Pacific Journal of Mathematics

TOPOLOGICAL MOLINO'S THEORY

JESÚS A. ÁLVAREZ LÓPEZ AND MANUEL F. MOREIRA GALICIA

Volume 280 No. 2 February 2016

PACIFIC JOURNAL OF MATHEMATICS
Vol. 280, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.280.257

TOPOLOGICAL MOLINO'S THEORY

JESÚS A. ÁLVAREZ LÓPEZ AND MANUEL F. MOREIRA GALICIA

Molino's description of Riemannian foliations on compact manifolds is generalized to the setting of compact equicontinuous foliated spaces, in the case where the leaves are dense. In particular, a structural local group is associated to such a foliated space. As an application, we obtain a partial generalization of results by Carrière and Breuillard–Gelander, relating the structural local group to the growth of the leaves.

1. Introduction

Riemannian foliations were introduced by Reinhart [1959] by requiring isometric transverse dynamics. It was pointed out by Ghys in [Molino 1988, Appendix E] (see also [Kellum 1993]) that equicontinuous foliated spaces should be considered as the "topological Riemannian foliations," and therefore many of the results about Riemannian foliations should have versions for equicontinuous foliated spaces. Some steps in this direction were given by Álvarez and Candel [2009; 2010], showing that, under reasonable conditions, their leaf closures are minimal foliated spaces, and their generic leaves are quasi-isometric to each other, like in the case of Riemannian foliations. In the same direction, Matsumoto [2010] proved that any minimal equicontinuous foliated space has a nontrivial transverse invariant measure, which is unique up to scaling if the space is compact — observe that this unicity

Álvarez is partially supported by the MICINN grant MTM2011-25656. Moreira was partially supported by the grant ERASMUS MUNDUS LOTE 20B, for the period 2010–2012.
MSC2010: 57R30.
Keywords: equicontinuous foliated space, equicontinuous pseudogroup, groupoid, germ, partial map, compact-open topology, local group, local action, growth.

implies ergodicity. The magnitude of the generalization from Riemannian foliations to equicontinuous foliated spaces was made precise by Álvarez and Candel [2010] (see also [Tarquini 2004]), giving a topological description of Riemannian foliations within the class of equicontinuous foliated spaces.

Most of the known properties of Riemannian foliations follow from a description due to Molino [1982; 1988]. However, so far, there was no version of Molino's description for equicontinuous foliated spaces — the indicated properties of equicontinuous foliated spaces were obtained by other means. The goal of our work is to develop such a version of Molino's theory, and use it to study the growth of their leaves, following the study of the growth of Riemannian foliations by Carrière [1988] and Breuillard and Gelander [2007]. To understand our results better, let us briefly recall Molino's theory.

1A. ***Molino's theory for Riemannian foliations.*** The necessary basic concepts from foliation theory can be seen in [Hector and Hirsch 1981; 1987; Candel and Conlon 2000].

Let $\mathcal{F}$ be a (smooth) foliation of codimension q on a manifold M. Let $T\mathcal{F} \subset TM$ denote the vector subbundle of vectors tangent to the leaves, and $N\mathcal{F} = TM/T\mathcal{F}$ its normal bundle. Recall that there is a natural flat leafwise partial connection on $N\mathcal{F}$ such that any local normal vector field is leafwise parallel if and only if it is locally projectable by the distinguished submersions; terms like "leafwise flat," "leafwise parallel" and "leafwise horizontal" will refer to this partial connection. It is said that $\mathcal{F}$ is

- *Riemannian* if $N\mathcal{F}$ has a leafwise parallel Riemannian structure;
- *transitive* if the group of its foliated diffeomorphisms acts transitively on M;
- *transversely parallelizable* (*TP*) if there is a leafwise parallel global frame of $N\mathcal{F}$, called *transverse parallelism*; and a
- *Lie foliation* if moreover the transverse parallelism is a basis of a Lie algebra with the operation induced by the vector field bracket.

These conditions are successively stronger. Molino's theory describes Riemannian foliations on compact manifolds in terms of minimal Lie foliations, and using TP foliations as an intermediate step:

1st step: If $\mathcal{F}$ is Riemannian and M compact, then there is an $\mathrm{O}(q)$-principal bundle $\hat{\pi} : \widehat{M} \to M$, with an $\mathrm{O}(q)$-invariant TP foliation $\widehat{\mathcal{F}}$, such that $\hat{\pi}$ is a foliated map whose restrictions to the leaves are the holonomy covers of the leaves of $\mathcal{F}$.

2nd step: If $\mathcal{F}$ is TP and M compact, then there is a fiber bundle $\pi : M \to W$ whose fibers are the leaf closures of $\mathcal{F}$, and the restriction of $\mathcal{F}$ to each fiber is a Lie foliation.

Since the structure of Lie foliation is unique in the minimal case, we end up with a Lie algebra associated to $\mathcal{F}$, called the *structural Lie algebra*. The proofs of the above statements strongly use the differential structure of $\mathcal{F}$. In the first step, $\hat{\pi} : \widehat{M} \to M$ is the $\mathrm{O}(q)$-principal bundle of orthonormal frames for some leafwise parallel metric on $N\mathcal{F}$, and $\widehat{\mathcal{F}}$ is given by the corresponding flat leafwise horizontal distribution. Then $\widehat{\mathcal{F}}$ is TP by a standard argument. In the second step, foliated flows are used to produce the fiber bundle trivializations whose fibers are the leaf closures; this works because there are foliated flows in any transverse direction since $\mathcal{F}$ is TP.

When $\mathcal{F}$ is minimal (the leaves are dense), any leaf closure $\widehat{M}_0$ of $\widehat{\mathcal{F}}$ is a principal subbundle of $\hat{\pi} : \widehat{M} \to M$, obtaining the following:

Minimal case: If $\mathcal{F}$ is minimal and Riemannian and M is compact, then, for some closed subgroup $H \subset \mathrm{O}(q)$, there is an H-principal bundle $\hat{\pi}_0 : \widehat{M}_0 \to M$ with an H-invariant minimal Lie foliation $\widehat{\mathcal{F}}_0$, such that $\hat{\pi}_0$ is a foliated map whose restrictions to the leaves are the holonomy covers of the leaves of $\mathcal{F}$.

A useful description of Lie foliations was also given by Fedida [1971; 1978], but it will not be considered here.

The differential structure cannot be used in our generalization; instead, we use the holonomy pseudogroup. Thus let us briefly indicate the holonomy properties of Riemannian foliations that will play an important role.

1B. ***Holonomy of Riemannian foliations.*** Recall that a *pseudogroup* is a maximal collection of local transformations of a space, which contains the identity map, and is closed under the operations of composition, inversion, restriction and combination. It can be considered as a generalized dynamical system, and all basic dynamical concepts have pseudogroup versions. They are relevant in foliation theory because the holonomy pseudogroup of a foliation $\mathcal{F}$ describes the transverse dynamics of $\mathcal{F}$. Such a pseudogroup is well determined up to certain *equivalence* of pseudogroups introduced by Haefliger [1985; 1988]. We may say that $\mathcal{F}$ is *transversely modeled* by a class of local transformations of some space if its holonomy pseudogroup can be generated by that type of local transformations. Riemannian, TP and Lie foliations can be respectively characterized by being transversely modeled by local isometries of some Riemannian manifold, by local parallelism preserving diffeomorphisms of some parallelizable manifold, and by local left translations of a Lie group. In this sense, Riemannian foliations are the transversely rigid ones, and TP foliations have a stronger type of transverse rigidity.

When the ambient manifold M is compact, Haefliger [2002] has observed that the holonomy pseudogroup $\mathcal{H}$ of $\mathcal{F}$ satisfies a property called compact generation. If moreover $\mathcal{F}$ is Riemannian, then Haefliger [1988; 2002] has also strongly used the following properties of $\mathcal{H}$: completeness, quasianalyticity, and existence of a closure $\overline{\mathscr{H}}$, which is also complete and quasianalytic. Here, $\overline{\mathscr{H}}$ is defined by taking the closure of the set of 1-jets of maps in $\mathscr{H}$ in the space of 1-jets.

For a compactly generated pseudogroup $\mathscr{H}$ of local isometries of a Riemannian manifold T, Salem has given a version of Molino's theory [Salem 1988; Molino 1988, Appendix D] (see also [Álvarez and Masa 2008]). In particular, in the minimal case, it turns out that there is a Lie group G, a compact subgroup $K \subset G$ and a dense finitely generated subgroup $\Gamma \subset G$ such that $\mathscr{H}$ is equivalent to the pseudogroup generated by the action of Γ on the homogeneous space G/K (this was also observed by Haefliger [1988]).

1C. ***Growth of Riemannian foliations.*** Molino's theory has many consequences for a Riemannian foliation $\mathscr{F}$ on a compact manifold M: classification in particular cases, growth, cohomology, tautness, tenseness and global analysis. In all of them, Molino's theory is used to reduce the study to the case of Lie foliations with dense leaves, where it usually becomes a problem of Lie theory. We concentrate on the consequences about growth of the leaves and their holonomy covers. This study was begun by Carrière [1988], and recently continued by Breuillard and Gelander [2007], as a consequence of their study of a topological Tits alternative. Their results state the following, where $\mathfrak{g}$ is the structural Lie algebra of $\mathscr{F}$:

Carrière's theorem. *The holonomy covers of the leaves are Følner if and only if* $\mathfrak{g}$ *is solvable*, *and of polynomial growth if and only if* $\mathfrak{g}$ *is nilpotent. In the second case*, *the degree of polynomial growth is bounded by the nilpotence degree of* $\mathfrak{g}$.

Breuillard and Gelander's theorem. *The growth of the holonomy covers of the leaves is either polynomial or exponential.*

1D. ***Equicontinuous foliated spaces.*** A *foliated space* $X \equiv (X, \mathscr{F})$ is a topological space X equipped with a partition $\mathscr{F}$ into connected manifolds (*leaves*), which can be locally described as the fibers of topological submersions. It will be assumed that X is locally compact and Polish. A foliated space should be considered as a "topological foliation". In this sense, all topological notions of foliations have obvious versions for foliated spaces. In particular, the *holonomy pseudogroup* $\mathscr{H}$ of X is defined on a locally compact Polish space T. Many basic results about foliations also have straightforward generalizations; for example, $\mathscr{H}$ is compactly generated if X is compact. Even leafwise differential concepts are easy to extend. However this task may be difficult or impossible for transverse differential concepts. For instance, the normal bundle of a foliated space does not make any sense in general; it would be the tangent bundle of a topological space in the case of a space foliated by points. Thus the concept of Riemannian foliation cannot be extended by using the normal bundle. Instead, this can be done via the holonomy pseudogroup as follows.

The transverse rigidity of a Riemannian foliation can be translated to the foliated space X by requiring equicontinuity of $\mathcal{H}$. In fact, the equicontinuity condition is not compatible with combinations of maps, and therefore it is indeed required for some generating subset $S \subset \mathcal{H}$ which is closed by the operations of composition and inversion. Such an S is called a pseudo$*$group with the terminology of Matsumoto [2010]. This gives rise to the concept of equicontinuous foliated space.

In the topological setting, the quasianalyticity of $\mathcal{H}$ does not follow from the equicontinuity assumption. Thus it will be required as an additional assumption when needed. Indeed, it does not work well enough when T is not locally connected, so we use a property called strong quasianalyticity, which is stronger than quasianalyticity only when T is not locally connected.

Álvarez and Candel [2009] have proved that, if $\mathcal{H}$ is compactly generated, equicontinuous and strongly quasianalytic, then it is complete and has a closure $\overline{\mathcal{H}}$. Here, $\overline{\mathcal{H}}$ is the pseudogroup generated by the homeomorphisms on small enough open subsets O of T that are limits in the compact-open topology of maps in $\mathcal{H}$ defined on those sets O.

Transitive and Lie foliations have the following topological versions. It is said that the foliated space X is

- *homogeneous* if its group of foliated transformations acts transitively on X;
- a *G-foliated space* if it is transversely modeled by local left translations in some locally compact Polish local group G (if X is minimal).

1E. *Topological Molino's theory.* Our first main result is the following topological version of the minimal case in Molino's theory.

Theorem A. *Let $X \equiv (X, \mathcal{F})$ be a compact Polish foliated space, and $\mathcal{H}$ its holonomy pseudogroup. Suppose that X is minimal and equicontinuous, and $\overline{\mathcal{H}}$ is strongly quasianalytic. Then there is a compact Polish minimal foliated space $\widehat{X}_0 \equiv (\widehat{X}_0, \widehat{\mathcal{F}}_0)$, an open continuous foliated map $\hat{\pi}_0 : \widehat{X}_0 \to X$, and a locally compact Polish local group G such that $\widehat{X}_0$ is a G-foliated space, the fibers of $\hat{\pi}_0$ are homeomorphic to each other, and the restrictions of $\hat{\pi}_0$ to the leaves of $\widehat{\mathcal{F}}_0$ are the holonomy covers of the leaves of $\mathcal{F}$.*

The proof of Theorem A is different from Molino's proof in the Riemannian foliation case because there may not be the normal bundle of $\mathcal{F}$. To define $\widehat{X}_0$, we first construct what should be its holonomy pseudogroup, $\widehat{\mathcal{H}}_0$ on a space $\widehat{T}_0$. To some extent, this was achieved by Álvarez and Candel [2010], proving that, with the assumptions of Theorem A, there is a locally compact Polish local group G, a compact subgroup $K \subset G$ and a dense finitely generated sub-local group $\Gamma \subset G$ such that $\mathcal{H}$ is equivalent to the pseudogroup generated by the local action of Γ on G/K, like in the Riemannian foliation case. Hence $\widehat{\mathcal{H}}_0$ should be the pseudogroup generated by the local action of Γ on G. This may look like a big step towards the proof, but the realization of compactly generated pseudogroups as holonomy pseudogroups of compact foliated spaces is impossible in general, as shown by Meigniez [Meigniez 2010]. This difficulty is overcome as follows.

Take a "good" cover of X by distinguished open sets $\{U_i\}$, with corresponding distinguished submersions $p_i : U_i \to T_i$, and elementary holonomy transformations $h_{ij} : T_{ij} \to T_{ji}$, where $T_{ij} = p_i(U_i \cap U_j)$. Let $\mathcal{H}$ denote the corresponding representative of the holonomy pseudogroup on $T = \bigsqcup_i T_i$, generated by the maps h_{ij}. Then the construction of $\widehat{\mathcal{H}}_0$ must be associated to $\mathcal{H}$ in a natural way, so that it becomes induced by some "good" cover by distinguished open sets of a compact foliated space. In the Riemannian foliation case, the good choices of $\widehat{T}_0$ and $\widehat{\mathcal{H}}_0$ are the following:

- Let P be the bundle of orthonormal frames for any $\mathcal{H}$-invariant metric on T. Fix $x_0 \in T$ and $\hat{x}_0 \in P_{x_0}$. Then, as a subspace of P,

$$\widehat{T}_0 = \overline{\{h_*(\hat{x}_0) \mid h \in \mathcal{H}, x_0 \in \operatorname{dom} h\}} = \{g_*(\hat{x}_0) \mid g \in \overline{\mathcal{H}}, x_0 \in \operatorname{dom} g\}. \tag{1}$$

- $\widehat{\mathcal{H}}_0$ is generated by the differentials of the maps in $\mathcal{H}$.

These differential concepts can be modified in the following way. In (1), each $g_*(\hat{x}_0)$ determines the germ $\boldsymbol{\gamma}(g, x_0)$ of g at x_0, by the strong quasianalyticity of $\overline{\mathcal{H}}$. Therefore it also determines $\boldsymbol{\gamma}(f, x)$, where $f = g^{-1}$ and $x = g(x_0)$ — this little change, using $\boldsymbol{\gamma}(f, x)$ instead of $\boldsymbol{\gamma}(g, x_0)$, is not really necessary, but it helps to simplify the notation in some involved arguments. So

$$\widehat{T}_0 \equiv \{\boldsymbol{\gamma}(f, x) \mid f \in \overline{\mathcal{H}}, x \in \operatorname{dom} f, f(x) = x_0\}. \tag{2}$$

The projection $\hat{\pi}_0 : \widehat{T}_0 \to T$ corresponds via (2) to the source map $\boldsymbol{\gamma}(f, x) \mapsto x$. The differentials of maps $h \in \mathcal{H}$, acting on orthonormal references, correspond via (2) to the maps $\hat{h}$ defined by

$$\hat{h}(\boldsymbol{\gamma}(f, x)) = \boldsymbol{\gamma}(fh^{-1}, h(x)).$$

Let us describe the topology of $\widehat{T}_0$ using (2). Let $\overline{S}$ be a pseudo∗group generating $\overline{\mathcal{H}}$ and satisfying the equicontinuity and strong quasianalyticity conditions. Endow $\overline{S}$ with the compact-open topology on partial maps with open domains, as defined by Abd-Allah and Brown [1980], and consider the subspace

$$\overline{S} * T = \{(f, x) \in \overline{S} \mid x \in \operatorname{dom} f\} \subset \overline{S} \times T.$$

Then the topology of $\widehat{T}_0$ corresponds via (2) to the quotient topology by the germ map $\boldsymbol{\gamma} : \overline{S} * T \to \boldsymbol{\gamma}(\overline{S} * T) \equiv \widehat{T}_0$, which is of course different from the sheaf topology on germs. This point of view, replacing orthonormal frames by germs, can be readily translated to the foliated space setting, obtaining good choices of $\widehat{T}_0$ and $\widehat{\mathcal{H}}_0$ under the conditions of Theorem A.

Now, consider triples (x, γ, i) with $x \in U_i$, $\gamma \in \widehat{T}_{i,0} := \hat{\pi}_0^{-1}(T_i)$ and $p_i(x) = \hat{\pi}_0(\gamma)$. Declare (x, γ, i) equivalent to (y, δ, j) if $x = y$ and $\widehat{h_{ij}}(\gamma) = \delta$. Then $\widehat{X}_0$ is defined as the corresponding quotient space. Let $[x, \gamma, i]$ denote the equivalence class of each triple (x, γ, i). The foliated structure $\widehat{\mathcal{F}}_0$ on $\widehat{X}_0$ is determined by requiring that, for each fixed index i, the elements of the type $[x, \gamma, i]$ form a distinguished open set $\widehat{U}_{i,0}$, with distinguished submersion $\hat{p}_{i,0} : \widehat{U}_{i,0} \to \widehat{T}_{i,0}$ given by $\hat{p}_{i,0}([x, \gamma, i]) = \gamma$. The projection $\hat{\pi}_0 : \widehat{X}_0 \to X$ is defined by $\hat{\pi}_0([x, \gamma, i]) = x$. The properties stated in Theorem A are satisfied with these definitions.

It is also proved that, up to foliated homeomorphisms (respectively, local isomorphisms), $\widehat{X}_0$ (respectively, G) is independent of the choices involved. Hence G can be called the *structural local group* of $\mathcal{F}$.

1F. *Growth of equicontinuous foliated spaces.* Our second main result is the following weak topological version of the above theorems of Carrière and Breuillard–Gelander.

Theorem B. *Let X be a foliated space satisfying the conditions of Theorem A, and let G be its structural local group. Then one of the following properties holds*:

- *G can be approximated by nilpotent local Lie groups*; *or*
- *the holonomy covers of all leaves of X have exponential growth.*

(The definition of *approximation* of a local group is given in Definition 2.25.) Like in the case of Riemannian foliations, Theorem A reduces the proof of Theorem B to the case of minimal G-foliated spaces, where it becomes a problem about local groups. Then, since any locally compact Polish local group can be approximated by local Lie groups in the above sense, the result follows by applying the same arguments as Breuillard and Gelander.

The paper concludes by indicating some examples where Theorems A and B may have interesting applications, and proposing some open problems.

2. Preliminaries on equicontinuous pseudogroups

2A. *Compact-open topology on partial maps with open domains.* (See [Abd-Allah and Brown 1980].) Given spaces X and Y, let $C(X, Y)$ be the space of all continuous maps $X \to Y$; the notation $C_{\text{c-o}}(X, Y)$ may be used to indicate that $C(X, Y)$ is equipped with the compact-open topology. Let $Y^* = Y \cup \{\omega\}$, where $\omega \notin Y$, endowed with the topology in which $U \subset Y^*$ is open if and only if $U = Y^*$ or U is open in Y. A *partial map* $X \rightarrowtail Y$ is a continuous map of a subset of X to Y; the set of all partial maps $X \rightarrowtail Y$ is denoted by $\operatorname{Par}(X, Y)$. A partial map $X \rightarrowtail Y$ with open domain is called a *paro map*, and the set of all paro maps $X \rightarrowtail Y$ is denoted by $\mathrm{Paro}(X, Y)$. There is a bijection $\mu : \mathrm{Paro}(X, Y) \to C(X, Y^*)$ defined by

$$\mu(f)(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{dom} f, \\ \omega & \text{if } x \notin \operatorname{dom} f. \end{cases}$$

The topology on $\mathrm{Paro}(X, Y)$ which makes $\mu : \mathrm{Paro}(X, Y) \to C_{\text{c-o}}(X, Y^*)$ a homeomorphism is called the *compact-open topology*, and the notation $\mathrm{Paro}_{\text{c-o}}(X, Y)$ may be used for the corresponding space. This topology has a subbasis of open sets of the form

$$\mathcal{N}(K, O) = \{h \in \mathrm{Paro}(X, Y) \mid K \subset \operatorname{dom} h, h(K) \subset O\},$$

where $K \subset X$ is compact and $O \subset Y$ is open.

Proposition 2.1. *If X is second countable and locally compact, and Y is second countable, then* $\mathrm{Paro}_{\text{c-o}}(X, Y)$ *is second countable.*

Proof. By hypothesis, there are countable bases of open sets, $\mathcal{V}$ of X and $\mathcal{W}$ of Y, such that $\overline{V}$ is compact for all $V \in \mathcal{V}$. Then the sets $\mathcal{N}(\overline{V}, W)$ ($V \in \mathcal{V}$ and $W \in \mathcal{W}$) form a countable subbasis of open sets of $\mathrm{Paro}_{\text{c-o}}(X, Y)$. □

The following result is elementary.

Proposition 2.2. *For any open subset $U \subset X$, the restriction of the topology of* $\mathrm{Paro}_{\text{c-o}}(X, Y)$ *to the subset $C(U, Y)$ is its usual compact-open topology.*

Since paro maps are not globally defined, let us make precise the definition of their composition. Given spaces X, Y and Z, the *composition* of two paro maps, $f \in \mathrm{Paro}(X, Y)$ and $g \in \mathrm{Paro}(Y, Z)$, is the paro map $gf \in \mathrm{Paro}(X, Z)$ defined as the usual composition of the maps

$$f^{-1}(\operatorname{dom} g) \xrightarrow{\ f\ } \operatorname{dom} g \xrightarrow{\ g\ } Z.$$

Proposition 2.3 [Abd-Allah and Brown 1980, Proposition 3]. *The following properties hold*:

(i) *Let $h : T \rightarrowtail X$ and $g : Y \rightarrowtail Z$ be paro maps. Then the maps*

$$g_* : \mathrm{Paro}_{\text{c-o}}(X, Y) \to \mathrm{Paro}_{\text{c-o}}(X, Z), \quad f \mapsto gf,$$
$$h^* : \mathrm{Paro}_{\text{c-o}}(X, Y) \to \mathrm{Paro}_{\text{c-o}}(T, Y), \quad f \mapsto fh,$$

are continuous.

(ii) *Let $X' \subset X$ and $Y' \subset Y$ be subspaces such that X' is open in X. Then the map*

$$\mathrm{Paro}_{\text{c-o}}(X', Y') \to \mathrm{Paro}_{\text{c-o}}(X, Y),$$

mapping a paro map $X' \rightarrowtail Y'$ to the paro map $X \rightarrowtail Y$ with the same graph, is an embedding.

Proposition 2.4 [Abd-Allah and Brown 1980, Proposition 7]. *If Y is locally compact, then the evaluation partial map*

$$\mathrm{ev}: \mathrm{Paro}_{\text{c-o}}(Y, Z) \times Y \rightarrowtail Z, \quad (f, y) \mapsto f(y),$$

is a paro map; in particular, its domain is open.

Proposition 2.5 [Abd-Allah and Brown 1980, Proposition 9]. *If X and Y are locally compact, then the composition mapping*

$$\mathrm{Paro}_{\text{c-o}}(X, Y) \times \mathrm{Paro}_{\text{c-o}}(Y, Z) \to \mathrm{Paro}_{\text{c-o}}(X, Y), \quad (f, g) \mapsto gf,$$

is continuous.

Let $\mathrm{Loct}(T)$ be the family of all homeomorphisms between open subsets of a space T, which are called *local transformations*. For $h, h' \in \mathrm{Loct}(T)$, the composition $h'h \in \mathrm{Loct}(T)$ is the composition of maps

$$h^{-1}(\operatorname{im} h \cap \operatorname{dom} h') \xrightarrow{\ h\ } \operatorname{im} h \cap \operatorname{dom} h' \xrightarrow{\ h'\ } h'(\operatorname{im} h \cap \operatorname{dom} h').$$

Each $h \in \mathrm{Loct}(T)$ can be identified with the paro map $T \rightarrowtail T$ with the same graph. This gives rise to a canonical injection $\mathrm{Loct}(T) \to \mathrm{Paro}(T, T)$ compatible with composition. The corresponding restriction of the compact-open topology of $\mathrm{Paro}(T, T)$ to $\mathrm{Loct}(T)$ is also called *compact-open topology*, and the notation $\mathrm{Loct}_{\text{c-o}}(T)$ may be used for the corresponding space. The *bi-compact-open topology* is the smallest topology on $\mathrm{Loct}(X)$ such that the identity and inversion maps

$$\mathrm{Loct}(T) \to \mathrm{Loct}_{\text{c-o}}(T), \quad f \mapsto f^{\pm 1},$$

are continuous, and the notation $\mathrm{Loct}_{\text{b-c-o}}(T)$ will be used for the corresponding space. The following result is elementary.

Proposition 2.6 [Abd-Allah and Brown 1980, Proposition 10]. *If T is locally compact, then the composition and inversion maps,*

$$\mathrm{Loct}_{\text{b-c-o}}(T) \times \mathrm{Loct}_{\text{b-c-o}}(T) \to \mathrm{Loct}_{\text{b-c-o}}(T), \quad (g, f) \mapsto gf,$$
$$\mathrm{Loct}_{\text{b-c-o}}(T) \to \mathrm{Loct}_{\text{b-c-o}}(T), \quad f \mapsto f^{-1},$$

are continuous.

2B. *Pseudogroups.*

Definition 2.7 [Sacksteder 1965; Haefliger 2002]. A *pseudogroup* on a space T is a collection $\mathcal{H} \subset \mathrm{Loct}(T)$ such that

- the identity map of T belongs to $\mathcal{H}$ ($\mathrm{id}_T \in \mathcal{H}$);
- if $h, h' \in \mathcal{H}$, then the composite $h'h$ is in $\mathcal{H}$ ($\mathcal{H}^2 \subset \mathcal{H}$);
- $h \in \mathcal{H}$ implies that $h^{-1} \in \mathcal{H}$ ($\mathcal{H}^{-1} \subset \mathcal{H}$);

- if $h \in \mathcal{H}$ and U is open in $\operatorname{dom} h$, then the restriction $h : U \to h(U)$ is in $\mathcal{H}$; and
- if a combination (union) of maps in $\mathcal{H}$ is defined and is a homeomorphism, then it is in $\mathcal{H}$.

Remark 1. The following properties hold:

- $\mathrm{id}_U \in \mathcal{H}$ for every open subset $U \subset T$.
- A local transformation $h \in \mathrm{Loct}(T)$ belongs to $\mathcal{H}$ if and only if it locally belongs to $\mathcal{H}$ (any point $x \in \operatorname{dom} h$ has a neighborhood $V_x \subset \operatorname{dom} h$ such that $h|_{V_x} \in \mathcal{H}$).
- Any intersection of pseudogroups on T is a pseudogroup on T.

Example 2.8. $\mathrm{Loct}(T)$ is the pseudogroup that contains every other pseudogroup on T.

