B}_n}(z)^2} = \frac{c|1-z_1|^4}{(1-\|z\|^2)^2}.$$

Putting $z = r \cdot e_1$ gives

$$\|G(re_1)\| \leq \frac{c(1-r)^4}{(1-r^2)^2} = \frac{c(1-r)^2}{(1+r)^2}.$$

From this it follows immediately that

$$\lim_{(0,1)\ni r\to 1} G(re_1) = 0 \quad \text{and} \quad \lim_{(0,1)\ni r\to 1} \frac{G_1(re_1)}{r-1} = 0.$$

Theorem 0.3 in [Bracci et al. 2010] implies that e_1 is a boundary regular fixed point for the generated semigroup with boundary dilatation coefficient 1. Hence we can apply Julia's lemma and obtain (4-7).

Now assume that $H_t(z)$ is piecewise constant with respect to time. By using the previous case, we see that (4-7) also holds in this case.

Finally, for a general $\mathcal{K}(\mathbb{H}_n, c)$-Herglotz vector field $H_t(z)$, we can approximate the solution φ_t by a sequence $\varphi_{t,n}$ such that for each n, the family $\{\varphi_{t,n}\}_{t\geq 0}$ solves (4-6) with a piecewise constant $\mathcal{K}(\mathbb{H}_n, c)$-Herglotz vector field. By using the continuity of $u_{\mathbb{H}_n}(z)$, we see that (4-7) also holds for φ_t.

Let $z = (z_1, z_2) \in \mathbb{H}_n$ and write $\varphi_t = (\varphi_{1,t}, \tilde{\varphi}_t)$, $H_t = (H_{1,t}, \widetilde{H}_t)$. The mapping φ_t satisfies the integral equation

$$\varphi_t(z) = z + \int_0^t H_s(\varphi_s(z))\, ds.$$

Similarly to the proof of Theorem 4.4, (4-4), we deduce from the fact that $H_t \in \mathcal{K}(\mathbb{H}_n, c)$ for almost every $t \geq 0$ and equations (3-8) and (3-10) that

$$\|\widetilde{H}_t(\varphi_t(z))\| \leq \frac{c}{2|u_{\mathbb{H}_n}(z)|^{3/2}} \tag{4-8}$$

for every $z \in \mathbb{H}_n$ and almost every $t \geq 0$, and similarly to (4-2), we deduce that

$$\left\|(H_{1,t}(\varphi_t(z)) - 2i\bar{\tilde{\varphi}}_t^{\,T}\, \widetilde{H}_t(\varphi_t(z)), 0)\right\|_{\mathbb{H}_n,z} \leq \frac{c}{u_{\mathbb{H}_n}(z)^2} \tag{4-9}$$

for every $z \in \mathbb{H}_n$ and almost every $t \geq 0$.

First we get

$$\|\tilde{\varphi}_s - \tilde{z}\| \leq \int_0^s \|\widetilde{H}_\tau(\varphi_\tau(z))\|\, d\tau \leq \int_0^s \frac{c}{2|u_{\mathbb{H}_n}(z)|^{3/2}}\, d\tau = \frac{cs}{2|u_{\mathbb{H}_n}(z)|^{3/2}}. \tag{4-10}$$