Definition 2.9. A *subpseudogroup* of a pseudogroup $\mathcal{H}$ on T is a pseudogroup on T contained in $\mathcal{H}$. The *restriction* of $\mathcal{H}$ to an open subset $U \subset T$ is the pseudogroup

$$\mathcal{H}|_U = \{ h \in \mathcal{H} \mid \operatorname{dom} h \cup \operatorname{im} h \subset U \}.$$

The pseudogroup *generated* by a set $S \subset \mathrm{Loct}(T)$ is the intersection of all pseudogroups that contain S (the smallest pseudogroup on T containing S).

Definition 2.10. Let $\mathcal{H}$ be a pseudogroup on T. The *orbit* of each $x \in T$ is the set

$$\mathcal{H}(x) = \{ h(x) \mid h \in \mathcal{H}, x \in \operatorname{dom} h \}.$$

The orbits form a partition of T. The space of orbits, equipped with the quotient topology, is denoted by $T/\mathcal{H}$. It is said that $\mathcal{H}$ is

- (*topologically*) *transitive* if some orbit is dense; and
- *minimal* when all orbits are dense.

The following notion, less restrictive than the concept of pseudogroup, is useful to study some properties of pseudogroups.

Definition 2.11 [Matsumoto 2010]. A *pseudo$*$group* on a space T is a family $S \subset \mathrm{Loct}(T)$ that is closed by the operations of composition and inversion.

Remark 2. Any intersection of pseudo$*$groups on T is a pseudo$*$group.

Definition 2.12. Any pseudo$*$group contained in another pseudo$*$group is called a *subpseudo$*$group*. The pseudo$*$group *generated* by a subset S_0 of $\mathrm{Loct}(T)$ is the intersection of all pseudo$*$groups containing S_0 (the smallest pseudo$*$group containing S_0).

Remark 3. Let S be a pseudo$*$group on T, and let S_1 be the collection of restrictions of all maps in S to all open subsets of their domains. Then S_1 is also a pseudo$*$group on T, and S is a subpseudo$*$group of S_1.

Definition 2.13. In Remark 3, it will be said that S_1 is the *localization* of S. If $S = S_1$, then the pseudo$*$group S is called *local*.

Remark 4. Let $S_0 \subset \mathrm{Loct}(T)$. The pseudo$*$group S generated by S_0 consists of all compositions of maps in S_0 and their inverses. The pseudogroup $\mathcal{H}$ generated by S_0 consists of all $h \in \mathrm{Loct}(T)$ that locally belong to the localization of S.

Remark 5. If two local pseudo$*$groups, S_1 and S_2, generate the same pseudogroup $\mathcal{H}$, then $S_1 \cap S_2$ is also a local pseudo$*$group that generates $\mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{H}'$ be pseudogroups on respective spaces T and T'.

Definition 2.14 [Haefliger 1985; 1988]. A *morphism*[1] $\Phi\colon \mathcal{H} \to \mathcal{H}'$ is a maximal collection of homeomorphisms of open sets of T to open sets of T' such that

- if $\phi \in \Phi$, $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$, then $h'\phi h \in \Phi$ ($\mathcal{H}'\Phi\mathcal{H} \subset \Phi$);
- the family of the domains of maps in Φ cover T; and
- if $\phi, \phi' \in \Phi$, then $\phi'\phi^{-1} \in \mathcal{H}'$ ($\Phi\Phi^{-1} \subset \mathcal{H}'$).

A morphism Φ is called an *equivalence* if the family $\Phi^{-1} = \{\phi^{-1} \mid \phi \in \Phi\}$ is also a morphism.

Remark 6. An equivalence $\Phi : \mathcal{H} \to \mathcal{H}'$ can be characterized as a maximal family of homeomorphisms of open sets of T to open sets of T' such that $\mathcal{H}'\Phi\mathcal{H} \subset \Phi$, and $\Phi^{-1}\Phi$ and $\Phi\Phi^{-1}$ generate $\mathcal{H}'$ and $\mathcal{H}'$, respectively.

Remark 7. Any morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ induces a map between the corresponding orbit spaces, $T/\mathcal{H} \to T/\mathcal{H}'$. This map is a homeomorphism if Φ is an equivalence.

Definition 2.15. Let Φ_0 be a family of homeomorphisms of open subsets of T to open subsets of T' such that

- the union of domains of maps in Φ_0 meet all $\mathcal{H}$-orbits; and
- $\Phi_0\mathcal{H}\Phi_0^{-1} \subset \mathcal{H}'$.

Then there is a unique morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ containing Φ_0, which is said to be *generated* by Φ_0. If, moreover,

- the union of images of maps in Φ_0 meet all $\mathcal{H}'$-orbits; and
- $\Phi_0^{-1}\mathcal{H}\Phi_0 \subset \mathcal{H}$;

then Φ is an equivalence.

[1] This is usually called *étale morphism*. We simply call it morphism because no other type of morphism will be considered here.

Definition 2.16 [Haefliger 2002]. A pseudogroup $\mathcal{H}$ on a locally compact space T is said to be *compactly generated* if

- there is a relatively compact open subset $U \subset T$ meeting all $\mathcal{H}$-orbits;
- there is a finite set $S = \{h_1, \dots, h_n\} \subset \mathcal{H}|_U$ that generates $\mathcal{H}|_U$; and
- each h_i is the restriction of some $\tilde{h}_i \in \mathcal{H}$ with $\overline{\operatorname{dom} h_i} \subset \operatorname{dom} \tilde{h}_i$.

Remark 8. Compact generation is very subtle (see [Ghys 1985; Meigniez 1995]). Haefliger asked when compact generation implies realizability as a holonomy pseudogroup of a compact foliated space. The answer is not always affirmative [Meigniez 2010].

Definition 2.17 [Haefliger 1985]. A pseudogroup $\mathcal{H}$ is called *quasianalytic* if every $h \in \mathcal{H}$ is the identity around some $x \in \operatorname{dom} h$ whenever h is the identity on some open set whose closure contains x.

If a pseudogroup $\mathcal{H}$ on a space T is quasianalytic, then every $h \in \mathcal{H}$ with connected domain is the identity on $\operatorname{dom} h$ if it is the identity on some nonempty open set. Because of this, quasianalyticity is interesting when T is locally connected, but local connectivity is too restrictive in our setting. Then, instead of requiring local connectivity, the following stronger version of quasianalyticity will be used.

Definition 2.18 [Álvarez and Candel 2009]. A pseudogroup $\mathcal{H}$ on a space T is said to be *strongly quasianalytic* if it is generated by some subpseudo$*$group $S \subset \mathcal{H}$ such that any transformation in S is the identity on its domain if it is the identity on some nonempty open subset of its domain.

Remark 9. In [Álvarez and Candel 2009], the term used for the above property is "quasieffective". However the term "strongly quasianalytic" seems to be more appropriate.

Remark 10. If the condition on $\mathcal{H}$ to be strongly quasianalytic is satisfied with a subpseudo$*$group S, it is also satisfied with the localization of S. It follows that this property is hereditary by taking subpseudogroups and restrictions to open subsets.

Definition 2.19 [Haefliger 1985]. A pseudogroup $\mathcal{H}$ on a space T is said to be *complete* if, for all $x, y \in T$, there are relatively compact open neighborhoods, U_x of x and V_y of y, such that, for all $h \in \mathcal{H}$ and $z \in U_x \cap \operatorname{dom} h$ with $h(z) \in V_y$, there is some $g \in \mathcal{H}$ such that $\operatorname{dom} g = U_x$ and with the same germ as h at z.

Since any pseudo$*$group S on T is a subpseudo$*$group of $\operatorname{Loct}(T)$, it can be endowed with the restriction of the (bi-)compact-open topology, also called the *(bi-)compact-open topology* of S, and the notation $S_{\text{(b-)c-o}}$ may be used for the corresponding space. In this way, according to Proposition 2.6, if T is locally compact, then $S_{\text{b-c-o}}$ becomes a *topological pseudo$*$group* in the sense that the composition and inversion maps of S are continuous. In particular, this applies to a pseudogroup $\mathscr{H}$ on T, obtaining $\mathscr{H}_{\text{(b-)c-o}}$; thus $\mathscr{H}_{\text{b-c-o}}$ is a *topological pseudogroup* in the above sense if T is locally compact.

Remark 11. $S_{\text{(b-)c-o}} \hookrightarrow S'_{\text{(b-)c-o}}$ is continuous for pseudo$*$groups $S \subset S'$.

The pseudogroups considered from now on will be assumed to act on locally compact Polish[2] spaces; i.e., locally compact, Hausdorff and second countable spaces [Kechris 1991, Theorem 5.3].

2C. *The groupoid of germs of a pseudogroup.*

Definition 2.20. A *groupoid* $\mathfrak{G}$ is a small category where every morphism is an isomorphism. This means that $\mathfrak{G}$ is a set (of *morphisms*) equipped with the structure defined by an additional set T (of *objects*), and the following *structural* maps:

- the *source* and *target* maps $s, t : \mathfrak{G} \to T$;
- the *unit* map $T \to \mathfrak{G}$, $x \mapsto 1_x$;
- the *operation* (or *multiplication*) map $\mathfrak{G} \times_T \mathfrak{G} \to \mathfrak{G}$, $(\delta, \gamma) \mapsto \delta\gamma$, where
$$\mathfrak{G} \times_T \mathfrak{G} = \{(\delta, \gamma) \in \mathfrak{G} \times \mathfrak{G} \mid t(\gamma) = s(\delta)\} \subset \mathfrak{G} \times \mathfrak{G};$$
- and the *inversion* map $\mathfrak{G} \to \mathfrak{G}$, $\gamma \mapsto \gamma^{-1}$;

such that the following conditions are satisfied:

- $s(\delta\gamma) = s(\gamma)$ and $t(\delta\gamma) = t(\delta)$ for all $(\delta, \gamma) \in \mathfrak{G} \times_T \mathfrak{G}$;
- for all $\gamma, \delta, \epsilon \in \mathfrak{G}$ with $t(\gamma) = s(\delta)$ and $t(\delta) = s(\epsilon)$, we have $\epsilon(\delta\gamma) = (\epsilon\delta)\gamma$ (associativity);
- $1_{t(\gamma)}\gamma = \gamma 1_{s(\gamma)} = \gamma$ (units or identity elements); and
- $s(\gamma) = t(\gamma^{-1})$, $t(\gamma) = s(\gamma^{-1})$, $\gamma^{-1}\gamma = 1_{s(\gamma)}$ and $\gamma\gamma^{-1} = 1_{t(\gamma)}$ for all $\gamma \in \mathfrak{G}$ (inverse elements).

If moreover $\mathfrak{G}$ and T are equipped with topologies such that all of the above structural maps are continuous, then $\mathfrak{G}$ is called a *topological groupoid*.

Remark 12. For a groupoid $\mathfrak{G}$, observe that $s(1_x) = t(1_x) = x$ for all $x \in T$, and therefore the source and target maps $s, t : \mathfrak{G} \to T$ are surjective, and the unit map $T \to \mathfrak{G}$ is injective. If moreover $\mathfrak{G}$ is a topological groupoid, then the unit map $T \to \mathfrak{G}$ is a topological embedding, and therefore the topology of T is determined by the topology of $\mathfrak{G}$; indeed, we can consider T as a subspace of $\mathfrak{G}$ if desired.

Definition 2.21. A topological groupoid is called *étale* if the source and target maps are local homeomorphisms.

[2]Recall that a space is called *Polish* if it is separable and completely metrizable.

Let $\mathcal{H}$ be a pseudogroup on a space T. Note that the domain of the evaluation partial map $\operatorname{ev} : \mathcal{H} \times T \rightarrowtail T$ is

$$\mathcal{H} * T = \{ (h, x) \in \mathcal{H} \times T \mid x \in \operatorname{dom} h \} \subset \mathcal{H} \times T.$$

Define an equivalence relation on $\mathcal{H} * T$ by setting $(h, x) \sim (h', x')$ if $x = x'$ and $h = h'$ on some neighborhood of x in $\operatorname{dom} h \cap \operatorname{dom} h'$. The equivalence class of each $(h, x) \in \mathcal{H} * T$ is called the *germ* of h at x, which will be denoted by $\boldsymbol{\gamma}(h, x)$. The corresponding quotient set is denoted by $\mathfrak{G}$, and the quotient map, $\boldsymbol{\gamma} : \mathcal{H} * T \to \mathfrak{G}$, is called the *germ map*. It is well known that $\mathfrak{G}$ is a groupoid with set of units T, where the source and target maps $s, t : \mathfrak{G} \to T$ are given by $s(\boldsymbol{\gamma}(h, x)) = x$ and $t(\boldsymbol{\gamma}(h, x)) = h(x)$, the unit map $T \to \mathfrak{G}$ is defined by $1_x = \boldsymbol{\gamma}(\mathrm{id}_T, x)$, the operation map $\mathfrak{G} \times_T \mathfrak{G} \to \mathfrak{G}$ is given by $\boldsymbol{\gamma}(g, h(x))\, \boldsymbol{\gamma}(h, x) = \boldsymbol{\gamma}(gh, x)$, and the inversion map is defined by $\boldsymbol{\gamma}(h, x)^{-1} = \boldsymbol{\gamma}(h^{-1}, h(x))$.

For $x, y \in T$, let us use the notation $\mathfrak{G}_x = s^{-1}(x)$, $\mathfrak{G}^y = t^{-1}(y)$ and $\mathfrak{G}_x^y = \mathfrak{G}_x \cap \mathfrak{G}^y$; in particular, the group $\mathfrak{G}_x^x$ will be called the *germ group* of $\mathcal{H}$ at x. Points in the same $\mathcal{H}$-orbit have isomorphic germ groups (if $y \in \mathcal{H}(x)$, an isomorphism $\mathfrak{G}_y^y \to \mathfrak{G}_x^x$ is given by conjugation with any element in $\mathfrak{G}_x^y$); hence the germ groups of the orbits make sense up to isomorphism. Under pseudogroup equivalences, corresponding orbits have isomorphic germ groups. The set $\mathfrak{G}_x$ will be called the *germ cover* of the orbit $\mathcal{H}(x)$ with base point x. The target map restricts to a surjective map $\mathfrak{G}_x \to \mathcal{H}(x)$ whose fibers are bijective to $\mathfrak{G}_x^x$ (if $y \in \mathcal{H}(x)$, a bijection $\mathfrak{G}_x^x \to \mathfrak{G}_x^y$ is given by left product with any element in $\mathfrak{G}_x^y$); thus $\mathfrak{G}_x$ is finite if and only if both $\mathfrak{G}_x^x$ and $\mathcal{H}(x)$ are finite. Moreover germ covers based on points in the same orbit are also bijective (if $y \in \mathcal{H}(x)$, a bijection $\mathfrak{G}_y \to \mathfrak{G}_x$ is given by right product with any element in $\mathfrak{G}_x^y$); therefore the germ covers of the orbits make sense up to bijections.

Definition 2.22. It is said that $\mathcal{H}$ is

- *locally free* if all of its germ groups are trivial; and
- *strongly locally free* if $\mathcal{H}$ is generated by a subpseudo$*$group $S \subset \mathcal{H}$ such that, for all $h \in S$ and $x \in \operatorname{dom} h$, if $h(x) = x$ then $h = \mathrm{id}_{\operatorname{dom} h}$.

Remark 13. The condition of being (strongly) locally free is stronger than the condition of being (strongly) quasianalytic. If $\mathcal{H}$ is locally free and satisfies the condition of strong quasianalyticity with a subpseudo$*$group $S \subset \mathcal{H}$ generating $\mathcal{H}$, then $\mathcal{H}$ also satisfies the condition of being strongly locally free with S.

Remark 14. If $\mathcal{H}$ being strongly locally free is witnessed by a subpseudo$*$group S, then it is also witnessed by the localization of S. It follows that this property is hereditary by taking subpseudogroups and restrictions to open subsets.

The *sheaf topology* on $\mathfrak{G}$ has a basis consisting of the sets $\{\boldsymbol{\gamma}(h, x) \mid x \in \operatorname{dom} h\}$ for $h \in \mathcal{H}$. Equipped with the sheaf topology, $\mathfrak{G}$ is an étale groupoid.

Let us define another topology on $\mathfrak{G}$. Suppose that $\mathcal{H}$ is generated by some subpseudo$*$group $S \subset \mathcal{H}$. The set $S * T = (\mathcal{H} * T) \cap (S \times T)$ is open in $S_{\text{(b-)c-o}} \times T$ by Proposition 2.4. It will be denoted by $S_{\text{(b-)c-o}} * T$ when endowed with the restriction of the topology of $S_{\text{(b-)c-o}} \times T$. The induced quotient topology on $\mathfrak{G}$, via the germ map $\boldsymbol{\gamma} : S_{\text{(b-)c-o}} * T \to \mathfrak{G}$, will also be called the *(bi-)compact-open topology*. The corresponding space will be denoted by $\mathfrak{G}_{\text{(b-)c-o}}$, or by $\mathfrak{G}_{S,\text{(b-)c-o}}$ if reference to S is needed. It follows from Proposition 2.6 that $\mathfrak{G}_{\text{b-c-o}}$ is a topological groupoid if T is locally compact. We get a commutative diagram

$$\begin{array}{ccc} S_{\text{(b-)c-o}} * T & \xrightarrow{\text{inclusion}} & \mathcal{H}_{\text{(b-)c-o}} * T \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathfrak{G}_{S,\text{(b-)c-o}} & \xrightarrow{\text{identity}} & \mathfrak{G}_{\mathcal{H},\text{(b-)c-o}} \end{array}$$

where the top map is an embedding and the vertical maps are identifications. Hence the identity map $\mathfrak{G}_{S,\text{(b-)c-o}} \to \mathfrak{G}_{\mathcal{H},\text{(b-)c-o}}$ is continuous. Similarly, the identity map $\mathfrak{G}_{S,\text{b-c-o}} \to \mathfrak{G}_{S,\text{c-o}}$ is continuous.

Question 2.23. When are $\mathfrak{G}_{S,\text{(b-)c-o}} = \mathfrak{G}_{\mathcal{H},\text{(b-)c-o}}$ and $\mathfrak{G}_{S,\text{b-c-o}} = \mathfrak{G}_{S,\text{c-o}}$?

For the second equality, a partial answer will be given in Section 3B.

2D. *Local groups and local actions.* (See [Jacoby 1957].)

Definition 2.24. A *local group* is a quintuple $G \equiv (G, e, \cdot, ', \mathfrak{D})$ satisfying the following conditions:

(1) $(G, \mathfrak{D})$ is a topological space.

(2) $\cdot$ is a function from a subset of $G \times G$ to G.

(3) $'$ is a function from a subset of G to G.

(4) There is a subset O of G such that
- O is an open neighborhood of e in G;
- $O \times O$ is a subset of the domain of $\cdot$;
- O is a subset of the domain of $'$;
- for all $a, b, c \in O$, if $a \cdot b, b \cdot c \in O$, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
- for all $a \in O$, we have $a' \in O$, $a \cdot e = e \cdot a = a$ and $a' \cdot a = a \cdot a' = e$;
- the map $\cdot : O \times O \to G$ is continuous; and
- the map $' : O \to G$ is continuous.

(5) The set $\{e\}$ is closed in G.

Asserting that a local group satisfies some topological property usually means that the property is satisfied on some open neighborhood of e.

A *local homomorphism* of a local group G to a local group H is a continuous partial map $\phi : G \rightarrowtail H$, whose domain is a neighborhood of e in G, which is compatible in the usual sense with the identity elements, the operations and inversions. If moreover ϕ restricts to a homeomorphism between some neighborhoods of the identities in G and H, then it is called a *local isomorphism*, and G and H are said to be *locally isomorphic*. A local group locally isomorphic to a Lie group is called a *local Lie group*.

The collection of all sets O satisfying (4) is denoted by ΨG. This is a neighborhood basis of e in G; all of these neighborhoods are symmetric with respect to the inverse operation (3). Let $\Phi(G, n)$ denote the collection of subsets A of G such that the product of any collection of at most n elements of A is defined, and the set A^n of such products is contained in some $O \in \Psi G$.

Let $H \subset G$. It is said that H is a *subgroup* of G if $H \in \Phi(G, 2)$, $e \in H$, $H' = H$ and $H^2 = H$; and H is a *sub-local group* of G if H is itself a local group with respect to the induced operations and topology.

Let ΥG denote the set of all pairs (H, V) of subsets of G so that $e \in H$, $V \in \Psi G$, $a \cdot b \in H$ for all $a, b \in V \cap H$, and $c' \in H$ for all $c \in V \cap H$. Then a subset $H \subset G$ is a sub-local group if and only if there exists some V such that $(H, V) \in \Upsilon G$ [Jacoby 1957, Theorem 26].

Let ΠG denote the family of pairs (H, V) of subsets of G such that

$$\begin{aligned} &e \in H, && V \in \Psi G \cap \Phi(G, 6), \\ &a \cdot b \in H && \text{for all } a, b \in V^6 \cap H, \\ &c' \in H && \text{for all } c \in V^6 \cap H, \\ &V^2 \setminus H \text{ is open.} && \end{aligned}$$

Given $(H, V) \in \Pi G$, there is a (completely regular, Hausdorff) space $G/(V, H)$ and a continuous open surjection $T : V^2 \to G/(V, H)$ such that $T(a) = T(b)$ if and only if $a' \cdot b \in H$ (cf. [Jacoby 1957, Theorem 29]). For another pair in ΠG of the form (H, W), the spaces $G/(H, V)$ and $G/(H, W)$ are locally homeomorphic at the identity class. Thus the concept of coset space of H is well defined in this sense, as "a germ of a topological space". The notation G/H may be used in this sense. It will be said that G/H has a certain topological property when some $G/(H, V)$ has that property around $T(e)$.

Let ΔG be the set of pairs (H, U) such that $(H, U) \in \Pi G$ and $b' \cdot (a \cdot b) \in H$ for all $a \in H \cap U^4$ and $b \in U^2$. A subset $H \subset G$ is called a normal sub-local group of G if there exists U such that $(H, U) \in \Delta G$. If $(H, U) \in \Delta G$ then the quotient space $G/(H, U)$ admits the structure of a local group (see [Jacoby 1957, Theorem 35] for details) and the natural projection $T : U^2 \to G/(H, U)$ is a local homomorphism. As before, another such pair (H, V) produces a locally isomorphic quotient local group.

As usual, $a \cdot b$ and a' will be denoted by ab and a^{-1}.

Local groups were first studied by Jacoby [1957], giving local versions of important theorems for topological groups. For instance, Jacoby characterized local Lie groups as the locally compact local groups without small subgroups[3] [Jacoby 1957, Theorem 96]. Also, any finite dimensional metrizable locally compact local group is locally isomorphic to the direct product of a Lie group and a compact zero-dimensional topological group [Jacoby 1957, Theorem 107]. In particular, this property shows that any locally Euclidean local group is a local Lie group, which is an affirmative answer to a local version of Hilbert's 5th problem. However the proof of Jacoby is incorrect because he did not realize that, in local groups, associativity for three elements does not imply associativity for any finite sequence of elements [Plaut 1993; Olver 1996]. Fortunately, a completely new proof of the local Hilbert's 5th problem was given by Goldbring [2010]. Moreover van den Dries and Goldbring [2010; 2012] proved that any locally compact local group is locally isomorphic to a topological group, and therefore all other theorems for local groups of Jacoby hold as well because they are known for locally compact topological groups [Montgomery and Zippin 1955].

Definition 2.25. It is said that a local group G can be *approximated* by a class $\mathscr{C}$ of local groups if, for all $W \in \Psi G \cap \Phi(G, 2)$, there is some $V \in \Psi G$ and a sequence of compact normal subgroups $F_n \subset V$ such that $V \subset W$, $F_{n+1} \subset F_n$, $\bigcap_n F_n = \{e\}$, $(F_n, V) \in \Delta G$ and $G/(F_n, V) \in \mathscr{C}$.

Theorem 2.26 [Jacoby 1957, Theorems 97–103; van den Dries and Goldbring 2010; 2012]. *Any locally compact second countable local group G can be approximated by local Lie groups.*

Definition 2.27. A *local action* of a local group G on a space X is a paro map $G \times X \rightarrowtail X$, $(g, x) \mapsto gx$, defined on some open neighborhood of $\{e\} \times X$, such that $ex = x$ for all $x \in X$, and $g_1(g_2x) = (g_1g_2)x$, provided both sides are defined.

Remark 15. The local transformations given by any local action of a local group on a space generate a pseudogroup.

A local action of a local group G on a space X is called *locally transitive* at some point $x \in X$ if there is a neighborhood W of e in G such that the local action is defined on $W \times \{x\}$, and $Wx := \{gx \mid g \in W\}$ is a neighborhood of x in X. Given another local action of G on a space Y, a paro map $\phi : X \rightarrowtail Y$ is called *equivariant* if $\phi(gx) = g\phi(x)$ for all $x \in X$ and $g \in G$, provided both sides are defined.

[3] A local group is said to have no small subgroups when some neighborhood of the identity element contains no nontrivial subgroup.

Example 2.28. Let H be a sub-local group of G. If $(H, V) \in \Pi G$, and if the map $T : V^2 \to G/(H, V)$ is the natural projection, then the map

$$V \times G/(H, V) \to G/(H, V), \quad (v, T(g)) \mapsto T(vg)$$

defines a local action of G on $G/(H, V)$.

Remark 16. If G is a local group locally acting on X and the local action is locally transitive at $x \in X$, then there is a sub-local group H of G such that $(H, V) \in \Pi G$ for some V and the orbit paro map $G \rightarrowtail X$, $g \mapsto gx$, induces an equivariant paro map $G/(H, V) \rightarrowtail X$, which restricts to a homeomorphism between neighborhoods of $T(e)$ and x.

2E. *Equicontinuous pseudogroups.* Álvarez and Candel [2009] introduced the following structure to define equicontinuity for pseudogroups. Let[4] $\{T_i, d_i\}$ be a family of metric spaces such that $\{T_i\}$ is a covering of a set T, each intersection $T_i \cap T_j$ is open in (T_i, d_i) and (T_j, d_j), and, for all $\epsilon > 0$, there is some $\delta(\epsilon) > 0$ such that the following property holds: for all i, j and $z \in T_i \cap T_j$, there is some open neighborhood $U_{i,j,z}$ of z in $T_i \cap T_j$ (with respect to the topology induced by d_i and d_j) such that

$$d_i(x, y) < \delta(\epsilon) \Longrightarrow d_j(x, y) < \epsilon$$

for all $\epsilon > 0$ and all $x, y \in U_{i,j,z}$. Such a family is called a *cover of T by quasilocally equal metric spaces*. Two such families are *quasilocally equal* when their union is also a cover of T by quasilocally equal metric spaces. This is an equivalence relation whose equivalence classes are called *quasilocal metrics* on T. For each quasilocal metric $\mathfrak{Q}$ on T, the pair $(T, \mathfrak{Q})$ is called a *quasilocal metric space*. Such a $\mathfrak{Q}$ induces a topology[5] on T so that, for each $\{T_i, d_i\}_{i\in I} \in \mathfrak{Q}$, the family of open balls of all metric spaces (T_i, d_i) form a basis of open sets. Any topological concept or property of $(T, \mathfrak{Q})$ refers to this underlying topology. $(T, \mathfrak{Q})$ is locally compact and Polish if and only if it is Hausdorff, paracompact and separable [Álvarez and Candel 2009].

Definition 2.29 [Álvarez and Candel 2009]. Let $\mathscr{H}$ be a pseudogroup on a quasilocal metric space $(T, \mathfrak{Q})$. Then $\mathscr{H}$ is said to be (*strongly*[6]) *equicontinuous* if there exists some $\{T_i, d_i\}_{i\in I} \in \mathfrak{Q}$ and some subpseudo$*$group $S \subset \mathscr{H}$ generating $\mathscr{H}$, such that, for every $\epsilon > 0$, there is some $\delta(\epsilon) > 0$ such that

$$d_i(x, y) < \delta(\epsilon) \Longrightarrow d_j(h(x), h(y)) < \epsilon$$

for all $h \in S$, $i, j \in I$ and $x, y \in T_i \cap h^{-1}(T_j \cap \operatorname{im} h)$.

[4] The notation will be simplified by using, for instance, $\{T_i, d_i\}$ instead of $\{(T_i, d_i)\}$.

[5] In fact, it induces a uniformity. We could even use any uniformity to define equicontinuity, but such generality will not be used here.

[6] This adverb, used in [Álvarez and Candel 2009], will be omitted for the sake of simplicity.

A pseudogroup $\mathcal{H}$ acting on a space T will be called (*strongly*) *equicontinuous* when it is equicontinuous with respect to some quasilocal metric inducing the topology of T.

Remark 17. If the equicontinuity of $\mathcal{H}$ is witnessed by a subpseudo$*$group S, then it is also witnessed by the localization of S. It follows that equicontinuity is hereditary by taking subpseudogroups and restrictions to open subsets.

Lemma 2.30 [Álvarez and Candel 2009, Lemma 8.8]. *Let $\mathcal{H}$ and $\mathcal{H}'$ be equivalent pseudogroups on locally compact Polish spaces. Then $\mathcal{H}$ is equicontinuous if and only if $\mathcal{H}'$ is equicontinuous.*

Proposition 2.31 [Álvarez and Candel 2009, Proposition 8.9]. *Let $\mathcal{H}$ be a compactly generated and equicontinuous pseudogroup on a locally compact Polish quasilocal metric space $(T, \mathfrak{Q})$, and let U be any relatively compact open subset of $(T, \mathfrak{Q})$ that meets every $\mathcal{H}$-orbit. Suppose that $\{T_i, d_i\}_{i\in I} \in \mathfrak{Q}$ satisfies the condition of equicontinuity. Let E be any system of compact generation of $\mathcal{H}$ on U, and let $\bar{g}$ be an extension of each $g \in E$ with $\overline{\operatorname{dom} g} \subset \operatorname{dom} \bar{g}$. Also, let $\{T_i'\}_{i\in I}$ be any shrinking*[7] *of $\{T_i\}_{i\in I}$. Then there is a finite family $\mathcal{V}$ of open subsets of $(T, \mathfrak{Q})$ whose union contains U and such that, for any $V \in \mathcal{V}$, $x \in U \cap V$ and $h \in \mathcal{H}$ with $x \in \operatorname{dom} h$ and $h(x) \in U$, the domain of $\tilde{h} = \bar{g}_n \cdots \bar{g}_1$ contains V for any composite $h = g_n \cdots g_1$ defined around x with $g_1, \dots, g_n \in E$. Moreover, $V \subset T'_{i_0}$ and $\tilde{h}(V) \subset T'_{i_1}$ for some $i_0, i_1 \in I$.*

Remark 18. The statement of Proposition 2.31 is stronger than the completeness of $\mathcal{H}|_U$. Since we can choose U large enough to contain two arbitrarily given points of T, it follows $\mathcal{H}$ is complete.

Proposition 2.32 [Álvarez and Candel 2009, Proposition 9.9]. *Let $\mathcal{H}$ be a compactly generated, equicontinuous and strongly quasianalytic pseudogroup on a locally compact Polish space T. Suppose that the conditions of equicontinuity and strong quasianalyticity are satisfied with a subpseudo$*$group $S \subset \mathcal{H}$ generating $\mathcal{H}$. Let A, B be open subsets of T such that $\overline{A}$ is compact and contained in B. If x and y are close enough points in T, then*

$$f(x) \in A \Rightarrow f(y) \in B$$

for all $f \in S$ whose domain contains x and y.

Theorem 2.33 [Álvarez and Candel 2009, Theorem 11.11]. *Let $\mathcal{H}$ be a compactly generated and equicontinuous pseudogroup on a locally compact Polish space T. If $\mathcal{H}$ is transitive, then $\mathcal{H}$ is minimal.*

---

[7] Recall that a *shrinking* of an open cover $\{U_i\}$ of a space X is an open cover $\{U_i'\}$ of X, with the same index set, such that $\overline{U_i'} \subset U_i$ for all i. Similarly, if $\{U_i\}$ is a cover of a subset $A \subset X$ by open subsets of X, a *shrinking* of $\{U_i\}$, as a cover of A by open subsets of X, is a cover $\{U_i'\}$ of A by open subsets of X, with the same index set, such that $\overline{U_i'} \subset U_i$ for all i.

Theorem 2.33 can be restated by saying that the orbit closures form a partition of the space. The following result states that indeed the orbit closures are orbits of a pseudogroup if strong quasianalyticity is also assumed.

Theorem 2.34 [Álvarez and Candel 2009, Theorem 12.1]. *Let $\mathcal{H}$ be a strongly quasianalytic, compactly generated and equicontinuous pseudogroup on a locally compact Polish space T. Let $S \subset \mathcal{H}$ be a subpseudo$*$group generating $\mathcal{H}$ such that $\mathcal{H}$ satisfies the conditions of equicontinuity and strong quasianalyticity with S. Let $\widetilde{\mathcal{H}}$ be the set of maps h between open subsets of T that satisfy the property that for every $x \in \operatorname{dom} h$, there exists a neighborhood O_x of x in $\operatorname{dom} h$ such that the restriction $h|_{O_x}$ is in the closure of $C(O_x, T) \cap S$ in $C_{\text{c-o}}(O_x, T)$. Then*

(i) *$\widetilde{\mathcal{H}}$ is closed by composition, combination and restriction to open sets;*

(ii) *any map in $\widetilde{\mathcal{H}}$ is a homeomorphism around every point of its domain;*

(iii) *$\overline{\mathcal{H}} = \widetilde{\mathcal{H}} \cap \operatorname{Loct}(T)$ is a pseudogroup that contains $\mathcal{H}$;*

(iv) *$\overline{\mathcal{H}}$ is equicontinuous;*

(v) *the orbits of $\overline{\mathcal{H}}$ are equal to the closures of the orbits of $\mathcal{H}$; and*

(vi) *$\widetilde{\mathcal{H}}$ and $\overline{\mathcal{H}}$ are independent of the choice of S.*

Remark 19. In Theorem 2.34, let $\overline{S}$ be the set of local transformations that are in the union of the closures of $C(O, T) \cap S$ in $C_{\text{c-o}}(O, T)$ with O running on the open sets of T. According to the proof of [Álvarez and Candel 2009, Theorem 12.1], $\overline{S}$ is a pseudo$*$group that generates $\overline{\mathcal{H}}$. Moreover, if $\mathcal{H}$ satisfies the equicontinuity condition with S and some representative $\{T_i, d_i\}$ of a quasilocal metric, then $\overline{\mathcal{H}}$ satisfies the equicontinuity condition with $\overline{S}$ and $\{T_i, d_i\}$.

Remark 20. From the proof of [Álvarez and Candel 2009, Theorem 12.1], it also follows that, with the notation of Remark 19, any $x \in \overline{U}$ has a neighborhood O in T such that the closure of

$$\{h \in C(O, T) \cap S \mid h(O) \cap \overline{U} \neq \varnothing\}$$

in $C_{\text{c-o}}(O, T)$ is contained in $\operatorname{Loct}(T)$, and therefore in $\overline{S}$.

Example 2.35. Let G be a locally compact Polish local group with a left invariant metric, let $\Gamma \subset G$ be a dense sub-local group, and let $\mathcal{H}$ be the minimal pseudogroup generated by the local action of Γ by local left translations on G. The local left and right translations in G by each $g \in G$ will be denoted by L_g and R_g. The restrictions of the local left translations L_γ ($\gamma \in \Gamma$) to open subsets of their domains form a subpseudo$*$group $S \subset \mathcal{H}$ that generates $\mathcal{H}$. Obviously, $\mathcal{H}$ satisfies with S the condition of being strongly locally free, and therefore strongly quasianalytic. Moreover $\mathcal{H}$ satisfies with S the condition of being equicontinuous (indeed isometric) by considering any left invariant metric on G. Observe that any local right translation R_g ($g \in G$) generates an equivalence $\mathcal{H} \to \mathcal{H}$.

Now, suppose that $\mathcal{H}$ is compactly generated. Then $\overline{\mathcal{H}}$ is generated by the local action of G on itself by local left translations. The subpseudo$*$group $\overline{S} \subset \overline{\mathcal{H}}$ consists of the restrictions of the local left translations L_g ($g \in G$) to open subsets of their domains. Observe that $\overline{\mathcal{H}}$ satisfies the condition of being strongly locally free, and therefore strongly quasianalytic, with $\overline{S}$.

Lemma 2.36. *Let G and G' be locally compact Polish local groups with left invariant metrics, let $\Gamma \subset G$ and $\Gamma' \subset G'$ be dense sub-local groups, and let $\mathcal{H}$ and $\mathcal{H}'$ be the pseudogroups generated by the local actions of Γ and Γ' by local left translations on G and G'. Suppose that $\mathcal{H}$ and $\mathcal{H}'$ are compactly generated. Then $\mathcal{H}$ and $\mathcal{H}'$ are equivalent if and only if G is locally isomorphic to G'.*

Proof. Consider the notation and observations of Example 2.35 for both G and G'; in particular, $S \subset \mathcal{H}$ and $S' \subset \mathcal{H}'$ denote the subpseudo$*$groups of restrictions of local translations L_γ and $L_{\gamma'}$ ($\gamma \in \Gamma$ and $\gamma' \in \Gamma'$) to open subsets of their domains. Let e and e' denote the identity elements of G and G'. Let $\Phi : \mathcal{H} \to \mathcal{H}'$ be en equivalence. Since $\mathcal{H}'$ is minimal, after composing Φ with the equivalence generated by some local right translation in G if necessary, we can assume that $\phi(e) = e'$ for some $\phi \in \Phi$ with $e \in \operatorname{dom}\phi$.

Let U be a relatively compact open symmetric neighborhood of e in G with $\overline{U} \subset \operatorname{dom}\phi$. Let $\{f_1, \dots, f_n\}$ be a symmetric system of compact generation of $\mathcal{H}$ on U. Thus each f_i has an extension $\tilde{f}_i \in \mathcal{H}$ such that $\overline{\operatorname{dom} f_i} \subset \operatorname{dom} \tilde{f}_i \subset \operatorname{dom}\phi$.

Claim 1. *We can assume that $\tilde{f}_i \in S$ and $\phi \tilde{f}_i \phi^{-1} \in S'$ for all i.*

Each point in $\operatorname{dom} \tilde{f}_i \cap \operatorname{dom}\phi$ has an open neighborhood O such that $O \subset \operatorname{dom} \tilde{f}_i$, $\tilde{f}_i|_O \in S$ and $\phi \tilde{f}_i \phi^{-1}|_{\phi(O)} \in S'$. Take a finite covering $\{O_{ij}\}$ ($j \in \{1, \dots, k_i\}$) of the compact set $\overline{\operatorname{dom} f_i}$ by sets of this type. Let $\{P_{ij}\}$ be a shrinking of $\{O_{ij}\}$, as a cover of $\overline{\operatorname{dom} f_i}$ by open subsets of $\operatorname{dom} \tilde{f}_i$. Then the restrictions $g_{ij} = f_i|_{P_{ij} \cap U}$ ($i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k_i\}$) generate $\mathcal{H}|_U$, each $\tilde{g}_{ij} = \tilde{f}_i|_{O_{ij}}$ is in S and extends g_{ij}, $\overline{\operatorname{dom} g_{ij}} \subset \operatorname{dom} \tilde{g}_{ij}$, and $\phi \tilde{g}_{ij} \phi^{-1} \in S'$, showing Claim 1.

According to Claim 1, the maps $f'_i = \phi f_i \phi^{-1}$ form a symmetric system of compact generation of $\mathcal{H}'$ on $U' = \phi(U)$, which can be checked with the extensions $\tilde{f}'_i = \phi \tilde{f}_i \phi^{-1}$. Let $S_0 \subset S$ and $S'_0 \subset S'$ be the subpseudo$*$groups consisting of the restrictions of compositions of maps f_i and f'_i to open subsets of their domains, respectively. They generate $\mathcal{H}$ and $\mathcal{H}'$. It follows from Claim 1 that $\phi f \phi^{-1} \in S'$ for all $f \in S_0$. On the other hand, by Proposition 2.31, there is a smaller open neighborhood of the identity, $V \subset U$, such that, for all $h \in \mathcal{H}$ and all $x \in V \cap \operatorname{dom} h$ with $h(x) \in U$, there is some $f \in S_0$ such that $\operatorname{dom} f = V$ and $\boldsymbol{\gamma}(f, x) = \boldsymbol{\gamma}(h, x)$.

Let W be another symmetric open neighborhood of the identity such that $W^2 \subset V$. Let us show that $\phi : W \to \phi(W)$ is a local isomorphism. Let $\gamma \in W \cap \Gamma$. The restriction $L_\gamma : W \to \gamma W$ is well defined and belongs to S. Hence there is some $f \in S_0$ such that $\operatorname{dom} f = V$ and $\boldsymbol{\gamma}(f, e) = \boldsymbol{\gamma}(L_\gamma, e)$. Since f is also a restriction of a local left translation in G, it follows that $f = L_\gamma$ on W. So $\phi L_\gamma \phi^{-1}|_{\phi(W)} \in S'$; i.e., there is some $\gamma' \in \Gamma'$ such that $\phi L_\gamma \phi^{-1} = L_{\gamma'}$ on $\phi(W)$. In fact,

$$\phi(\gamma) = \phi L_\gamma(e) = \phi L_\gamma \phi^{-1}(e') = L_{\gamma'}(e') = \gamma'.$$

Hence, for all $\gamma, \delta \in \Gamma$,

$$\phi(\gamma\delta) = \phi L_\gamma(\delta) = L_{\phi(\gamma)}\phi(\delta) = \phi(\gamma)\phi(\delta),$$

$$\begin{aligned}\phi(\gamma)^{-1} &= L^{-1}_{\phi(\gamma)}(e') = (\phi L_\gamma \phi^{-1})^{-1}(e') \\ &= \phi L_{\gamma^{-1}} \phi^{-1}(e') = L_{\phi(\gamma^{-1})}(e') = \phi(\gamma^{-1}).\end{aligned}$$

Since ϕ and the product and inversion maps are continuous, it follows that, for all $g, h \in W$, we have $\phi(gh) = \phi(g)\phi(h)$ and $\phi(g^{-1}) = \phi(g)^{-1}$. □

Example 2.37. This generalizes Example 2.35. Let G be a locally compact Polish local group with a left invariant metric, $K \subset G$ a compact subgroup, and $\Gamma \subset G$ a dense sub-local group. Take some V such that $(H, V) \in \Pi(G)$. The left invariant metric on G can be assumed to be also K-right invariant by the compactness of K, and therefore it defines a metric on $G/(K, V)$. Then the canonical local action of Γ on some neighborhood of the identity class in $G/(K, V)$ induces a transitive equicontinuous pseudogroup $\mathcal{H}$ on a locally compact Polish space; in fact, this is a pseudogroup of local isometries.

Assume that $\mathcal{H}$ is compactly generated. Then $\overline{\mathcal{H}}$ is generated by the canonical local action of G on some neighborhood of the identity class in $G/(K, V)$. Moreover the subpseudo∗group $\overline{S} \subset \overline{\mathcal{H}}$ consists of the local translations of the local action of G on $G/(K, V)$.

Examples 2.35 and 2.37 are particular cases of pseudogroups induced by local actions (Remark 15). The following result indicates their relevance.

Theorem 2.38 [Álvarez and Candel 2010, Theorem 5.2]. *Let $\mathcal{H}$ be a transitive, compactly generated and equicontinuous pseudogroup on a locally compact Polish space, and suppose that $\overline{\mathcal{H}}$ is strongly quasianalytic. Then $\mathcal{H}$ is equivalent to a pseudogroup of the type described in Example 2.37.*

Remark 21. From the proof of [Álvarez and Candel 2010, Theorems 3.3 and 5.2], it also follows that, in Theorem 2.38, if moreover $\overline{\mathcal{H}}$ is strongly locally free, then $\mathcal{H}$ is equivalent to a pseudogroup of the type described in Example 2.35.

3. Molino's theory for equicontinuous pseudogroups

3A. *Conditions on $\mathcal{H}$.* Let $\mathcal{H}$ be a pseudogroup of local transformations of a locally compact Polish space T. Suppose that $\mathcal{H}$ is compactly generated, complete and equicontinuous, and that $\overline{\mathcal{H}}$ is also strongly quasianalytic.

Let U be a relatively compact open set in T that meets all the orbits of $\mathcal{H}$. The condition of compact generation is satisfied with U. Consider a representative $\{T_i, d_i\}$ of a quasilocal metric on T satisfying the condition of equicontinuity of $\mathcal{H}$ with some subpseudo$*$group $S \subset \mathcal{H}$ that generates $\mathcal{H}$. We can also suppose that the condition of strong quasianalyticity of $\mathcal{H}$ is satisfied with S.

Remark 22. According to Theorem 2.34 and Remark 19, there is a mapping $\epsilon \mapsto \delta(\epsilon) > 0$ ($\epsilon > 0$) such that

$$d_i(x, y) < \delta(\epsilon) \Longrightarrow d_j(h(x), h(y)) < \epsilon$$

for all indices i and j, every $h \in \overline{S}$, and $x, y \in T_i \cap h^{-1}(T_j \cap \operatorname{im} h)$.

Remark 23. By Remark 20 and refining $\{T_i\}$ if necessary, we can assume that $\overline{U}$ is covered by a finite collection $\{T_{i_1}, \dots, T_{i_r}\}$ of the sets T_i, such that the closure of

$$\{h \in C(T_{i_k}, T) \cap S \mid h(T_{i_k}) \cap \overline{U} \neq \varnothing\}$$

in $C_{\text{c-o}}(T_{i_k}, T)$ is contained in $\overline{S}$ for all $k \in \{1, \dots, r\}$.

Remark 24. By Proposition 2.31 and Remark 23, and refining $\{T_i\}$ if necessary, we can assume that, for all $h \in \overline{\mathcal{H}}$ and $x \in T_{i_k} \cap U \cap \operatorname{dom} h$ with $h(x) \in U$, there is some $\tilde{h} \in \overline{S}$ with $\operatorname{dom} \tilde{h} = T_{i_k}$ and $\boldsymbol{\gamma}(h, x) = \boldsymbol{\gamma}(\tilde{h}, x)$.

Remark 25. By Remarks 5, 10 and 17, and refining $\{T_i\}$ if necessary, we can assume that the strong quasianalyticity of $\overline{\mathcal{H}}$ is satisfied with $\overline{S}$.

3B. *Coincidence of topologies.*

Proposition 3.1. $\overline{S}_{\text{b-c-o}} = \overline{S}_{\text{c-o}}$.

Proof. (This is inspired by [Arens 1946].) For each $g \in \overline{S}$, take any index i and open sets $V, W \subset T$ such that $\overline{V} \subset W$ and $\overline{W} \subset \operatorname{im} g$. By Proposition 2.32, there is some $\epsilon(i, V, W) > 0$ such that, for all $x, y \in T_i$, if $d_i(x, y) < \epsilon(i, V, W)$, then

$$f(x) \in \overline{V} \Longrightarrow f(y) \in W$$

for all $f \in \overline{S}$ with $x, y \in \operatorname{dom} f$. Let $\mathcal{K}(g, i, V, W)$ be the family of compact subsets $K \subset T_i \cap \operatorname{dom} g$ such that

$$\mathring{K} \neq \varnothing, \quad \operatorname{diam}_{d_i}(K) < \epsilon(i, V, W), \quad g(K) \subset V,$$

where $\mathring{K}$ and $\operatorname{diam}_{d_i}(K)$ denote the interior and d_i-diameter of K. Moreover let $\mathcal{K}(g)$ denote the union of the families $\mathcal{K}(g, i, V, W)$ as above. Then a subbasis

$\mathcal{N}(g)$ of open neighborhoods of each g in $\overline{S}_{\text{c-o}}$ is given by the sets $\mathcal{N}(K, O) \cap \overline{S}$, where $K \in \mathcal{K}(g)$ and O is an open neighborhood of $g(K)$ in T.

We have to prove the continuity of the inversion map $\overline{S}_{\text{c-o}} \to \overline{S}_{\text{c-o}}$, $h \mapsto h^{-1}$. Let $h \in \overline{S}$ and let $\mathcal{N}(K, O) \in \mathcal{N}(h^{-1})$ with $K \in \mathcal{K}(h^{-1}, i, V, W)$, and fix any point $x \in \mathring{K}$. Then

$$\mathcal{V} = \mathcal{N}(\{h^{-1}(x)\}, \mathring{K}) \cap \mathcal{N}(\overline{W} \setminus O, T \setminus K)$$

is an open neighborhood of h in $\mathcal{H}_{\text{c-o}}$. We have $d_i(fh^{-1}(x), y) < \epsilon(i, V, W)$ for all $f \in \mathcal{V} \cap \overline{S}$ and $y \in K$ since $fh^{-1}(x) \in \mathring{K}$ and $\operatorname{diam}_{d_i}(K) < \epsilon(i, V, W)$. So $f^{-1}(y) \in W$ by the definition of $\epsilon(i, V, W)$ since $f^{-1} \in \overline{S}$ and $h^{-1}(x) \in h^{-1}(K) \subset V$. Thus, if $f^{-1}(y) \notin O$, we get $f^{-1}(y) \in \overline{W} \setminus O$, obtaining $y \in T \setminus K$, which is a contradiction. Hence $f^{-1} \in \mathcal{N}(K, O)$ for all $f \in \mathcal{V} \cap \overline{S}$. □

Let $\overline{\mathfrak{G}}$ denote the groupoid of germs of $\overline{\mathcal{H}}$. The following direct consequence of Proposition 3.1 gives a partial answer to Question 2.23.

Corollary 3.2. $\overline{\mathfrak{G}}_{\overline{S},\text{b-c-o}} = \overline{\mathfrak{G}}_{\overline{S},\text{c-o}}$; *i.e.*, $\overline{\mathfrak{G}}_{\overline{S},\text{c-o}}$ *is a topological groupoid.*

3C. ***The space*** $\widehat{T}$**.** Recall that $s, t : \overline{\mathfrak{G}}_{\overline{S},\text{c-o}} \to T$ denote the source and target projections. Let $\widehat{T} = \mathfrak{G}_{\overline{S},\text{c-o}}$, where the following subsets are open:

$$\widehat{T}_U = s^{-1}(U) \cap t^{-1}(U), \quad \widehat{T}_{k,l} = s^{-1}(T_{i_k,i_l}) \cap t^{-1}(T_{i_k,i_l}), \quad \widehat{T}_{U,k,l} = \widehat{T}_U \cap \widehat{T}_{k,l}.$$

Observe that $\widehat{T}_U$ is an open subspace of $\widehat{T}$, and the family of sets $\widehat{T}_{U,k,l}$ form an open covering of $\widehat{T}_U$.

Let $\boldsymbol{\gamma}(h, x) \in \widehat{T}_{U,k,l}$. We can assume that $h \in \overline{S}$ and $\operatorname{dom} h = T_{i_k}$ according to Remark 24. Since $x \in T_{i_k} \cap U$ and $h(x) \in T_{i_l} \cap U$, there are relatively compact open neighborhoods, V of x and W of $h(x)$, such that $\overline{V} \subset T_{i_k} \cap U$, $\overline{W} \subset T_{i_l} \cap U$ and $h(\overline{V}) \subset W$.

By Remark 24, for each $f \in \overline{S}$ with $x \in \operatorname{dom} f$, there is some $\tilde{f} \in \overline{S}$ with $\operatorname{dom} \tilde{f} = T_{i_k}$ and $\boldsymbol{\gamma}(\tilde{f}, x) = \boldsymbol{\gamma}(f, x)$.

Lemma 3.3. *We have* $f = \tilde{f}$ *on* V.

Proof. The composition $f|_V \tilde{f}^{-1}$ is defined on $\tilde{f}(V)$, belongs to $\overline{S}$, and is the identity on some neighborhood of $\tilde{f}(x) = f(x)$. So $f|_V \tilde{f}^{-1}$ is the identity on $\tilde{f}(V)$ because $\overline{\mathcal{H}}$ satisfies strong quasianalyticity with $\overline{S}$. Hence $f = \tilde{f}$ on V. □

Let

$$\overline{S}_0 = \{ f \in \overline{S} \mid \overline{V} \subset \operatorname{dom} f,\ f(\overline{V}) \subset W \}, \tag{3}$$

$$\overline{S}_1 = \{ f \in \overline{S} \mid \overline{V} \subset \operatorname{dom} f,\ f(\overline{V}) \subset \overline{W} \}, \tag{4}$$

equipped with the restriction of the compact-open topology. Notice that $\overline{S}_0$ is an open neighborhood of h in $\overline{S}_{\text{c-o}}$. Consider the compact-open topology on $C(\overline{V}, \overline{W})$.

Lemma 3.4. *The restriction map* $\mathcal{R} : \overline{S}_1 \to C(\overline{V}, \overline{W})$, $\mathcal{R}(f) = f|_{\overline{V}}$, *defines an identification* $\mathcal{R} : \overline{S}_1 \to \mathcal{R}(\overline{S}_1)$.

Proof. The continuity of $\mathcal{R}$ is elementary.

Let $G \subset \mathcal{R}(\overline{S}_1)$ such that $\mathcal{R}^{-1}(G)$ is open in $\overline{S}_1$. For each $g_0 \in G$, there is some $g_0' \in \mathcal{R}^{-1}(G)$ such that $\mathcal{R}(g_0') = g_0$. Since $\mathcal{R}^{-1}(G)$ is open in $\overline{S}_1$, there are finite collections $\{K_1, \dots, K_p\}$ of compact subsets and $\{O_1, \dots, O_p\}$ of open subsets, such that

$$g_0' \in \left\{ f \in \overline{S}_1 \;\middle|\; \bigcup_{i=1}^p K_i \subset \operatorname{dom} f \text{ and } f(K_i) \subset O_i \text{ for each } i \right\} \subset \mathcal{R}^{-1}(G).$$

Then

$$g_0 \in \left\{ g \in \overline{S}_1 \;\middle|\; \bigcup_{i=1}^p K_i \cap \overline{V} \subset \operatorname{dom} g \text{ and } g(K_i \cap \overline{V}) \subset O_i \cap \overline{W} \text{ for each } i \right\} \subset G.$$

Since $K_1 \cap \overline{V}, \dots, K_p \cap \overline{V}$ are compact in $\overline{V}$ and $O_1 \cap \overline{W}, \dots, O_p \cap \overline{W}$ are open in $\overline{W}$, it follows that g_0 is in the interior of G in $\mathcal{R}(\overline{S}_1)$. Hence G is open in $\mathcal{R}(\overline{S}_1)$. □

Lemma 3.5. $\mathcal{R}(\overline{S}_1)$ *is closed in* $C(\overline{V}, \overline{W})$.

Proof. Observe that $C(\overline{V}, \overline{W})$ is second countable because T is Polish. Take a sequence g_n in $\mathcal{R}(\overline{S}_1)$ converging to g in $C(\overline{V}, \overline{W})$. Then it easily follows that $g_n|_V$ converges to $g|_V$ in $C(V, T)$ with the compact-open topology. Thus $g|_V \in \overline{S}$ according to Remark 23. Let $f = \widetilde{g|_V}$. By Lemma 3.3, we have $g = f|_{\overline{V}}$. Therefore $f \in \overline{S}_1$ and $g = \mathcal{R}(f)$. □

Corollary 3.6. $\mathcal{R}(\overline{S}_1)$ *is compact in* $C(\overline{V}, \overline{W})$.

Proof. This follows by the Arzelà–Ascoli theorem and Lemma 3.5, because $\overline{V}$ and $\overline{W}$ are compact, and $\mathcal{R}(\overline{S}_1)$ is equicontinuous since $\mathcal{H}$ satisfies the equicontinuity condition with $\overline{S}$ and $\{T_i, d_i\}$. □

Let V_0 be an open subset of T such that $x \in V_0$ and $\overline{V_0} \subset V$. Since $\overline{V_0} \subset \operatorname{dom} f$ for all $f \in \overline{S}_1$, we can consider the restriction $\overline{S}_1 \times \overline{V_0} \to \widehat{T}$ of the germ map.

Lemma 3.7. *The image* $\boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0})$ *is compact in* $\widehat{T}$.