Suppose $|u_{\mathbb{H}_n}(z)| \geq 1$. Then we have

$$\begin{aligned}
\|\varphi_t(z)-z\|_{\mathbb{H}_n,z} &\leq \int_0^t \|H_s(\varphi_s(z))\|_{\mathbb{H}_n,z}\,ds \\
&\leq \int_0^t \left\| \begin{pmatrix} H_{1,s}(\varphi_s(z)) - 2i\bar{\tilde{\varphi}}_s^{\,T}\,\widetilde{H}_s(\varphi_s(z)) \\ 0 \end{pmatrix} \right\|_{\mathbb{H}_n,z} ds \\
&\qquad + \int_0^t \left\| \begin{pmatrix} 2i\bar{\tilde{\varphi}}_s^{\,T}\,\widetilde{H}_s(\varphi_s(z)) \\ \widetilde{H}_s(\varphi_s(z)) \end{pmatrix} \right\|_{\mathbb{H}_n,z} ds \\
&\underset{(4\text{-}9),(3\text{-}5)}{\leq} \int_0^t \frac{c}{u_{\mathbb{H}_n}(z)^2}\,ds + \int_0^t 2\frac{\|\widetilde{H}_s(\varphi_s(z))\|}{|u_{\mathbb{H}_n}(z)|}\sqrt{|u_{\mathbb{H}_n}(z)| + \|\tilde{\varphi}_s - \tilde{z}\|^2}\,ds \\
&\underset{(4\text{-}8),(4\text{-}10)}{\leq} \int_0^t \frac{c}{u_{\mathbb{H}_n}(z)^2}\,ds + \int_0^t \frac{c}{|u_{\mathbb{H}_n}(z)|^{5/2}}\sqrt{|u_{\mathbb{H}_n}(z)| + \frac{c^2s^2}{4|u_{\mathbb{H}_n}(z)|^3}}\,ds \\
&= \frac{ct}{u_{\mathbb{H}_n}(z)^2} + \int_0^t \frac{c}{|u_{\mathbb{H}_n}(z)|^2}\sqrt{1 + \frac{c^2s^2}{4|u_{\mathbb{H}_n}(z)|^4}}\,ds \\
&\leq \frac{ct}{u_{\mathbb{H}_n}(z)^2} + \int_0^t \frac{c}{|u_{\mathbb{H}_n}(z)|^2}\sqrt{1+c^2s^2}\,ds \\
&= c\cdot\frac{t + \int_0^t \sqrt{1+c^2s^2}\,ds}{u_{\mathbb{H}_n}(z)^2}.
\end{aligned}$$

The case $|u_{\mathbb{H}_n}(z)| \leq 1$ is treated similarly, compare with the proof of Theorem 4.4, and we conclude that for every $t \geq 0$, there exists $C > 0$ such that $\|\varphi_t(z) - z\|_{\mathbb{H}_n} \leq C/u_{\mathbb{H}_n}(z)^2$ for all $z \in \mathbb{H}_n$. Together with Remark 4.2, this implies that $\varphi_t \in \mathfrak{P}_n$. ☐

Question 4.6. Let $f \in \mathfrak{P}_1$. In [Goryaĭnov and Ba 1992, Section 4], it is shown that there exists a $\mathcal{K}(\mathbb{H}, c)$-Herglotz vector field H_t and a time $T \geq 0$ such that $f = \varphi_T$, where $\{\varphi_t\}_{t\geq 0}$ is the solution of (4-6). What can be said in the higher-dimensional case?

On the behavior of iterates. Let $F : \mathbb{B}_n \to \mathbb{B}_n$ be holomorphic. We say that $p \in \overline{\mathbb{B}}_n$ is the Denjoy–Wolff point of F if $F^n \to p$ for $n \to \infty$ locally uniformly. The basic results about the behavior of the iterates F^n for $n \to \infty$ can be found in [Abate 1989, Chapter 2.2]. In particular we have (Theorem 2.2.31)

(4-11)
$$F \text{ has a Denjoy–Wolff point on the boundary } \partial\mathbb{B}_n \iff F \text{ has no fixed points.}$$

Now let $f \in \mathfrak{P}_n$. For $n = 1$, f has the Denjoy–Wolff point ∞ if f is not the identity: As f is not an elliptic automorphism, the classical Denjoy–Wolff theorem implies that f has a Denjoy–Wolff point. This point has to be ∞, e.g., because of Proposition 4.3 (c).

Next we will show that this is also true in higher dimensions, provided that f extends smoothly to the boundary point ∞. There are different possible definitions of smoothness of f near ∞. We will use the following one: Let $H(z) = f(z) - z$, and denote by $G : \mathbb{B}_n \to \mathbb{C}^n$ the corresponding generator on $\mathbb{B}_n$; i.e., we have

$$H(z) = (C^{-1})'(C(z)) \cdot G(C(z))$$

and a small computation shows

$$H_1(z) = -\frac{i}{2}(z_1 + i)^2 \cdot G_1(C(z)).$$

Our smoothness condition will be that G_1 has a C^3-extension to e_1; i.e., we can write

$$G_1(z) = \sum_{\substack{k_1+\cdots+k_n \le 3\\ k_1,\ldots,k_n \ge 0}} a_{k_1,\ldots,k_n}(z_1 - 1)^{k_1} \cdot z_2^{k_2} \cdot \cdots \cdot z_n^{k_n} + o(\|z - e_1\|^3),$$