Proof. For each $g \in C(\overline{V}, \overline{W})$ and $y \in \overline{V}$, let $\bar{\boldsymbol{\gamma}}(g, y)$ denote the germ of g at y, defining a germ map

$$\bar{\boldsymbol{\gamma}} : C(\overline{V}, \overline{W}) \times \overline{V} \to \bar{\boldsymbol{\gamma}}(C(\overline{V}, \overline{W}) \times \overline{V}).$$

Since $\overline{V_0} \subset V$, we get that $\boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0}) = \bar{\boldsymbol{\gamma}}(\mathcal{R}(\overline{S}_1) \times \overline{V_0})$ and the diagram

$$\begin{array}{ccc} \overline{S}_1 \times \overline{V_0} & \xrightarrow{\mathcal{R}\times\mathrm{id}} & \mathcal{R}(\overline{S}_1) \times \overline{V_0} \\ {\scriptstyle\boldsymbol{\gamma}}\downarrow & & \downarrow{\scriptstyle\bar{\boldsymbol{\gamma}}} \\ \boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0}) & = & \bar{\boldsymbol{\gamma}}(\mathcal{R}(\overline{S}_1) \times \overline{V_0}) \end{array} \tag{5}$$

is commutative. Then

$$\bar{\boldsymbol{\gamma}} : \mathcal{R}(\bar{S}_1) \times \overline{V_0} \to \bar{\boldsymbol{\gamma}}(\mathcal{R}(\bar{S}_1) \times \overline{V_0})$$

is continuous because

$$\mathcal{R} \times \mathrm{id} : \bar{S}_1 \times \overline{V_0} \to \mathcal{R}(\bar{S}_1) \times \overline{V_0}$$

is an identification by Lemma 3.4, and

$$\boldsymbol{\gamma} : \bar{S}_1 \times \overline{V_0} \to \boldsymbol{\gamma}(\bar{S}_1 \times \overline{V_0})$$

is continuous. Hence $\boldsymbol{\gamma}(\bar{S}_1 \times \overline{V_0})$ is compact by Corollary 3.6. □

Lemma 3.8. *The image $\boldsymbol{\gamma}(\bar{S}_0 \times V_0)$ is open in $\widehat{T}$.*

Proof. This holds because $\bar{S}_0 \times V_0$ is open in $\bar{S}_{\text{c-o}} * T$ and saturated by the fibers of $\boldsymbol{\gamma} : \bar{S}_{\text{c-o}} * T \to \widehat{T}$. □

Remark 26. Observe that the proof of Lemma 3.8 does not require $\overline{V_0} \subset V$; it holds for any open $V_0 \subset V$.

Corollary 3.9. *$\widehat{T}_U$ is locally compact.*

Proof. We have that $\boldsymbol{\gamma}(\bar{S}_1 \times \overline{V_0})$ is compact by Lemma 3.7 and contains $\boldsymbol{\gamma}(\bar{S}_0 \times V_0)$, which is an open neighborhood of $\boldsymbol{\gamma}(h, x)$ by Lemma 3.8. Then the result follows because $\boldsymbol{\gamma}(h, x) \in \widehat{T}_U$ is arbitrary. □

Lemma 3.10. *The map $\bar{\boldsymbol{\gamma}} : \mathcal{R}(\bar{S}_1) \times \overline{V_0} \to \widehat{T}$ is injective.*

Proof. For $f_1, f_2 \in \bar{S}_1$ with $\bar{\boldsymbol{\gamma}}(\mathcal{R}(f_1), y_1) = \bar{\boldsymbol{\gamma}}(\mathcal{R}(f_2), y_2)$, suppose

$$(\mathcal{R}(f_1), y_1), (\mathcal{R}(f_2), y_2) \in \mathcal{R}(\bar{S}_1) \times \overline{V_0},$$

Thus, $y_1 = y_2 =: y$ and $\boldsymbol{\gamma}(f_1, y_1) = \boldsymbol{\gamma}(f_2, y_2)$; i.e., $f_1 = f_2$ on some neighborhood O of y in $\operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Then $f_1(O) \subset \operatorname{dom}(f_2 f_1^{-1})$ and $f_2 f_1^{-1} = \mathrm{id}_T$ on $f_1(O)$. Since $f_2 f_1^{-1} \in \bar{S}$, we get $f_2 f_1^{-1} = \mathrm{id}_T$ on $\operatorname{dom}(f_2 f_1^{-1}) = f_1(\operatorname{dom} f_1 \cap \operatorname{dom} f_2)$ by the strong quasianalyticity of $\bar{S}$. Since $\bar{V} \subset \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, it follows that $f_2 f_1^{-1} = \mathrm{id}_T$ on $f_1(\bar{V})$, and therefore $f_1 = f_2$ on $\bar{V}$; i.e., $\mathcal{R}(f_1) = \mathcal{R}(f_2)$. □

Let $\hat{\pi} := (s, t) : \widehat{T} \to T \times T$, which is continuous.

Corollary 3.11. *The restriction $\hat{\pi} : \widehat{T}_U \to U \times U$ is proper.*

Proof. Since $U \times U$ can be covered by sets of the form $V_0 \times W$, for V_0 and W as above, it is enough to prove that $\hat{\pi}^{-1}(K_1 \times K_2)$ is compact for all compact sets $K_1 \subset V_0$ and $K_2 \subset W$. Then, with the above notation,

$$\hat{\pi}^{-1}(K_1 \times K_2) \subset \boldsymbol{\gamma}(\bar{S}_1 \times K_1) \subset \boldsymbol{\gamma}(\bar{S}_1 \times \overline{V_0}),$$

and the result follows from Lemma 3.7. □

Corollary 3.12. *The closure of $\widehat{T}_U$ in $\widehat{T}$ is compact.*

Proof. Take a relatively compact open subset $U' \subset T$ containing $\overline{U}$. By applying Corollary 3.11 to U', it follows that $\hat{\pi} : \widehat{T}_{U'} \to U' \times U'$ is proper. Therefore $\hat{\pi}^{-1}(\overline{U} \times \overline{U})$ is compact and contains the closure of $\widehat{T}_U$ in $\widehat{T}$. □

Lemma 3.13. *$\widehat{T}_U$ is Hausdorff.*

Proof. Let $\boldsymbol{\gamma}(h_1, x_1) \neq \boldsymbol{\gamma}(h_2, x_2)$ in $\widehat{T}_U$.

Suppose first that $x_1 \neq x_2$. Since T is Hausdorff, there are disjoint open subsets V_1 and V_2 such that $x_1 \in V_1$ and $x_2 \in V_2$. Then $\widehat{V}_1 = \widehat{T}_U \cap s^{-1}(V_1)$ and $\widehat{V}_2 = \widehat{T}_U \cap s^{-1}(V_2)$ are disjoint and open in $\widehat{T}_U$, and $\boldsymbol{\gamma}(h_1, x_1) \in \widehat{V}_1$ and $\boldsymbol{\gamma}(h_2, x_2) \in \widehat{V}_2$.

Now, assume that $x_1 = x_2 =: x$ but $h_1(x) \neq h_2(x)$. Take disjoint open subsets $W_1, W_2 \subset U$ such that $h_1(x) \in W_1$ and $h_2(x) \in W_2$. Then $\widehat{W}_1 = \widehat{T}_U \cap t^{-1}(W_1)$ and $\widehat{W}_2 = \widehat{T}_U \cap t^{-1}(W_2)$ are disjoint and open in $\widehat{T}_U$, and $\boldsymbol{\gamma}(h_1, x) \in \widehat{W}_1$ and $\boldsymbol{\gamma}(h_2, x) \in \widehat{W}_2$.

Finally, suppose that $x_1 = x_2 =: x$ and $h_1(x) = h_2(x) =: y$. Then $x \in T_{i_k} \cap U$ and $y \in T_{i_l} \cap U$ for some indices k and l. Take open neighborhoods V of x and W of y, such that $\overline{V} \subset T_{i_k} \cap U$, $\overline{W} \subset T_{i_l} \cap U$ and $h_1(\overline{V}) \cup h_2(\overline{V}) \subset W$. Define $\overline{S}_0$ and $\overline{S}_1$ by using V and W like in (3) and (4), and take an open subset $V_0 \subset T$ such that $x \in V_0$ and $\overline{V_0} \subset V$, as above. We can assume that $h_1, h_2 \in \overline{S}_1$. Then

$$\bar{\boldsymbol{\gamma}}(\mathcal{R}(h_1), x) = \boldsymbol{\gamma}(h_1, x_1) \neq \boldsymbol{\gamma}(h_2, x_2) = \bar{\boldsymbol{\gamma}}(\mathcal{R}(h_2), x),$$

and therefore $\mathcal{R}(h_1) \neq \mathcal{R}(h_2)$ in $\mathcal{R}(\overline{S}_1)$ by Lemma 3.10. Since $\mathcal{R}(\overline{S}_1)$ is Hausdorff (because it is a subspace of $C_{\text{c-o}}(\overline{V}, \overline{W})$), it follows that there are disjoint open subsets $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{R}(\overline{S}_1)$ such that $\mathcal{R}(h_1) \in \mathcal{N}_1$ and $\mathcal{R}(h_2) \in \mathcal{N}_2$. So $\mathcal{R}^{-1}(\mathcal{N}_1)$ and $\mathcal{R}^{-1}(\mathcal{N}_2)$ are disjoint open subsets of $\overline{S}_1$ with $h_1 \in \mathcal{R}^{-1}(\mathcal{N}_1)$ and $h_2 \in \mathcal{R}^{-1}(\mathcal{N}_2)$. Hence $\mathcal{M}_1 = \mathcal{R}^{-1}(\mathcal{N}_1) \cap \overline{S}_0$ and $\mathcal{M}_2 = \mathcal{R}^{-1}(\mathcal{N}_2) \cap \overline{S}_0$ are disjoint and open in $\overline{S}_0$, and therefore they are open in $\overline{S}$. Moreover $\mathcal{M}_1 \times V_0$ and $\mathcal{M}_2 \times V_0$ are saturated by the fibers of $\boldsymbol{\gamma} : \overline{S}_0 \times V_0 \to \boldsymbol{\gamma}(\overline{S}_0 \times V_0)$; in fact, if $(f, z) \in \overline{S}_0 \times V_0$ satisfies $\boldsymbol{\gamma}(f, z) = \boldsymbol{\gamma}(f', z)$ for some $f' \in \mathcal{M}_a$ ($a \in \{1, 2\}$), then

$$\bar{\boldsymbol{\gamma}}(\mathcal{R}(f), z) = \boldsymbol{\gamma}(f, z) = \boldsymbol{\gamma}(f', z) = \bar{\boldsymbol{\gamma}}(\mathcal{R}(f'), z),$$

giving $\mathcal{R}(f) = \mathcal{R}(f') \in \mathcal{N}_a$ by Lemma 3.10. Therefore $f \in \mathcal{R}^{-1}(\mathcal{N}_a) \cap \overline{S}_0 = \mathcal{M}_a$. It follows that $\boldsymbol{\gamma}(\mathcal{M}_1 \times V_0)$ and $\boldsymbol{\gamma}(\mathcal{M}_2 \times V_0)$ are open in $\boldsymbol{\gamma}(\overline{S}_0 \times V_0)$, because the map $\boldsymbol{\gamma} : \overline{S}_0 \times V_0 \to \boldsymbol{\gamma}(\overline{S}_0 \times V_0)$ is an identification as $\overline{S}_0 \times V_0$ is open in $\overline{S}_{\text{c-o}} * T$ and saturated by the fibers of $\boldsymbol{\gamma} : \overline{S}_{\text{c-o}} * T \to \widehat{T}$. Furthermore, by the commutativity of the diagram (5),

$$\begin{aligned}\boldsymbol{\gamma}(\mathcal{M}_1 \times V_0) \cap \boldsymbol{\gamma}(\mathcal{M}_2 \times V_0) &= \bar{\boldsymbol{\gamma}}(\mathcal{N}_1 \times V_0) \cap \bar{\boldsymbol{\gamma}}(\mathcal{N}_2 \times V_0)\\ &= \bar{\boldsymbol{\gamma}}((\mathcal{N}_1 \cap \mathcal{N}_1) \times V_0) = \varnothing,\end{aligned}$$

and $\boldsymbol{\gamma}(h_1, x) \in \boldsymbol{\gamma}(\mathcal{M}_1 \times V_0)$ and $\boldsymbol{\gamma}(h_2, x) \in \boldsymbol{\gamma}(\mathcal{M}_2 \times V_0)$. □

Corollary 3.14. *The map* $\bar{\boldsymbol{\gamma}} : \mathcal{R}(\overline{S}_1) \times \overline{V_0} \to \bar{\boldsymbol{\gamma}}(\mathcal{R}(\overline{S}_1) \times \overline{V_0})$ *is a homeomorphism.*

Lemma 3.15. $\widehat{T}_U$ *is second countable.*

Proof. $\widehat{T}_U$ can be covered by a countable collection of open subsets of the type $\boldsymbol{\gamma}(\overline{S}_0 \times V_0)$ as above. But $\boldsymbol{\gamma}(\overline{S}_0 \times V_0)$ is second countable because it is a subspace of $\boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0}) = \bar{\boldsymbol{\gamma}}(\mathcal{R}(\overline{S}_1) \times \overline{V_0})$, which is homeomorphic to $\mathcal{R}(\overline{S}_1) \times \overline{V_0}$ by Corollary 3.14, and this space is second countable as a subspace of the second countable space $C(\overline{V_0}, \overline{W_0}) \times \overline{V_0}$. □

Corollary 3.16. $\widehat{T}_U$ *is Polish.*

Proof. This follows from Corollary 3.9, Lemmas 3.13 and 3.15, and [Kechris 1991, Theorem 5.3]. □

Proposition 3.17. $\widehat{T}$ *is Polish and locally compact.*

Proof. First, let us prove that $\widehat{T}$ is Hausdorff. Take different points $\boldsymbol{\gamma}(g, x)$ and $\boldsymbol{\gamma}(g', x')$ in $\widehat{T}$. Let O, O', P and P' be relatively compact open neighborhoods of x, x', $g(x)$ and $g(x')$, respectively. Then $U_1 = U \cup O \cup O' \cup P \cup P'$ is a relatively compact open subset of T that meets all $\mathcal{H}$-orbits. By Lemma 3.13, $\widehat{T}_{U_1}$ is a Hausdorff open subset of $\widehat{T}$ that contains $\boldsymbol{\gamma}(g, x)$ and $\boldsymbol{\gamma}(g', x')$. Hence $\boldsymbol{\gamma}(g, x)$ and $\boldsymbol{\gamma}(g', x')$ can be separated in $\widehat{T}_{U_1}$ by disjoint open neighborhoods in $\widehat{T}_{U_1}$, and therefore also in $\widehat{T}$.

Second, let us show that $\widehat{T}$ is locally compact. For $\boldsymbol{\gamma}(g, x) \in \widehat{T}$, let O and P be relatively compact open neighborhoods of x and $g(x)$, respectively. Then $U_1 = U \cup O \cup P$ is a relatively compact open set of T that meets all $\mathcal{H}$-orbits. By Corollary 3.9, it follows that $\widehat{T}_{U_1}$ is a locally compact open neighborhood of $\boldsymbol{\gamma}(g, x)$ in $\widehat{T}$. Hence $\boldsymbol{\gamma}(g, x)$ has a compact neighborhood in $\widehat{T}_{U_1}$, and therefore also in $\widehat{T}$.

Finally, let us show that $\widehat{T}$ is second countable. Since T is second countable (it is Polish) and locally compact, it can be covered by countably many relatively compact open subsets $O_n \subset T$. Then each $U_{n,m} = O_n \cup O_m \cup U$ is a relatively compact open set of T that meets all $\mathcal{H}$-orbits. Hence, by Lemma 3.15, the sets $\widehat{T}_{U_{n,m}}$ are second countable and open in $\widehat{T}$. Moreover these sets form a countable cover of $\widehat{T}$ because, for any $\boldsymbol{\gamma}(g, x) \in \widehat{T}$, we have $x \in O_n$ and $g(x) \in O_m$ for some n and m, obtaining $\boldsymbol{\gamma}(g, x) \in \widehat{T}_{U_{n,m}}$. So $\widehat{T}$ is second countable.

Now the result follows by [Kechris 1991, Theorem 5.3]. □

Proposition 3.18. *The map* $\hat{\pi} : \widehat{T} \to T \times T$ *is proper.*

Proof. Take any compact $K \subset T \times T$ and any relatively compact open $U' \subset T$ meeting all $\mathcal{H}$-orbits and such that $K \subset U' \times U'$. By applying Corollary 3.11 to U', we get that $\hat{\pi}^{-1}(K)$ is compact in $\widehat{T}_{U'}$, and therefore in $\widehat{T}$. □

3D. ***The space*** $\widehat{T}_0$**.** From now on, assume that $\mathcal{H}$ is minimal, and therefore $\overline{\mathcal{H}}$ has only one orbit, the whole of T. Fix a point $x_0 \in U$, and let[8]

$$\widehat{T}_0 = t^{-1}(x_0) = \{\boldsymbol{\gamma}(g,x) \in \widehat{T} \mid g(x) = x_0\}, \quad \widehat{T}_{0,U} = \widehat{T}_0 \cap \widehat{T}_U.$$

Observe that $\widehat{T}_0$ is closed in $\widehat{T}$, whereas $\widehat{T}_{0,U}$ is open in $\widehat{T}_0$. Moreover, we have $\hat{\pi}(\widehat{T}_0) = T \times \{x_0\} \equiv T$ and $\hat{\pi}(\widehat{T}_{0,U}) = U \times \{x_0\} \equiv U$ because T is the unique $\overline{\mathcal{H}}$-orbit; indeed, $\hat{\pi}(\boldsymbol{\gamma}(h,x)) = x$ for each $x \in T$ and any $h \in \overline{S}$ with $x \in \operatorname{dom} h$ and $h(x) = x_0$. Let $\hat{\pi}_0 := s : \widehat{T}_0 \to T$, which is continuous and surjective.

The following two corollaries are direct consequences of Proposition 3.17 (see [Kechris 1991, Theorem 3.11]) and Corollary 3.12.

Corollary 3.19. *$\widehat{T}_0$ is Polish and locally compact.*

Corollary 3.20. *The closure of $\widehat{T}_{0,U}$ in $\widehat{T}_0$ is compact.*

The following corollary is a direct consequence of Proposition 3.18 because $\hat{\pi}_0 : \widehat{T}_0 \to T$ can be identified with the restriction $\hat{\pi} : \widehat{T}_0 \to T \times \{x_0\} \equiv T$.

Corollary 3.21. *The map $\hat{\pi}_0 : \widehat{T}_0 \to T$ is proper.*

Proposition 3.22. *The fibers of $\hat{\pi}_0 : \widehat{T}_0 \to T$ are homeomorphic to each other.*

Proof. For each $x \in T$, there is some $f \in \overline{S}$ with $f(x) = x_0$. Then the mapping $\boldsymbol{\gamma}(g,x) \mapsto \boldsymbol{\gamma}(gf^{-1}, x_0)$ defines a homeomorphism $\hat{\pi}_0^{-1}(x) \to \hat{\pi}_0^{-1}(x_0)$ whose inverse is given by $\boldsymbol{\gamma}(g_0,x_0) \mapsto \boldsymbol{\gamma}(g_0 f, x)$. □

Question 3.23. When is $\hat{\pi}_0$ a fiber bundle?

3E. ***The pseudogroup*** $\widehat{\mathcal{H}}_0$**.** For $h \in S$, define

$$\hat{h} : \hat{\pi}_0^{-1}(\operatorname{dom} h) \to \hat{\pi}_0^{-1}(\operatorname{im} h), \quad \hat{h}(\boldsymbol{\gamma}(g,x)) = \boldsymbol{\gamma}(gh^{-1}, h(x)),$$

for $g \in S$ and $x \in \operatorname{dom} g \cap \operatorname{dom} h$ with $g(x) = x_0$. The following two results are elementary.

Lemma 3.24. *For any $h \in S$, we have $\hat{\pi}_0(\operatorname{dom} \hat{h}) = \operatorname{dom} h$ and $\hat{\pi}_0(\operatorname{im} \hat{h}) = \operatorname{im} h$, and the following diagram is commutative*:

$$\begin{array}{ccc} \operatorname{dom} \hat{h} & \xrightarrow{\hat{h}} & \operatorname{im} \hat{h} \\ {\scriptstyle \hat{\pi}_0}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\pi}_0} \\ \operatorname{dom} h & \xrightarrow{h} & \operatorname{im} h \end{array}$$

Lemma 3.25. *If $O \subset T$ is open with $\operatorname{id}_O \in S$, then $\widehat{\operatorname{id}_O} = \operatorname{id}_{\hat{\pi}_0^{-1}(O)}$.*

Lemma 3.26. *For $h, h' \in S$, we have $\widehat{h'h} = \widehat{h'}\hat{h}$.*

[8]The definition $\widehat{T}_0 = s^{-1}(x_0)$ would be valid too, of course, but it seems that the proofs in Sections 3D and 3E have a simpler notation with the choice $\widehat{T}_0 = t^{-1}(x_0)$.

Proof. By Lemma 3.24, we have

$$\begin{aligned}\operatorname{dom}(\widehat{h'}\hat{h}) &= \hat{h}^{-1}(\operatorname{dom}\widehat{h'} \cap \operatorname{im}\hat{h}) = \hat{h}^{-1}(\hat{\pi}_0^{-1}(\operatorname{dom}h' \cap \operatorname{im}h)) \\ &= \hat{\pi}_0^{-1}(h^{-1}(\operatorname{dom}h' \cap \operatorname{im}h)) = \hat{\pi}_0^{-1}(\operatorname{dom}(h'h)) = \operatorname{dom}\widehat{h'h}.\end{aligned}$$

Now let $\boldsymbol{\gamma}(g,x) \in \operatorname{dom}(\widehat{h'}\hat{h}) = \operatorname{dom}\widehat{h'h}$; therefore $g \in \overline{S}$, $x \in \operatorname{dom}g \cap \operatorname{dom}h$, $h(x) \in \operatorname{dom}h'$ and $g(x) = x_0$. Then

$$\begin{aligned}\widehat{h'h}(\boldsymbol{\gamma}(g,x)) &= \boldsymbol{\gamma}(g(h'h)^{-1}, h'h(x)) = \boldsymbol{\gamma}(gh^{-1}(h')^{-1}, h'h(x)) \\ &= \widehat{h'}\big(\boldsymbol{\gamma}(gh^{-1}, h(x))\big) = \widehat{h'}\hat{h}(\boldsymbol{\gamma}(g,x)).\end{aligned}$$ □

The following is a direct consequence of Lemmas 3.25 and 3.26.

Corollary 3.27. *For $h \in S$, the map $\hat{h}$ is bijective with $\hat{h}^{-1} = \widehat{h^{-1}}$.*

Lemma 3.28. *The map $\hat{h}$ is a homeomorphism for all $h \in S$.*

Proof. By Corollary 3.27, it is enough to prove that $\hat{h}$ is continuous, which holds because it can be expressed as the composition of continuous maps

$$\begin{aligned}\hat{\pi}_0^{-1}(\operatorname{dom}h) &\xrightarrow{(\mathrm{id},\mathrm{const},h\hat{\pi}_0)} \hat{\pi}_0^{-1}(\operatorname{dom}h) \times \{h^{-1}\} \times \operatorname{im}h \\ &\xrightarrow{\mathrm{id}\times\boldsymbol{\gamma}} \hat{\pi}_0^{-1}(\operatorname{dom}h) \times \boldsymbol{\gamma}(\{h^{-1}\} \times \operatorname{im}h) \\ &\xrightarrow{\text{product}} \hat{\pi}_0^{-1}(\operatorname{im}h).\end{aligned}$$

This can be checked on elements:

$$\begin{aligned}\boldsymbol{\gamma}(g,x) &\mapsto (\boldsymbol{\gamma}(g,x), h^{-1}, h(x)) \\ &\mapsto \big(\boldsymbol{\gamma}(g,x), \boldsymbol{\gamma}(h^{-1}, h(x))\big) \\ &\mapsto \boldsymbol{\gamma}(gh^{-1}, h(x)) = \hat{h}(\boldsymbol{\gamma}(g,x)).\end{aligned}$$ □

Set $\widehat{S}_0 = \{\hat{h} \mid h \in S\}$, and let $\widehat{\mathcal{H}}_0$ be the pseudogroup on $\widehat{T}_0$ generated by $\widehat{S}_0$. Lemmas 3.26 and 3.28 and Corollary 3.27 give the following.

Corollary 3.29. *$\widehat{S}_0$ is a pseudo∗group on $\widehat{T}_0$.*

Lemma 3.30. *$\widehat{T}_{0,U}$ meets all orbits of $\widehat{\mathcal{H}}_0$.*

Proof. Let $\boldsymbol{\gamma}(g,x) \in \widehat{T}_0$ with $g \in \overline{S}$; then $x \in \operatorname{dom}g$ and $g(x) = x_0$. Since U meets all orbits of $\mathcal{H}$, there is some $h \in S$ such that $x \in \operatorname{dom}h$ and $h(x) \in U$. Then $\boldsymbol{\gamma}(g,x) \in \operatorname{dom}\hat{h}$ and $\hat{h}(\boldsymbol{\gamma}(g,x)) = \boldsymbol{\gamma}(gh^{-1}, h(x))$ satisfies

$$\hat{\pi}_0\big(\hat{h}(\boldsymbol{\gamma}(g,x))\big) = \hat{\pi}_0\big(\boldsymbol{\gamma}(gh^{-1}, h(x))\big) = h(x) \in U.$$

Hence $\hat{h}(\boldsymbol{\gamma}(g,x)) \in \widehat{T}_{0,U}$ as desired. □

Lemma 3.31. *The map $S_{\text{c-o}} \to \widehat{S}_{0,\text{c-o}}$, $h \mapsto \hat{h}$, is a homeomorphism.*

Proof. If $\widehat{h_1} = \widehat{h_2}$ for some $h_1, h_2 \in S$, then $h_1 = h_2$ by Lemma 3.24. So the stated map is injective, and therefore it is bijective by the definition of $\widehat{S}_0$.

Take a subbasic open set of $S_{\text{c-o}}$, which is of the form $S \cap \mathcal{N}(K, O)$ for some compact K and open O in T. The set $\hat{\pi}_0^{-1}(K)$ is compact by Corollary 3.21, and $\hat{\pi}_0^{-1}(O)$ is open. Then the map of the statement is open because

$$\{\hat{h} \mid h \in \mathcal{N}(K, O) \cap S\} = \widehat{\mathcal{N}}\big(\hat{\pi}_0^{-1}(K), \hat{\pi}_0^{-1}(O)\big) \cap \widehat{S}_0$$

by Lemma 3.24, which is open in $\widehat{S}_{0,\text{c-o}}$.

To prove its continuity, let us first show that its restriction to $S_U = S \cap \mathcal{H}|_U$ is continuous. Fix $h_0 \in S_U$, and take relatively compact open subsets

$$V, V_0, W, V', V_0', W' \subset U,$$

and indices k and k' such that

$$\overline{V_0} \subset V, \quad \overline{V} \subset T_{i_k} \cap \operatorname{dom} h_0, \tag{6}$$

$$\overline{V_0'} \subset V', \quad \overline{V'} \subset T_{i_{k'}} \cap \operatorname{im} h_0, \tag{7}$$

$$\overline{W} \subset W', \quad \overline{W'} \subset T_{i_{k_0}}, \tag{8}$$

$$h_0^{-1}(\overline{V'}) \subset V, \tag{9}$$

$$h_0(\overline{V_0}) \subset V'. \tag{10}$$

Let $\overline{S}_0$ and $\overline{S}_1$ (respectively, $\overline{S}_0'$ and $\overline{S}_1'$) be defined like in (3) and (4), by using V and W (respectively, V' and W'). Then $\widehat{K} = \boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0})$ is compact in $\widehat{T}$ by Lemma 3.7, and $\widehat{O} = \boldsymbol{\gamma}(\overline{S}_0' \times V')$ is open in $\widehat{T}$ by Lemma 3.8 and Remark 26. Thus $\widehat{K}_0 = \widehat{K} \cap \widehat{T}_0$ is compact and $\widehat{O}_0 = \widehat{O} \cap \widehat{T}_0$ is open in $\widehat{T}_0$. So $\widehat{\mathcal{N}}(\widehat{K}_0, \widehat{O}_0) \cap \widehat{S}_0$ is a subbasic open set of $\widehat{S}_{0,\text{c-o}}$.

Claim 1. $\widehat{h_0} \in \widehat{\mathcal{N}}(\widehat{K}_0, \widehat{O}_0)$.

Let $\boldsymbol{\gamma}(g, x) \in \widehat{K}_0$; thus $g \in \overline{S}_1$, $x \in \overline{V}_0 \cap \operatorname{dom} g$ and $g(x) = x_0$. The condition $g \in \overline{S}_1$ means that $g \in \overline{S}$, $\overline{V} \subset \operatorname{dom} g$ and $g(\overline{V}) \subset \overline{W}$. By (7)–(9), it follows that $\overline{V'} \subset \operatorname{dom} g h_0^{-1}$ and

$$g h_0^{-1}(\overline{V'}) \subset g(\overline{V}) \subset \overline{W} \subset W'.$$

Hence $g h_0^{-1} \in \overline{S}_0'$, obtaining that

$$\widehat{h_0}(\boldsymbol{\gamma}(g, x)) = \boldsymbol{\gamma}(g h_0^{-1}, h_0(x)) \in \widehat{O},$$

which completes the proof of Claim 1.

Claim 2. *The sets* $\widehat{\mathcal{N}}(\widehat{K}_0, \widehat{O}_0) \cap \widehat{S}_0$*, constructed as above, form a local subbasis of* $\widehat{S}_{0,\text{c-o}}$ *at* $\widehat{h_0}$.

This assertion follows by Claim 1 and because the sets of the type $\widehat{O}_0$ form a basis of the topology of $\operatorname{im}\widehat{h_0}$, and any compact subset of $\operatorname{dom}\widehat{h_0}$ is contained in a finite union of sets of the type of $\widehat{K}_0$.

The sets

$$\mathcal{N} = \mathcal{N}(\overline{V_0}, V') \cap \mathcal{N}(\overline{V'}, V)^{-1} \cap S_U$$

are open neighborhoods of h_0 by (9), (10), and Propositions 2.6 and 3.1.

Claim 3. *We have $\hat{h} \in \widehat{\mathcal{N}}(\widehat{K}_0, \widehat{O}_0)$ for all $h \in \mathcal{N}$.*

Given $h \in \mathcal{N}$, we have $\overline{V'} \subset \operatorname{im} h$ and $h^{-1}(\overline{V'}) \subset V$. Let $\boldsymbol{\gamma}(g, x) \in \widehat{K}_0$; thus $x \in \overline{V_0} \cap \operatorname{dom} g$, $g(x) = x_0$, and we can assume that $g \in \overline{S}_1$, which means that $g \in \overline{S}$, $\overline{V} \subset \operatorname{dom} g$ and $g(\overline{V}) \subset \overline{W}$. Then $\overline{V'} \subset \operatorname{dom}(gh^{-1})$, $gh^{-1}(\overline{V'}) \subset \overline{W} \subset W'$ and $h(x) \in h(\overline{V_0}) \subset V'$. Therefore

$$\hat{h}(\boldsymbol{\gamma}(g, x)) = \boldsymbol{\gamma}(gh^{-1}, h(x)) \in \boldsymbol{\gamma}(\overline{S}'_0 \times V') \cap \widehat{T}_0 = \widehat{O}_0,$$

proving Claim 3.

Claims 2 and 3 show that the map $S_{U,\text{c-o}} \to \widehat{S}_{0,\text{c-o}}$, $h \mapsto \hat{h}$, is continuous at h_0. Now, let us prove that the whole map $S_{\text{c-o}} \to \widehat{S}_{0,\text{c-o}}$, $h \mapsto \hat{h}$, is continuous. Since the sets $\mathcal{N}(\widehat{K}, \widehat{O}) \cap \widehat{S}_0$, for small enough compact subsets $\widehat{K} \subset \widehat{T}_0$ and small enough open subsets $\widehat{O} \subset \widehat{T}_0$, form a subbasis of $\widehat{S}_{0,\text{c-o}}$, it is enough to prove that the inverse image of these subbasic sets are open in $S_{\text{c-o}}$. We can assume that $\widehat{K}, \widehat{O} \subset \hat{\pi}_0^{-1}(U')$ for some relatively compact open subset $U' \subset T$ that meets all $\mathcal{H}$-orbits. Consider the inclusion map $\iota : U' \hookrightarrow T$, and the paro map $\phi : T \rightarrowtail U'$ with $\operatorname{dom}\phi = U'$, where it is the identity map. According to Proposition 2.3, we get a continuous map $\phi_*\iota^* : \operatorname{Paro}_{\text{c-o}}(T, T) \to \operatorname{Paro}_{\text{c-o}}(U', U')$, which restricts to a continuous map $\phi_*\iota^* : S_{\text{c-o}} \to S_{U',\text{c-o}}$. Observe that $\phi_*\iota^*(h)$ is the restriction $h : U' \cap h^{-1}(U') \to h(U') \cap U'$ for each $h \in S$. Hence, since $\widehat{K}, \widehat{O} \subset \hat{\pi}_0^{-1}(U')$, it follows from Lemma 3.24 that $\mathcal{N}(\widehat{K}, \widehat{O}) \cap \widehat{S}_0$ has the same inverse image by the map $S_{\text{c-o}} \to \widehat{S}_{0,\text{c-o}}$, $h \mapsto \hat{h}$, and by the composition

$$S_{\text{c-o}} \xrightarrow{\phi_*\iota^*} S_{U',\text{c-o}} \longrightarrow \widehat{S}_{0,\text{c-o}},$$

where the second map is given by $h \mapsto \hat{h}$. This composition is continuous by the above case applied to U', and therefore the inverse image of $\mathcal{N}(\widehat{K}, \widehat{O}) \cap \widehat{S}_0$ by $S_{\text{c-o}} \to \widehat{S}_{0,\text{c-o}}$, $h \mapsto \hat{h}$, is open in $S_{\text{c-o}}$. □

Since the compact generation of $\mathcal{H}$ is satisfied with the relatively compact open set U, there is a symmetric finite set $\{f_1, \dots, f_m\}$ generating $\mathcal{H}|_U$, which can be chosen in S, such that each f_a has an extension $\tilde{f}_a$ with $\overline{\operatorname{dom} f_a} \subset \operatorname{dom} \tilde{f}_a$. We can also assume that $\tilde{f}_a \in S$. Let $\widehat{\mathcal{H}}_{0,U} = \widehat{\mathcal{H}}|_{\widehat{T}_{0,U}}$. Obviously, each $\widehat{\tilde{f}_a}$ is an extension of $\widehat{f_a}$. Moreover,

$$\overline{\operatorname{dom} \widehat{f_a}} = \overline{\hat{\pi}_0^{-1}(\operatorname{dom} f_a)} \subset \hat{\pi}_0^{-1}(\overline{\operatorname{dom} f_a}) \subset \hat{\pi}_0^{-1}(\operatorname{dom} \tilde{f}_a) = \operatorname{dom} \widehat{\tilde{f}_a}.$$

Lemma 3.32. *The maps $\widehat{f}_a$ ($a \in \{1, \dots, m\}$) generate $\widehat{\mathscr{H}}_{0,U}$.*

Proof. $\widehat{\mathscr{H}}_{0,U}$ is generated by the maps of the form $\hat{h}$ with $h \in S_U$, and any such $\hat{h}$ can be written as a composition of maps $\widehat{f}_a$ around any $\boldsymbol{\gamma}(g, x) \in \operatorname{dom} \hat{h} = \hat{\pi}_0^{-1}(\operatorname{dom} h)$ by Lemma 3.26. □

Corollary 3.33. *$\widehat{\mathscr{H}}_0$ is compactly generated.*

Proof. We saw that $\widehat{T}_{0,U}$ is relatively compact in $\widehat{T}_0$ (Corollary 3.20) and meets all $\widehat{\mathscr{H}}_0$-orbits (Lemma 3.30), the maps $\widehat{f}_a$ generate $\widehat{\mathscr{H}}_{0,U}$ (Lemma 3.32), and each $\widehat{\tilde{f}}_a$ is an extension of each $\widehat{f}_a$ with $\overline{\operatorname{dom} \widehat{f}_a} \subset \operatorname{dom} \widehat{\tilde{f}}_a$. □

Recall that the sets T_{i_k} form a finite covering of $\overline{U}$ by open sets of T. Fix some index k_0 such that $x_0 \in T_{i_{k_0}}$. Let $\{W_k\}$ be a shrinking of $\{T_{i_k}\}$ as cover of $\overline{U}$ by open subsets of T; i.e., $\{W_k\}$ is a cover of $\overline{U}$ by open subsets of T and $\overline{W_k} \subset T_{i_k}$ for all k. By applying Proposition 2.32 several times, we get finite covers, $\{V_a\}$ and $\{V'_u\}$, of $\overline{U}$ by open subsets of T, and shrinkings, $\{W_{0,k}\}$ of $\{W_k\}$ and $\{V_{0,a}\}$ of $\{V_a\}$, as covers of $\overline{U}$ by open subsets of T, such that the following properties hold:

- For all $h \in \mathscr{H}$ and $x \in \operatorname{dom} h \cap U \cap V_a \cap W_{0,k}$ with $h(x) \in U \cap W_{0,l}$, there is some $\tilde{h} \in S$ such that
$$\overline{V_a} \subset \operatorname{dom} \tilde{h} \cap W_k, \quad \boldsymbol{\gamma}(\tilde{h}, x) = \boldsymbol{\gamma}(h, x), \quad \tilde{h}(\overline{V_a}) \subset W_l.$$
- For all $h \in \mathscr{H}$ and $x \in \operatorname{dom} h \cap U \cap V'_u \cap V_{0,a}$ with $h(x) \in U \cap V_{0,b}$, there is some $\tilde{h} \in S$ such that
$$\overline{V'_u} \subset \operatorname{dom} \tilde{h} \cap V_a, \quad \boldsymbol{\gamma}(\tilde{h}, x) = \boldsymbol{\gamma}(h, x), \quad \tilde{h}(\overline{V'_u}) \subset V_b.$$

By the definition of $\overline{\mathscr{H}}$ and $\overline{S}$, it follows that these properties also hold for all $h \in \overline{\mathscr{H}}$ with $\tilde{h} \in \overline{S}$. Let $\{V'_{0,u}\}$ be a shrinking of $\{V'_u\}$ as a cover of $\overline{U}$ by open subsets of T. We have $x_0 \in W_{0,k_0} \cap V_{0,a_0} \cap V'_{0,u_0}$ for some indices k_0, a_0 and u_0. For each a, let $\overline{S}_{0,a}, \overline{S}_{1,a} \subset \overline{S}$ be defined like $\overline{S}_0$ and $\overline{S}_1$ in (3) and (4) by using V_a and W_{k_0} instead of V and W. Take an index u such that $\overline{V'_u} \subset V_a$. The sets $V_{0,a} \cap V'_{0,u}$, defined in this way, form a cover of $\overline{U}$, so that the sets $\widehat{T}_{a,u} = \boldsymbol{\gamma}(\overline{S}_{0,a} \times (V_{0,a} \cap V'_{0,u}))$ form a cover of $\overline{\widehat{T}_U}$ by open subsets of $\widehat{T}$ (Lemma 3.8), and thus the sets $\widehat{T}_{0,a,u} = \widehat{T}_{a,u} \cap \widehat{T}_0$ form a cover of $\overline{\widehat{T}_{0,U}}$ by open subsets of $\widehat{T}_0$. Let $\widehat{T}_{0,U,a,u} = \widehat{T}_{0,U} \cap \widehat{T}_{a,u}$. Like in Section 3C, let $\bar{\boldsymbol{\gamma}}$ denote the germ map defined on $C(\overline{V_a}, \overline{W_{k_0}}) \times \overline{V_a}$, and let $\mathscr{R}_a : \overline{S}_{1,a} \to C(\overline{V_a}, \overline{W_{k_0}})$ be the restriction map $f \mapsto f|_{\overline{V_a}}$. Then

$$\bar{\boldsymbol{\gamma}} : \mathscr{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}} \to \bar{\boldsymbol{\gamma}}\big(\mathscr{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}\big) \tag{11}$$

is a homeomorphism by Corollary 3.14. Since $\overline{V_a}$ is compact, the compact-open topology on $\mathscr{R}_a(\overline{S}_{1,a})$ equals the topology induced by the supremum metric d_a

on $C(\overline{V_a}, \overline{W_{k_0}})$, defined with the metric $d_{i_{k_0}}$ on $T_{i_{k_0}}$. Take some index k such that $\overline{V_a} \subset W_k$. Then the topology of

$$\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}$$

is induced by the metric $d_{a,u,k}$ given by

$$d_{a,u,k}((g, y), (g', y')) = d_{i_k}(y, y') + d_a(g, g')$$

(recall that $\overline{W_k} \subset T_{i_k}$). Let $\hat{d}_{a,u,k}$ be the metric on $\bar{\boldsymbol{\gamma}}(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}})$ that corresponds to $d_{a,u,k}$ by the homeomorphism (11); it induces the topology of $\bar{\boldsymbol{\gamma}}(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}})$. By the commutativity of (5),

$$\bar{\boldsymbol{\gamma}}\big(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}\big) = \boldsymbol{\gamma}\big(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}\big),$$

which is contained in $\widehat{T}$. Then the restriction $\hat{d}_{0,a,u,k}$ of $\hat{d}_{a,u,k}$ to

$$\bar{\boldsymbol{\gamma}}\big(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}\big) \cap \widehat{T}_0$$

induces the topology of this space. Moreover, by the proof of Corollary 3.9, we get

$$\widehat{T}_{a,u} \subset \bar{\boldsymbol{\gamma}}\big(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}\big),$$

and therefore

$$\widehat{T}_{0,a,u} \subset \bar{\boldsymbol{\gamma}}\big(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}\big) \cap \widehat{T}_0.$$

For any index v, define $\overline{S}'_{0,v}$ and $\overline{S}'_{1,v}$ like $\overline{S}_0$ and $\overline{S}_1$ in (3) and (4) by using V'_v and W_{k_0} instead of V and W. Let $\mathcal{R}'_v : \overline{S}'_{1,v} \to C(\overline{V'_v}, \overline{W_{k_0}})$ denote the restriction map. Again, the compact-open topology on $\mathcal{R}'_v(\overline{S}'_{1,v})$ equals the topology induced by the supremum metric d'_v on $C(\overline{V'_v}, \overline{W_{k_0}})$, defined with the metric $d_{i_{k_0}}$ on $T_{i_{k_0}}$ (recall that $\overline{W_{k_0}} \subset T_{i_{k_0}}$). Take indices b and l such that $\overline{V'_v} \subset V_b$ and $\overline{V_b} \subset W_l$. Then we can consider the restriction map

$$\mathcal{R}^v_b : C(\overline{V_b}, \overline{W_{k_0}}) \to C(\overline{V'_v}, \overline{W_{k_0}}).$$

Its restriction $\mathcal{R}^v_b : \mathcal{R}_b(\overline{S}_{1,b}) \to \mathcal{R}'_v(\overline{S}'_{1,v})$ is injective by Remark 25, and surjective by Remark 24. So $\mathcal{R}^v_b : \mathcal{R}_b(\overline{S}_{1,b}) \to \mathcal{R}'_v(\overline{S}'_{1,v})$ is a continuous bijection between compact Hausdorff spaces, giving that it is a homeomorphism. Then, by compactness, it is a uniform homeomorphism with respect to the supremum metrics d_b and d'_v. Since b and v run in finite families of indices, there is a mapping $\epsilon \mapsto \delta_1(\epsilon) > 0$ ($\epsilon > 0$) such that

$$d'_v(\mathcal{R}^v_b\mathcal{R}_b(f), \mathcal{R}^v_b\mathcal{R}_b(f')) < \delta_1(\epsilon) \Longrightarrow d_b(\mathcal{R}_b(f), \mathcal{R}_b(f')) < \epsilon \tag{12}$$

for all indices v and b, and maps $f, f' \in \overline{S}_{1,b}$.

Lemma 3.34. *$\widehat{\mathcal{H}}_{0,U}$ satisfies the equicontinuity condition with $\widehat{S}_{0,U} = \widehat{S}_0 \cap \widehat{\mathcal{H}}_{0,U}$ and the quasilocal metric represented by the family $\{\widehat{T}_{0,U,a,u}, \hat{d}_{0,a,u,k}\}$.*

Proof. Let $h \in S$, and

$$\boldsymbol{\gamma}(g, y), \boldsymbol{\gamma}(g', y') \in \widehat{T}_{0,U,a,u} \cap \hat{h}^{-1}(\widehat{T}_{0,U,b,v}),$$

where $g, g' \in \overline{S}_{0,a}$ and $y, y' \in V_{0,a} \cap V'_{0,u}$ with $g(y) = g(y') = x_0$. Take some indices k and l such that $\overline{V_a} \subset W_k$ and $\overline{V_b} \subset W_l$ (recall that $\overline{W_k} \subset T_{i_k}$ and $\overline{W_l} \subset T_{i_l}$). By Remark 24, we can assume that $\operatorname{dom} h = T_{i_k}$. Then

$$\hat{h}(\boldsymbol{\gamma}(g, y)) = \boldsymbol{\gamma}(gh^{-1}, h(y)), \quad \hat{h}(\boldsymbol{\gamma}(g', y')) = \boldsymbol{\gamma}(g'h^{-1}, h(y'))$$

both belong to $\widehat{T}_{0,U,b,v}$, which means that $h(y), h(y') \in V_{0,b} \cap V'_{0,v}$ and there are $f, f' \in \overline{S}_{0,b}$ such that

$$\boldsymbol{\gamma}(f, h(y)) = \boldsymbol{\gamma}(gh^{-1}, h(y)), \quad \boldsymbol{\gamma}(f', h(y')) = \boldsymbol{\gamma}(g'h^{-1}, h(y')). \tag{13}$$

In particular, $\overline{V_b} \subset \operatorname{dom} f \cap \operatorname{dom} f'$. In fact, we can assume $\operatorname{dom} f = \operatorname{dom} f' = T_{i_l}$ by Remark 24. Observe that the image of h may not be included in T_{i_l}, and the images of f, f', g and g' may not be included in $T_{i_{k_0}}$.

Claim 1. $\overline{V'_v} \subset \operatorname{im} h$ *and* $h^{-1}(\overline{V'_v}) \subset V_a$.

By the assumptions on $\{V'_w\}$, since

$$h(y) \in U \cap V'_v \cap V_{0,b} \cap \operatorname{dom} h^{-1}, \quad h^{-1}h(y) = y \in U \cap V'_u \cap V_{0,a},$$

there is some $\widetilde{h^{-1}} \in S$ such that

$$\overline{V'_v} \subset \operatorname{dom} \widetilde{h^{-1}} \cap V_b, \quad \widetilde{h^{-1}}(\overline{V'_v}) \subset V_a, \quad \boldsymbol{\gamma}(\widetilde{h^{-1}}, h(y)) = \boldsymbol{\gamma}(h^{-1}, h(y));$$

indeed, we can suppose that $\operatorname{dom} \widetilde{h^{-1}} = T_{i_{k_0}}$ by Remark 24. Then

$$\widetilde{h^{-1}}(\overline{V'_v}) \subset V_a \subset T_{i_k} = \operatorname{dom} h,$$

obtaining $\overline{V'_v} \subset \operatorname{dom}(h\widetilde{h^{-1}})$. Moreover

$$\boldsymbol{\gamma}(h\widetilde{h^{-1}}, h(y)) = \boldsymbol{\gamma}(\mathrm{id}_T, h(y)).$$

Therefore $h\widetilde{h^{-1}} = \mathrm{id}_{\operatorname{dom}(h\widetilde{h^{-1}})}$ because $h\widetilde{h^{-1}} \in S$ since $h, \widetilde{h^{-1}} \in S$. So $h\widetilde{h^{-1}} = \mathrm{id}_T$ on some neighborhood of $\overline{V'_v}$, and therefore $\overline{V'_v} \subset \operatorname{im} h$ and $h^{-1} = \widetilde{h^{-1}}$ on $\overline{V'_v}$. Thus $h^{-1}(\overline{V'_v}) = \widetilde{h^{-1}}(\overline{V'_v}) \subset V_a$, which shows Claim 1.

By Claim 1 and since $\overline{V_a} \subset \operatorname{dom} g \cap \operatorname{dom} g'$ because $g, g' \in \overline{S}_{0,a}$, we get

$$\overline{V'_v} \subset \operatorname{dom}(gh^{-1}) \cap \operatorname{dom}(g'h^{-1}). \tag{14}$$

Since $f, f' \in \overline{S}_{0,b}$, we have $\overline{V_b} \subset \operatorname{dom} f \cap \operatorname{dom} f'$ and $f(\overline{V_b}) \cup f'(\overline{V_b}) \subset W_{k_0}$. On the other hand, it follows from (13) that $fh(y) = f'h(y') = x_0$ and

$$\boldsymbol{\gamma}(gh^{-1}f^{-1}, x_0) = \boldsymbol{\gamma}(g'h^{-1}f'^{-1}, x_0) = \boldsymbol{\gamma}(\mathrm{id}_T, x_0).$$

Moreover,

$$f(\overline{V_v'}) \subset \operatorname{dom}(gh^{-1}f^{-1}), \quad f'(\overline{V_v'}) \subset \operatorname{dom}(g'h^{-1}f'^{-1})$$

by (14). So, by Remark 25, $gh^{-1}f^{-1} = \mathrm{id}_T$ on some neighborhood of $f(\overline{V_v'})$, and $g'h^{-1}f'^{-1} = \mathrm{id}_T$ on some neighborhood of $f'(\overline{V_v'})$. Thus $gh^{-1} = f$ and $g'h^{-1} = f'$ on some neighborhood of $\overline{V_v'}$; in particular,

$$\mathcal{R}_b^v \mathcal{R}_b(f) = gh^{-1}|_{\overline{V_v'}}, \quad \mathcal{R}_b^v \mathcal{R}_b(f') = g'h^{-1}|_{\overline{V_v'}}.$$

Consider the mappings $\epsilon \mapsto \delta(\epsilon) > 0$ and $\epsilon \mapsto \delta_1(\epsilon) > 0$ satisfying Remark 22 and (12). Then, for each $\epsilon > 0$, define

$$\hat{\delta}(\epsilon) = \min\{\delta(\epsilon/2), \delta_1(\epsilon/2)\}.$$

Given any $\epsilon > 0$, suppose that

$$\hat{d}_{0,a,u,k}(\boldsymbol{\gamma}(g, y), \boldsymbol{\gamma}(g', y')) < \hat{\delta}(\epsilon).$$

This means that

$$d_{a,u,k}((\mathcal{R}_a(g), y), (\mathcal{R}_a(g'), y')) < \hat{\delta}(\epsilon),$$

or, equivalently,

$$d_{i_k}(y, y') + \sup_{x \in \overline{V_a}} d_{i_{k_0}}(g(x), g'(x)) < \hat{\delta}(\epsilon).$$

Therefore

$$d_{i_k}(y, y') < \delta(\epsilon/2), \tag{15}$$

$$\sup_{x \in \overline{V_a}} d_{i_{k_0}}(g(x), g'(x)) < \delta_1(\epsilon/2). \tag{16}$$

From (15) and Remark 22, it follows that

$$d_{i_l}(h(y), h(y')) < \epsilon/2 \tag{17}$$

since $h \in S \subset \overline{S}$ and $y, y' \in T_{i_k} \cap h^{-1}(T_{i_l} \cap \operatorname{im} h)$. On the other hand, by Claim 1 and (16), we get

$$\begin{aligned} d_v'\big(\mathcal{R}_b^v \mathcal{R}_b(f), \mathcal{R}_b^v \mathcal{R}_b(f')\big) &= \sup_{z \in \overline{V_v'}} d_{i_{k_0}}(gh^{-1}(z), g'h^{-1}(z)) = \sup_{x \in h^{-1}(\overline{V_v'})} d_{i_{k_0}}(g(x), g'(x)) \\ &\le \sup_{x \in \overline{V_a}} d_{i_{k_0}}(g(x), g'(x)) = d_a(\mathcal{R}_a(g), \mathcal{R}_a(g')) < \delta_1(\epsilon/2). \end{aligned}$$

So, by (12),

$$d_b(\mathcal{R}_b(f), \mathcal{R}_b(f')) < \epsilon/2. \tag{18}$$

From (17) and (18), we get

$$\begin{aligned}\hat{d}_{0,b,v,l}\big(\hat{h}(\boldsymbol{\gamma}(g,y)),\hat{h}(\boldsymbol{\gamma}(g',y'))\big) &= \hat{d}_{0,b,v,l}\big(\boldsymbol{\gamma}(f,h(y)),\boldsymbol{\gamma}(f',h(y'))\big)\\ &= d_{b,v,l}\big((\mathcal{R}_b(f),h(y)),(\mathcal{R}_b(f'),h(y'))\big)\\ &= d_{i_l}(h(y),h(y'))+d_b(\mathcal{R}_b(f),\mathcal{R}_b(f'))<\epsilon. \quad\square\end{aligned}$$

Corollary 3.35. *$\widehat{\mathcal{H}}_0$ is equicontinuous.*

Proof. $\widehat{\mathcal{H}}_0$ is equivalent to $\widehat{\mathcal{H}}_{0,U}$ by Lemma 3.30. Thus the result follows from Lemma 3.34 because equicontinuity is preserved by equivalences. □

Lemma 3.36. *$\widehat{\mathcal{H}}_0$ is minimal.*

Proof. By Lemma 3.30, it is enough to prove that $\widehat{\mathcal{H}}_{0,U}$ is minimal. Let the germs $\boldsymbol{\gamma}(g,y),\boldsymbol{\gamma}(g',y')$ be in $\widehat{T}_{0,U}$ with $g,g'\in\overline{S}$, $y\in\operatorname{dom}g\cap U$, $y'\in\operatorname{dom}g'\cap U$ and $g(y)=g'(y')=x_0$. Take indices k and k' such that $y\in T_{i_k}$ and $y'\in T_{i_{k'}}$. We can assume that $\operatorname{dom}g=T_{i_k}$ and $\operatorname{dom}g'=T_{i_{k'}}$ by Remark 24.

Let $f=g^{-1}g'\in\overline{S}$. We have $y'\in\operatorname{dom}f$ and $f(y')=y$. By Remark 24, there exists $\tilde{f}\in\overline{S}$ with $\operatorname{dom}\tilde{f}=T_{i_{k'}}$ and $\boldsymbol{\gamma}(\tilde{f},y')=\boldsymbol{\gamma}(f,y')$. By the definition of $\overline{S}$, there is a sequence f_n in S with $\operatorname{dom}f_n=T_{i_{k'}}$ and $f_n\to f$ in $C_{\text{c-o}}(T_{i_k},T)$ as $n\to\infty$; in particular, $f_n(y')\to f(y')=y$. So we can assume that $f_n(y')\in T_{i_k}$ for all n.

Take some relatively compact open neighborhood V of y' such that

$$\overline{V}\subset\operatorname{dom}(g\tilde{f})\cap\operatorname{dom}(gf)$$

and $\tilde{f}=f$ in some neighborhood of $\overline{V}$. Since $f_n\to\tilde{f}$ in $\overline{S}_{\text{c-o}}$ as $n\to\infty$, we get $gf_n\to g\tilde{f}$ and $f_n^{-1}\to\tilde{f}^{-1}$ by Propositions 2.6 and 3.1. So $\overline{V}\subset\operatorname{dom}(gf_n)$ and $y\in\operatorname{dom}f_n^{-1}=\operatorname{im}f_n$ for n large enough, and $f_n^{-1}(y)\to\tilde{f}^{-1}(y)=y'$. Moreover $gf_n|_V\to g\tilde{f}|_V=gf|_V=g'|_V$ in $C_{\text{c-o}}(V,T)$. So $\boldsymbol{\gamma}(gf_n,f_n^{-1}(y))\to\boldsymbol{\gamma}(g',y')$ in $\widehat{T}_{0,U}$ by Proposition 2.2 and the definition of the topology of $\widehat{T}$. Thus, with $h_n=f_n^{-1}\in S$, we get

$$\widehat{h_n}(\boldsymbol{\gamma}(g,y))=\boldsymbol{\gamma}(gh_n^{-1},h_n(y))=\boldsymbol{\gamma}(gf_n,f_n^{-1}(y))\to\boldsymbol{\gamma}(g',y'),$$

and therefore $\boldsymbol{\gamma}(g',y')$ is in the closure of the $\widehat{\mathcal{H}}_{0,U}$-orbit of $\boldsymbol{\gamma}(g,y)$. □

Remark 27. By Lemma 3.24, the map $\hat{\pi}_0:\widehat{T}_0\to T$ generates a morphism of pseudogroups $\widehat{\mathcal{H}}_0\to\mathcal{H}$ in the sense of [Álvarez and Masa 2008] — this morphism is not étale.