which translates to

$$H_1(z) = -\frac{i}{2}(z_1 + i)^2 \cdot \sum_{k_1+\cdots+k_n \le 3} a_{k_1,\ldots,k_n}\left(\frac{-2i}{z_1 + i}\right)^{k_1} \cdot \left(\frac{2z_2}{z_1 + i}\right)^{k_2} \cdot \cdots \cdot \left(\frac{2z_n}{z_1 + i}\right)^{k_n} + o(\|C(z) - e_1\|^3),$$

or

(4-12)
$$\begin{aligned} H_1(z) = b_{0,\ldots,0}\cdot(z_1+i)^2 + (z_1+i)\cdot \sum_{k_1+\cdots+k_n=1} b_{k_1,\ldots,k_n} z_2^{k_2}\cdots z_n^{k_n} \\ + \sum_{k_1+\cdots+k_n=2} b_{k_1,\ldots,k_n} z_2^{k_2}\cdots z_n^{k_n} + (z_1+i)^{-1}\cdot \sum_{k_1+\cdots+k_n=3} b_{k_1,\ldots,k_n} z_2^{k_2}\cdots z_n^{k_n} \\ + o\big(|z_1+i|^{-1}\cdot\|(1, z_2, \ldots, z_n)\|^3\big) \end{aligned}$$

for some coefficients $b_{k_1,\ldots,k_n} \in \mathbb{C}$.

Theorem 4.7. *Let $f \in \mathfrak{P}_n$, $f \neq \mathrm{id}$, and assume that* (4-12) *is satisfied. Then ∞ is the Denjoy–Wolff point of f.*

Proof. Write $f(z) = z + H(z)$, where $H \in \mathcal{K}(\mathbb{H}_n, c)$ and $H = (H_1, \widetilde{H})$. Let $\gamma \in \mathbb{C}^{n-1}$. If we can show that the slice $h_\gamma(\zeta) = H_1(\varphi(\zeta)) - 2i\bar{\gamma}^T \widetilde{H}(\varphi_\gamma(\zeta))$ has no zeros, then we are done:

This implies that H has no zeros because of (3-7) and (3-8). Hence, f has no fixed points and (4-11) implies that f has a Denjoy–Wolff point. This point has to be ∞ because of Proposition 4.3 (c).

Similarly to the proof of Theorem 4.4, (4-4), we have

$$\|\widetilde{H}(z)\| \le \frac{c}{2|u_{\mathbb{H}_n}(z)|^{3/2}},$$

and thus

$$\|\widetilde{H}(\varphi_\gamma(\zeta))\| \le \frac{c}{2|u_{\mathbb{H}_n}(\varphi_\gamma(\zeta))|^{3/2}} = \frac{c}{2\operatorname{Im}(\zeta)^{3/2}}.$$

Consequently,

$$\lim_{y\to\infty} y|\bar{\gamma}^T \widetilde{H}(\varphi_\gamma(iy))| = 0.$$

On the other hand, we know from Proposition 3.4 that $h_\gamma \in \mathcal{K}(\mathbb{H}, c)$, which implies (see Remark 1.5)

$$\limsup_{y\to\infty} y|h_\gamma(iy)| = \limsup_{y\to\infty} y\big|H_1(\varphi(iy)) - 2i\bar{\gamma}^T \widetilde{H}(\varphi_\gamma(iy))\big| \le c,$$

which gives us

$$\limsup_{y\to\infty} |iy \cdot H_1(\varphi_\gamma(iy))| \le c. \tag{4-13}$$

Now we use the assumption of the smoothness of H_1:

Because of (4-13), all coefficients $b_{k_1,\dots,k_n}$ from (4-12) with $k_1 + \dots + k_n \le 2$ have to be 0. Thus,

$$\lim_{y\to\infty} iy \cdot H_1(\varphi_\gamma(iy)) =: K(\gamma)$$

exists and is a polynomial in $\gamma = (\gamma_2, \dots, \gamma_n)$:

$$K(\gamma) = \sum_{k_1+\dots+k_n=3} b_{k_1,\dots,k_n} \gamma_2^{k_2} \cdot \dots \cdot \gamma_n^{k_n}.$$

As $K(\gamma)$ is bounded, it has to be constant.