The following result is elementary.

Proposition 3.37. *In Example 2.37, if $\mathcal{H}$ is compactly generated and $\overline{\mathcal{H}}$ is strongly quasianalytic, then $\widehat{\mathcal{H}}_0$ is equivalent to the pseudogroup generated by the local action of Γ on G by local left translations, so that $\hat{\pi}_0:\widehat{T}_0\to T$ corresponds to the projection $T:V^2\to G/(K,V)$.*

Corollary 3.38. *The map $\hat{\pi}_0 : \widehat{T}_0 \to T$ is open.*

Proof. This follows from Theorem 2.38 and Proposition 3.37 since, in Example 2.37, the projection $T : V^2 \to G/(K, V)$ is open. □

3F. *The closure of $\widehat{\mathcal{H}}_0$.* Let $\widehat{\overline{\mathcal{H}}}_0$ be the pseudogroup on $\widehat{T}_0$ defined like $\widehat{\mathcal{H}}$ by taking the maps h in $\overline{S}$ instead of S; thus it is generated by $\widehat{\overline{S}}_0 = \{\hat{h} \mid h \in \overline{S}\}$. Observe that $\widehat{\overline{\mathcal{H}}}_0$, $\overline{S}$ and $\widehat{\overline{S}}_0$ satisfy the obvious versions of Lemmas 3.24–3.26, 3.28, 3.30 and 3.31, and Corollaries 3.27 and 3.29 (Section 3E). In particular, $\widehat{\overline{S}}_0$ is a pseudo$*$group, and $\widehat{T}_{0,U}$ meets all the orbits of $\widehat{\overline{\mathcal{H}}}_0$. The restriction of $\widehat{\overline{\mathcal{H}}}_0$ to $\widehat{T}_{0,U}$ will be denoted by $\widehat{\overline{\mathcal{H}}}_{0,U}$.

Lemma 3.39. $\overline{\widehat{\mathcal{H}}_0} = \widehat{\overline{\mathcal{H}}}_0$.

Proof. By the version of Lemma 3.31 for $\overline{S}$ and $\widehat{\overline{S}}_0$, the set $\widehat{S}_0$ is dense in $\widehat{\overline{S}}_{0,\text{c-o}}$. Then the result follows easily by Proposition 2.2 and the definition of $\overline{\widehat{\mathcal{H}}_0}$ (see Theorem 2.34 and Remark 19). □

Lemma 3.40. *$\widehat{\overline{\mathcal{H}}}_0$ is strongly locally free.*

Proof. Let $\hat{h} \in \widehat{\overline{S}}_0$ for $h \in \overline{S}$, and $\boldsymbol{\gamma}(g, x) \in \operatorname{dom} \hat{h}$ for $g \in \overline{S}$ and $x \in \operatorname{dom} g \cap \operatorname{dom} h$ with $g(x) = x_0$. Suppose that $\hat{h}(\boldsymbol{\gamma}(g, x)) = \boldsymbol{\gamma}(g, x)$. This means

$$\boldsymbol{\gamma}(gh^{-1}, h(x)) = \boldsymbol{\gamma}(g, x).$$

So $h(x) = x$ and $gh^{-1} = g$ on some neighborhood of x, and therefore $h = \mathrm{id}_T$ on some neighborhood of x. Then $h = \mathrm{id}_{\operatorname{dom} h}$ by the strong quasianalyticity condition of $\overline{\mathcal{H}}$ since $h \in \overline{S}$. Hence $\hat{h} = \mathrm{id}_{\operatorname{dom} \hat{h}}$ by Lemma 3.25. □

Proposition 3.41. *There is a locally compact Polish local group G and some dense finitely generated sub-local group $\Gamma \subset G$ such that $\widehat{\mathcal{H}}_0$ is equivalent to the pseudogroup defined by the local action of Γ on G by local left translations.*

Proof. This follows from Remark 21 (see also Theorem 2.38) since $\widehat{\mathcal{H}}_0$ is compactly generated (Corollary 3.33) and equicontinuous (Corollary 3.35), and $\overline{\widehat{\mathcal{H}}_0}$ is strongly locally free (Lemma 3.40). □

3G. *Independence of the choices involved.* First, let us prove that $\widehat{T}_0$ and $\widehat{\mathcal{H}}_0$ are independent of the choice of the point x_0 up to an equivalence generated by a homeomorphism. Let x_1 be another point of T, and let $\widehat{T}_1$, $\hat{\pi}_1$, $\widehat{S}_1$ and $\widehat{\mathcal{H}}_1$ be constructed like $\widehat{T}_0$, $\hat{\pi}_0$, $\widehat{S}_0$ and $\widehat{\mathcal{H}}_0$ by using x_1 instead of x_0. Now, for each $h \in S$, let us use the notation $\hat{h}_0 := \hat{h} \in \widehat{S}_0$, and let $\hat{h}_1 : \hat{\pi}_1^{-1}(\operatorname{dom} h) \to \hat{\pi}_1^{-1}(\operatorname{im} h)$ be the map in $\widehat{S}_1$ defined like $\hat{h}$.

Proposition 3.42. *There is a homeomorphism $\theta : \widehat{T}_0 \to \widehat{T}_1$ that generates an equivalence $\Theta : \widehat{\mathcal{H}}_0 \to \widehat{\mathcal{H}}_1$ and such that $\hat{\pi}_0 = \hat{\pi}_1 \theta$.*

Proof. Since $\mathscr{H}$ is minimal, there is some $f_0 \in \overline{S}$ such that $x_0 \in \operatorname{dom} f_0$ and $f_0(x_0) = x_1$. Let $\theta : \widehat{T}_0 \to \widehat{T}_1$ be defined by $\theta(\boldsymbol{\gamma}(f, x)) = \boldsymbol{\gamma}(f_0 f, x)$. This map is continuous because $\theta(\boldsymbol{\gamma}(f, x)) = \boldsymbol{\gamma}(f_0, x)\,\boldsymbol{\gamma}(f, x)$. So θ is a homeomorphism because f_0^{-1} defines θ^{-1} in the same way. We also have $\hat{\pi}_0 = \hat{\pi}_1\theta$ since θ preserves the source of each germ. For each $h \in S$, we have $\operatorname{dom}\hat{h}_1 = \theta(\operatorname{dom}\hat{h}_0)$ because $\hat{\pi}_0 = \hat{\pi}_1\theta$, and $\hat{h}_1\theta = \theta$ since

$$\begin{aligned}\hat{h}_1\theta(\boldsymbol{\gamma}(f, x)) &= \hat{h}_1(\boldsymbol{\gamma}(f_0 f, x)) = \boldsymbol{\gamma}(f_0 f h^{-1}, h(x))\\ &= \theta\big(\boldsymbol{\gamma}(f h^{-1}, h(x))\big) = \theta\big(\hat{h}_0(\boldsymbol{\gamma}(f, x))\big)\end{aligned}$$

for all $\boldsymbol{\gamma}(f, x) \in \operatorname{dom}\hat{h}_0$. It follows easily that θ generates an étale morphism $\Theta : \widehat{\mathscr{H}}_0 \to \widehat{\mathscr{H}}_1$, which is an equivalence since θ^{-1} generates Θ^{-1}. □

Now, let us show that the topology of $\widehat{T}$ is independent of the choice of S. Therefore the topology of $\widehat{T}_0$ will be independent of the choice of S as well. Let $S', S'' \subset \mathscr{H}$ be two subpseudo∗groups generating $\mathscr{H}$ and satisfying the conditions of Section 3A. With the notation of Section 3B, we have to prove the following.

Proposition 3.43. $\overline{\mathfrak{G}}_{\overline{S'},\text{c-o}} = \overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}$.

Proof. First, up to solving the case where $S' \subset S''$, we can assume that S' and S'' are local by Remarks 10 and 17. Second, if S' and S'' are local, then the subpseudo∗group $S' \cap S''$ of $\mathscr{H}$ also generates $\mathscr{H}$. Moreover $S' \cap S''$ obviously satisfies all other properties required in Section 3A; note that a refinement of $\{T_i\}$ may be necessary to get the properties stated in Remarks 22–25 with $S' \cap S''$. Hence the result follows from the special case where $S' \subset S''$. With this assumption, the identity map $\overline{\mathfrak{G}}_{\overline{S'},\text{c-o}} \to \overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}$ is continuous because the diagram

$$\begin{array}{ccc}\overline{S'}_{\text{c-o}} & \xrightarrow{\text{inclusion}} & \overline{S''}_{\text{c-o}}\\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\gamma}\\ \overline{\mathfrak{G}}_{\overline{S'},\text{c-o}} & \xrightarrow{\text{identity}} & \overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}\end{array}$$

is commutative, where the vertical maps are identifications and the top map is continuous.

For any compact subset $Q \subset T$, let $s^{-1}(Q)_{\overline{S'},\text{c-o}}$ and $s^{-1}(Q)_{\overline{S''},\text{c-o}}$ denote the spaces obtained by endowing $s^{-1}(Q)$ with the restriction of the topologies of $\overline{\mathfrak{G}}_{\overline{S'},\text{c-o}}$ and $\overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}$, respectively. They are compact and Hausdorff by Propositions 3.17 and 3.18. It follows that $s^{-1}(Q)_{\overline{S'},\text{c-o}} = s^{-1}(Q)_{\overline{S''},\text{c-o}}$ because the identity map $s^{-1}(Q)_{\overline{S'},\text{c-o}} \to s^{-1}(Q)_{\overline{S''},\text{c-o}}$ is continuous. Hence, for any $\boldsymbol{\gamma}(f, x) \in \overline{\mathfrak{G}}$ and a compact neighborhood Q of x in T, the set $s^{-1}(Q)$ is a neighborhood of $\boldsymbol{\gamma}(f, x)$ in $\overline{\mathfrak{G}}_{\overline{S'},\text{c-o}}$ and $\overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}$ with $s^{-1}(Q)_{\overline{S'},\text{c-o}} = s^{-1}(Q)_{\overline{S''},\text{c-o}}$. This shows that the identity map $\overline{\mathfrak{G}}_{\overline{S'},\text{c-o}} \to \overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}$ is a local homeomorphism, and therefore a homeomorphism. □

Let T' be an open subset of T containing x_0, which meets all orbits because $\mathcal{H}$ is minimal. Then use T', $\mathcal{H}' = \mathcal{H}|_{T'}$ and $S' = S \cap \mathcal{H}'$ to define $\widehat{T}'_0$, $\hat{\pi}'_0$, $\widehat{S}'_0$ and $\widehat{\mathcal{H}}'_0$ like $\widehat{T}_0$, $\hat{\pi}_0$, $\widehat{S}_0$ and $\widehat{\mathcal{H}}_0$. The proof of the following result is elementary.

Proposition 3.44. *There is a canonical identity of topological spaces, $\widehat{T}'_0 \equiv \hat{\pi}_0^{-1}(T')$, such that $\hat{\pi}'_0 \equiv \hat{\pi}_0|_{\widehat{T}'_0}$ and $\widehat{\mathcal{H}}'_0 = \widehat{\mathcal{H}}_0|_{\widehat{T}'_0}$.*

Corollary 3.45. *Let $\mathcal{H}$ and $\mathcal{H}'$ be minimal equicontinuous compactly generated pseudogroups on locally compact Polish spaces such that $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}'}$ are strongly quasianalytic. If $\mathcal{H}$ is equivalent to $\mathcal{H}'$, then $\widehat{\mathcal{H}}_0$ is equivalent to $\widehat{\mathcal{H}}'_0$.*

Proof. This is a direct consequence of Propositions 3.42–3.44. □

The following definition makes sense by Lemma 2.36, Propositions 3.42 and 3.43, and Corollary 3.45.

Definition 3.46. In Proposition 3.41, it is said that (the local isomorphism class of) G is the *structural local group* of (the equivalence class of) $\mathcal{H}$.

4. Molino's theory for equicontinuous foliated spaces

4A. ***Preliminaries on equicontinuous foliated spaces.*** (See [Moore and Schochet 1988; Candel and Conlon 2000, Chapter 11; Ghys 1999].)

Let X and Z be locally compact Polish spaces. A *foliated chart* in X of *leaf dimension* n, *transversely modeled* on Z, is a pair (U, ϕ), where $U \subseteq X$ is open and $\phi : U \to B \times T$ is a homeomorphism for some open $T \subset Z$ and some open ball B in $\mathbb{R}^n$. It is said that U is a *distinguished open set*. The sets $P_y = \phi^{-1}(B \times \{y\})$ for $y \in T$ are called *plaques* of this foliated chart. For every $x \in B$, the set $S_x = \phi^{-1}(\{x\} \times T)$ is called a *transversal* of the foliated chart. This local product structure defines a local projection $p : U \to T$, called *distinguished submersion*, given as composition of ϕ with the second factor projection $\mathrm{pr}_2 : B \times T \to T$.

Let $\mathcal{U} = \{U_i, \phi_i\}$ be a family of foliated charts in X of leaf dimension n modeled transversally on Z and covering X. Assume further that the foliated charts are *coherently foliated* in the sense that, if P and Q are plaques in different charts of $\mathcal{U}$, then $P \cap Q$ is open both in P and Q. Then $\mathcal{U}$ is called a *foliated atlas* on X of *leaf dimension* n and *transversely modeled* on Z. A maximal foliated atlas $\mathcal{F}$ of leaf dimension n and transversely modeled on Z is called a *foliated structure* on X of *leaf dimension* n and *transversely modeled* on Z. Any foliated atlas $\mathcal{U}$ of this type is contained in a unique foliated structure $\mathcal{F}$; then it is said that $\mathcal{U}$ *defines* (or is an atlas of) $\mathcal{F}$. If $Z = \mathbb{R}^m$, then X is a manifold of dimension $n + m$, and $\mathcal{F}$ is traditionally called a *foliation* of *dimension* n and *codimension* m. The reference to Z will be omitted.

For a foliated structure $\mathcal{F}$ on X of dimension n, the plaques form a basis of a topology on X called the *leaf topology*. With the leaf topology, X becomes an

n-manifold whose connected components are called *leaves* of $\mathcal{F}$. $\mathcal{F}$ is determined by its leaves.

A foliated atlas $\mathcal{U} = \{U_i, \phi_i\}$ of $\mathcal{F}$ is called *regular* if

- each $\overline{U_i}$ is a compact subset of a foliated chart (W_i, ψ_i) and $\phi_i = \psi_i|_{U_i}$;
- the cover $\{U_i\}$ is locally finite; and,
- if (U_i, ϕ_i) and (U_j, ϕ_j) are elements of $\mathcal{U}$, then each plaque P of (U_i, ϕ_i) meets at most one plaque of (U_j, ϕ_j).

In this case, there are homeomorphisms $h_{ij} : T_{ij} \to T_{ji}$ such that $h_{ij} p_i = p_j$ on $U_i \cap U_j$, where $p_i : U_i \to T_i$ is the distinguished submersion defined by (U_i, ϕ_i) and $T_{ij} = p_i(U_i \cap U_j)$. Observe that the cocycle condition $h_{ik} = h_{jk} h_{ij}$ is satisfied on $T_{ijk} = p_i(U_i \cap U_j \cap U_k)$. For this reason, $\{U_i, p_i, h_{ij}\}$ is called a *defining cocycle* of $\mathcal{F}$ with values in Z — we only consider defining cocycles induced by regular foliated atlases. The equivalence class of the pseudogroup $\mathcal{H}$ generated by the maps h_{ij} on $T = \bigsqcup_{i \in I} T_i$ is called the *holonomy pseudogroup* of the foliated space $(X, \mathcal{F})$; $\mathcal{H}$ is the representative of the holonomy pseudogroup of $(X, \mathcal{F})$ induced by the defining cocycle $\{U_i, p_i, h_{ij}\}$. This T can be identified with a *total* (or *complete*) *transversal* to the leaves in the sense that it meets all leaves and is locally given by the transversals defined by foliated charts. All compositions of maps h_{ij} form a pseudo$*$group S that generates $\mathcal{H}$, called the *holonomy pseudo$*$group* of $\mathcal{F}$ induced by $\{U_i, p_i, h_{ij}\}$. There is a canonical identity between the space of leaves and the space of $\mathcal{H}$-orbits, $X/\mathcal{F} \equiv T/\mathcal{H}$.

A foliated atlas (respectively, defining cocycle) contained in another one is called a *subfoliated atlas* (respectively, *subfoliated cocycle*).

The *holonomy group* of each leaf L is defined as the germ group of the corresponding orbit. It can be considered as a quotient of $\pi_1(L)$ by taking "chains" of sets U_i along loops in L; this representation of $\pi_1(L)$ is called the *holonomy representation*. The kernel of the holonomy representation is equal to $q_* \pi_1(\widetilde{L})$ for a regular covering space $q : \widetilde{L} \to L$, which is called the *holonomy cover* of L. If $\mathcal{F}$ admits a countable defining cocycle, then the leaves in some dense G_δ subset of M have trivial holonomy groups [Hector and Hirsch 1981; 1987; Candel and Conlon 2000], and therefore they can be identified with their holonomy covers.

It is said that a foliated space is (*topologically*) *transitive* or *minimal* if any representative of its holonomy pseudogroup is such. Transitivity (respectively, minimality) of a foliated space means that some leaf is dense (respectively, all leaves are dense).

Haefliger [2002] has observed that, if X is compact, then $\mathcal{H}$ is compactly generated, which can be seen as follows. There is some defining cocycle $\{U'_i, p'_i, h'_{ij}\}$, with $p'_i : U'_i \to T'_i$, such that $\overline{U_i} \subset U'_i$, $T_i \subset T'_i$ and p'_i extends p_i. Therefore each h'_{ij} is an extension of h_{ij} so that $\overline{\operatorname{dom} h_{ij}} \subset \operatorname{dom} h'_{ij}$. Moreover $\mathcal{H}$ is the restriction to T of the pseudogroup $\mathcal{H}'$ on $T' = \bigsqcup_i T_i'$ generated by the maps h_{ij}', and T is a relatively compact open subset of T' that meets all $\mathcal{H}'$-orbits.

Definition 4.1. It is said that a foliated space is *equicontinuous* if any representative of its holonomy pseudogroup is equicontinuous.

Remark 28. The above definition makes sense by Lemma 2.30.

Definition 4.2. Let G be a locally compact Polish local group. A minimal foliated space is called a *G-foliated space* if its holonomy pseudogroup can be represented by a pseudogroup given by Example 2.35 on a local group locally isomorphic to G.

4B. ***Molino's theory for equicontinuous foliated spaces.*** Let $(X, \mathcal{F})$ be a compact minimal foliated space that is equicontinuous and such that the closure of its holonomy pseudogroup is strongly quasianalytic. Let $\{U_i, p_i, h_{ij}\}$ be a defining cocycle of $\mathcal{F}$ induced by a regular foliated atlas, where $p_i : U_i \to T_i$. Let $\mathcal{H}$ denote the corresponding representative of the holonomy pseudogroup on $T = \bigsqcup_i T_i$, which satisfies the conditions of Section 3A. Let S be the localization of the holonomy pseudo$*$group induced by $\{U_i, p_i, h_{ij}\}$. Fix an index i_0 and a point $x_0 \in U_{i_0}$. Let $\hat{\pi}_0 : \widehat{T}_0 \to T$ and $\widehat{\mathcal{H}}_0$ be defined like in Sections 3D and 3E, by using T, $\mathcal{H}$, the point $p_{i_0}(x_0) \in T_{i_0} \subset T$, and a local subpseudo$*$group $S \subset \mathcal{H}$.

With the notation $\widehat{T}_{i,0} = \hat{\pi}_0^{-1}(T_i) \subset \widehat{T}_0$, let

$$\check{X}_0 = \bigsqcup_i U_i \times \widehat{T}_{i,0} = \bigcup_i U_i \times \widehat{T}_{i,0} \times \{i\},$$

equipped with the corresponding topological sum of the product topologies, and consider its closed subspace

$$\widetilde{X}_0 = \{ (x, \gamma, i) \in \check{X}_0 \mid p_i(x) = \hat{\pi}_0(\gamma) \} \subset \check{X}_0.$$

For $(x, \gamma, i), (y, \delta, j) \in \widetilde{X}_0$, write $(x, \gamma, i) \sim (y, \delta, j)$ if $x = y$ and $\gamma = \widehat{h_{ji}}(\delta)$. Since $h_{ij} p_i(x) = p_j(x)$, $h_{ji}^{-1} = h_{ij}$ and $h_{ik} = h_{jk} h_{ij}$, it follows that this defines an equivalence relation $\sim$ on $\widetilde{X}_0$. Let $\widehat{X}_0$ be the corresponding quotient space, $q : \widetilde{X}_0 \to \widehat{X}_0$ the quotient map, and $[x, \gamma, i]$ the equivalence class of each triple (x, γ, i). For each i, let

$$\check{U}_{i,0} = U_i \times \widehat{T}_{i,0} \times \{i\}, \quad \widetilde{U}_{i,0} = \check{U}_{i,0} \cap \widetilde{X}_0, \quad \widehat{U}_{i,0} = q(\widetilde{U}_{i,0}).$$

Lemma 4.3. *$\widehat{U}_{i,0}$ is open in $\widehat{X}_0$.*

Proof. We have to check that $q^{-1}(\widehat{U}_{i,0}) \cap \widetilde{U}_{j,0}$ is open in $\widetilde{U}_{j,0}$ for all j, which is true because

$$q^{-1}(\widehat{U}_{i,0}) \cap \widetilde{U}_{j,0} = ((U_i \cap U_j) \times \widehat{T}_{j,0} \times \{j\}) \cap \widetilde{X}_0. \qquad \square$$

Lemma 4.4. *The quotient map $q : \widetilde{U}_{i,0} \to \widehat{U}_{i,0}$ is a homeomorphism.*

Proof. This map is surjective by the definition of $\widehat{U}_{i,0}$. On the other hand, two equivalent triples in $\widetilde{U}_{i,0}$ are of the form (x,γ,i) and (x,δ,i) with $\gamma=\widehat{h_{ii}}(\delta)=\delta$. So $q:\widetilde{U}_{i,0}\to\widehat{U}_{i,0}$ is also injective. Since $q:\widetilde{U}_{i,0}\to\widehat{U}_{i,0}$ is continuous, it only remains to prove that this map is open. A basis of the topology of $\widetilde{U}_{i,0}$ consists of the sets of the form $(V\times W\times\{i\})\cap\widetilde{X}_0$, where V and W are open in U_i and $\widehat{T}_{i,0}$, respectively. These basic sets satisfy

$$\widetilde{U}_{j,0}\cap q^{-1}q\big((V\times W\times\{i\})\cap\widetilde{X}_0\big)=\widetilde{U}_{j,0}\cap\big(V\times\widehat{h_{ij}}(W\cap\operatorname{dom}\widehat{h_{ij}})\times\{j\}\big)$$

for all j, which is open in $\widetilde{U}_{j,0}$. So $q^{-1}q((V\times W\times\{i\})\cap\widetilde{X}_0)$ is open in $\widetilde{X}_0$ and therefore $q((V\times W\times\{i\})\cap\widetilde{X}_0)$ is open in $\widehat{X}_0$. □

Proposition 4.5. *$\widehat{X}_0$ is compact and Polish.*

Proof. Let $\{U_i',p_i',h_{ij}'\}$ be a shrinking of $\{U_i,p_i,h_{ij}\}$; i.e., it is a defining cocycle of $\mathcal{F}$ such that $\overline{U_i'}\subset U_i$ and $p_i':U_i'\to T_i'$ is the restriction of p_i for all i. Therefore each h_{ij}' is also a restriction of h_{ij} and T_i' is a relatively compact open subset of T_i. Then $\hat{\pi}_0^{-1}(\overline{T_i'})$ is a compact subset of $\widehat{T}_{i,0}$ by Corollary 3.21. Moreover $\widehat{X}_0$ is the union of the sets $q(\overline{U_i'}\times\hat{\pi}_0^{-1}(\overline{T_i'})\times\{i\})$. So $\widehat{X}_0$ is compact because it is a finite union of compact sets.

On the other hand, since $\widetilde{X}_0$ is closed in $\check{X}_0$, and $\check{U}_{i,0}$ is Polish and locally compact by Corollary 3.19, it follows that $\widetilde{U}_{i,0}$ is Polish and locally compact, and therefore $\widehat{U}_{i,0}$ is Polish and locally compact by Lemma 4.4. Then, by the compactness of $\widehat{X}_0$, Lemma 4.3 and [Kechris 1991, Theorem 5.3], it only remains to prove that $\widehat{X}_0$ is Hausdorff.

Let $[x,\gamma,i]\neq[y,\delta,j]$ in $\widehat{X}_0$. So $x\in U_i$ and $y\in U_j$. If $x=y$, then we have $[y,\delta,j]=[x,\widehat{h_{ji}}(\delta),i]\in\widehat{U}_{i,0}$. Thus, in this case, $[x,\gamma,i]$ and $[y,\delta,j]$ can be separated by open subsets of $\widehat{U}_{i,0}$ because $\widehat{U}_{i,0}$ is Hausdorff.

Now suppose that $x\neq y$. Then take disjoint open neighborhoods, V of x in U_i and W of y in U_j. Let

$$\begin{aligned}
\check{V}&=V\times\widehat{T}_{i,0}\times\{i\}\subset\check{U}_{i,0}, &\quad \check{W}&=V\times\widehat{T}_{j,0}\times\{j\}\subset\check{U}_{j,0},\\
\widetilde{V}&=\check{V}\cap\widetilde{X}_0\subset\widetilde{U}_{i,0}, & \widetilde{W}&=\check{W}\cap\widetilde{X}_0\subset\widetilde{U}_{j,0},\\
\widehat{V}&=q(\widetilde{V})\subset\widehat{U}_{i,0}, & \widehat{W}&=q(\widetilde{W})\subset\widehat{U}_{j,0}.
\end{aligned}$$

The sets $\widehat{V}$ and $\widehat{W}$ are open neighborhoods of $[x,\gamma,i]$ and $[y,\delta,j]$ in $\widehat{X}_0$. Suppose that $\widehat{V}\cap\widehat{W}\neq\varnothing$. Then there is a point $(x',\gamma',i)\in\widetilde{V}$ which is equivalent to some point $(y',\delta',j)\in\widetilde{W}$. This implies that $x'=y'\in V\cap W$, which is a contradiction because $V\cap W=\varnothing$. Therefore $\widehat{V}\cap\widehat{W}=\varnothing$. □

According to the above equivalence relation of triples, a map $\hat{\pi}_0:\widehat{X}_0\to X$ is defined by $\hat{\pi}_0([x,\gamma,i])=x$.

Proposition 4.6. *The map $\hat\pi_0 : \widehat{X}_0 \to X$ is continuous and surjective, and its fibers are homeomorphic to each other.*

Proof. Since each map $\hat\pi_0 : \widehat{T}_{i,0} \to T_i$ is surjective, we have $\hat\pi_0(\widehat{U}_{i,0}) = U_i$, obtaining that $\hat\pi_0 : \widehat{X}_0 \to X$ is surjective. Moreover the composition

$$\widetilde{U}_{i,0} \xrightarrow{\ q\ } \widehat{U}_{i,0} \xrightarrow{\ \hat\pi_0\ } U_i$$

is the restriction of the first factor projection $\check{U}_{i,0} \to U_i$, $(x, \gamma, i) \mapsto x$. Therefore, $\hat\pi_0 : \widehat{X}_0 \to X$ is continuous by Lemmas 4.3 and 4.4.