If $K(\gamma) \equiv 0$, then all slices of H are zero; hence $H = 0$ by Remark 3.5 and f is the identity, a contradiction.

Hence $K(\gamma)$ is a nonzero constant and $h_\gamma(\zeta)$ is not identically zero, which implies (e.g., by using the representation (1-5)) that $h_\gamma(\zeta)$ has no zeros. □

Question 4.8. Is ∞ the Denjoy–Wolff point for every $f \in \mathfrak{P}_n$?

Appendix: Proof of Lemma 3.2

Lemma 3.2. *Let $a \in \mathbb{C}$, $p, v \in \mathbb{C}^{n-1}$ and $z = (z_1, \tilde{z}) \in \mathbb{H}_n$. Then the following formulas hold*:

$$\left\| \begin{pmatrix} a \\ 0 \end{pmatrix} \right\|_{\mathbb{H}_n, z} = \frac{|a|}{|u_{\mathbb{H}_n}(z)|}, \tag{3-4}$$

$$\left\| \begin{pmatrix} 2i\,\bar{p}^T v \\ v \end{pmatrix} \right\|_{\mathbb{H}_n, z} = 2\frac{\sqrt{\|v\|^2\,|u_{\mathbb{H}_n}(z)| + |\overline{(p-\tilde{z})}^T v|^2}}{|u_{\mathbb{H}_n}(z)|}, \tag{3-5}$$

$$\left\| \begin{pmatrix} a - 2i\bar{\tilde{z}}^T v \\ 0 \end{pmatrix} + \begin{pmatrix} 2i\bar{\tilde{z}}^T v \\ v \end{pmatrix} \right\|^2_{\mathbb{H}_n, z} = \left\| \begin{pmatrix} a - 2i\bar{\tilde{z}}^T v \\ 0 \end{pmatrix} \right\|^2_{\mathbb{H}_n, z} + \left\| \begin{pmatrix} 2i\bar{\tilde{z}}^T v \\ v \end{pmatrix} \right\|^2_{\mathbb{H}_n, z}. \tag{3-6}$$

Proof. We write $\tilde{z} = (z_2, \dots, z_n)$, $v = (v_2, \dots, v_n)$, $p = (p_2, \dots, p_n)$.

An explicit formula of the Kobayashi metric for the unit ball is given in [Abate 2004, Theorem 3.4].[4] It coincides with the Bergman metric and by using the Cayley map, we get the following formula for the upper half-space:

$$\|w\|^2_{\mathbb{H}_n, z} = w^T \cdot (g_{j,k})_{j,k} \cdot \bar{w},$$

where $w \in \mathbb{C}^n$ and $(g_{j,k})_{j,k}$ is an $n \times n$-matrix with

$$g_{j,k} = -4\frac{\partial^2}{\partial z_j\,\partial \bar{z}_k} \log\left(\operatorname{Im}(z_1) - \sum_{l=2}^{n} |z_l|^2\right),$$

and we get for $j, k \geq 2$,

$$g_{1,1} = \frac{1}{u_{\mathbb{H}_n}(z)^2}, \qquad g_{1,k} = \frac{2i\,z_k}{u_{\mathbb{H}_n}(z)^2}, \qquad g_{j,1} = \frac{-2i\,\bar{z}_j}{u_{\mathbb{H}_n}(z)^2},$$

$$g_{j,j} = 4\frac{\operatorname{Im}(z_1) - \sum_{l=2, l\neq j}^{n} |z_l|^2}{u_{\mathbb{H}_n}(z)^2}, \qquad g_{j,k} = \frac{4 z_k \bar{z}_j}{u_{\mathbb{H}_n}(z)^2}, \quad k \neq j.$$

The formulas (3-4) and (3-5) are now straightforward calculations. We obtain

$$\|(a,0)\|_{\mathbb{H}_n, z} = \sqrt{(a,0) \cdot (g_{j,k})_{j,k} \cdot \overline{(a,0)^T}} = \sqrt{a \cdot g_{1,1} \cdot \bar{a}} = \frac{|a|}{|u_{\mathbb{H}_n}(z)|},$$

[4] Note, however, that the Kobayashi metric in [Abate 2004] differs by a factor of 2 from the one we are using here.