For $x \in U_i$, we have $\hat\pi_0^{-1}(x) \subset \widehat{U}_{i,0}$ and

$$\widetilde{U}_{i,0} \cap q^{-1}(\hat\pi_0^{-1}(x)) = \{x\} \times \hat\pi_0^{-1}(p_i(x)) \times \{i\} \equiv \hat\pi_0^{-1}(p_i(x)) \subset \widehat{T}_{i,0}.$$

So the last assertion follows from Lemma 4.4 and Proposition 3.22. □

Let $\tilde p_{i,0} : \widetilde{U}_{i,0} \to \widehat{T}_{i,0}$ denote the restriction of the second factor projection $\check p_{i,0} : \check U_{i,0} = U_i \times \widehat{T}_{i,0} \times \{i\} \to \widehat{T}_{i,0}$. By Lemma 4.4, $\tilde p_{i,0}$ induces a continuous map $\hat p_{i,0} : \widehat{U}_{i,0} \to \widehat{T}_{i,0}$.

Proposition 4.7. *$\{\widehat{U}_{0.i}, \hat p_{i,0}, \widehat{h_{ij}}\}$ is a defining cocycle of a foliated structure $\widehat{\mathscr{F}}_0$ on $\widehat{X}_0$.*

Proof. Let $\{U_i, \phi_i\}$ be a regular foliated atlas of $\mathscr{F}$ inducing the defining cocycle $\{U_i, p_i, h_{ij}\}$, where $\phi_i : U_i \to B_i \times T_i$ is a homeomorphism and B_i is a ball in $\mathbb{R}^n$ ($n = \dim \mathscr{F}$). Then we get a homeomorphism

$$\check\phi_{i,0} = \phi_i \times \mathrm{id} \times \mathrm{id} : \check U_{i,0} = U_i \times \widehat{T}_{i,0} \times \{i\} \to B_i \times T_i \times \widehat{T}_{i,0} \times \{i\}.$$

Observe that $\check\phi_{i,0}(\widetilde{U}_{i,0})$ consists of the elements (y, z, γ, i) with $\hat\pi_0(\gamma) = z$. So $\check\phi_{i,0}$ restricts to a homeomorphism

$$\tilde\phi_{i,0} : \widetilde{U}_{i,0} \to \check\phi_{i,0}(\widetilde{U}_{i,0}) \equiv B_i \times \widehat{T}_{i,0} \times \{i\} \equiv B_i \times \widehat{T}_{i,0}.$$

By Lemma 4.4, $\tilde\phi_{i,0}$ induces a homeomorphism $\hat\phi_{i,0} : \widehat{U}_{i,0} \to B_i \times \widehat{T}_{i,0}$. Moreover, $\check p_{i,0}$ corresponds to the third factor projection via $\check\phi_{i,0}$, obtaining that $\tilde p_{i,0}$ corresponds to the second factor projection via $\tilde\phi_{i,0}$, and therefore $\hat p_{i,0}$ also corresponds to the second factor projection via $\hat\phi_{i,0}$. Observe that $\hat p_{i,0} = \widehat{h_{ji}}\, \hat p_{j,0}$ on $\widehat{U}_{i,0} \cap \widehat{U}_{j,0}$ by the definition of $\sim$. The regularity of the foliated atlas $\{\widehat{U}_{0.i}, \hat\phi_{i,0}\}$ follows easily from the regularity of $\{U_i, \phi_i\}$. □

According to Proposition 4.7, the holonomy pseudogroup of $\widehat{\mathscr{F}}_0$ is represented by the pseudogroup on $\bigsqcup_i \widehat{T}_{i,0}$ generated by the maps $\widehat{h_{ij}}$, which is the pseudogroup $\widehat{\mathscr{H}}_0$ on $\widehat{T}_0$.

Corollary 4.8. *There is some locally compact Polish local group G such that $(\widehat{X}_0, \widehat{\mathscr{F}}_0)$ is a minimal G-foliated space; in particular, it is equicontinuous.*

Proof. This follows from Propositions 4.7 and 3.41, and Lemma 3.36. □

Proposition 4.9. *The map* $\hat\pi_0 : (\widehat{X}_0, \widehat{\mathcal{F}}_0) \to (X, \mathcal{F})$ *is foliated.*

Proof. According to Proposition 4.7, this follows by checking the commutativity of each diagram

$$\begin{array}{ccc} \widehat{U}_{i,0} & \xrightarrow{\hat p_{i,0}} & \widehat{T}_{i,0} \\ {\scriptstyle \hat\pi_0}\big\downarrow & & \big\downarrow{\scriptstyle \hat\pi_0} \\ U_i & \xrightarrow{p_i} & T_i \end{array}$$

By Lemma 4.4, and the definition of $\hat p_{i,0}$ and $\hat\pi_{i,0}$, this commutativity follows from the commutativity of

$$\begin{array}{ccc} \widetilde{U}_{i,0} & \longrightarrow & \widehat{T}_{i,0} \\ \big\downarrow & & \big\downarrow{\scriptstyle \hat\pi_0} \\ U_i & \xrightarrow{p_i} & T_i \end{array}$$

where the left vertical and the top horizontal arrows denote the restrictions of the first and second factor projections of $\breve{U}_{i,0} = U_i \times \widehat{T}_{i,0} \times \{i\}$. But the commutativity of this diagram holds by the definition of $\widetilde{X}_0$ and $\widetilde{U}_{i,0}$. □

Proposition 4.10. *The restrictions of* $\hat\pi_0 : \widehat{X}_0 \to X$ *to the leaves are the holonomy covers of the leaves of* $\mathcal{F}$.

Proof. With the notation of the proof of Proposition 4.7, the diagram

$$\begin{array}{ccc} \widehat{U}_{i,0} & \xrightarrow{\hat\phi_{i,0}} & B_i \times \widehat{T}_{i,0} \\ {\scriptstyle \hat\pi_0}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{B_i} \times \hat\pi_0} \\ U_i & \xrightarrow{\phi_i} & B_i \times T_i \end{array} \tag{19}$$

is commutative, and $\widehat{U}_{i,0} = \hat\pi_0^{-1}(U_i)$. Hence, for corresponding plaques in U_i and $\widehat{U}_{i,0}$, namely $P_z = \phi_0^{-1}(B_i \times \{\hat z\})$ and $\widehat{P}_{\hat z} = \hat\phi_0^{-1}(B_i \times \{z\})$ with $z \in T_i$ and $\hat z \in \hat\pi_0^{-1}(z) \subset \widehat{T}_{i,0}$, the restriction $\hat\pi_0 : \widehat{P}_{\hat z} \to P_z$ is a homeomorphism. It follows easily that $\hat\pi_0 : \widehat{X}_0 \to X$ restricts to covering maps of the leaves of $\widehat{\mathcal{F}}_0$ to the leaves of $\mathcal{F}$. In fact, these are the holonomy covers, which can be seen as follows.

According to the proof of Proposition 4.6 and the definition of the equivalence relation $\sim$ on $\widetilde{X}_0$, for each x in $U_i \cap U_j$, we have homeomorphisms

$$\hat\pi_0^{-1}(p_i(x)) \xleftarrow{\hat p_{i,0}} \hat\pi_0^{-1}(x) \xrightarrow{\hat p_{j,0}} \hat\pi_0^{-1}(p_j(x))$$

satisfying $\hat p_{j,0} \hat p_{i,0}^{-1} = \widehat{h_{ij}}$. This easily implies the following. Given $x \in U_i$ and $\hat x \in \hat\pi_0^{-1}(X)$, denoting by L and $\widehat{L}$ the leaves through x and $\hat x$, respectively, and given a loop c in L based at x inducing a local holonomy transformation $h \in S$

around $p_i(x)$ in T_i, the lift $\hat{c}$ of c to $\widehat{L}$ with $\hat{c}(0)=\hat{x}$ satisfies $\hat{p}_{i,0}\hat{c}(1)=\hat{h}\hat{p}_{i,0}(\hat{x})$. Writing $\hat{p}_{i,0}(\hat{x})=\boldsymbol{\gamma}(f,p_i(x))$, we obtain

$$\hat{p}_{i,0}\hat{c}(1)=\hat{h}\big(\boldsymbol{\gamma}(f,p_i(x))\big)=\boldsymbol{\gamma}(fh,p_i(x)).$$

Thus $\hat{c}$ is a loop if and only if $\boldsymbol{\gamma}(fh,p_i(x))=\boldsymbol{\gamma}(f,p_i(x))$, which means that $\boldsymbol{\gamma}(h,p_i(x))=\boldsymbol{\gamma}(\mathrm{id}_T,p_i(x))$. So $\widehat{L}$ is the holonomy cover of L. □

Proposition 4.11. *The map $\hat{\pi}_0:\widehat{X}_0\to X$ is open.*

Proof. This follows from Corollary 3.38 and the commutativity of (19). □

Theorem A is the combination of the results of this section.

4C. ***Independence of the choices involved.*** Let x_1 be another point of X, and let $\widehat{X}_1$, $\widehat{\mathcal{F}}_1$ and $\hat{\pi}_1:\widehat{X}_1\to X$ be constructed like $\widehat{X}_0$, $\widehat{\mathcal{F}}_0$ and $\hat{\pi}_0:\widehat{X}_0\to X$ by using x_1 instead of x_0.

Proposition 4.12. *There is a foliated homeomorphism $\hat{\theta}:(\widehat{X}_0,\widehat{\mathcal{F}}_0)\to(\widehat{X}_1,\widehat{\mathcal{F}}_1)$ such that $\hat{\pi}_1F=\hat{\pi}_0$.*

Proof. Take an index i_1 such that $x_1\in U_{i_1}$. Let $\widehat{S}_1$, $\widehat{T}_1$, $\widehat{\mathcal{H}}_1$ and $\hat{\pi}_1:\widehat{T}_1\to T$ be constructed like $\widehat{S}_0$, $\widehat{T}_0$, $\widehat{\mathcal{H}}_0$ and $\hat{\pi}_0:\widehat{T}_0\to T$ by using $p_{i_1}(x_1)$ instead of $p_{i_0}(x_0)$, and let $\widehat{T}_{i,1}=\hat{\pi}_1^{-1}(T_i)$. Then the construction of $\widehat{X}_1$, $\widehat{\mathcal{F}}_1$ and $\hat{\pi}_1:\widehat{X}_1\to X$ involves the objects $\check{X}_1$, $\widetilde{X}_1$, $\check{U}_{i,1}$, $\widetilde{U}_{i,1}$, $\widehat{U}_{i,1}$, $\check{p}_{i,1}$, $\tilde{p}_{i,1}$, $\hat{p}_{i,1}$, $\check{\phi}_{i,1}$, $\tilde{\phi}_{i,1}$ and $\hat{\phi}_{i,1}$, defined like $\check{X}_0$, $\widetilde{X}_0$, $\check{U}_{i,0}$, $\widetilde{U}_{i,0}$, $\widehat{U}_{i,0}$, $\check{p}_{i,0}$, $\tilde{p}_{i,0}$, $\hat{p}_{i,0}$, $\check{\phi}_{i,0}$, $\tilde{\phi}_{i,0}$ and $\hat{\phi}_{i,0}$, by using $\widehat{T}_{i,1}$ and $\hat{\pi}_1:\widehat{T}_{i,1}\to T_i$ instead of $\widehat{T}_{i,0}$ and $\hat{\pi}_0:\widehat{T}_{i,0}\to T_i$. Let $\theta:\widehat{T}_0\to\widehat{T}_1$ be the homeomorphism given by Proposition 3.42, which obviously restricts to homeomorphisms $\theta_i:\widehat{T}_{i,0}\to\widehat{T}_{i,1}$. Since $\hat{\pi}_0=\hat{\pi}_1\theta$, it follows that each homeomorphism

$$\check{\theta}_i=\mathrm{id}_{U_i}\times\theta_i\times\mathrm{id}:\check{U}_{i,0}=U_i\times\widehat{T}_{i,0}\times\{i\}\to\check{U}_{i,1}=U_i\times\widehat{T}_{i,1}\times\{i\}$$

restricts to a homeomorphism $\tilde{\theta}_i=\widetilde{U}_{i,0}\to\widetilde{U}_{i,1}$. The combination of the homeomorphisms $\tilde{\theta}_i$ is a homeomorphism $\tilde{\theta}:\widetilde{X}_0\to\widetilde{X}_1$.

For each $h\in S$, use the notation $\hat{h}_0\in\widehat{S}_0$ and $\hat{h}_1\in\widehat{S}_1$ for the map $\hat{h}$ defined with $p_{i_0}(x_0)$ and $p_{i_1}(x_1)$, respectively. From the proof of Proposition 3.42, we get $\hat{h}_1\theta=\theta\hat{h}_0$ for all $h\in S$; in particular, this holds with $h=h_{ij}$. So $\tilde{\theta}:\widetilde{X}_0\to\widetilde{X}_1$ is compatible with the equivalence relations used to define $\widehat{X}_0$ and $\widehat{X}_1$, and therefore it induces a homeomorphism $\hat{\theta}:\widehat{X}_0\to\widehat{X}_1$. Note that $\hat{\theta}$ restricts to homeomorphisms $\hat{\theta}_i:\widehat{U}_{i,0}\to\widehat{U}_{i,1}$. Obviously, $\check{p}_{i,1}\check{\theta}_i=\theta_i\check{p}_{i,1}$, yielding $\tilde{p}_{i,1}\tilde{\theta}_i=\theta_i\tilde{p}_{i,1}$, and therefore $\hat{p}_{i,1}\hat{\theta}_i=\theta_i\hat{p}_{i,1}$. It follows that $\hat{\theta}$ is a foliated map. □

Let $\{U'_a,p'_a,h'_{ab}\}$ be another defining cocycle of $\mathcal{F}$ induced by a regular foliated atlas. Then construct $\widehat{X}'_0$, $\widehat{\mathcal{F}}'_0$ and $\hat{\pi}'_0:\widehat{X}_0\to X$ like $\widehat{X}_0$, $\widehat{\mathcal{F}}_0$ and $\hat{\pi}_0:\widehat{X}_0\to X$ by using $\{U'_a,p'_a,h'_{ab}\}$ instead of $\{U_i,p_i,h_{ij}\}$.

Proposition 4.13. *There is a foliated homeomorphism* $F : (\widehat{X}_0, \widehat{\mathscr{F}}_0) \to (\widehat{X}'_0, \widehat{\mathscr{F}}'_0)$ *such that* $\hat{\pi}'_0 F = \hat{\pi}_0$.

Proof. By using a common refinement of the open coverings $\{U_i\}$ and $\{U'_a\}$, we can assume that $\{U'_a\}$ refines $\{U_i\}$. In this case, the union of the defining cocycles $\{U_i, p_i, h_{ij}\}$ and $\{U'_a, p'_a, h'_{ab}\}$ is contained in another defining cocycle induced by a regular foliated atlas. Thus the proof boils down to showing that a subdefining cocycle[9] $\{U_{i_k}, p_{i_k}, h_{i_k i_l}\}$ of $\{U_i, p_i, h_{ij}\}$ induces a foliated space homeomorphic to $(\widehat{X}_0, \widehat{\mathscr{F}}_0)$. But the pseudogroup $\mathscr{H}'$ induced by $\{U_{i_k}, p_{i_k}, h_{i_k i_l}\}$ is the restriction of $\mathscr{H}$ to an open subset $T' \subset T$, and the pseudo$*$group induced by $\{U_{i_k}, p_{i_k}, h_{i_k i_l}\}$ is $S' = S \cap \mathscr{H}'$. Then, by using the canonical identity given by Proposition 3.44, it easily follows that the foliated space $(\widehat{X}'_0, \widehat{\mathscr{F}}'_0)$ defined with $\{U_{i_k}, p_{i_k}, h_{i_k i_l}\}$ can be canonically identified with an open foliated subspace of $(\widehat{X}_0, \widehat{\mathscr{F}}_0)$, which indeed is the whole of $(\widehat{X}_0, \widehat{\mathscr{F}}_0)$ because $\{U_{i_k}\}$ covers X. □

The following definition makes sense by Propositions 4.12–4.13 and the results used to justify Definition 3.46.

Definition 4.14. In Corollary 4.8, (the local isomorphism class of) G is called the *structural local group* of $(X, \mathscr{F})$.

5. Growth of equicontinuous pseudogroups and foliated spaces

5A. ***Coarse quasi-isometries and growth of metric spaces.*** A *net* in a metric space M, with metric d, is a subset $A \subset M$ that satisfies $d(x, A) \le C$ for some $C > 0$ and all $x \in M$; the term *C-net* is also used. A *coarse quasi-isometry* between M and another metric space M' is a bi-Lipschitz bijection between nets of M and M'; in this case, M and M' are said to be *coarsely quasi-isometric* (in the sense of Gromov) [Gromov 1993]. If such a bi-Lipschitz bijection, as well as its inverse, has dilation $\le \lambda$, and it is defined between C-nets, then it will be said that the coarse quasi-isometry has *distortion* (C, λ). A family of coarse quasi-isometries with a common distortion will be called a family of *equicoarse quasi-isometries*, and the corresponding metric spaces are called *equicoarsely quasi-isometric*.

The version of growth for metric spaces given here is taken from [Álvarez and Candel 2015; Álvarez and Wolak 2013].

Recall that, given nondecreasing functions[10] $u, v : [0, \infty) \to [0, \infty)$, it is said that u is *dominated* by v, written $u \preccurlyeq v$, when there are $a, b \ge 1$ and $c \ge 0$ such that $u(r) \le a v(br)$ for all $r \ge c$. If $u \preccurlyeq v \preccurlyeq u$, then it is said that u and v represent the same *growth type* or have *equivalent growth*; this is an equivalence relation, and $\preccurlyeq$ defines a partial order relation between growth types called *domination*. For

[9] A subdefining cocycle is a defining cocycle contained in another one.

[10] Usually, growth types are defined by using nondecreasing functions $\mathbb{Z}^+ \to [0, \infty)$, but nondecreasing functions $[0, \infty) \to [0, \infty)$ give rise to an equivalent concept.

a family of pairs of nondecreasing functions $[0,\infty)\to[0,\infty)$, *equidomination* means that those pairs satisfy the above condition of domination with the same constants a, b, c. A family of functions $[0,\infty)\to[0,\infty)$ will be said to have *equiequivalent growth* if they equidominate one another.

For a complete connected Riemannian manifold L, the growth type of each mapping $r \mapsto \operatorname{vol} B(x,r)$ is independent of x, and is called the *growth type* of L. For metric spaces whose bounded sets are finite, a similar definition of *growth type* can be given where the number of points is used in place of the volume.

Let M be a metric space with metric d. A *quasilattice* Γ of M is a C-net of M for some $C \geq 0$ such that, for every $r \geq 0$, there is some $K_r \geq 0$ such that $\operatorname{card}(\Gamma \cap B(x,r)) \leq K_r$ for every $x \in M$. It is said that M is of *coarse bounded geometry* if it has a quasilattice. In this case, the *growth type* of M can be defined as the growth type of any quasilattice Γ of M; i.e., it is the growth type of the *growth function* $r \mapsto v_\Gamma(x,r) = \operatorname{card}(B(x,r)\cap\Gamma)$ for any $x \in \Gamma$. This definition is independent of Γ.

For a family of metric spaces, if they satisfy the above condition of coarse bounded geometry with the same constants C and K_r, then they are said to have *equicoarse bounded geometry*. If moreover the lattices involved in this condition have growth functions with equiequivalent growth, then these metric spaces are said to have *equiequivalent growth*.

The condition of coarse bounded geometry is satisfied by complete connected Riemannian manifolds of bounded geometry, and also by discrete metric spaces with a uniform upper bound on the number of points in all balls of each given radius [Block and Weinberger 1997]. In those cases, the two given definitions of growth type are equal.

Lemma 5.1 ([Álvarez and Candel 2009]; see also [Álvarez and Wolak 2013, Lemma 2.1]). *Two coarsely quasi-isometric metric spaces of coarse bounded geometry have the same growth type. Moreover, if a family of metric spaces are equicoarsely quasi-isometric to each other, then they have equiequivalent growth.*

5B. ***Quasi-isometry and growth types of orbits.*** Let $\mathcal{H}$ be a pseudogroup on a space T, and E a symmetric set of generators of $\mathcal{H}$. Let $\mathfrak{G}$ be the groupoid of germs of maps in $\mathcal{H}$.

For each $h \in \mathcal{H}$ and $x \in \operatorname{dom} h$, let $|h|_{E,x}$ be the length of the shortest expression of $\boldsymbol{\gamma}(h,x)$ as a product of germs of maps in E (being 0 if $\boldsymbol{\gamma}(h,x) = \boldsymbol{\gamma}(\mathrm{id}_T, x)$). For each $x \in T$, define metrics d_E on $\mathcal{H}(x)$ and $\mathfrak{G}_x$ by

$$d_E(y,z) = \min\{\,|h|_{E,y} \mid h \in \mathcal{H},\ y \in \operatorname{dom} h,\ h(y) = z\,\},$$
$$d_E(\boldsymbol{\gamma}(f,x), \boldsymbol{\gamma}(g,x)) = |fg^{-1}|_{E,g(x)}.$$

Notice that

$$d_E(f(x), g(x)) \le d_E(\boldsymbol{\gamma}(f, x), \boldsymbol{\gamma}(g, x)).$$

Moreover, on the germ covers, d_E is right invariant in the sense that, if $y \in \mathcal{H}(x)$, the bijection $\mathfrak{G}_y \to \mathfrak{G}_x$, given by right multiplication with any element in $\mathfrak{G}_x^y$, is isometric; so the isometry types of the germ covers of the orbits make sense without any reference to base points. In fact, the definition of d_E on $\mathfrak{G}_x$ is analogous to the definition of the right invariant metric d_S on a group Γ induced by a symmetric set of generators S: $d_S(\gamma, \delta) = |\gamma\delta^{-1}|$ for $\gamma, \delta \in \Gamma$, where $|\gamma|$ is the length of the shortest expression of γ as a product of elements of S (being 0 if $\gamma = e$).

Assume that $\mathcal{H}$ is compactly generated and T locally compact. Let $U \subset T$ be a relatively compact open subset that meets all $\mathcal{H}$-orbits, let $\mathcal{G} = \mathcal{H}|_U$, and let E be a symmetric system of compact generation of $\mathcal{H}$ on U. With these conditions, the quasi-isometry type of the $\mathcal{G}$-orbits with d_E may depend on E [Álvarez and Candel 2009, Section 6]. So the following additional condition on E is considered.

Definition 5.2 [Álvarez and Candel 2009, Definition 4.2]. With the above notation, it is said that E is *recurrent* if, for any relatively compact open subset $V \subset U$ that meets all $\mathcal{G}$-orbits, there exists some $R > 0$ such that $\mathcal{G}(x) \cap V$ is an R-net in $\mathcal{G}(x)$ with d_E for all $x \in U$.

The role played by V in Definition 5.2 can be played by any relatively compact open subset meeting all orbits [Álvarez and Candel 2009, Lemma 4.3]. Furthermore there exists a recurrent system of compact generation on U [Álvarez and Candel 2009, Corollary 4.5].

Theorem 5.3 [Álvarez and Candel 2009, Theorem 4.6]. *Let $\mathcal{H}$ and $\mathcal{H}'$ be compactly generated pseudogroups on locally compact spaces T and T', let U and U' be relatively compact open subsets of T and T' that meet all orbits of $\mathcal{H}$ and $\mathcal{H}'$, let $\mathcal{G}$ and $\mathcal{G}'$ denote the restrictions of $\mathcal{H}$ and $\mathcal{H}'$ to U and U', and let E and E' be recurrent symmetric systems of compact generation of $\mathcal{H}$ and $\mathcal{H}'$ on U and U', respectively. Suppose that there exists an equivalence $\mathcal{H} \to \mathcal{H}'$, and consider the induced equivalence $\mathcal{G} \to \mathcal{G}'$ and homeomorphism $U/\mathcal{G} \to U'/\mathcal{G}'$. Then the $\mathcal{G}$-orbits with d_E are equicoarsely quasi-isometric to the corresponding $\mathcal{G}'$-orbits with $d_{E'}$.*

An obvious modification of the arguments of the proof of [Álvarez and Candel 2009, Theorem 4.6] gives the following.

Theorem 5.4. *With the notation and conditions of Theorem 5.3, the germ covers of the $\mathcal{G}$-orbits with d_E are equicoarsely quasi-isometric to the germ covers of the corresponding $\mathcal{G}'$-orbits with $d_{E'}$.*

Corollary 5.5. *With the notation and conditions of Theorem 5.3, the corresponding orbits of $\mathcal{G}$ and $\mathcal{G}'$, as well as their germ covers, have equiequivalent growth.*

Proof. This follows from Lemma 5.1 and Theorems 5.3 and 5.4. □

Example 5.6. Let G be a locally compact Polish local group with a left-invariant metric, let $\Gamma \subset G$ be a dense finitely generated sub-local group, and let $\mathcal{H}$ denote the pseudogroup generated by the local action of Γ on G by local left translations. Suppose that $\mathcal{H}$ is compactly generated, and let $\mathcal{G} = \mathcal{H}|_U$ for some relatively compact open neighborhood U of the identity element e in G, which meets all $\mathcal{H}$-orbits because Γ is dense. For every $\gamma \in \Gamma$ with $\gamma U \cap U \neq \varnothing$, let h_γ denote the restriction $U \cap \gamma^{-1}U \to \gamma U \cap U$ of the local left translation by γ. There is a finite symmetric set $S = \{s_1, \dots, s_k\} \subset \Gamma$ such that $E = \{h_{s_1}, \dots, h_{s_k}\}$ is a recurrent system of compact generation of $\mathcal{H}$ on U; in fact, by reducing Γ if necessary, we can assume that S generates Γ. The recurrence of E means that there is some $N \in \mathbb{N}$ such that

$$U = \bigcup_{h \in E^N} h^{-1}(V \cap \operatorname{im} h), \tag{20}$$

where E^N is the family of compositions of at most N elements of E.

For each $x \in U$, let

$$\Gamma_{U,x} = \{\gamma \in \Gamma \mid \gamma x \in U\}.$$

Let $\mathfrak{G}$ denote the topological groupoid of germs of $\mathcal{G}$. The map $\Gamma_{U,x} \to \mathfrak{G}_x$, $\gamma \mapsto \boldsymbol{\gamma}(h_\gamma, x)$ is bijective. For $\gamma \in \Gamma_{U,x}$, let $|\gamma|_{S,U,x} := |h_\gamma|_{E,x}$. Thus $|e|_{S,U,x} = 0$, and if $\gamma \neq e$, then $|\gamma|_{S,U,x}$ equals the minimum $n \in \mathbb{N}$ such that there are indices $i_1, \dots, i_n \in \{1, \dots, k\}$ with $\gamma = s_{i_n} \cdots s_{i_1}$ and $s_{i_m} \cdots s_{i_1} \cdot x \in U$ for all $1 \le m \le n$. Moreover d_E on $\mathfrak{G}_x$ corresponds to the metric $d_{S,U,x}$ on $\Gamma_{U,x}$ given by

$$d_{S,U,x}(\gamma, \delta) = |\delta\gamma^{-1}|_{S,U,\gamma(x)}.$$

Observe that, for all $\gamma \in \Gamma_{U,x}$ and $\delta \in \Gamma_{U,\gamma\cdot x}$,

$$\delta\gamma \in \Gamma_{U,x}, \qquad |\delta\gamma|_{S,U,x} \le |\gamma|_{S,U,x} + |\delta|_{S,U,\gamma\cdot x}, \tag{21}$$

$$\gamma^{-1} \in \Gamma_{U,\gamma\cdot x}, \qquad |\gamma|_{S,U,x} = |\gamma^{-1}|_{S,U,\gamma\cdot x}. \tag{22}$$

In this example, we will be interested on the growth type of the orbits of $\mathcal{G}$ with d_E, or, equivalently, the growth type of the metric spaces $(\Gamma_{U,x}, d_{S,U,x})$. The following result was used by Breuillard and Gelander to study this growth type when G is a Lie group.