and

$$\begin{aligned}
&u_{\mathbb{H}_n}(z)^2\cdot\|(2i\,\bar{p}^T v, v)\|^2_{\mathbb{H}_n,z}\\
&\quad= u_{\mathbb{H}_n}(z)^2\cdot(2i\,\bar{p}^T v, v^T)\cdot(g_{j,k})_{j,k}\cdot\overline{(2i\,\bar{p}^T v, v^T)}^T\\
&\quad= u_{\mathbb{H}_n}(z)^2\cdot\Bigg(\sum_{j=2}^n g_{j,j}|v_j|^2+g_{1,1}|2i\,\bar{p}^T v|^2\\
&\qquad\qquad+\sum_{j=2}^n g_{j,1}v_j\overline{2i\,\bar{p}^T v}+\sum_{k=2}^n g_{1,k}\bar{v}_j 2i\,\bar{p}^T v+\sum_{j,k\ge 2, j\neq k}^n g_{j,k}v_j\bar{v}_k\Bigg)\\
&\quad= 4\sum_{j=2}^n(\mathrm{Im}(z_1)-\|\tilde{z}\|^2)\cdot|v_j|^2+4\sum_{j=2}^n|z_j|^2\cdot|v_j|^2+4\sum_{j,k\ge 2}^n p_j\bar{p}_k v_j\bar{v}_k\\
&\qquad\qquad-4\sum_{j,k\ge 2}^n\bar{z}_j p_k v_j\bar{v}_k-4\sum_{j,k\ge 2}^n z_j\bar{p}_k\bar{v}_j v_k+4\sum_{j,k\ge 2, j\neq k}^n\bar{z}_j z_k v_j\bar{v}_k\\
&\quad= 4\|v\|^2\cdot|u_{\mathbb{H}_n}(z)|+4\sum_{j=2}^n z_j\bar{z}_j v_j\bar{z}_j\\
&\qquad\qquad+4\sum_{j,k\ge 2}^n(p_j\bar{p}_k v_j\bar{v}_k-\bar{z}_j p_k v_j\bar{v}_k-z_j\bar{p}_k\bar{v}_j v_k)+4\sum_{j,k\ge 2, j\neq k}^n\bar{z}_j z_k v_j\bar{v}_k\\
&\quad= 4\|v\|^2\cdot|u_{\mathbb{H}_n}(z)|+4\sum_{j,k\ge 2}^n(p_j\bar{p}_k v_j\bar{v}_k-\bar{z}_j p_k v_j\bar{v}_k-z_j\bar{p}_k\bar{v}_j v_k+\bar{z}_j z_k v_j\bar{v}_k)\\
&\quad= 4\|v\|^2\cdot|u_{\mathbb{H}_n}(z)|+4|\overline{(p-\tilde{z})}^T v|^2.
\end{aligned}$$

For formula (3-6) we just need to show that

$$(2i\bar{\tilde{z}}^T v, v^T)\cdot(g_{j,k})_{j,k}\cdot\overline{(a-2i\bar{\tilde{z}}^T v, 0)}^T = 0.$$

Indeed, we have

$$\begin{aligned}
&u_{\mathbb{H}_n}(z)^2\cdot(g_{j,k})_{j,k}\cdot\overline{(a-2i\bar{\tilde{z}}^T v, 0)}^T\\
&\qquad\qquad= (\bar{a}+2i\tilde{z}^T\bar{v}, -2i\bar{z}_2\bar{a}+4\bar{z}_2\tilde{z}^T\bar{v},\dots,-2i\bar{z}_n\bar{a}+4\bar{z}_n\tilde{z}^T\bar{v})^T
\end{aligned}$$

and

$$\begin{aligned}
&(2i\bar{\tilde{z}}^T v, v^T)(\bar{a}+2i\tilde{z}^T\bar{v}, -2i\bar{z}_2\bar{a}+4\bar{z}_2\tilde{z}^T\bar{v},\dots,-2i\bar{z}_n\bar{a}+4\bar{z}_n\tilde{z}^T\bar{v})^T\\
&\qquad\qquad= 2i\bar{a}\bar{\tilde{z}}^T v-4|\tilde{z}^T\bar{v}|^2-2i\bar{a}\bar{\tilde{z}}^T v+4|\tilde{z}^T\bar{v}|^2 = 0. \quad\square
\end{aligned}$$

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