Proposition 5.7 [Breuillard and Gelander 2007, Proposition 10.5]. *Let G be a nonnilpotent connected real Lie group and Γ a finitely generated dense subgroup. For any finite set $S = \{s_1, \dots, s_k\}$ of generators of Γ, and any neighborhood B of e in G, there are elements $t_i \in \Gamma \cap s_i B$ ($i \in \{1, \dots, k\}$) which freely generate a free semigroup. If G is not solvable, then we can choose the elements t_i so that they generate a free group.*

5C. ***Growth of equicontinuous pseudogroups.*** Let G be a locally compact Polish local group with a left-invariant metric, let $\Gamma \subset G$ be a dense finitely generated sub-local group, and let $\mathcal{H}$ denote the pseudogroup generated by the local action of Γ on G by local left translations. Suppose that $\mathcal{H}$ is compactly generated. Let $\mathcal{G} = \mathcal{H}|_U$ for some relatively compact open neighborhood U of the identity element e in G, which meets all $\mathcal{H}$-orbits because Γ is dense. Let E be a recurrent symmetric system of compact generation of $\mathcal{H}$ on U. Let $\mathfrak{G}$ be the groupoid of germs of maps in $\mathcal{G}$.

Theorem 5.8. *With the above notation and conditions, one of the following properties hold*:

- *G can be approximated by nilpotent local Lie groups*; *or*
- *the germ covers of all $\mathcal{G}$-orbits have exponential growth with d_E.*

Proof. By Theorem 2.26, there is some $U_0 \in \Psi G$, contained in any given element of $\Psi G \cap \Phi(G, 2)$, and there exists a sequence of compact normal subgroups $F_n \subset U_0$ such that $F_{n+1} \subset F_n$, $\bigcap_n F_n = \{e\}$, $(F_n, U_0) \in \Delta G$, and $G/(F_n, U_0)$ is a local Lie group. Let $T_n : U_0^2 \to G/(F_n, U_0)$ denote the canonical projection. Take an open neighborhood U_1 of e such that $\overline{U_1} \subset U_0$. Then $F_n\overline{U_1} \subset U_0$ for n large enough by the properties of the sequence F_n. Let $U_2 = F_n U_1$ for such an n. Thus U_2 is saturated by the fibers of T_n, and $\overline{U_2} \subset U_0$. Then $U' := T_n(U_2)$ is a relatively compact open neighborhood of the identity in the local Lie group $G' := G/(F_n, U_0)$. Let $\Gamma' = T_n(\Gamma \cap U_0^2)$, which is a dense sub-local group of G', and let $\mathcal{H}'$ denote the pseudogroup on G' generated by the local action of Γ' by local left translations.

For every $\gamma \in \Gamma \cap U_0$ for which $\gamma U_2 \cap U_2 \neq \varnothing$, let h_γ denote the restriction $U_2 \cap \gamma^{-1} U_2 \to \gamma U_2 \cap U_2$ of the local left translation by γ. There is a finite symmetric set $S = \{s_1, \dots, s_k\} \subset \Gamma$ such that $E_2 = \{h_{s_1}, \dots, h_{s_k}\}$ is a recurrent system of compact generation of $\mathcal{H}$ on U_2. By reducing Γ if necessary, we can suppose that S generates Γ. For every $\delta \in \Gamma'$ with $\delta U' \cap U' \neq \varnothing$, let h'_δ denote the restriction $U' \cap \delta^{-1} U' \to \delta U' \cap U'$ of the local left translation by δ. We can assume that $s_1, \dots, s_k$ are in U_2, and therefore we can consider their images $s'_1, \dots, s'_k$ by T_n. Moreover each h_{s_i} induces via T_n the map h'_{s_i}, and $E' = \{h'_{s_1}, \dots, h'_{s_k}\}$ is a system of compact generation of $\mathcal{H}'$ on U'. By increasing E_2 if necessary, we can assume that E' is also recurrent. Fix any open set V' in G' with $\overline{V'} \subset U'$. Then $V = T_n^{-1}(V')$ satisfies $\overline{V} \subset U_2$.

Claim 1. *For each finite subset $F \subset \Gamma \cap U_2$, we have $U_2 \subset \bigcup_{\gamma \in \Gamma \setminus F} \gamma V$.*

Since U_2 and V are saturated by the fibers of T_n, Claim 1 follows by showing that $U' \subset \bigcup_{\gamma \in \Gamma' \setminus F'} \gamma V'$, where $F' = T_n(F)$. Suppose that this inclusion is false. Then there is some finite symmetric subset $F \subset \Gamma \cap U_2$ and some $x \in U'$ such that $((\Gamma' \setminus F')x) \cap V' = \varnothing$. By the recurrence of E', there is some $N \in \mathbb{N}$ satisfying (20) with U' and E'. Since $\Gamma'_{U',x}$ is infinite because Γ' is dense in G', it follows that there is some $\gamma \in \Gamma'_{U',x} \setminus F'$ such that

$$|\gamma|_{S',U',x} > N + \max\{|\epsilon|_{S',U',x} \mid \epsilon \in F' \cap \Gamma'_{U',x}\}. \tag{23}$$

By (20), there is some $h \in E'^N$ such that $\gamma x \in h^{-1}(V' \cap \operatorname{im} h')$. We have $h = h'_\delta$ for some $\delta \in \Gamma'$. Note that $\delta \in \Gamma'_{U',\gamma' x}$ and $|\delta|_{S',U',\gamma x} \leq N$. Hence

$$|\gamma|_{S',U',x} \leq |\delta\gamma|_{S',U',x} + |\delta^{-1}|_{S',U',\delta\gamma x} = |\delta\gamma|_{S',U',x} + |\delta|_{S',U',\gamma' x} \leq |\delta\gamma|_{S',U',x} + N$$

by (21) and (22), obtaining that $\delta\gamma \notin F'$ by (23). However, $\delta\gamma x \in V'$, obtaining a contradiction, which completes the proof of Claim 1.

Claim 2. *For each finite subset $F \subset \Gamma \cap U_2$, we have $\overline{U_2} \subset \bigcup_{\gamma \in \Gamma \setminus F} \gamma V$.*

Take a relatively compact open subset $O_1 \subset G$ such that $\overline{U_1} \subset O_1$ and $F_n \overline{O_1} \subset U_0$. Let $O_2 = F_n O_1$ and $\mathcal{K} = \mathcal{H}|_{O_2}$. Then Claim 2 follows by applying Claim 1 to O_2.

According to Claim 2, by increasing S if necessary, we can suppose that

$$\overline{U_2} \subset \bigcup_{i<j} (s_i \cdot V \cap s_j \cdot V) = \bigcup_{i<j} (s_i^{-1} \cdot V \cap s_j^{-1} \cdot V). \tag{24}$$

Suppose that G cannot be approximated by nilpotent local Lie groups. Then we can assume that the local Lie group G' is not nilpotent. Moreover we can suppose that G' is a sub-local Lie group of a simply connected Lie group L. Let Δ be the dense subgroup of L whose intersection with G' is Γ'. Then, by Proposition 5.7, there are elements $t'_1, \dots, t'_k$ in Δ, as close as desired to $s'_1, \dots, s'_k$, which are free generators of a free semigroup. If the elements t'_i are close enough to s'_i, then they are in U'. So there are elements $t_i \in U_2$ such that $T_n(t_i) = t'_i$. By the compactness of $\overline{U_2}$, and because U_2 and V are saturated by the fibers of T_n, if $t'_1, \dots, t'_k$ are close enough to $s'_1, \dots, s'_k$, then (24) gives

$$\overline{U_2} \subset \bigcup_{i<j} (t_i^{-1} V \cap t_j^{-1} V). \tag{25}$$

Now, we adapt the argument of the proof of [Breuillard and Gelander 2007, Lemma 10.6]. Let $\widehat{\Gamma} \subset \Gamma$ be the sub-local group generated by $t_1, \dots, t_k$; thus $\widehat{S} = \{t_1^{\pm 1}, \dots, t_k^{\pm 1}\}$ is a symmetric set of generators of $\widehat{\Gamma}$, and $S \cup \widehat{S}$ is a symmetric set of generators of Γ. With $\widehat{E} = \{h_{t_1}^{\pm 1}, \dots, h_{t_k}^{\pm 1}\}$, observe that $E_2 \cup \widehat{E}$ is a recurrent system of compact generation of $\mathcal{H}$ on U_2. Given $x \in U_2$, let $\mathsf{S}(n)$ be the sphere with center e and radius $n \in \mathbb{N}$ in $\widehat{\Gamma}_{U_2,x}$ with $d_{\widehat{S},U,x}$. By (25), for each $\gamma \in \mathsf{S}(n)$, there are indices $i < j$ such that $\gamma x \in t_i^{-1} V \cap t_j^{-1} V$. So the points $t_i \gamma x$ and $t_j \gamma x$ are in V, obtaining that $t_i\gamma, t_j\gamma \in \mathsf{S}(n+1)$. Moreover all elements obtained in this way from elements of $\mathsf{S}(n)$ are pairwise distinct because $t'_1, \dots, t'_k$ freely generate a free semigroup. Hence $\operatorname{card}(\mathsf{S}(n+1)) \geq 2 \operatorname{card}(\mathsf{S}(n))$, giving $\operatorname{card}(\mathsf{S}(n)) \geq 2^n$. So $(\widehat{\Gamma}_{U_2,x}, d_{\widehat{S},U_2,x})$ has exponential growth. Since $\widehat{\Gamma}_{U_2,x} \subset \Gamma_{U_2,x}$

and $d_{S\cup\widehat{S},U_2,x} \le d_{\widehat{S},U_2,x}$ on $\widehat{\Gamma}_{U_2,x}$, it follows that $(\Gamma_{U_2,x}, d_{S\cup\widehat{S},U_2,x})$ also has exponential growth. So $(\mathfrak{G}_x, d_{E_2\cup\widehat{E}})$ has exponential growth, obtaining that $(\mathfrak{G}_x, d_E)$ has exponential growth by Corollary 5.5. □

5D. *Growth of equicontinuous foliated spaces.* Let $X \equiv (X, \mathcal{F})$ be a compact Polish foliated space. Let $\{U_i, p_i, h_{ij}\}$ be a defining cocycle of $\mathcal{F}$, where $p_i : U_i \to T_i$ and $h_{ij} : T_{ij} \to T_{ji}$, and let $\mathcal{H}$ be the induced representative of the holonomy pseudogroup. As we saw in Section 4A, $\mathcal{H}$ can be considered as the restriction of some compactly generated pseudogroup $\mathcal{H}'$ to some relatively compact open subset, and $E = \{h_{ij}\}$ is a system of compact generation on T. Moreover, Álvarez and Candel [2009] observed that E is recurrent. According to Theorems 5.3 and 5.4, it follows that the quasi-isometry type of the $\mathcal{H}$-orbits and their germ covers with d_E are independent of the choice of $\{U_i, p_i, h_{ij}\}$ under the above conditions; thus they can be considered as quasi-isometry types of the corresponding leaves and their holonomy covers.

This has the following interpretation when X is a smooth manifold. In this case, given any Riemannian metric g on X, for each leaf L, the differentiable (and coarse) quasi-isometry type of $g|_L$ is independent of the choice of g; they depend only on $\mathcal{F}$ and L. In fact, it is coarsely quasi-isometric to the corresponding $\mathcal{H}$-orbit, and therefore they have the same growth type [Carrière 1988] (this is an easy consequence of the existence of a uniform bound of the diameter of the plaques). Similarly, the germ covers of the $\mathcal{H}$-orbits are also quasi-isometric to the holonomy covers of the corresponding leaves.

Theorem B follows from these observations and Theorem 5.8.

6. Examples and open problems

Theorems A and B may be relevant in the following examples; most of them are taken from [Candel and Conlon 2000, Chapter 11].

Example 6.1. Any locally free action of a connected Lie group on a locally compact Polish space, $\phi : H \times X \to X$, defines a foliated structure $\mathcal{F}$ on X whose leaves are the orbits [Candel and Conlon 2000, Theorem 11.3.14; Palais 1961]. Moreover $\mathcal{F}$ is equicontinuous if ϕ is equicontinuous.

Example 6.2. A *matchbox manifold* is a foliated continuum[11] $X \equiv (X, \mathcal{F})$ transversely modeled on a totally disconnected space. The case of a single leaf is discarded, and it is assumed that X is C^1 in the sense that the changes of foliated coordinates are C^1 along the leaves, with transversely continuous leafwise derivatives. An example of matchbox manifold is given by any inverse limit of smooth proper covering maps of compact n-manifolds, called an n-dimensional

[11] Recall that a *continuum* is a nonempty compact connected metrizable space.

solenoid; if moreover any composite of a finite number of bounding maps is a normal covering, then it is called a *McCord solenoid*. A matchbox manifold X is equicontinuous if and only if it is a solenoid [Clark and Hurder 2013, Theorem 7.9]; and X is homogeneous if and only if it is a McCord solenoid [Clark and Hurder 2013, Theorem 1.1]; this is the case where it is a G-foliated space. See [Alcalde Cuesta et al. 2011] for a generalization using inverse limits of compact branched manifolds.

Example 6.3. Let $C_b(\mathbb{R})$ be the space of continuous bounded functions $\mathbb{R} \to \mathbb{R}$, with the topology of uniform convergence. For a function $f \in C_b(\mathbb{R})$ and $t \in \mathbb{R}$, let $f_t \in C_b(\mathbb{R})$ be defined by $f_t(r) = f(r+t)$. It is said that f is *almost periodic* if $\{f_t \mid t \in \mathbb{R}\}$ is equicontinuous [Besicovitch 1955; Gottschalk 1946], which means that $\mathfrak{M}(f) := \overline{\{f_t \mid t \in \mathbb{R}\}}$ is compact in $C_b(\mathbb{R})$. An equicontinuous flow

$$\Phi : \mathbb{R} \times \mathfrak{M}(f) \to \mathfrak{M}(f)$$

is defined by $\Phi_t(g) = g_t$. If f is nonconstant, then Φ is nonsingular, defining an equicontinuous foliated structure $\mathscr{F}$ on $\mathfrak{M}(f)$. If f is nonperiodic, then $\mathscr{F}$ does not reduce to a single leaf. With more generality, we can consider an almost-periodic nonperiodic continuous function f on any connected Lie group with values in a Hilbert space.

Example 6.4. For each $n \in \mathbb{Z}_+$, let $\mathcal{M}_*(n)$ denote the set[12] of isometry classes $[M, x]$ of pointed complete connected Riemannian n-manifolds (M, x). The C^∞ convergence [Petersen 1998] defines a Polish topology on $\mathcal{M}_*(n)$ [Álvarez et al. 2016, Theorem 1.1]. The corresponding space is denoted by $\mathcal{M}_*^\infty(n)$, and its closure operator by Cl_∞. For any complete connected Riemannian manifold $M \equiv (M, g)$, a canonical continuous map $\iota : M \to \mathcal{M}_*^\infty(n)$ is defined by $\iota(x) = [M, x]$. A concept of *weak aperiodicity* of M was introduced in [Álvarez et al. 2016]. On the other hand, M is called *almost periodic* if, for all $m \in \mathbb{N}$, $\epsilon > 0$ and $x \in M$, there is a set H of diffeomorphisms of M such that $\sup |\nabla^k h^* g| < \epsilon$ for all $h \in H$ and $k \le m$, and $\{h(x) \mid h \in H\}$ is a net in M. If M is weakly aperiodic and almost periodic, then $\mathrm{Cl}_\infty(\iota(M))$ canonically becomes a compact minimal equicontinuous foliated space of dimension n, as follows from [Álvarez et al. 2016, Theorem 1.2 and Lemma 12.5(ii); Lessa 2015, Lemma 7.2 and Theorem 4.1] (see also [Petersen 1998, Chapter 10, Theorem 3.3; Cheeger 1970]).

Problem 6.5. In the Examples 6.1–6.4, understand the specific application of Theorems A and B.

Problem 6.6. Use Theorem A to classify particular classes of equicontinuous foliated spaces.

[12]The logical problems of this definition can be avoided because any complete connected Riemannian manifold is equipotent to some subset of $\mathbb{R}$.

Question 6.7. Is it possible to improve Theorem B for special types of structural local groups?

Question 6.8. Is there any consequence of Theorems A and B in usual foliation theory, assuming that the minimal sets are equicontinuous?

The following questions refer to extensions of known properties of Riemannian foliations, where Theorem A could play an important role.

Question 6.9. For compact minimal equicontinuous foliated spaces, does the leafwise heat flow of leafwise differential forms preserve transverse continuity at infinite time?

Question 6.10. Is it possible to give useful extensions of tautness and tenseness to equicontinuous foliated spaces, and relate them to some kind of basic cohomology?

Acknowledgement

We thank the referee for suggestions to improve the paper.

References

[Abd-Allah and Brown 1980] A. M. Abd-Allah and R. Brown, "A compact-open topology on partial maps with open domain", *J. London Math. Soc.* (2) **21**:3 (1980), 480–486. MR 81i:54010 Zbl 0436.54012

[Alcalde Cuesta et al. 2011] F. Alcalde Cuesta, Á. Lozano Rojo, and M. Macho Stadler, "Transversely Cantor laminations as inverse limits", *Proc. Amer. Math. Soc.* **139**:7 (2011), 2615–2630. MR 2784831 Zbl 1229.57029

[Álvarez and Candel 2009] J. A. Álvarez López and A. Candel, "Equicontinuous foliated spaces", *Math. Z.* **263**:4 (2009), 725–774. MR 2010i:53040 Zbl 1177.53026

[Álvarez and Candel 2010] J. A. Álvarez López and A. Candel, "Topological description of Riemannian foliations with dense leaves", *Pacific J. Math.* **248**:2 (2010), 257–276. MR 2012b:53043 Zbl 1207.53033

[Álvarez and Candel 2015] J. A. Álvarez López and A. Candel, "Generic coarse geometry of leaves", preprint, 2015. arXiv 1406.1746

[Álvarez and Masa 2008] J. A. Álvarez López and X. M. Masa, "Morphisms between complete Riemannian pseudogroups", *Topology Appl.* **155**:6 (2008), 544–604. MR 2009c:58028 Zbl 1147.57025

[Álvarez and Wolak 2013] J. A. Álvarez López and R. Wolak, "Growth of some transversely homogeneous foliations", preprint, 2013. arXiv 1304.1939

[Álvarez et al. 2016] J. A. Álvarez López, R. Barral Lijó, and A. Candel, "A universal Riemannian foliated space", *Topology Appl.* **198** (2016), 47–85. MR 3433188 Zbl 06521163

[Arens 1946] R. Arens, "Topologies for homeomorphism groups", *Amer. J. Math.* **68** (1946), 593–610. MR 8,479i Zbl 0061.24306

[Besicovitch 1955] A. S. Besicovitch, *Almost periodic functions*, Dover, New York, 1955. MR 16,817a Zbl 0065.07102

[Block and Weinberger 1997] J. Block and S. Weinberger, "Large scale homology theories and geometry", pp. 522–569 in *Geometric topology* (Athens, GA, 1993), edited by W. H. Kazez, AMS/IP Stud. Adv. Math. **2**, American Mathematical Society, Providence, RI, 1997. MR 98h:57001 Zbl 0898.55006

[Breuillard and Gelander 2007] E. Breuillard and T. Gelander, "A topological Tits alternative", *Ann. of Math.* (2) **166**:2 (2007), 427–474. MR 2009a:20077 Zbl 1149.20039

[Candel and Conlon 2000] A. Candel and L. Conlon, *Foliations I*, Graduate Studies in Mathematics **23**, American Mathematical Society, Providence, RI, 2000. MR 2002f:57058 Zbl 0936.57001

[Carrière 1988] Y. Carrière, "Feuilletages Riemanniens à croissance polynômiale", *Comment. Math. Helv.* **63**:1 (1988), 1–20. MR 89a:57033 Zbl 0661.53022

[Cheeger 1970] J. Cheeger, "Finiteness theorems for Riemannian manifolds", *Amer. J. Math.* **92** (1970), 61–74. MR 41 #7697 Zbl 0194.52902

[Clark and Hurder 2013] A. Clark and S. Hurder, "Homogeneous matchbox manifolds", *Trans. Amer. Math. Soc.* **365**:6 (2013), 3151–3191. MR 3034462 Zbl 1285.57017

[van den Dries and Goldbring 2010] L. van den Dries and I. Goldbring, "Globalizing locally compact local groups", *J. Lie Theory* **20**:3 (2010), 519–524. MR 2011m:22009 Zbl 1203.22006

[van den Dries and Goldbring 2012] L. van den Dries and I. Goldbring, "Erratum to 'Globalizing locally compact local groups'", *J. Lie Theory* **22**:2 (2012), 489–490. MR 2976929 Zbl 1242.22005

[Fedida 1971] E. Fedida, "Sur les feuilletages de Lie", *C. R. Acad. Sci. Paris Sér. A* **272** (1971), 999–1001. MR 44 #2249 Zbl 0218.57014

[Fedida 1978] E. Fedida, "Sur la théorie des feuilletages associée au repère mobile: cas des feuilletages de Lie", pp. 183–195 in *Differential topology, foliations and Gelfand–Fuks cohomology* (Rio de Janeiro, 1976), edited by P. A. Schweitzer, Lecture Notes in Math. **652**, Springer, Berlin, 1978. MR 80a:57012 Zbl 0393.57006

[Ghys 1985] É. Ghys, "Groupes d'holonomie des feuilletages de Lie", *Indag. Math.* **88**:2 (1985), 173–182. MR 87d:57021 Zbl 0571.57020

[Ghys 1999] É. Ghys, "Laminations par surfaces de Riemann", pp. 49–95 in *Dynamique et géométrie complexes* (Lyon, 1997), edited by D. Cerveau et al., Panor. Synthèses **8**, Société Mathématique de France, Paris, 1999. MR 2001g:37068 Zbl 1018.37028

[Goldbring 2010] I. Goldbring, "Hilbert's fifth problem for local groups", *Ann. of Math.* (2) **172**:2 (2010), 1269–1314. MR 2011f:22010 Zbl 1219.22004

[Gottschalk 1946] W. H. Gottschalk, "Almost periodicity, equi-continuity and total boundedness", *Bull. Amer. Math. Soc.* **52** (1946), 633–636. MR 8,34b Zbl 0063.01711

[Gromov 1993] M. Gromov, *Geometric group theory, 2: Asymptotic invariants of infinite groups* (Brighton, 1991), edited by G. A. Niblo and M. A. Roller, London Math. Soc. Lecture Note Ser. **182**, Cambridge University Press, 1993. MR 95m:20041 Zbl 0841.20039

[Haefliger 1985] A. Haefliger, "Pseudogroups of local isometries", pp. 174–197 in *Differential geometry* (Santiago de Compostela, 1984), edited by L. A. Cordero, Res. Notes in Math. **131**, Pitman, Boston, 1985. MR 88i:58174 Zbl 0656.58042

[Haefliger 1988] A. Haefliger, "Leaf closures in Riemannian foliations", pp. 3–32 in *A fête of topology*, edited by Y. Matsumoto et al., Academic Press, Boston, 1988. MR 89c:57033 Zbl 0667.57012

[Haefliger 2002] A. Haefliger, "Foliations and compactly generated pseudogroups", pp. 275–295 in *Foliations: geometry and dynamics* (Warsaw, 2000), edited by P. Walczak et al., World Scientific, River Edge, NJ, 2002. MR 2003g:58029 Zbl 1002.57059

[Hector and Hirsch 1981] G. Hector and U. Hirsch, *Introduction to the geometry of foliations, A: Foliations on compact surfaces, fundamentals for arbitrary codimension, and holonomy*, Aspects of Mathematics **1**, Vieweg, Braunschweig, 1981. MR 83d:57019 Zbl 0486.57002

[Hector and Hirsch 1987] G. Hector and U. Hirsch, *Introduction to the geometry of foliations, B: Foliations of codimension one*, 2nd ed., Aspects of Mathematics **3**, Vieweg, Braunschweig, 1987. MR 92f:57037 Zbl 0704.57001

[Jacoby 1957] R. Jacoby, "Some theorems on the structure of locally compact local groups", *Ann. of Math.* (2) **66** (1957), 36–69. MR 19,751c Zbl 0084.03202

[Kechris 1991] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics **156**, Springer, Berlin, 1991. MR 96e:03057 Zbl 0819.04002

[Kellum 1993] M. Kellum, "Uniformly quasi-isometric foliations", *Ergodic Theory Dynam. Systems* **13**:1 (1993), 101–122. MR 94c:57048 Zbl 0784.58060

[Lessa 2015] P. Lessa, "Reeb stability and the Gromov–Hausdorff limits of leaves in compact foliations", *Asian J. Math.* **19**:3 (2015), 433–463. MR 3361278 Zbl 06470466

[Matsumoto 2010] S. Matsumoto, "The unique ergodicity of equicontinuous laminations", *Hokkaido Math. J.* **39**:3 (2010), 389–403. MR 2011m:37038 Zbl 1213.37044

[Meigniez 1995] G. Meigniez, "Feuilletages de Lie résolubles", *Ann. Fac. Sci. Toulouse Math.* (6) **4**:4 (1995), 801–817. MR 99e:57042 Zbl 0867.57023

[Meigniez 2010] G. Meigniez, "A compactly generated pseudogroup which is not realizable", *J. Math. Soc. Japan* **62**:4 (2010), 1205–1218. MR 2011m:58028 Zbl 1207.57038

[Molino 1982] P. Molino, "Géométrie globale des feuilletages Riemanniens", *Indag. Math.* **85**:1 (1982), 45–76. MR 84j:53043 Zbl 0516.57016

[Molino 1988] P. Molino, *Riemannian foliations*, Progress in Mathematics **73**, Birkhäuser, Boston, 1988. MR 89b:53054 Zbl 0633.53001

[Montgomery and Zippin 1955] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience Publishers, New York, 1955. MR 17,383b Zbl 0068.01904

[Moore and Schochet 1988] C. C. Moore and C. Schochet, *Global analysis on foliated spaces*, Mathematical Sciences Research Institute Publications **9**, Springer, New York, 1988. MR 89h:58184 Zbl 0648.58034

[Olver 1996] P. J. Olver, "Non-associative local Lie groups", *J. Lie Theory* **6**:1 (1996), 23–51. MR 98d:22005 Zbl 0862.22005

[Palais 1961] R. S. Palais, "On the existence of slices for actions of non-compact Lie groups", *Ann. of Math.* (2) **73** (1961), 295–323. MR 23 #A3802 Zbl 0103.01802

[Petersen 1998] P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics **171**, Springer, New York, 1998. MR 98m:53001 Zbl 0914.53001

[Plaut 1993] C. Plaut, "Associativity and the local version of Hilbert's fifth problem", Notes, University of Tennessee, 1993.

[Reinhart 1959] B. L. Reinhart, "Foliated manifolds with bundle-like metrics", *Ann. of Math.* (2) **69** (1959), 119–132. MR 21 #6004 Zbl 0122.16604

[Sacksteder 1965] R. Sacksteder, "Foliations and pseudogroups", *Amer. J. Math.* **87** (1965), 79–102. MR 30 #4268 Zbl 0136.20903

[Salem 1988] É. Salem, "Une généralisation du théorème de Myers–Steenrod aux pseudogroupes d'isométries", *Ann. Inst. Fourier* (*Grenoble*) **38**:2 (1988), 185–200. MR 89i:58166 Zbl 0613.58041

[Tarquini 2004] C. Tarquini, "Feuilletages conformes", *Ann. Inst. Fourier* (*Grenoble*) **54**:2 (2004), 453–480. MR 2005f:53040 Zbl 1064.53014

Received May 22, 2014. Revised November 9, 2014.

Jesús A. Álvarez López
Departamento de Xeometría e Topoloxía
Facultade de Matemáticas
Universidade de Santiago de Compostela
15782 Santiago de Compostela
Spain

jesus.alvarez@usc.es

Manuel F. Moreira Galicia
Departamento de Xeometría e Topoloxía
Facultade de Matemáticas
Universidade de Santiago de Compostela
15782 Santiago de Compostela
Spain

manuel.moreira@usc.es

morgal2002@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2016 is US $440/year for the electronic version, and $600/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 280 No. 2 February 2016